Hyperbolic Random Graphs

Degree Sequence and Clustering

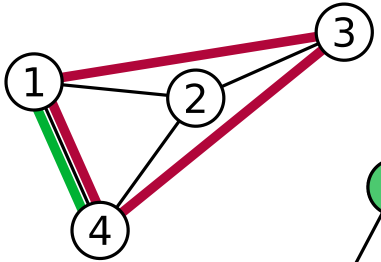
Algorithm Engineering Group

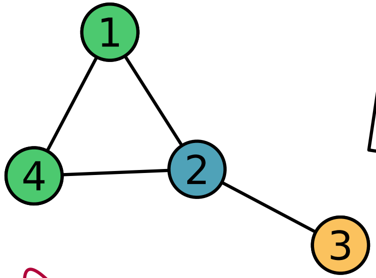
Janosch Ruff

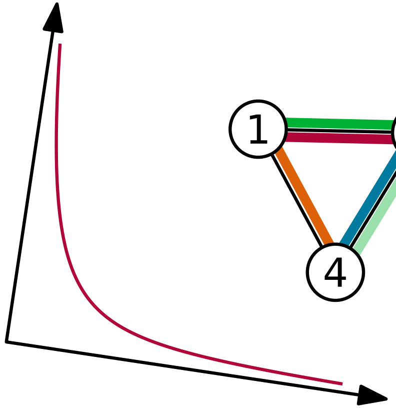
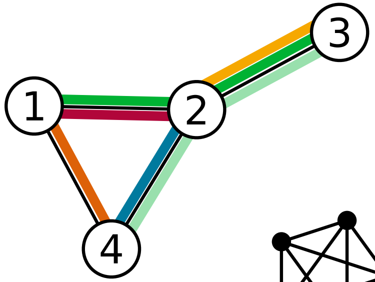
Network Features - Distributions

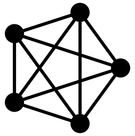
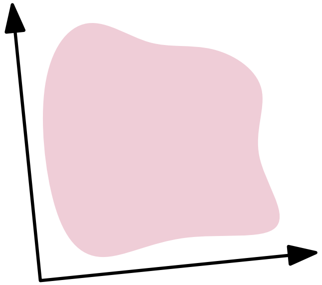
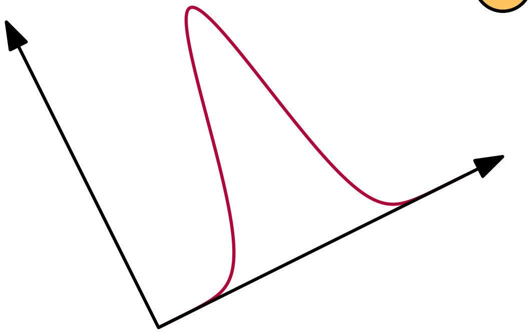
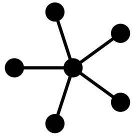
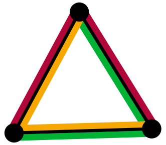
The most commonly considered network features are distributions over parts of the network.

- Degrees
- Assortativity
- Distances
- Centrality
- Clustering



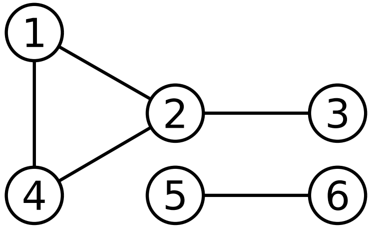


[The Structure and Function of Complex Networks, M. E. J. Newman, Computer Physics Communications 2003]

Network Features – Degree Distribution



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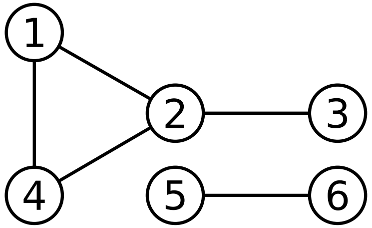

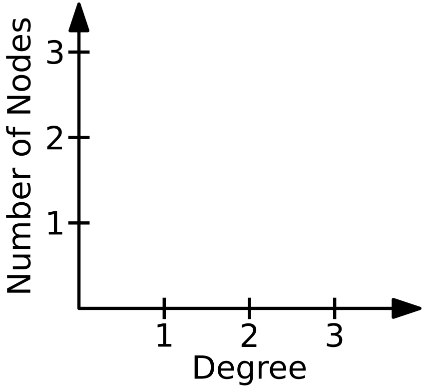

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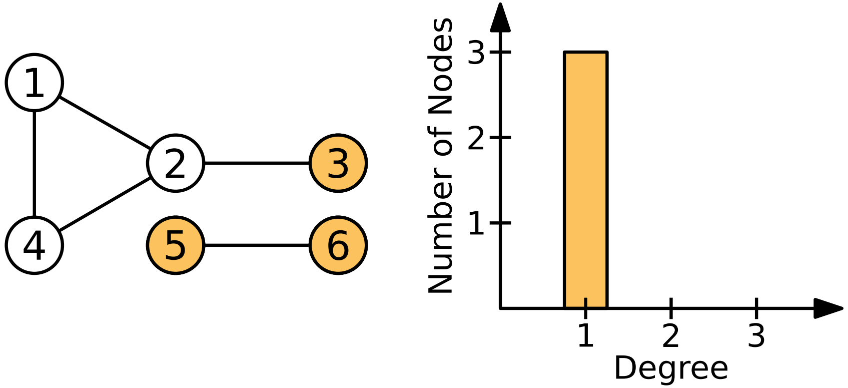

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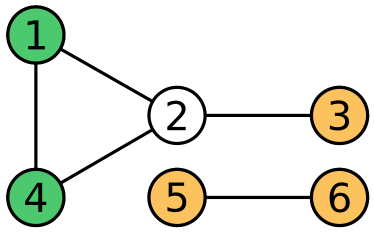

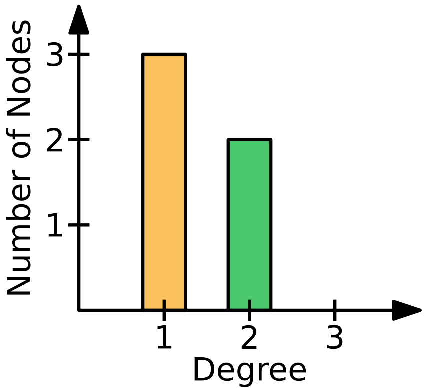

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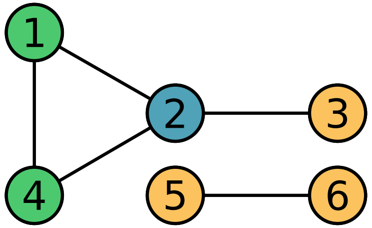

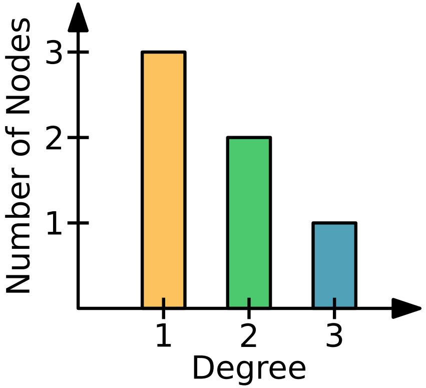

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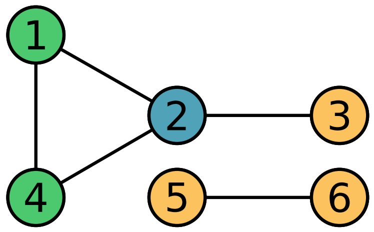

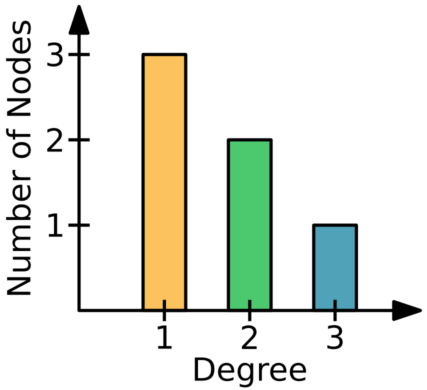

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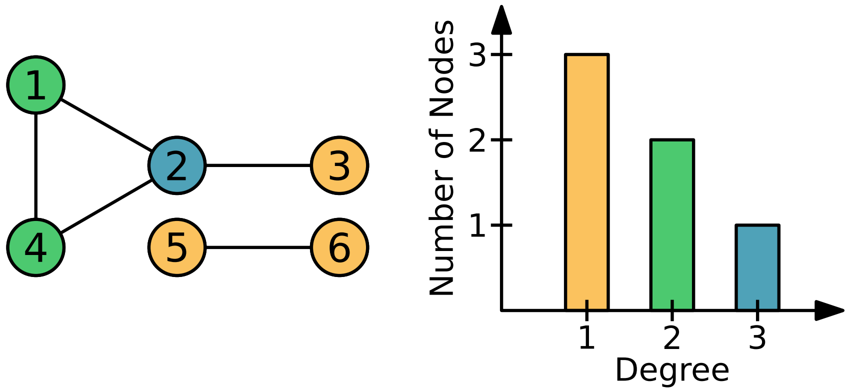

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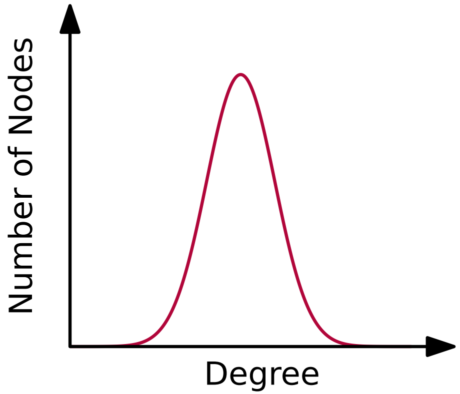

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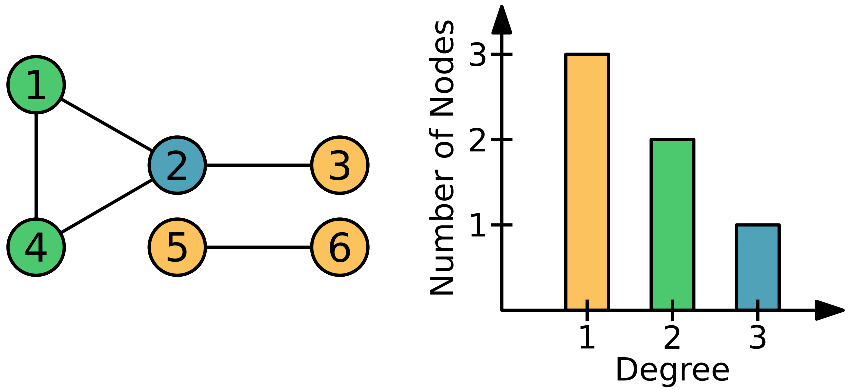

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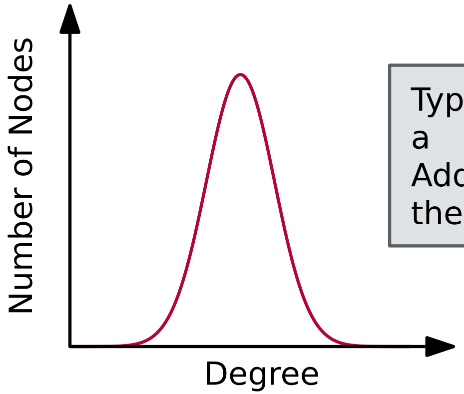

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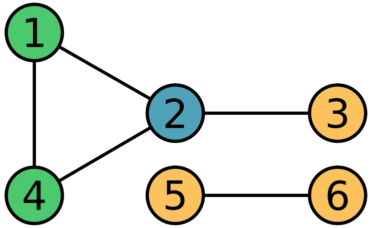

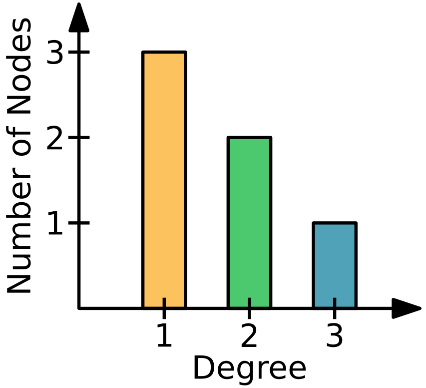

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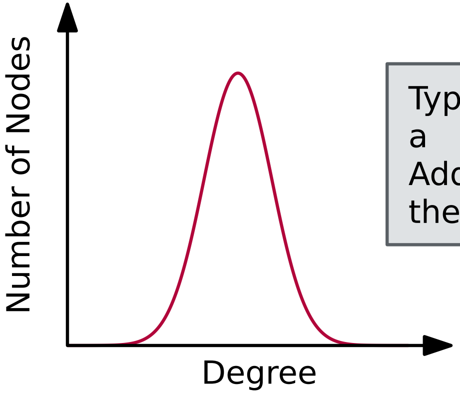

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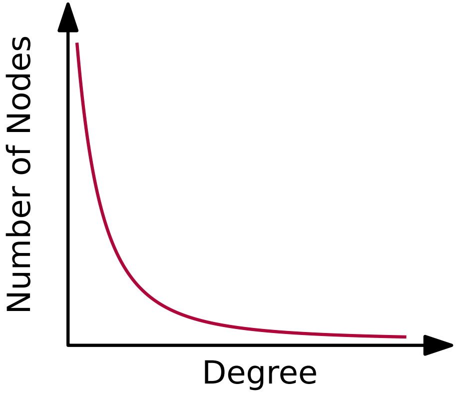

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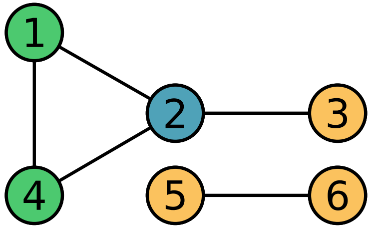

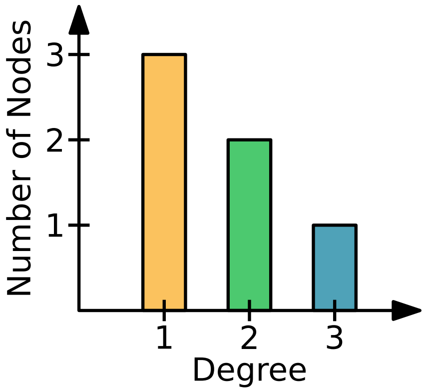

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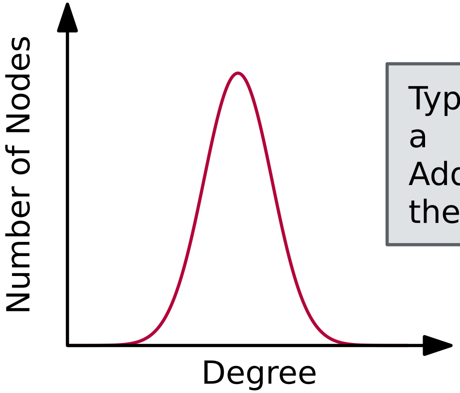

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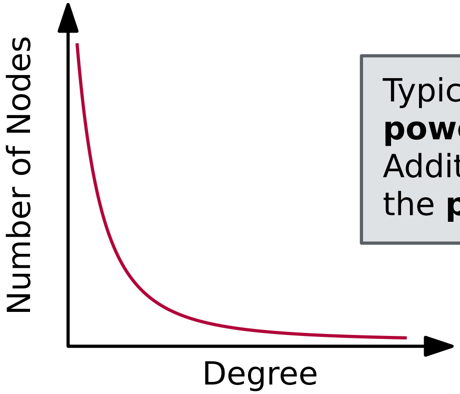

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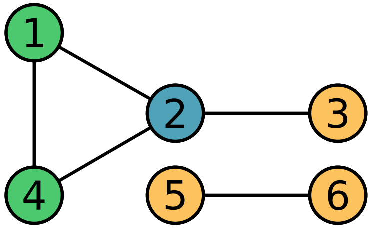

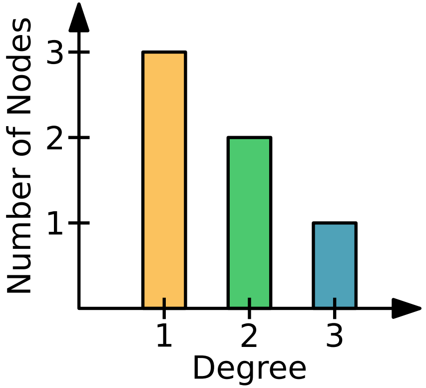

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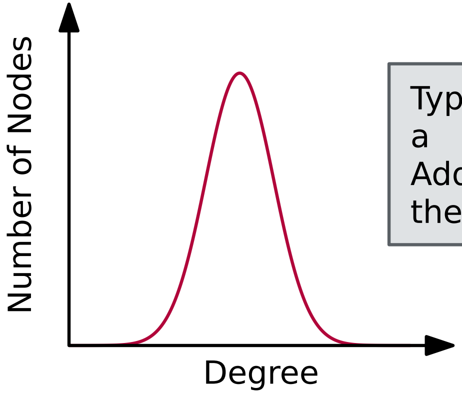

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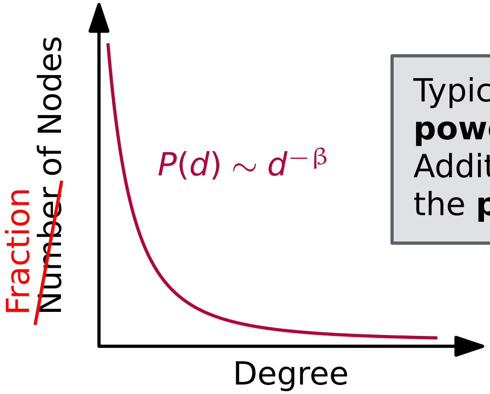

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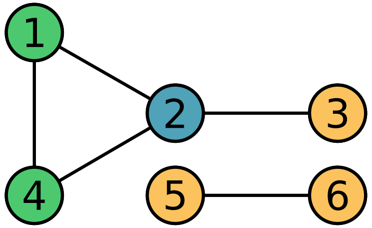

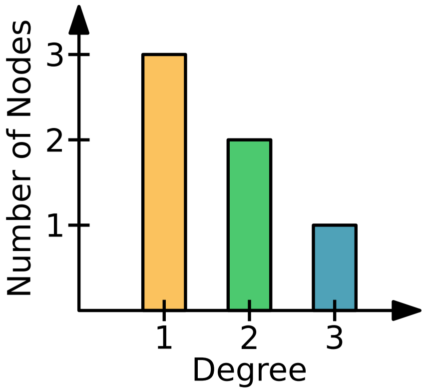

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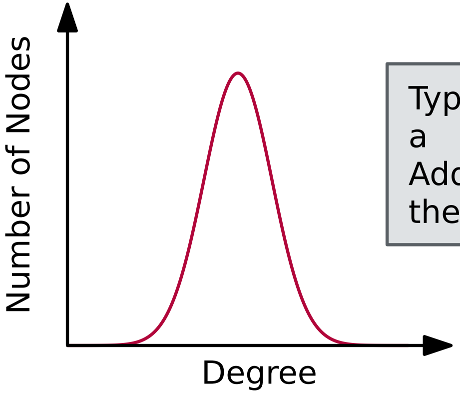

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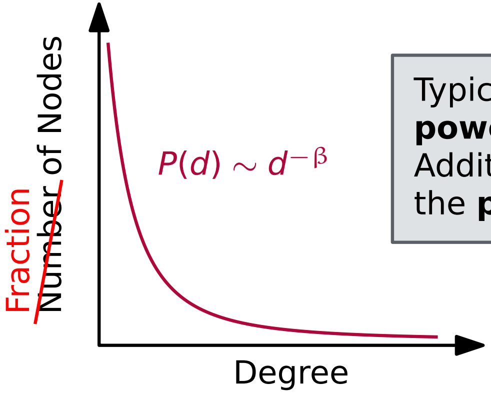

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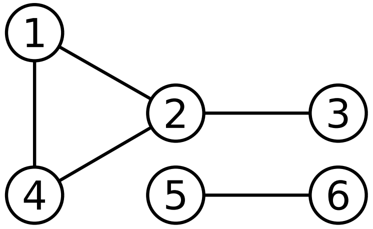

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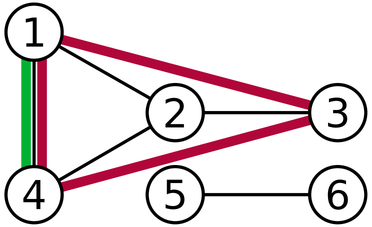

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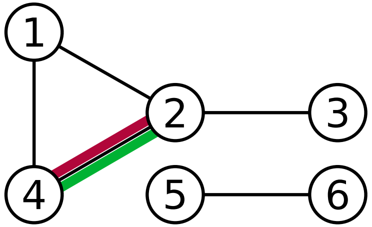

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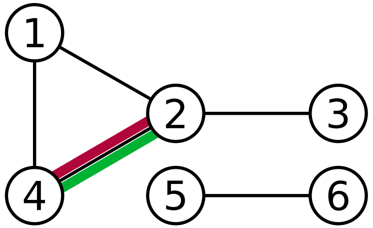

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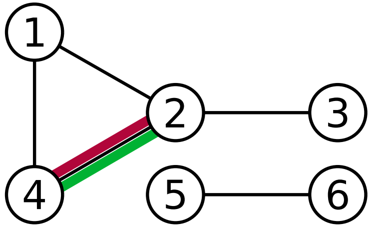

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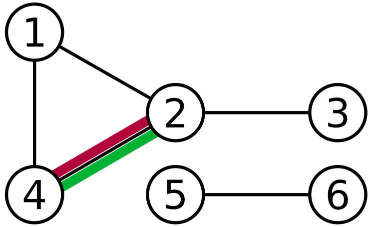

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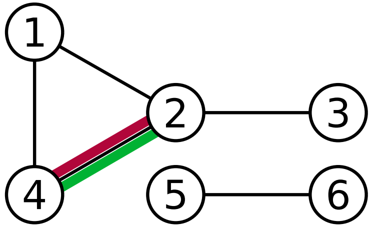

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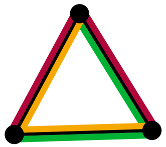

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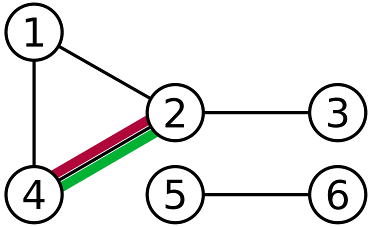

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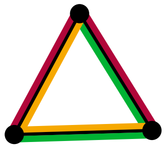

Compared to the average local clustering coefficient, the global one basically computes the ratio of the means rather than the mean of the ratios. It does not weight the contributions of low-degree nodes as much.

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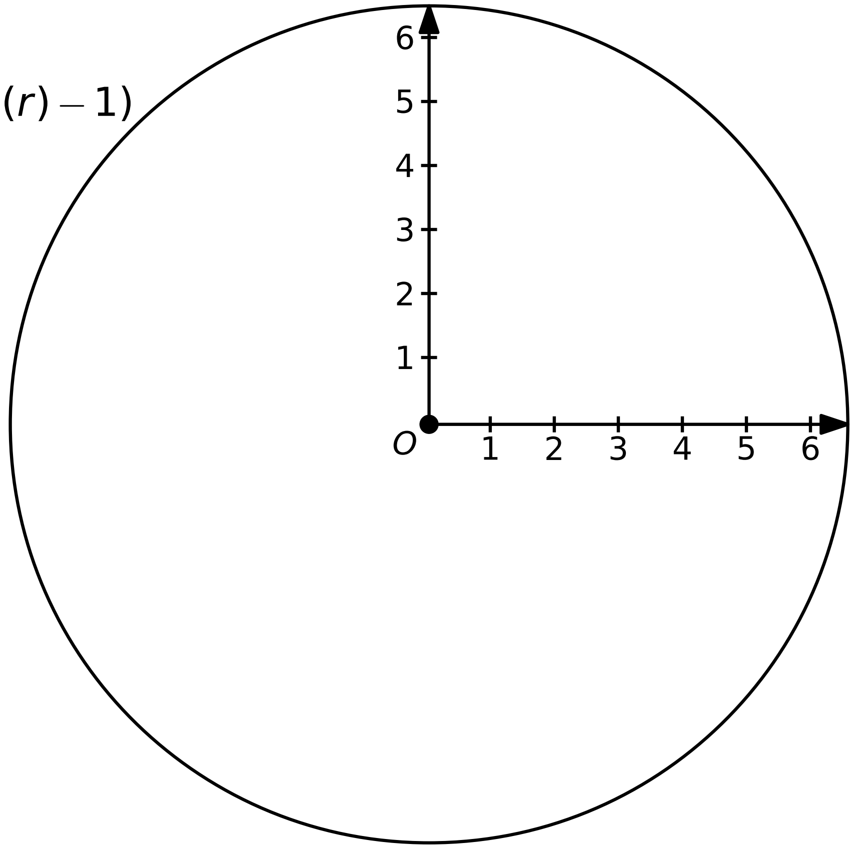

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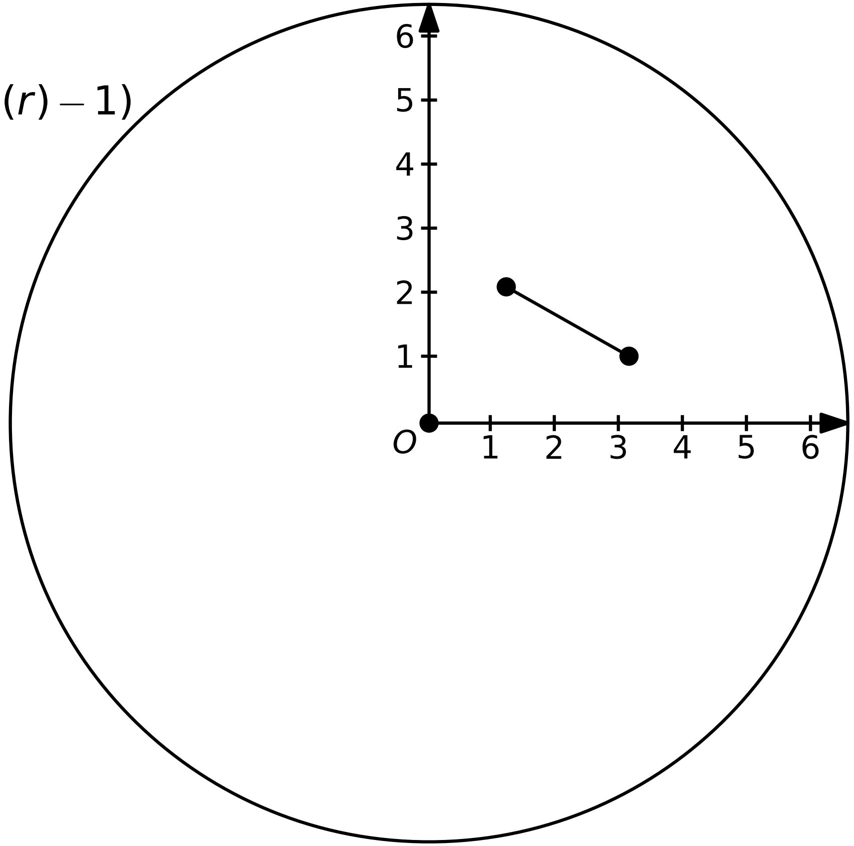

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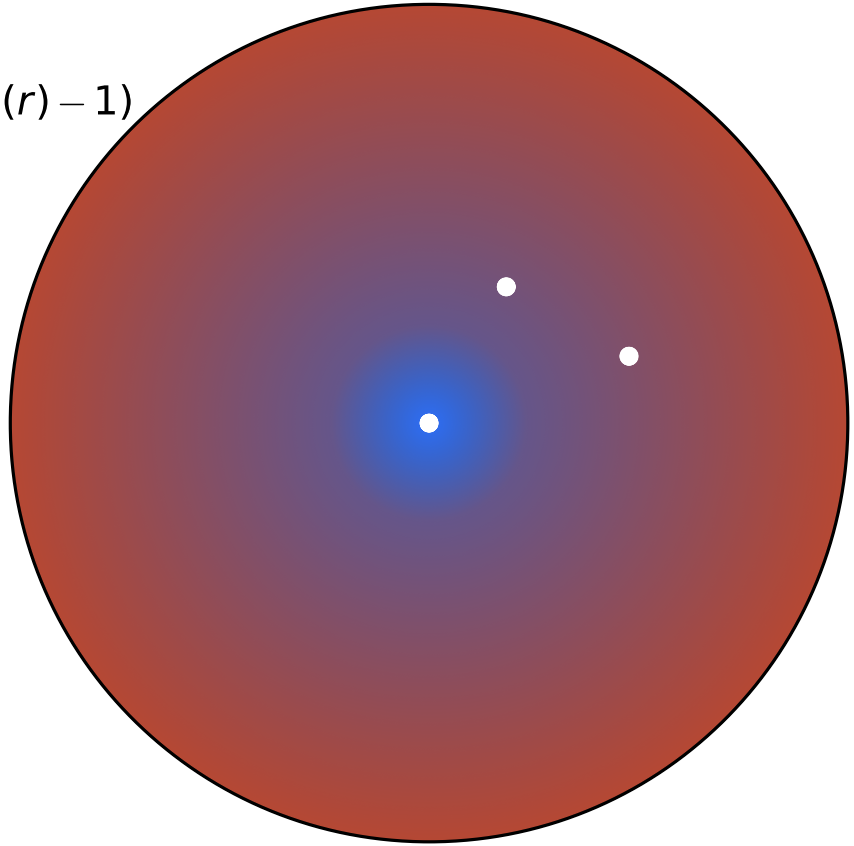

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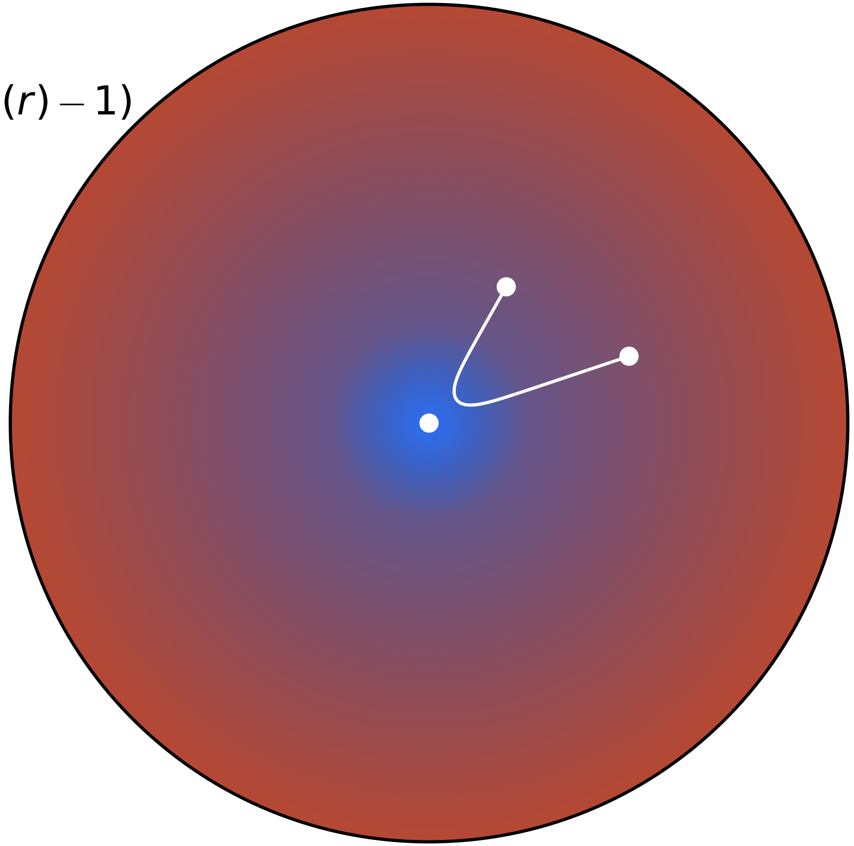

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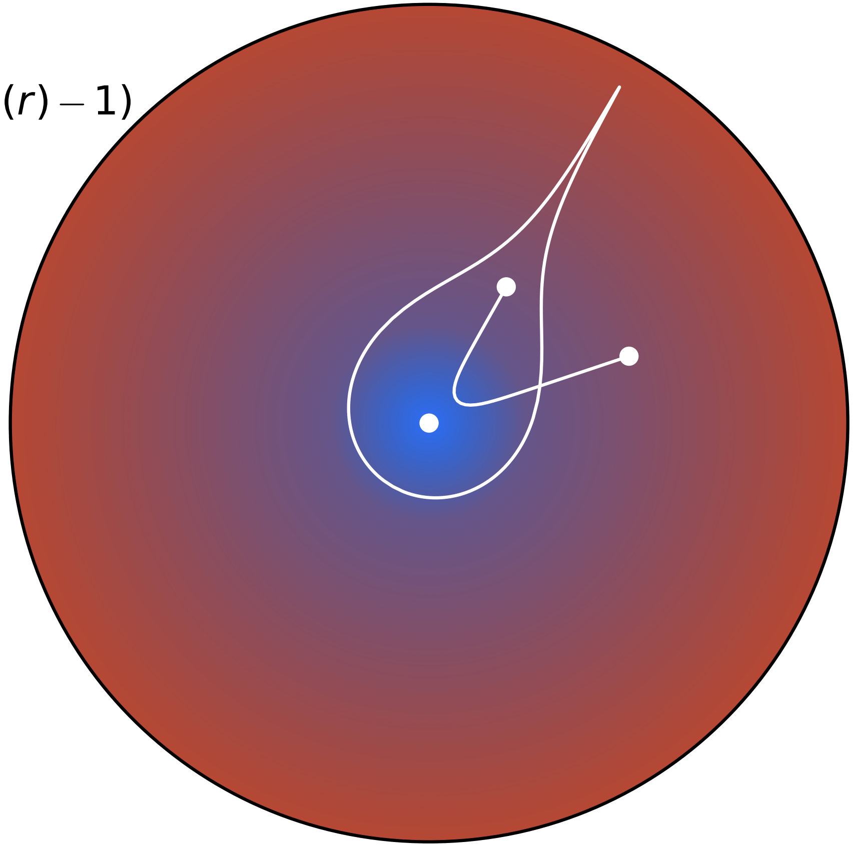

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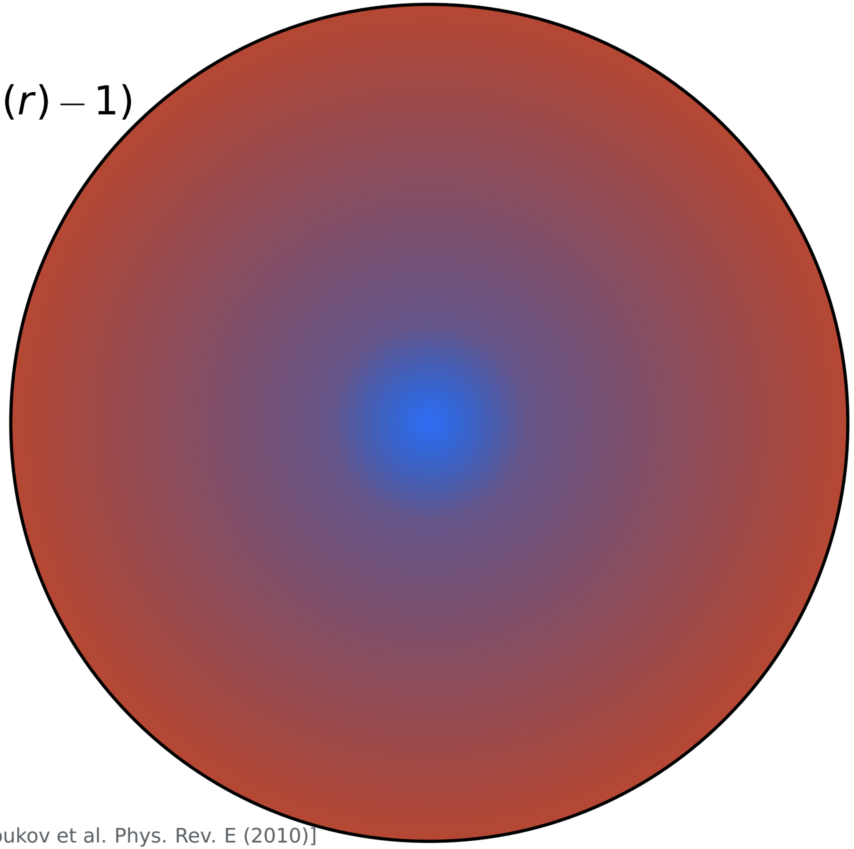

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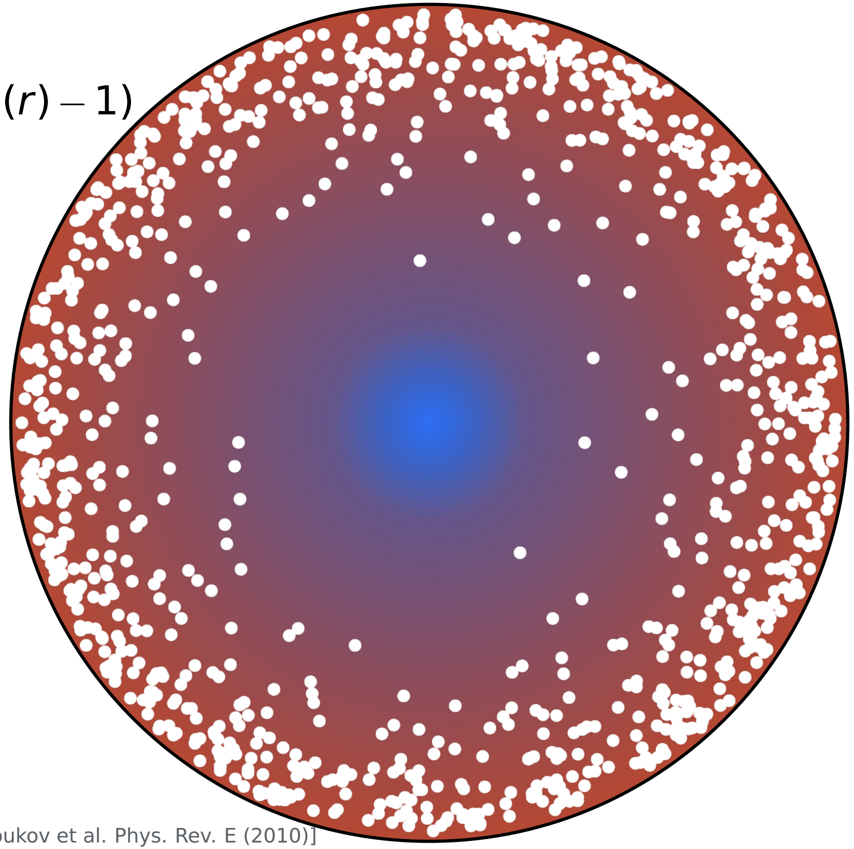

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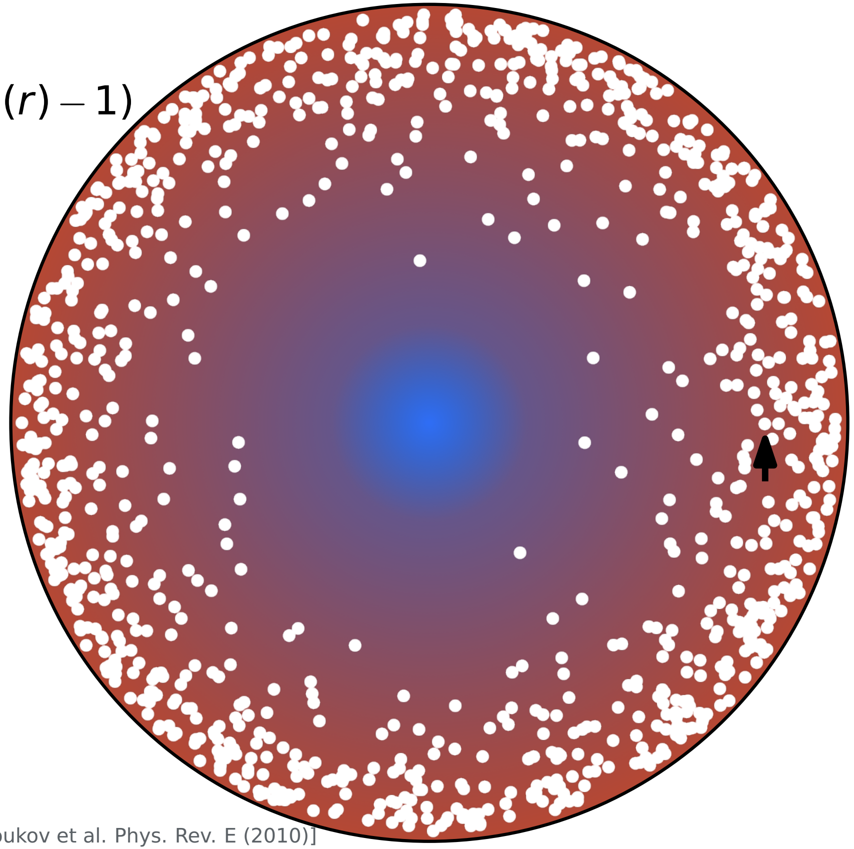

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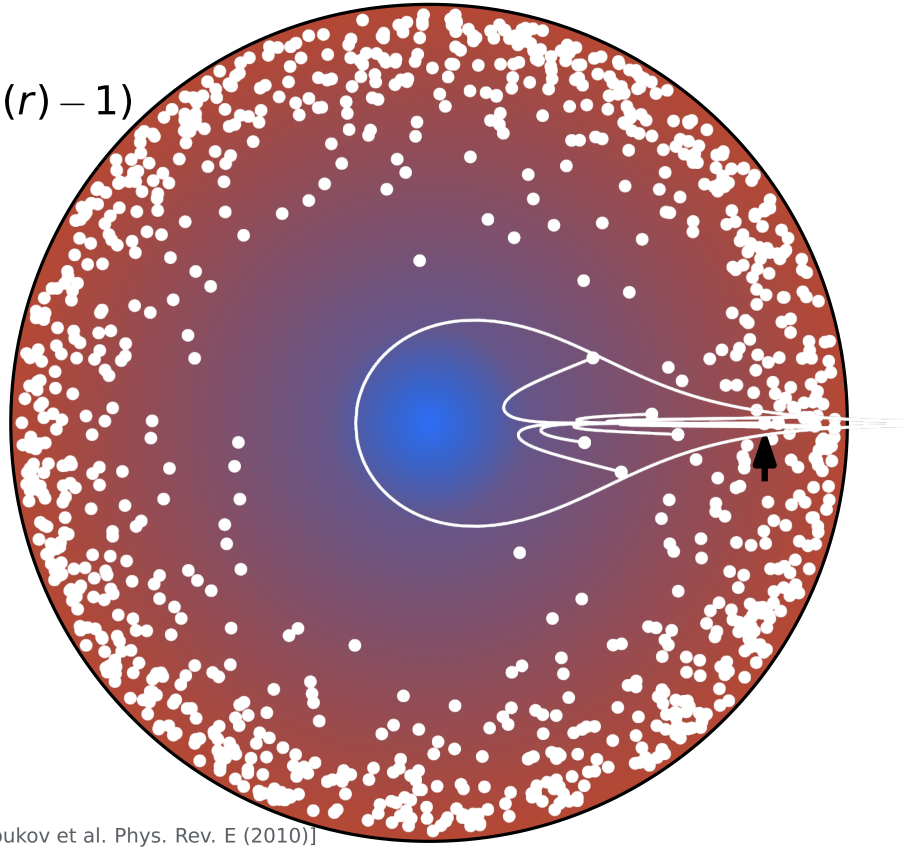

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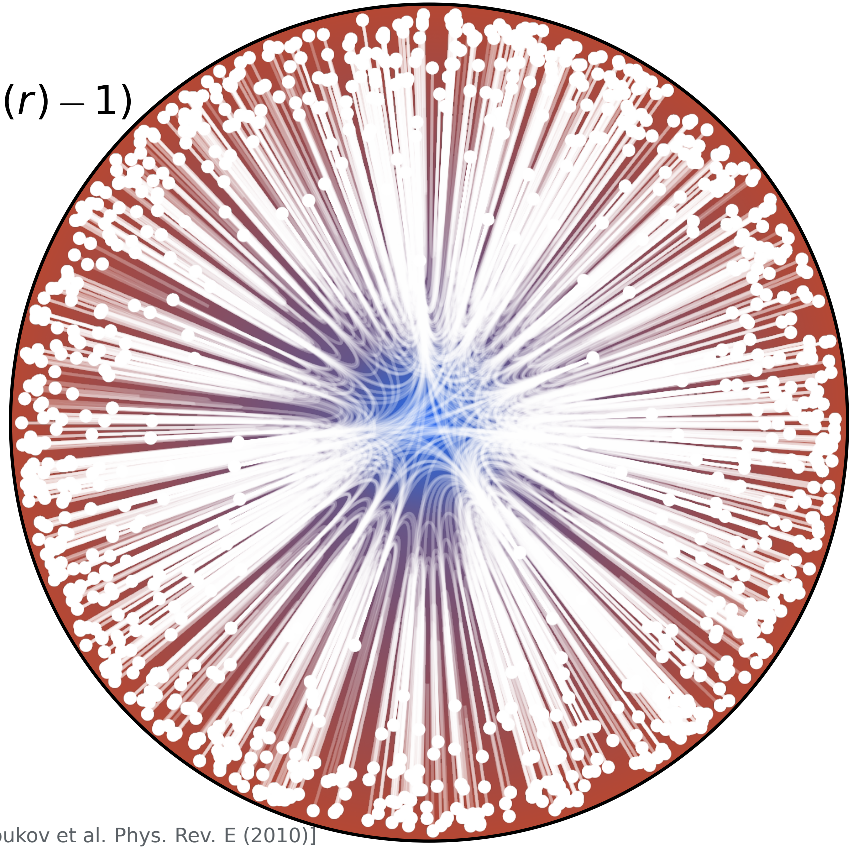

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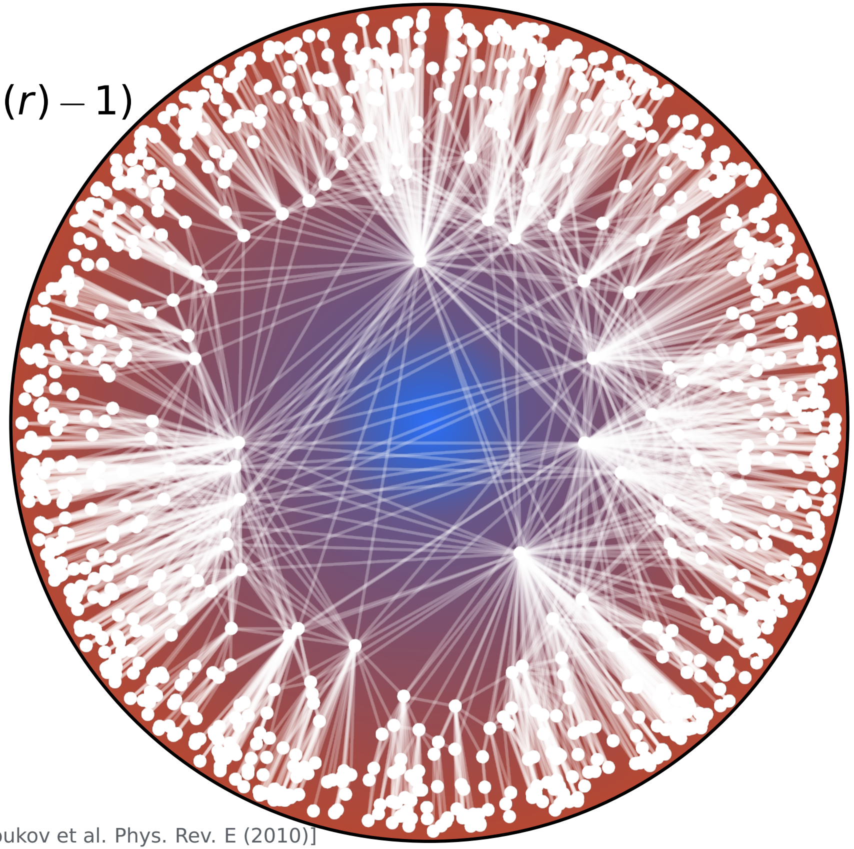

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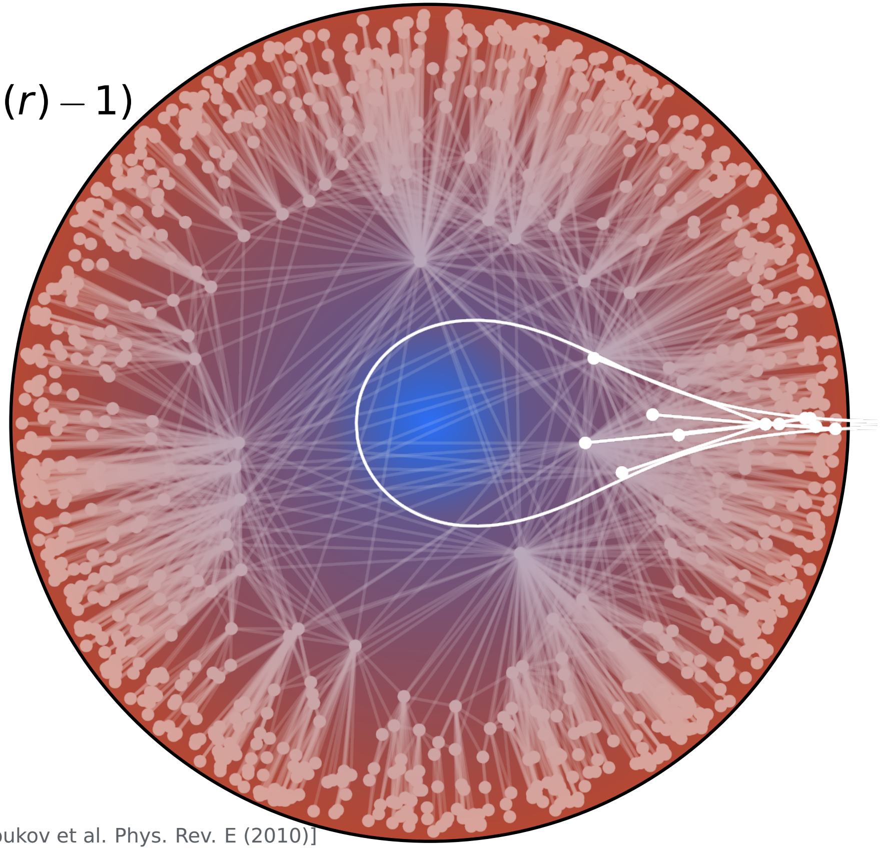

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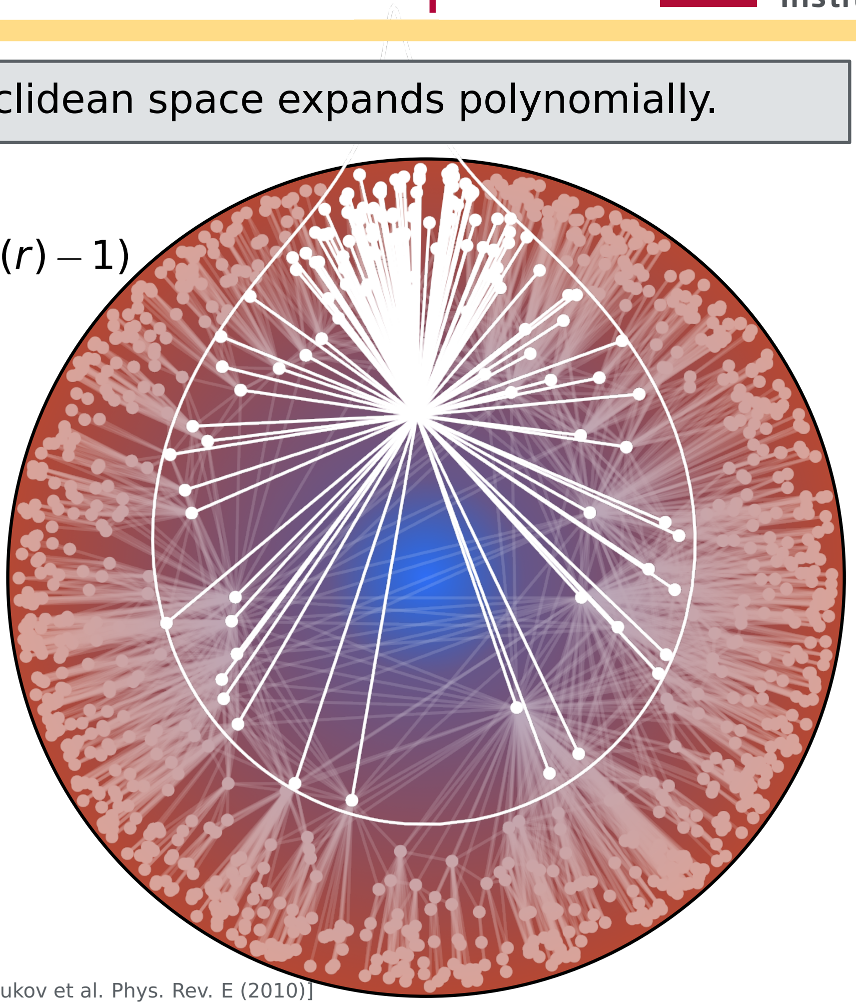

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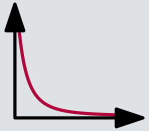

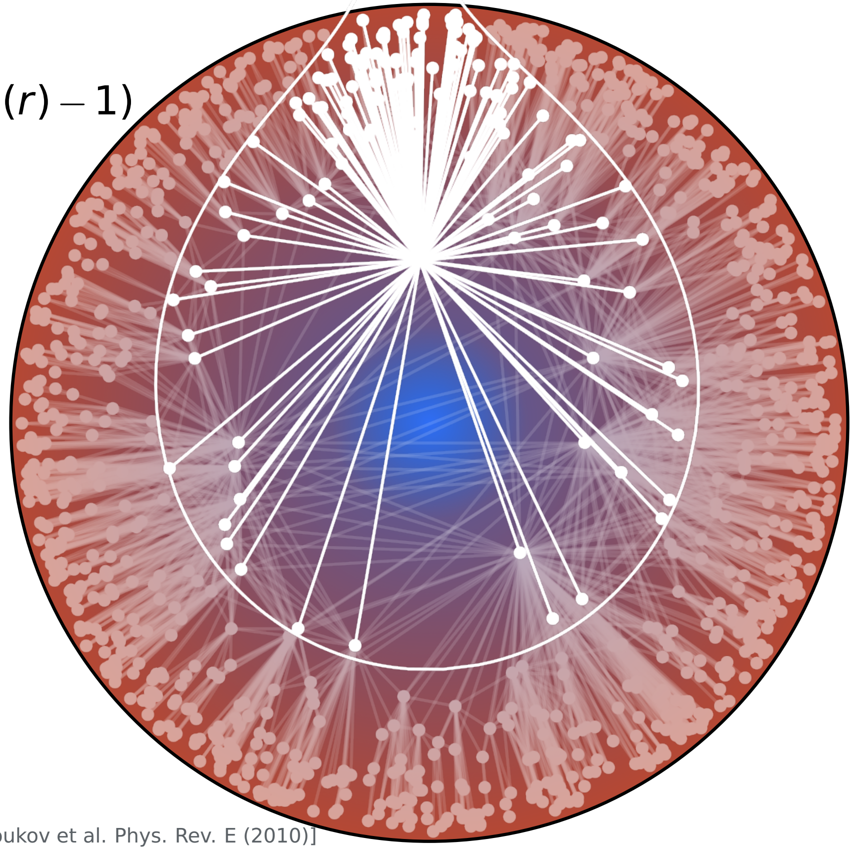

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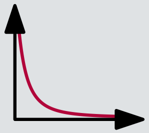

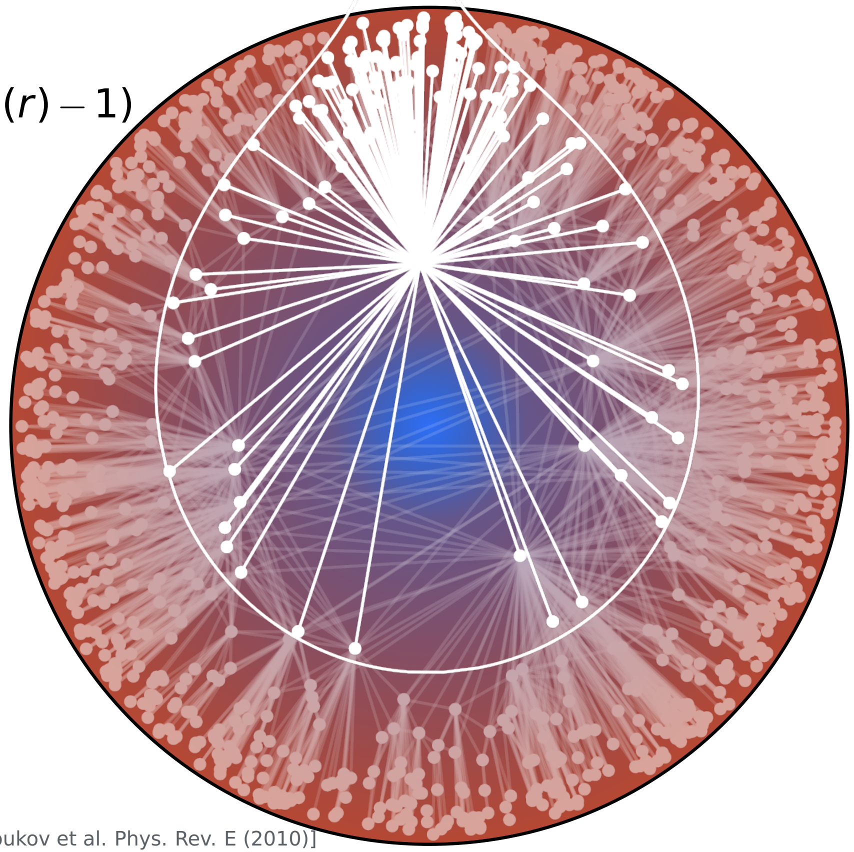

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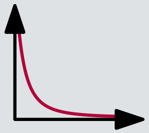

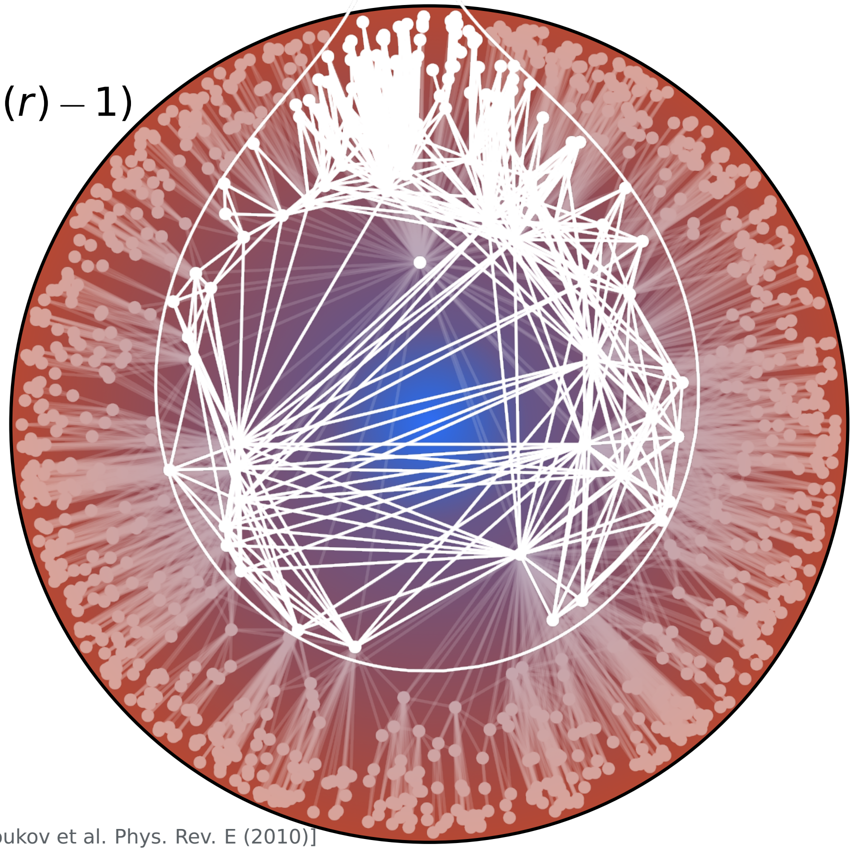

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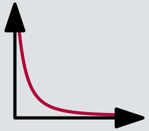
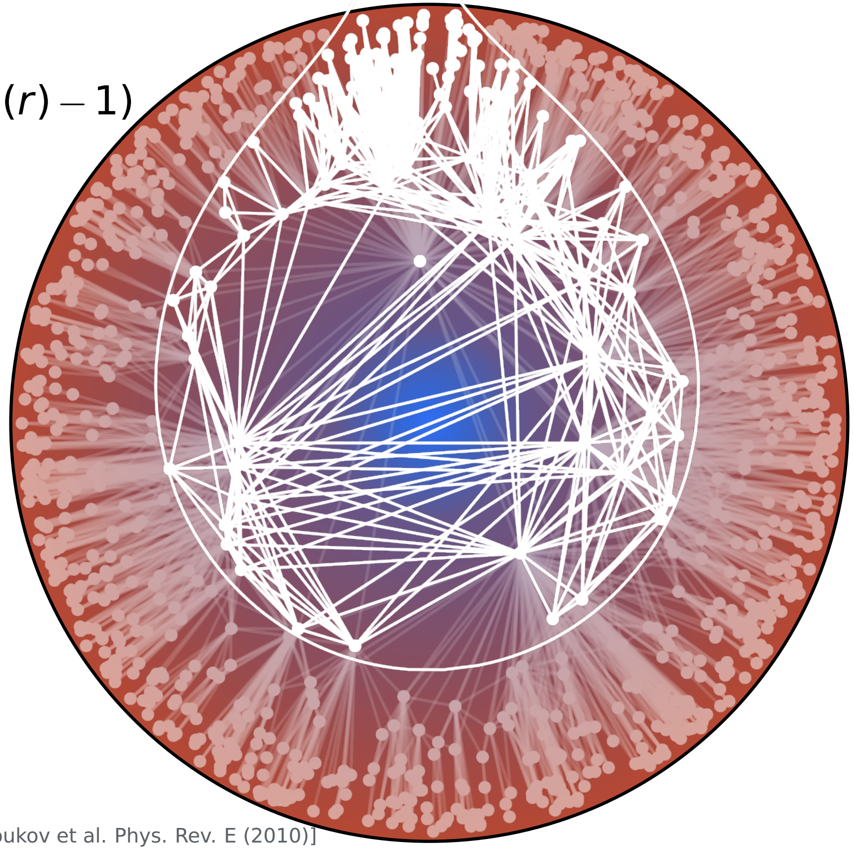
- ... a non-vanishing clustering coefficient
- ... a logarithmic diameter





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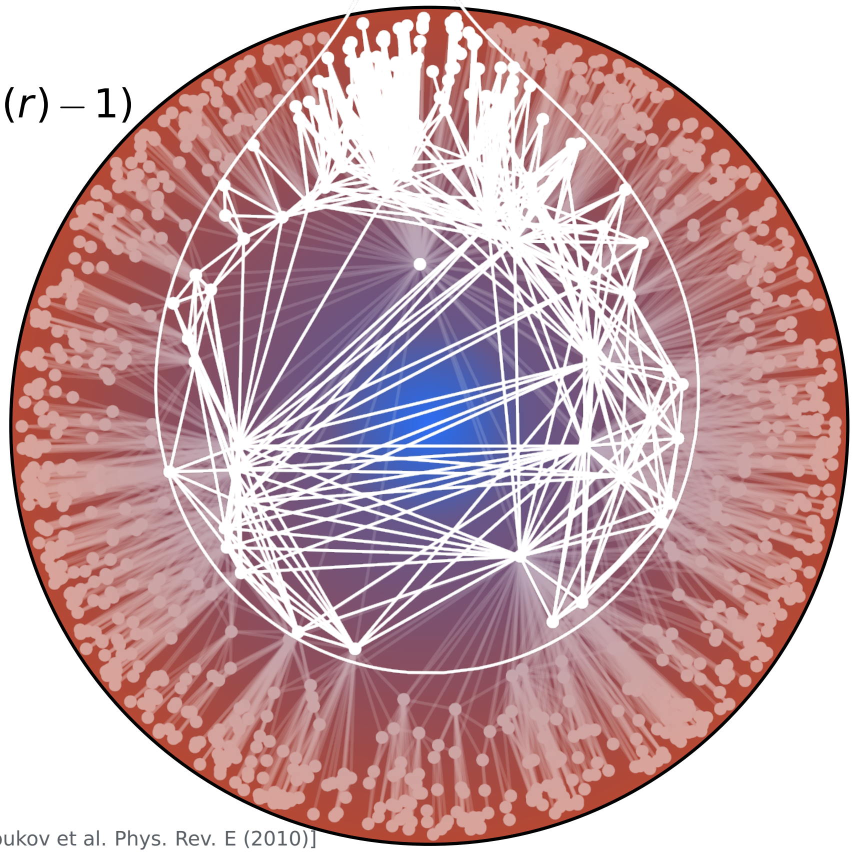

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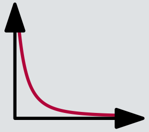
- ... a non-vanishing clustering coefficient
- ... a logarithmic diameter
- ... negative assortativity



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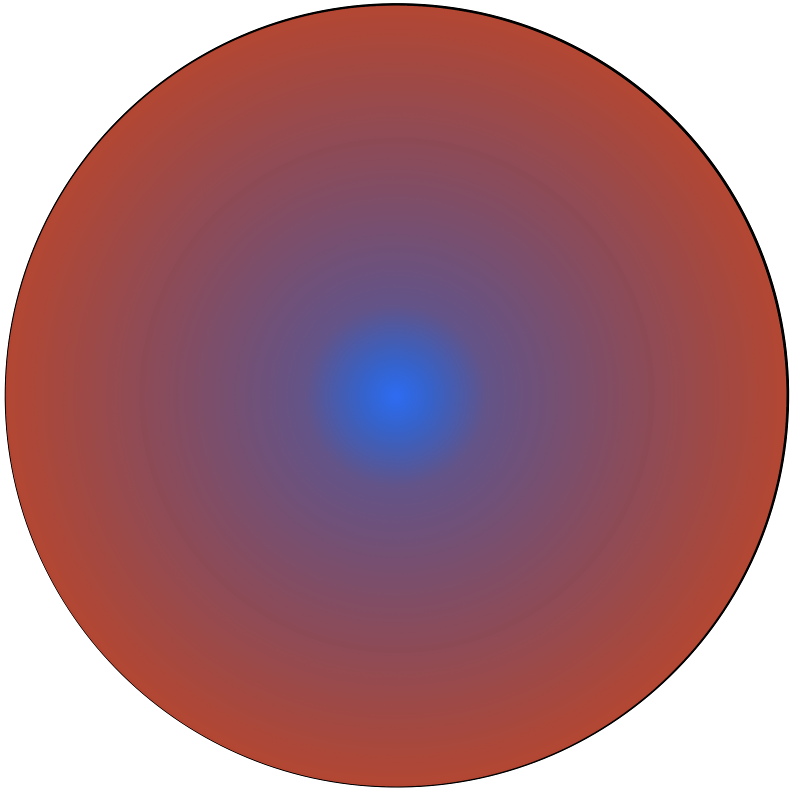

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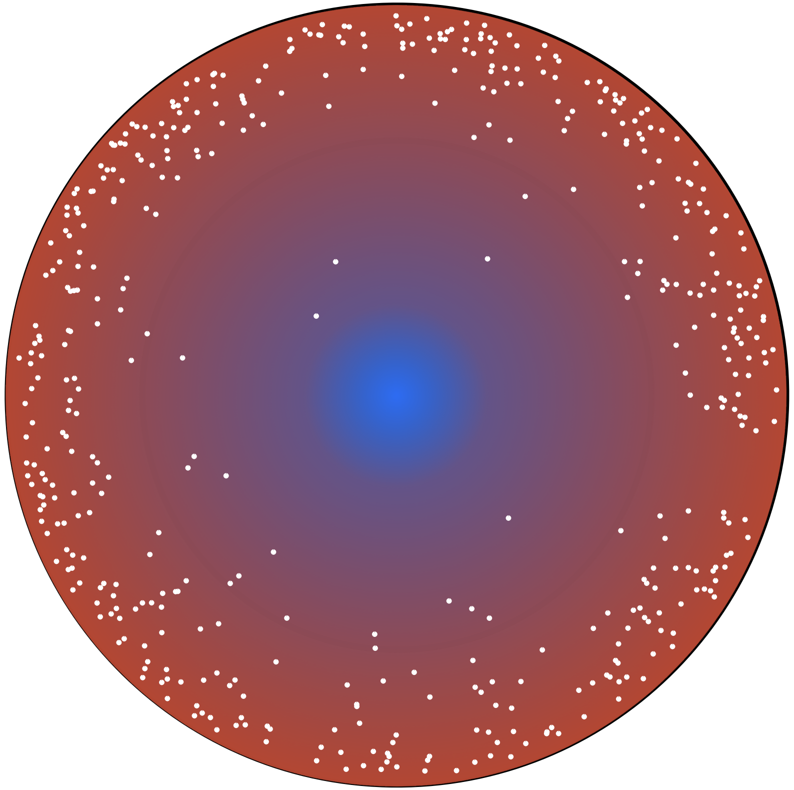

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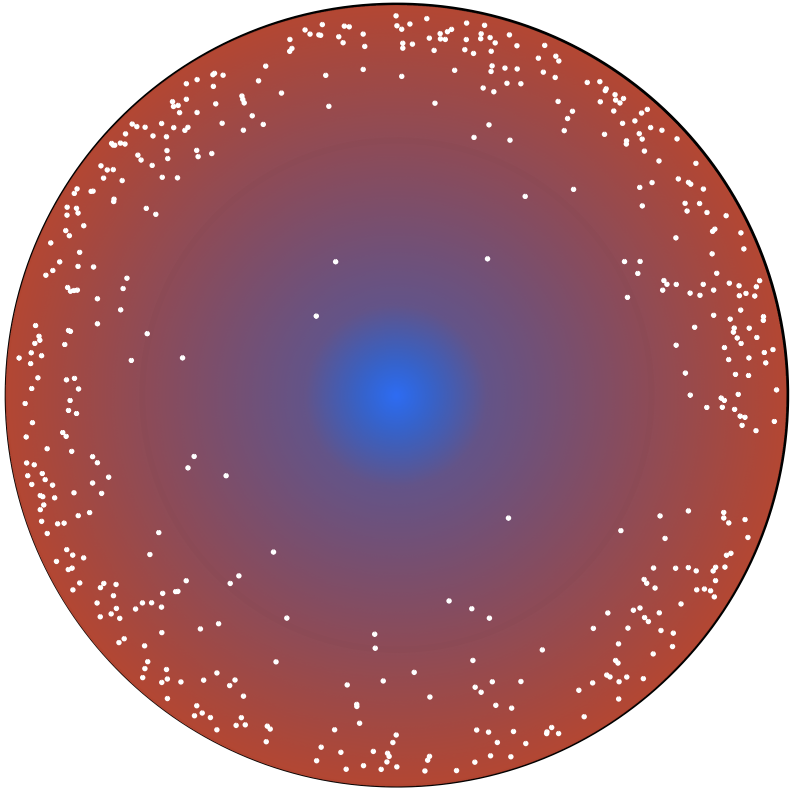

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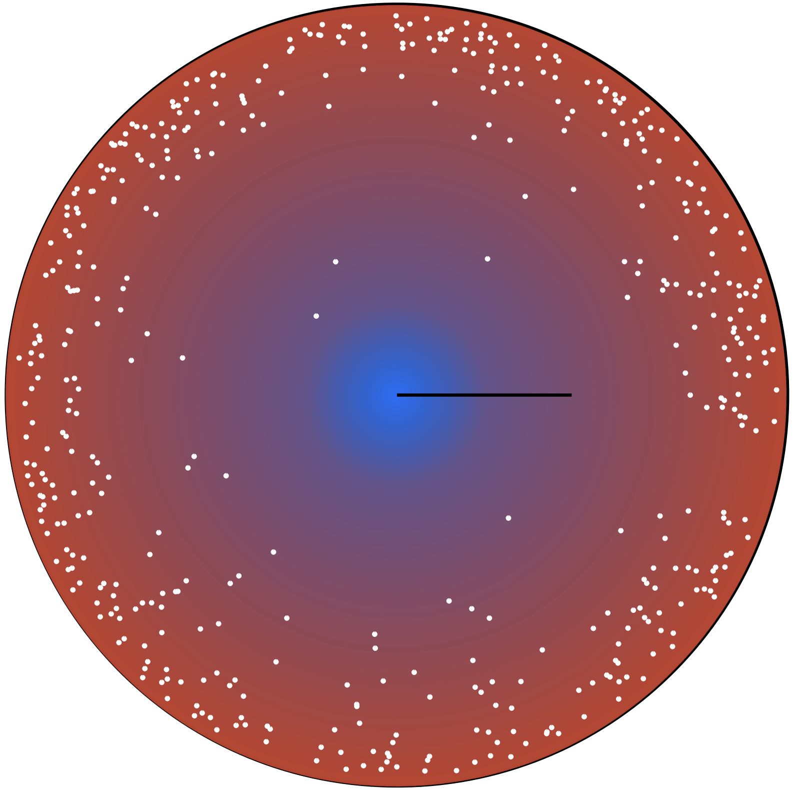

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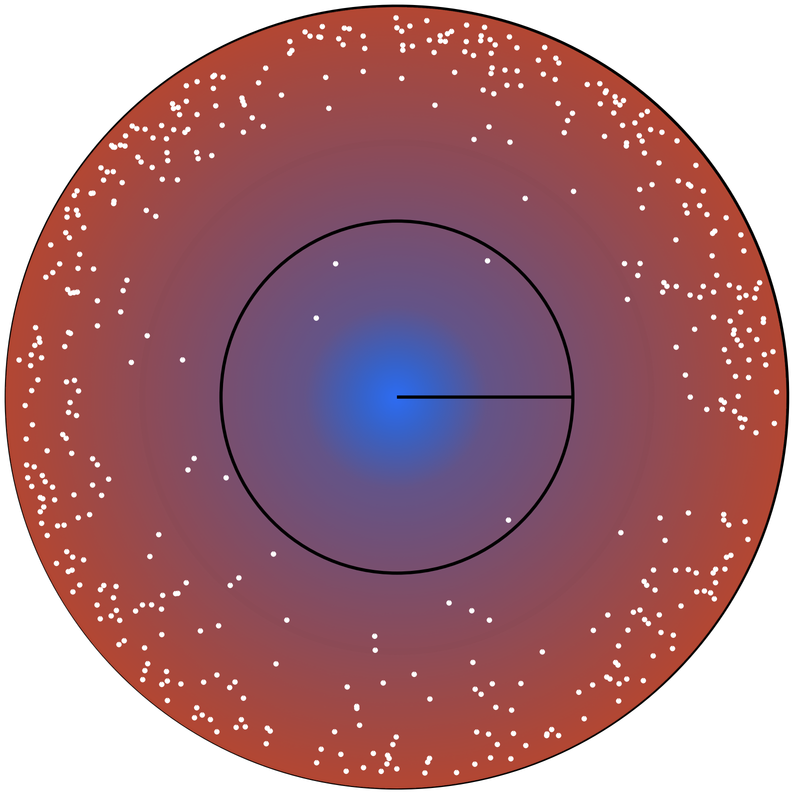

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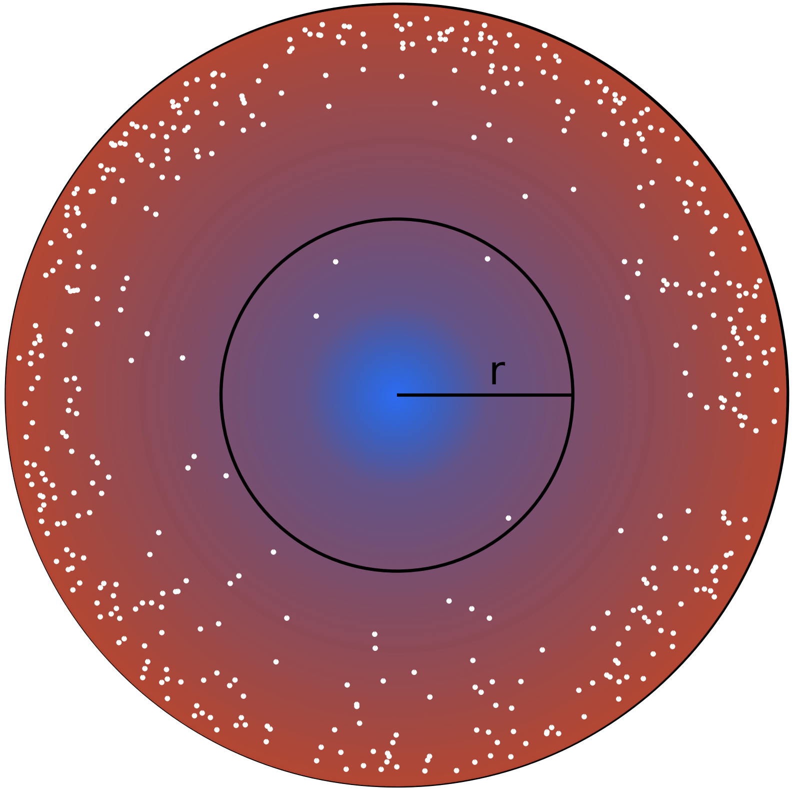

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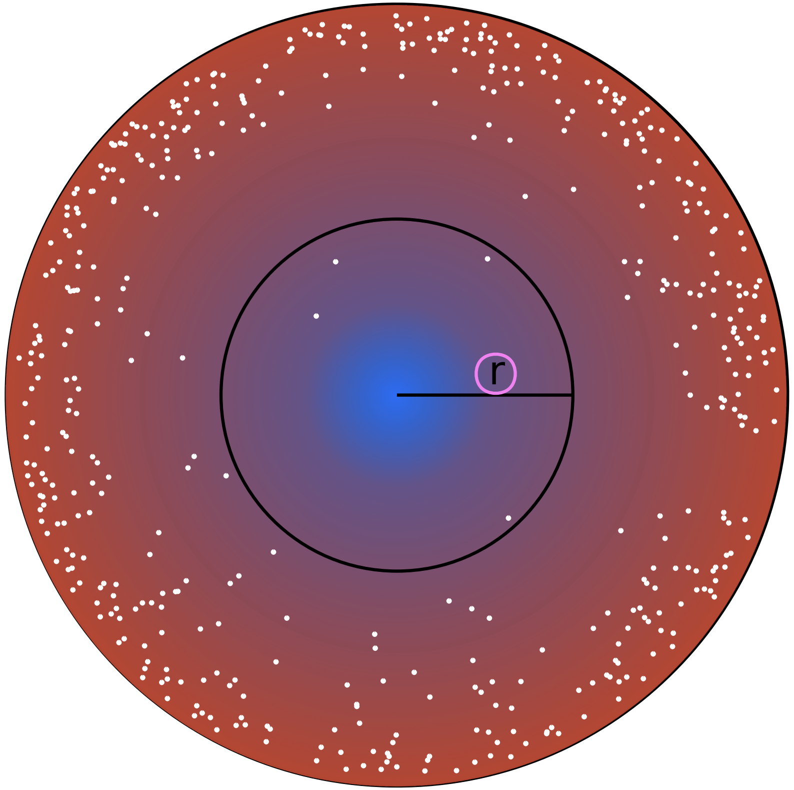

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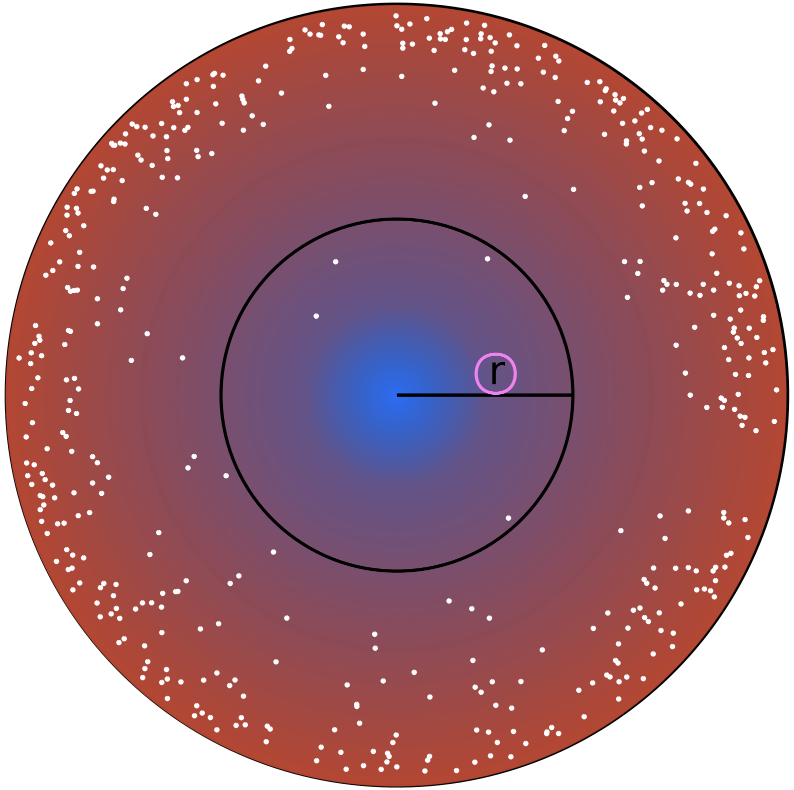

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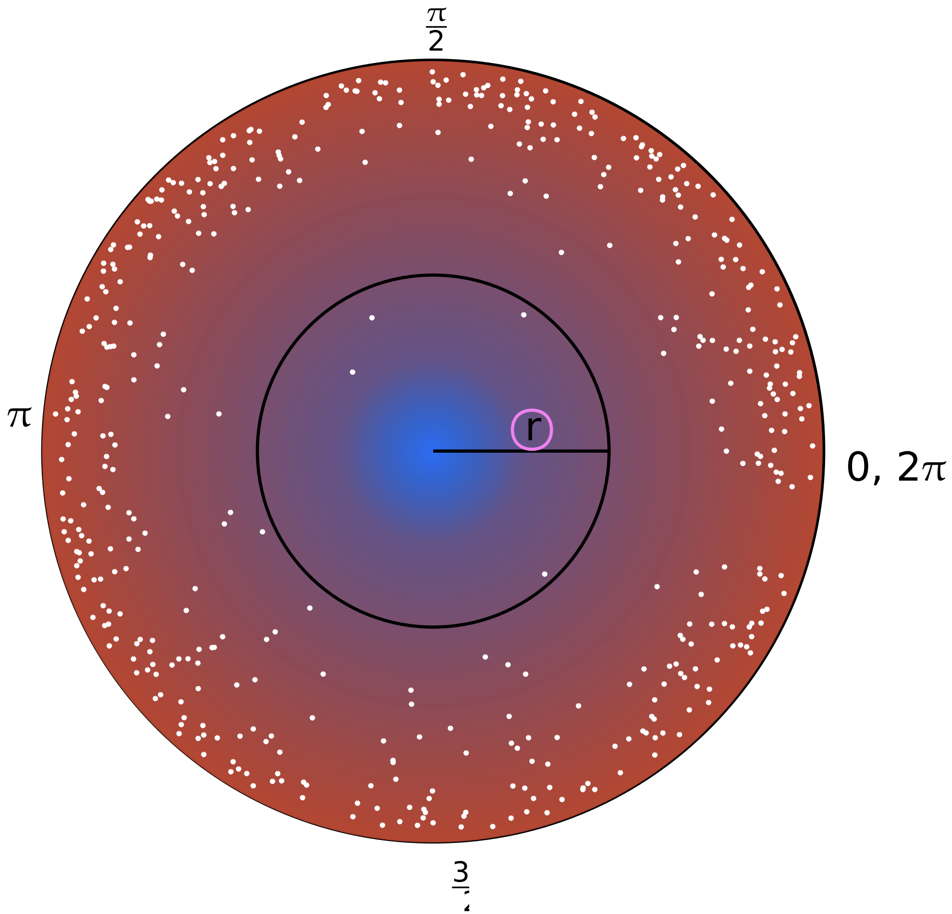

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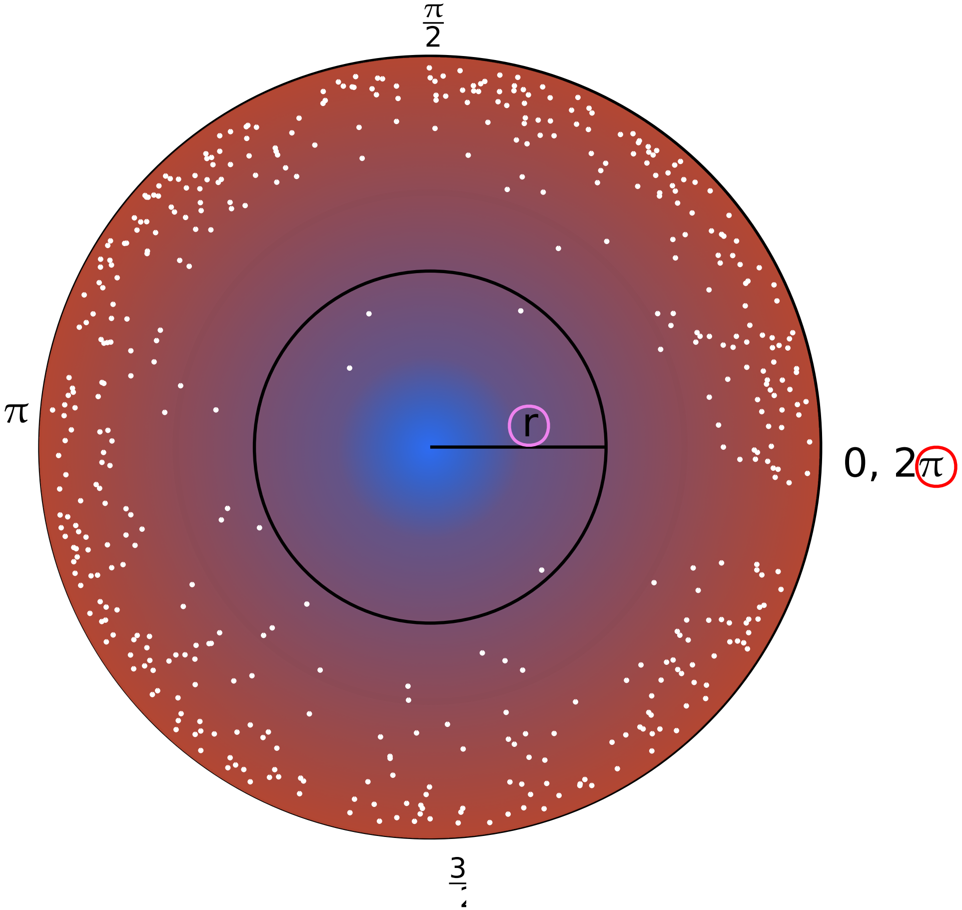

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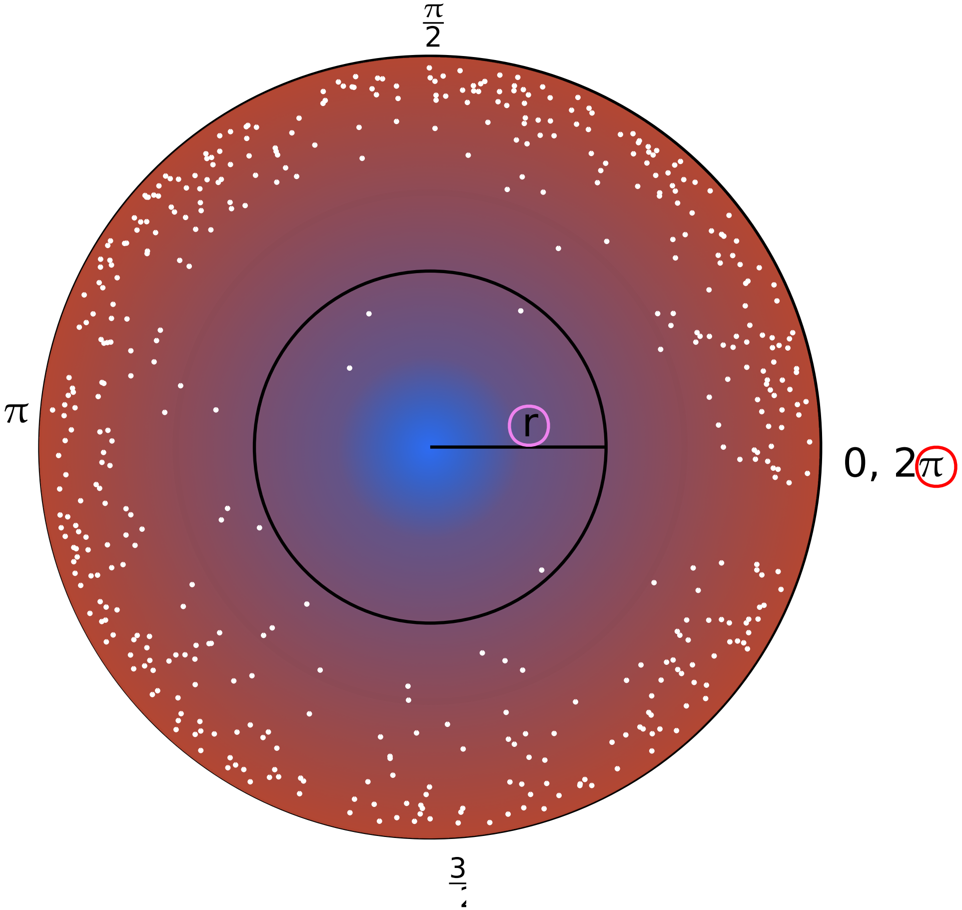

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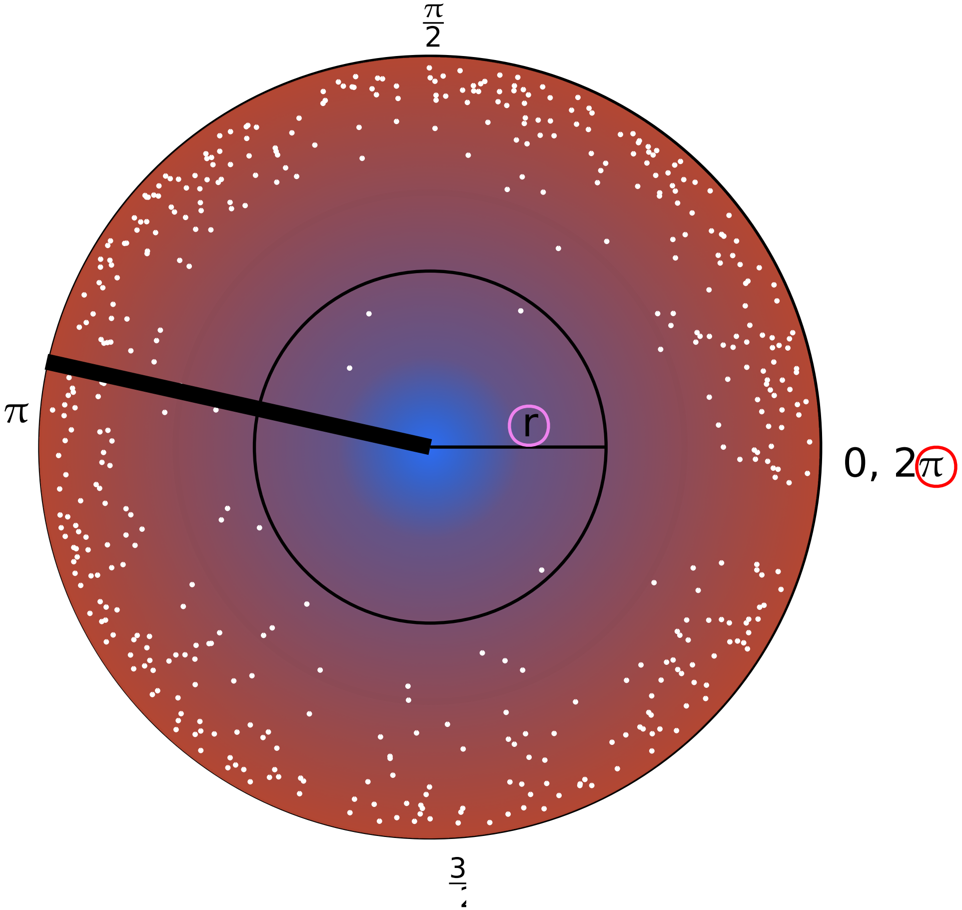

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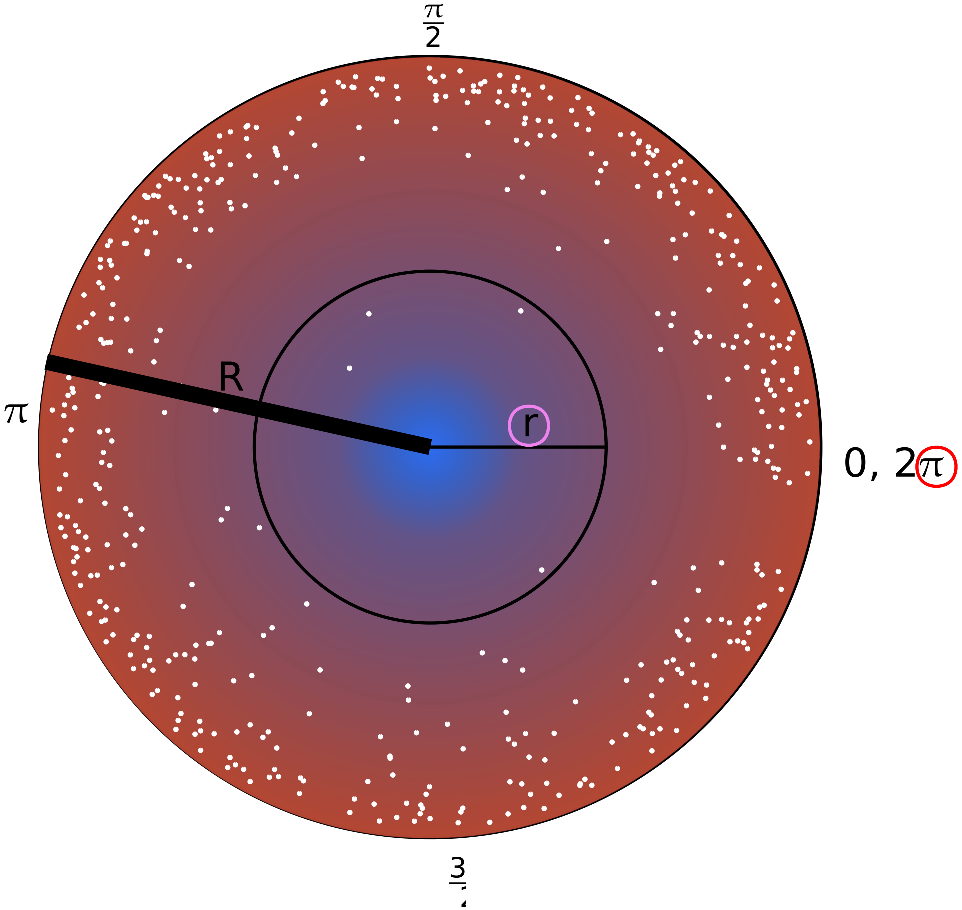

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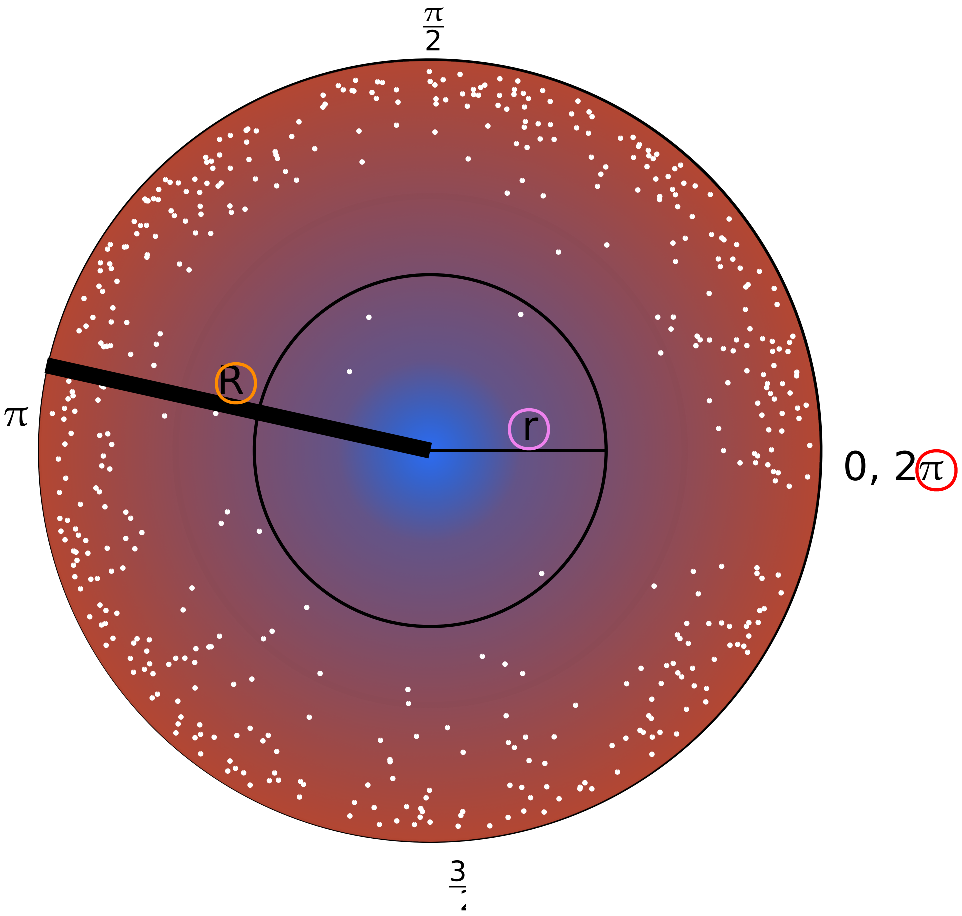

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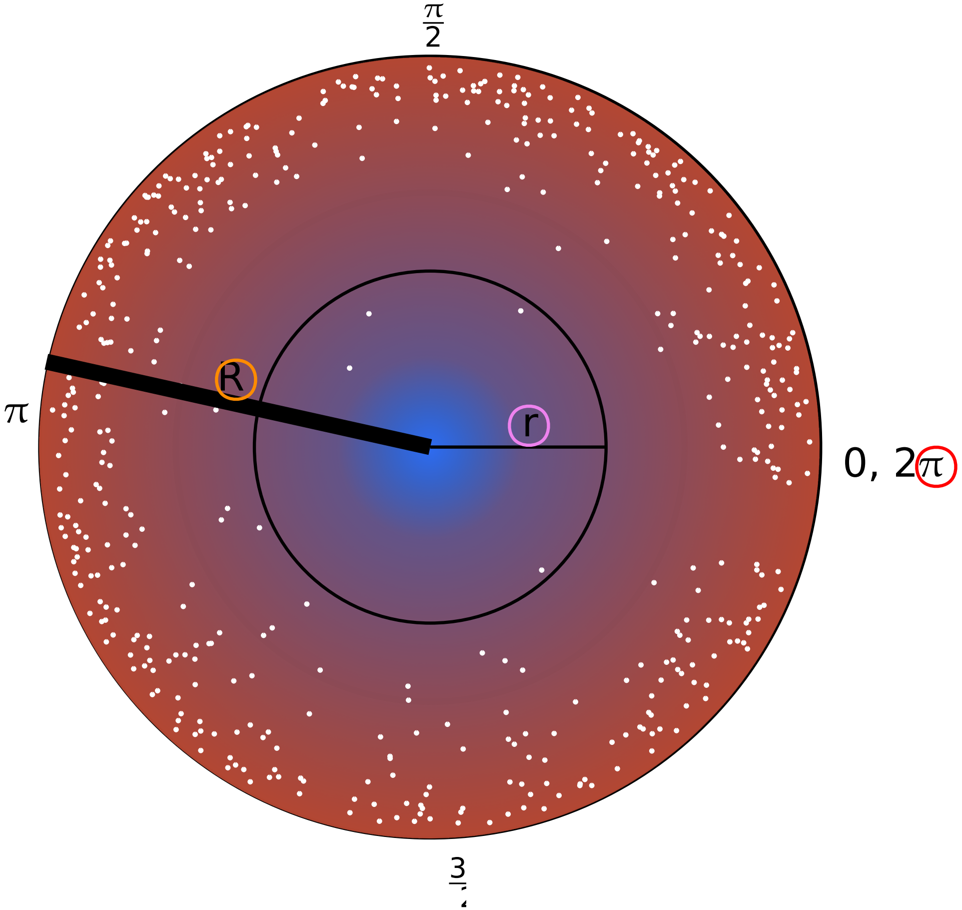

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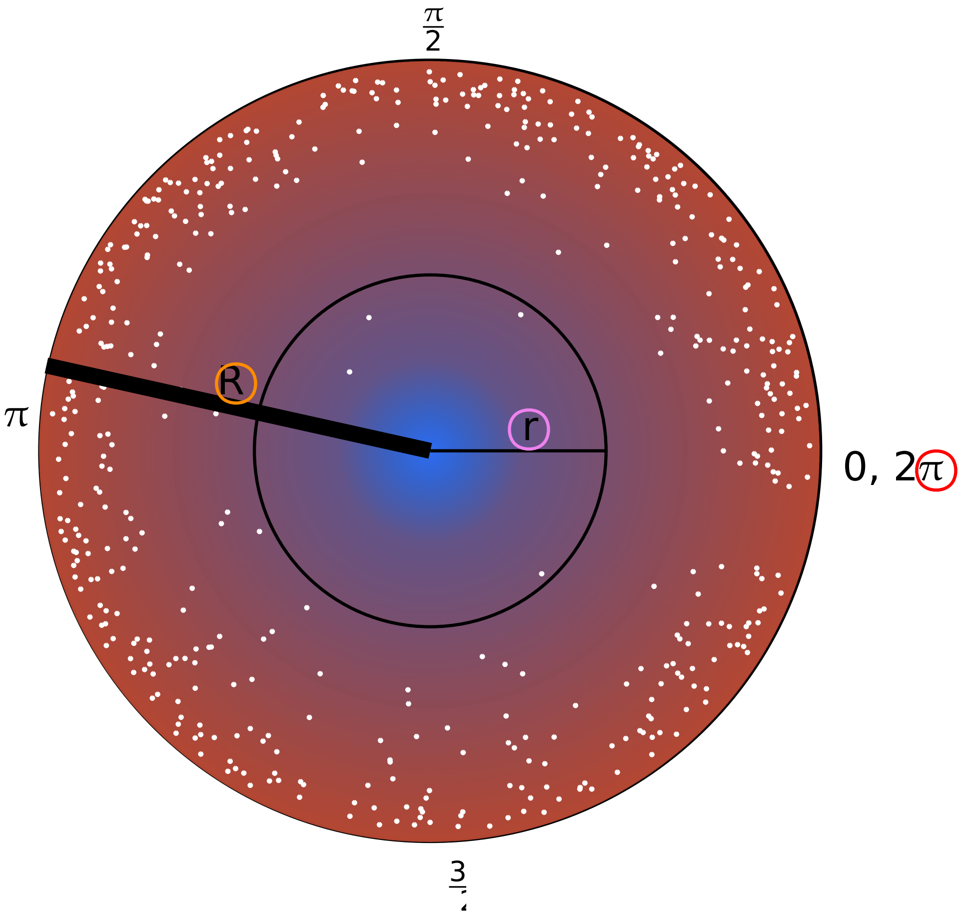

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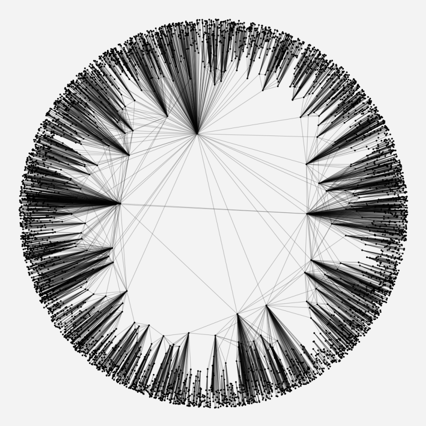

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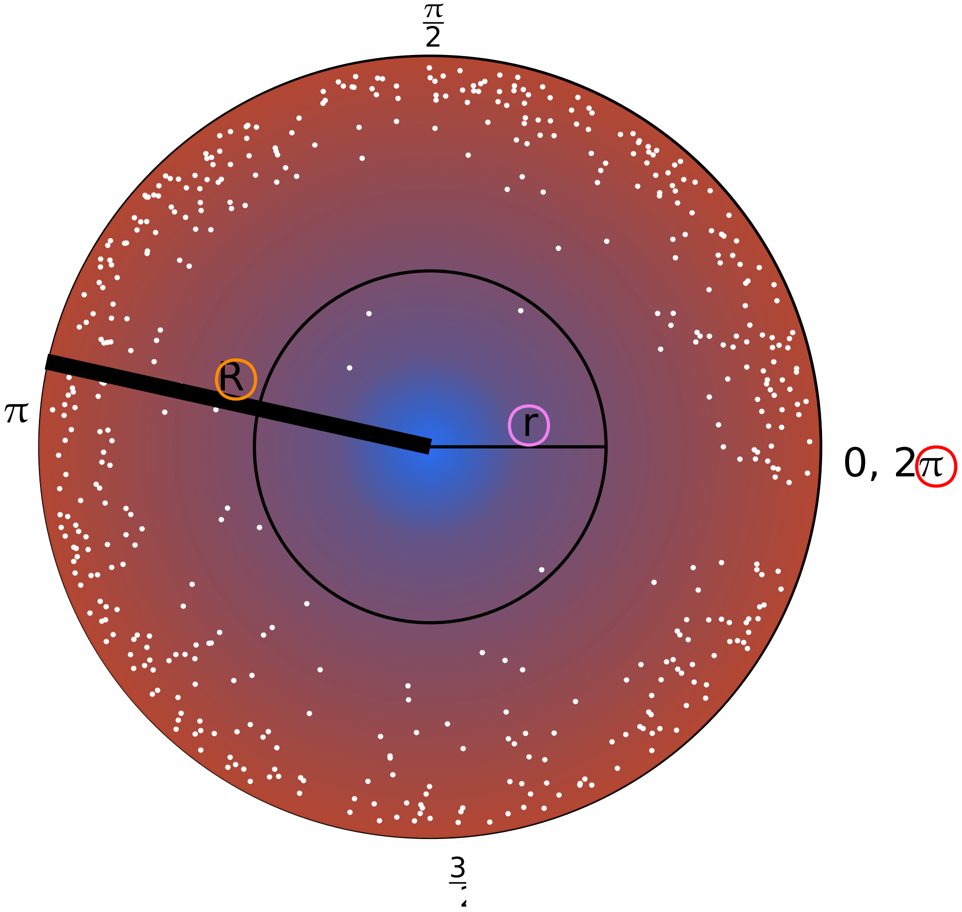

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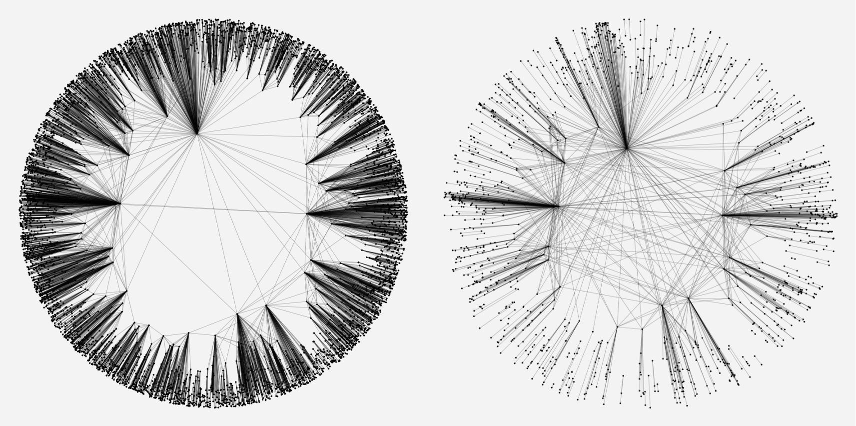

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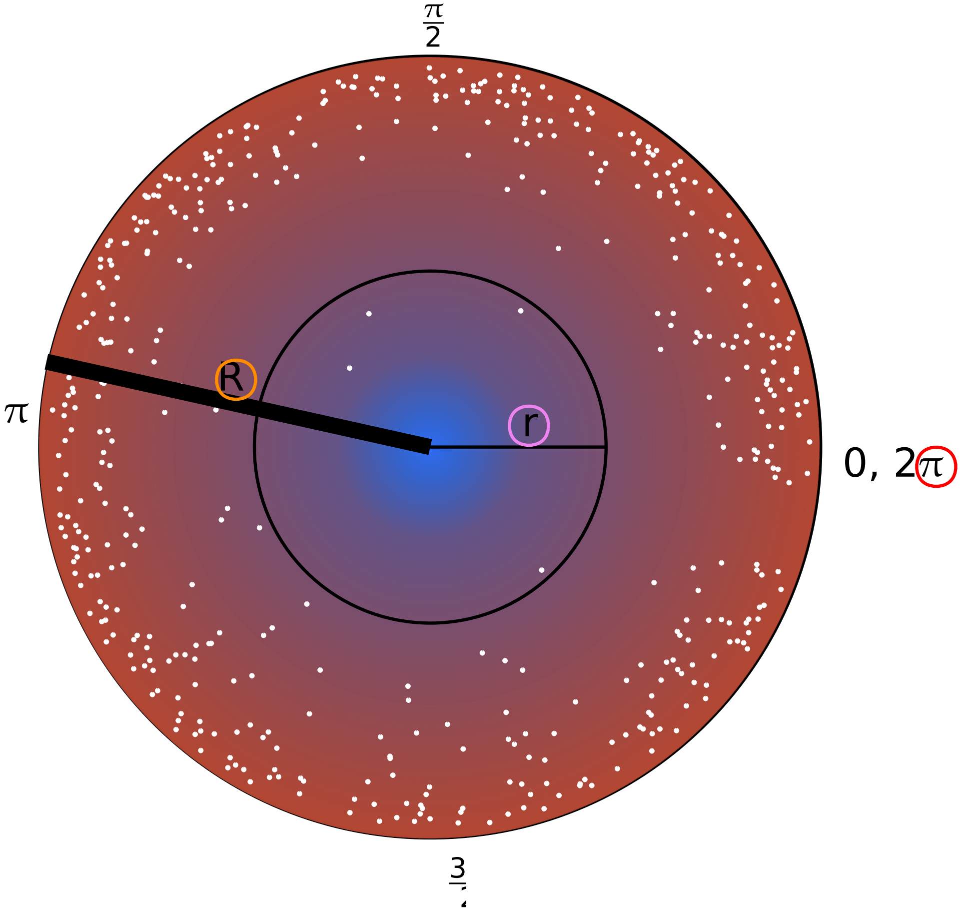

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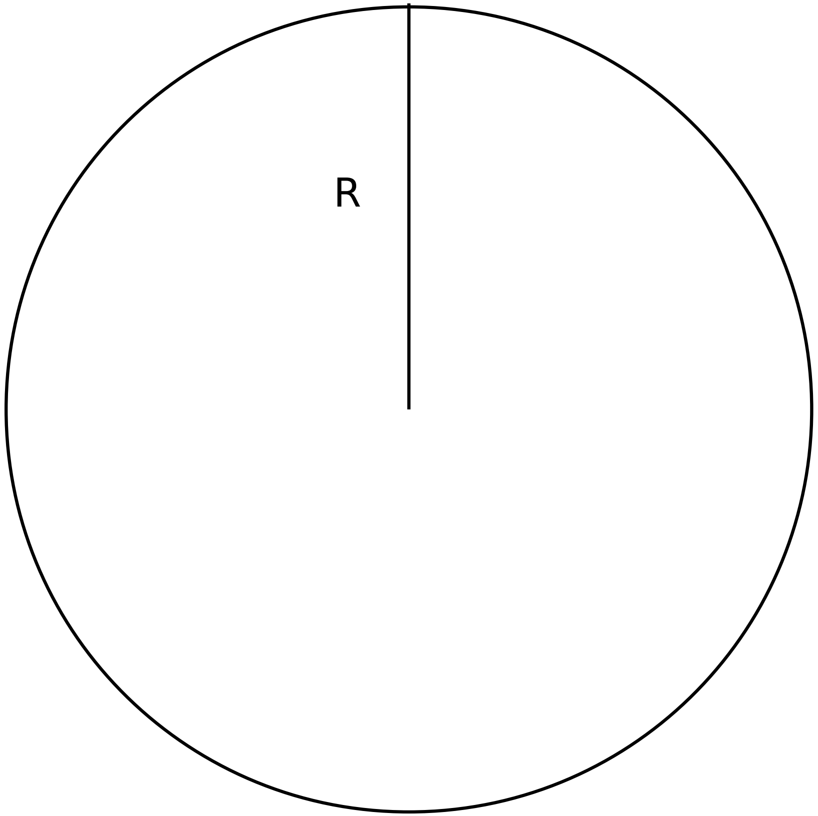

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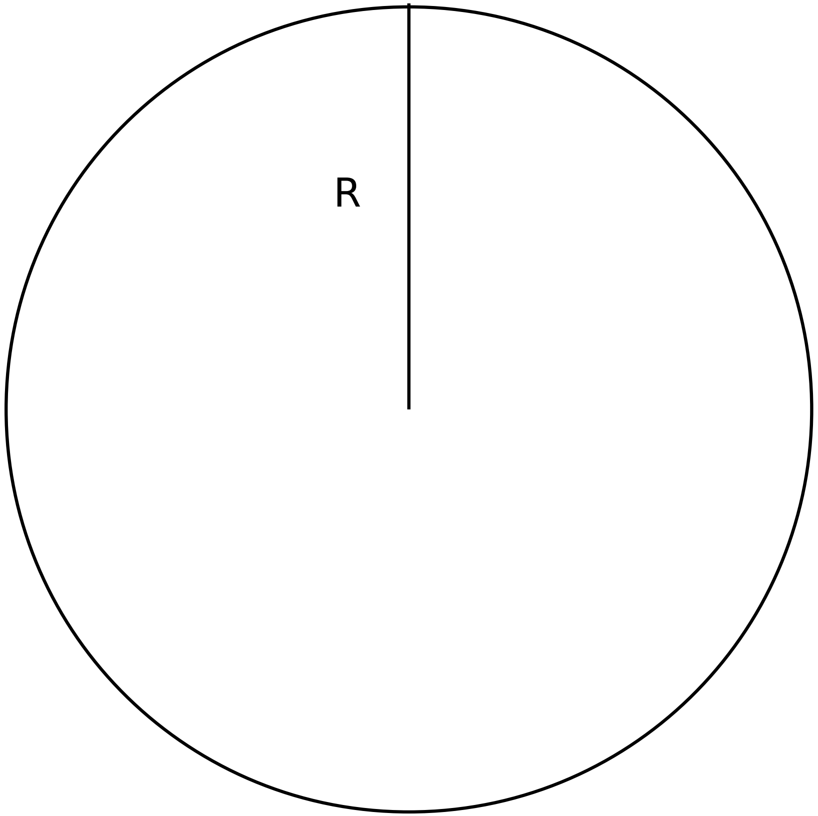

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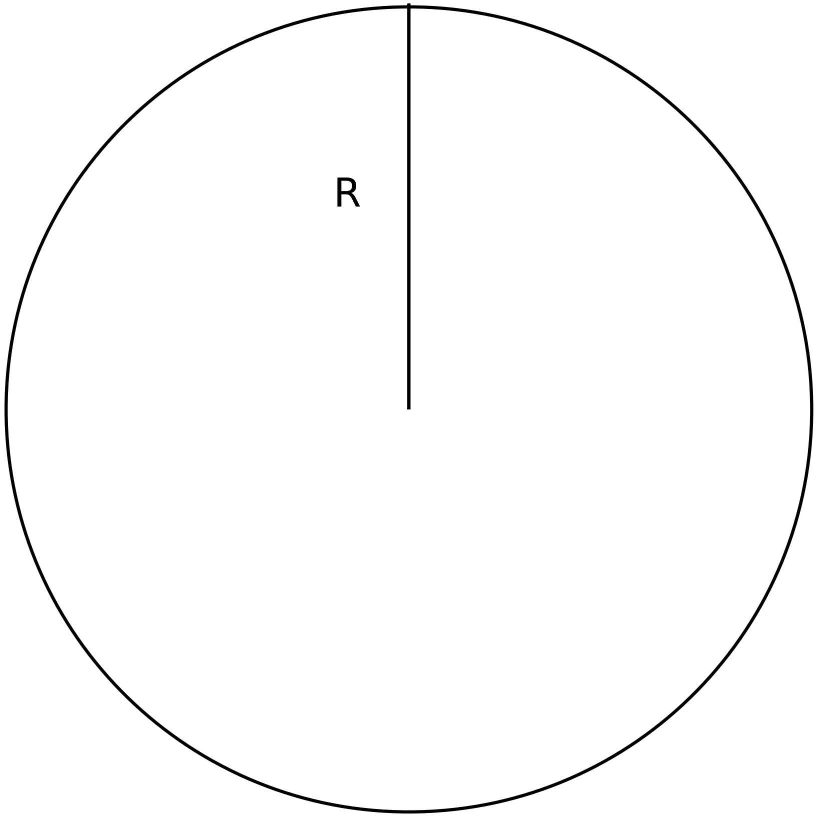

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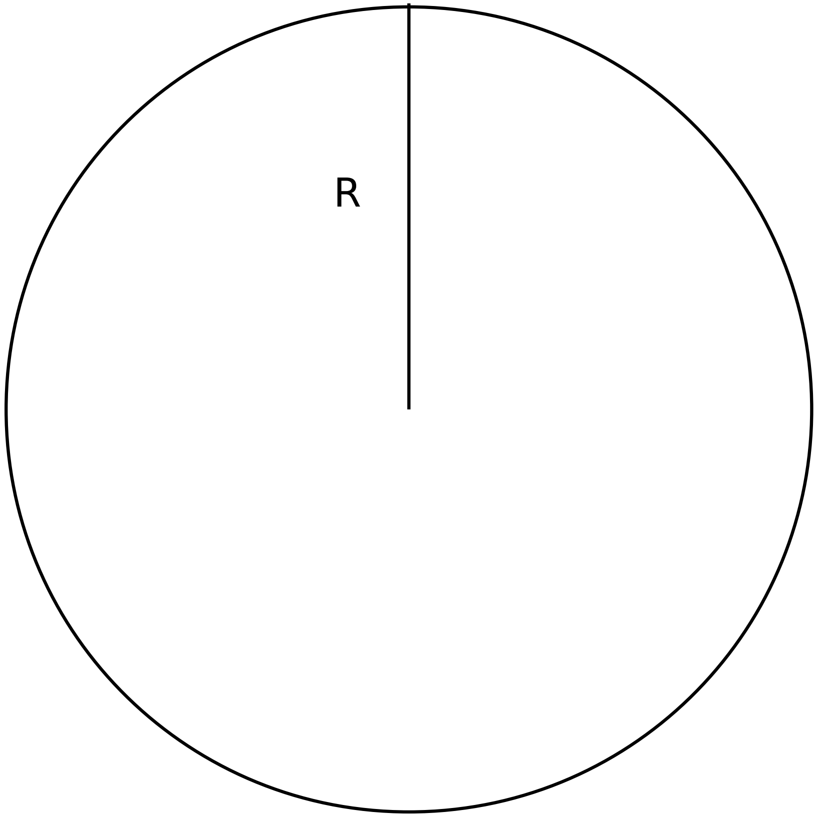

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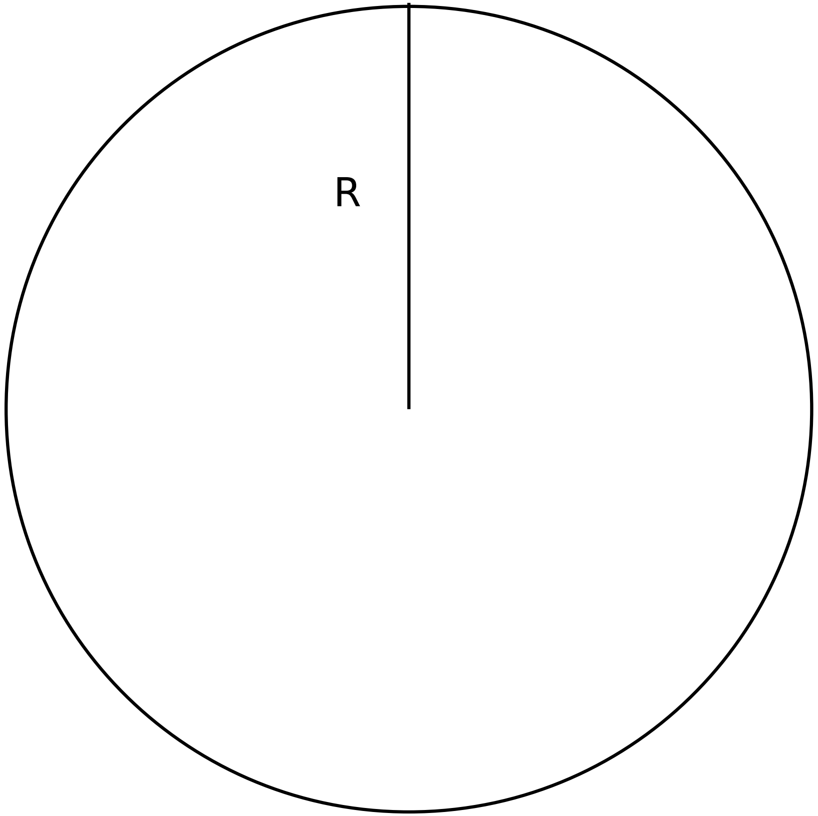

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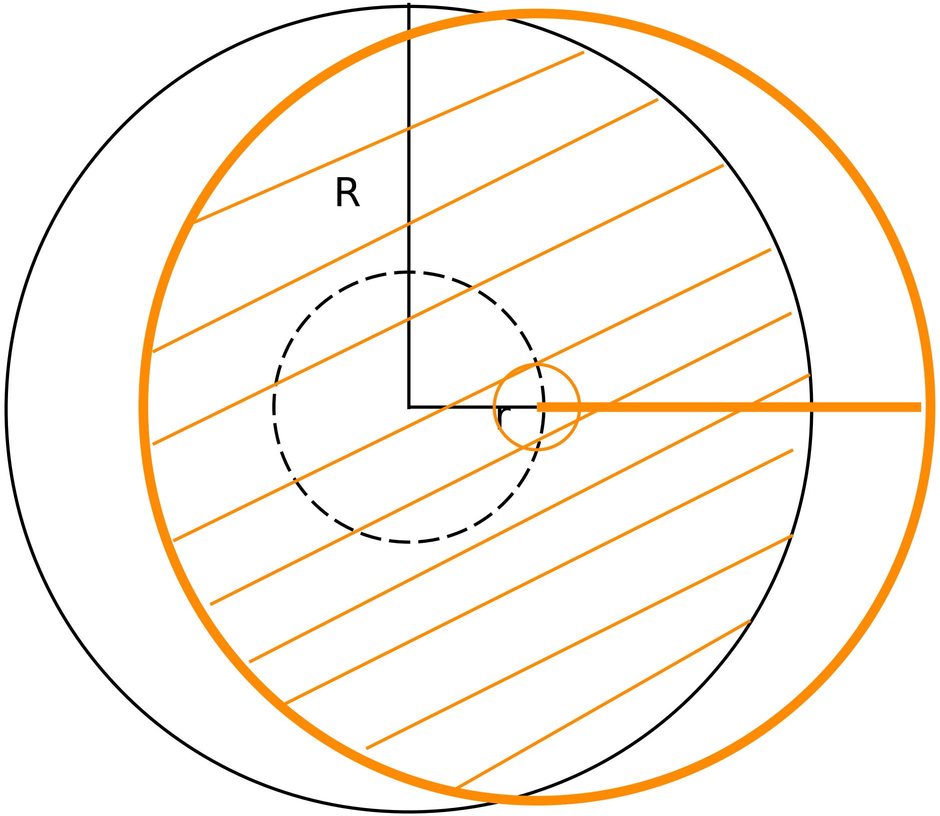

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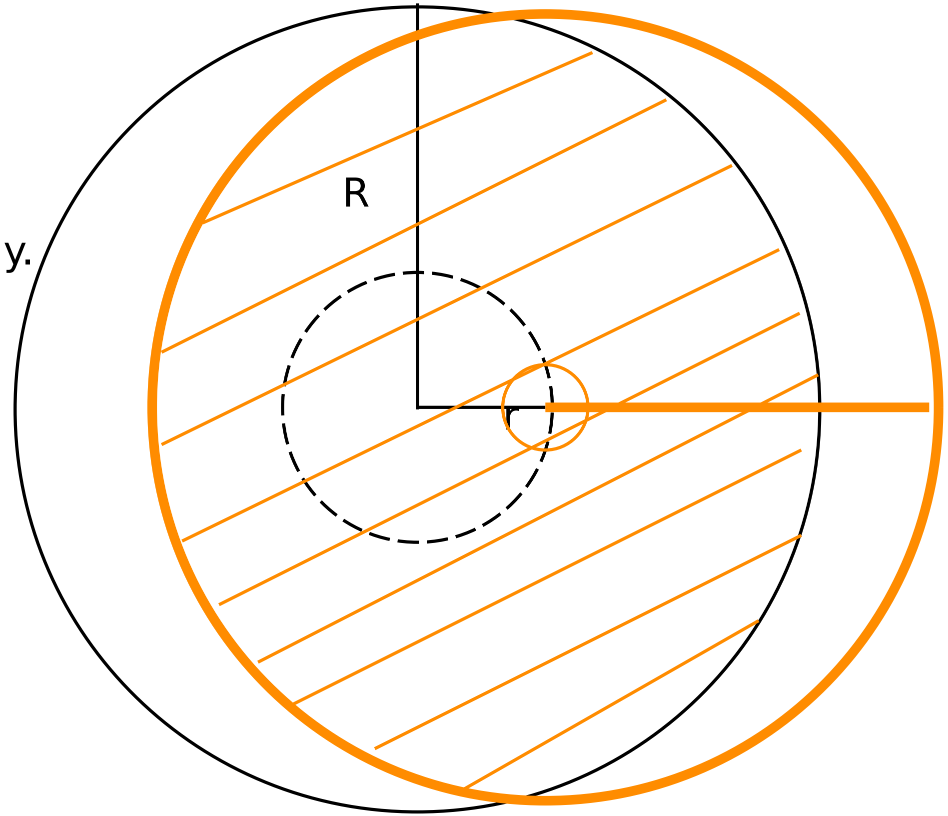

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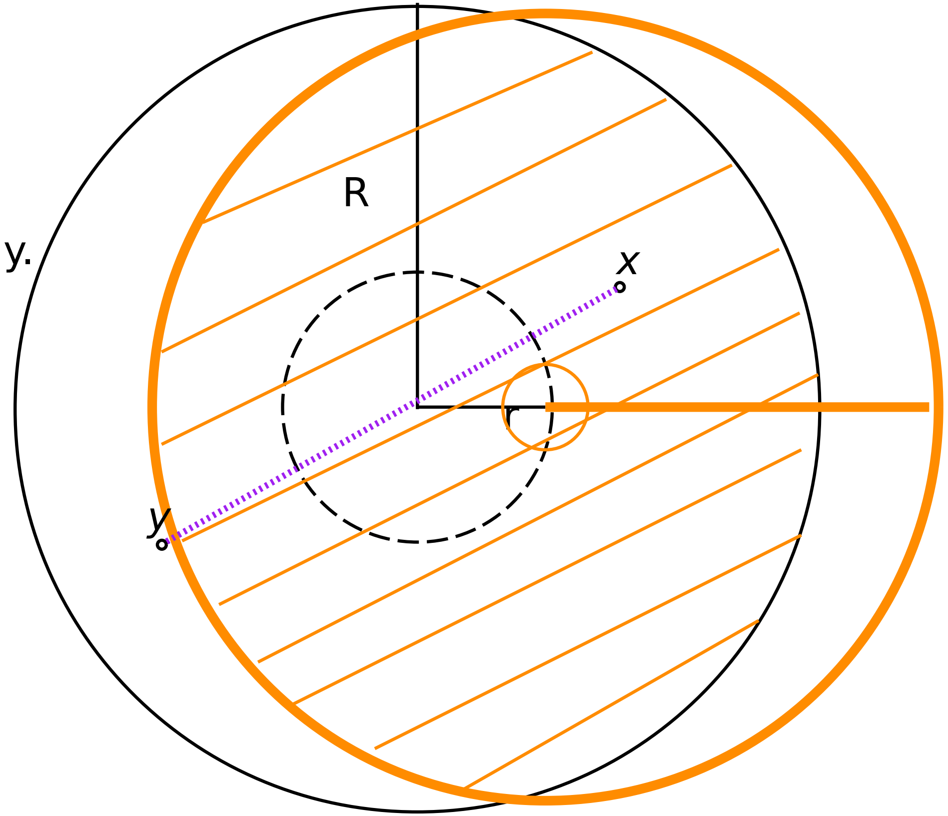

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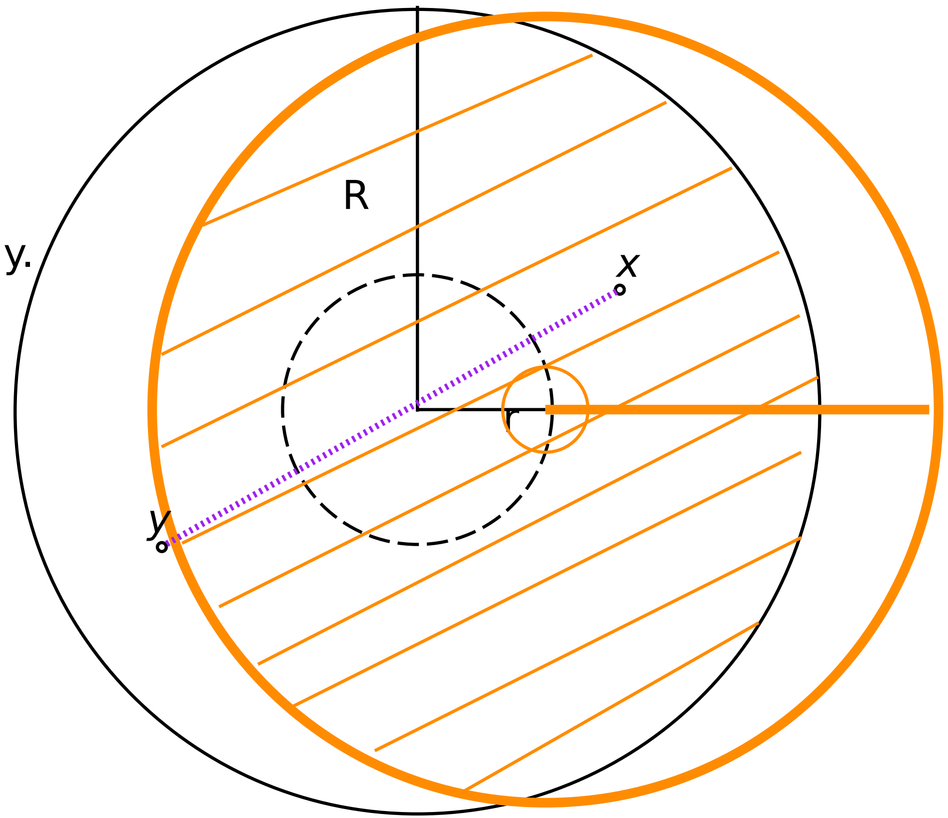

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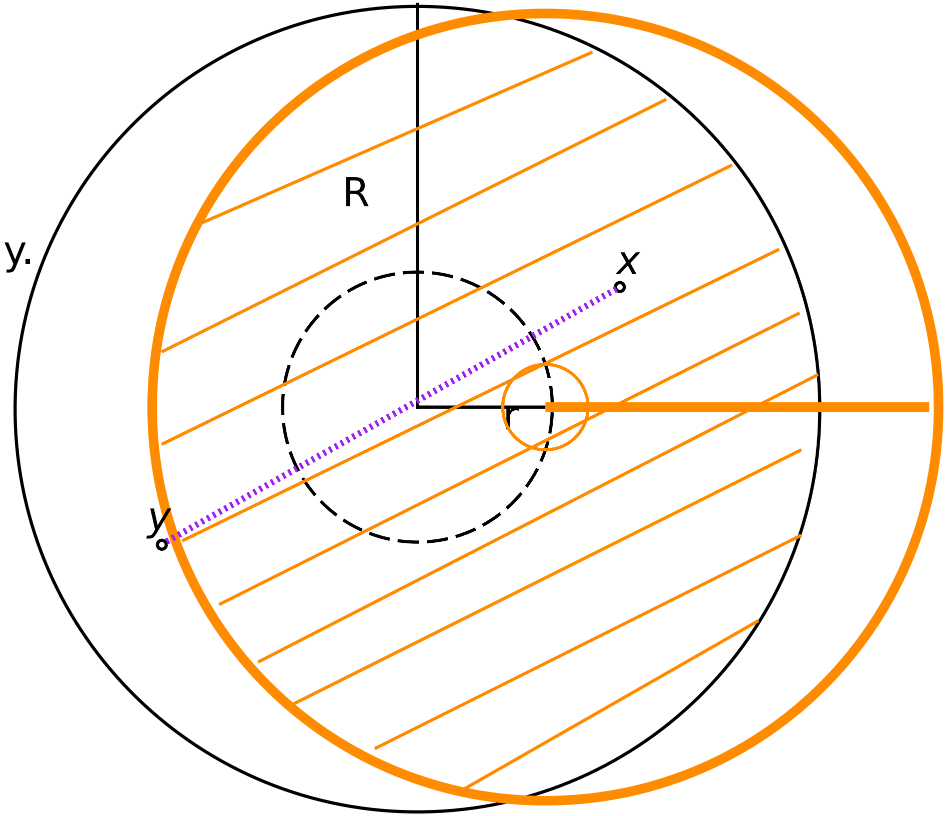

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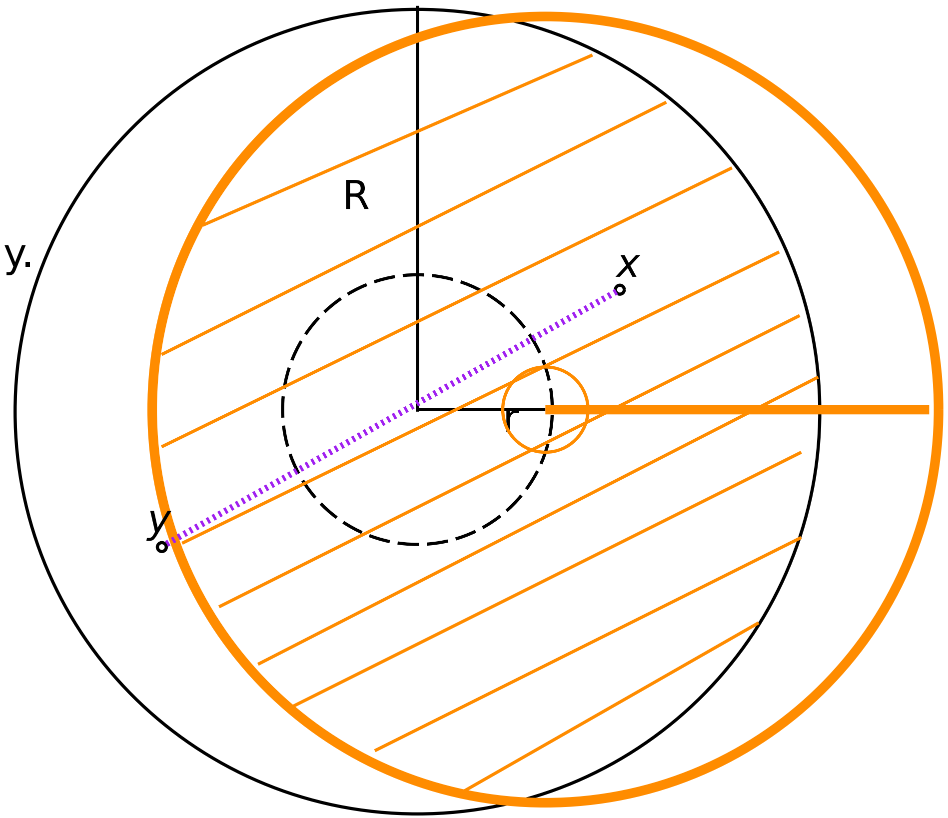

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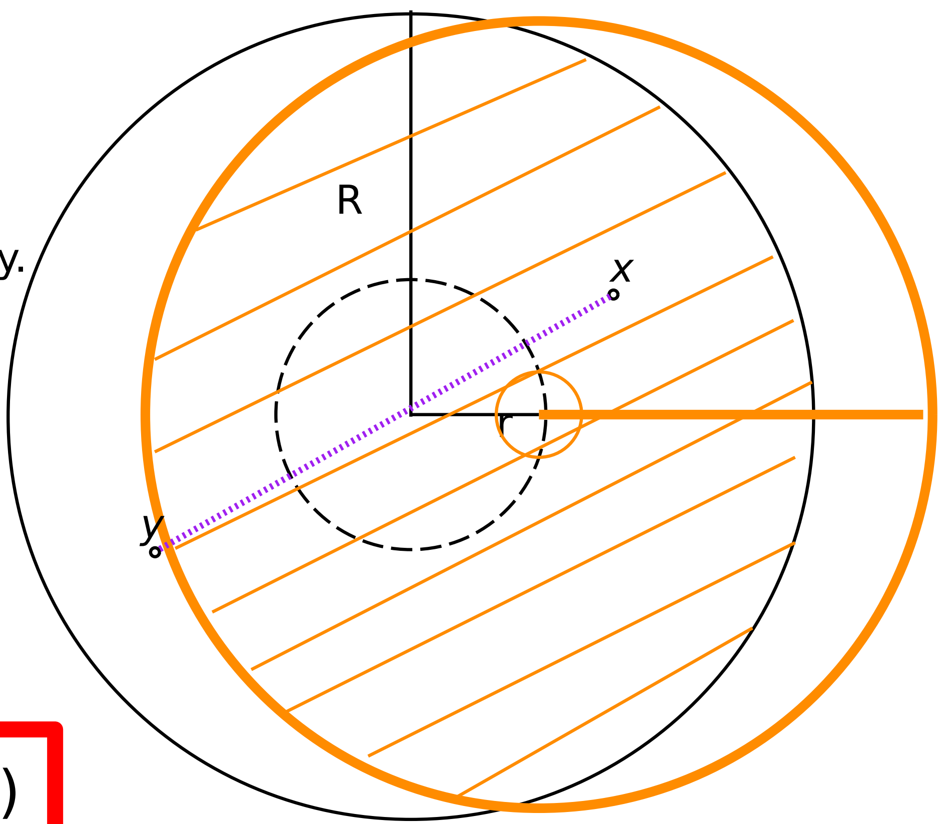

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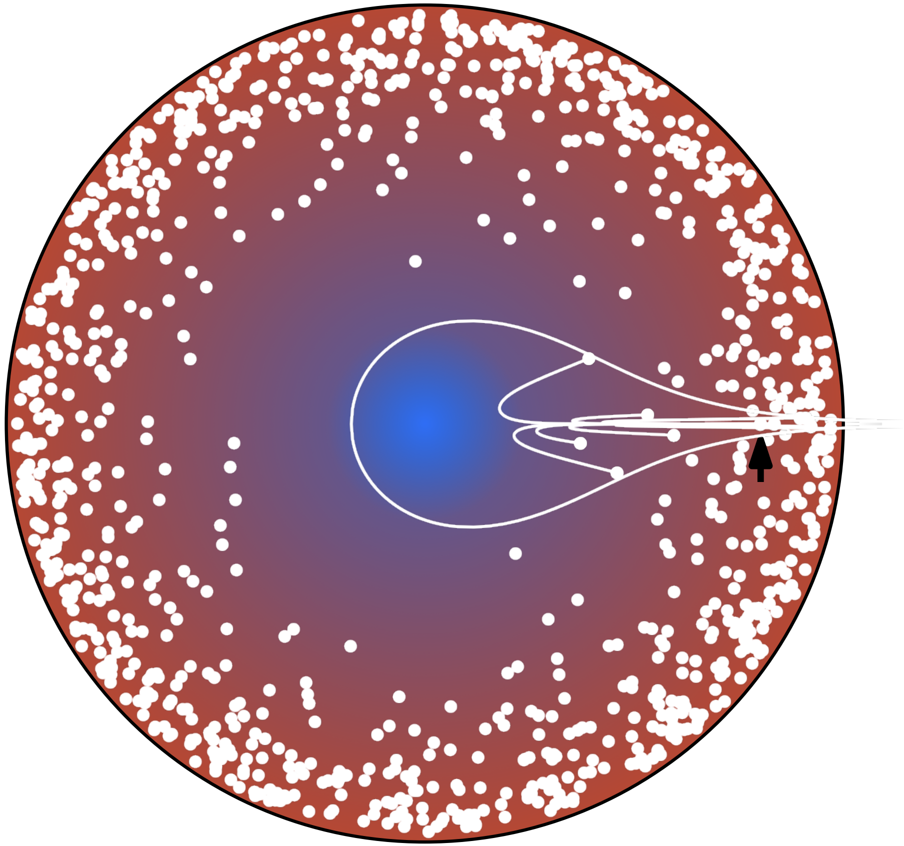

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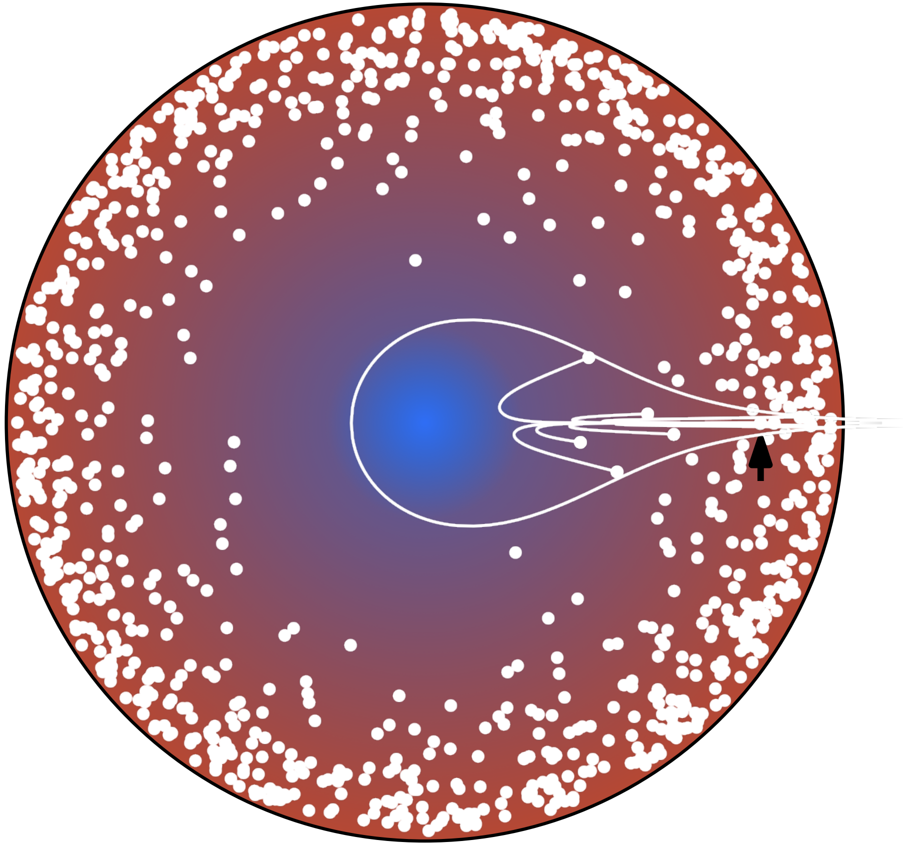

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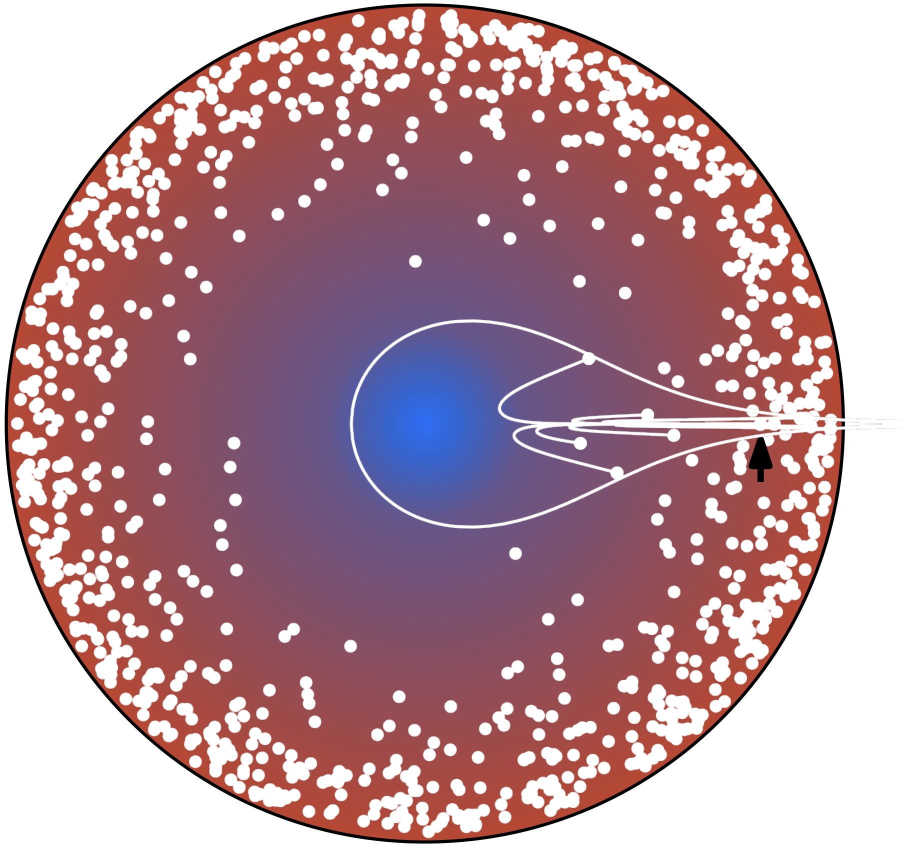

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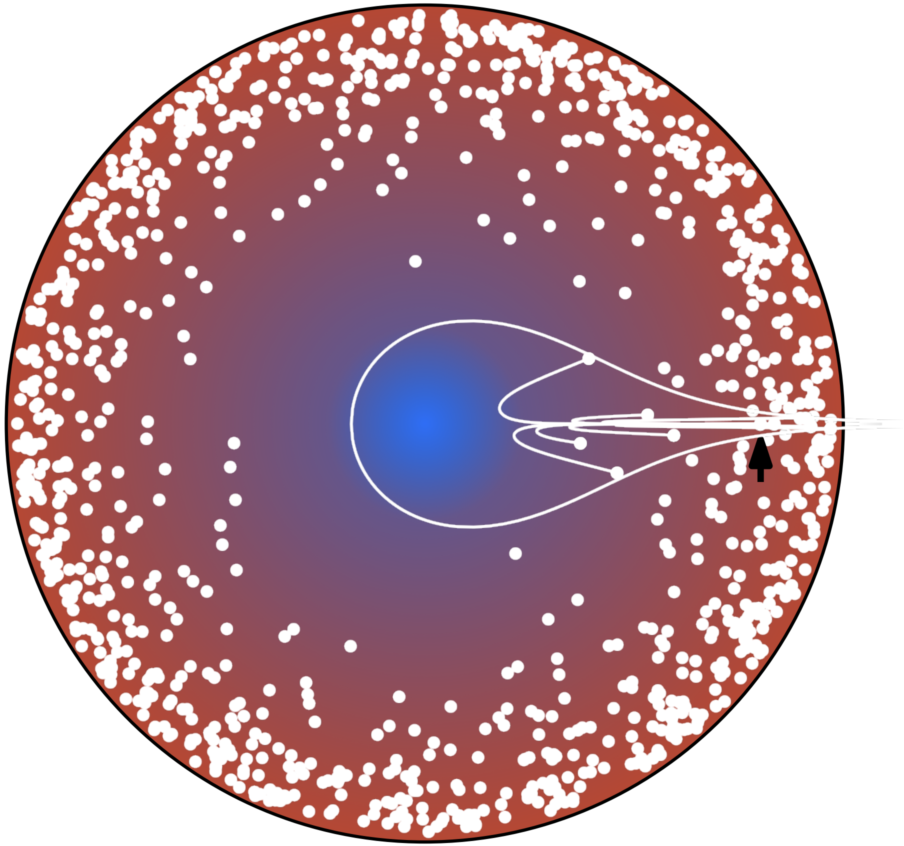

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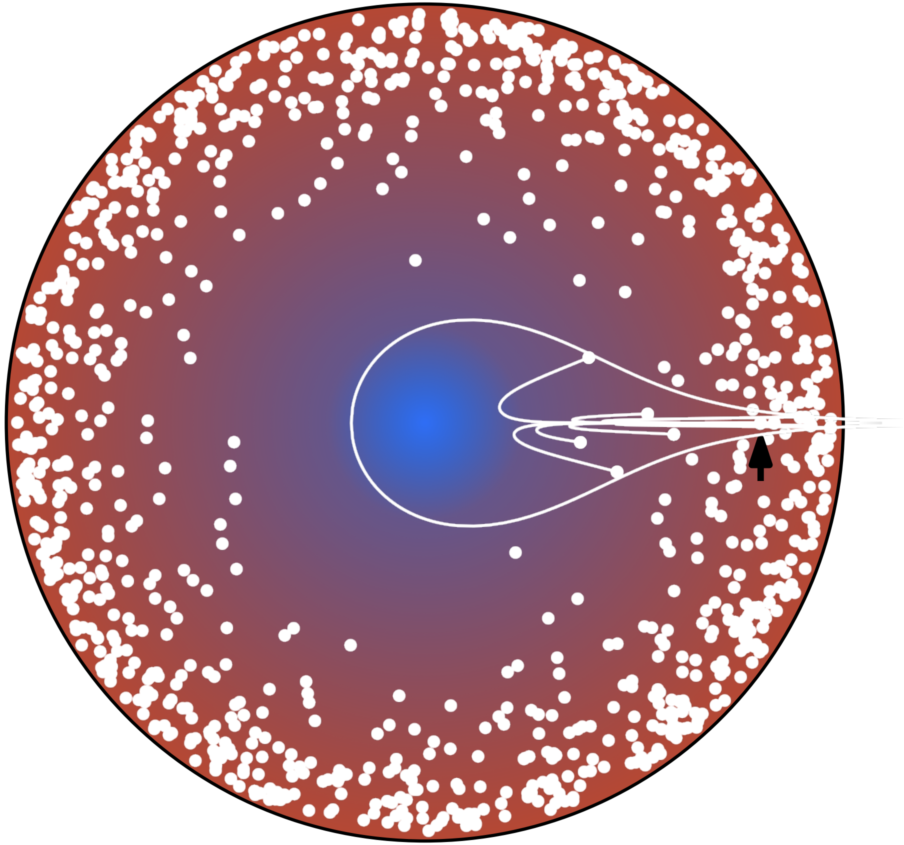

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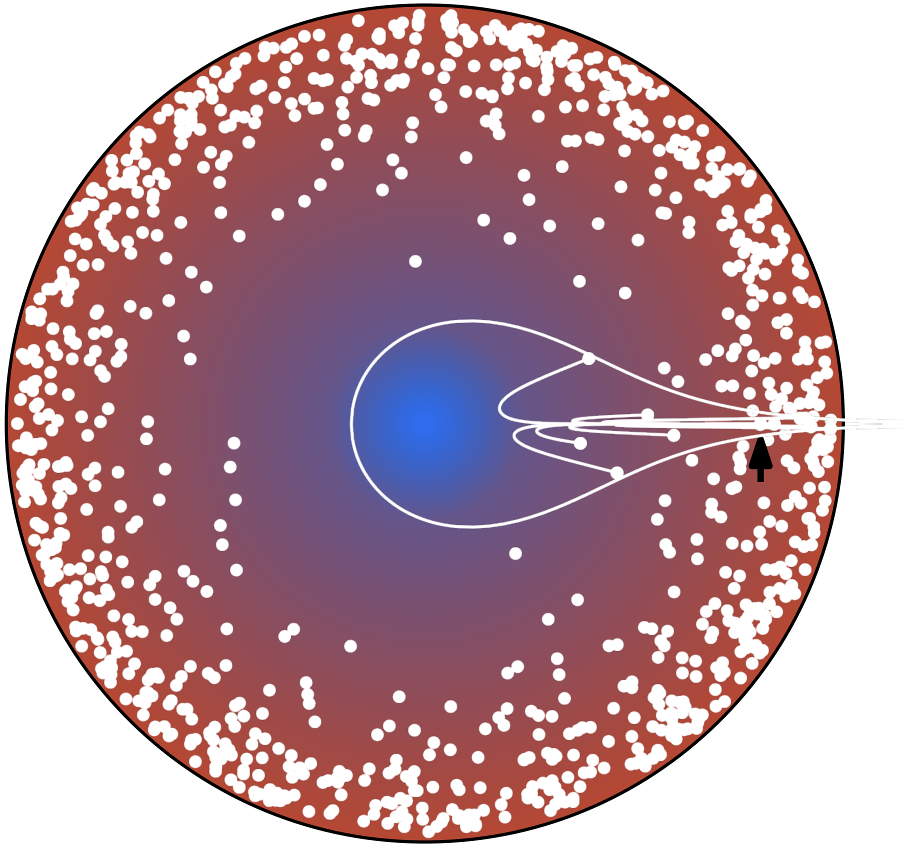

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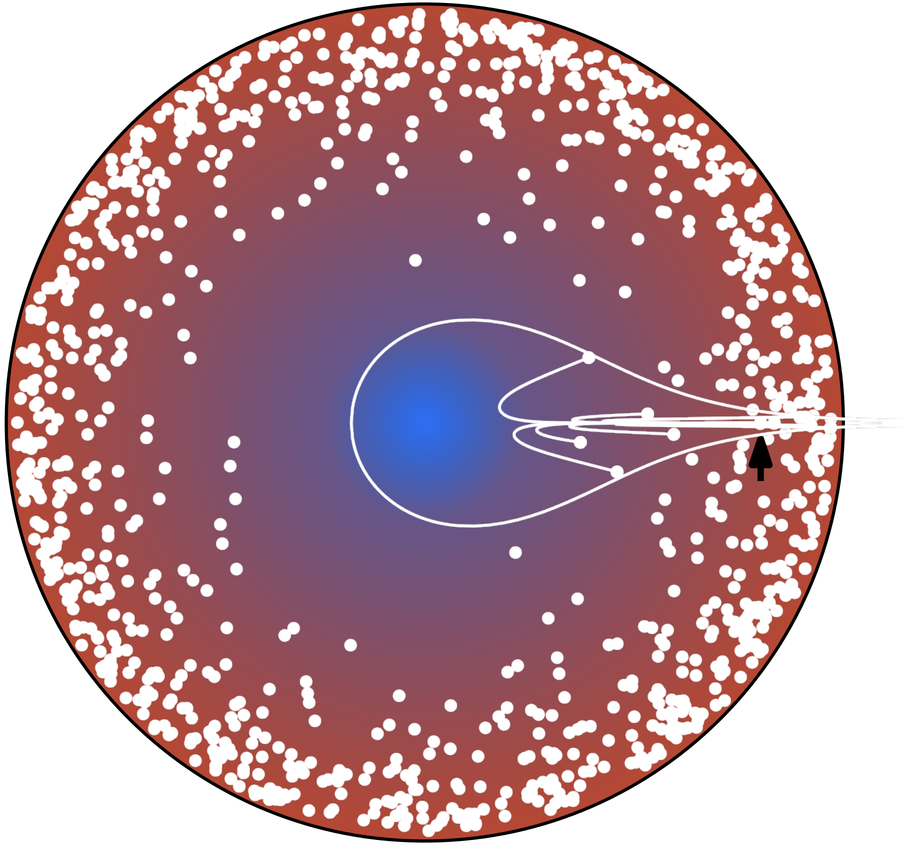

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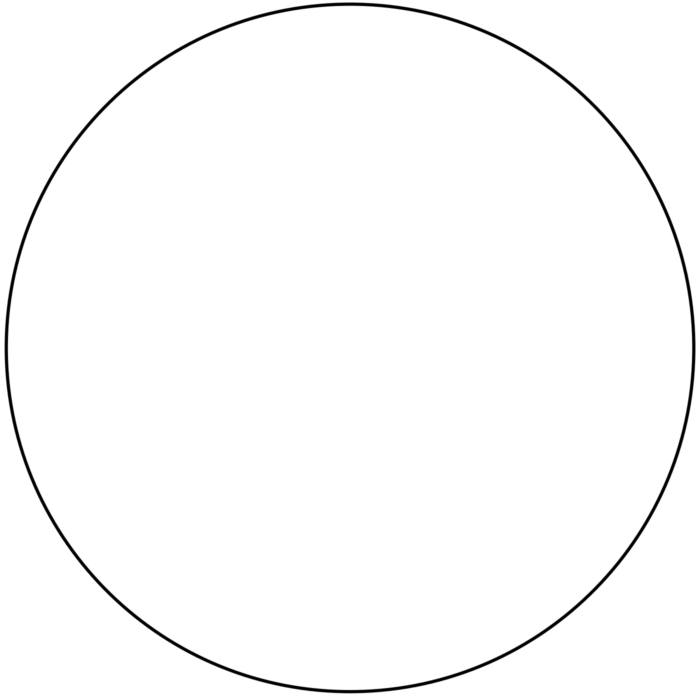

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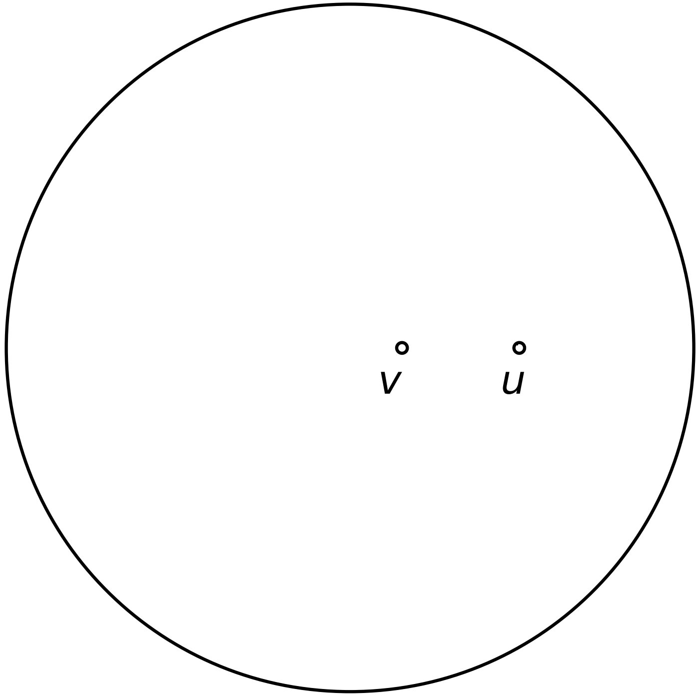

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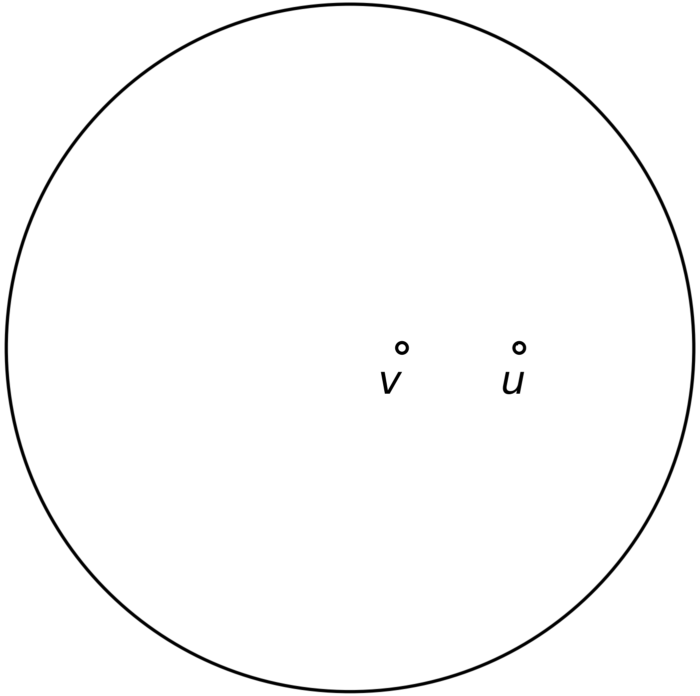

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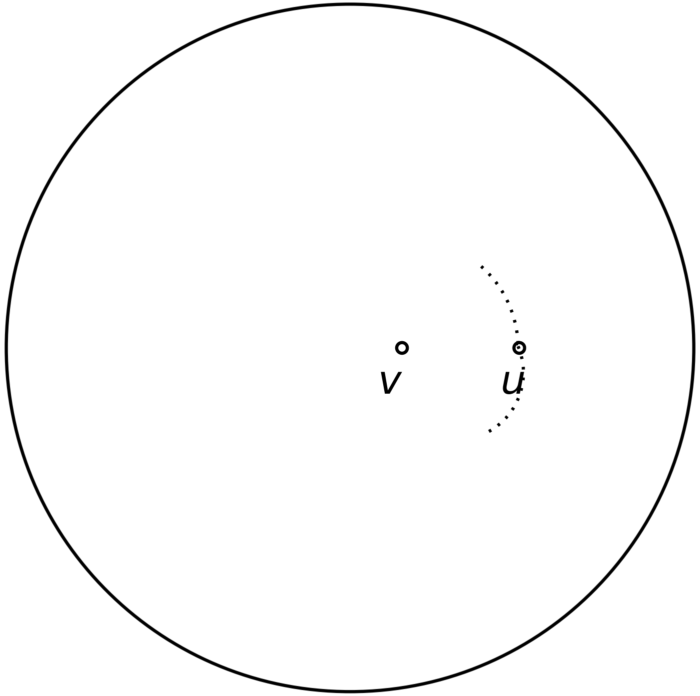

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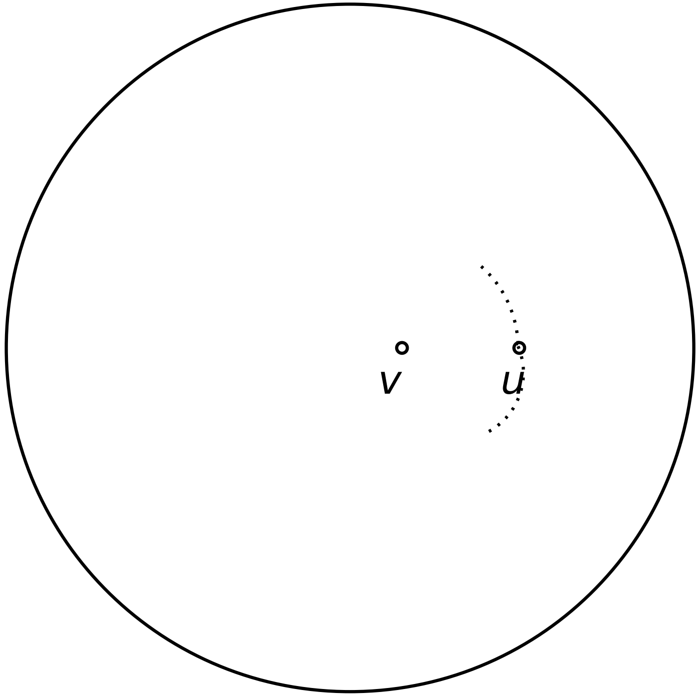

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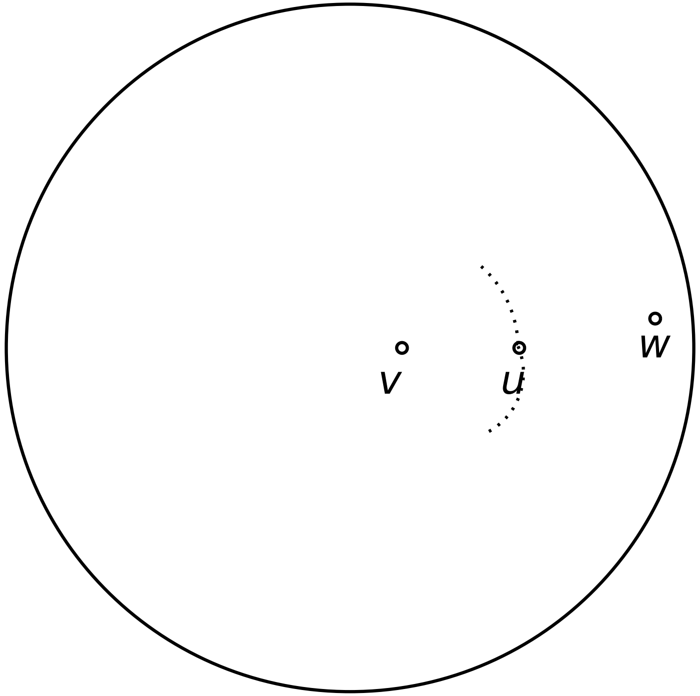

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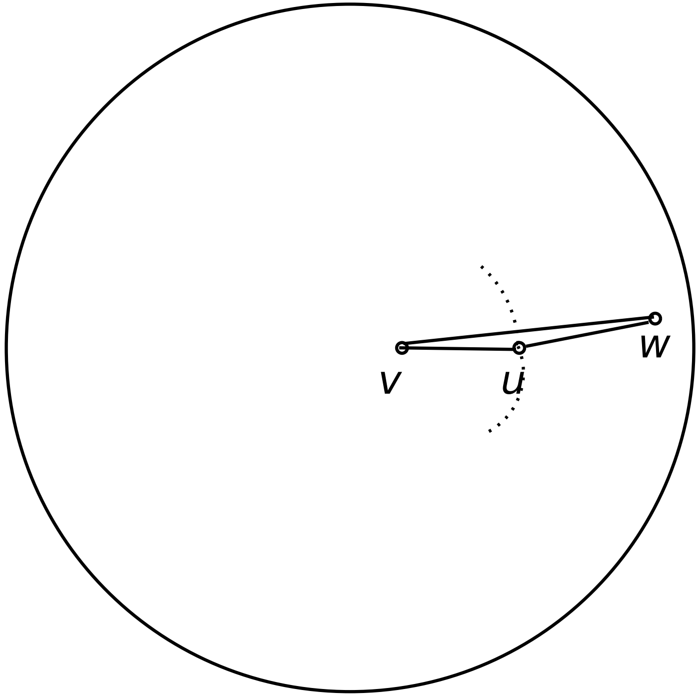

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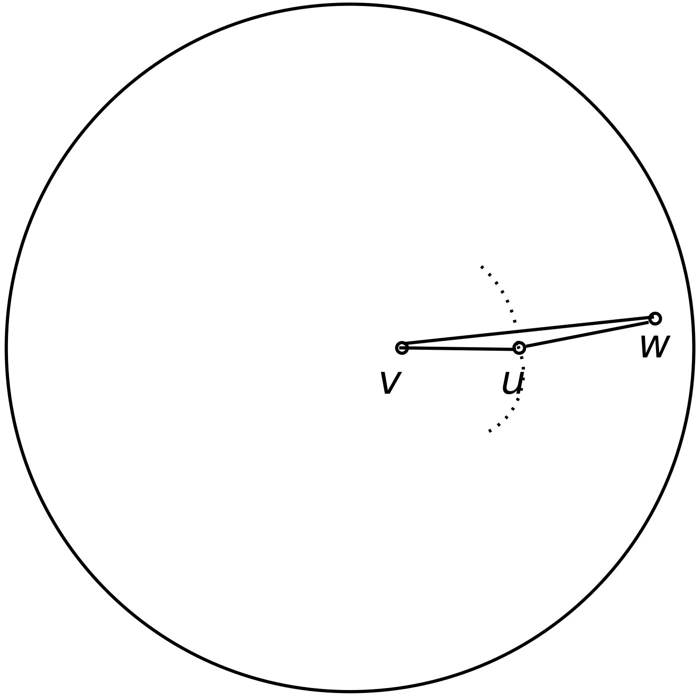

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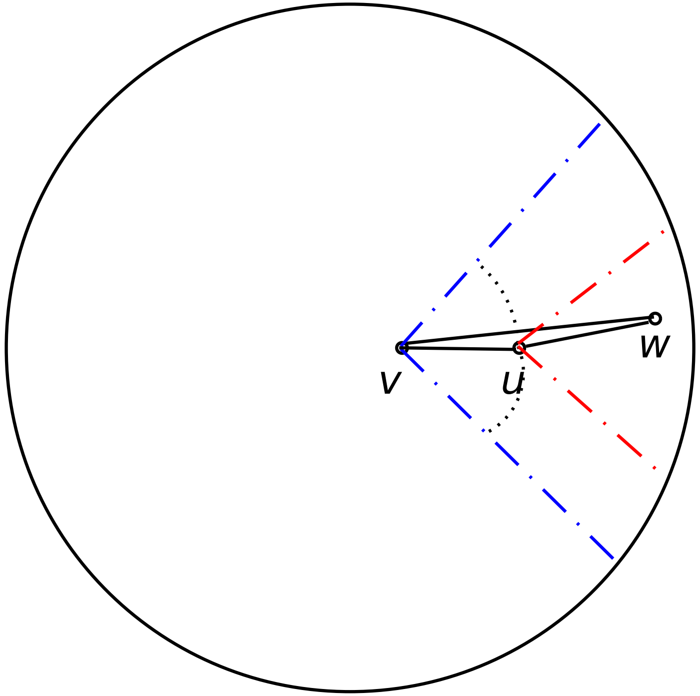

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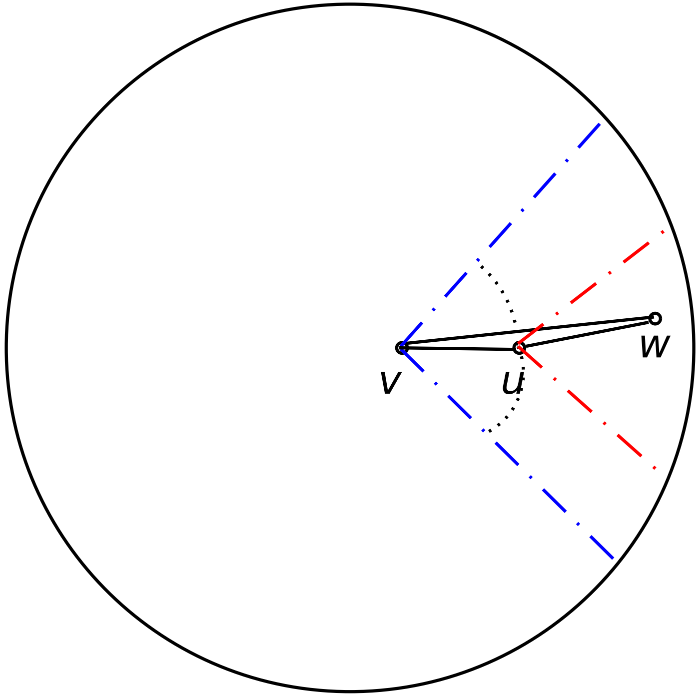

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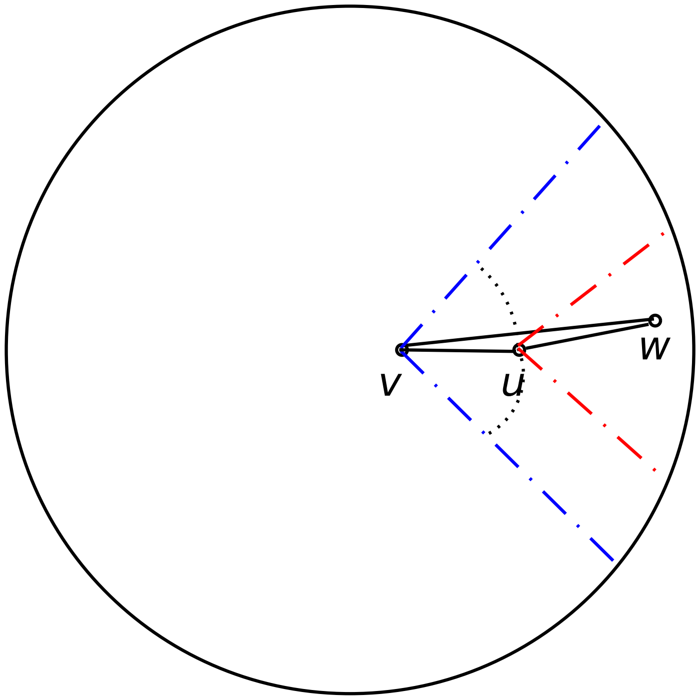

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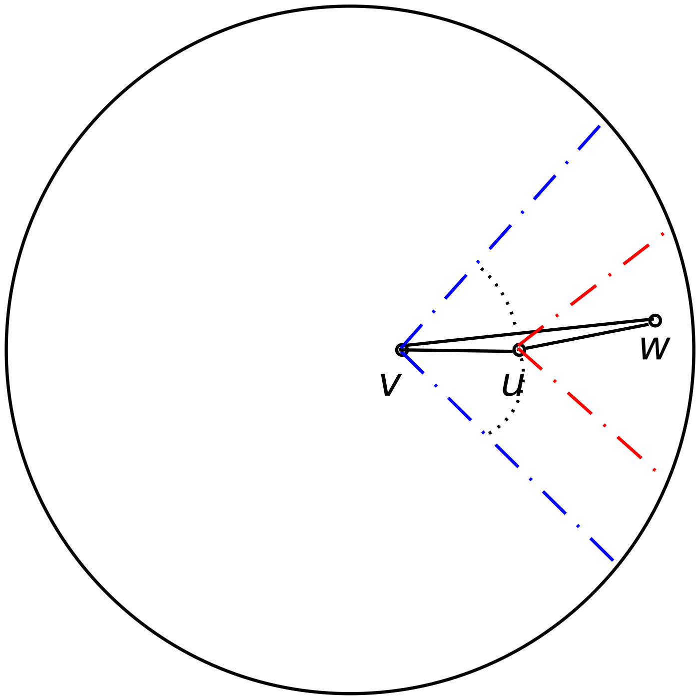

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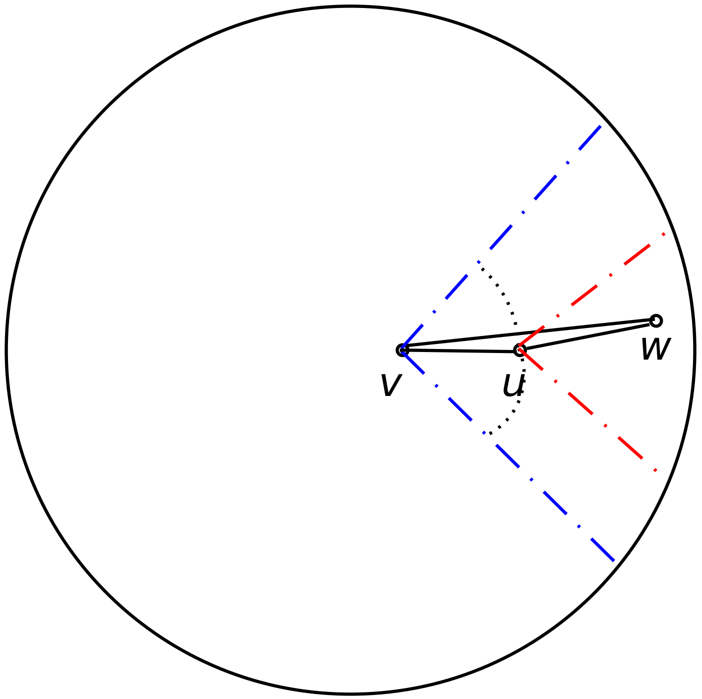

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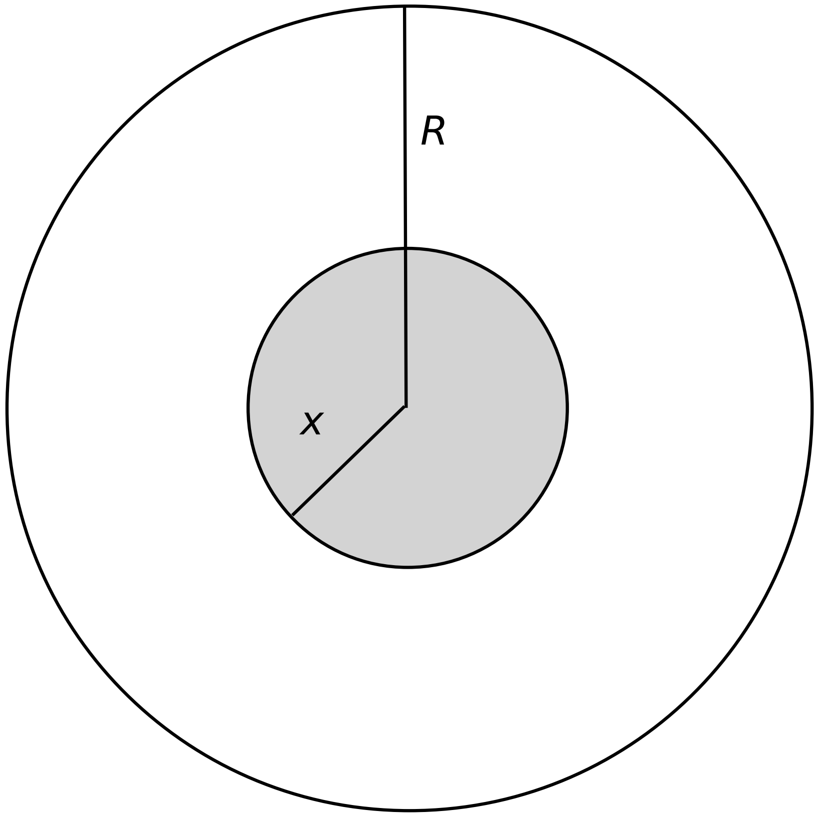

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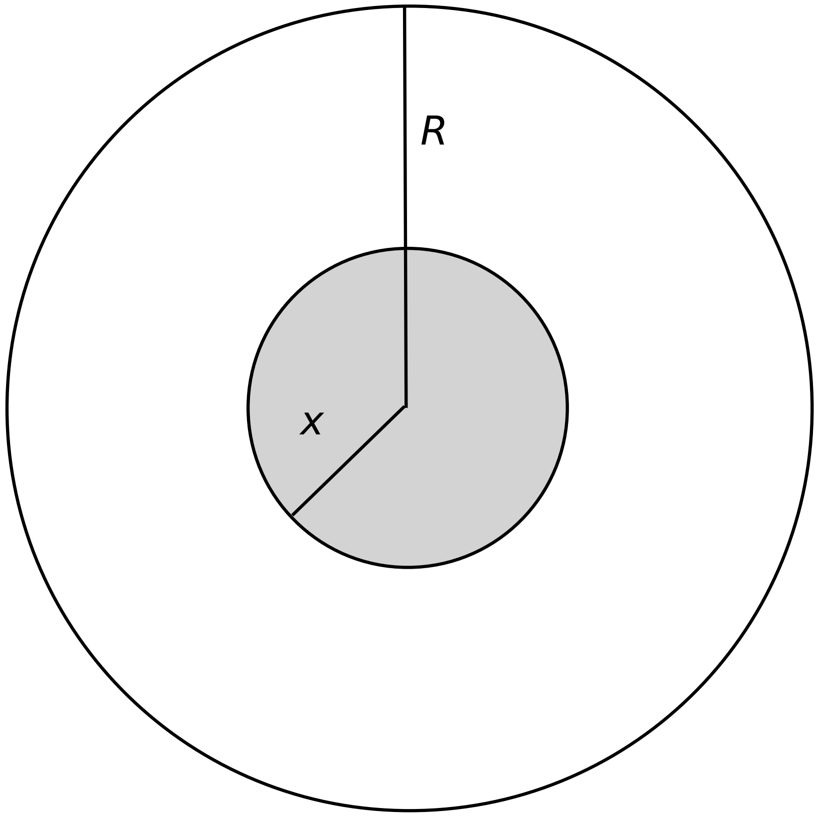

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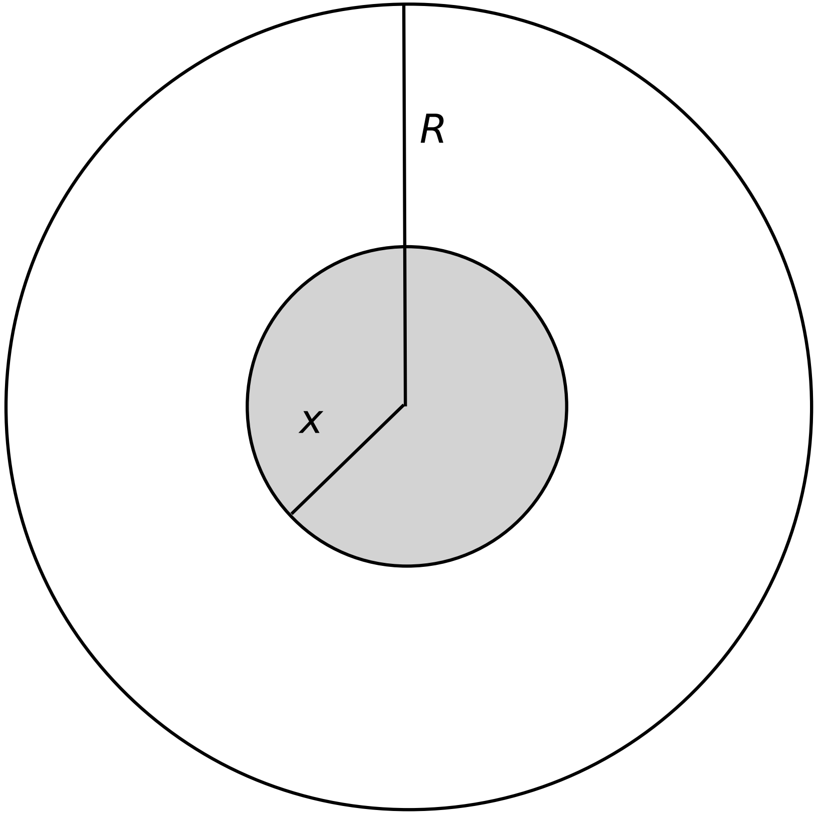

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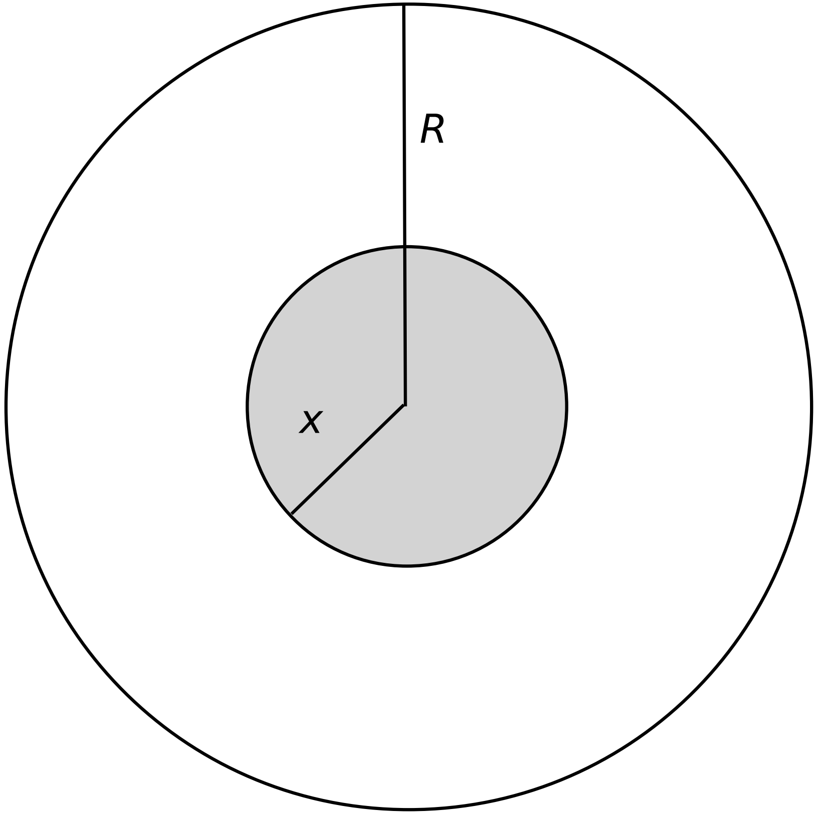

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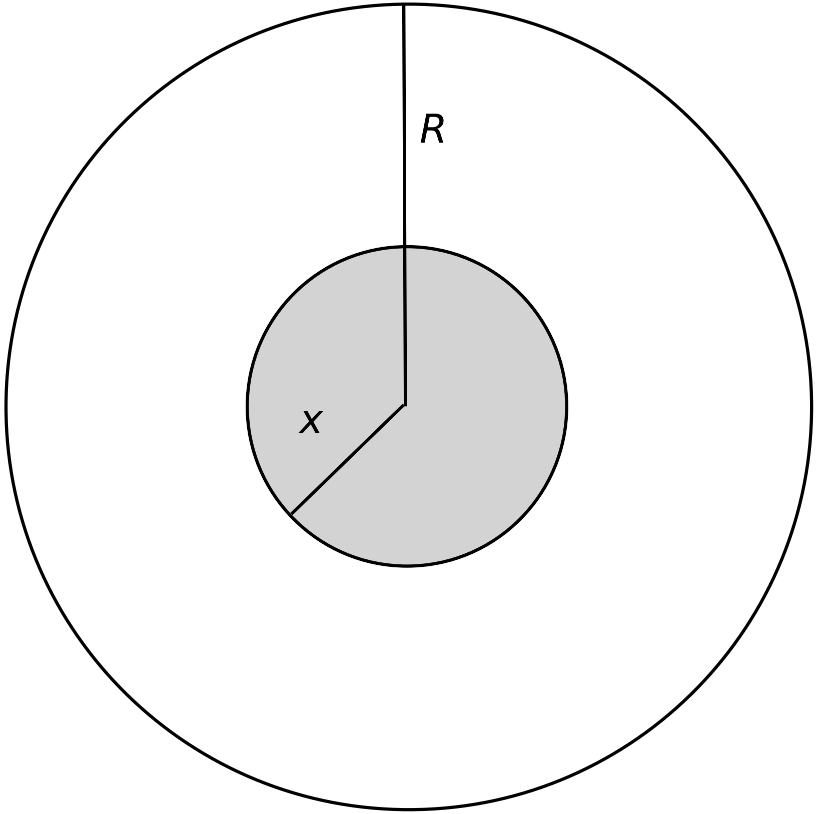

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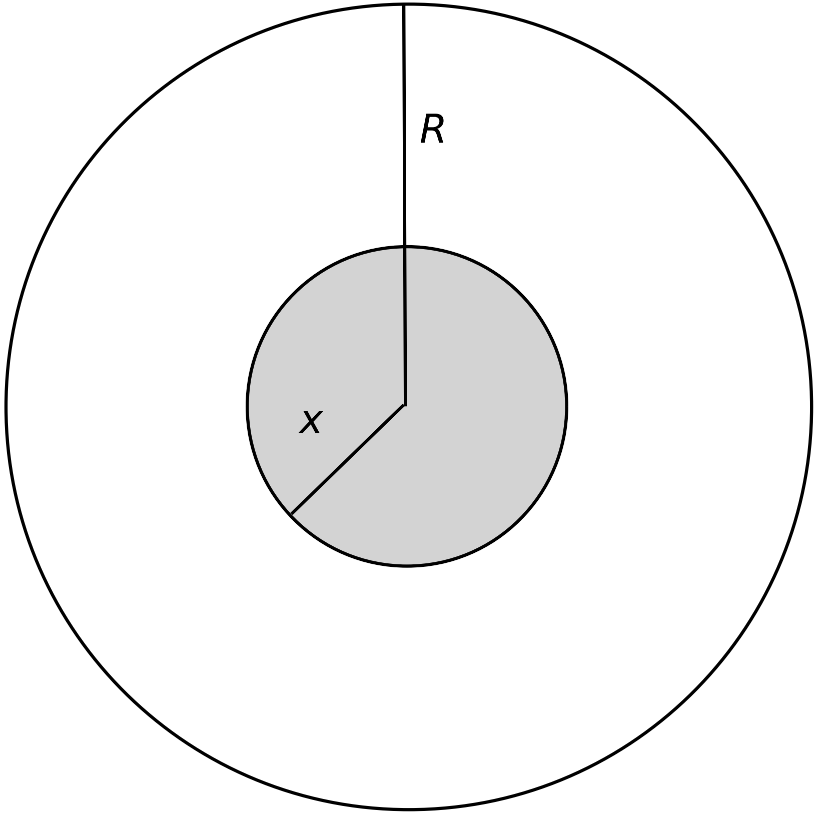

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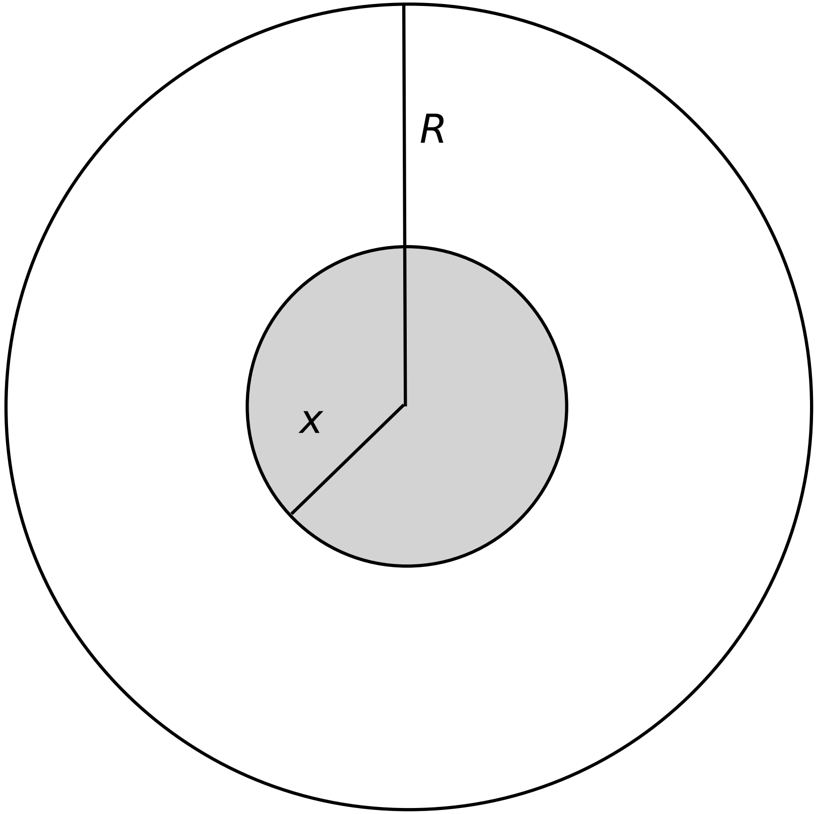

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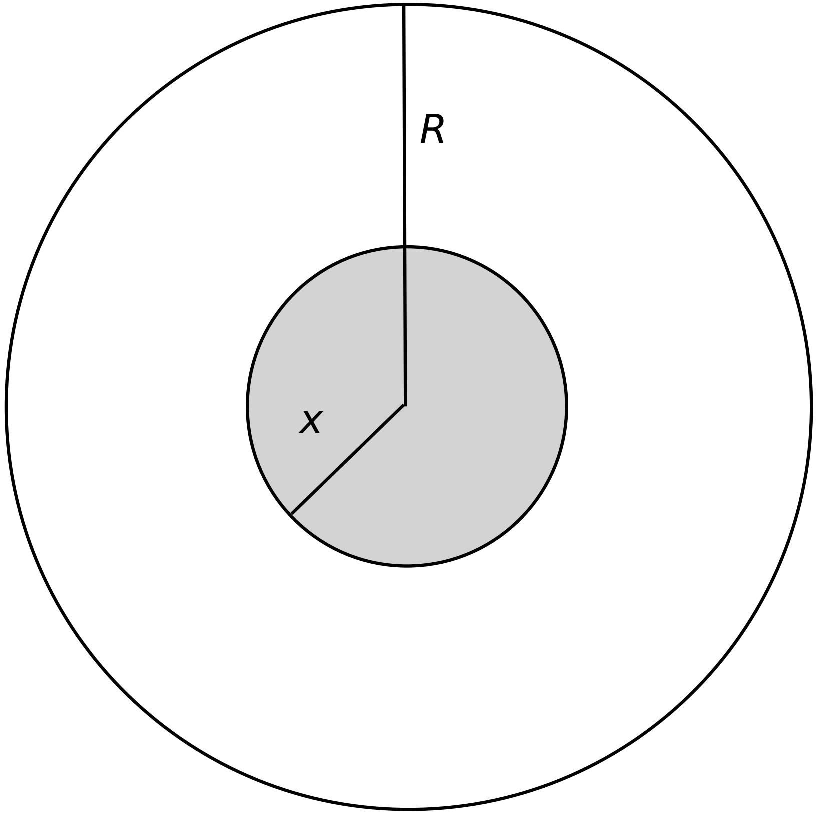

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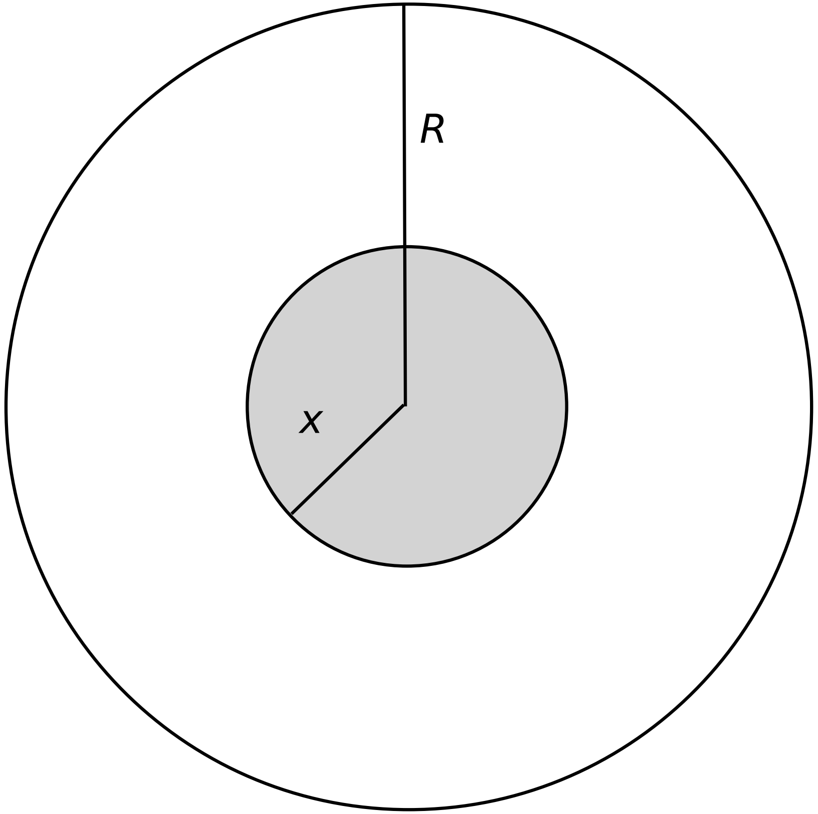

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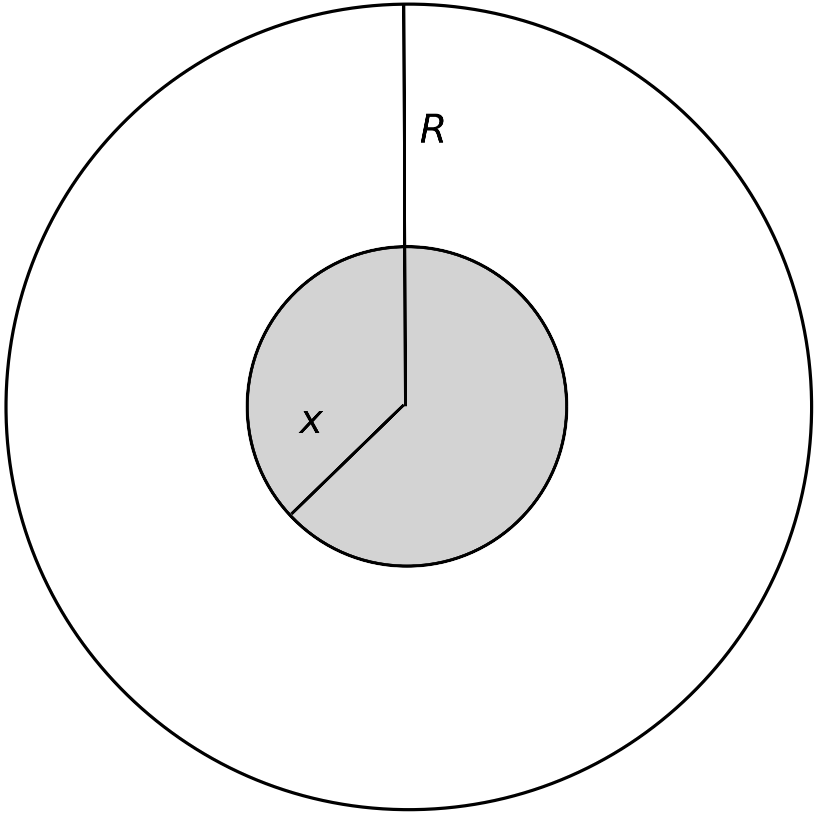

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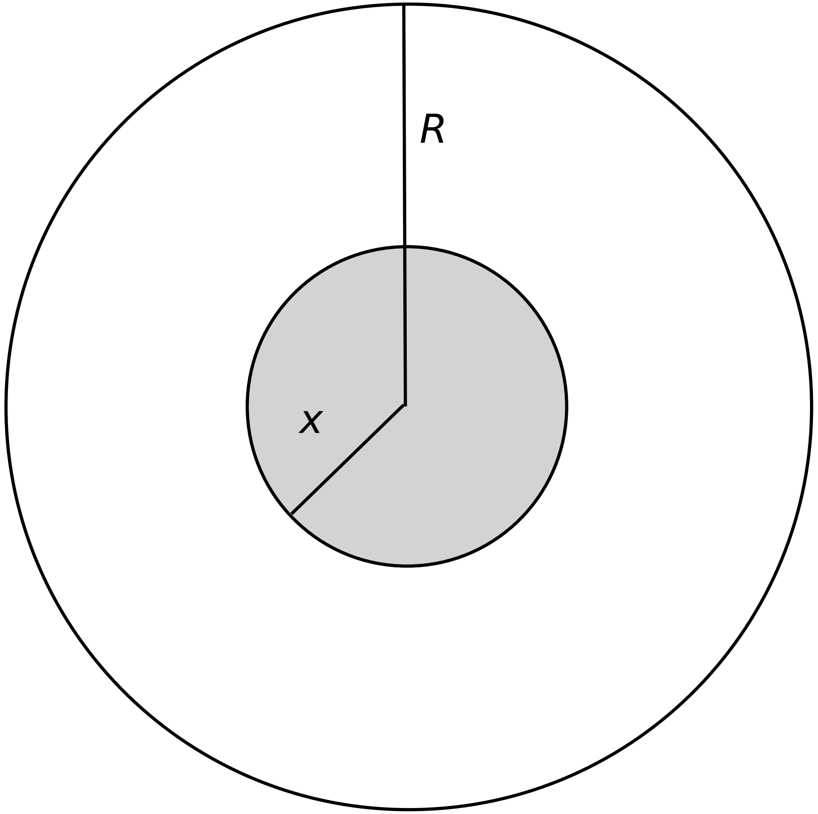

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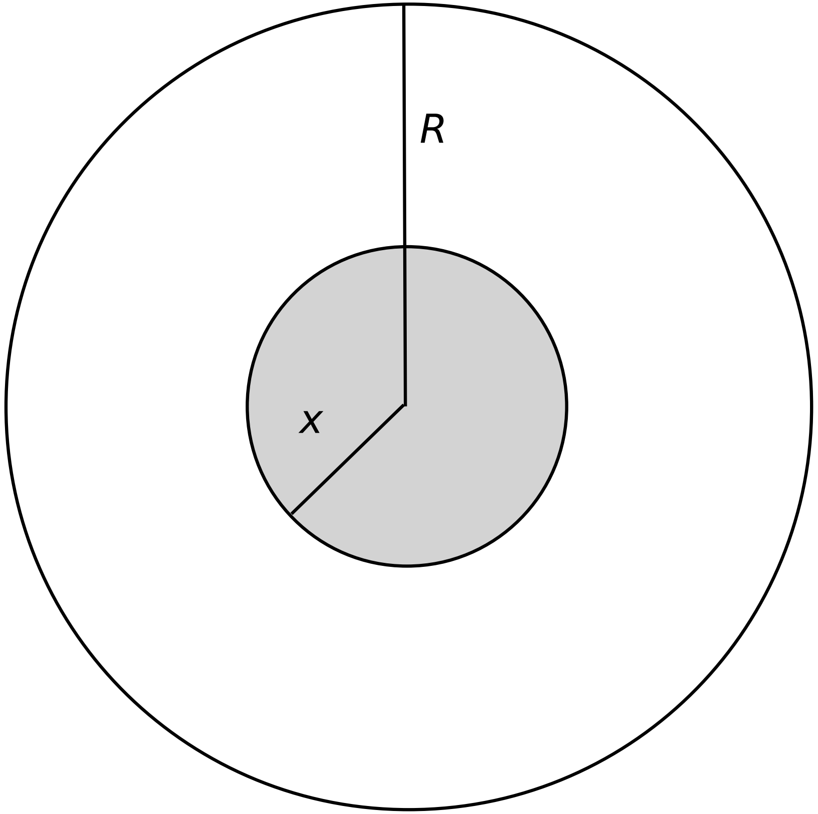

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Let x be a point with radius r.

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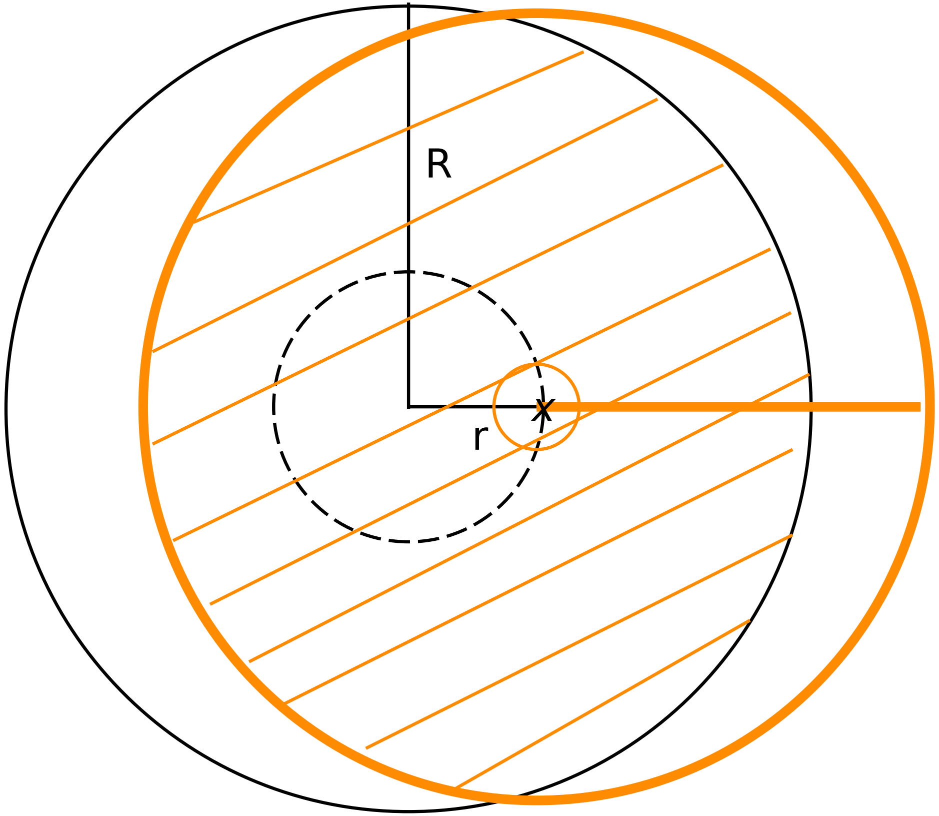

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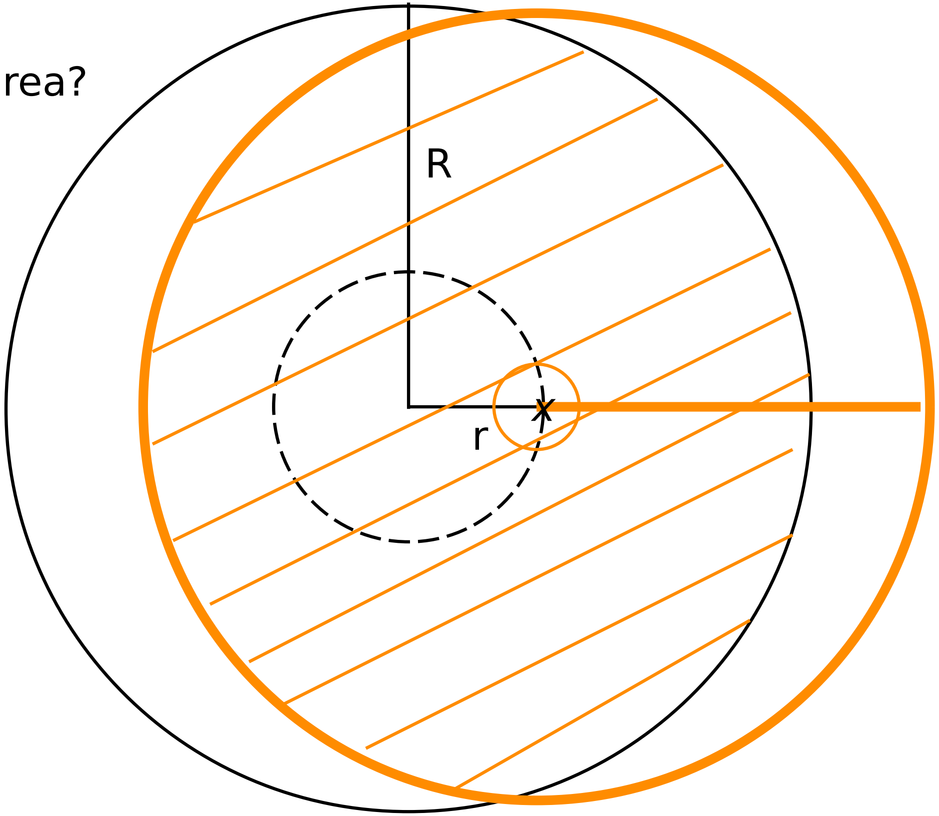

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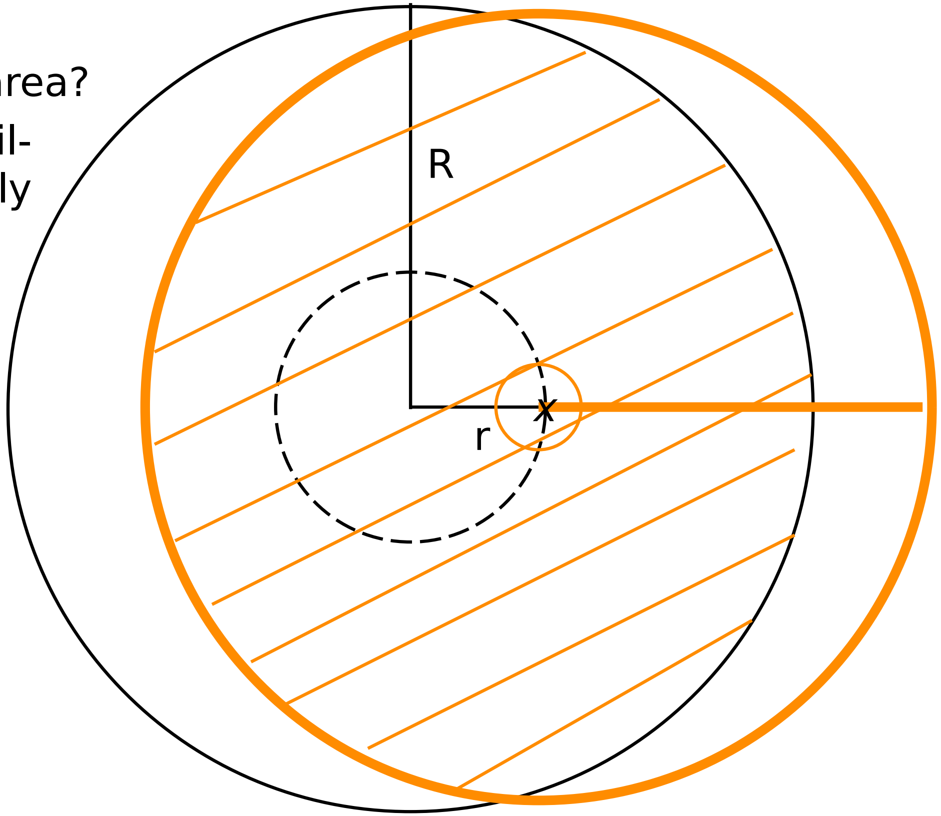

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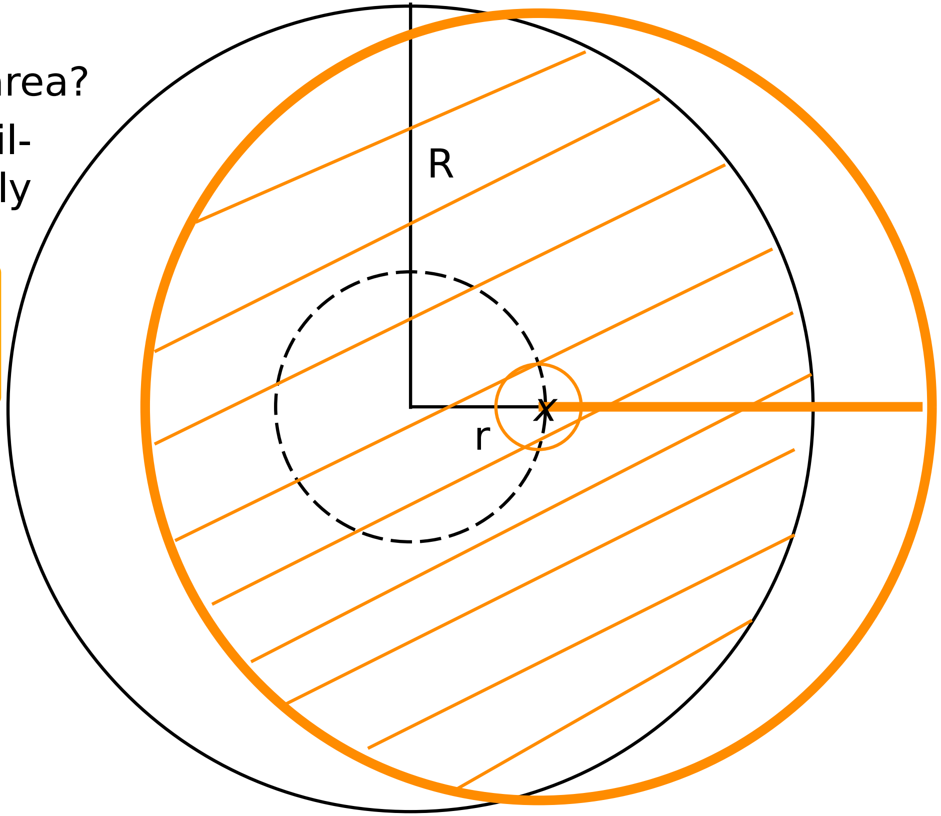

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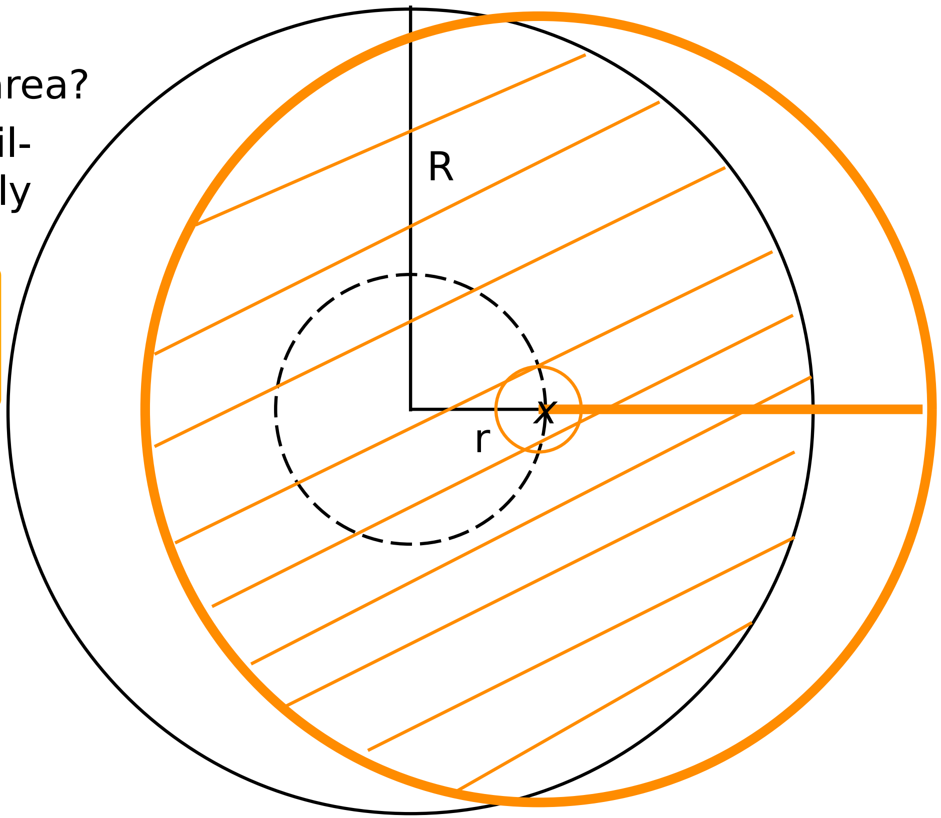

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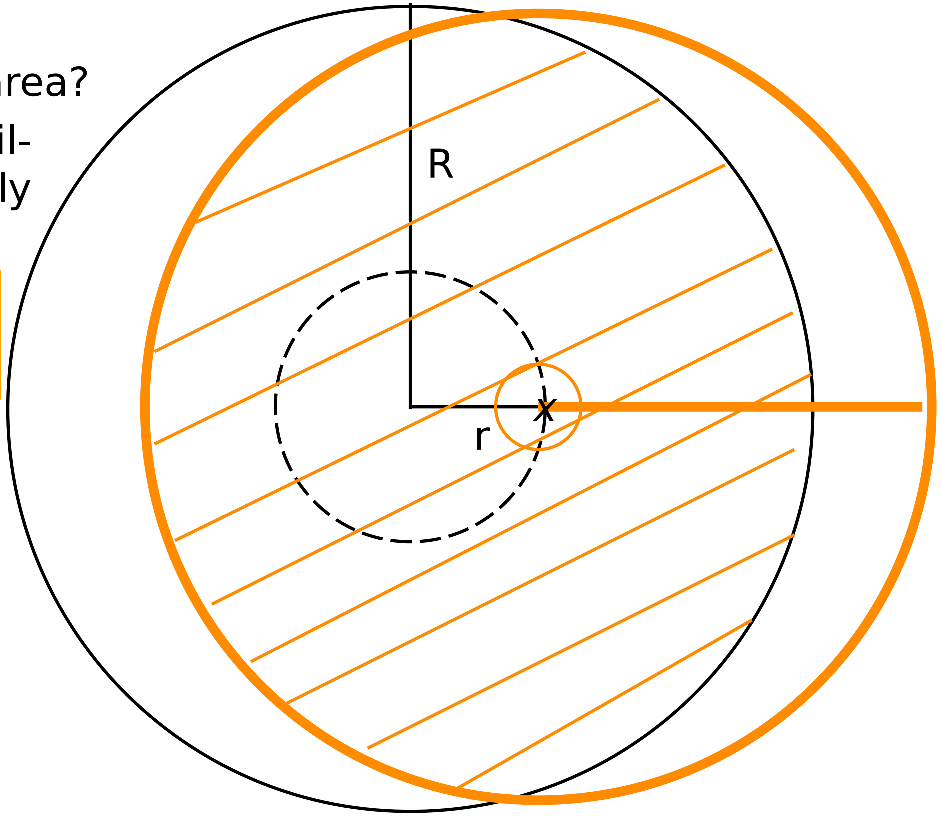

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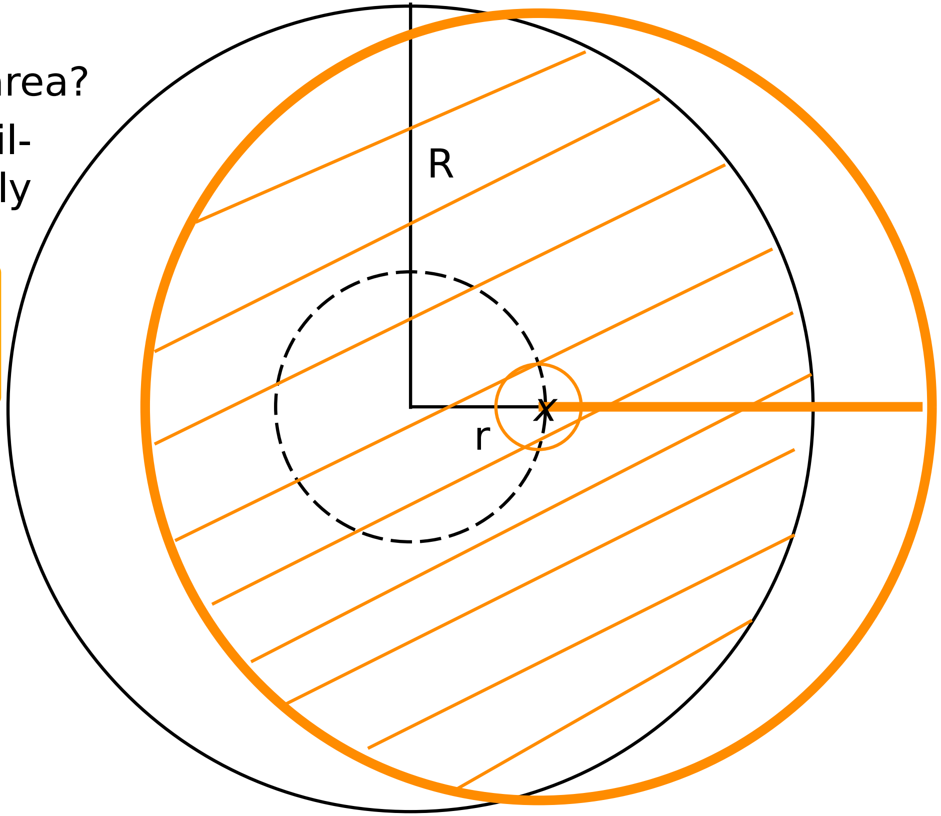

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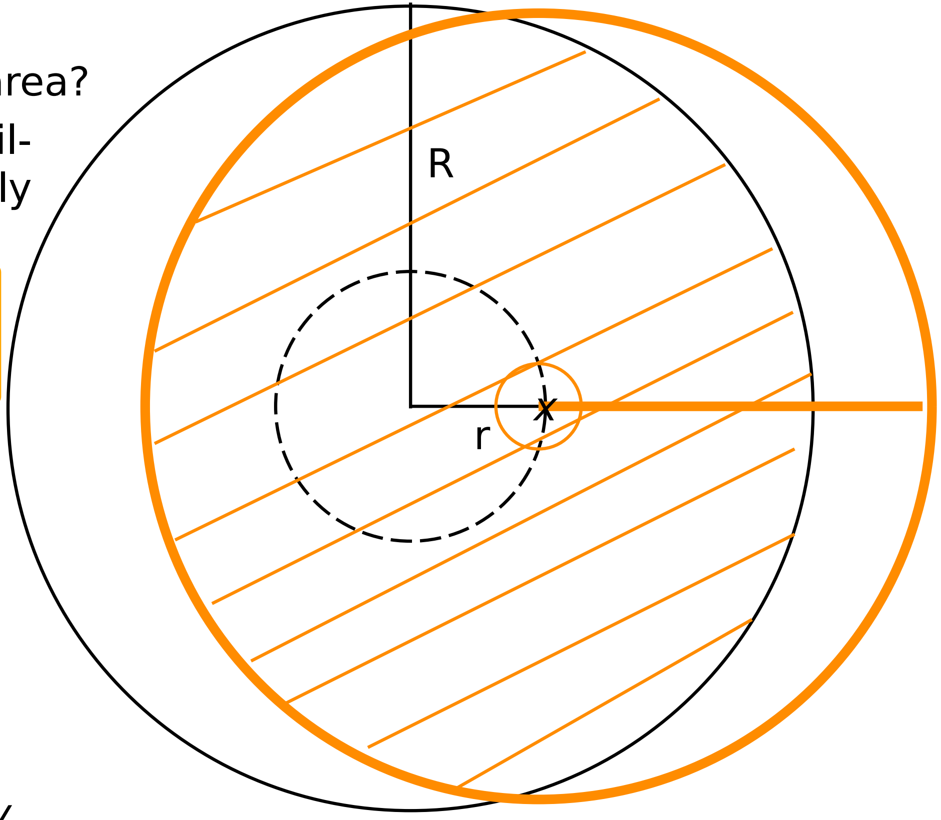

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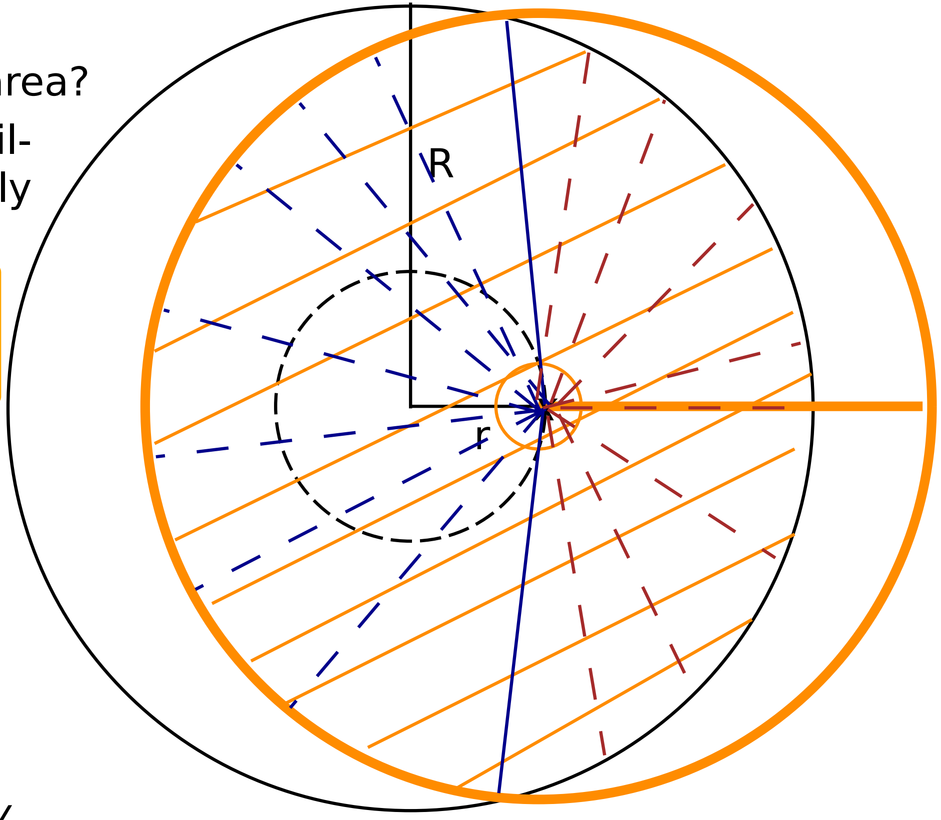

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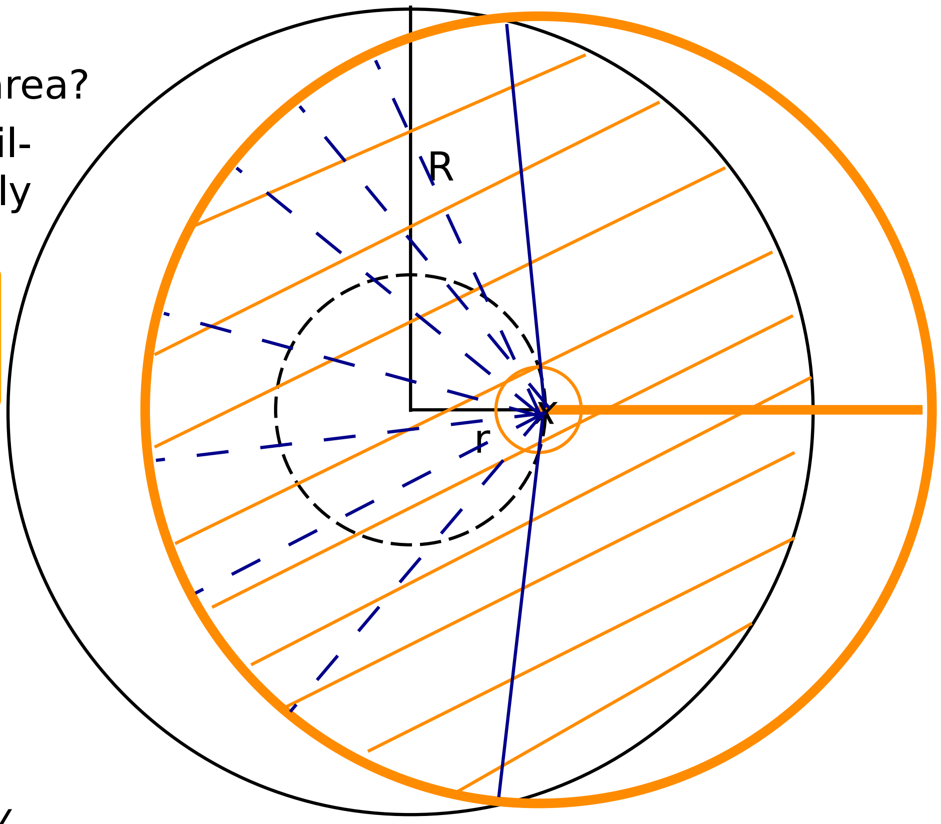

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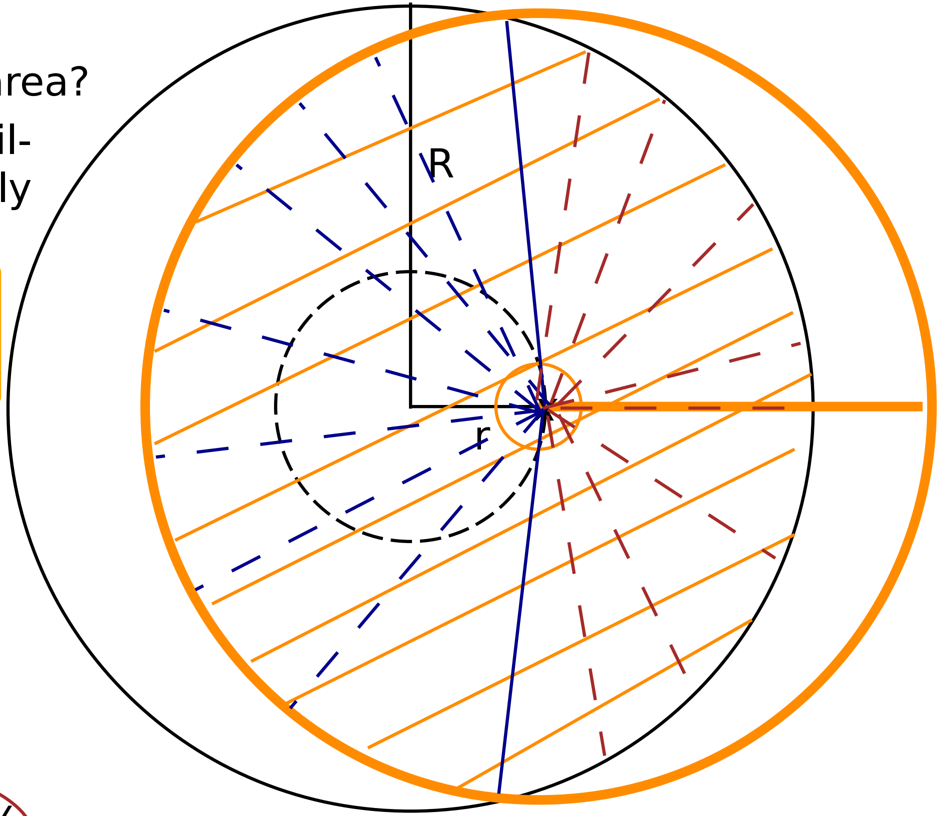

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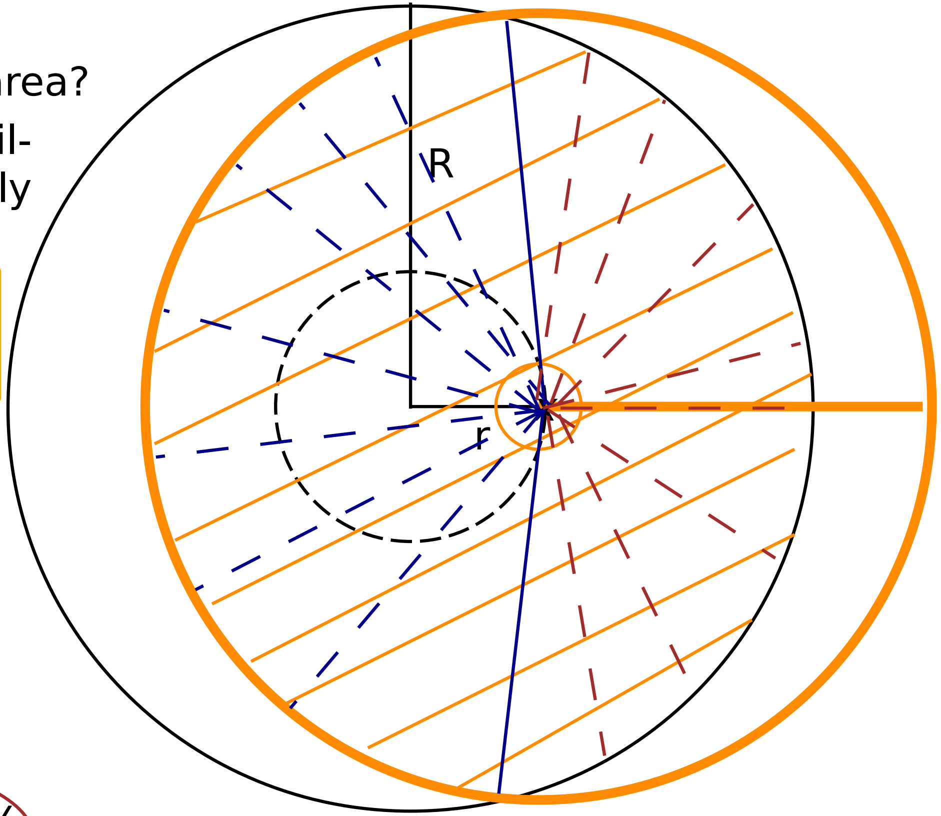

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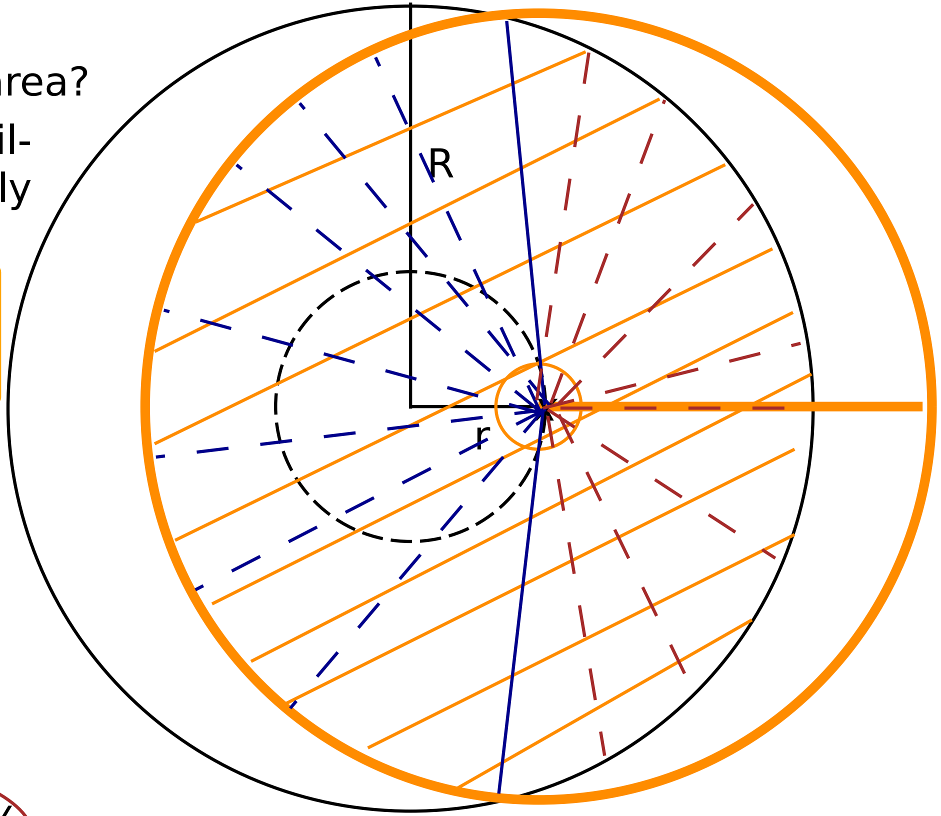

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For the details check the paper! For refinements of the lemma check out the cheat sheet!

A variant of the previous lemma

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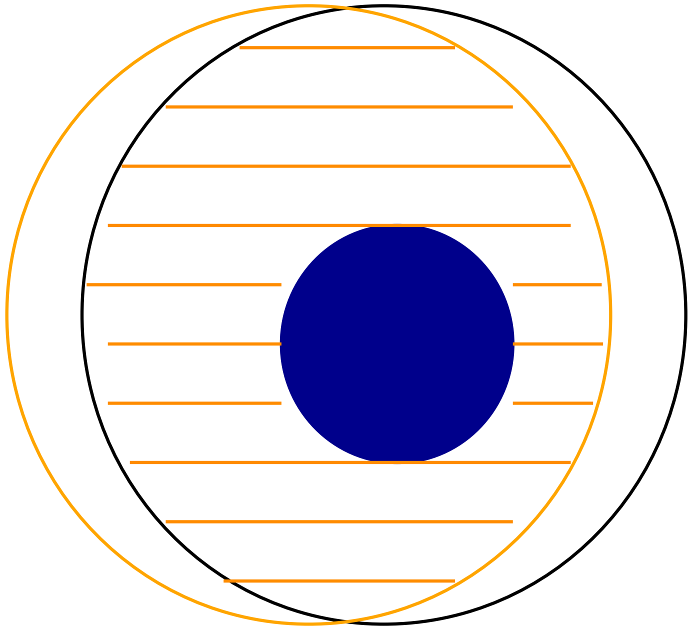

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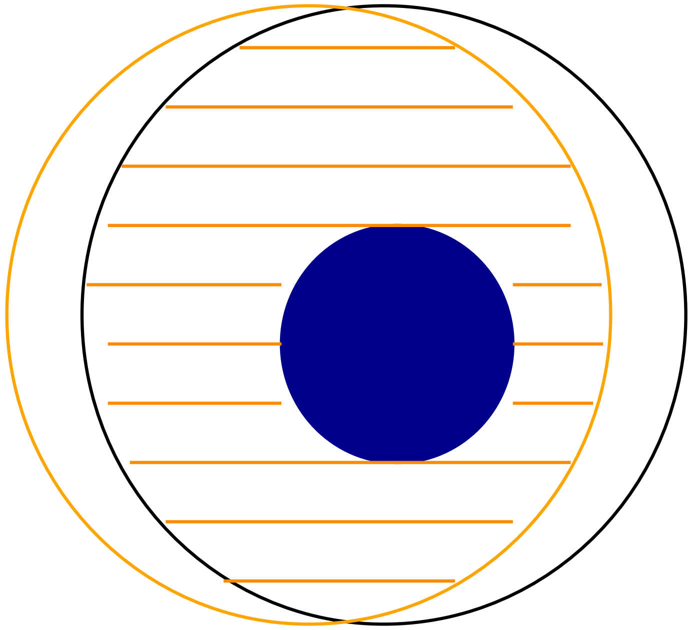

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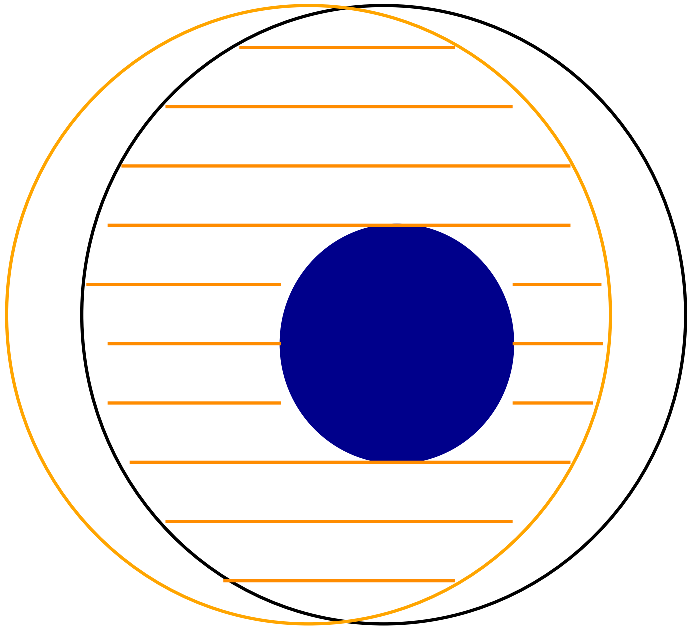

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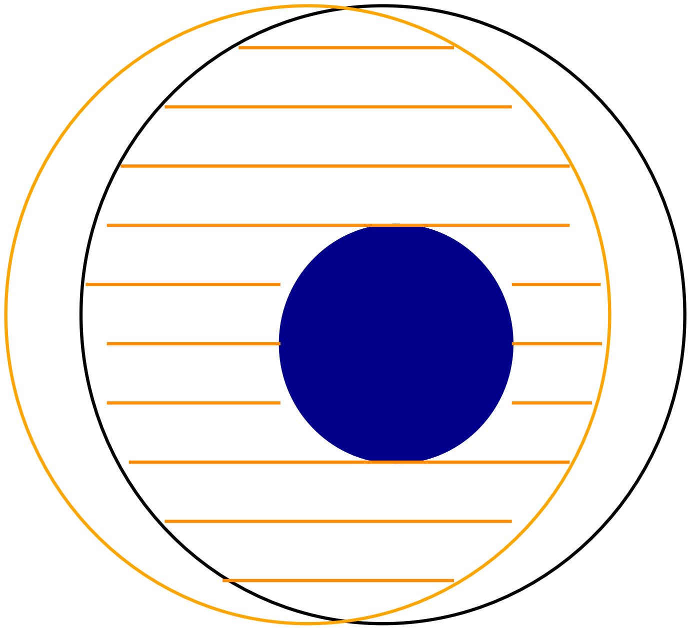

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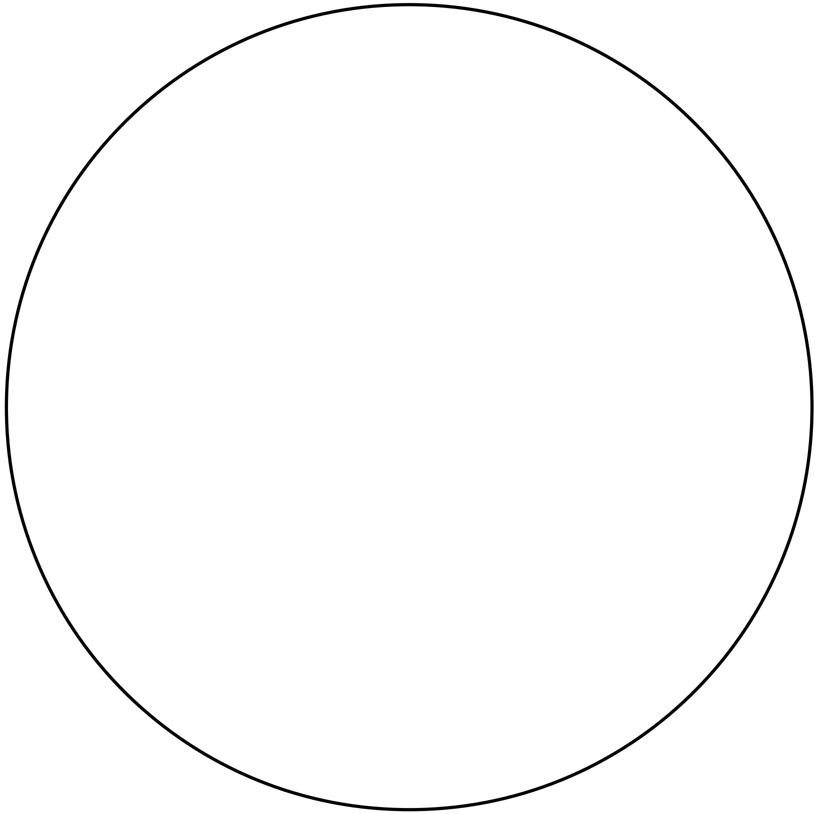

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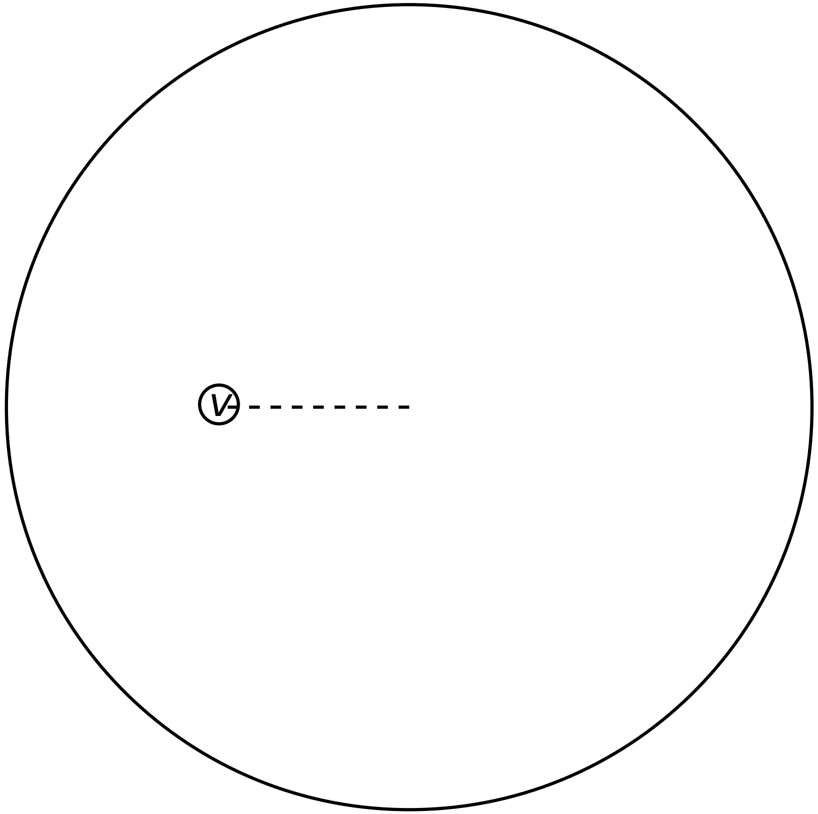

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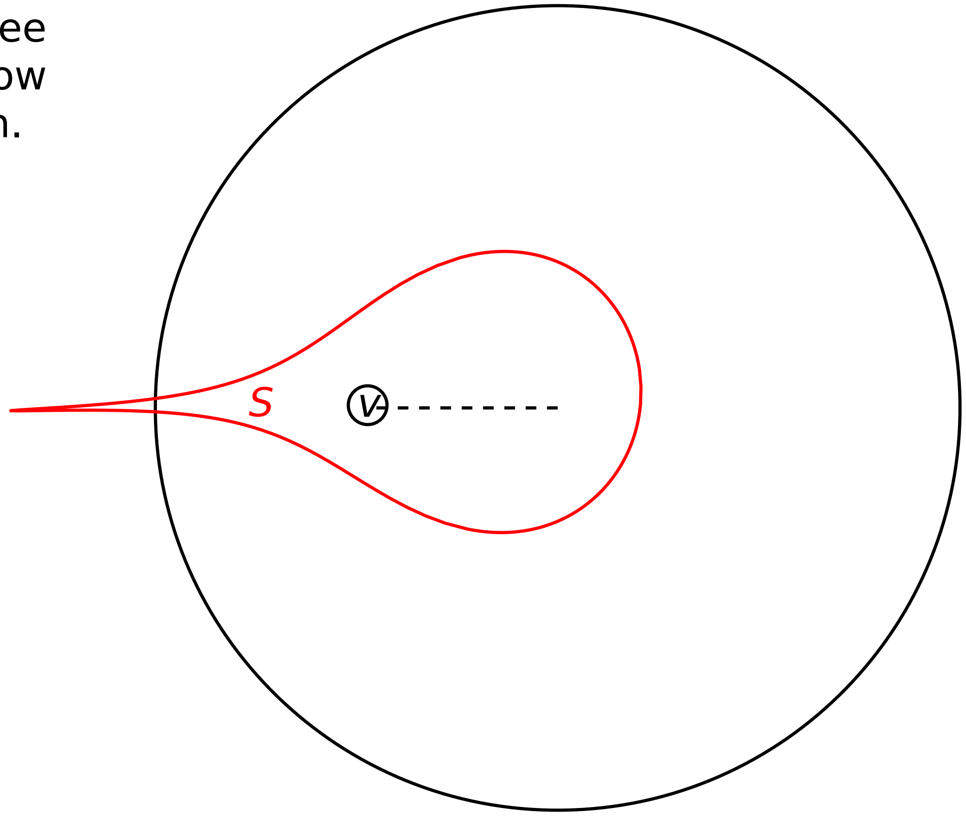
Notice: $S = \{u \in G(V) | u \in (B_{r_v}(R) \cap B_0(R))\}$ are exactly the points connected to v.
So we need to compute $\mathbb{E}[|S|] = \delta(v)$.



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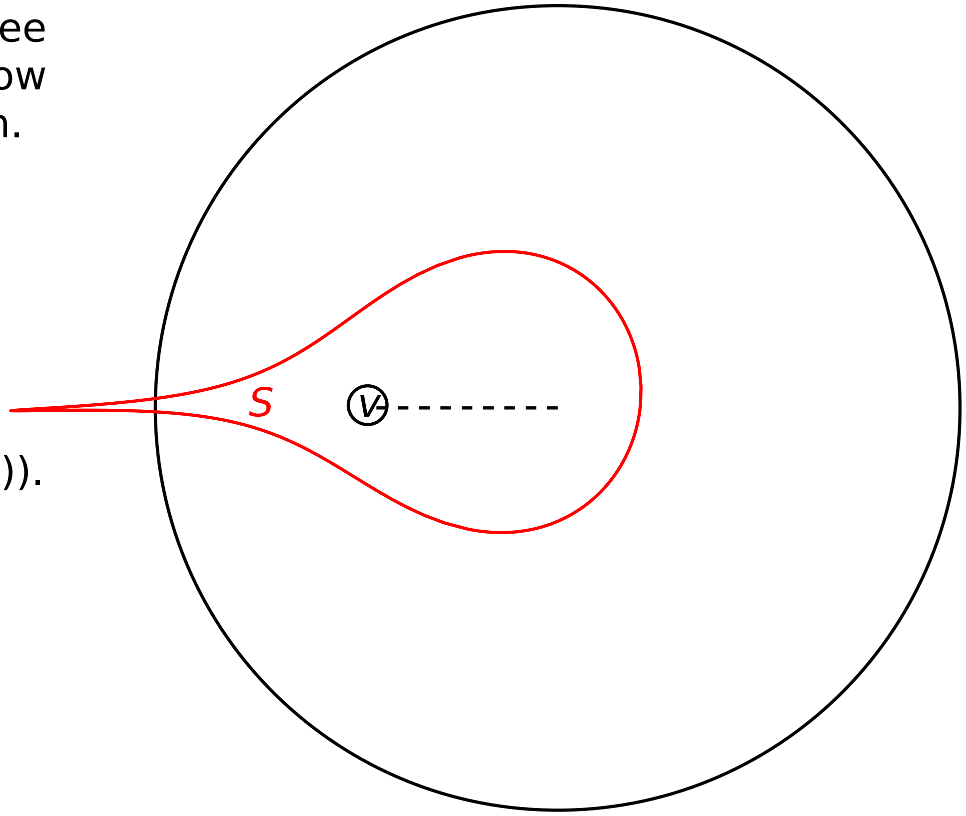

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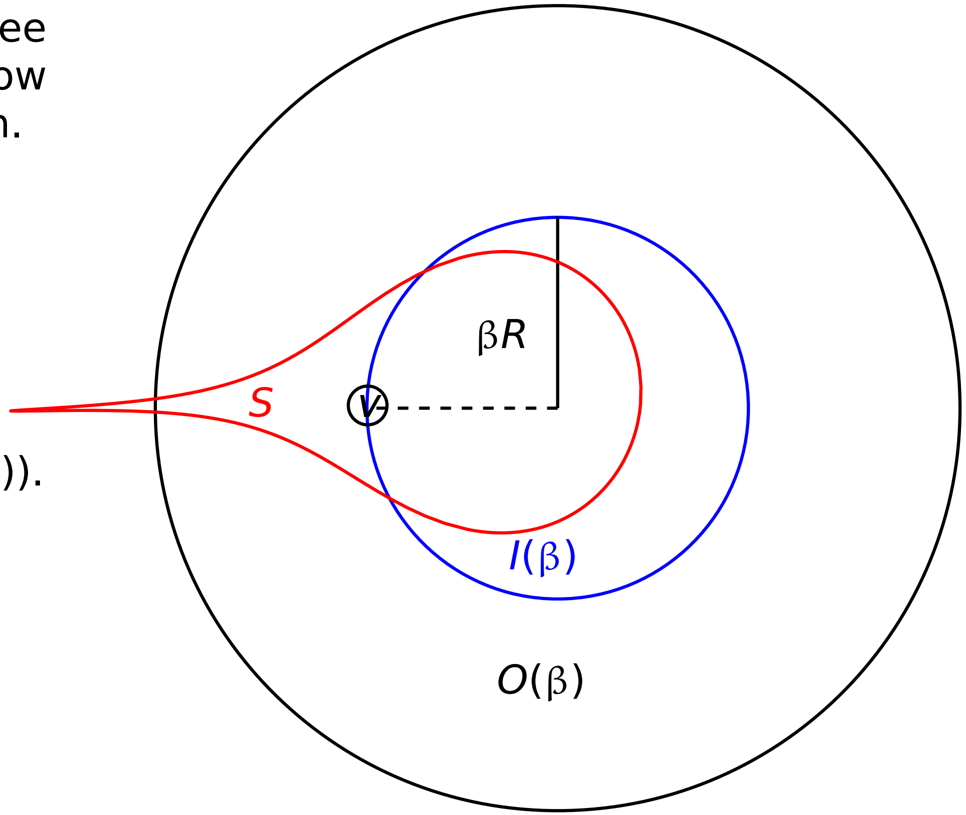

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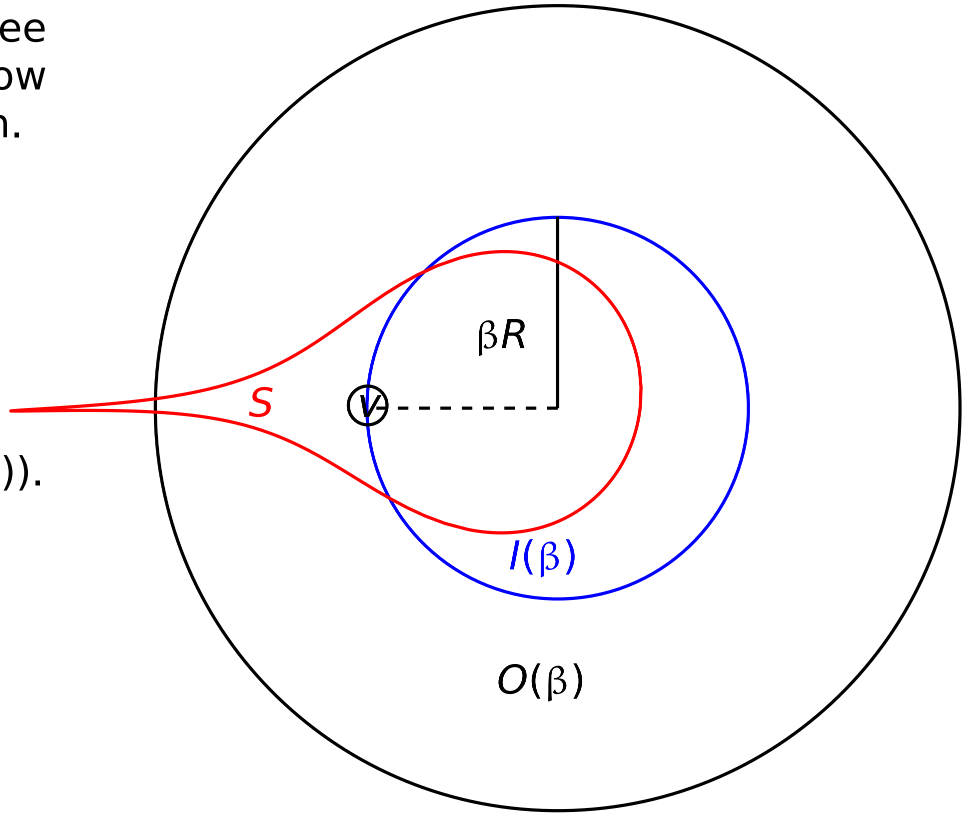

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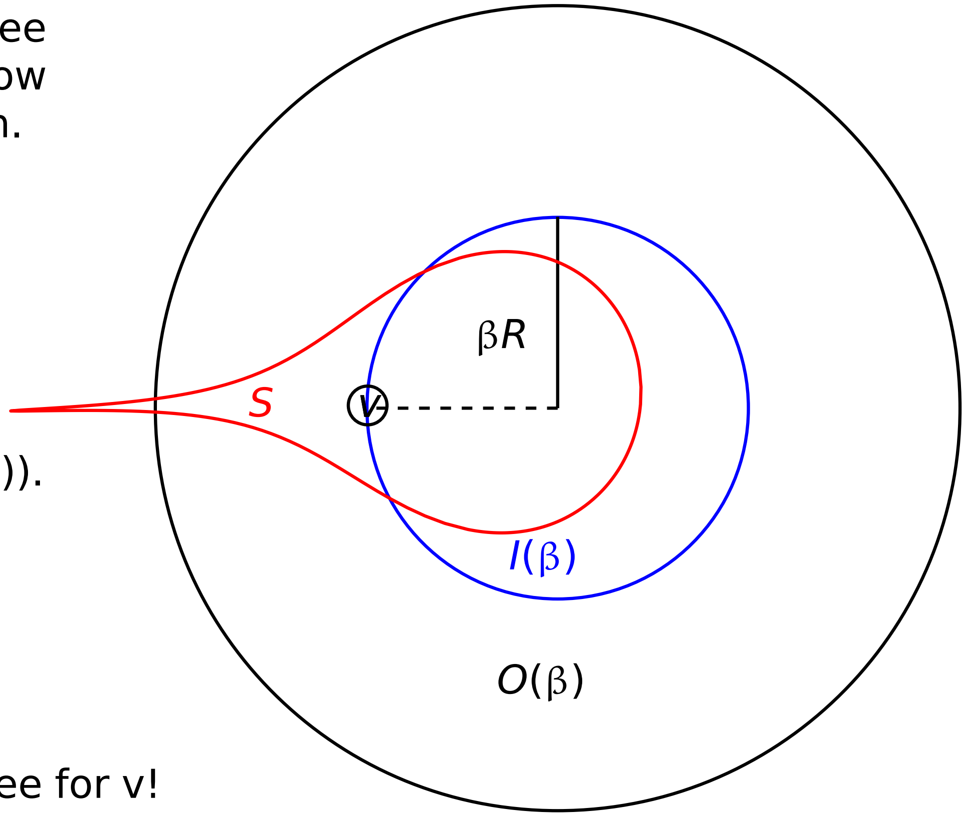

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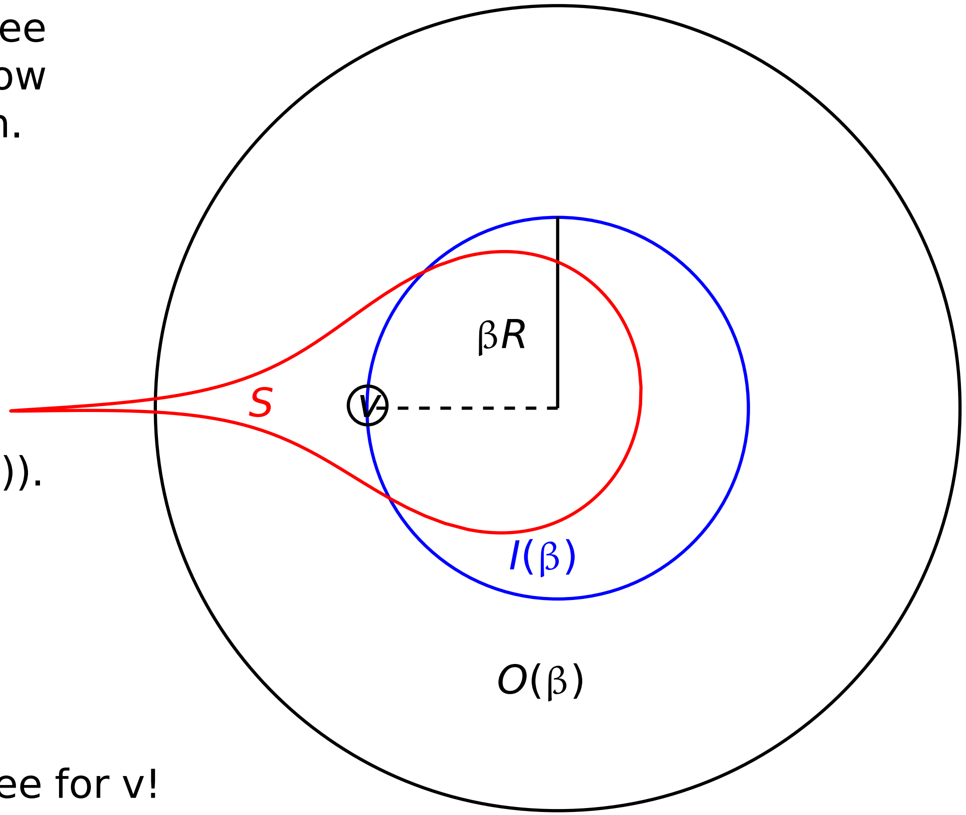

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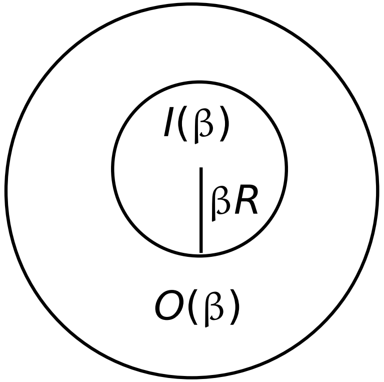

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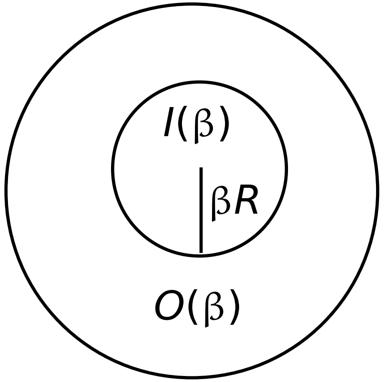

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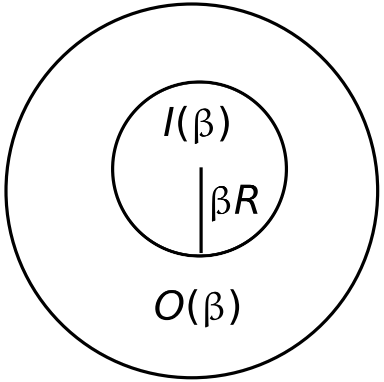

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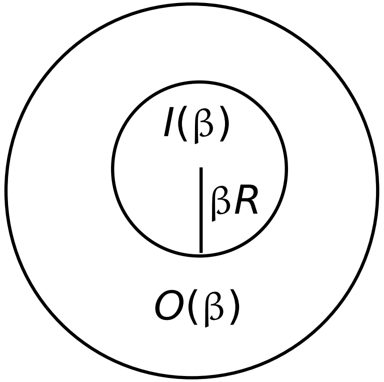

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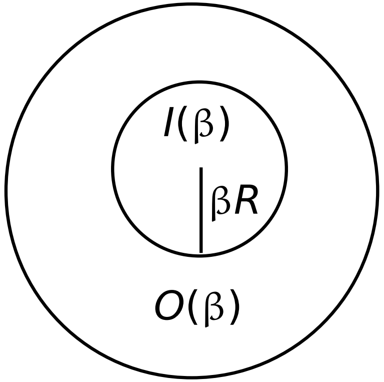

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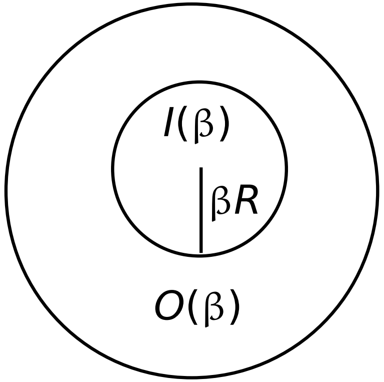

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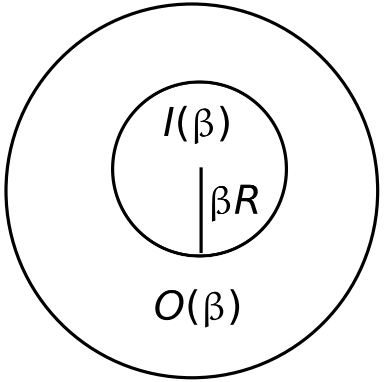

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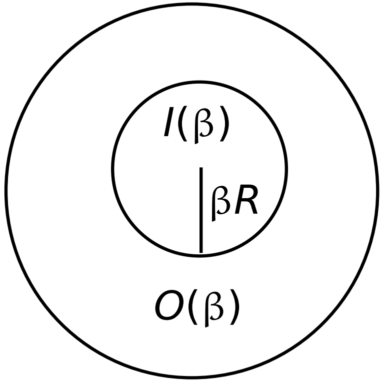

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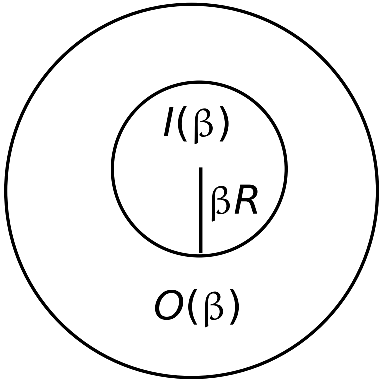

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Theorem: Let $\alpha > 1/2$ and $C \in \mathbb{R}$ for $G_{\alpha,C}(n)$.
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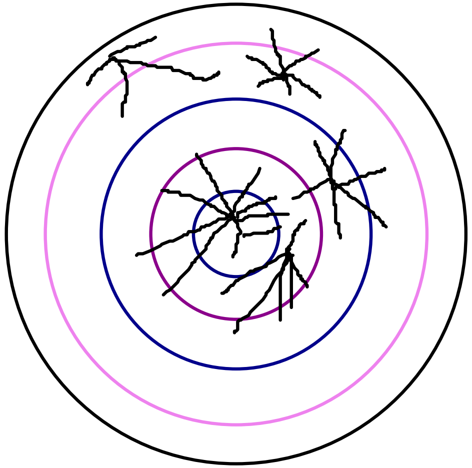

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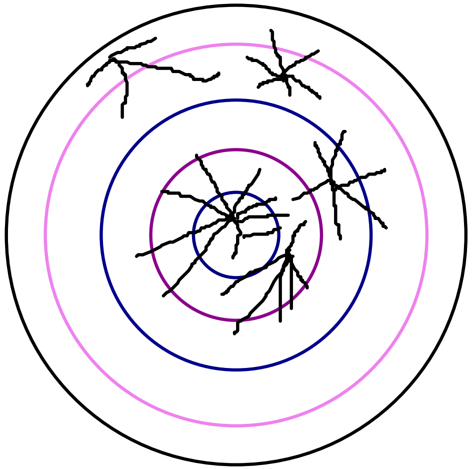

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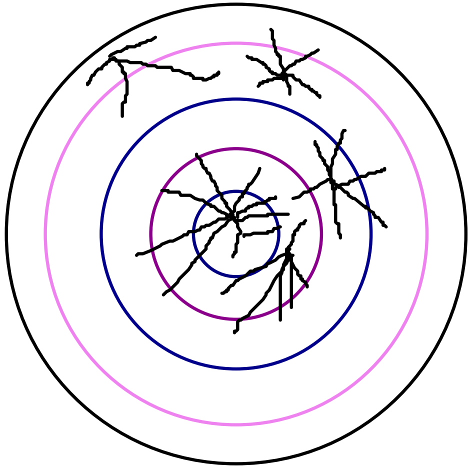

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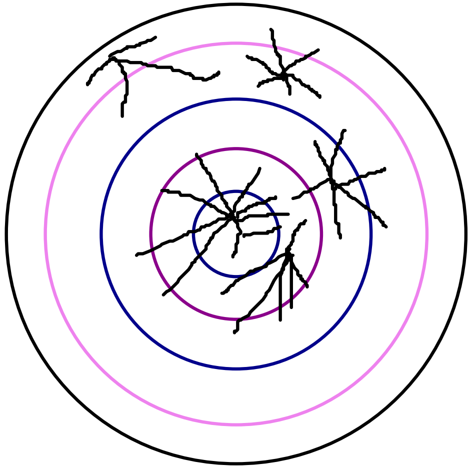

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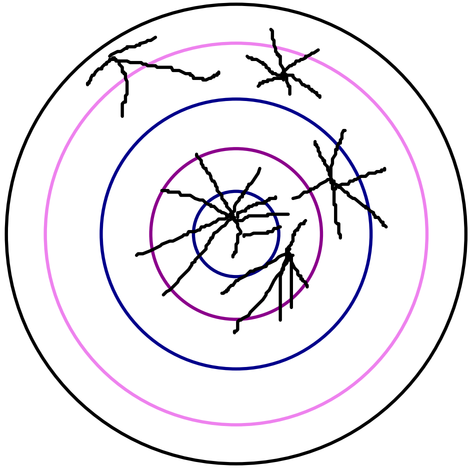

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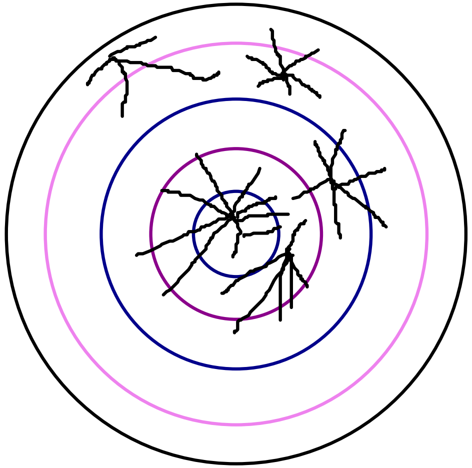

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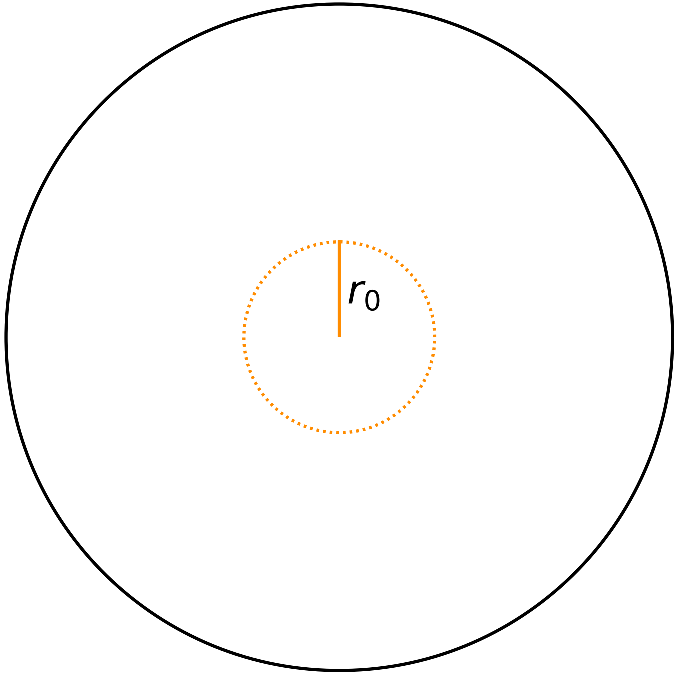

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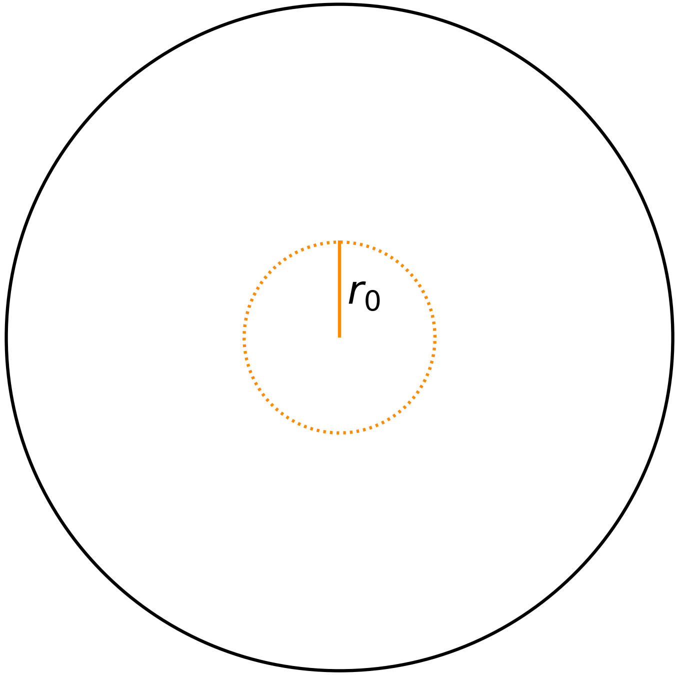

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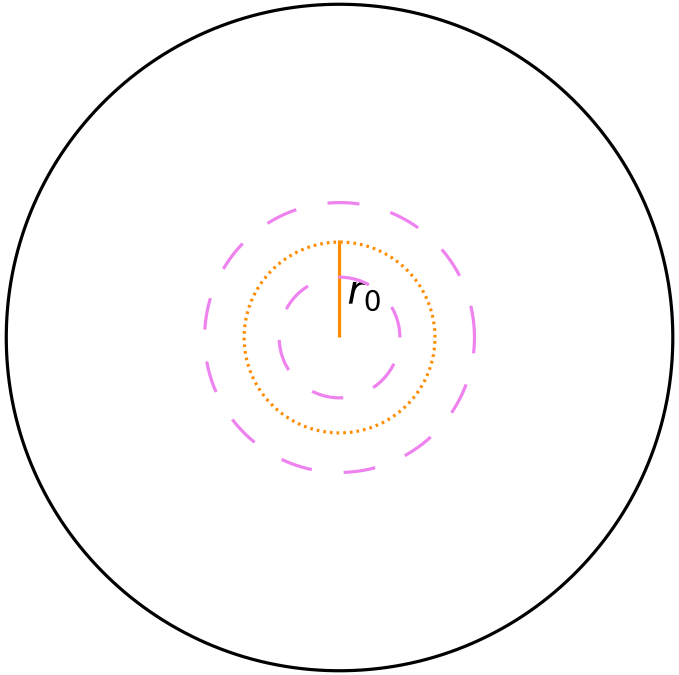

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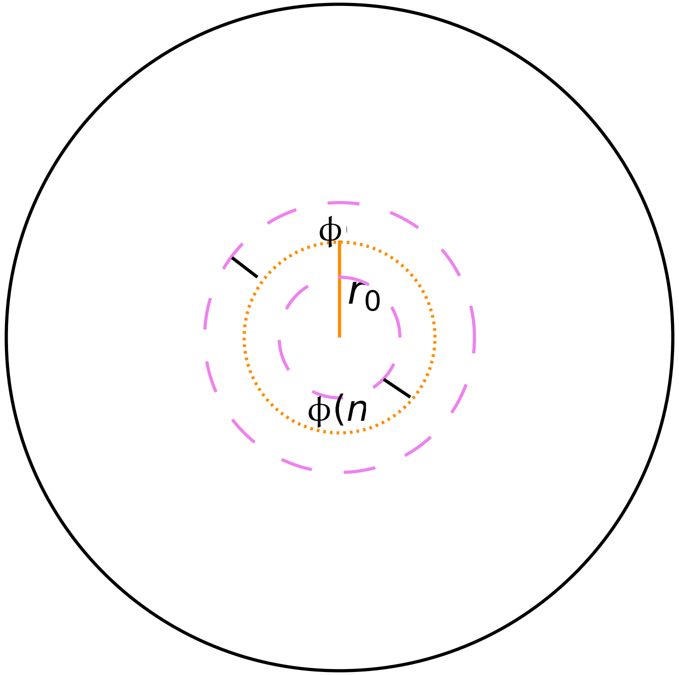

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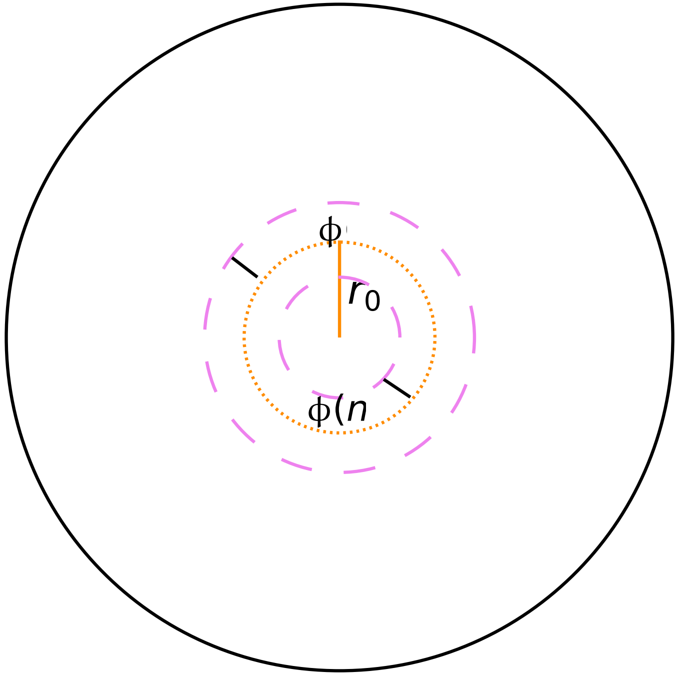

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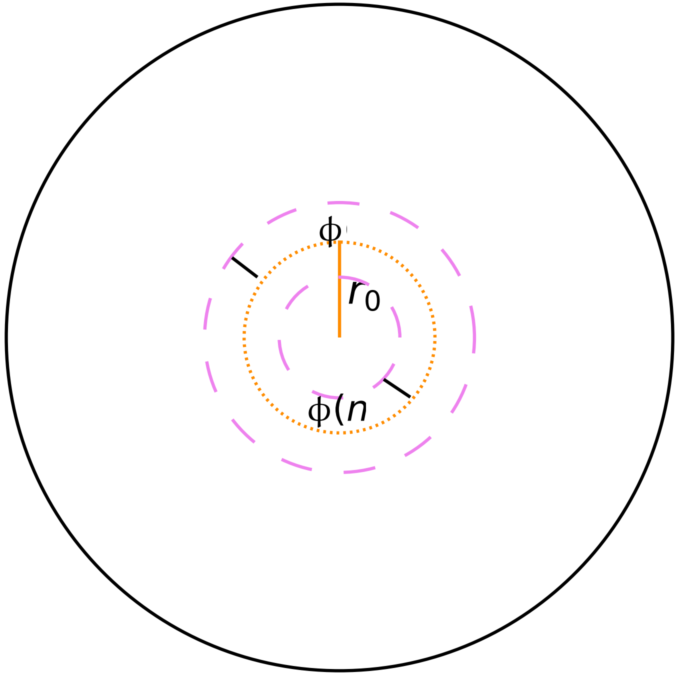

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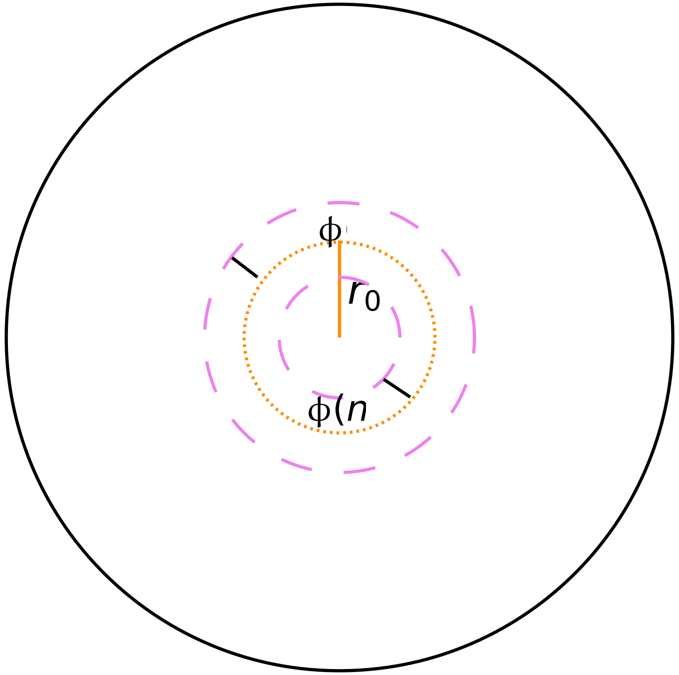

Maximum degree of a node of a HRG

Theorem: Let $\alpha > 1/2$ and $C \in \mathbb{R}$.
Then the maximum vertex degree of $G_{\alpha,C}(n)$ is with high probability $\Delta = n^{\frac{1}{2\alpha}+o(1)}$.

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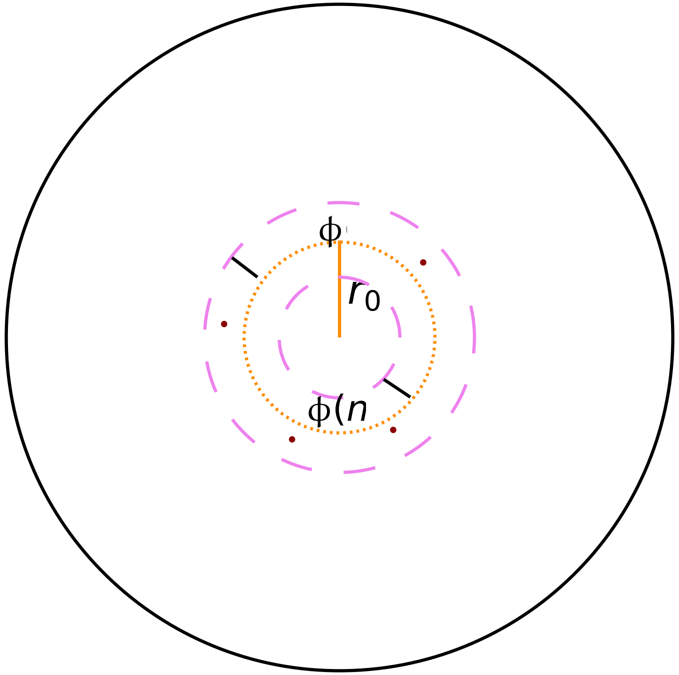

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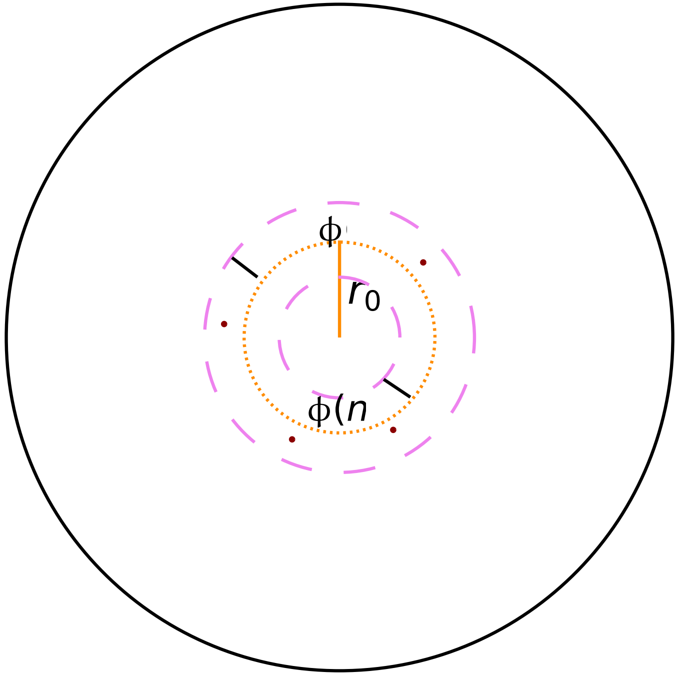

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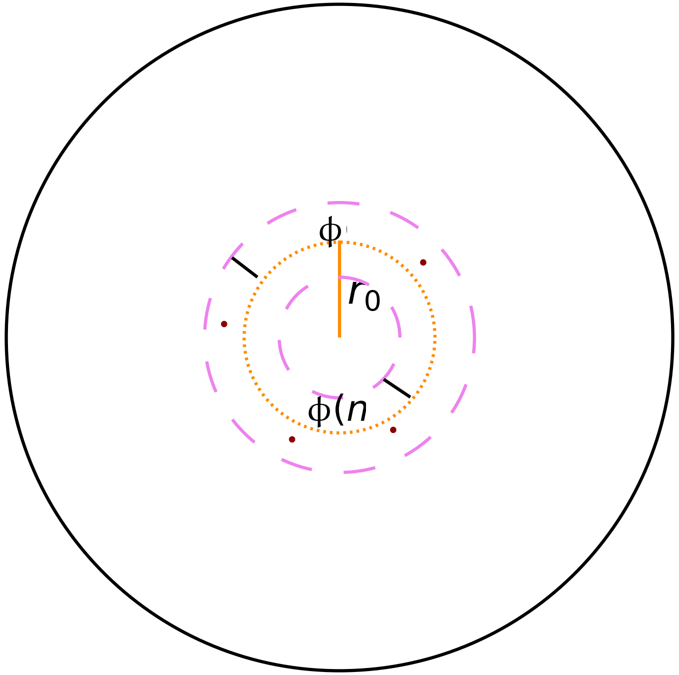

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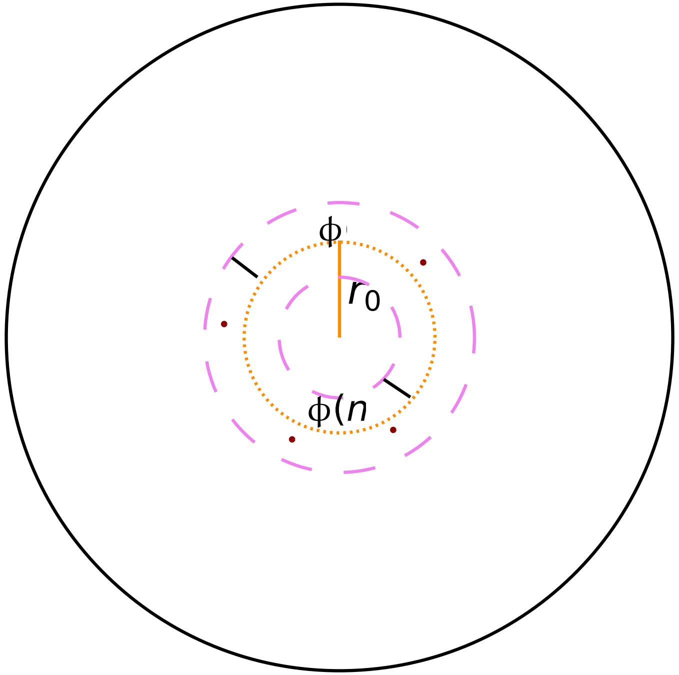

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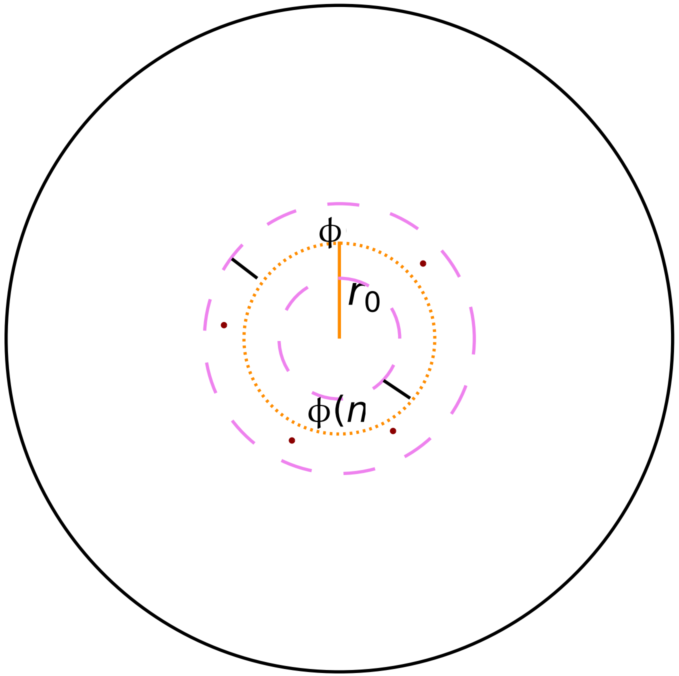

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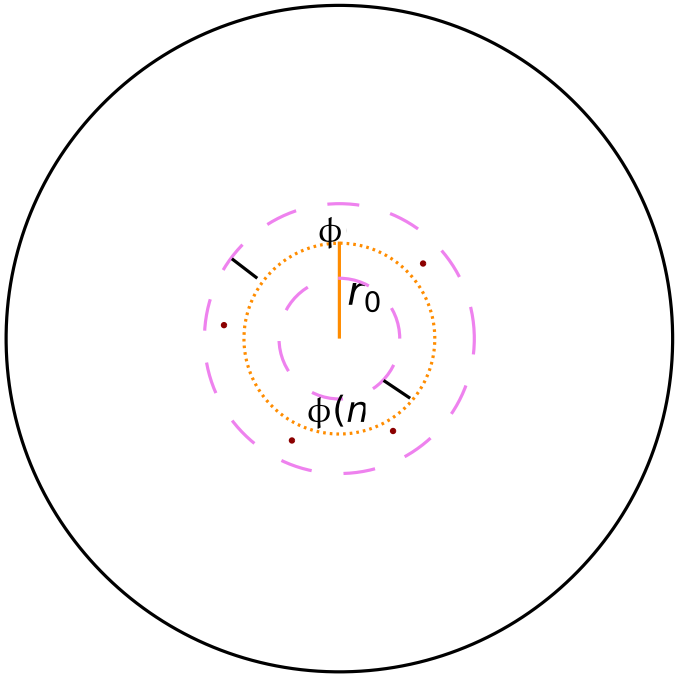

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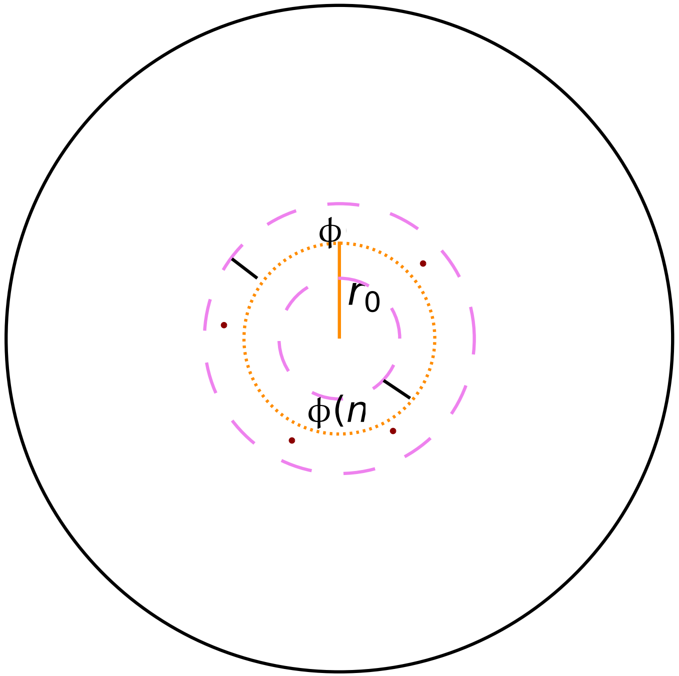

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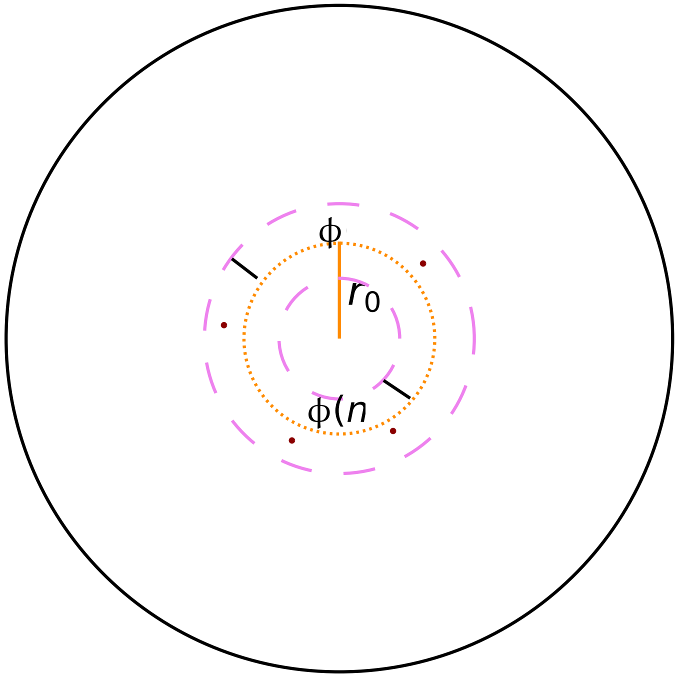

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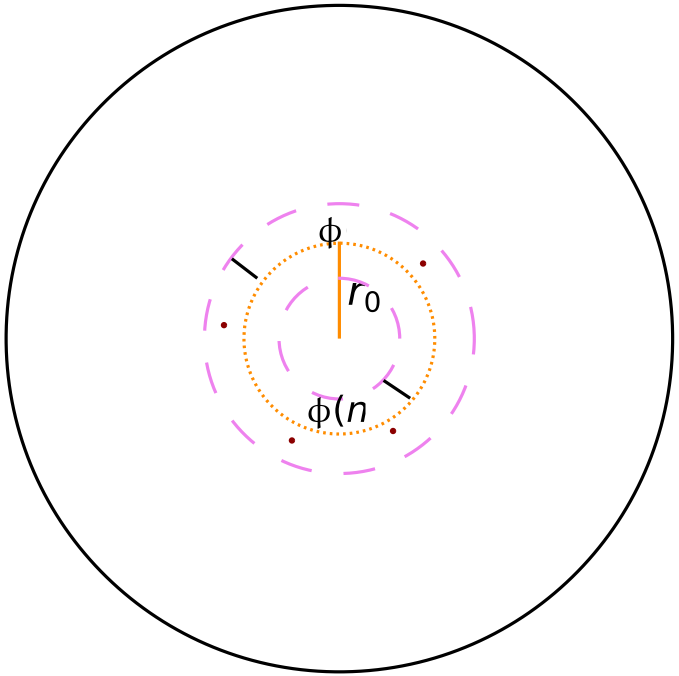

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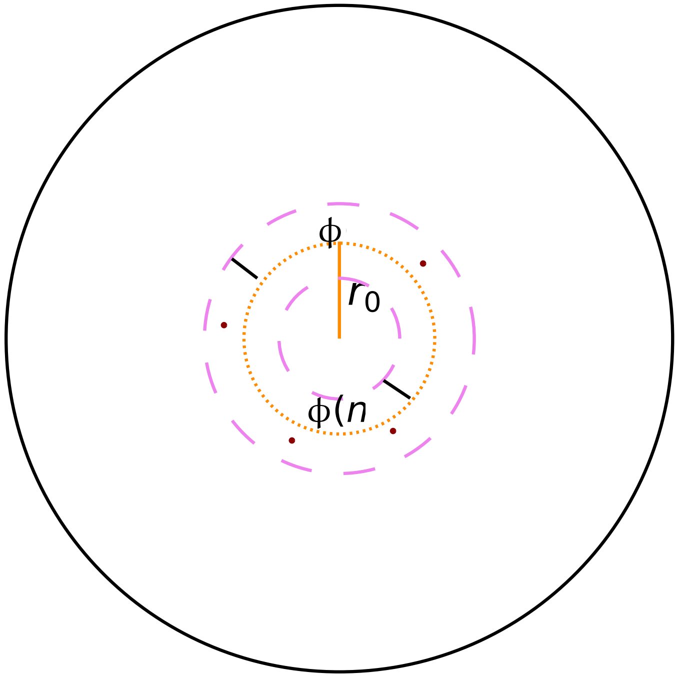

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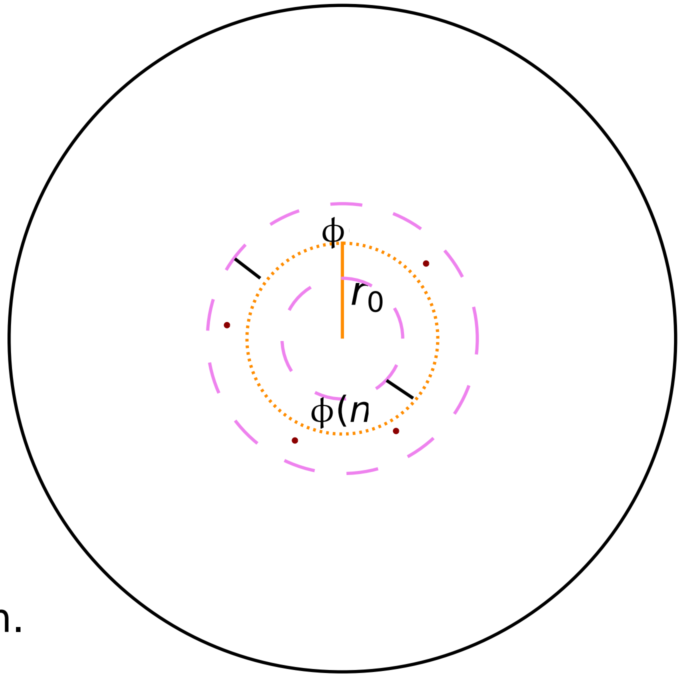

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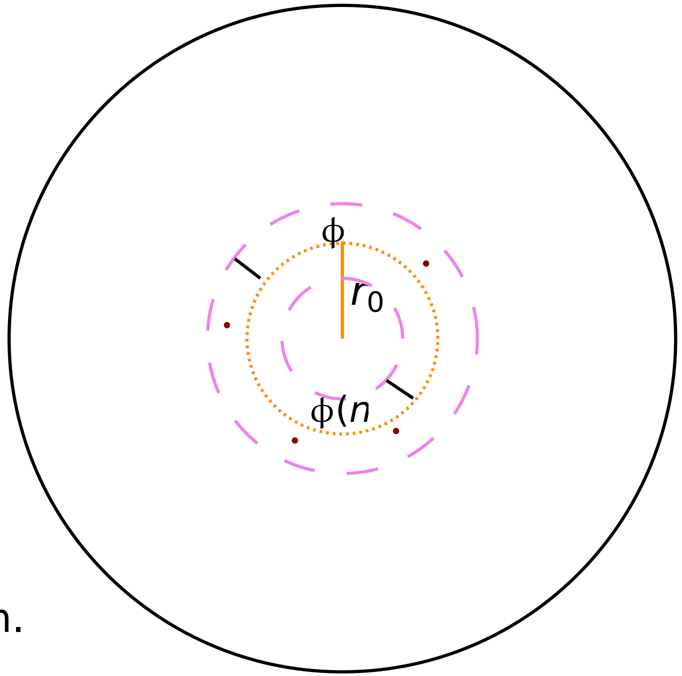

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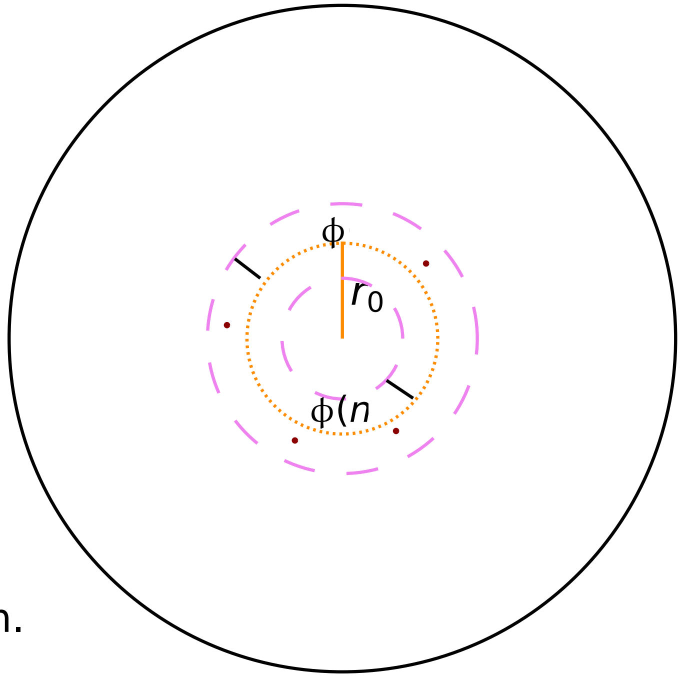

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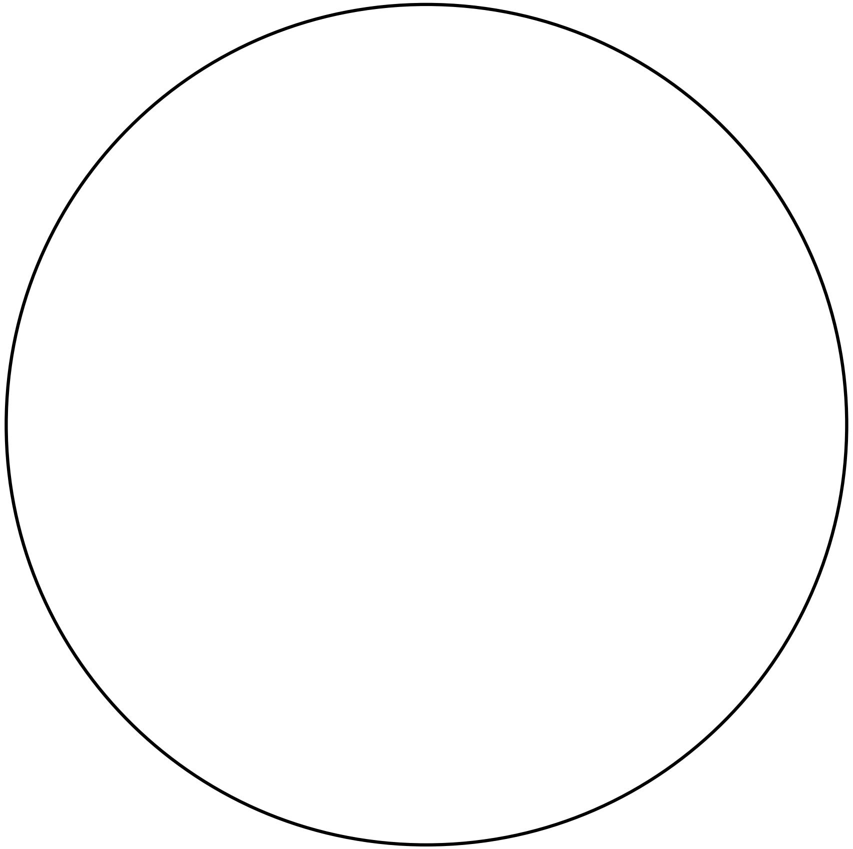

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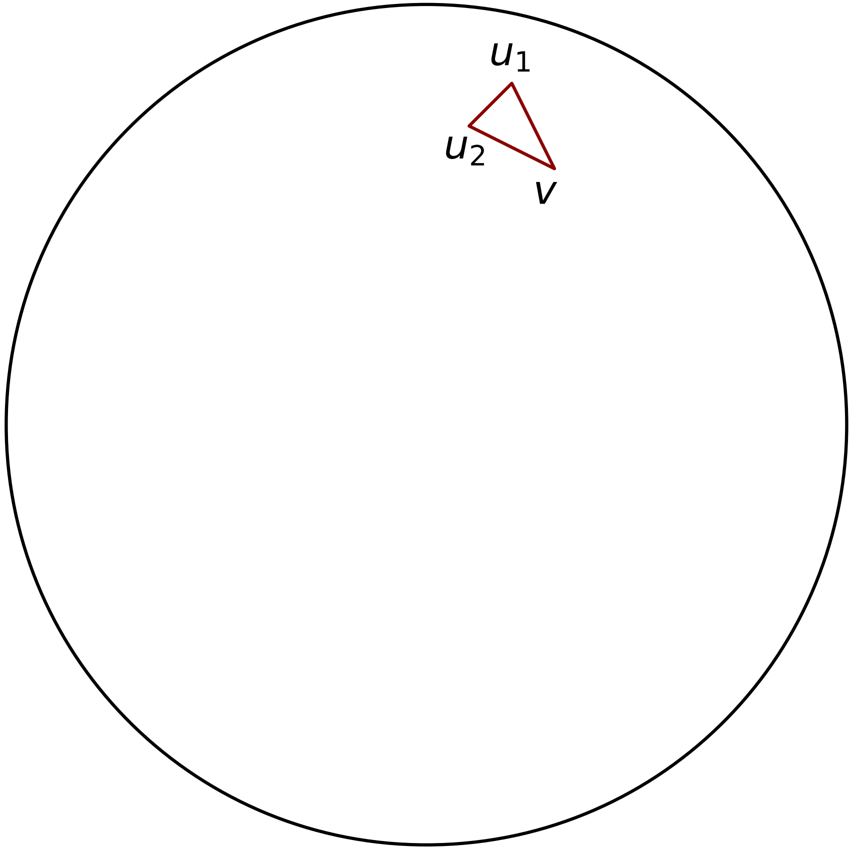

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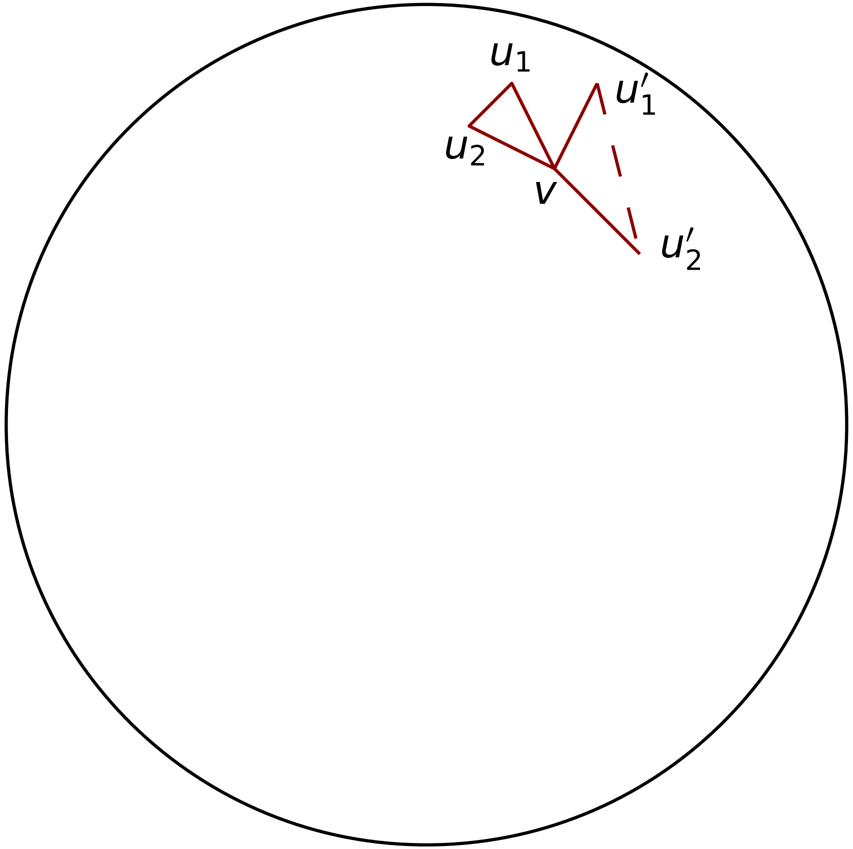

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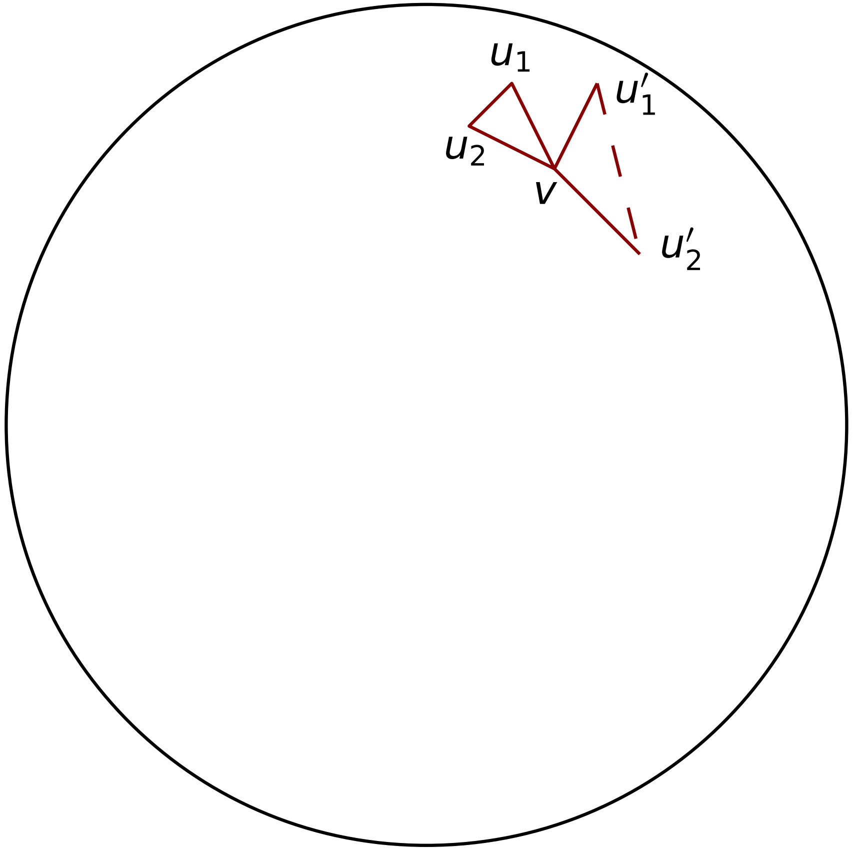

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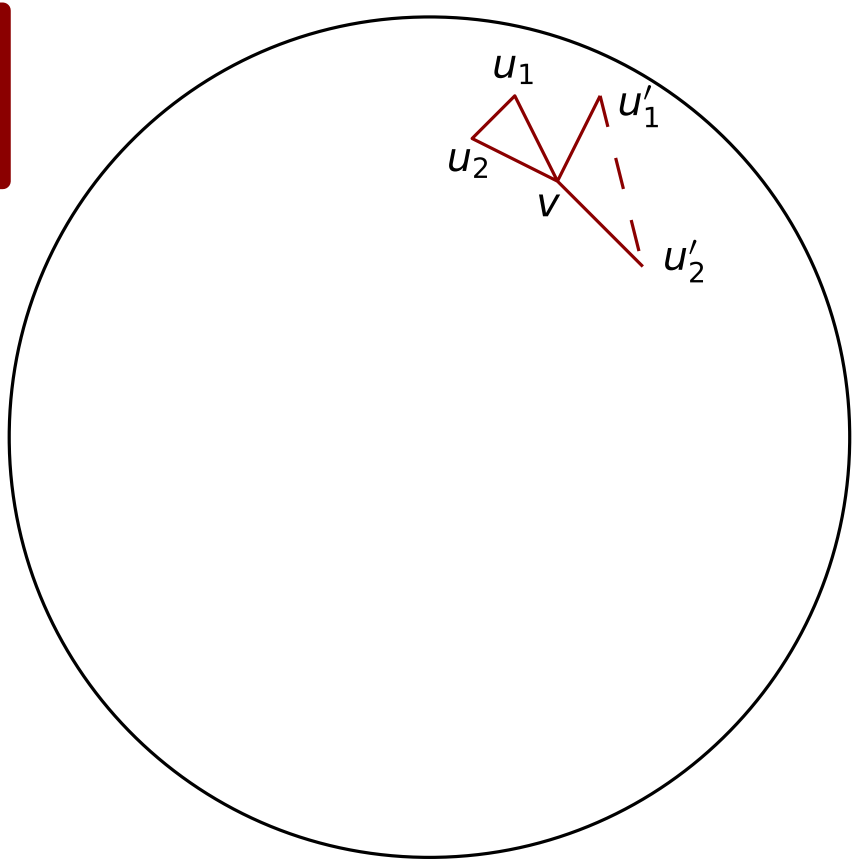

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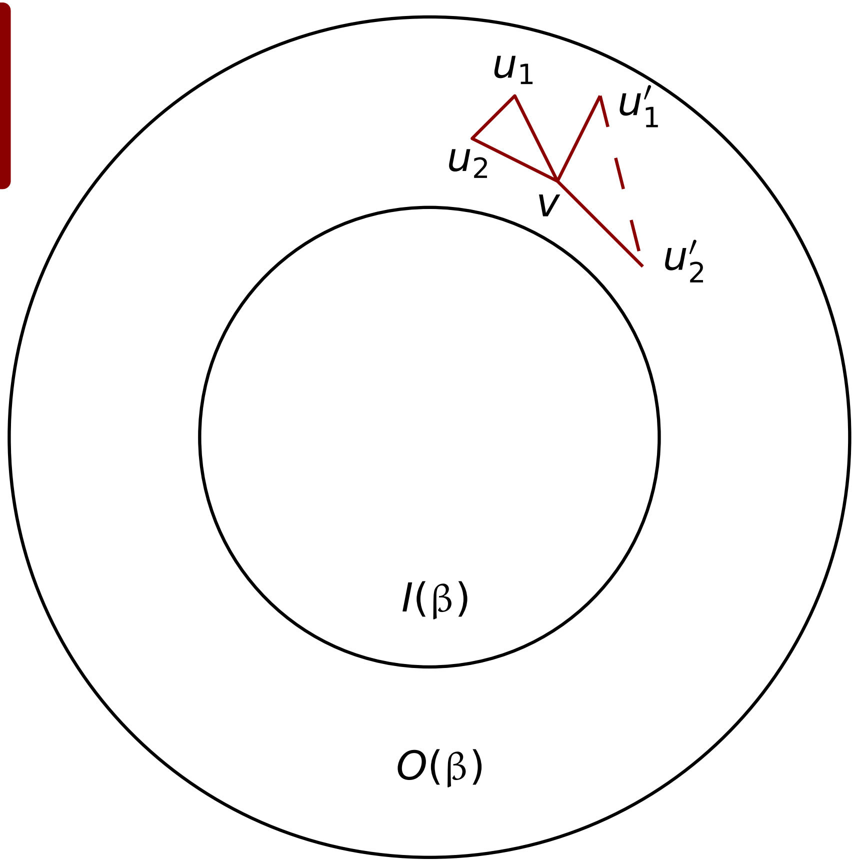

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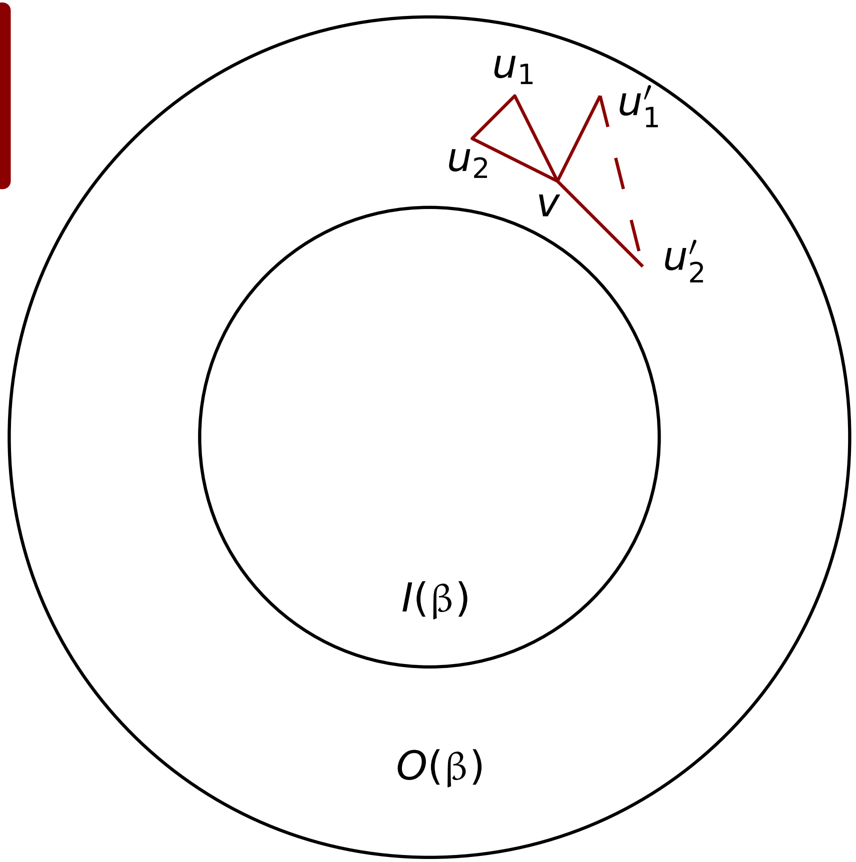

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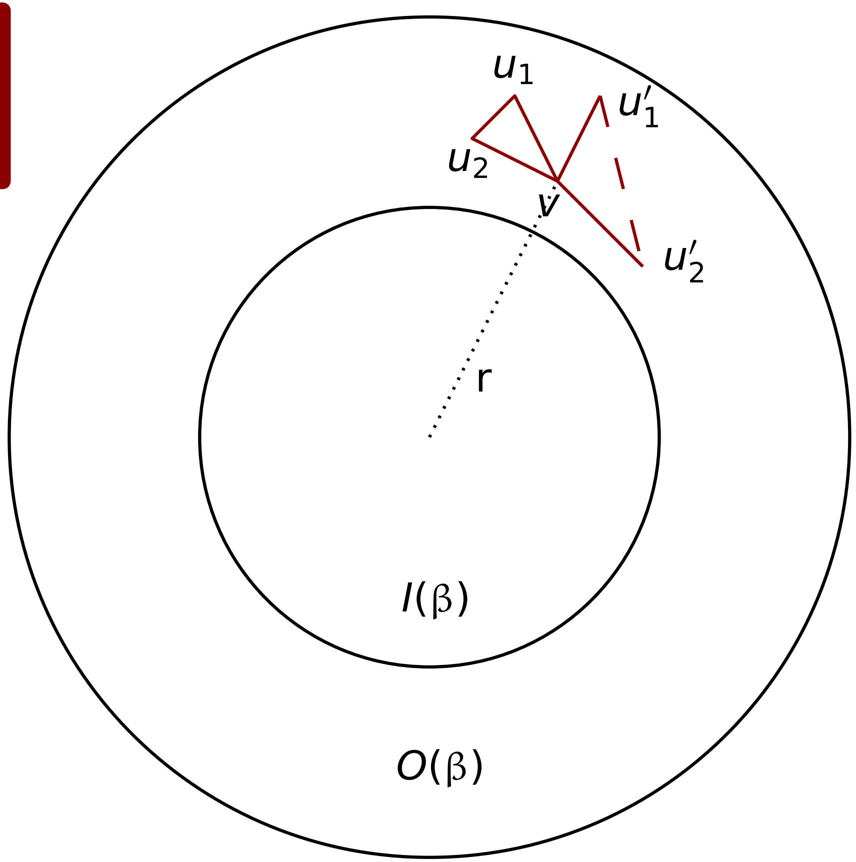

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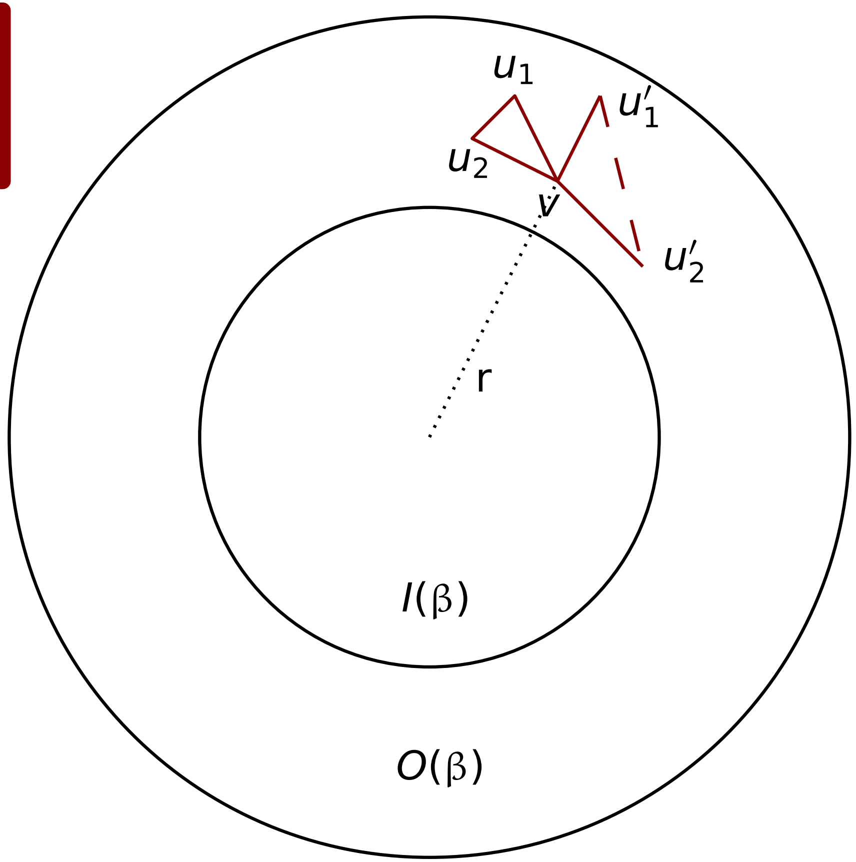

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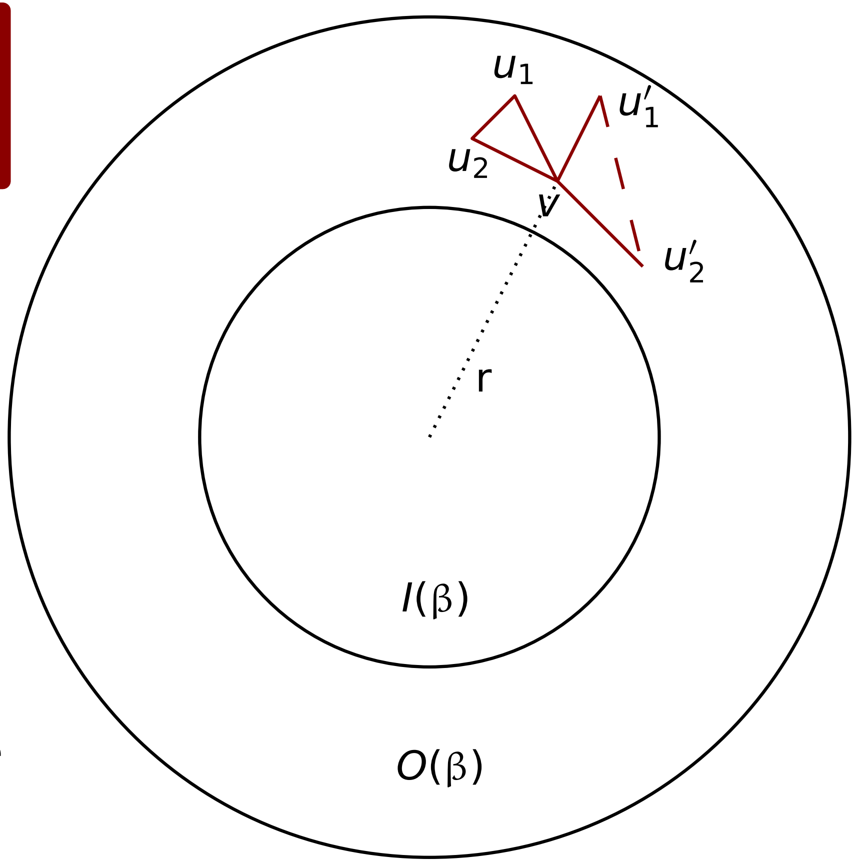

*Let $X := \sum_{v \in V} \frac{|\{\{u_1,u_2\} \in E | u_1,u_2 \in \Gamma(v)\}|}{\binom{\delta(v)}{2}}$. Then $\overline{c} = \frac{X}{n}$.

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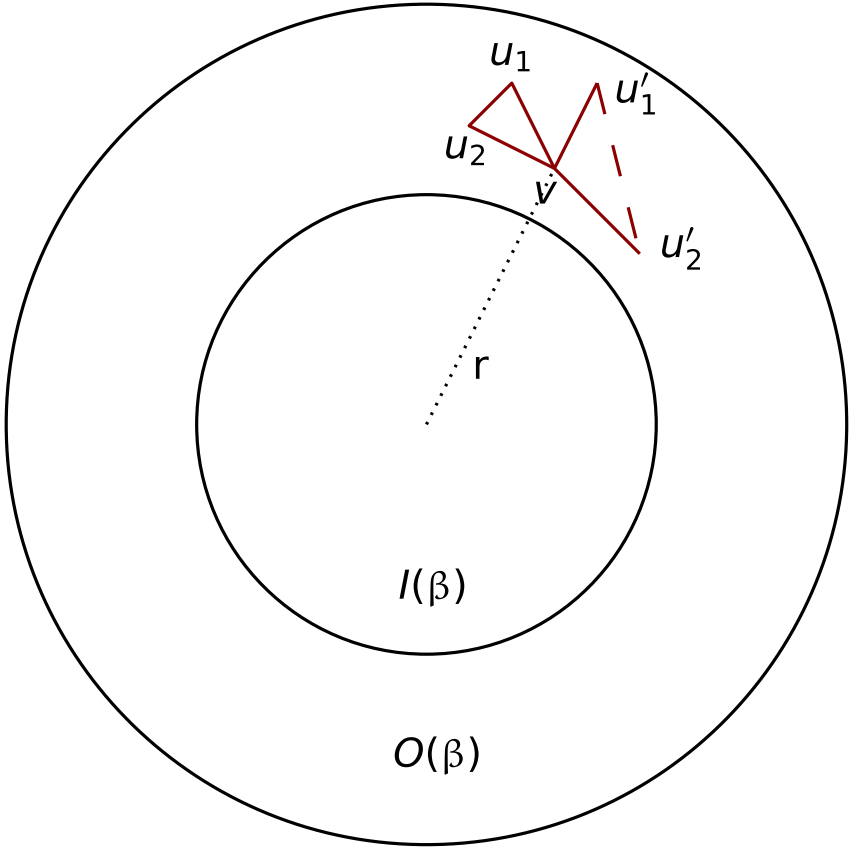

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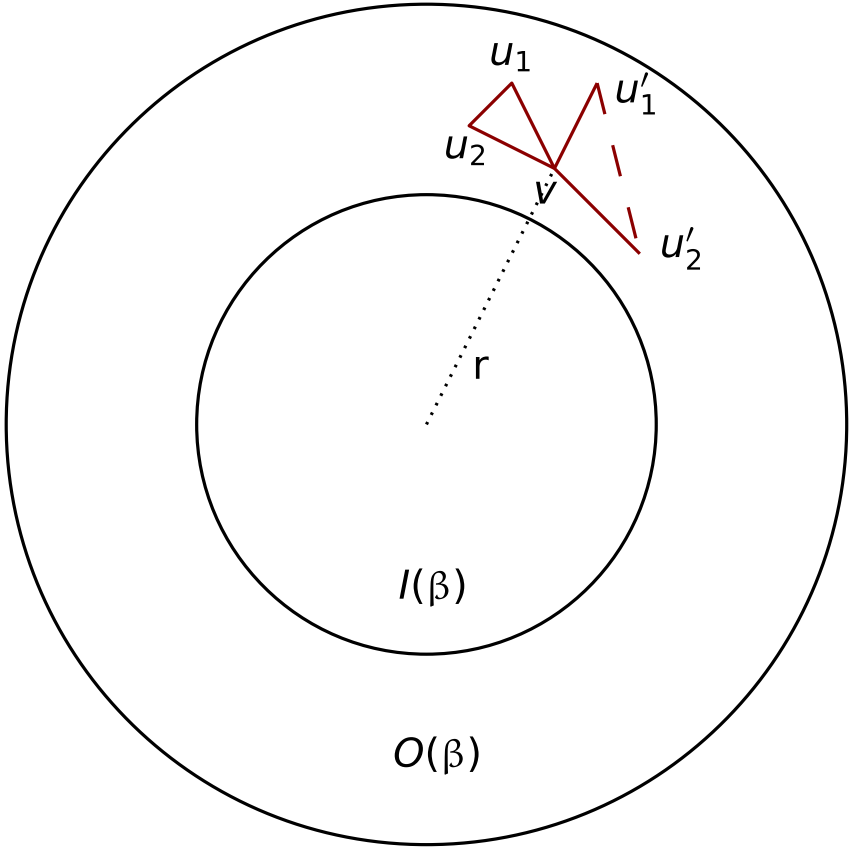

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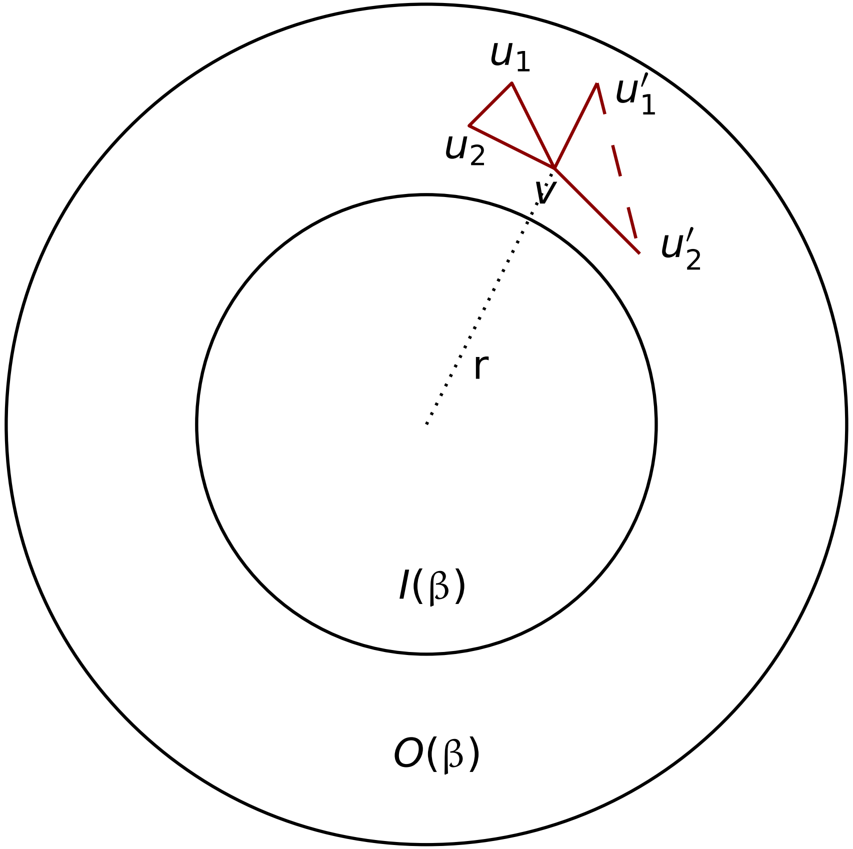

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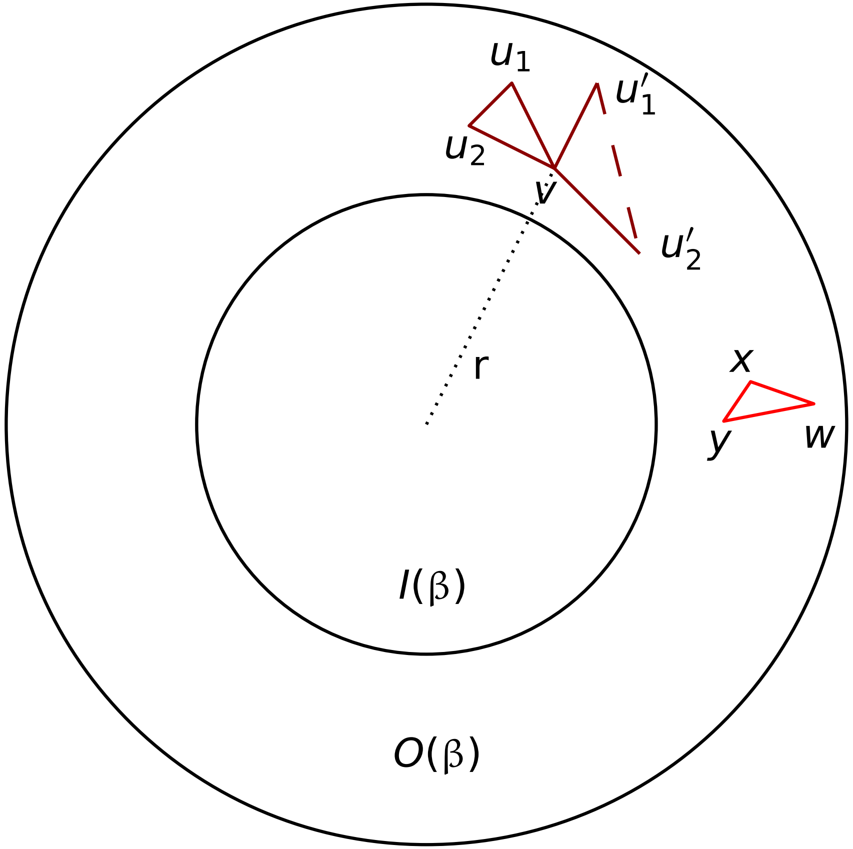

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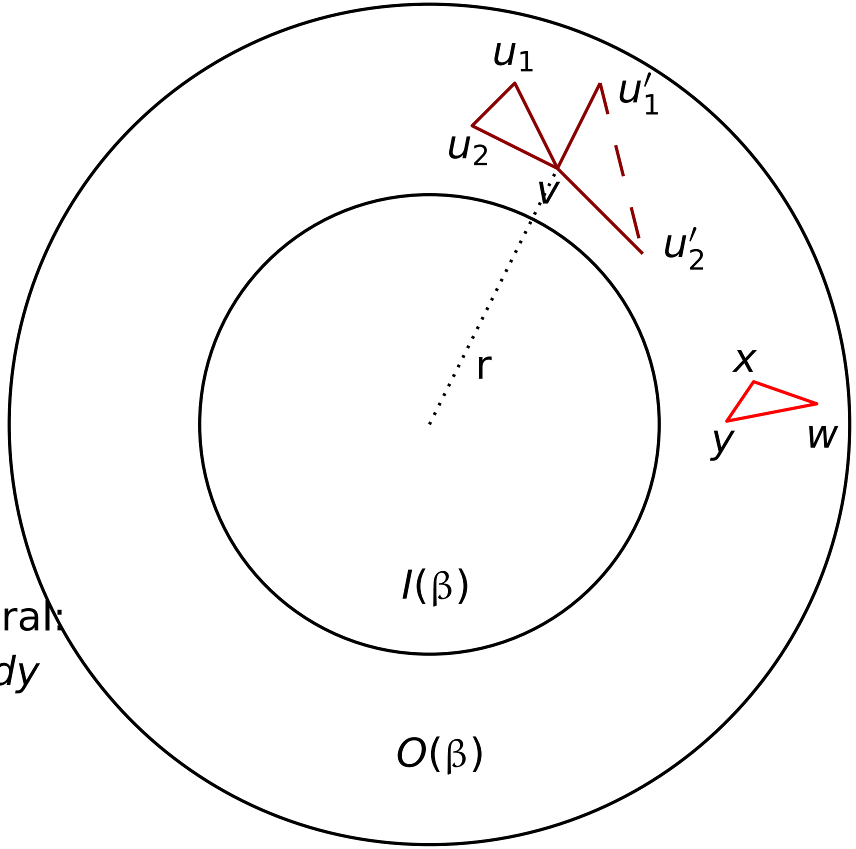

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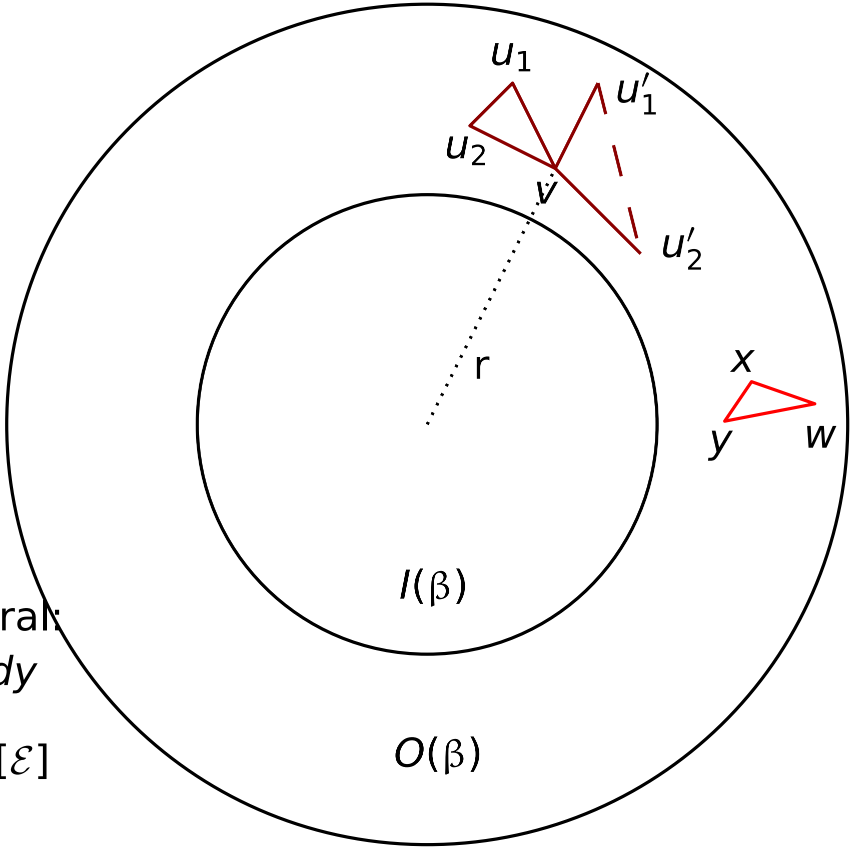
Lower bounding the integral and lower bounding $\mathbb{P}[\mathcal{E}]$ gives us with ”some” calculations:



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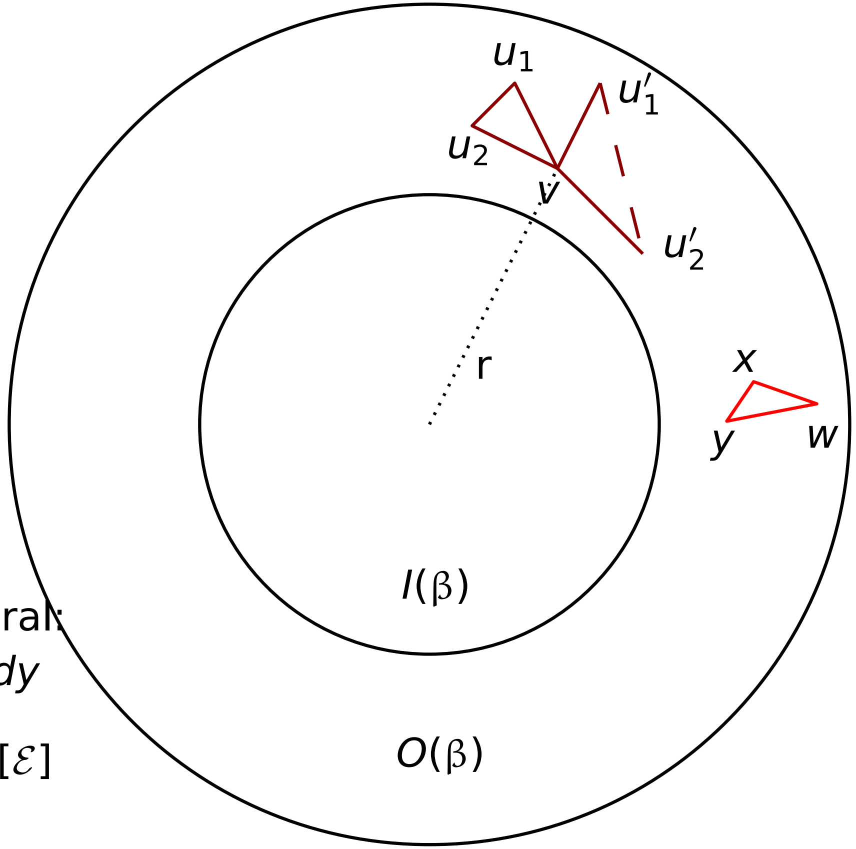

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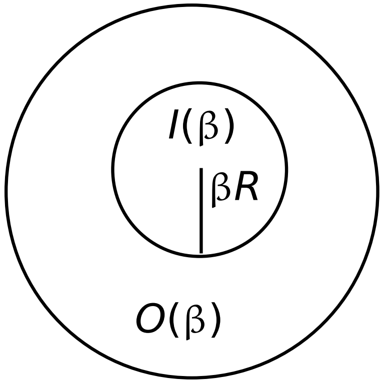

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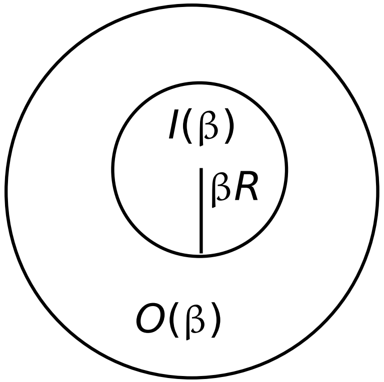

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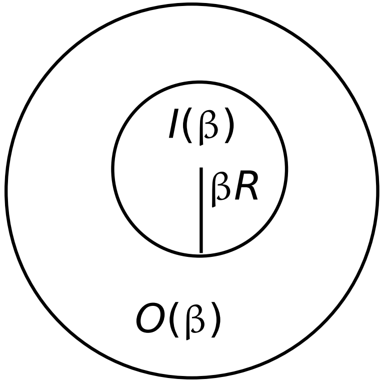

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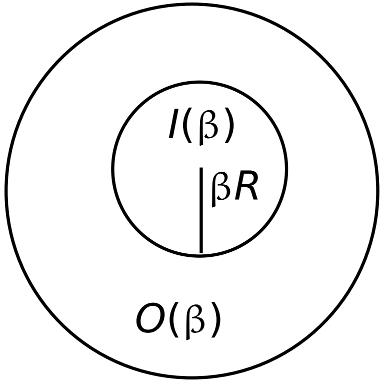

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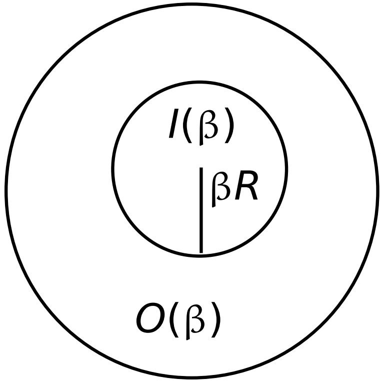

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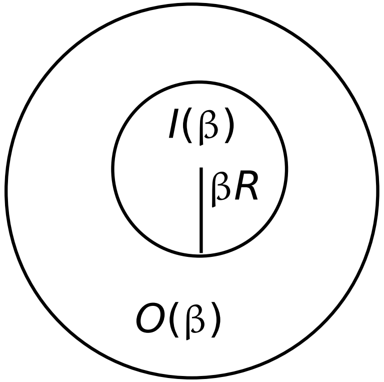

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