Hyperbolic Random Graphs

Degree Sequence and Clustering

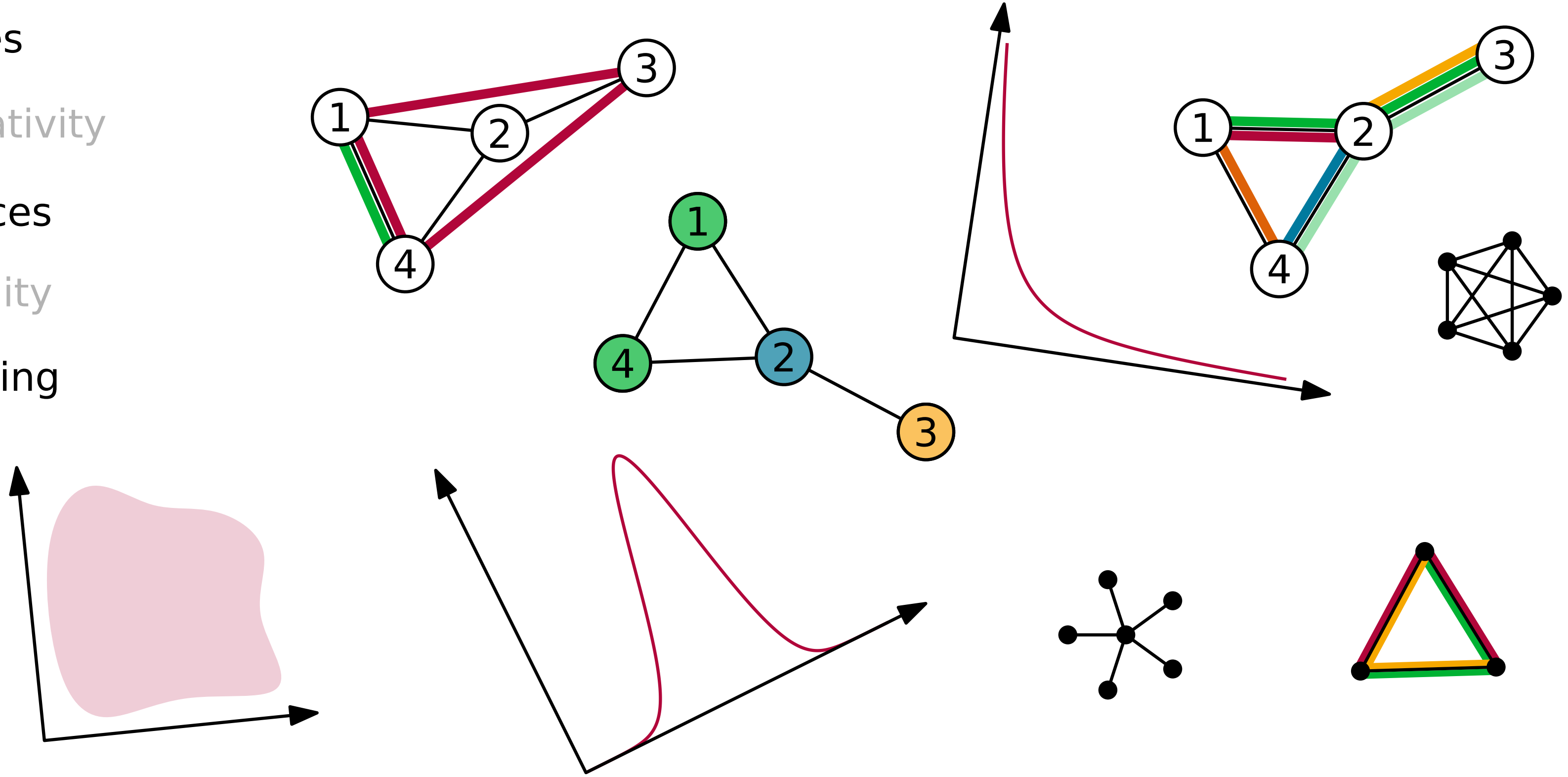
Algorithm Engineering Group

Janosch Ruff

Network Features - Distributions

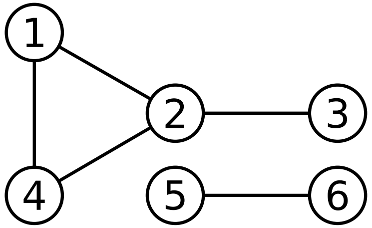
The most commonly considered network features are distributions over parts of the network.

- Degrees
- Assortativity
- Distances
- Centrality
- Clustering



[The Structure and Function of Complex Networks, M. E. J. Newman, Computer Physics Communications 2003]

Network Features – Degree Distribution



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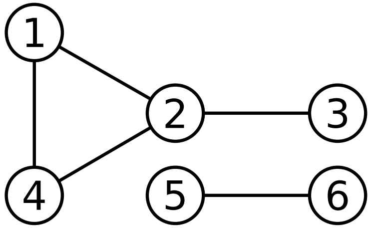

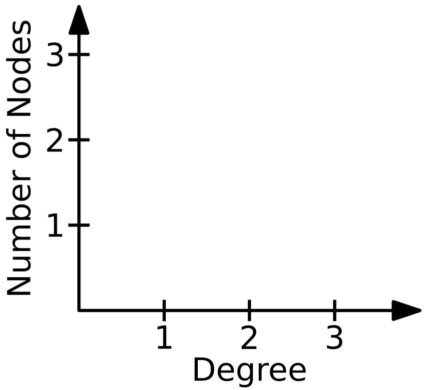

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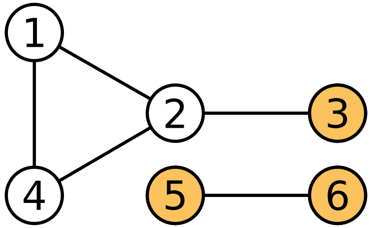

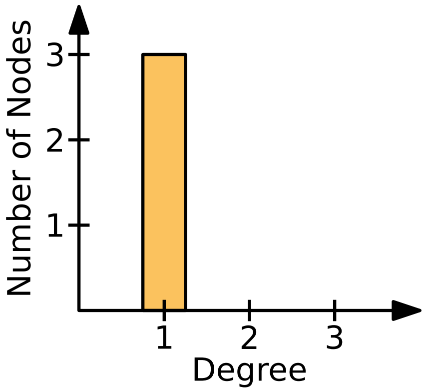

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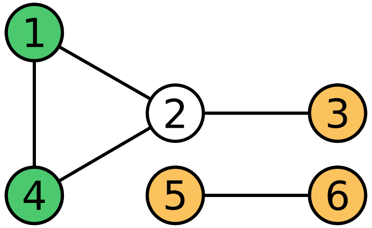

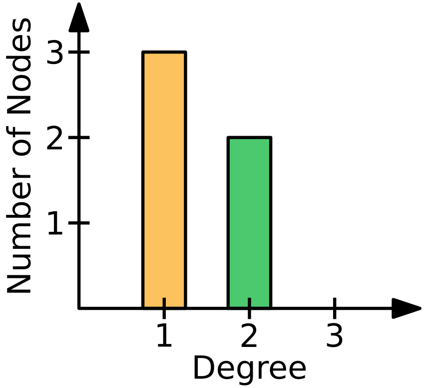

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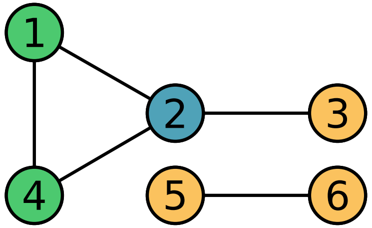

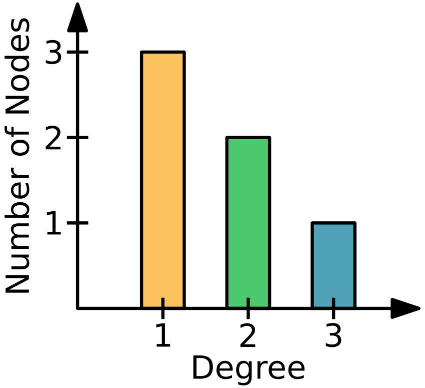

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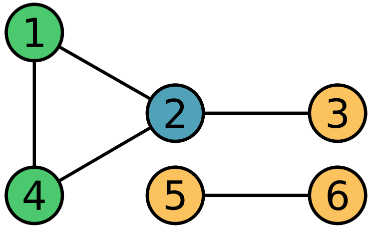

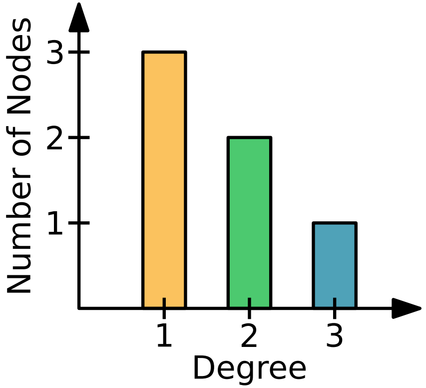

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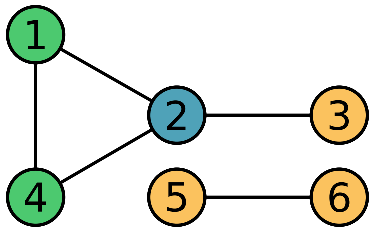

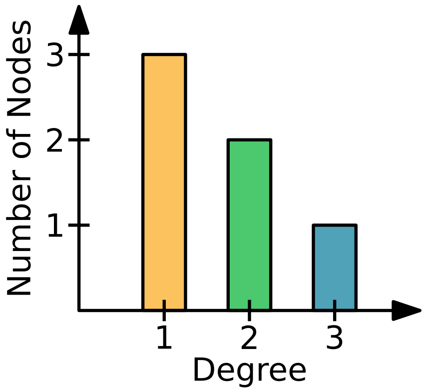

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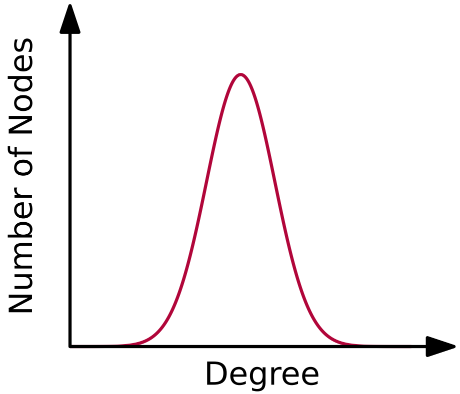

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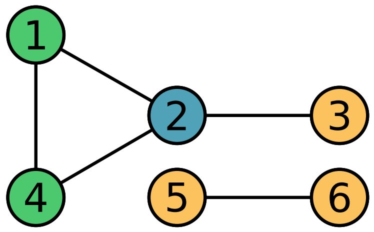

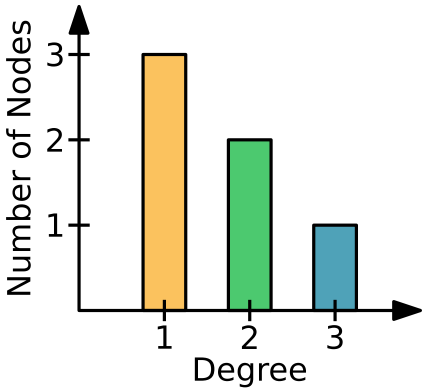

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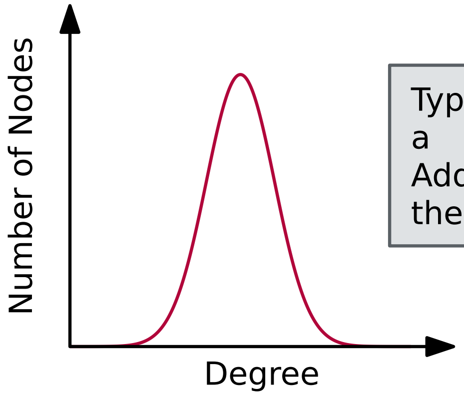

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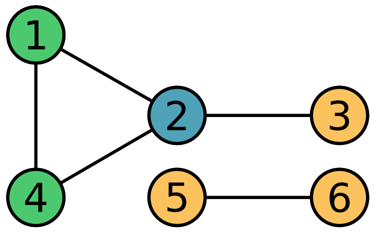

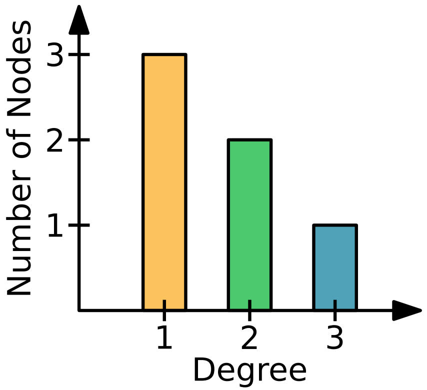

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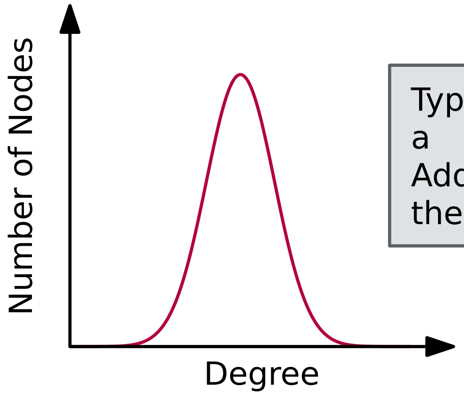

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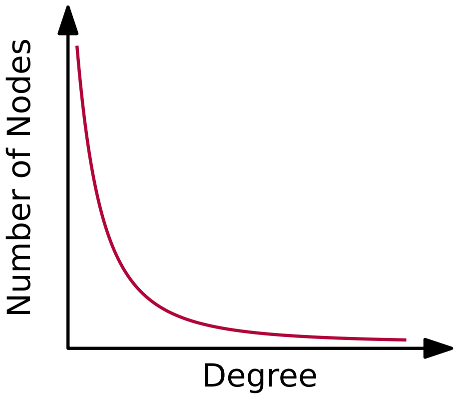

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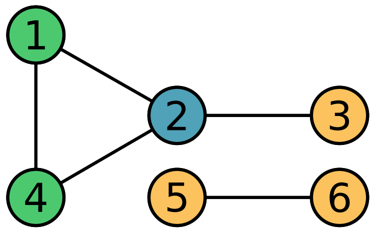

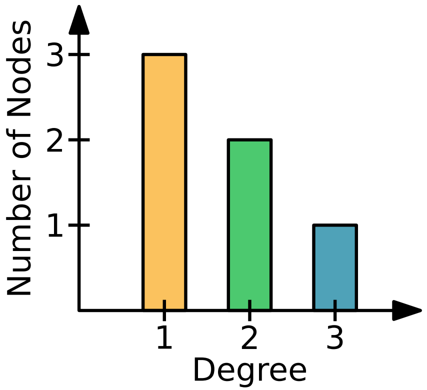

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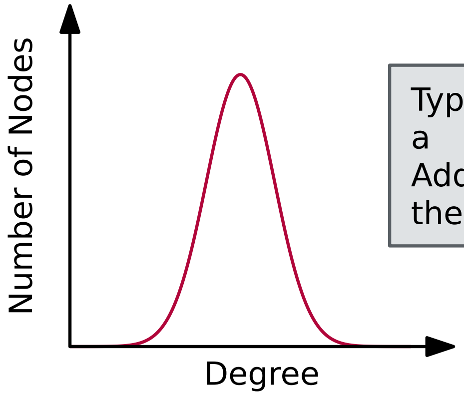

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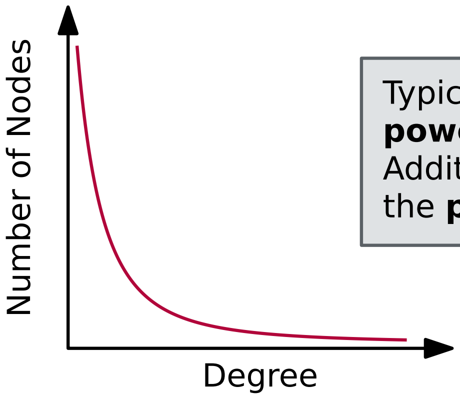

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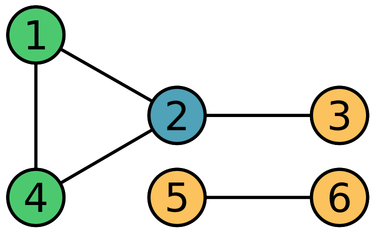

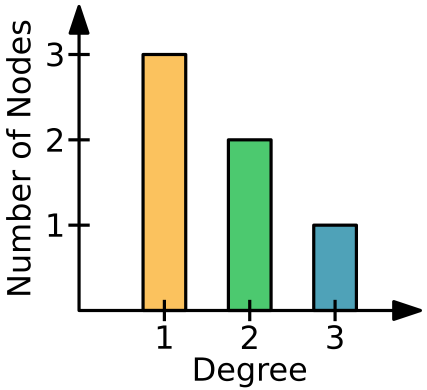

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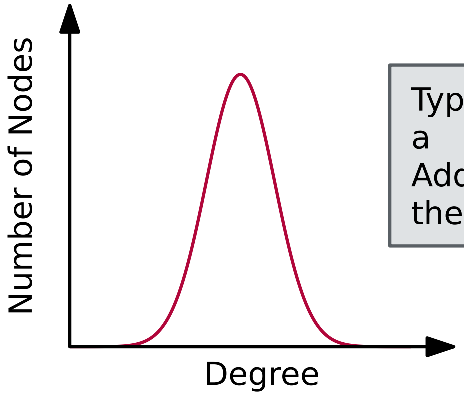

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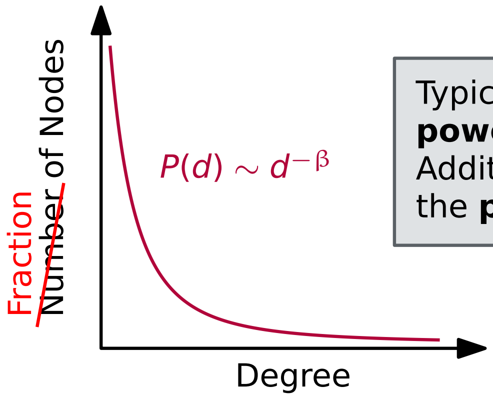

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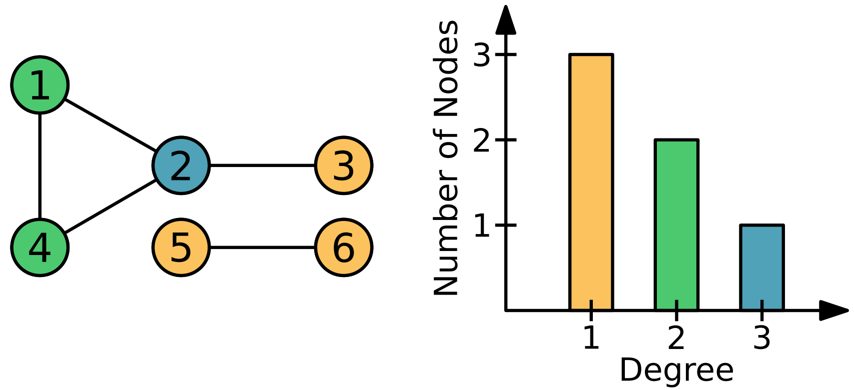

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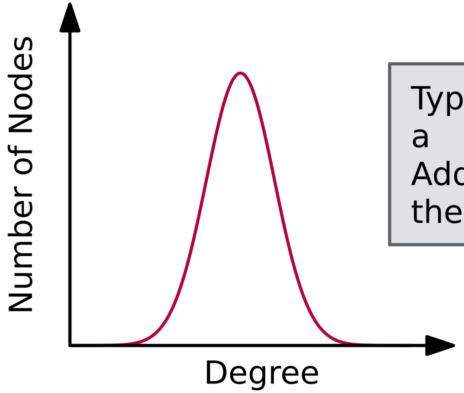

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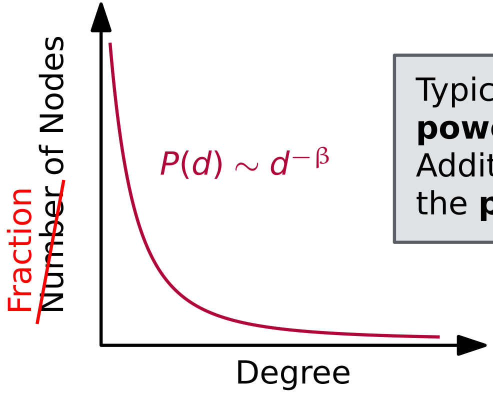

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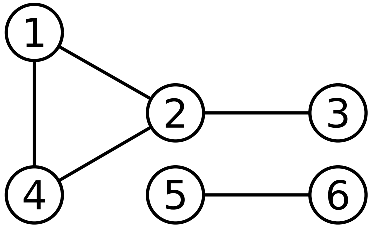

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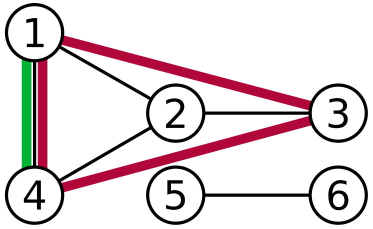

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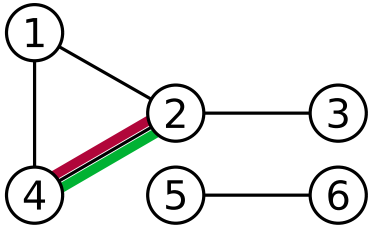

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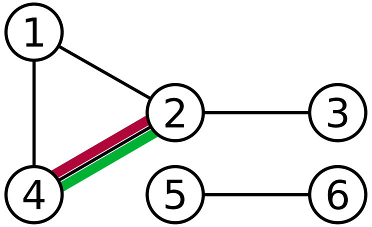

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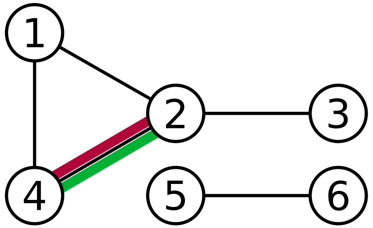

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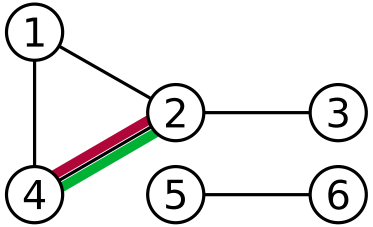

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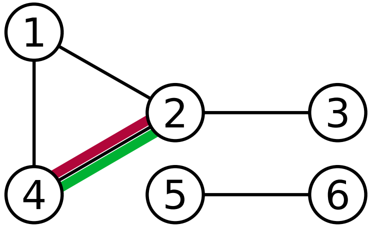

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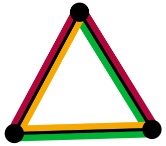

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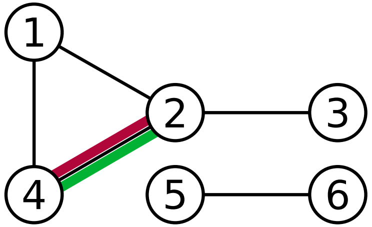

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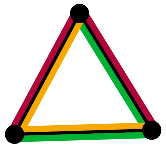

Compared to the average local clustering coefficient, the global one basically computes the ratio of the means rather than the mean of the ratios. It does not weight the contributions of low-degree nodes as much.

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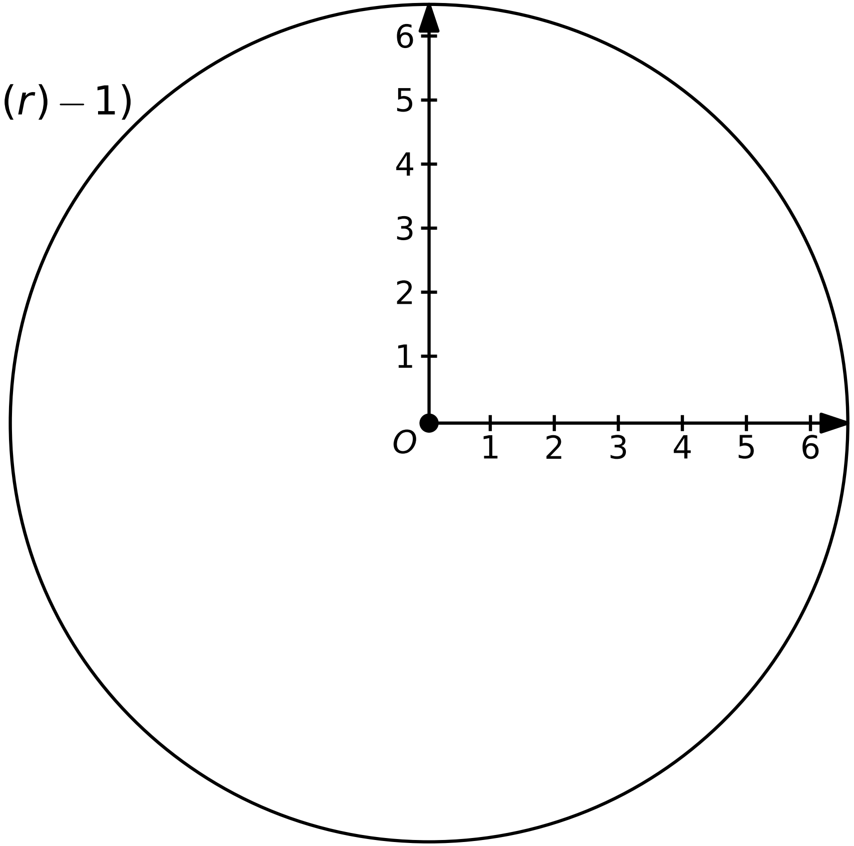

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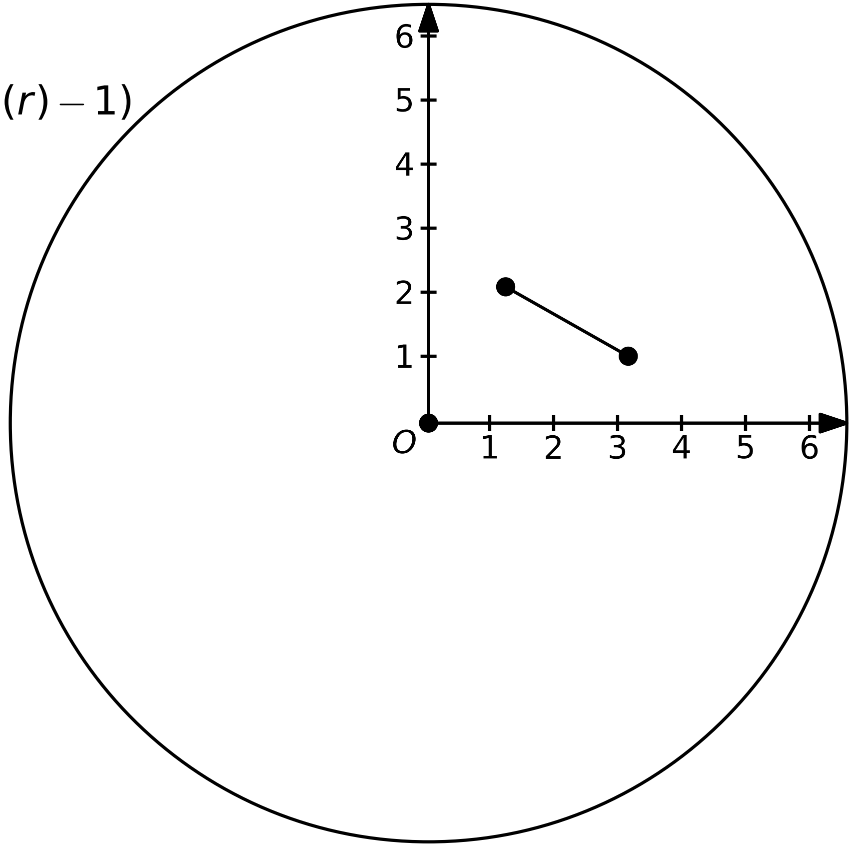

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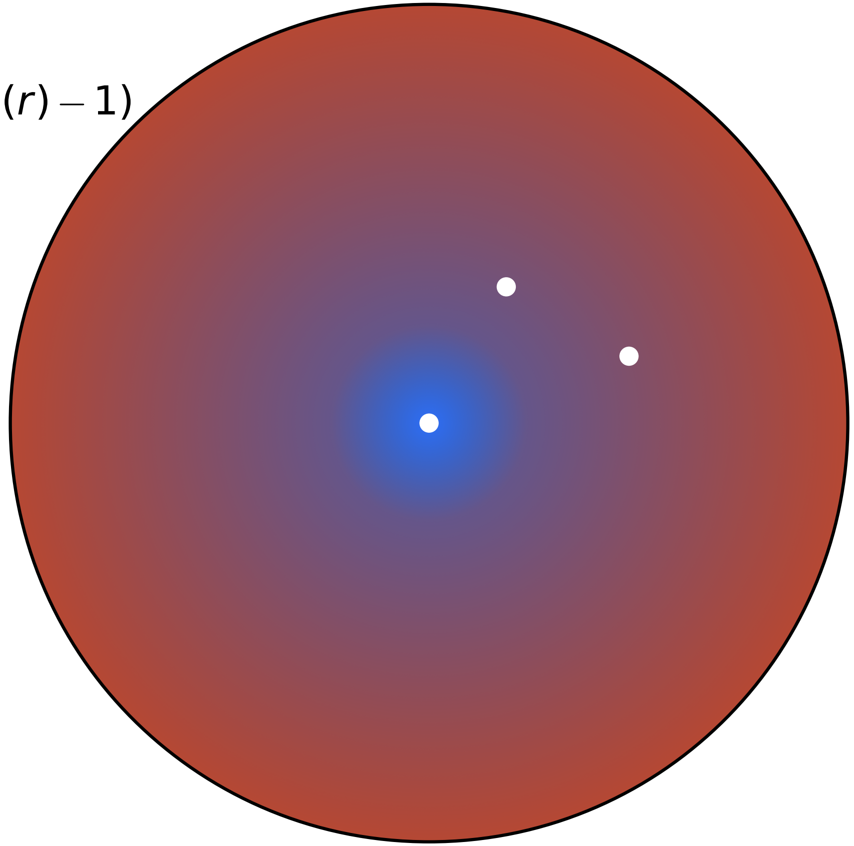

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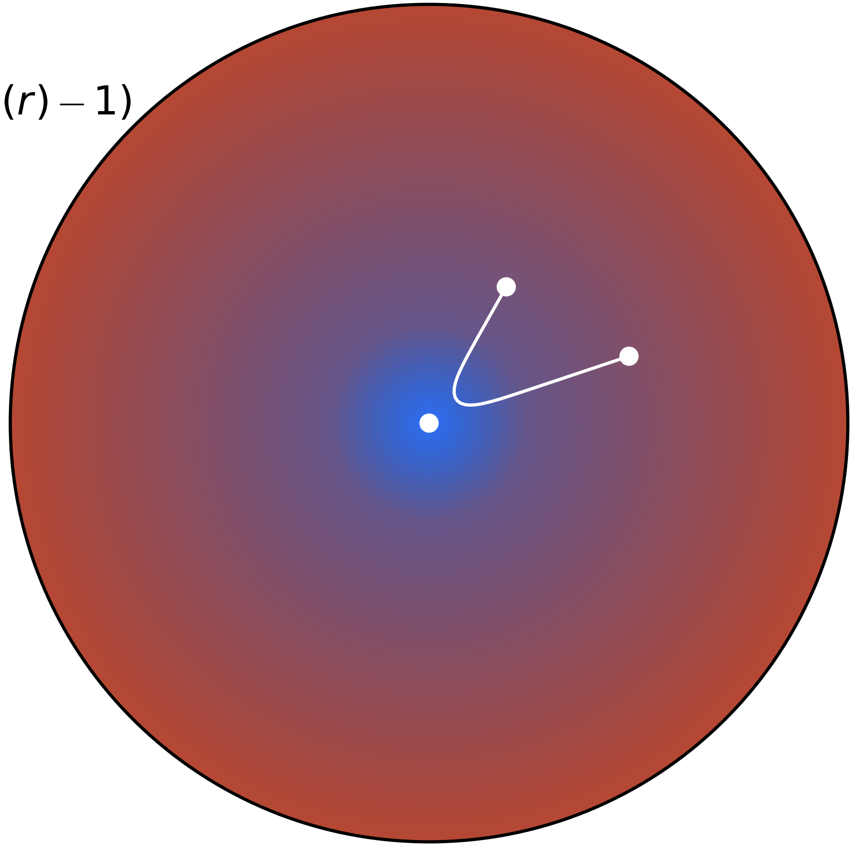

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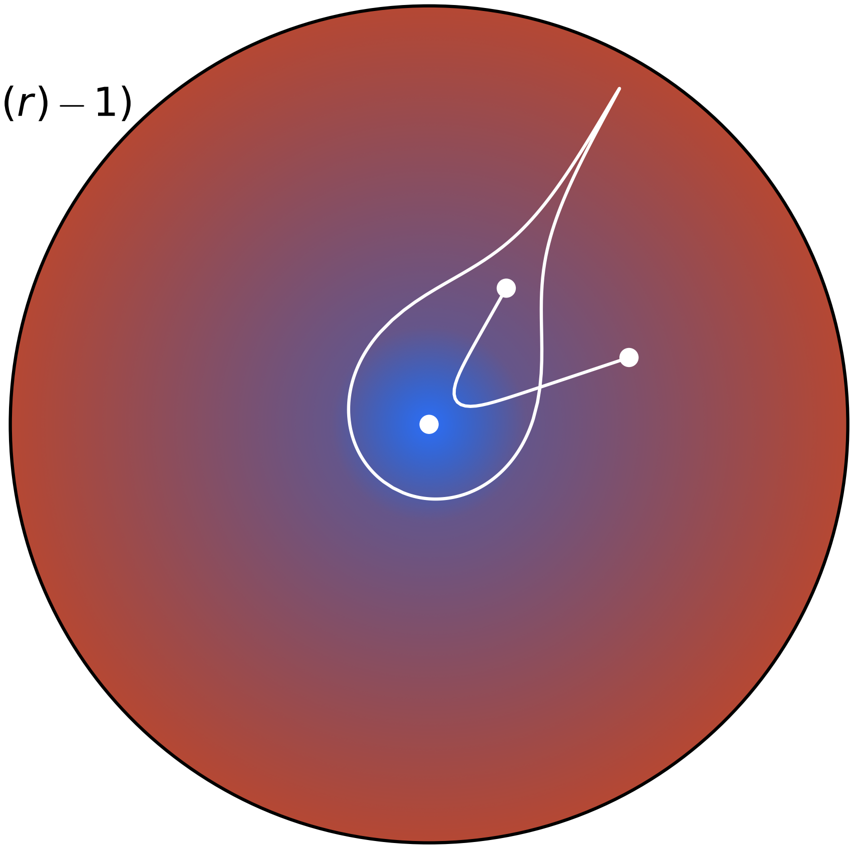

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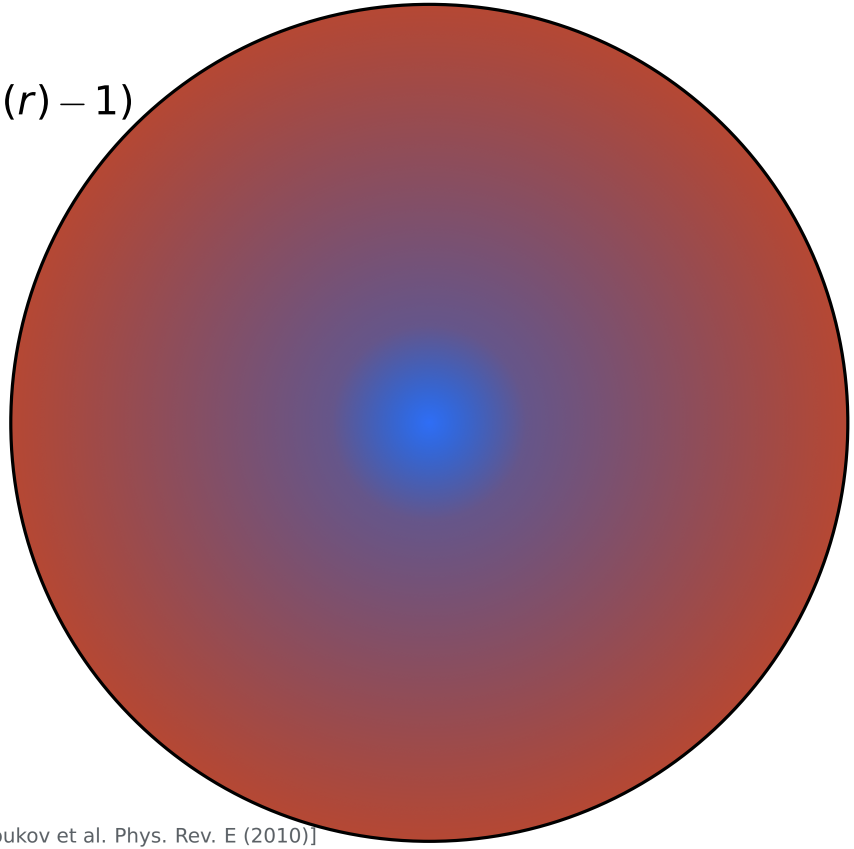
A **hyperbolic random graph** is obtained by distributing n nodes uniformly at random in a hyperbolic disk of radius R and connecting any two at distance $\leq R$.

$R \approx 2\log(n)$



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Hyperbolic space expands exponentially, while Euclidean space expands polynomially.

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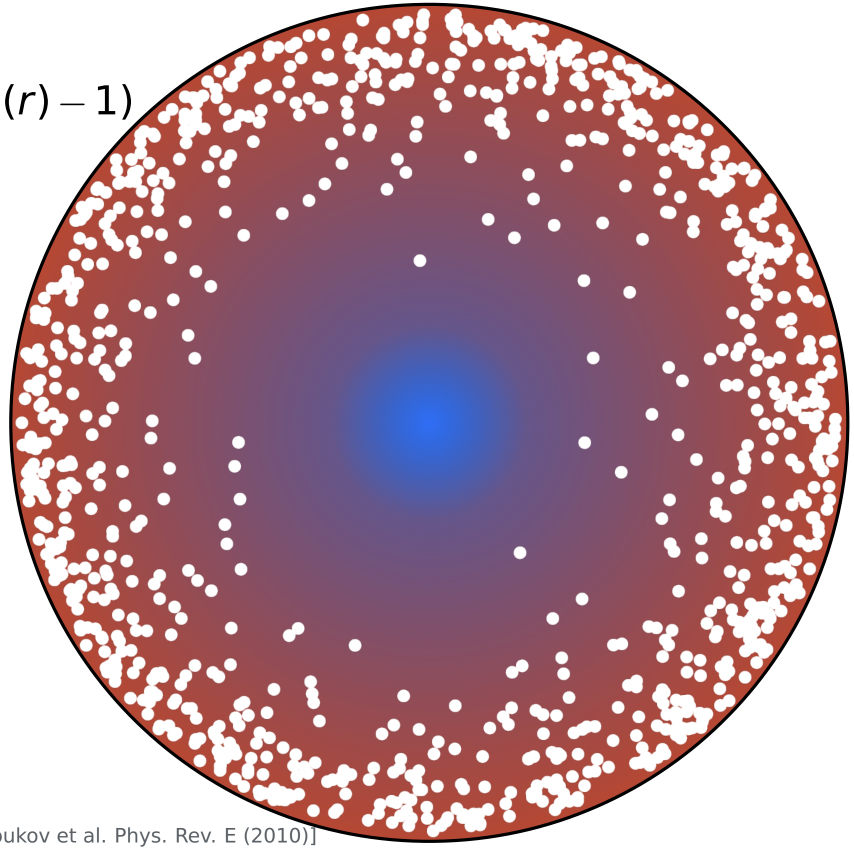

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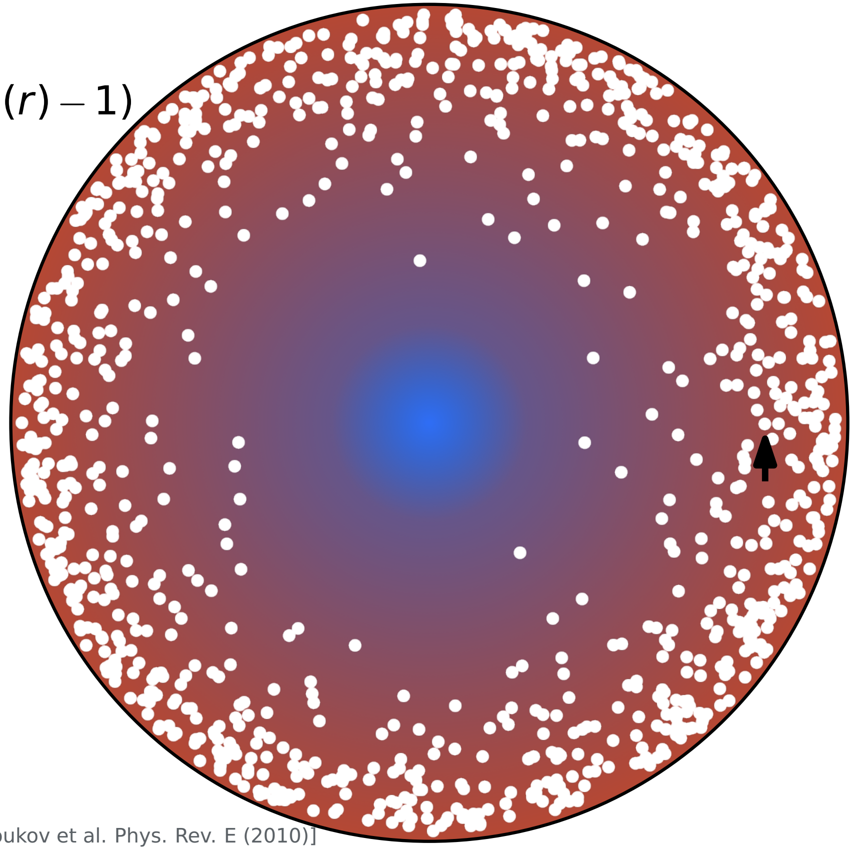

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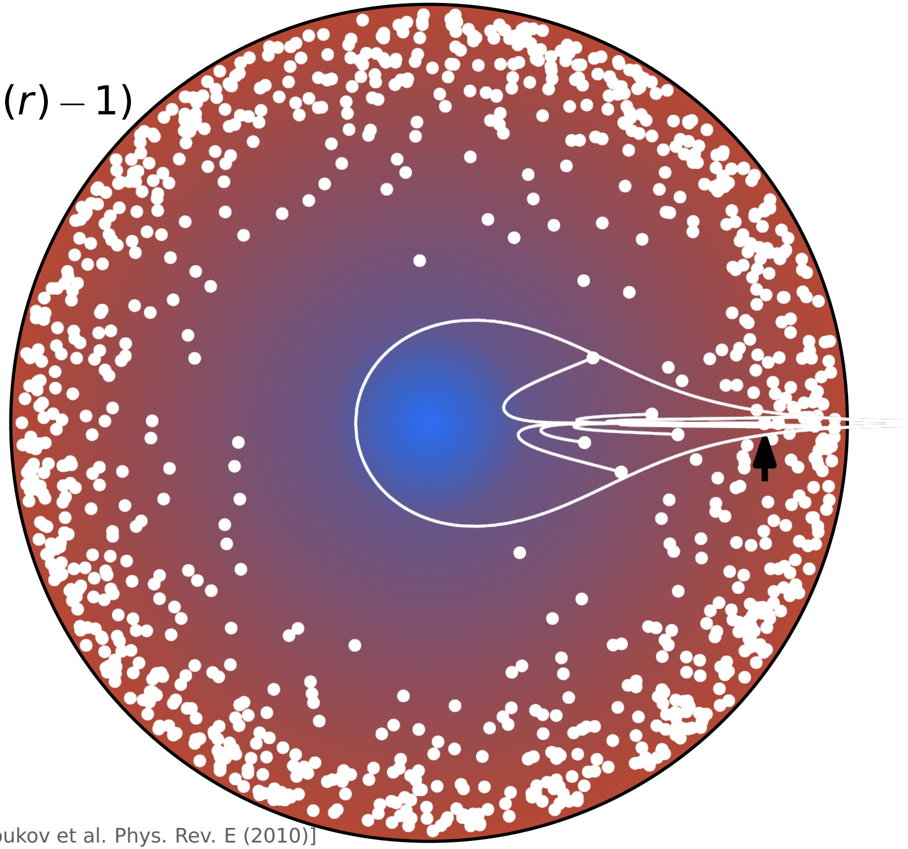

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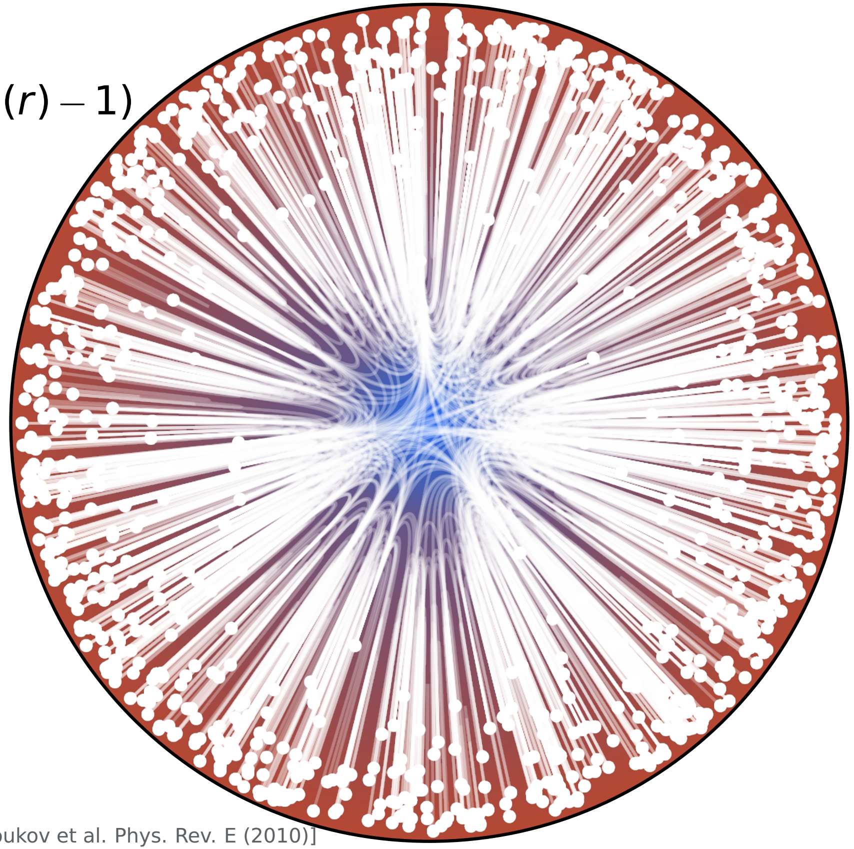

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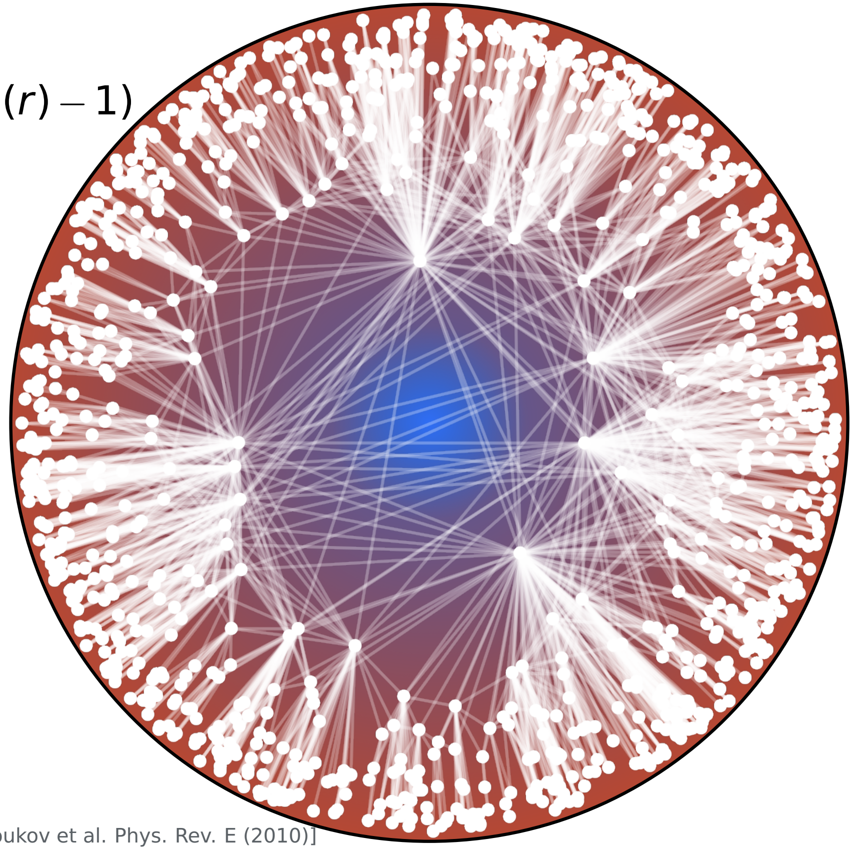

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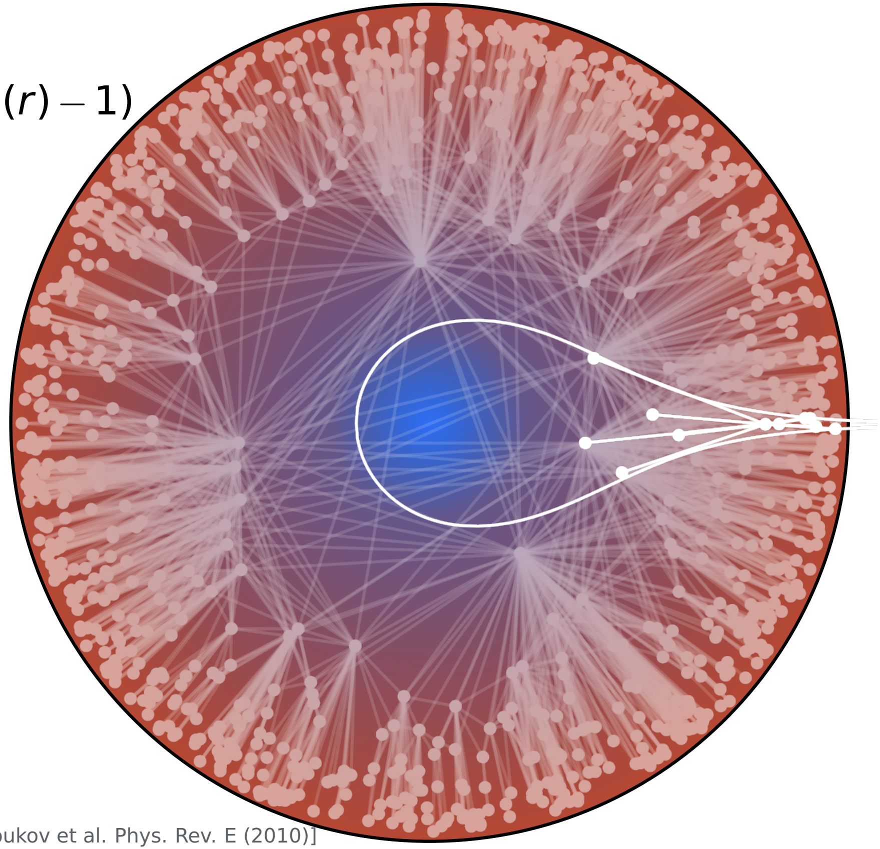

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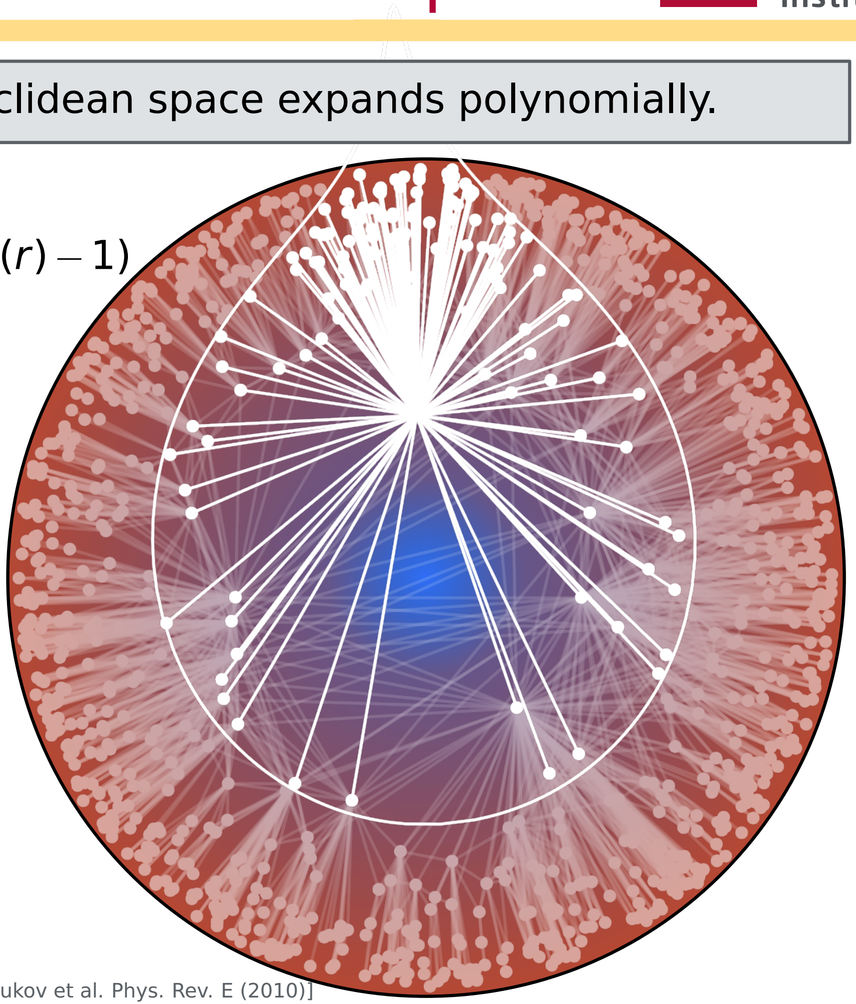

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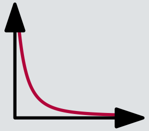
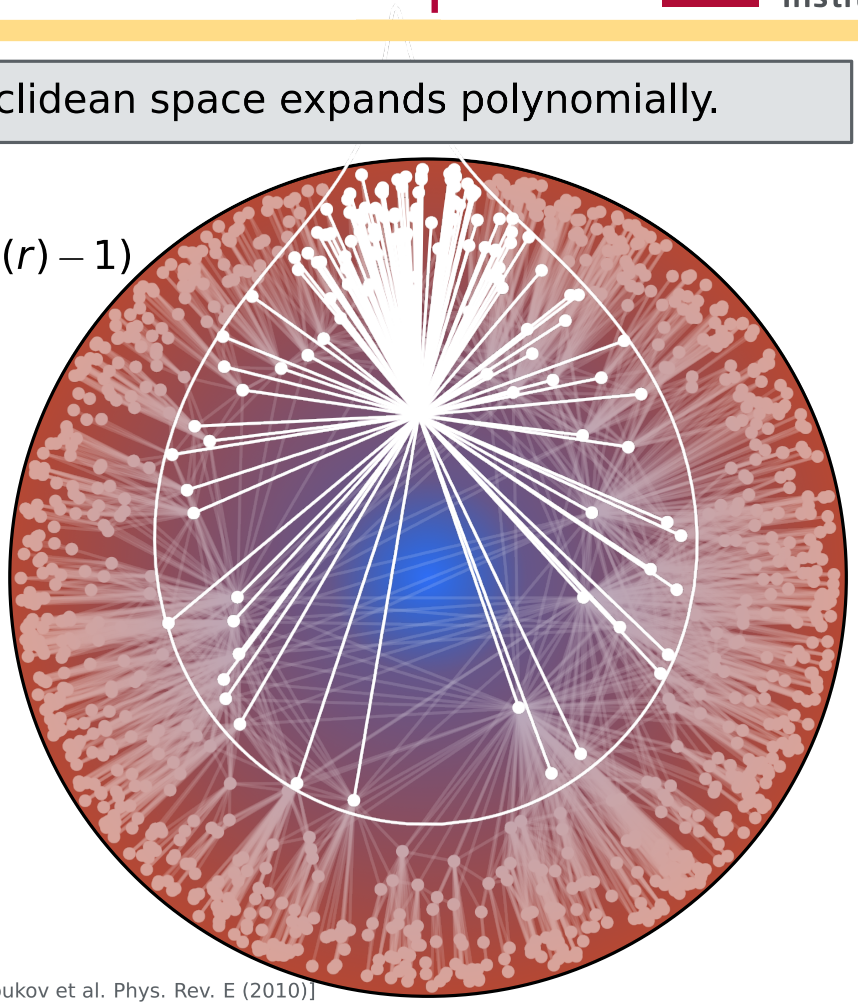
- ... a power-law degree distribution





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Hyperbolic random graphs have...

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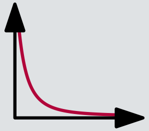
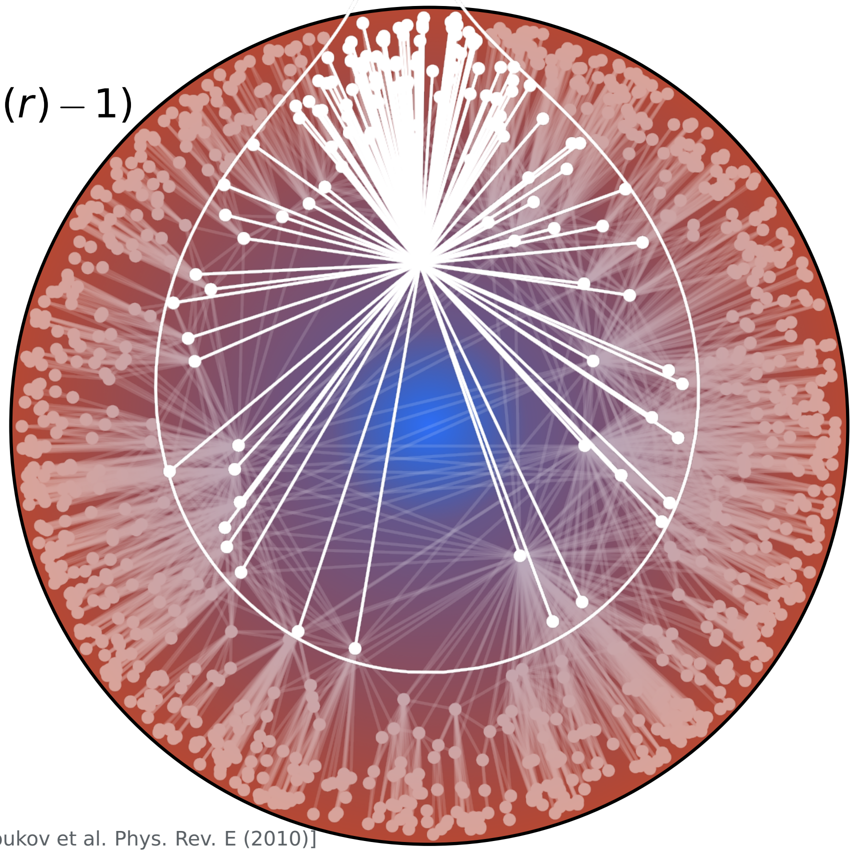
- ... a constant average degree





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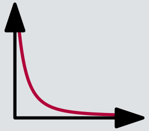
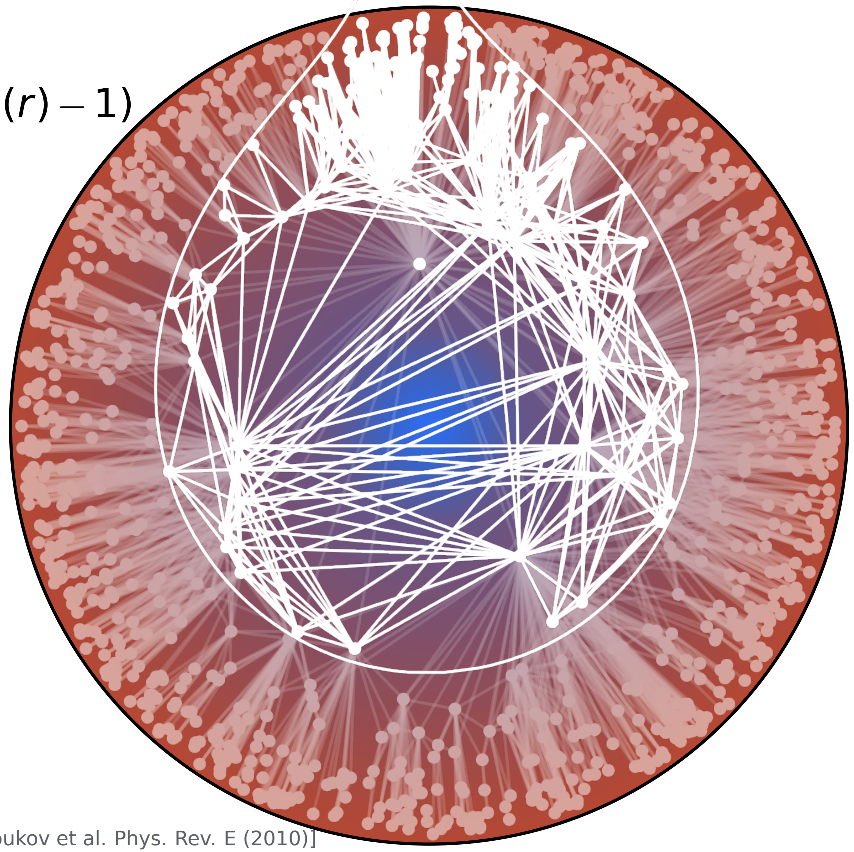
- ... a constant average degree
- ... a non-vanishing clustering coefficient





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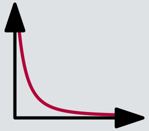
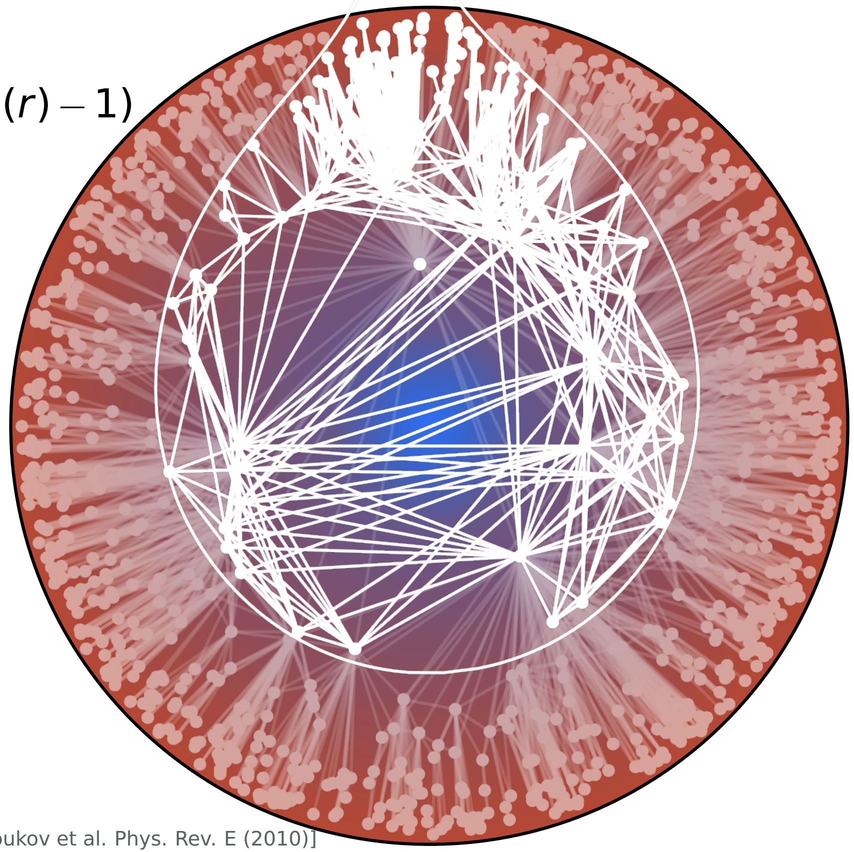
- ... a constant average degree
- ... a non-vanishing clustering coefficient
- ... a logarithmic diameter





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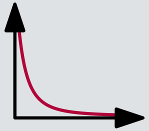
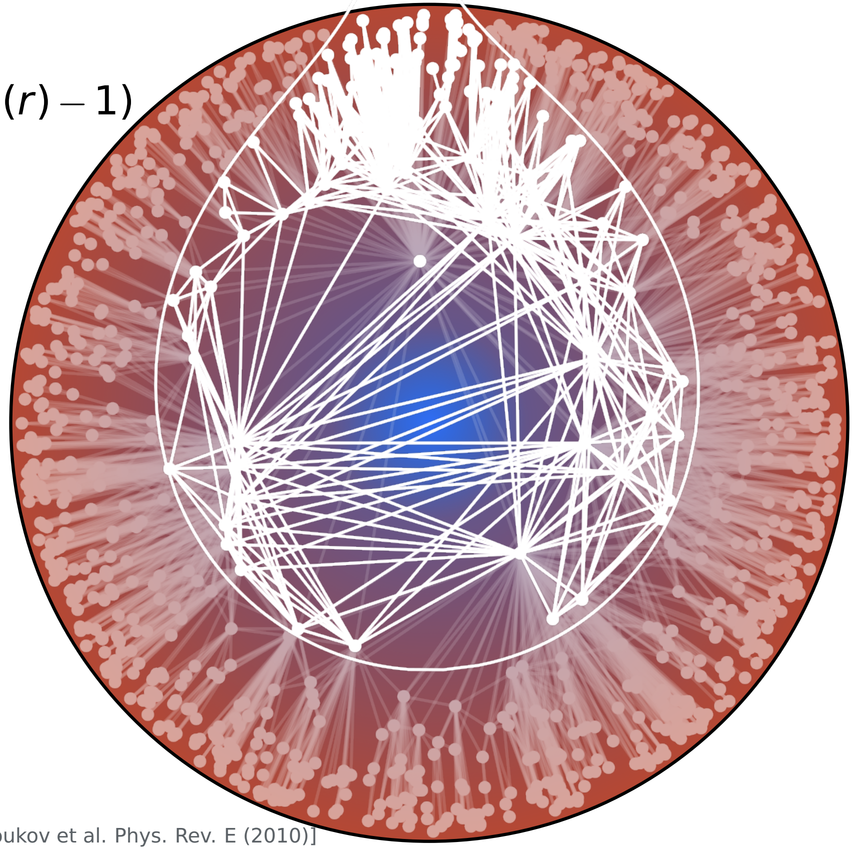
Hyperbolic random graphs have...

- ... a power-law degree distribution
- ... a constant average degree
- ... a non-vanishing clustering coefficient
- ... a logarithmic diameter
- ... negative assortativity





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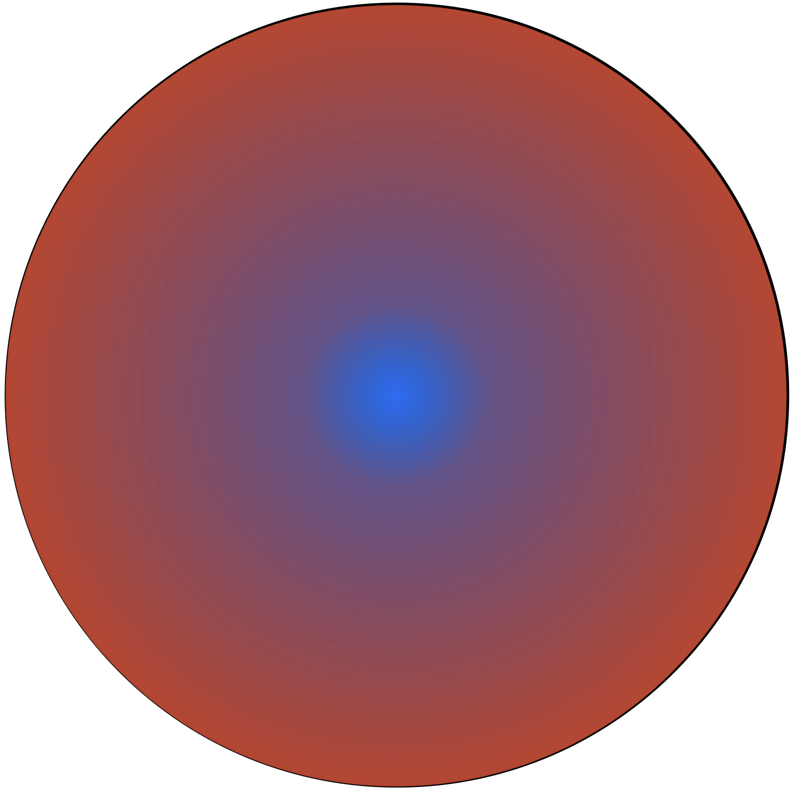

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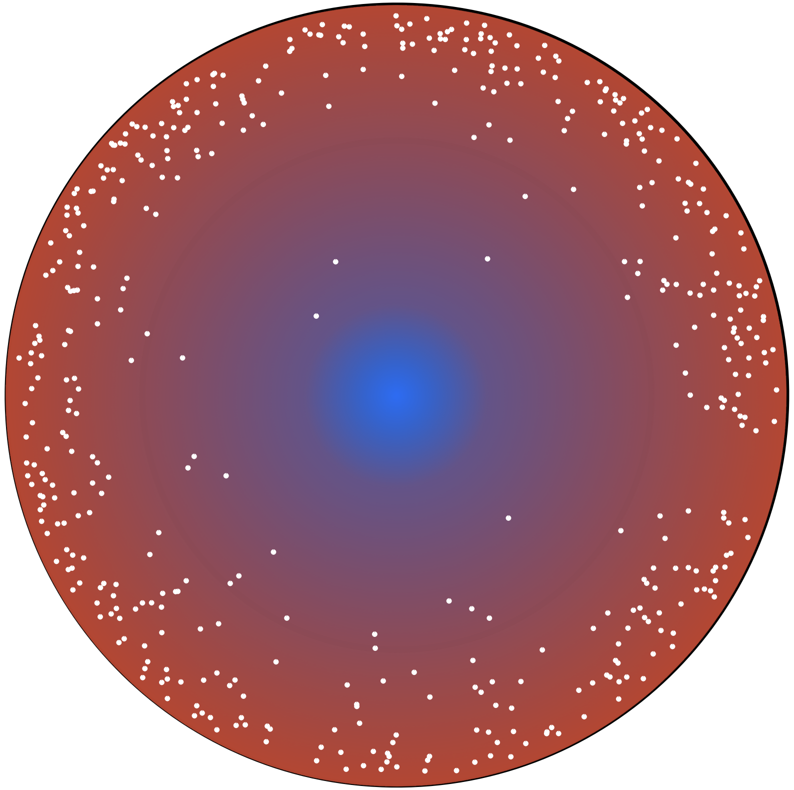

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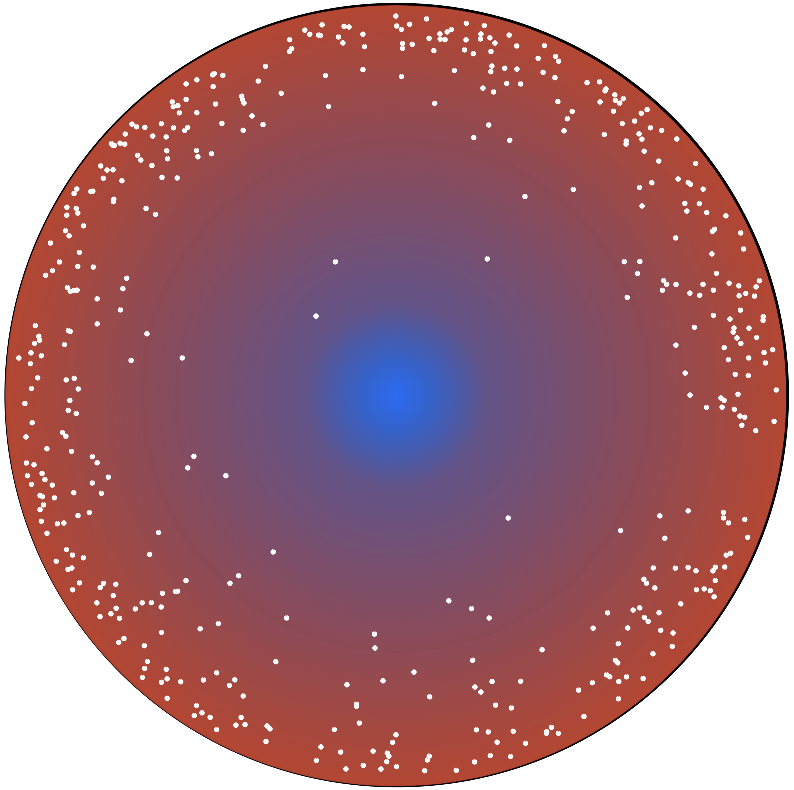

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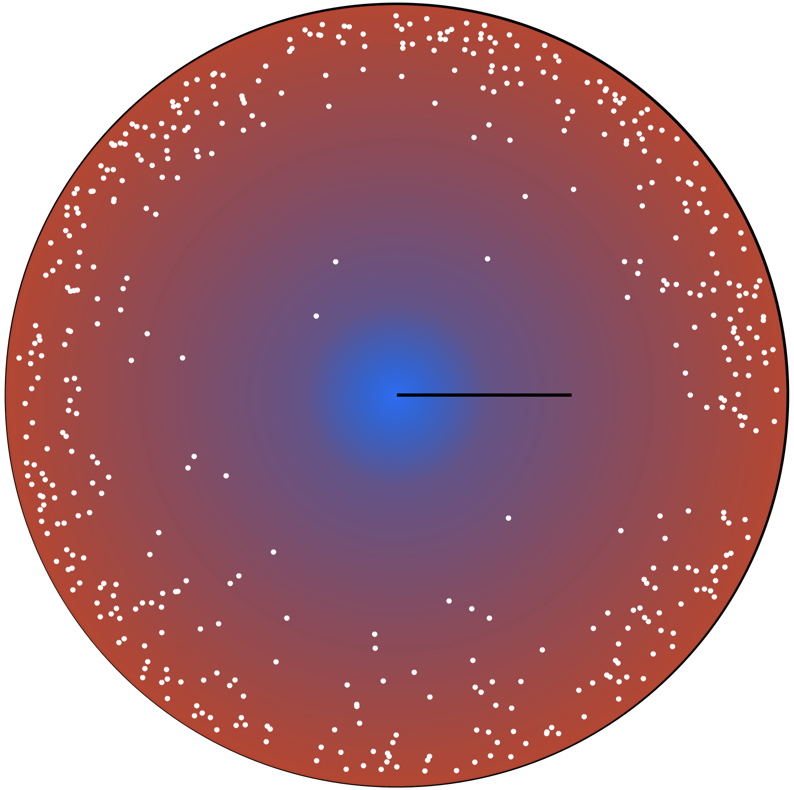

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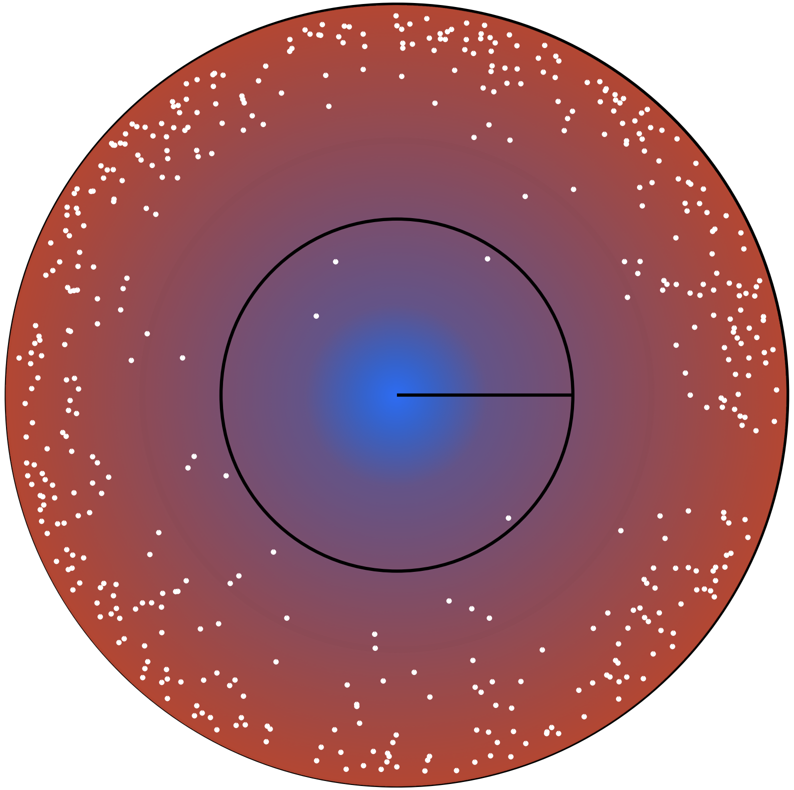

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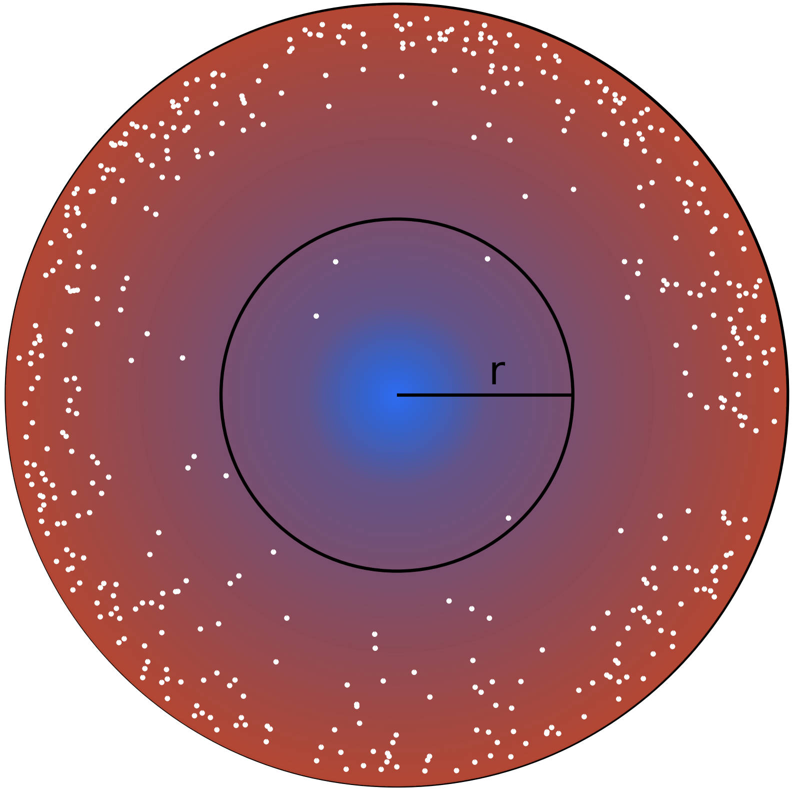

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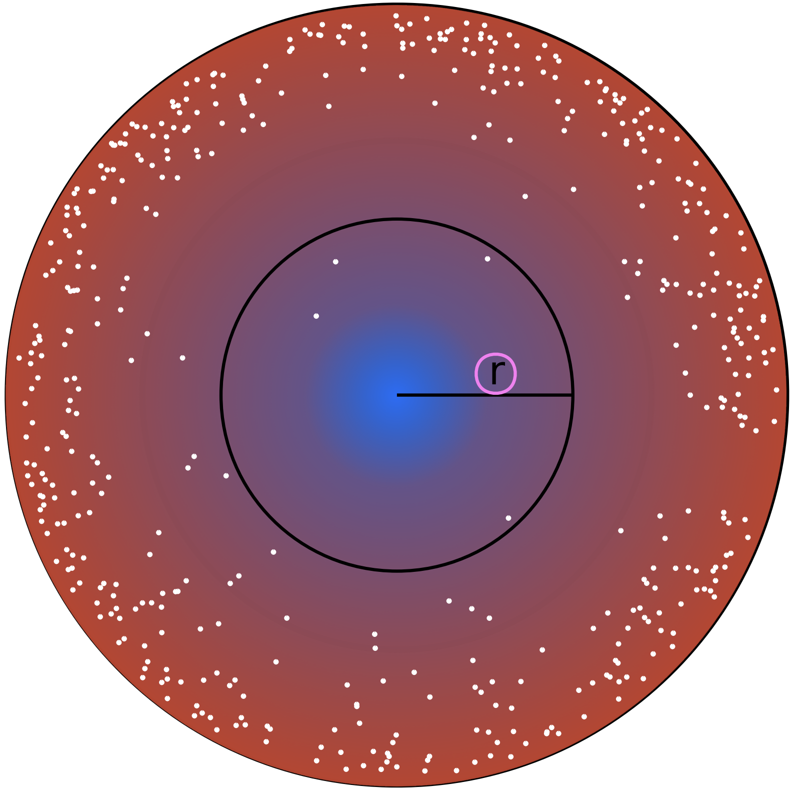

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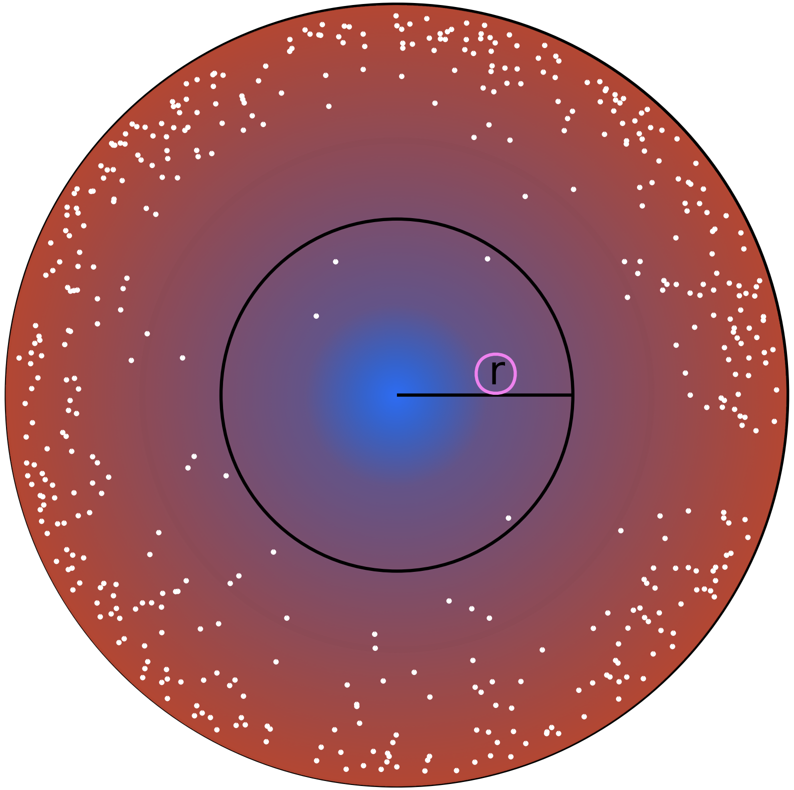

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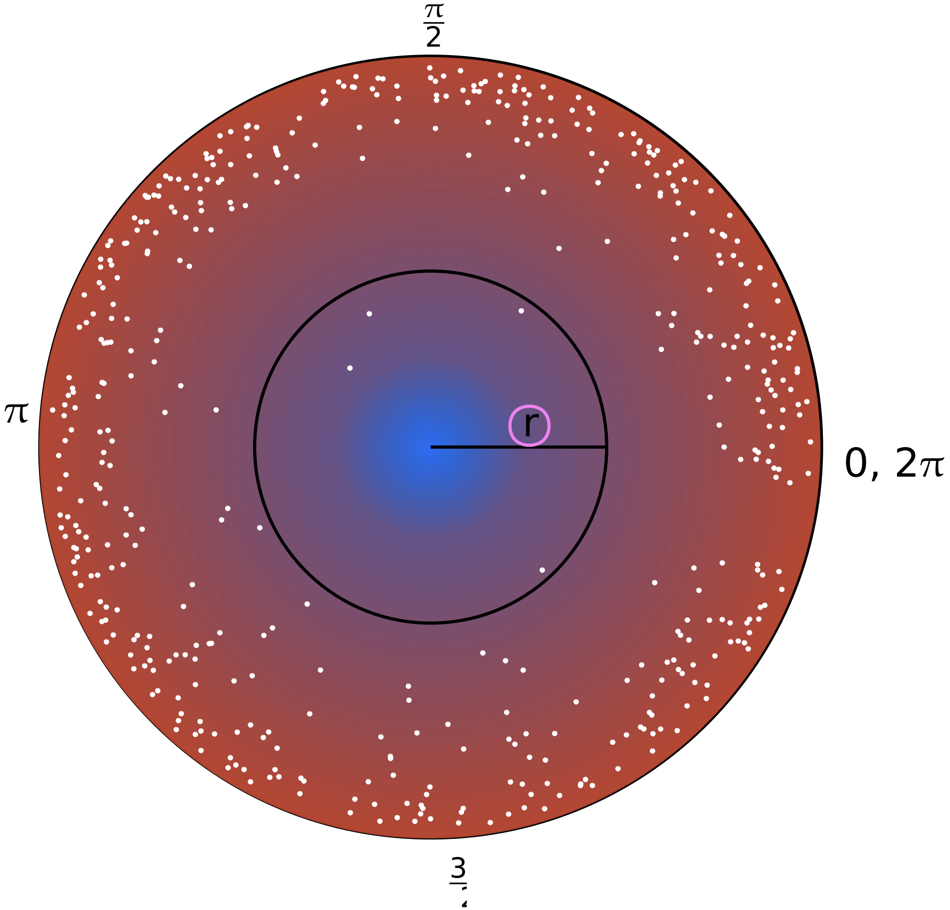

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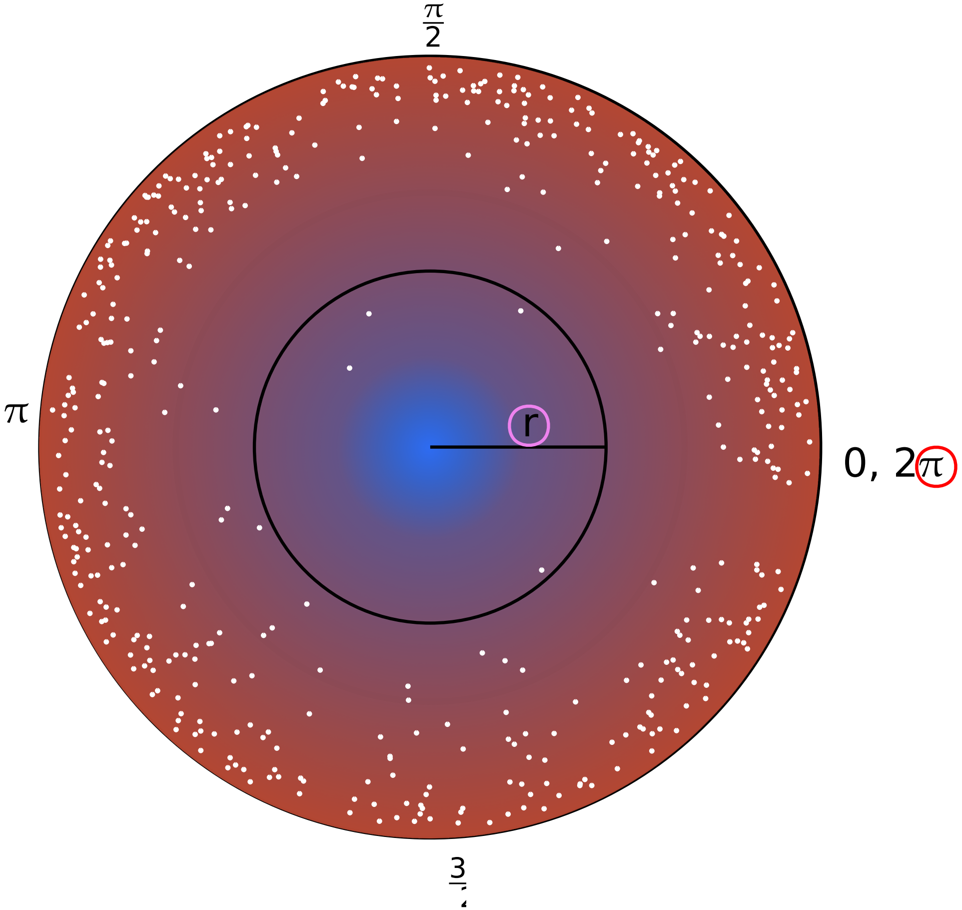

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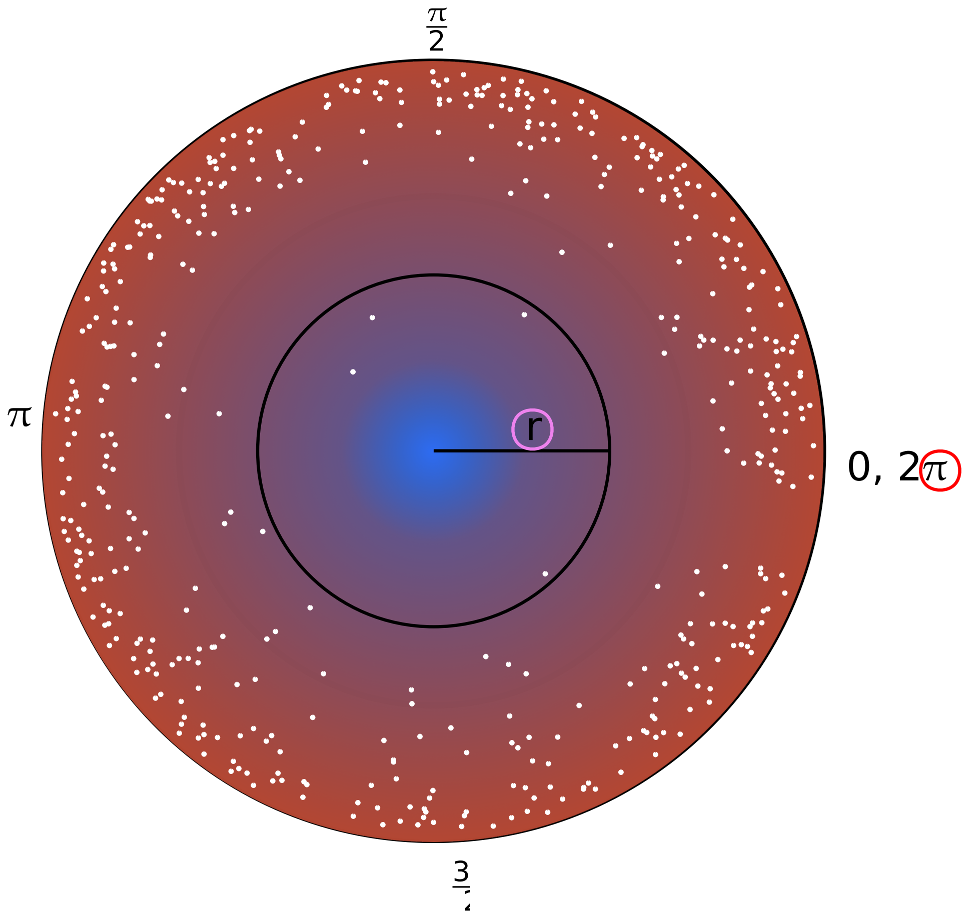

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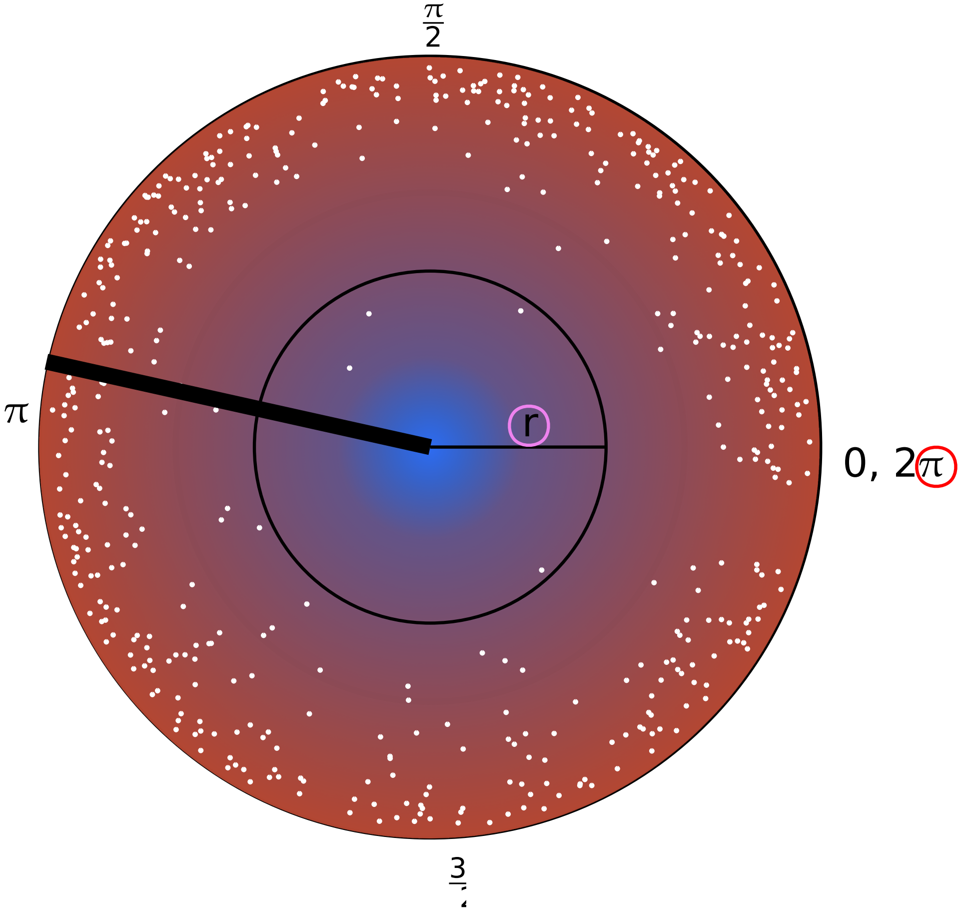

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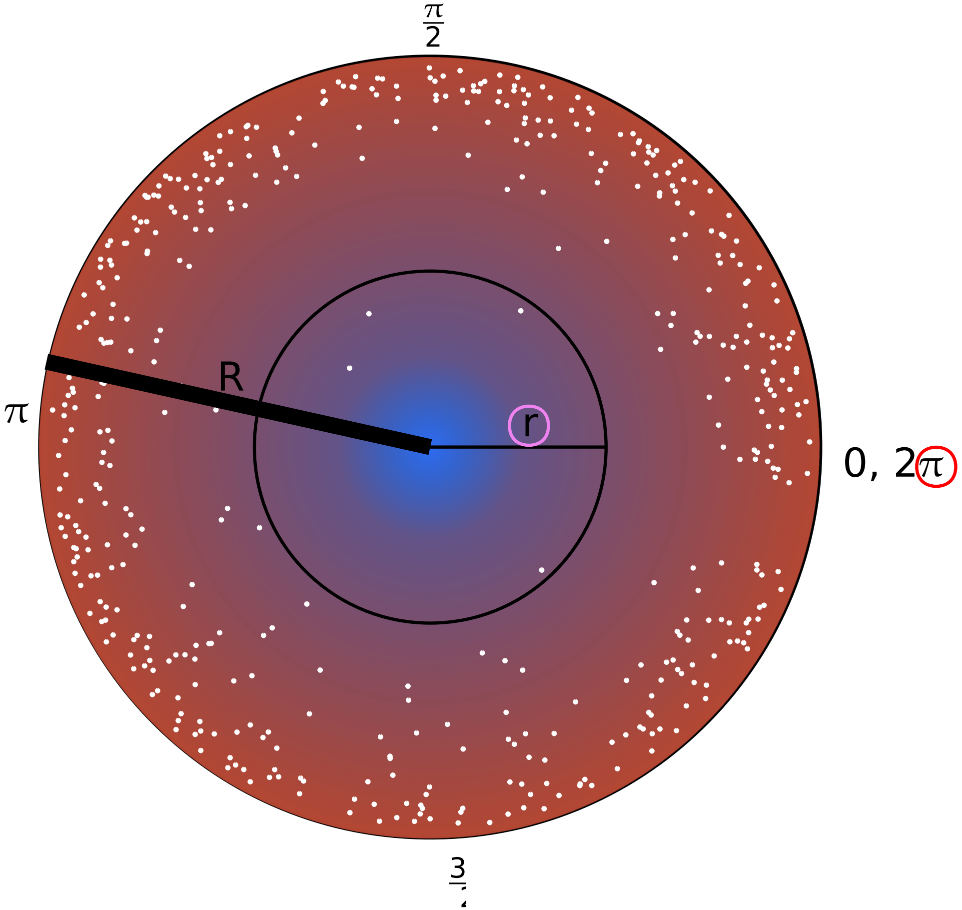

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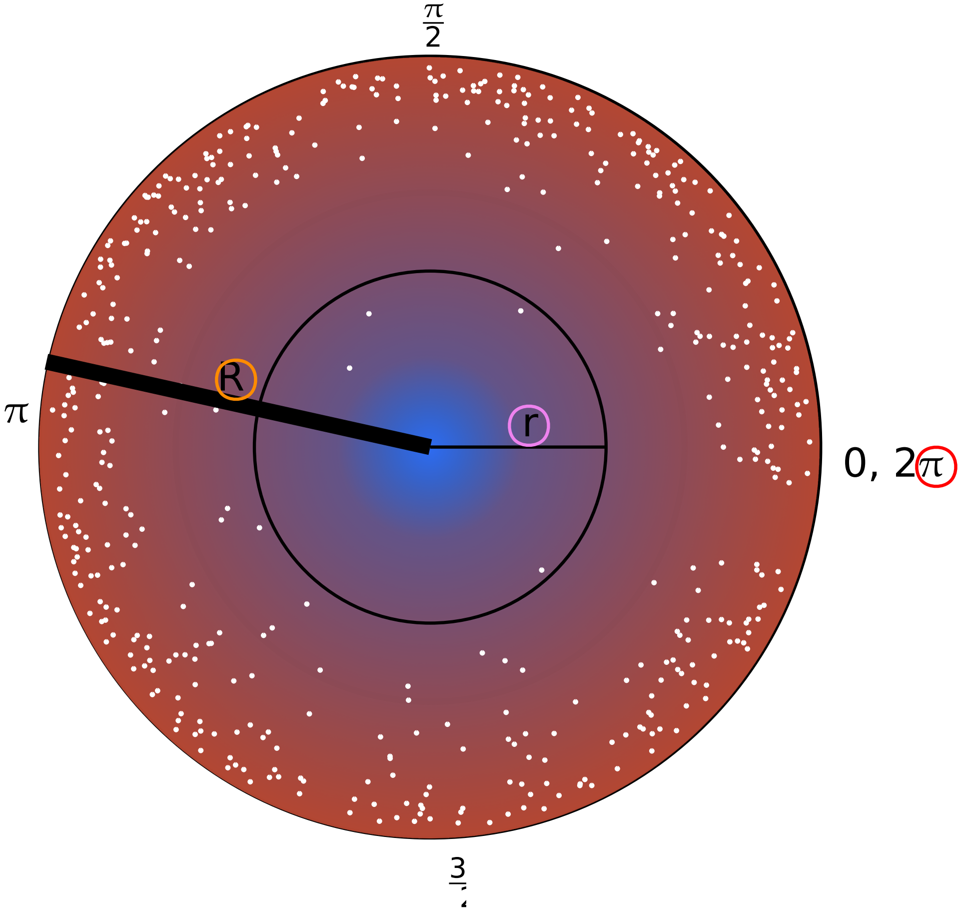

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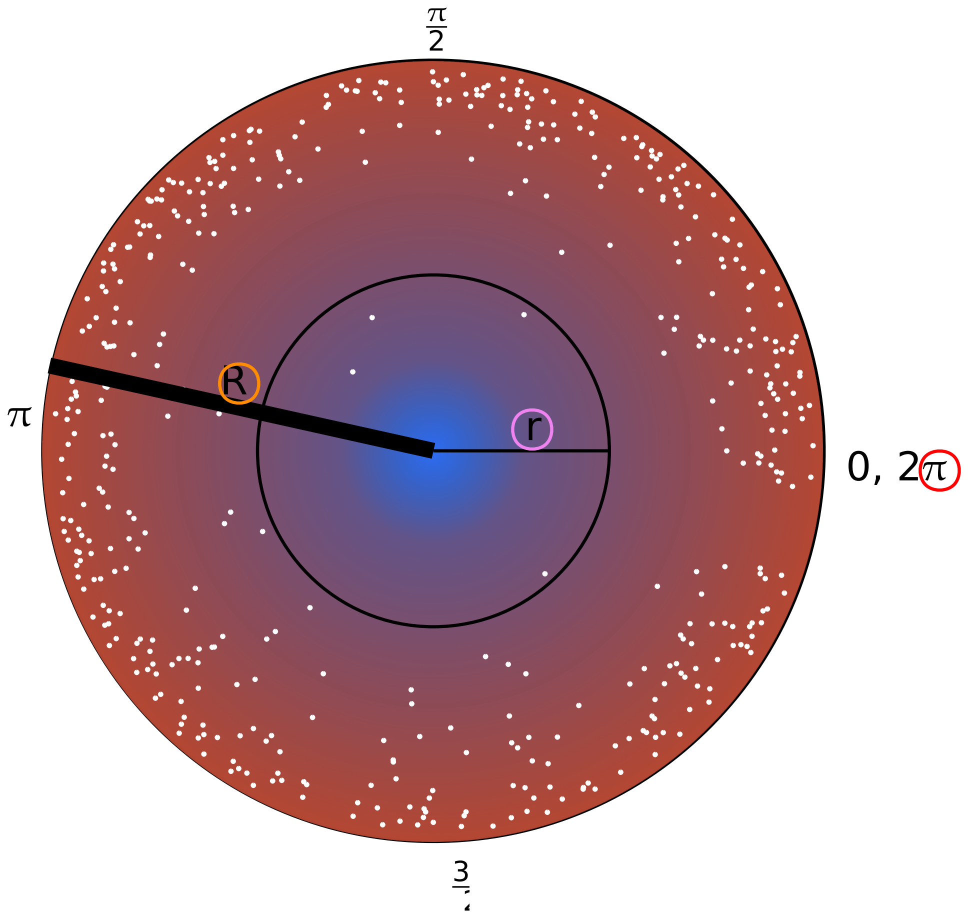

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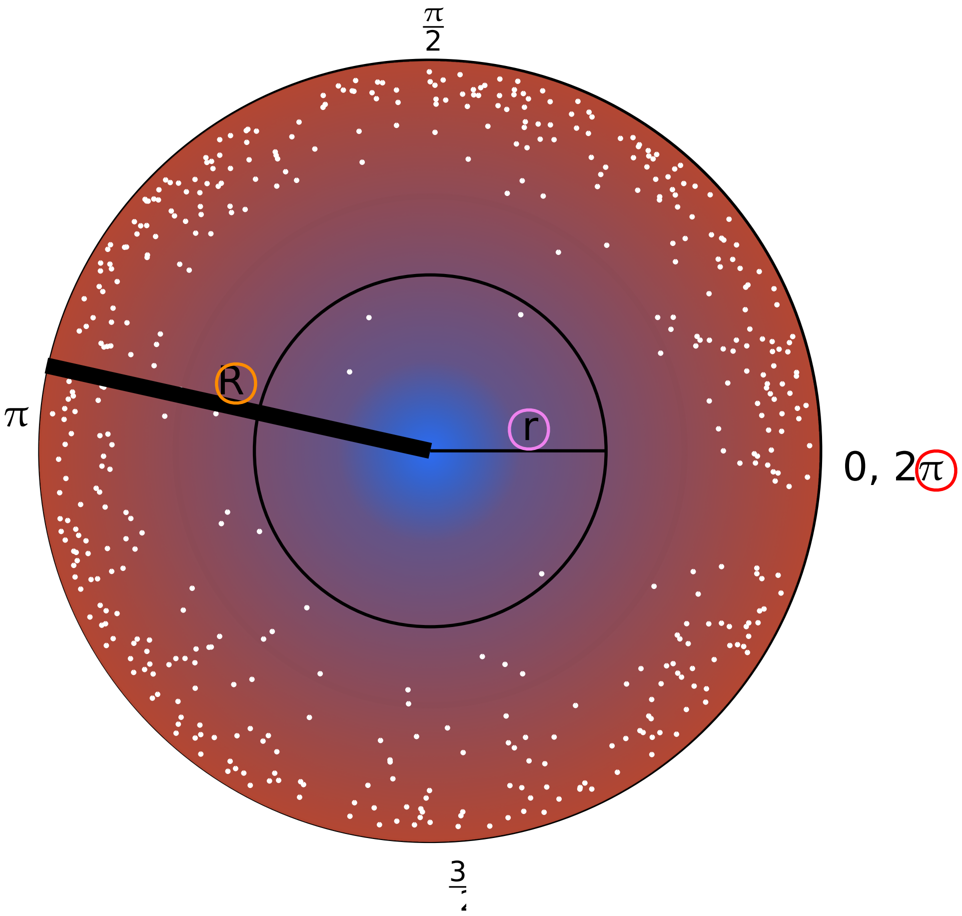

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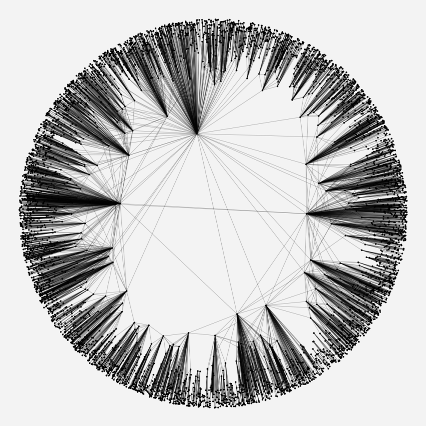

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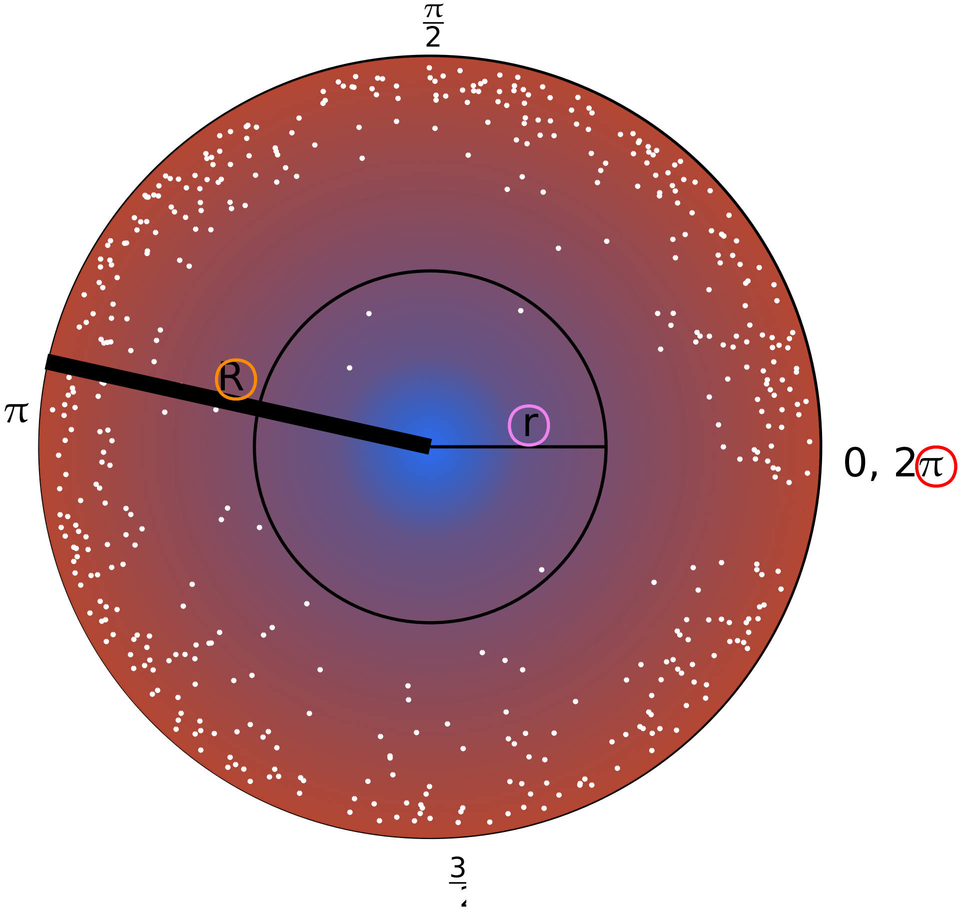

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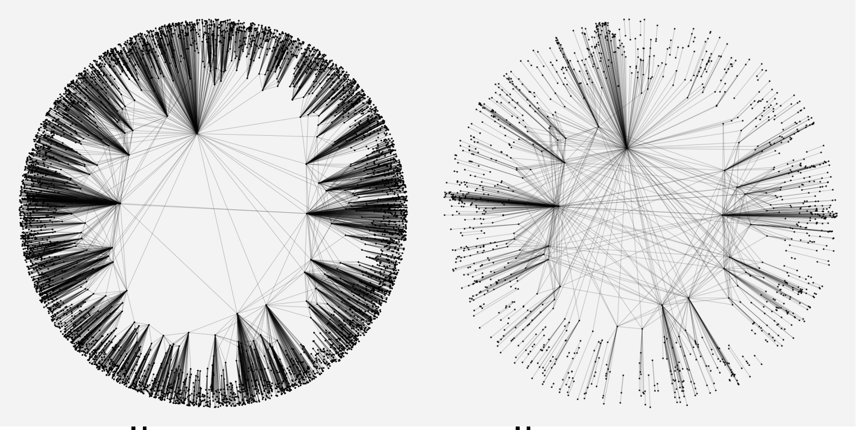
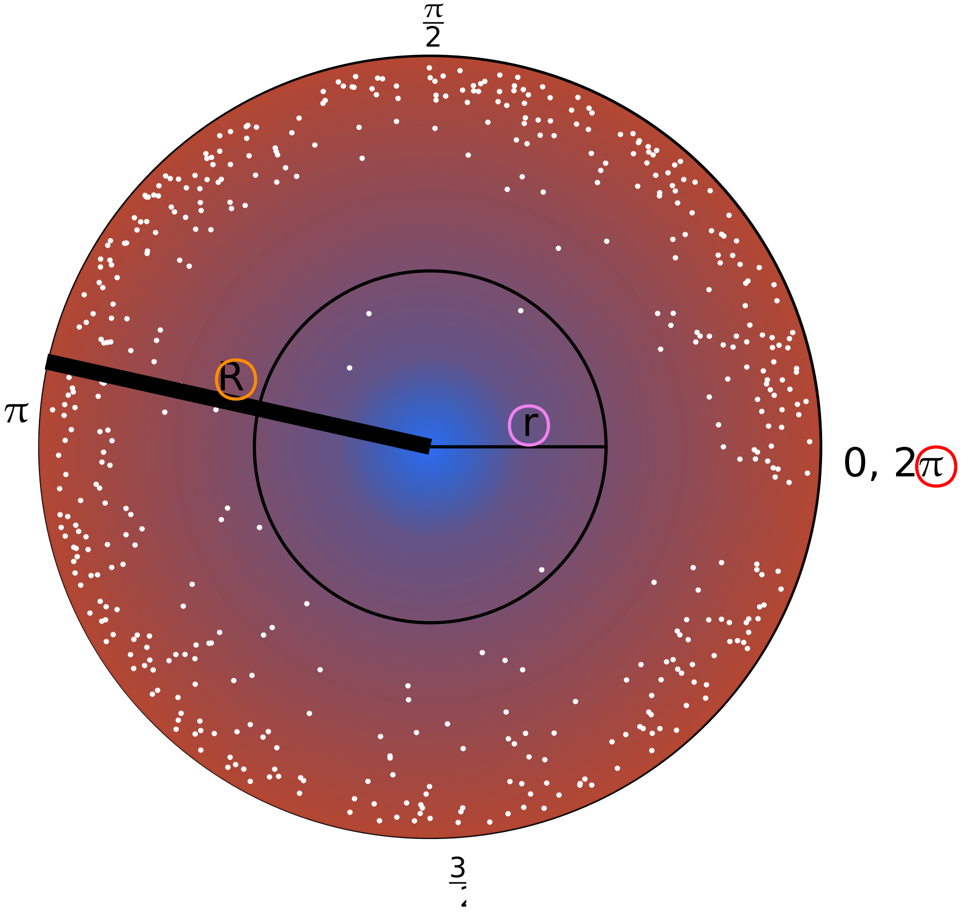


$\lim_{\alpha \uparrow 1}$ $\lim_{\alpha \downarrow 0.5}$



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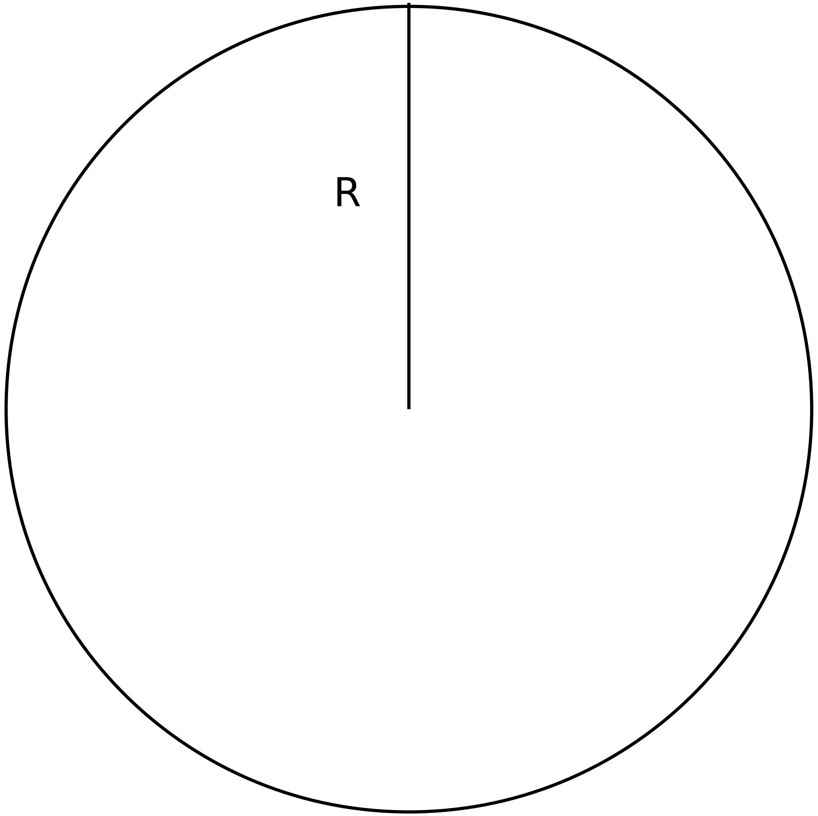

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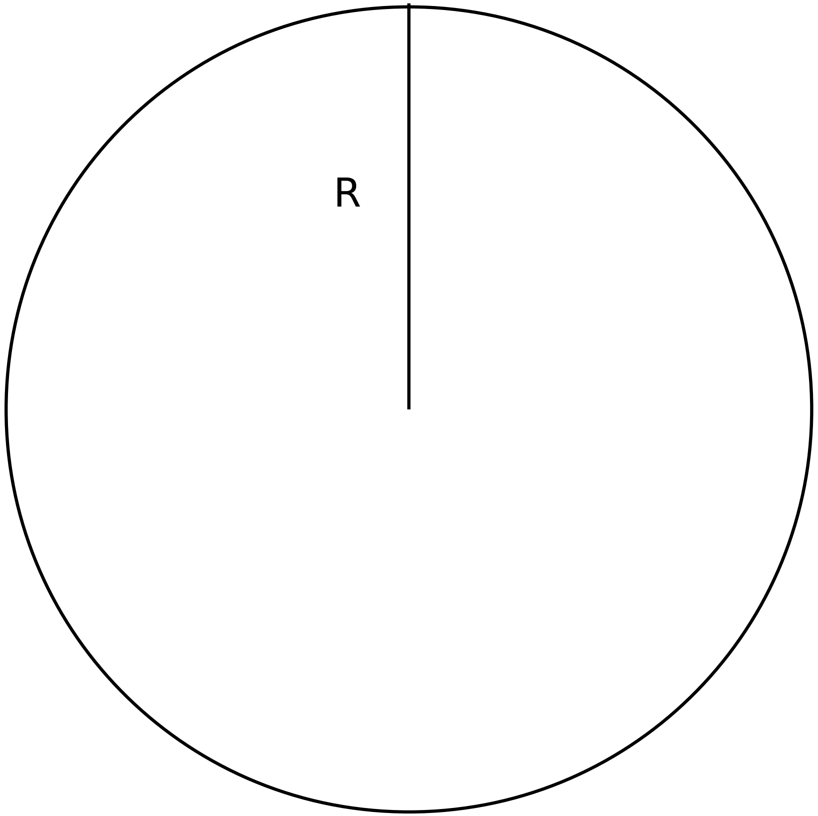

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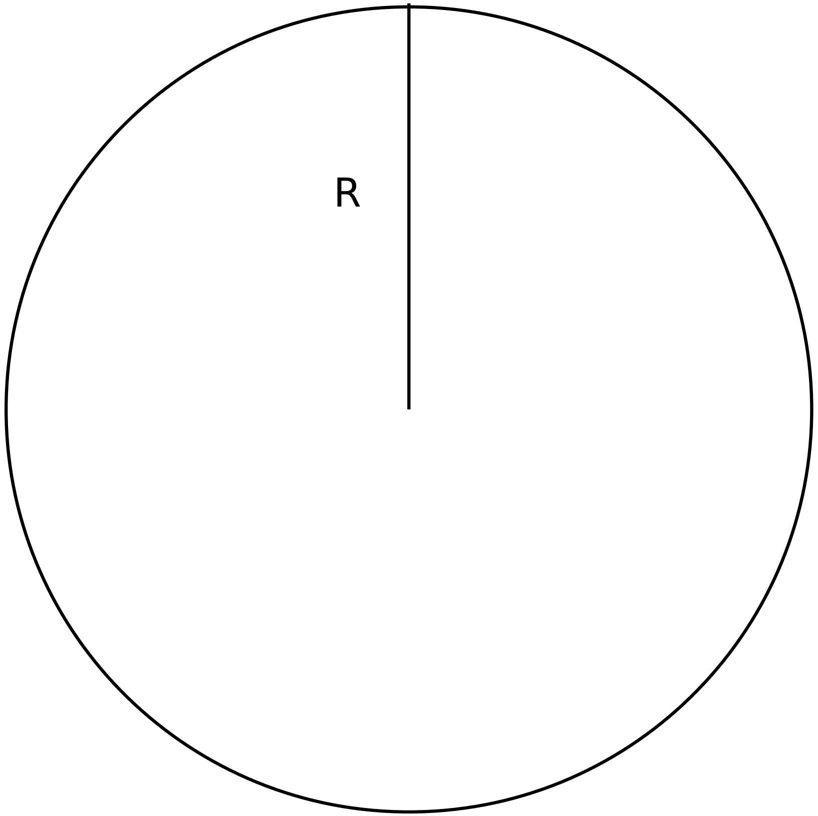

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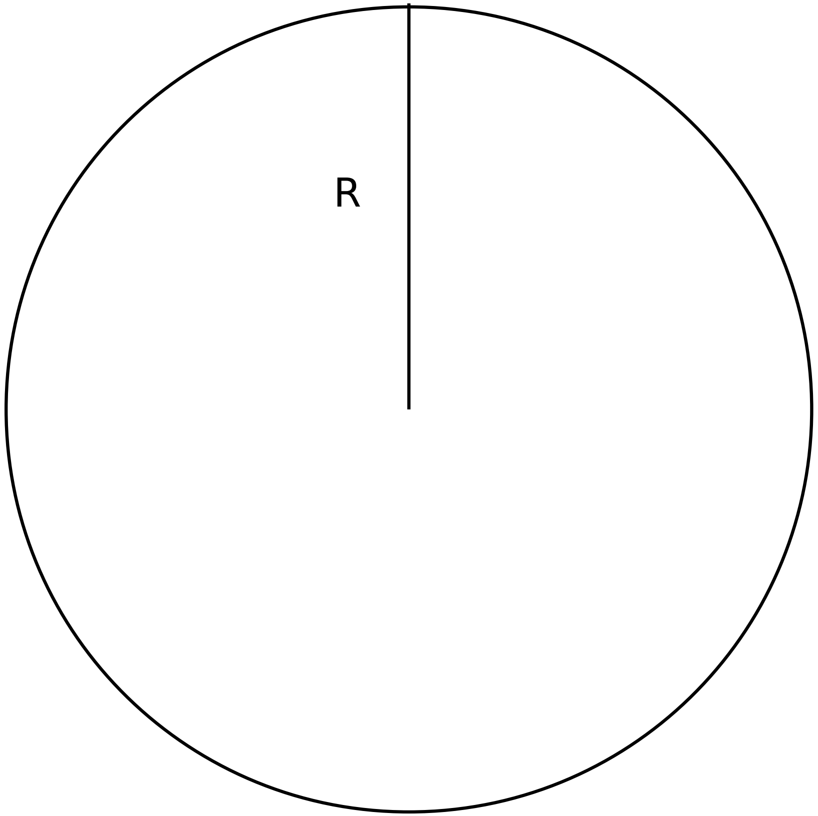
$E(G) = \{E_{i,j} | dist(i,j) \leq R\}$



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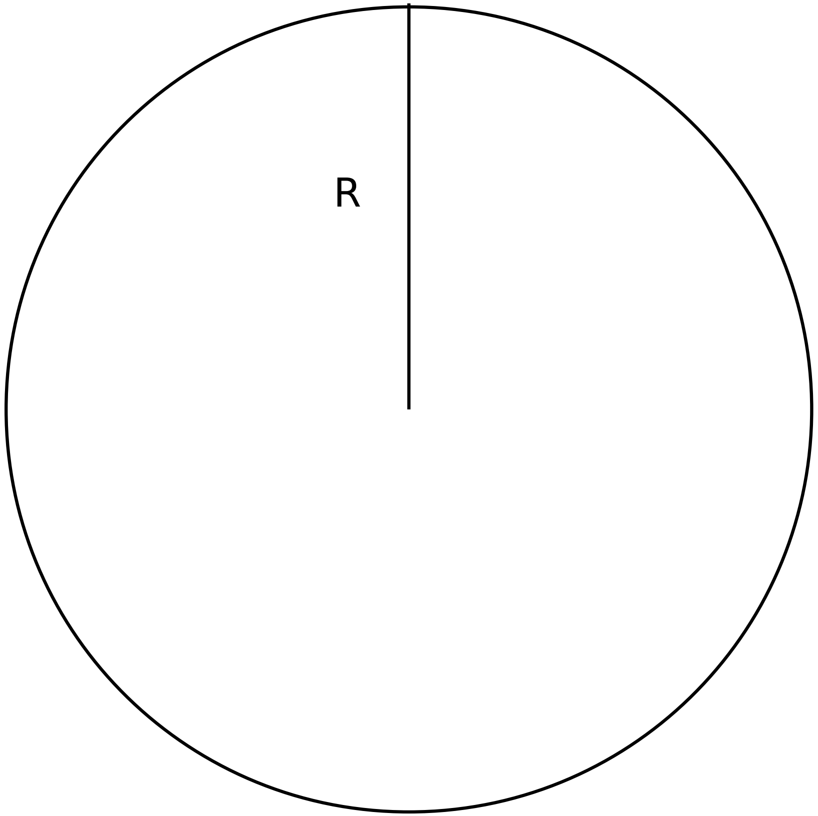

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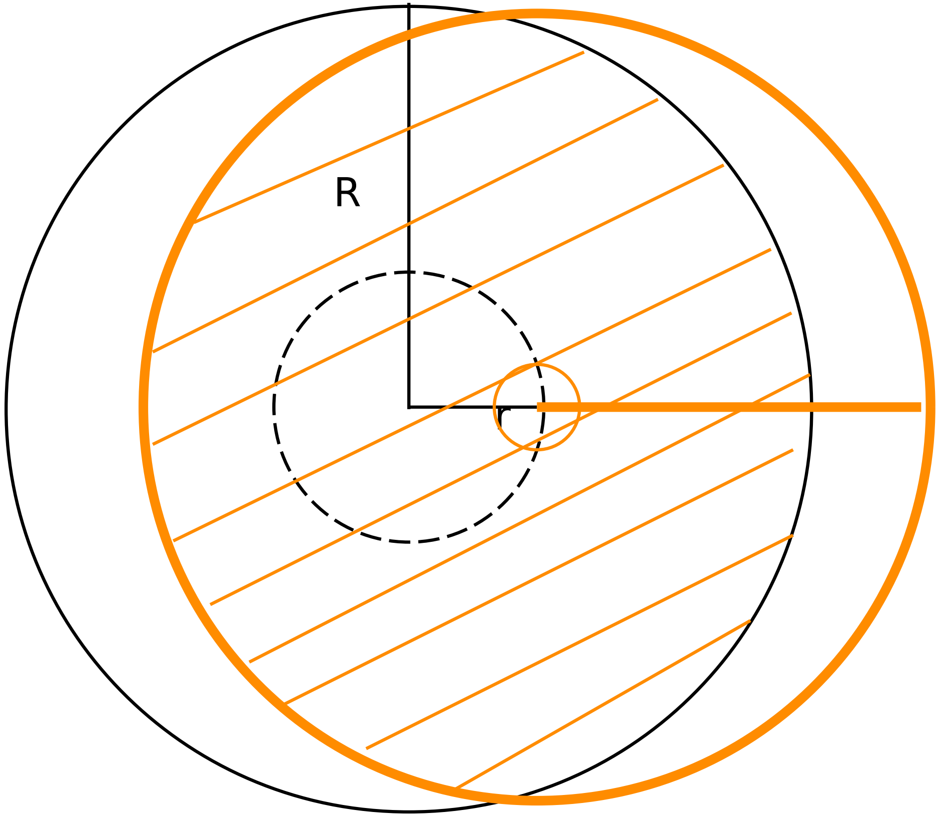

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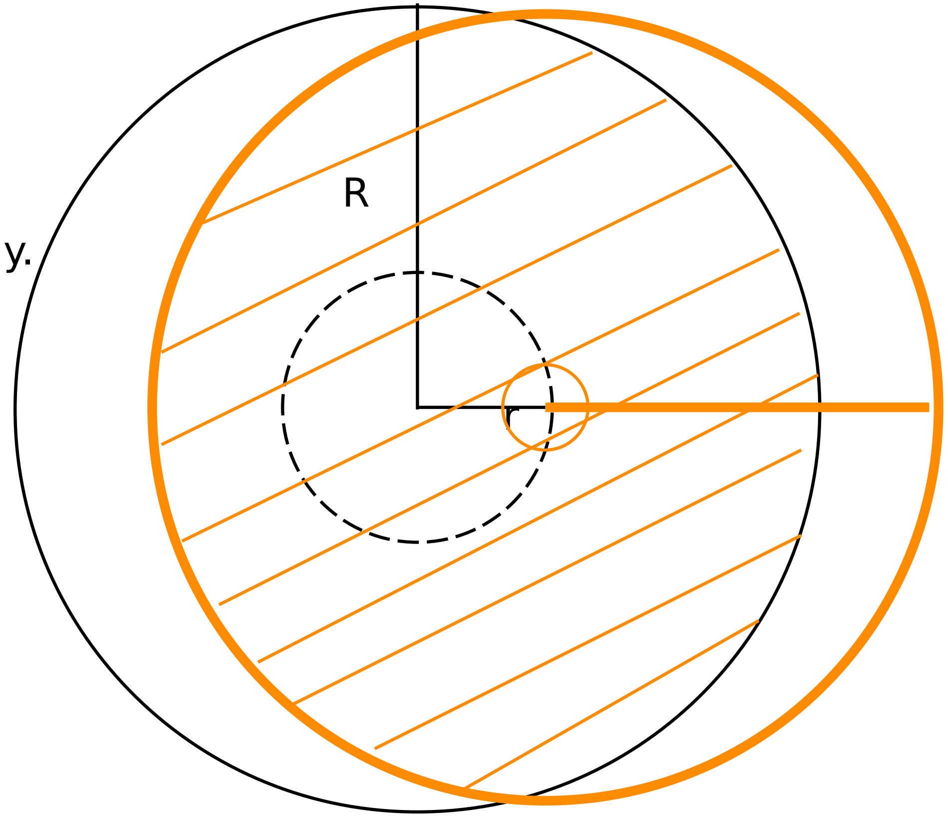
Consider two nodes with radial coordinates x, y.



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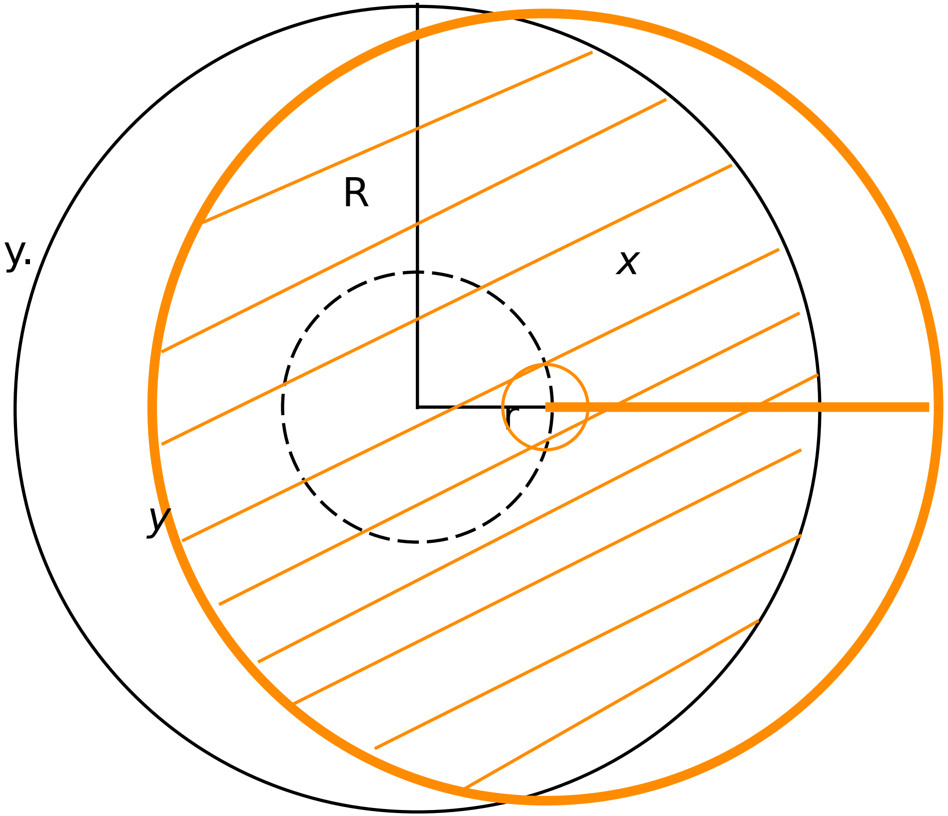

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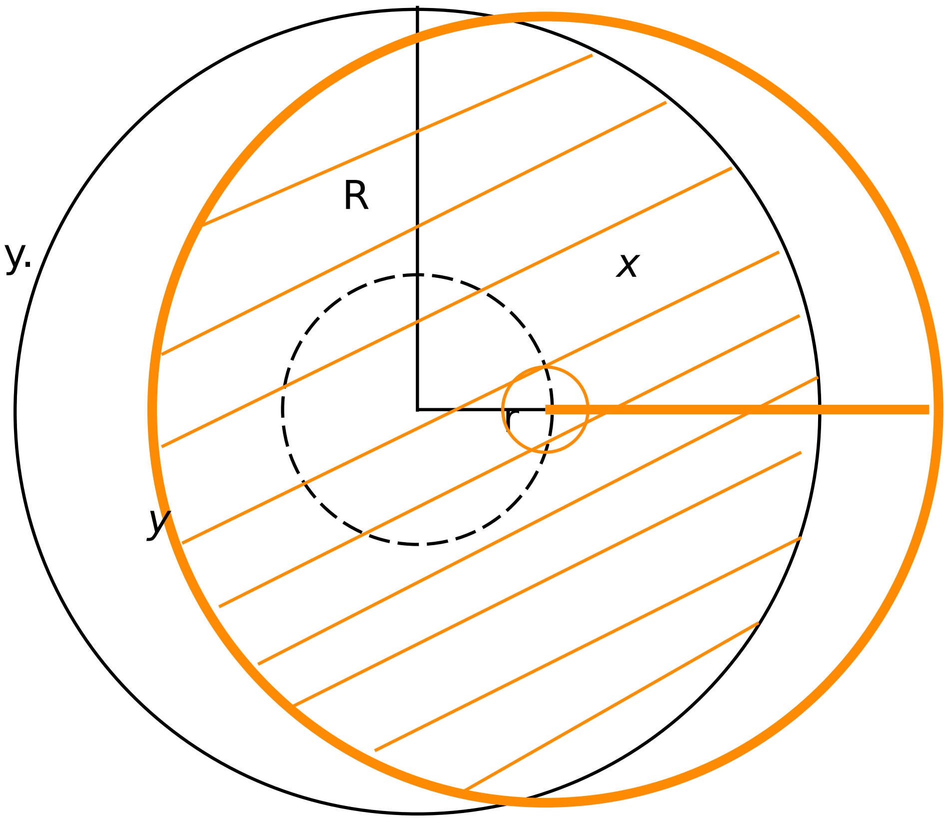

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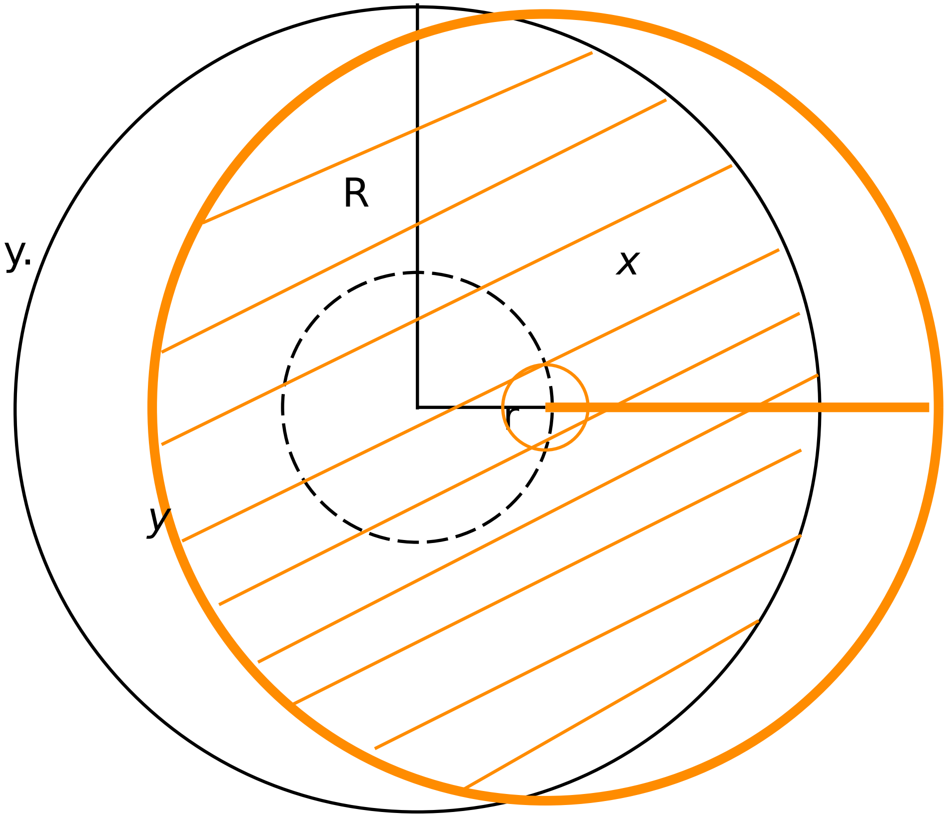

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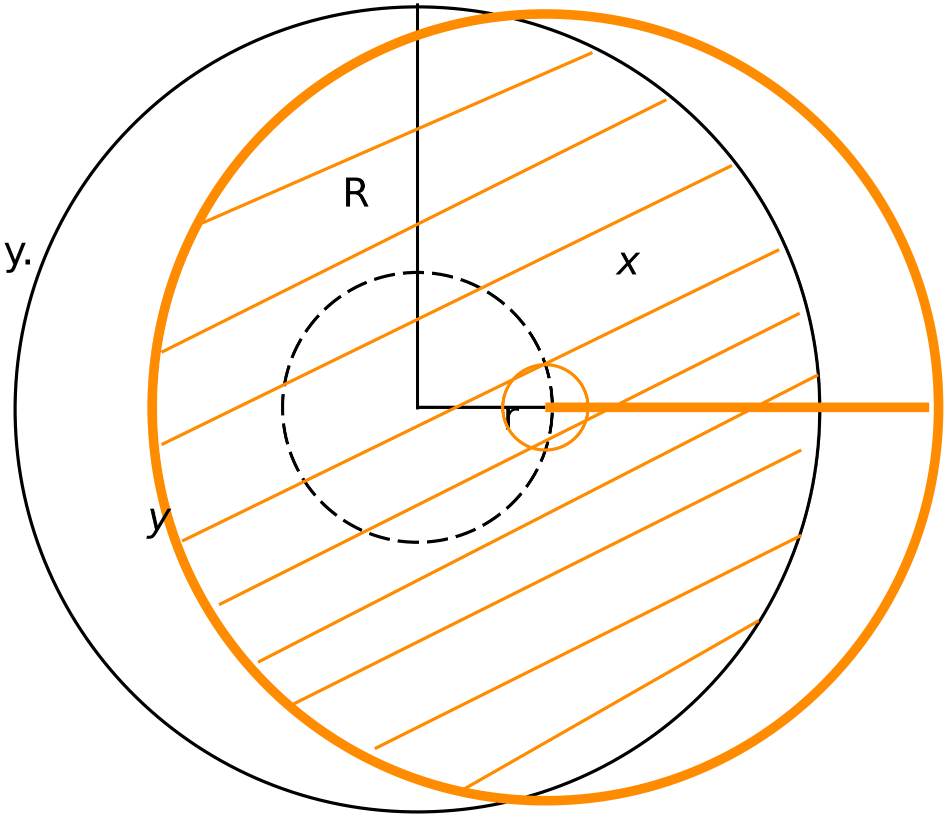

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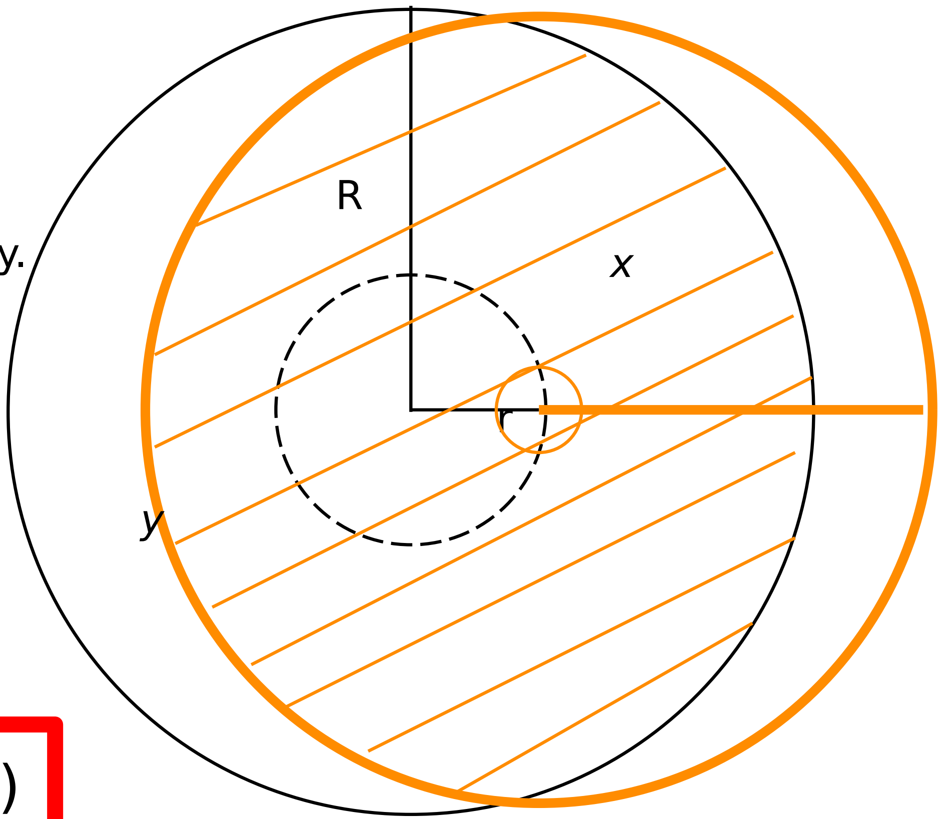

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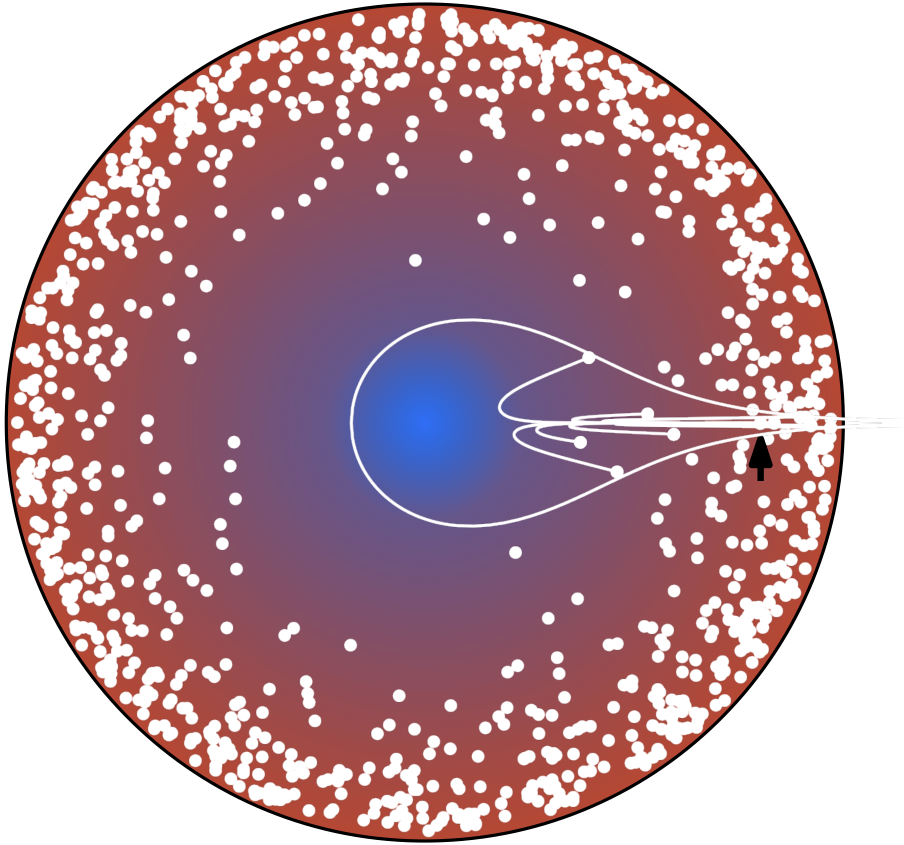

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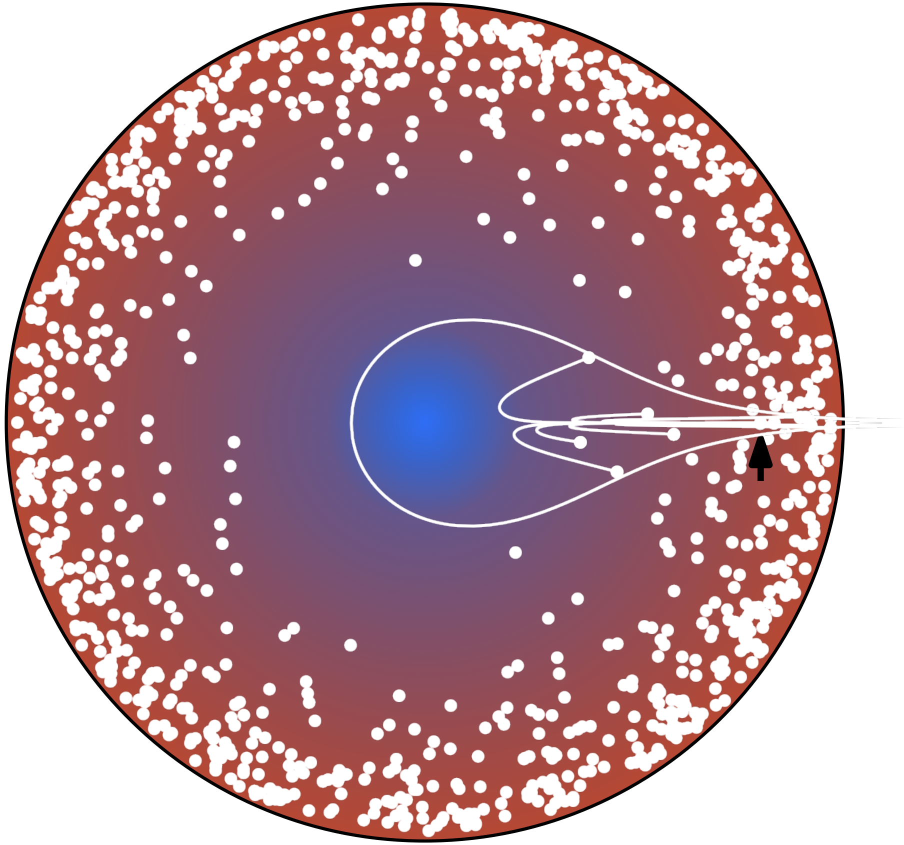

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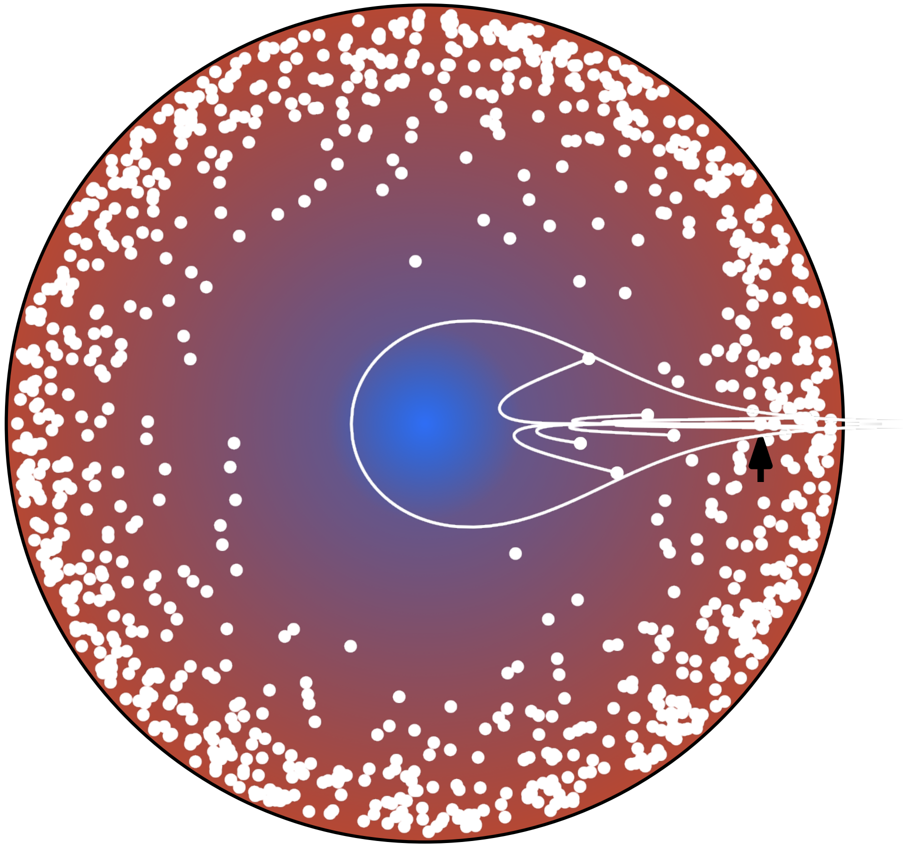

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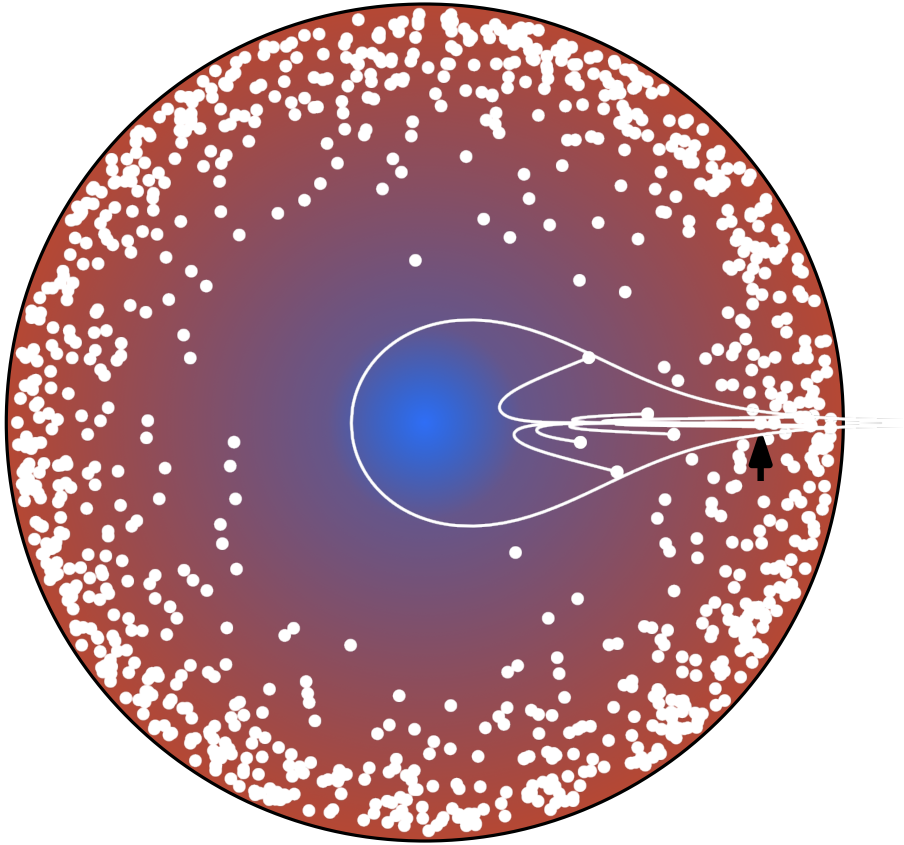

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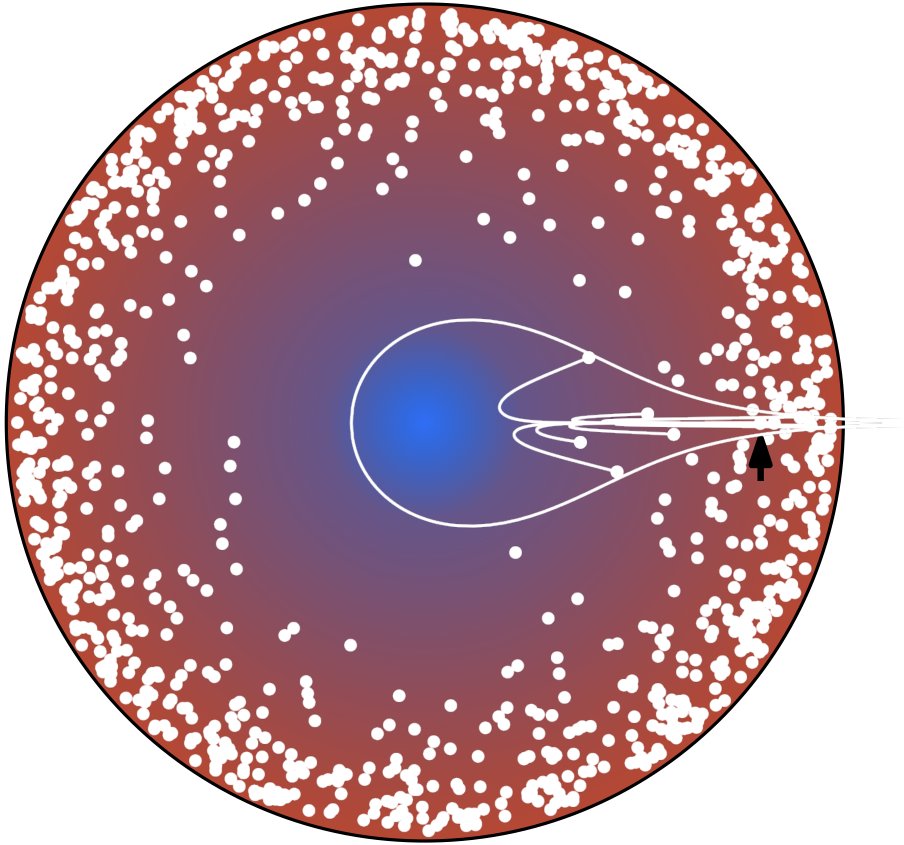

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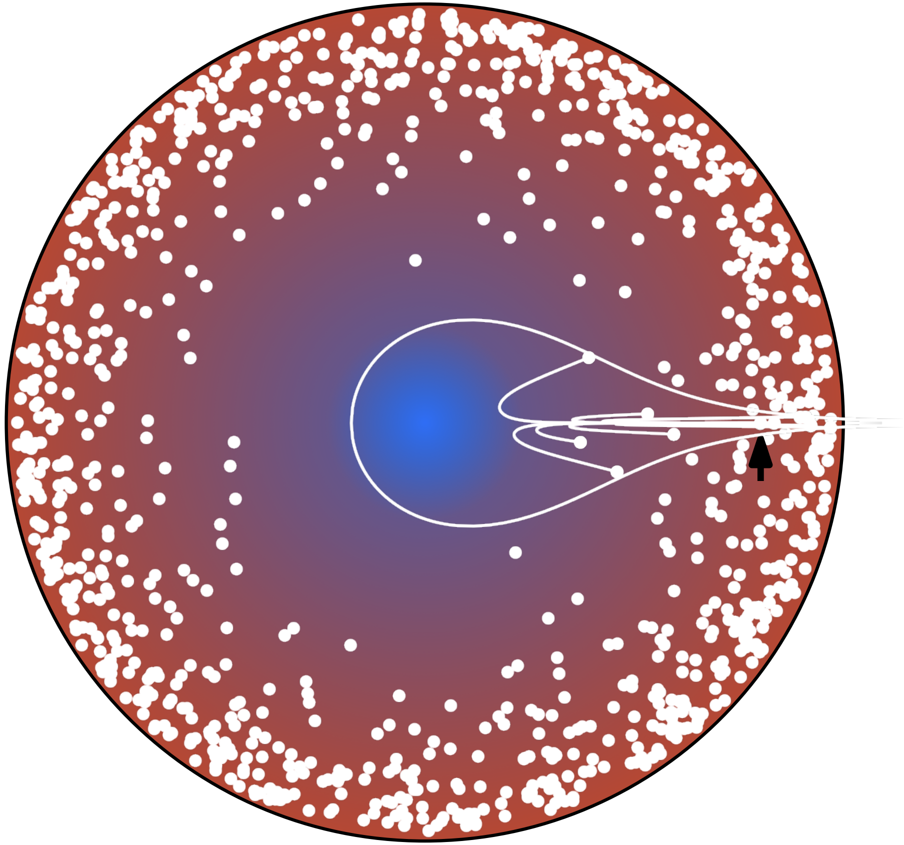

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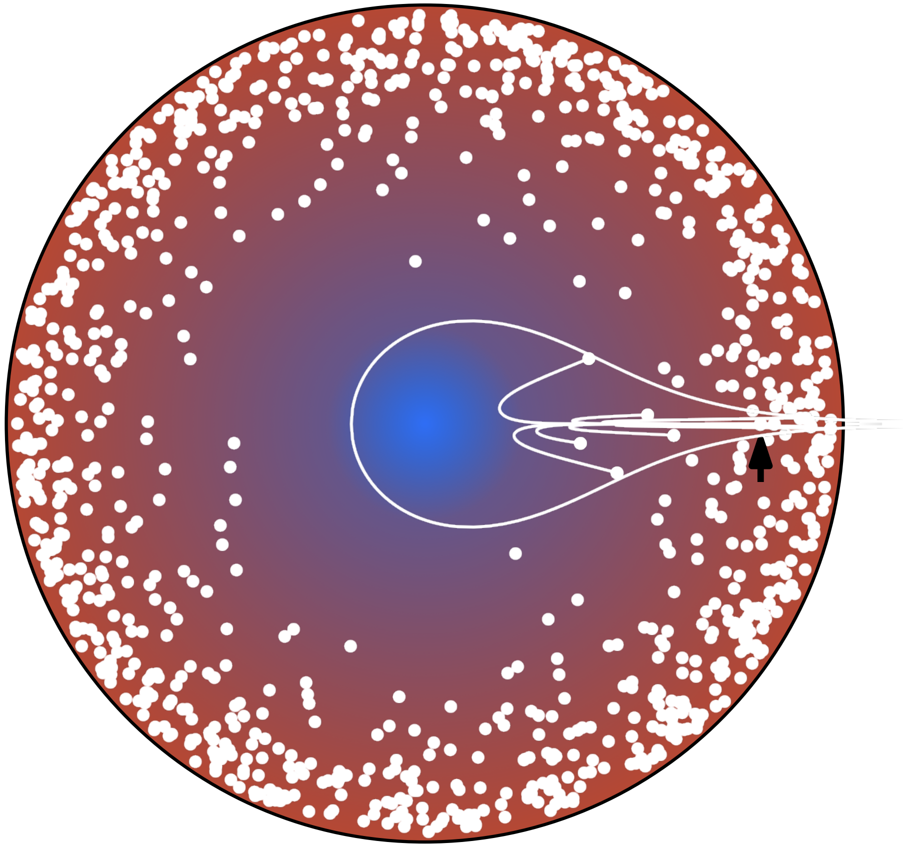

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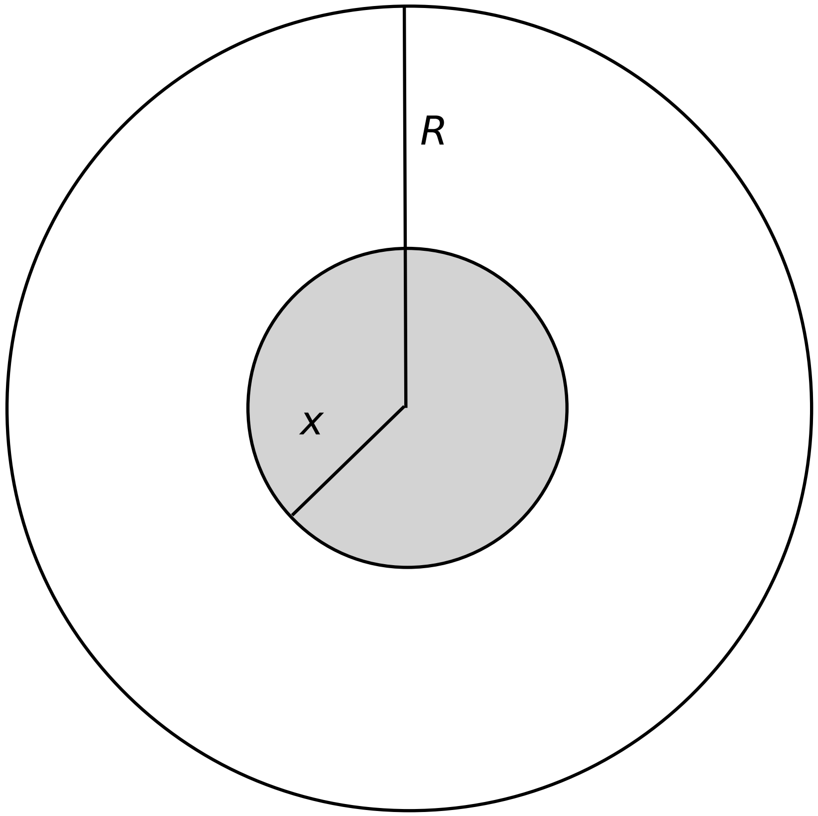

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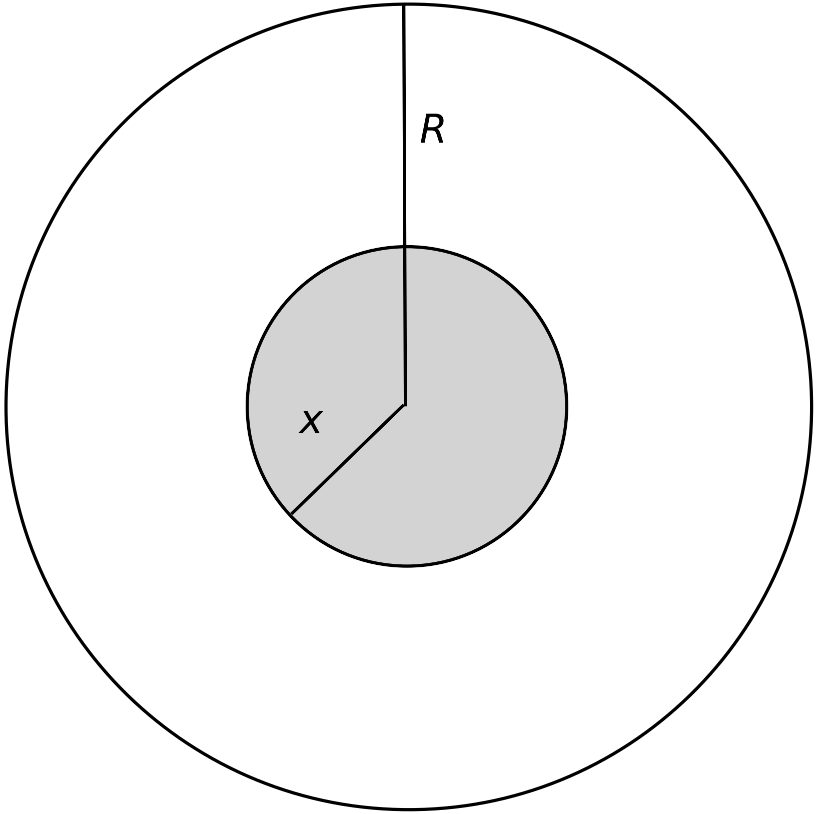

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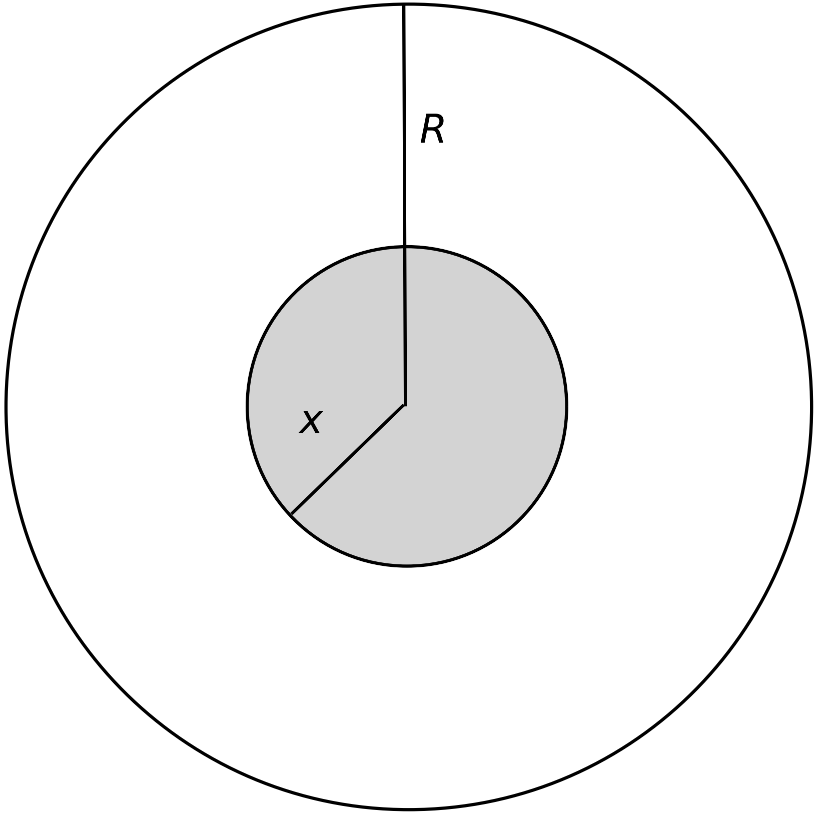

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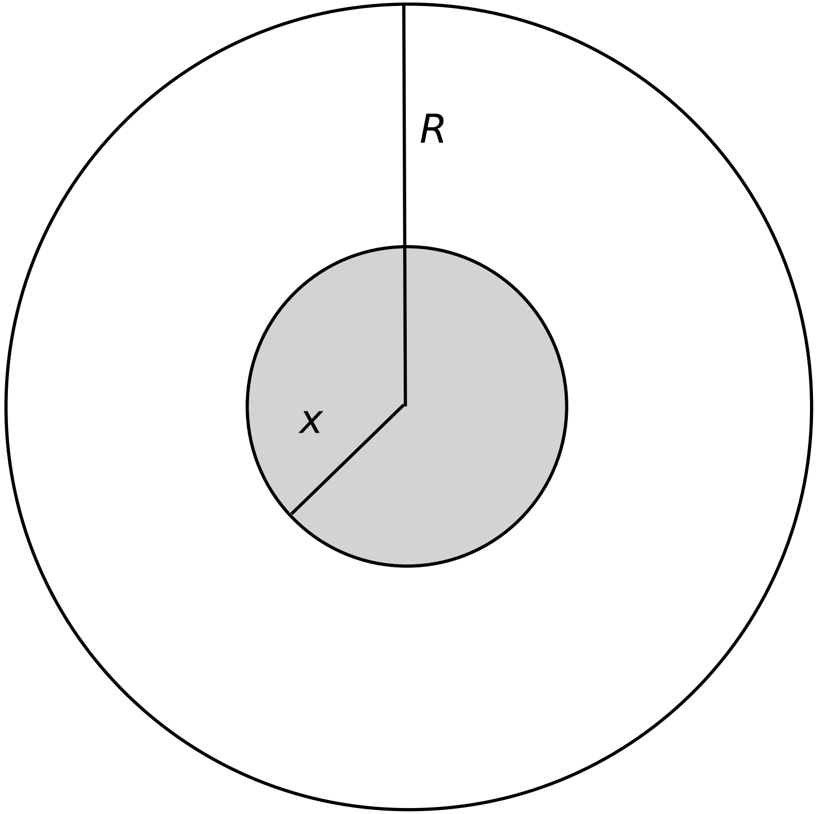

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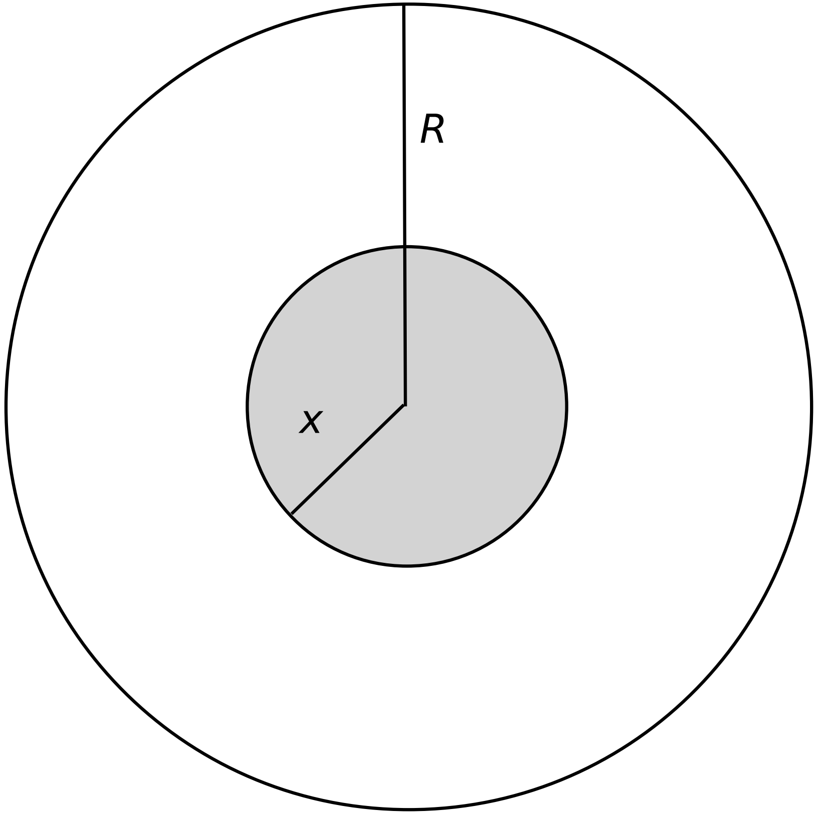

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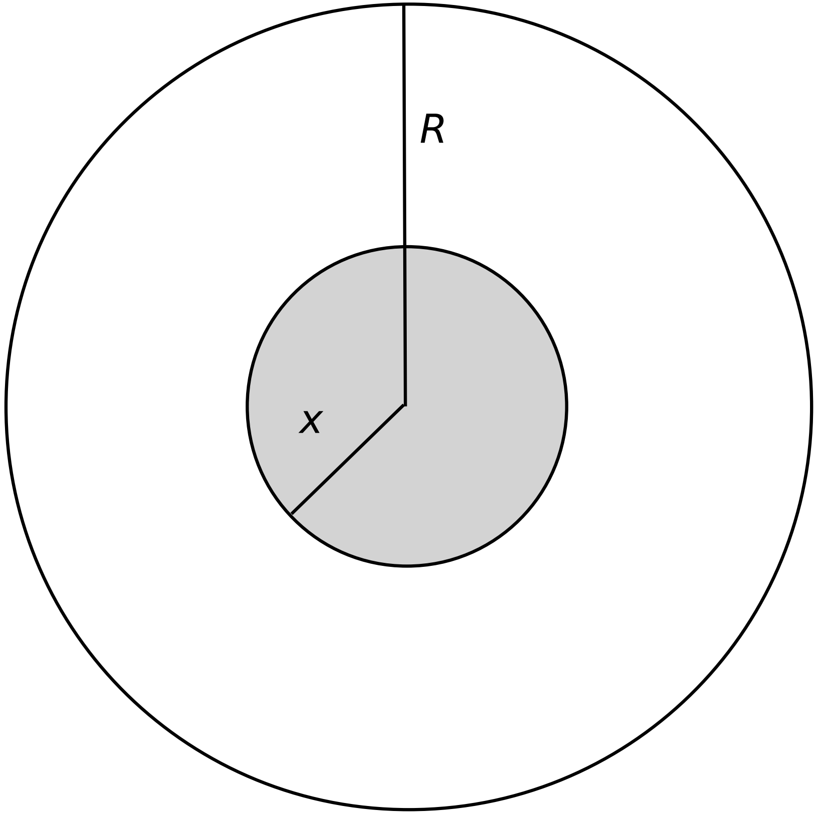

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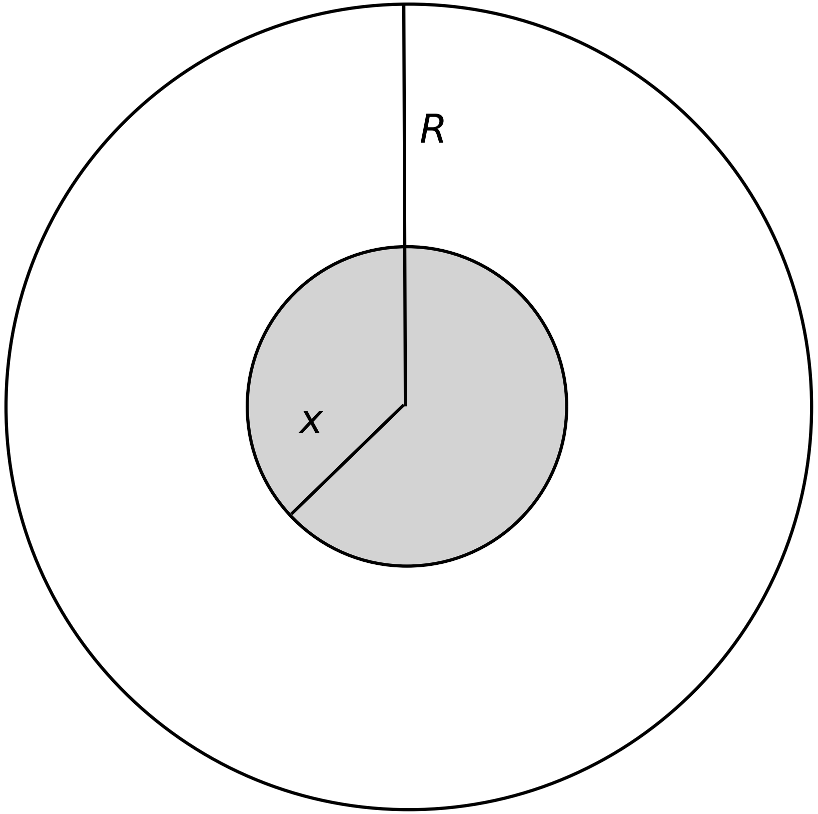

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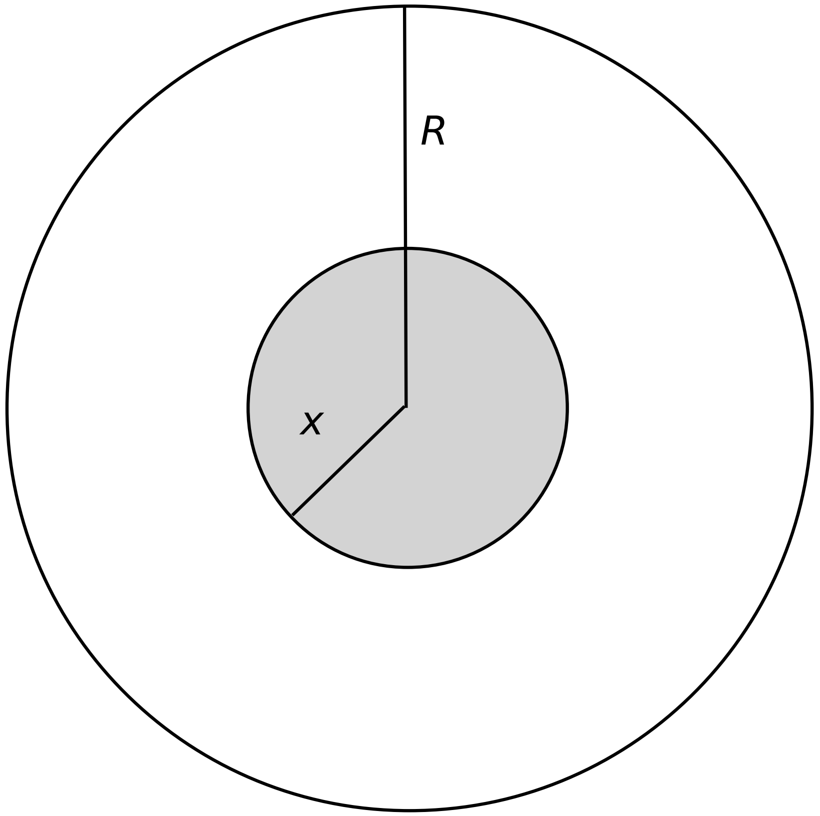

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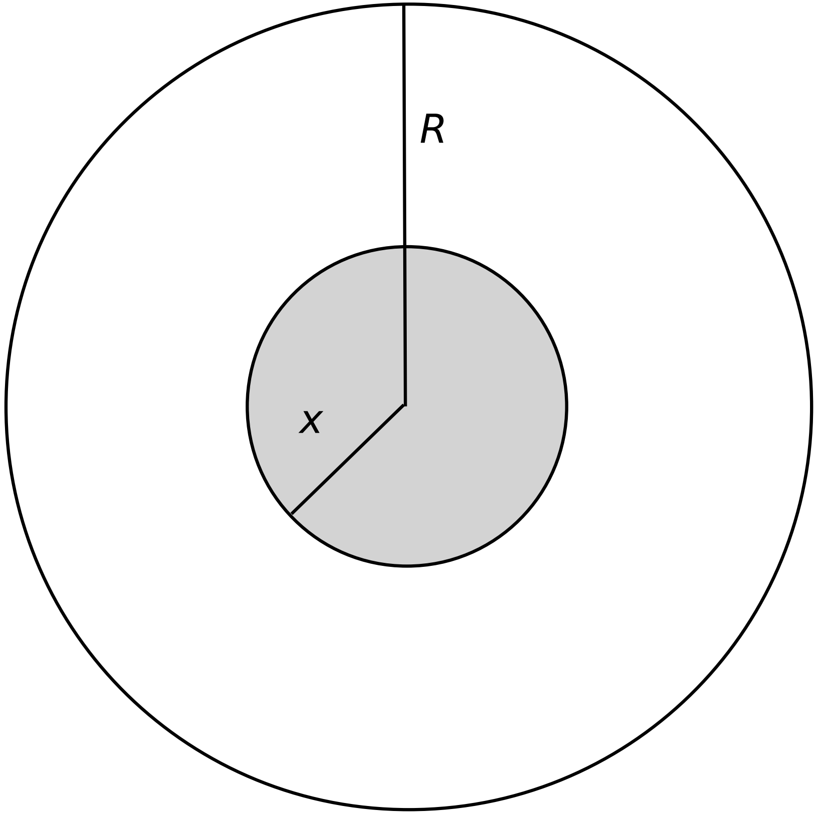

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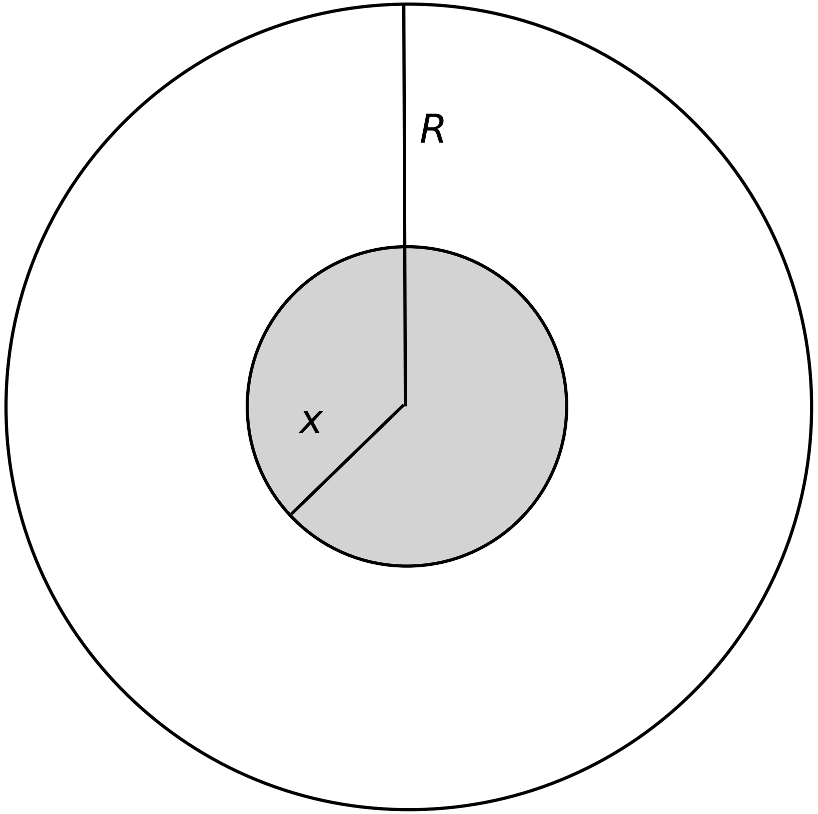

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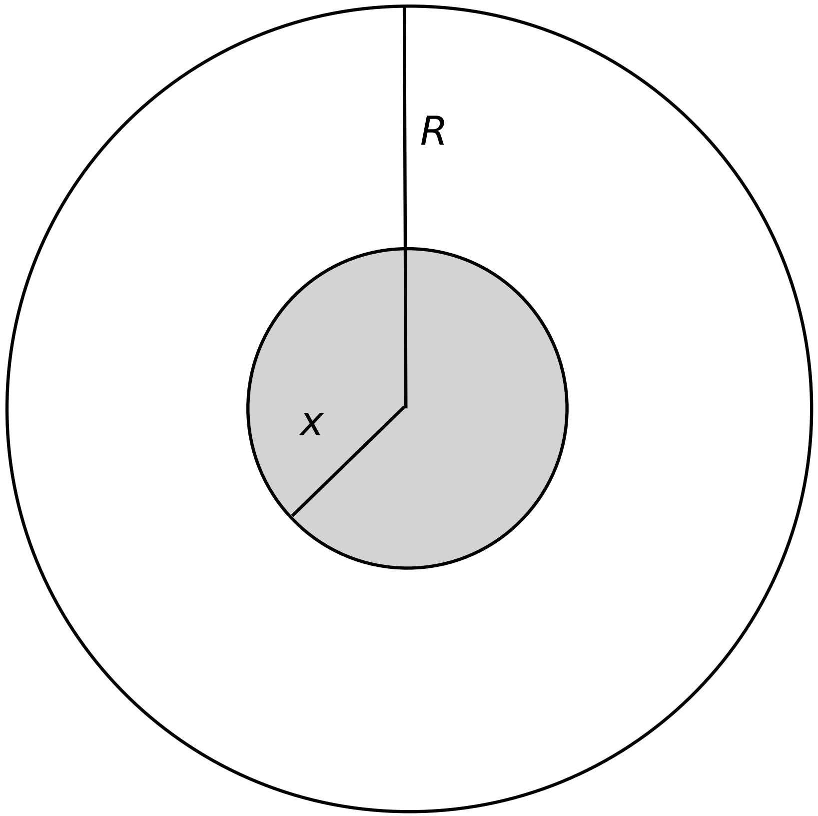

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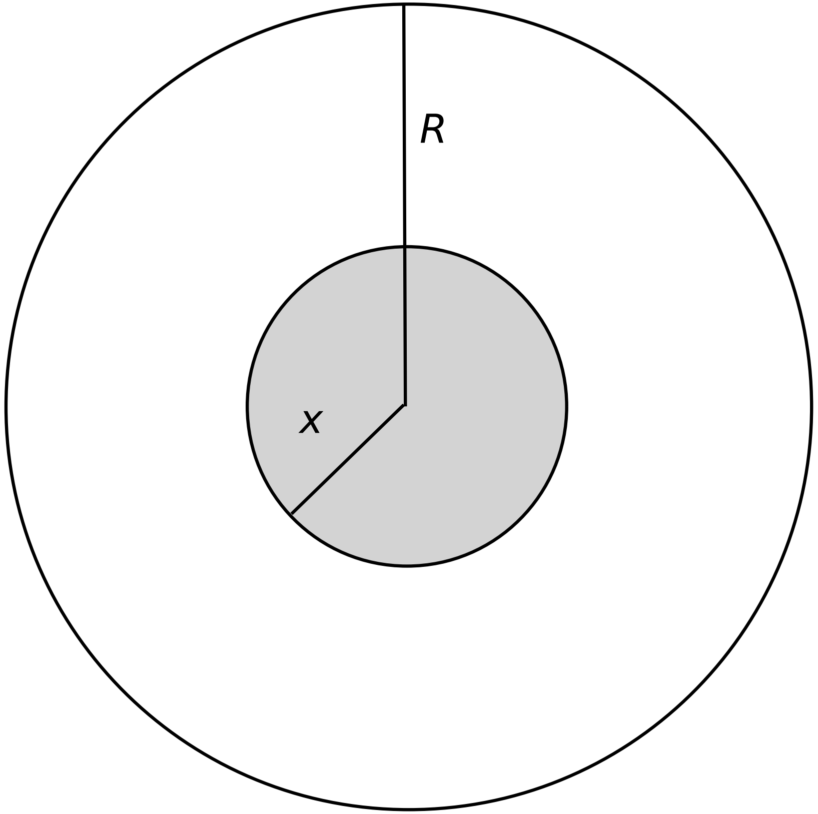

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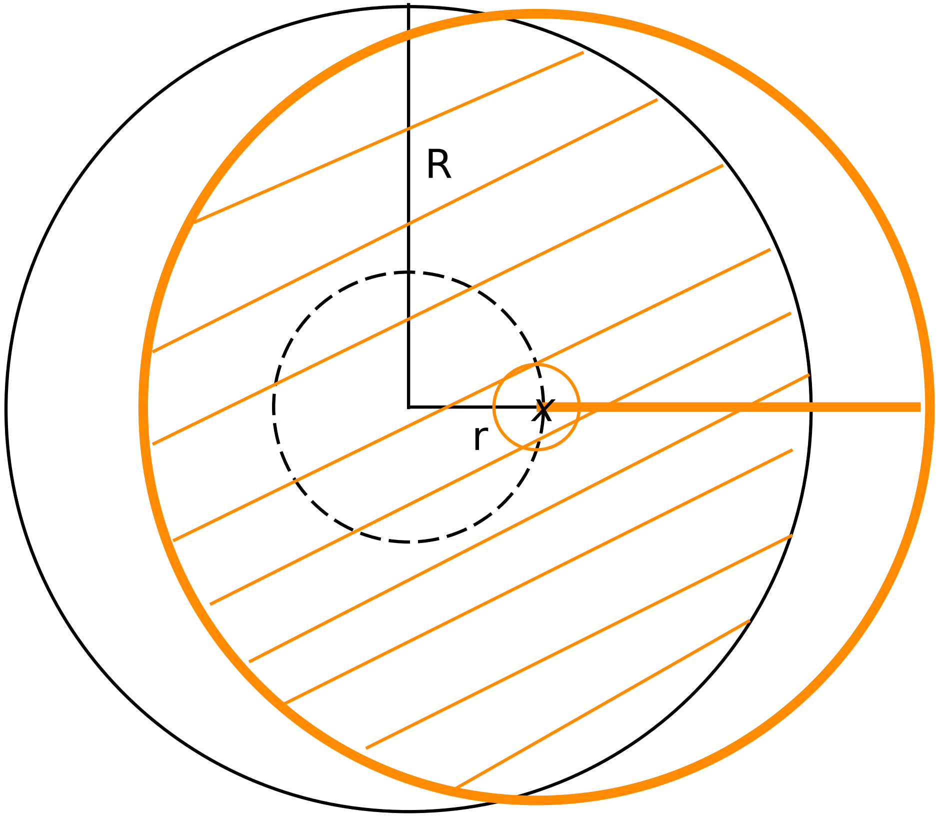

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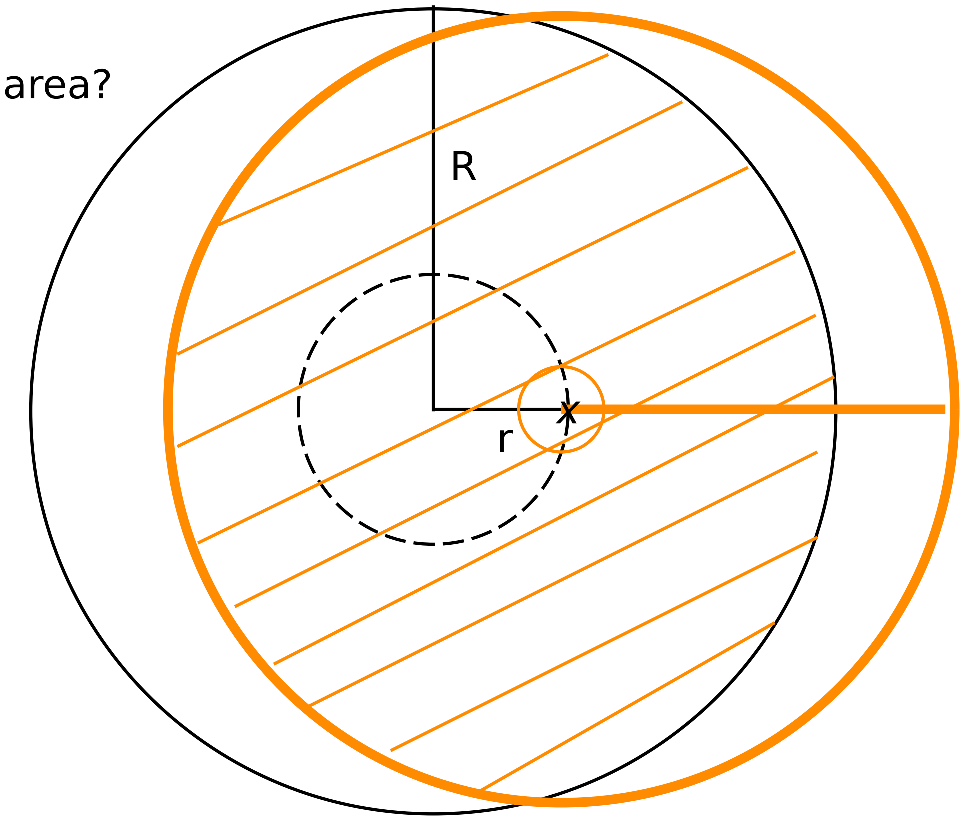

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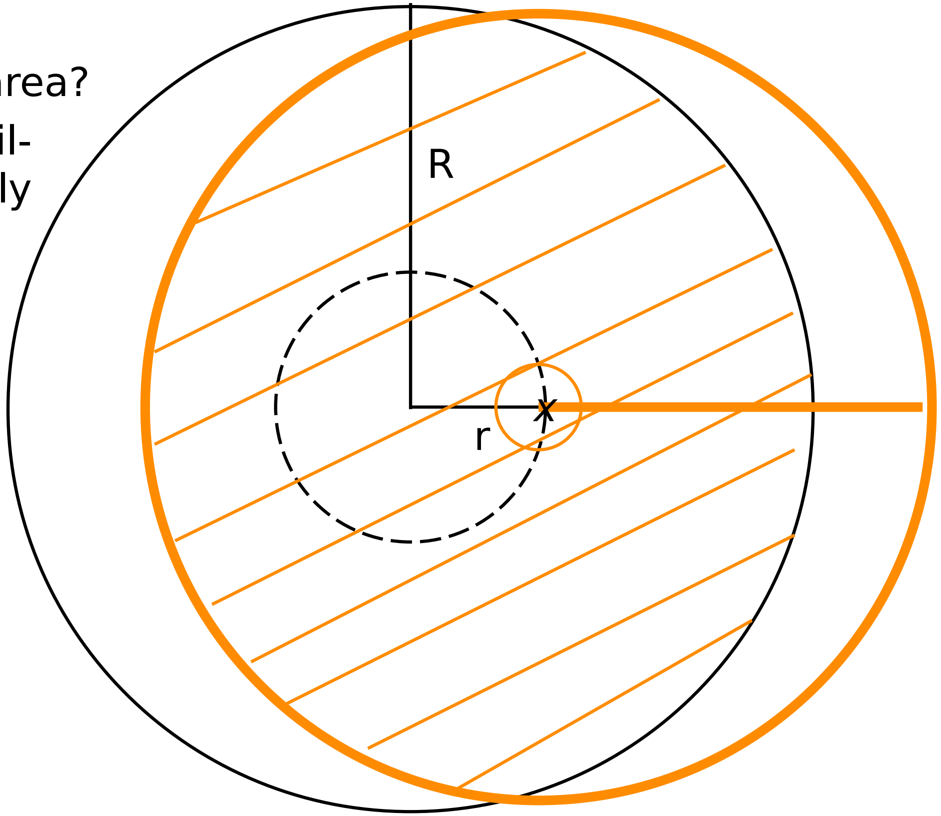
Notice: The orange dashed area is the probability measure that a point sampled by randomly drawn point is connected with x.



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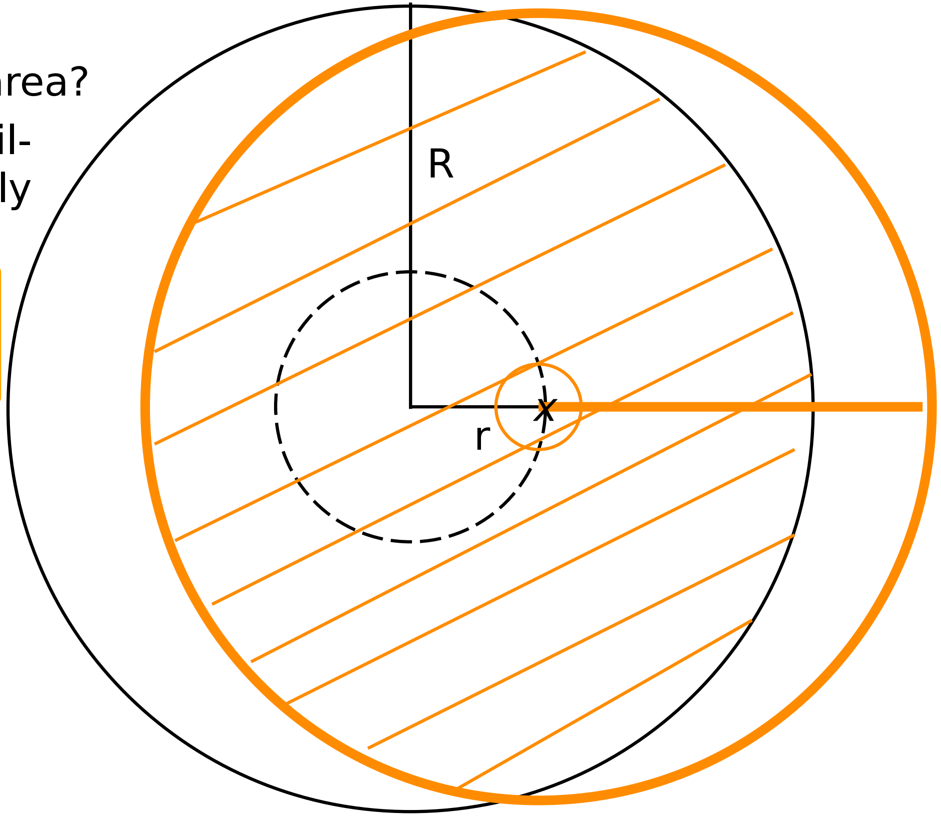

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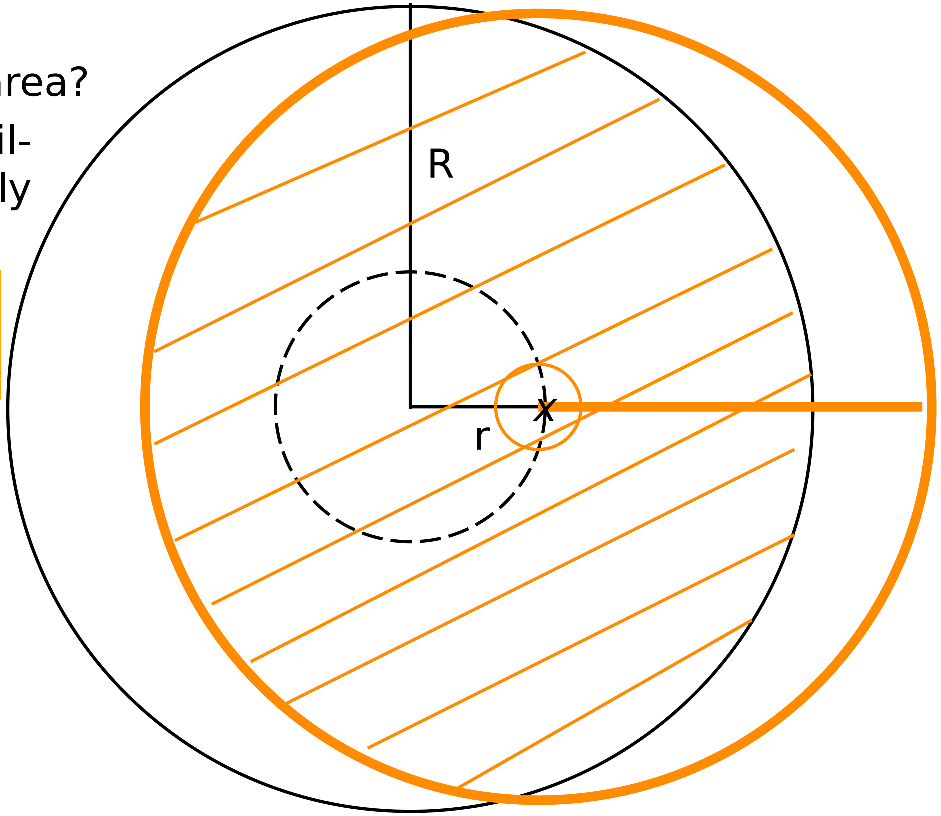

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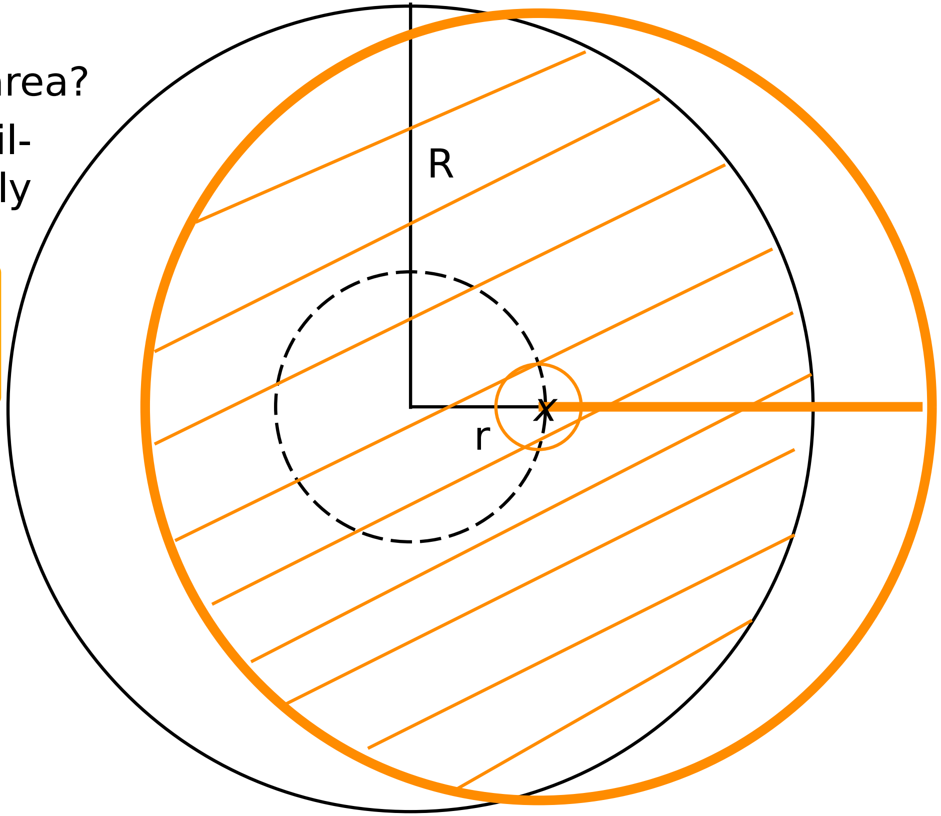

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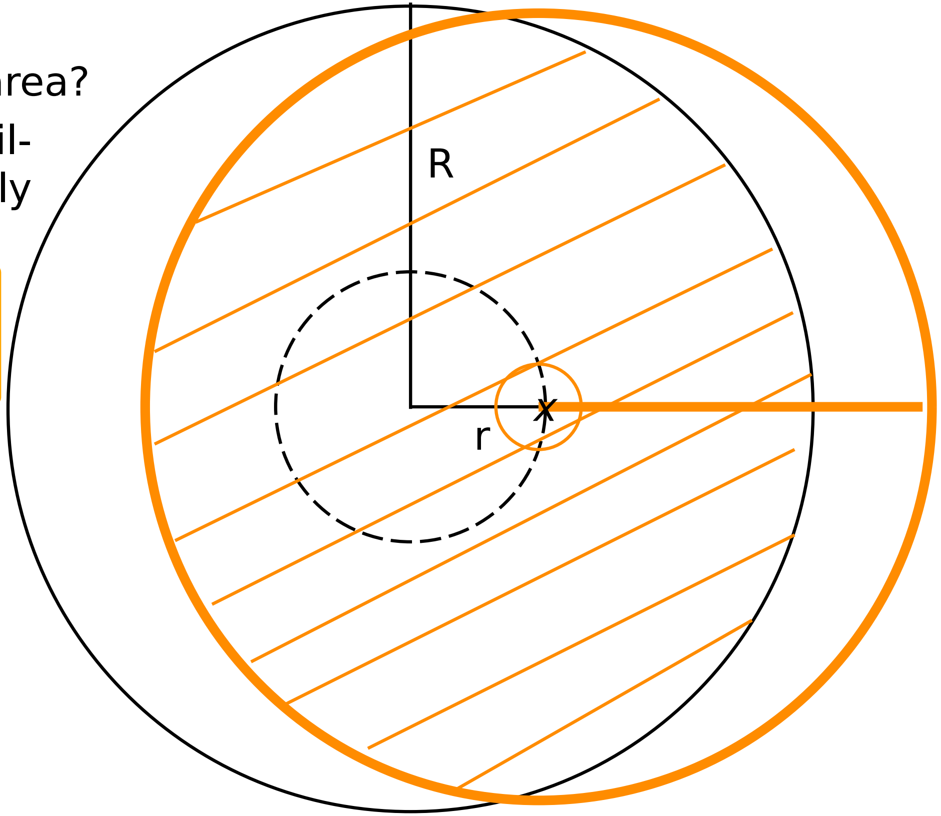
2. Just do it with the lemma for $\theta(x, y)$!



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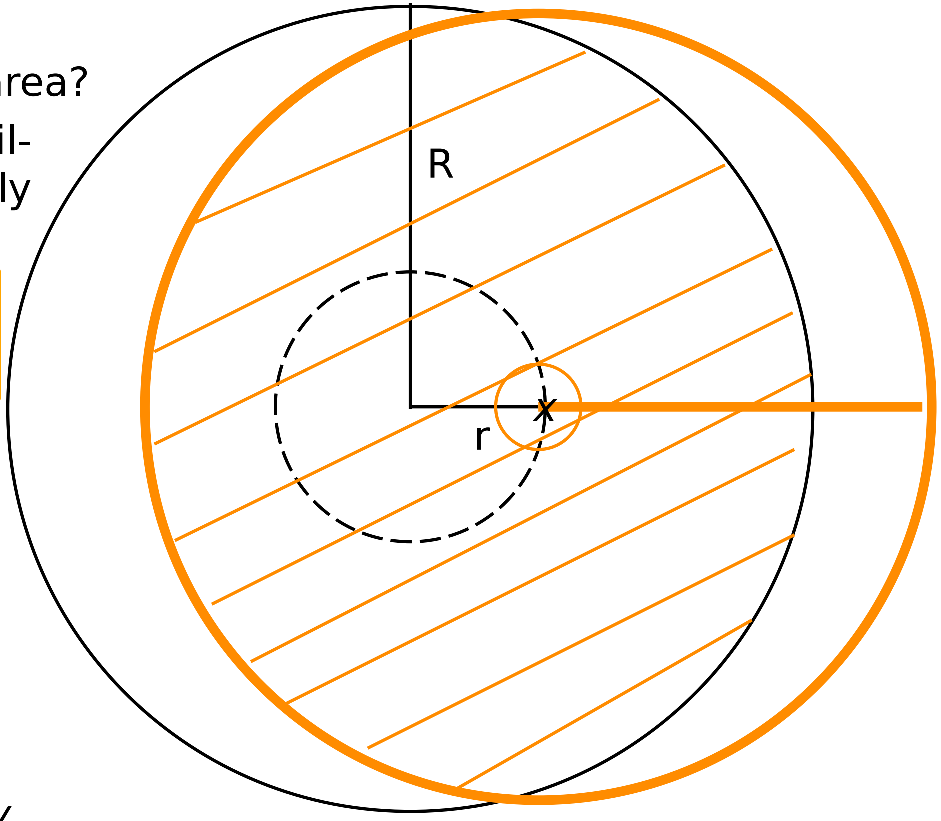
$$\mu(B_r(R) \cap B_o(R)) = \frac{2\alpha e^{-r/2}}{\pi(\alpha - \frac{1}{2})}(1 \pm \mathcal{O}(e^{-(\alpha-\frac{1}{2})r} + e^{-r}))$$



Proof ingredients:

1. Compute the integral:

$$\mu(B_r(R) \cap B_0(R)) = \int_0^R \int_{-\theta(r,y)}^{\theta(r,y)} f(y) d\theta dy$$
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2. Just do it with the lemma for $\theta(x, y)$!

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Let x be a point with radius r.

Question: What is the measure of the orange area?

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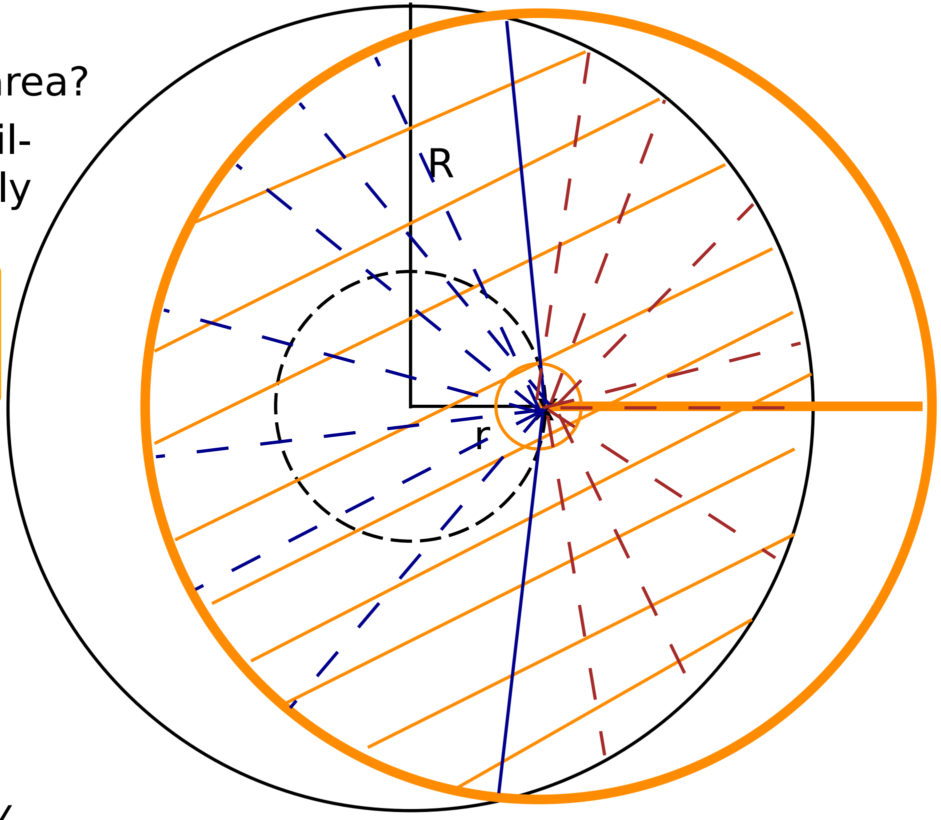

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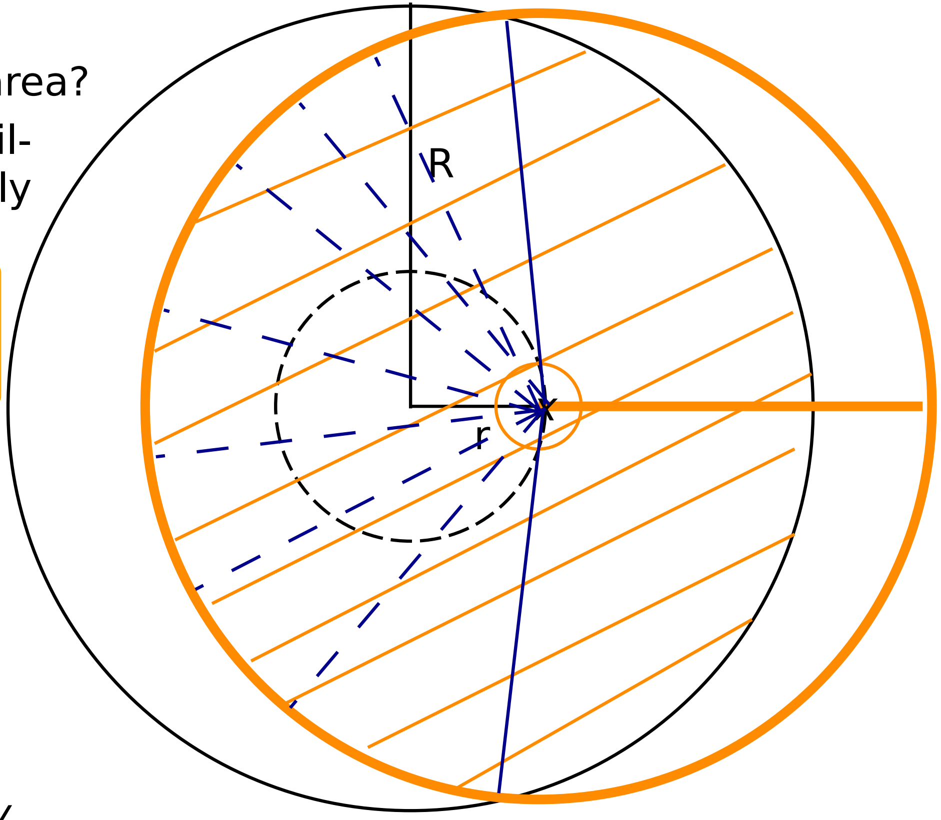

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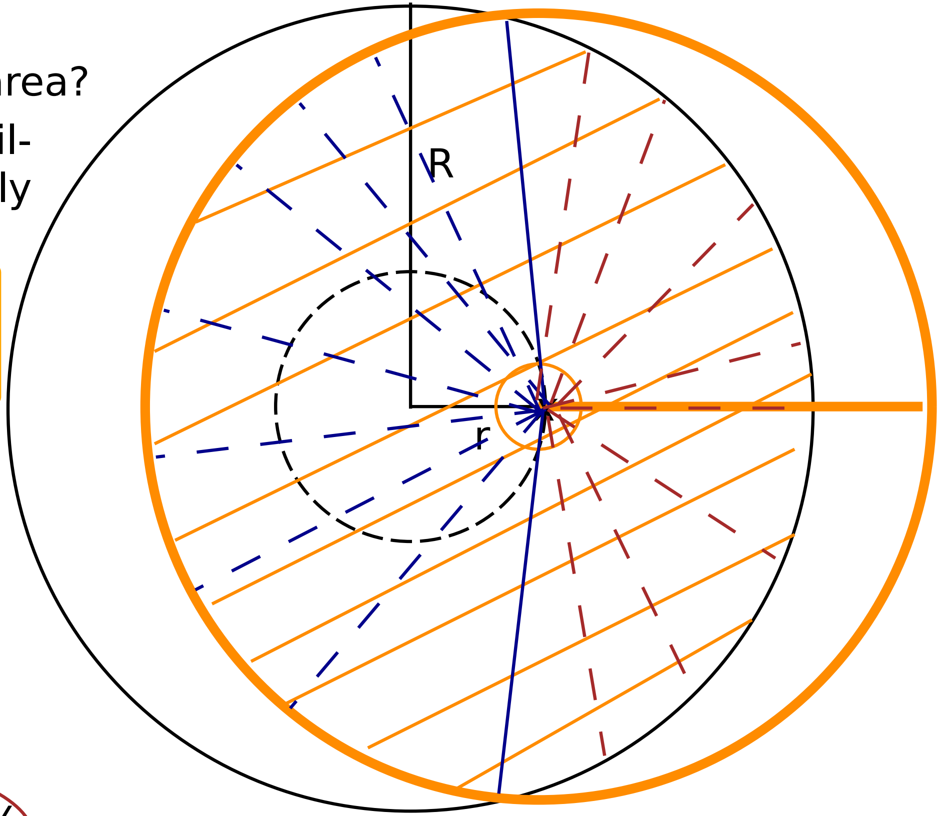

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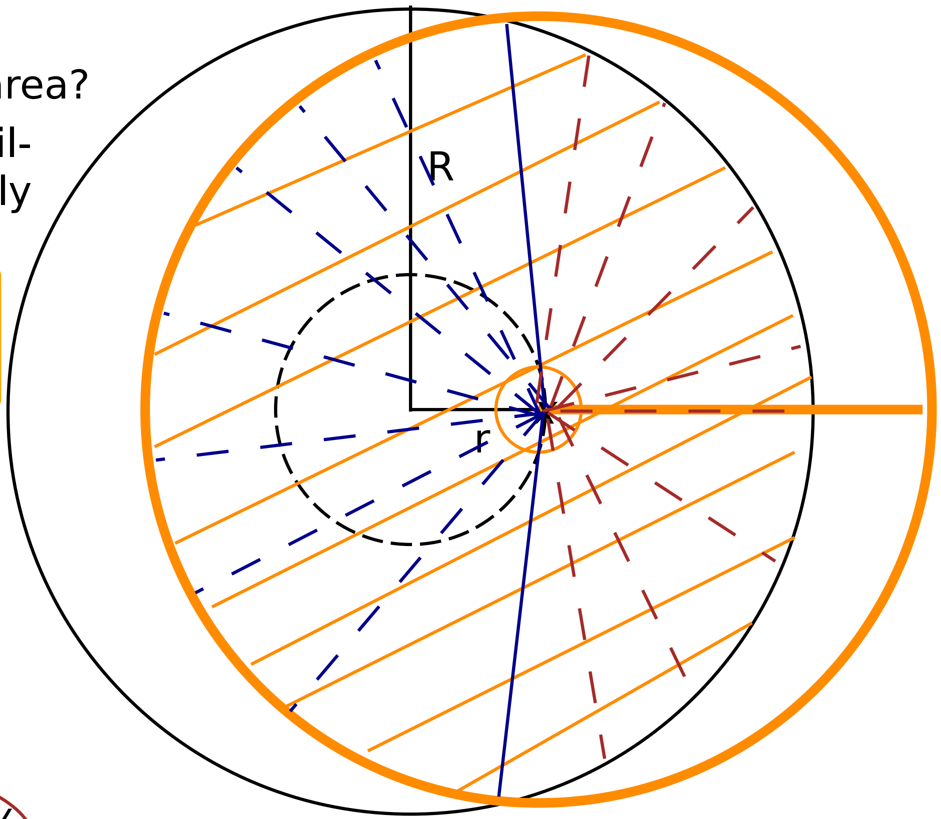

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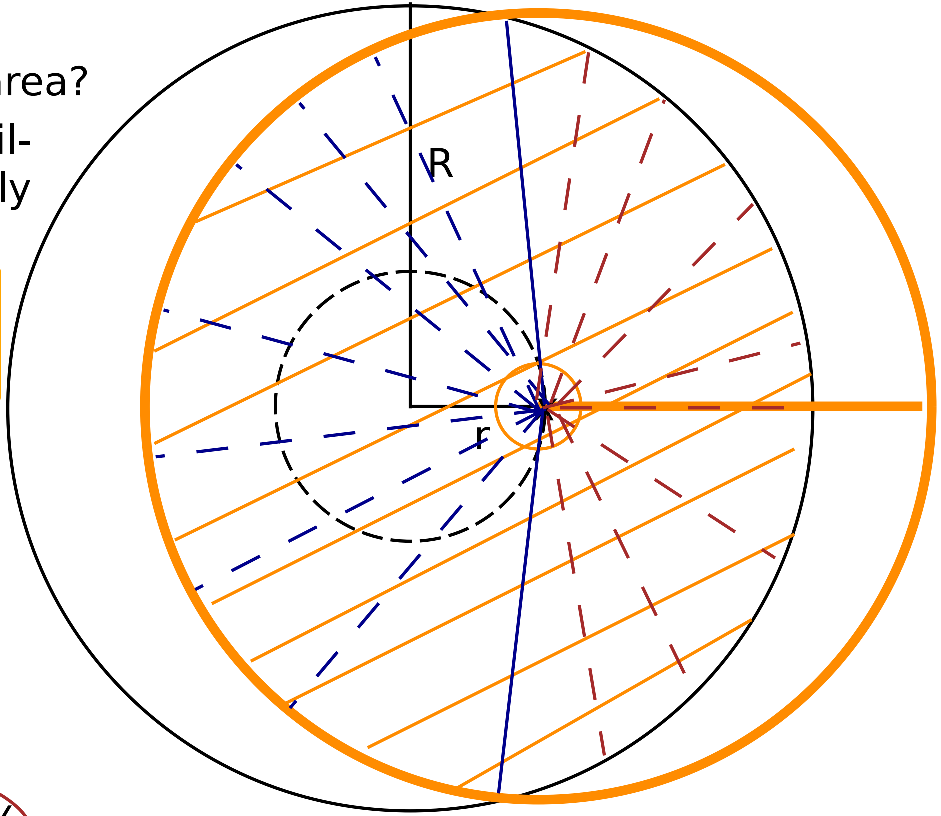

For the details check the paper! For refinements of the lemma check out the cheat sheet!