Hyperbolic Random Graphs

Degree Sequence and Clustering

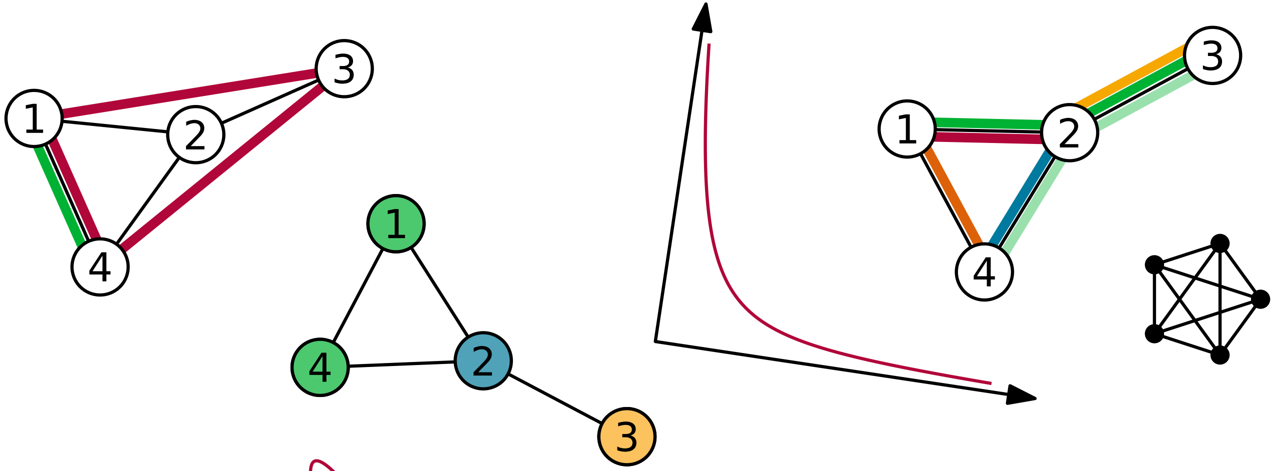
Algorithm Engineering Group

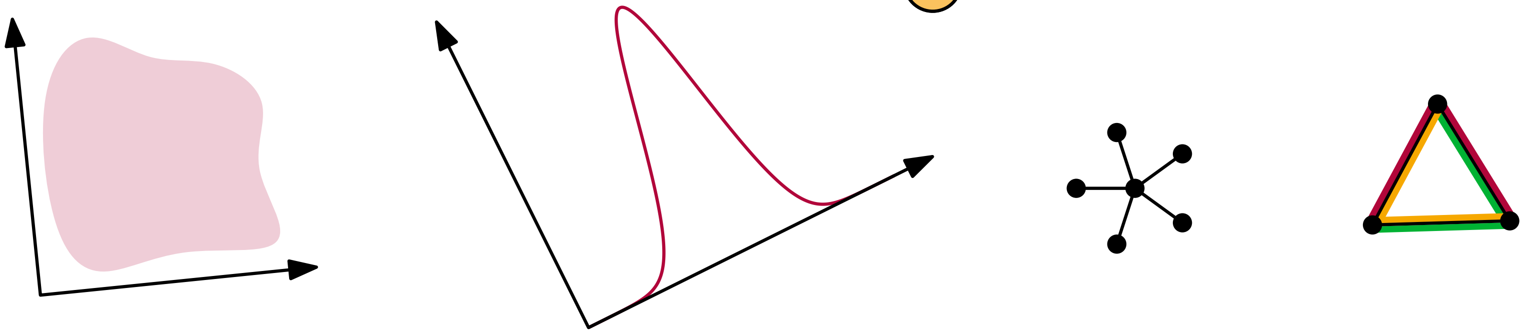
Janosch Ruff

Network Features - Distributions

The most commonly considered network features are distributions over parts of the network.

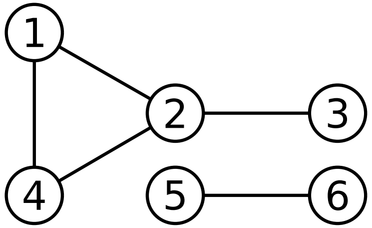
- Degrees
- Assortativity
- Distances
- Centrality
- Clustering





[The Structure and Function of Complex Networks, M. E. J. Newman, Computer Physics Communications 2003]

Network Features – Degree Distribution



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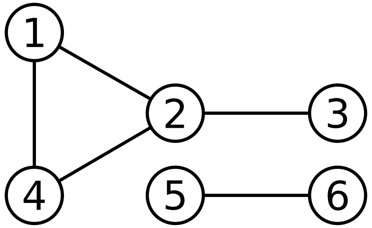

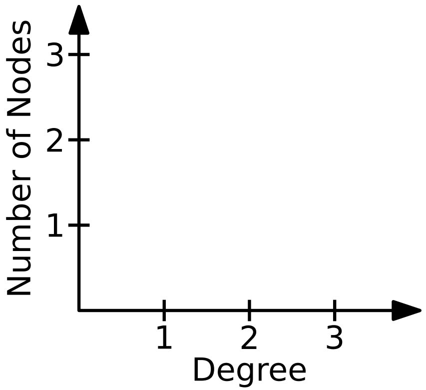

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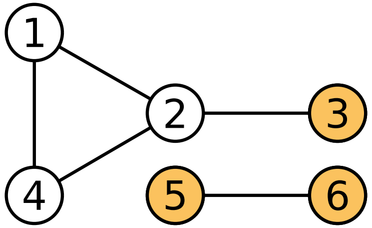

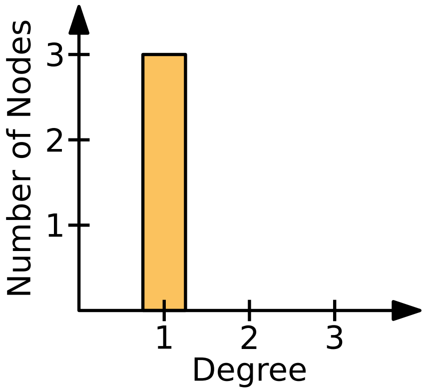

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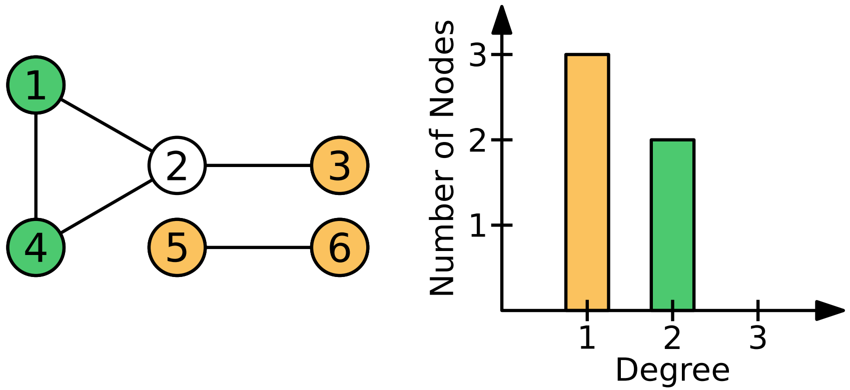

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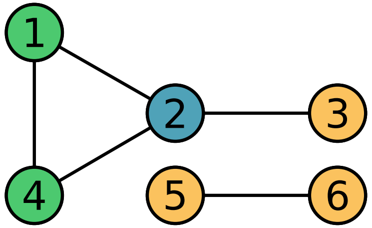

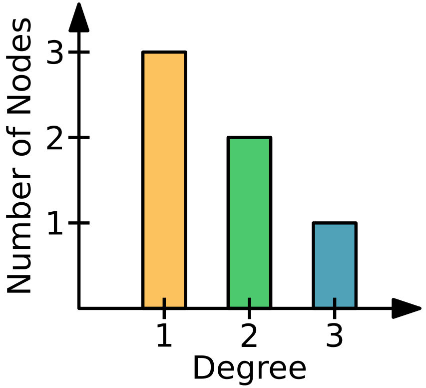

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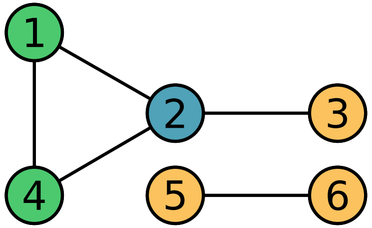

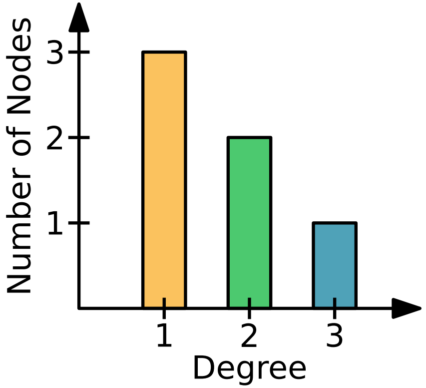

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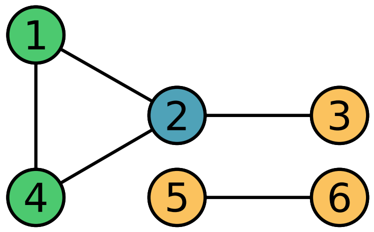

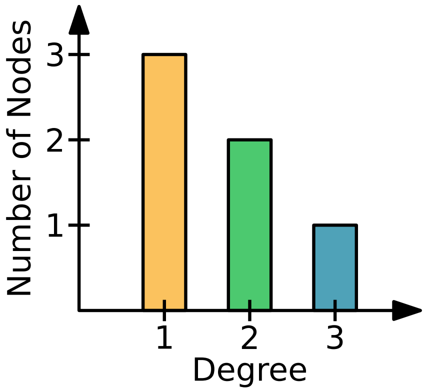

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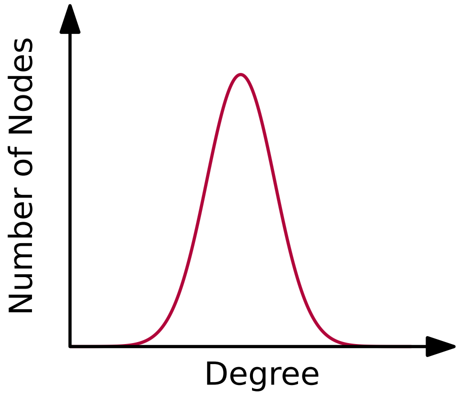

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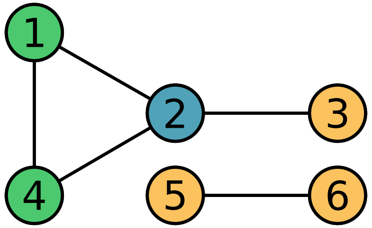

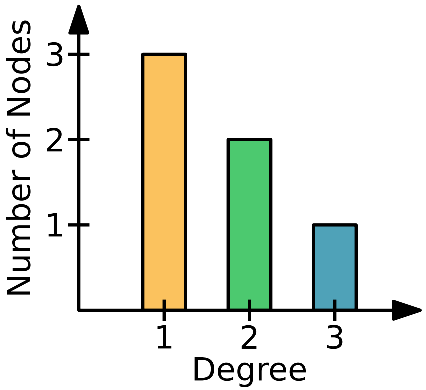

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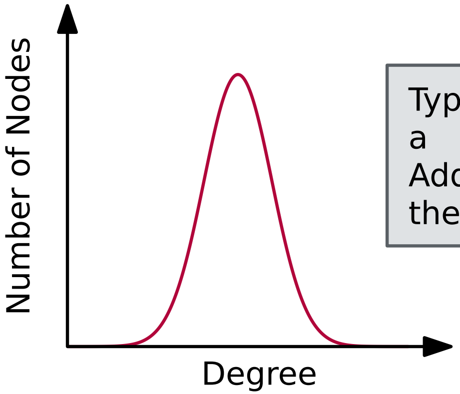

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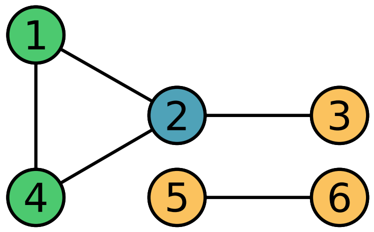

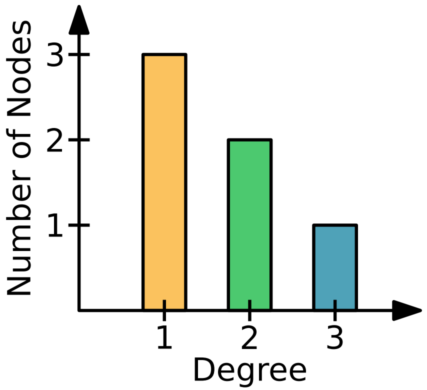

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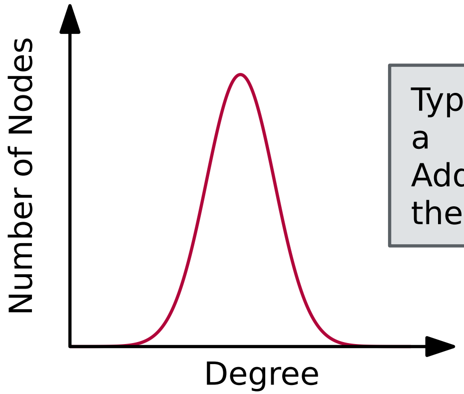

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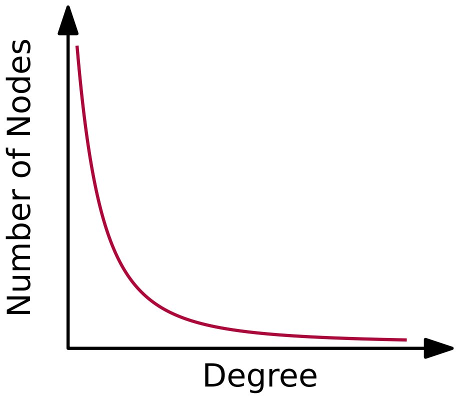

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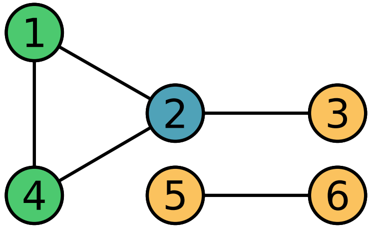

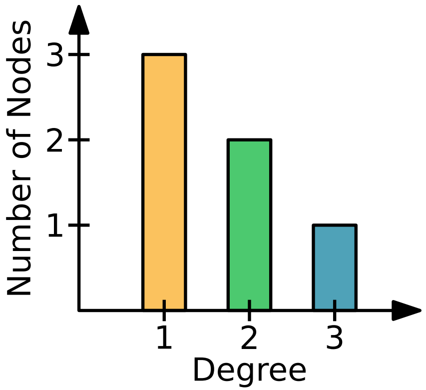

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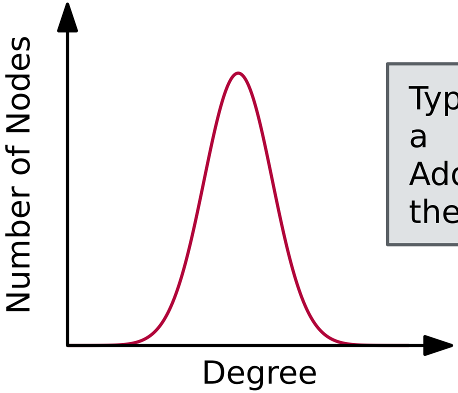

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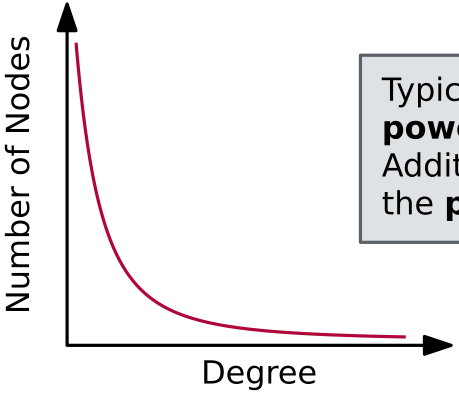

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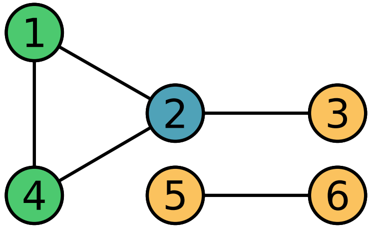

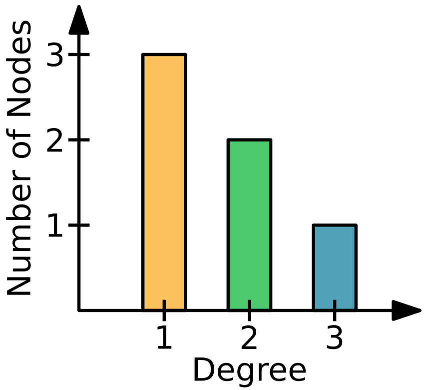

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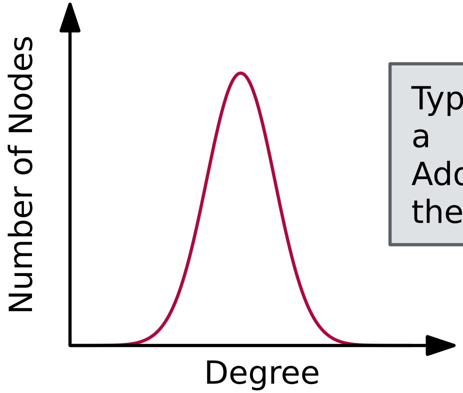

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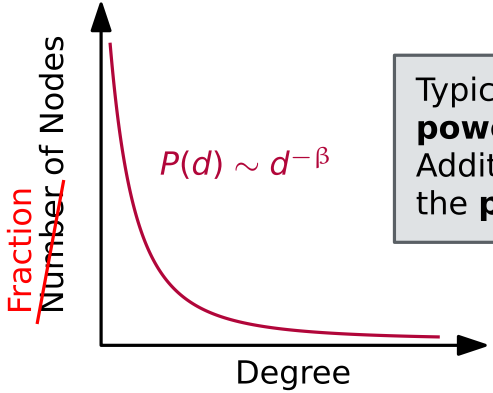

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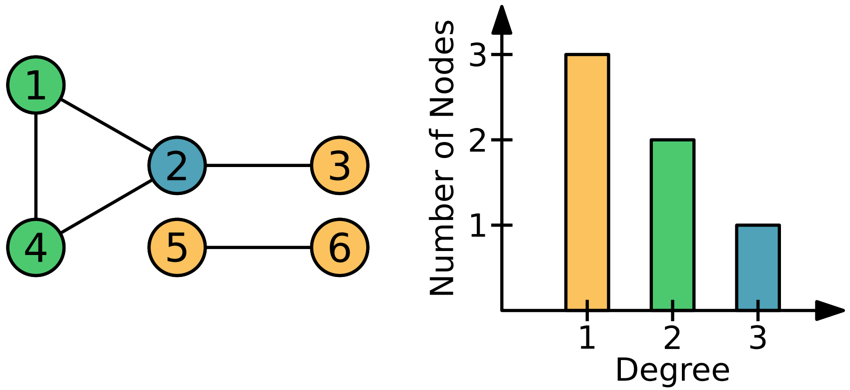

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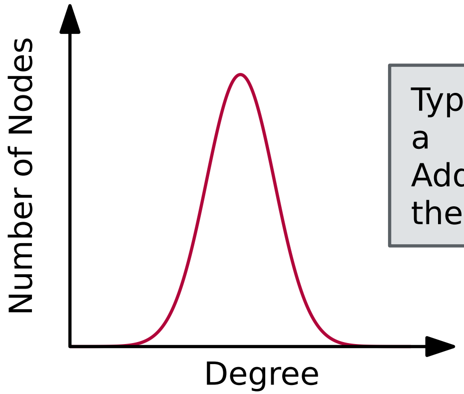

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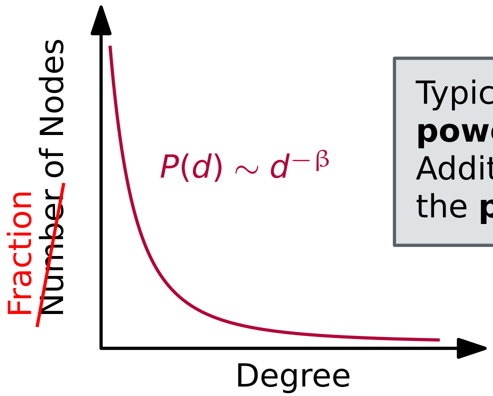

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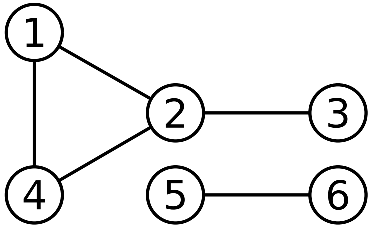

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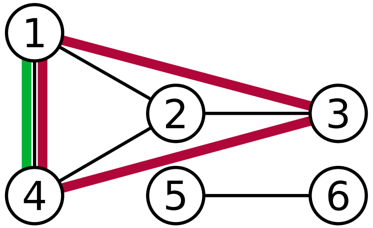

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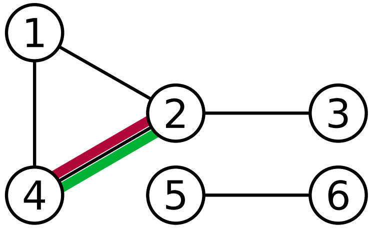

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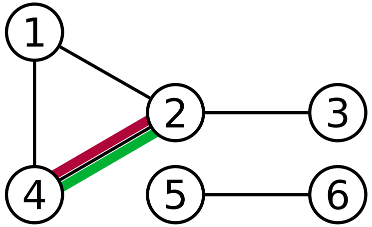

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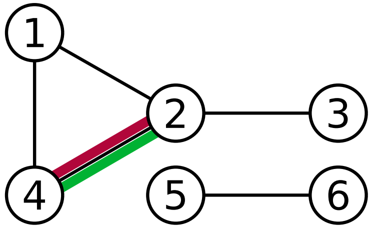

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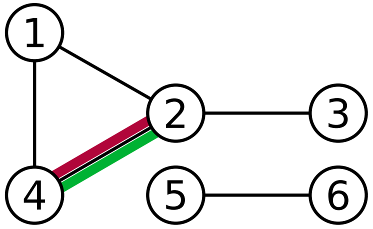

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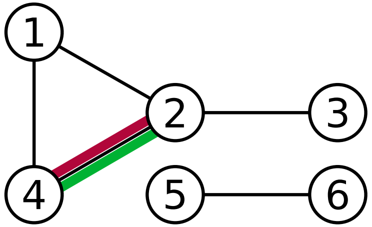

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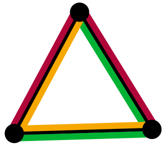

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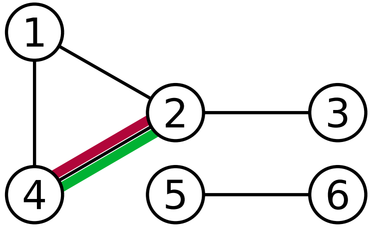

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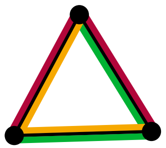

Compared to the average local clustering coefficient, the global one basically computes the ratio of the means rather than the mean of the ratios. It does not weight the contributions of low-degree nodes as much.

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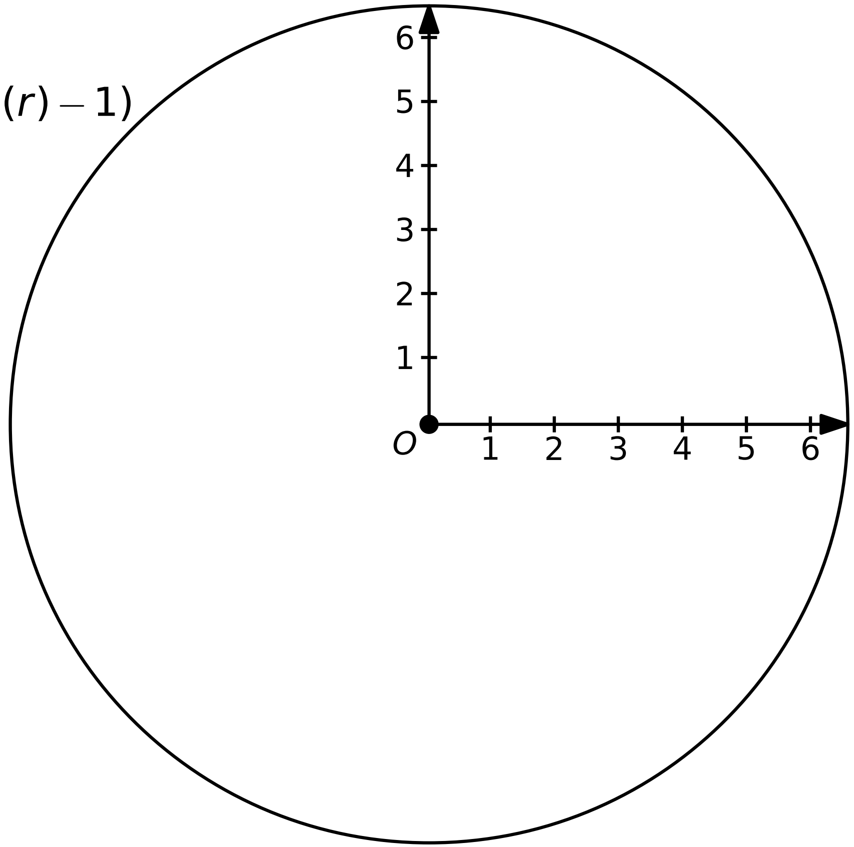

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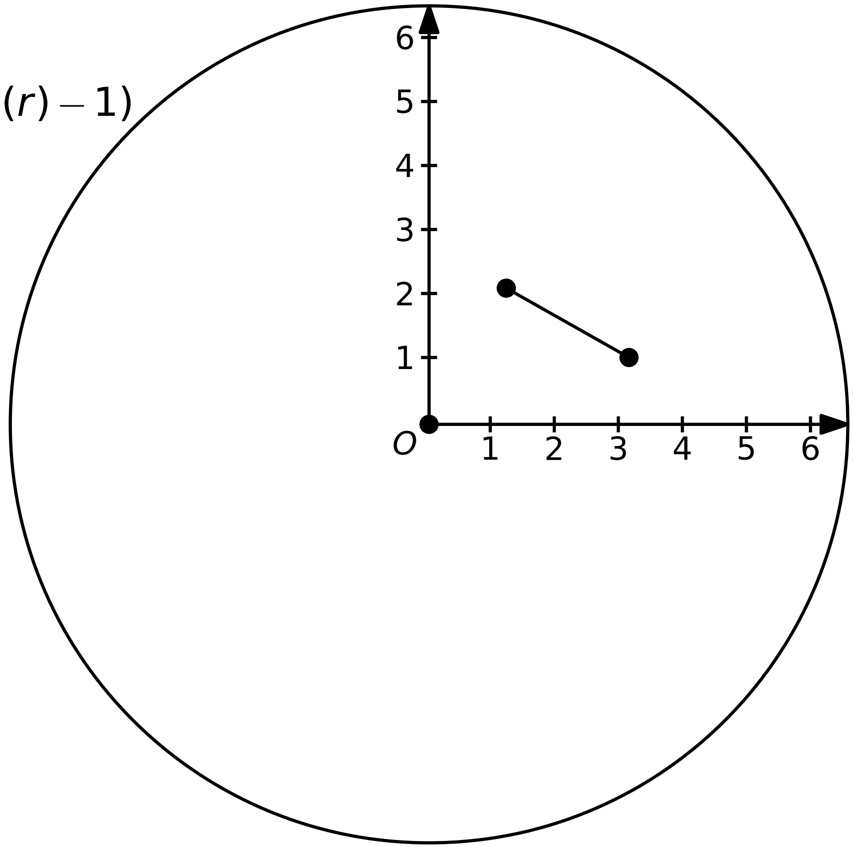

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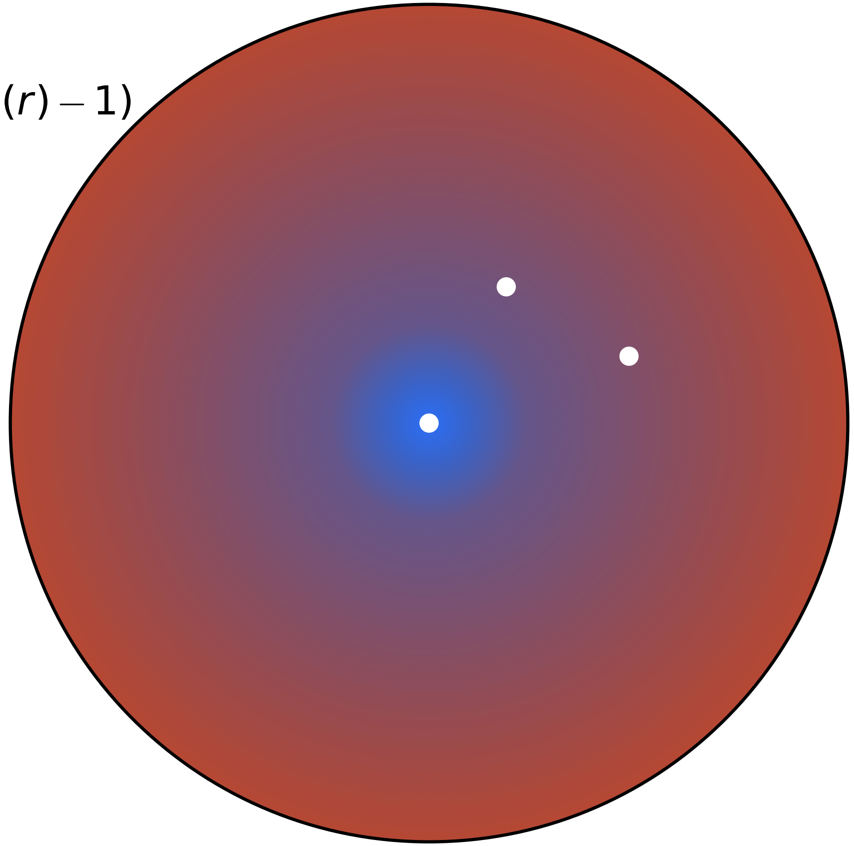

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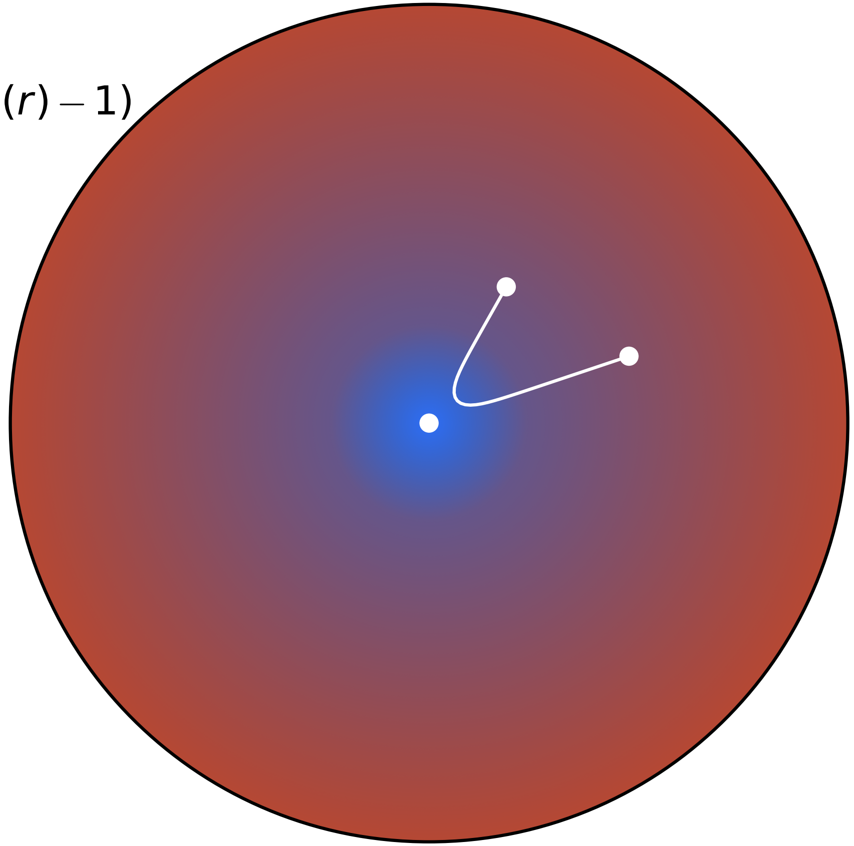

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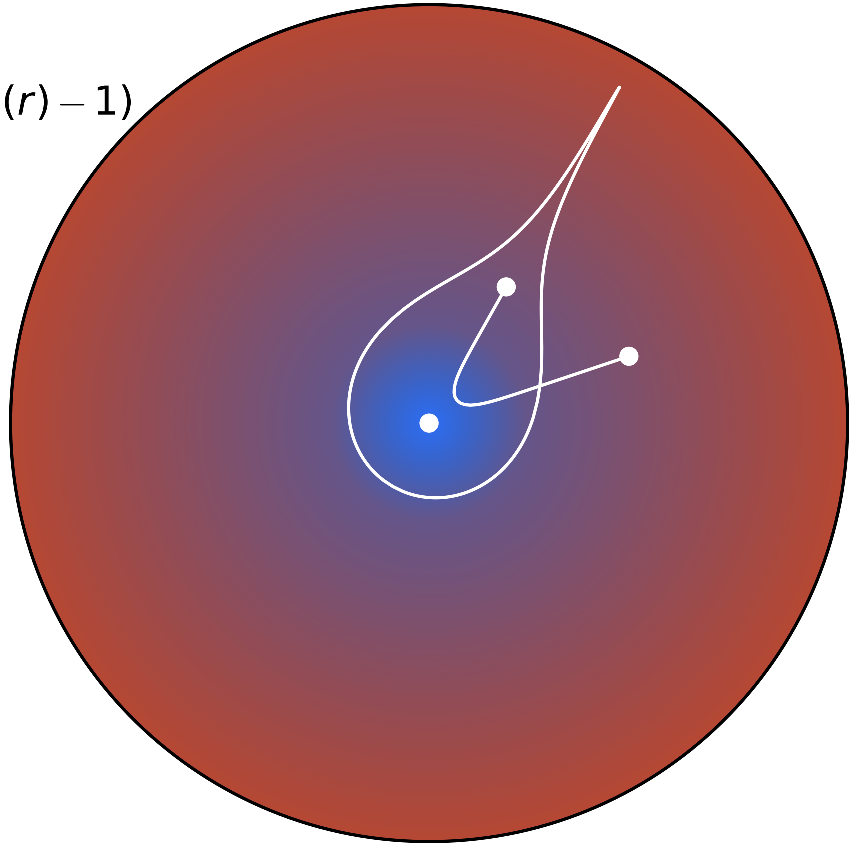

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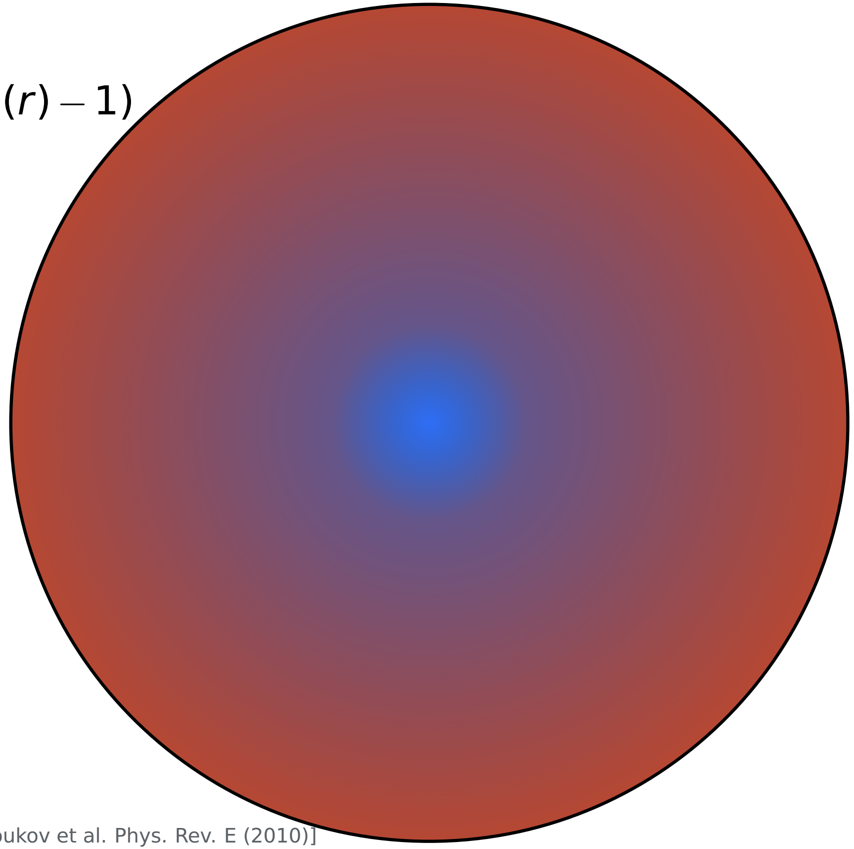
$R \approx 2\log(n)$



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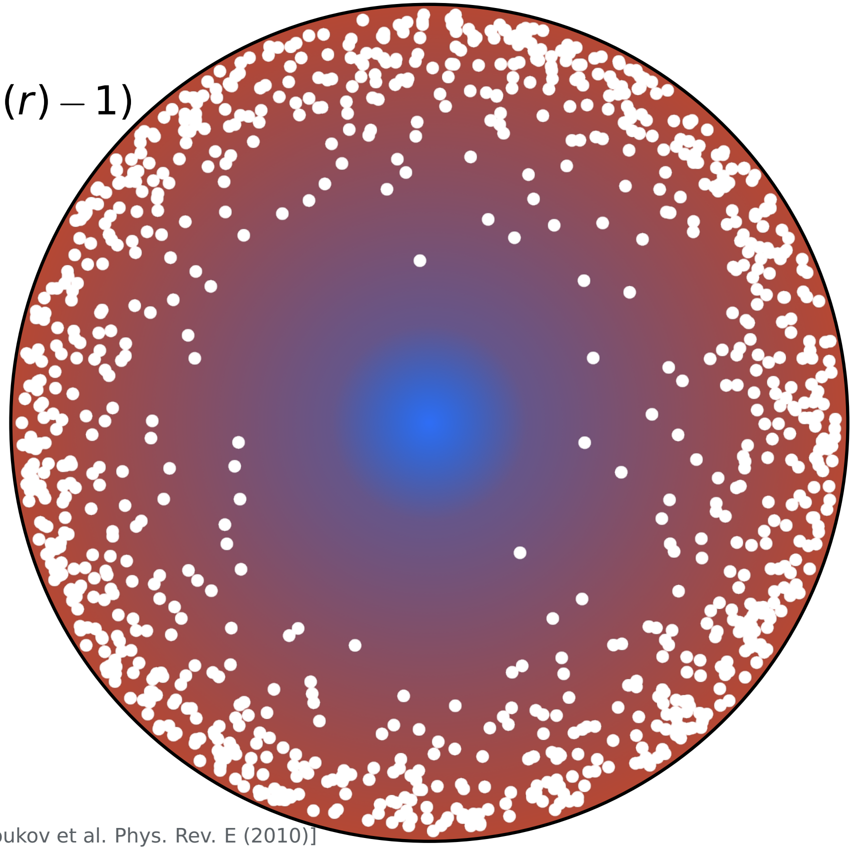

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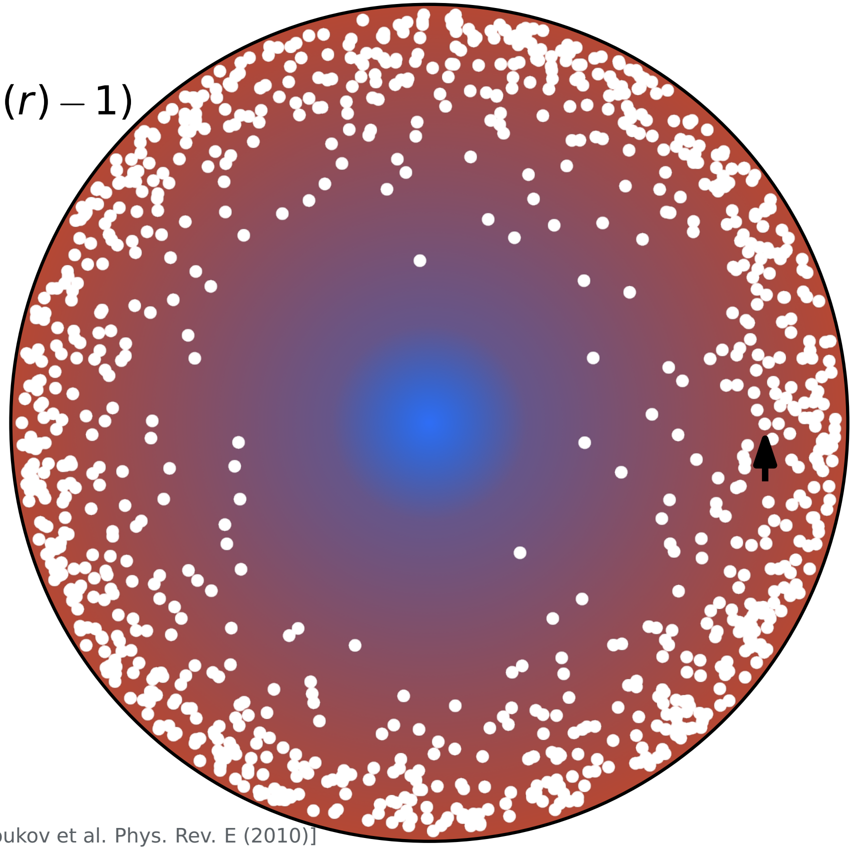

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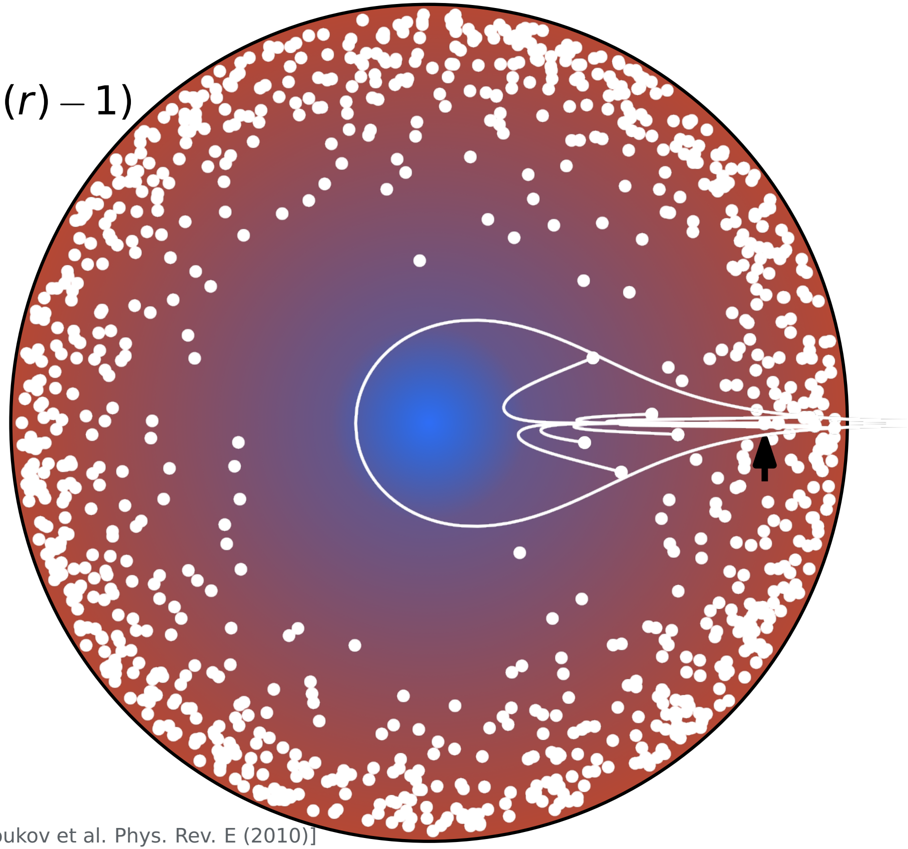

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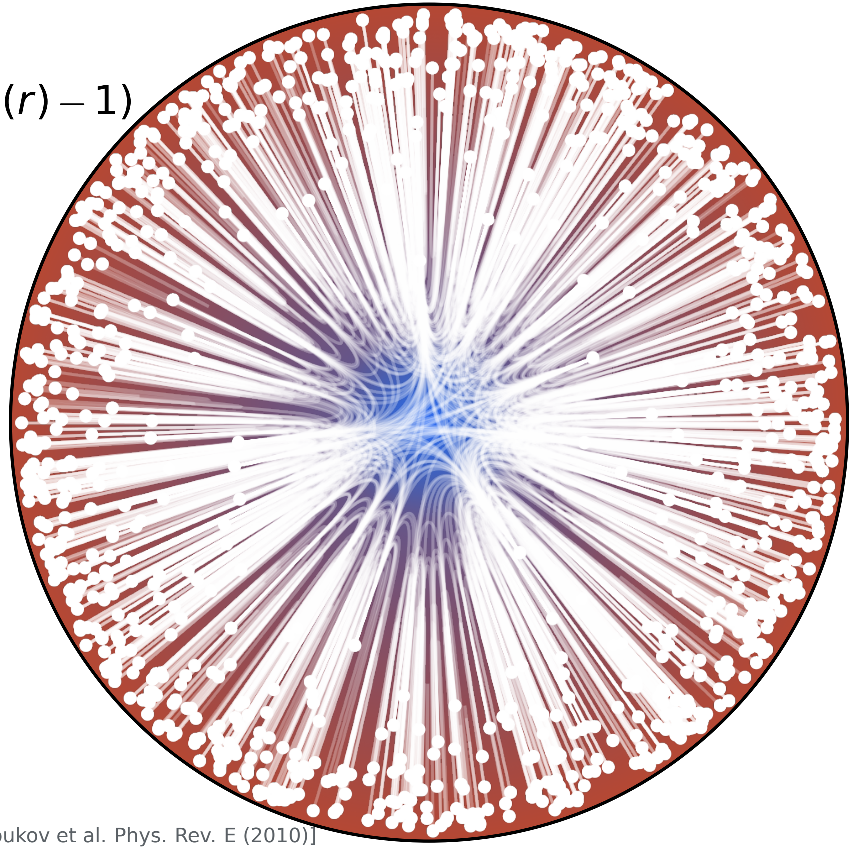

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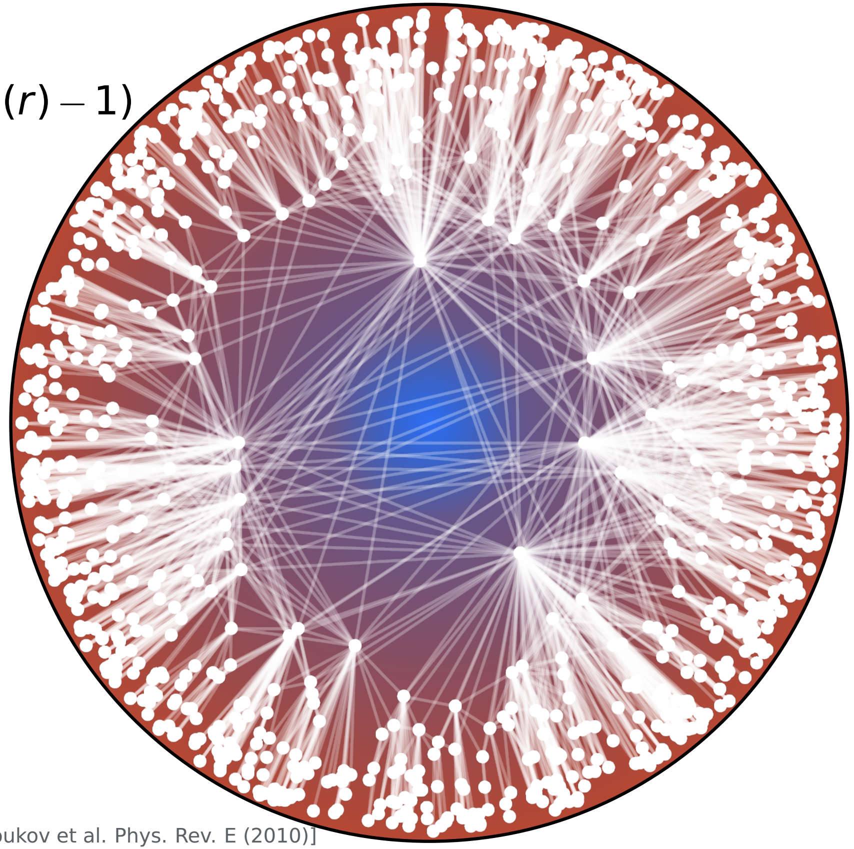

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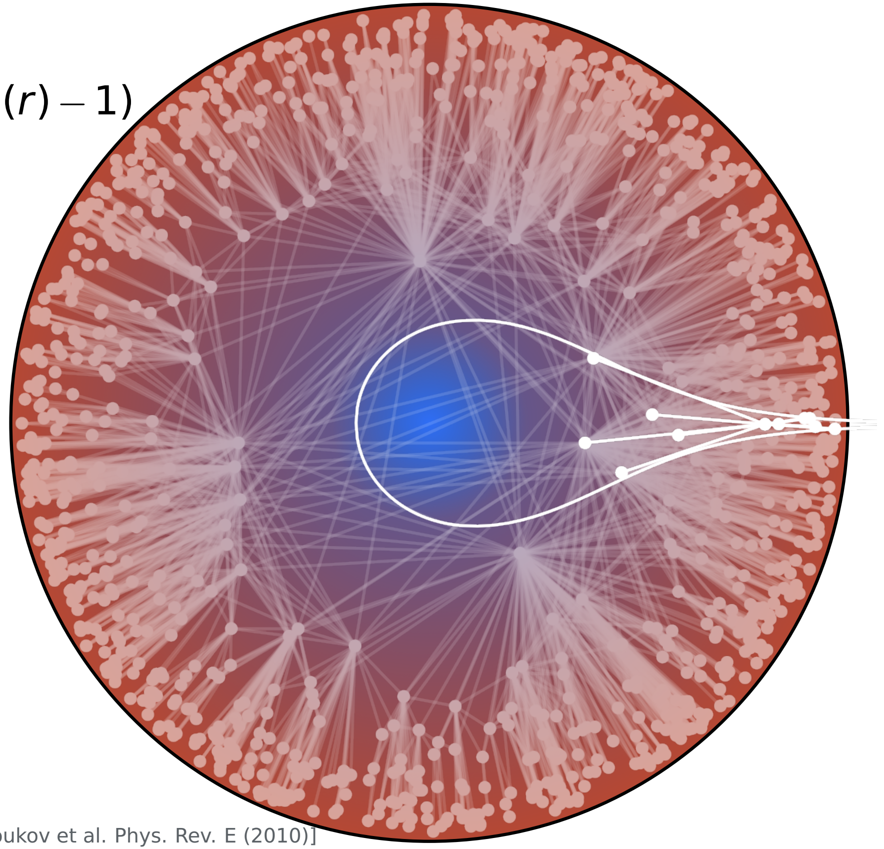

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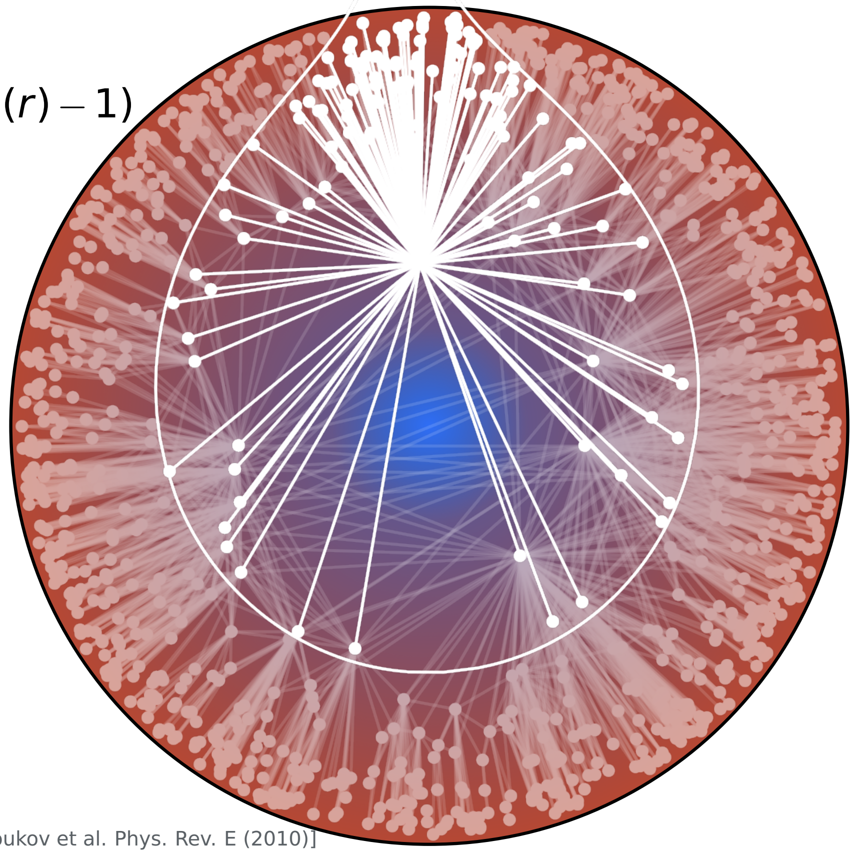

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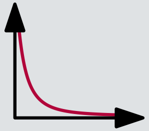

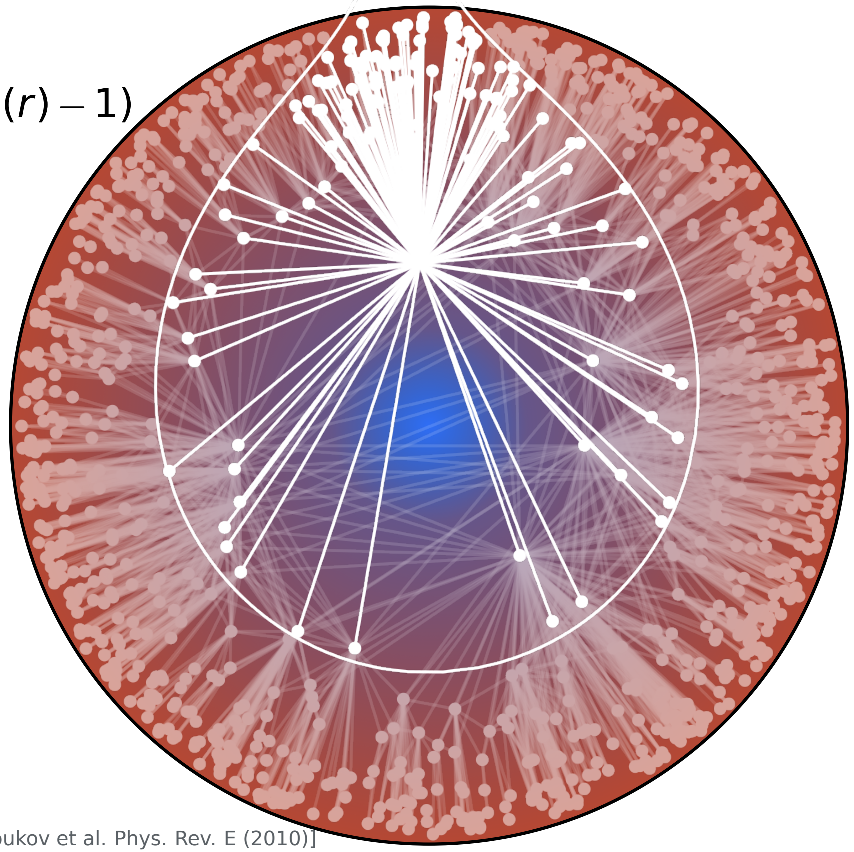

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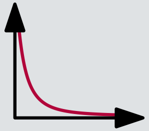

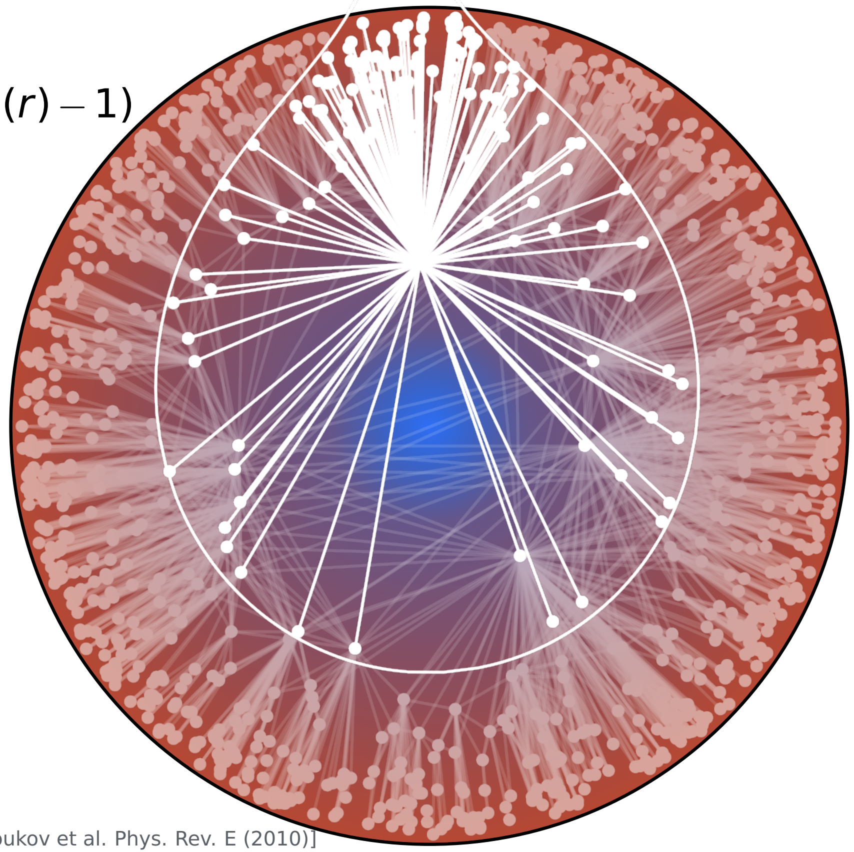

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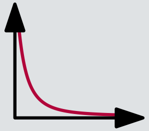

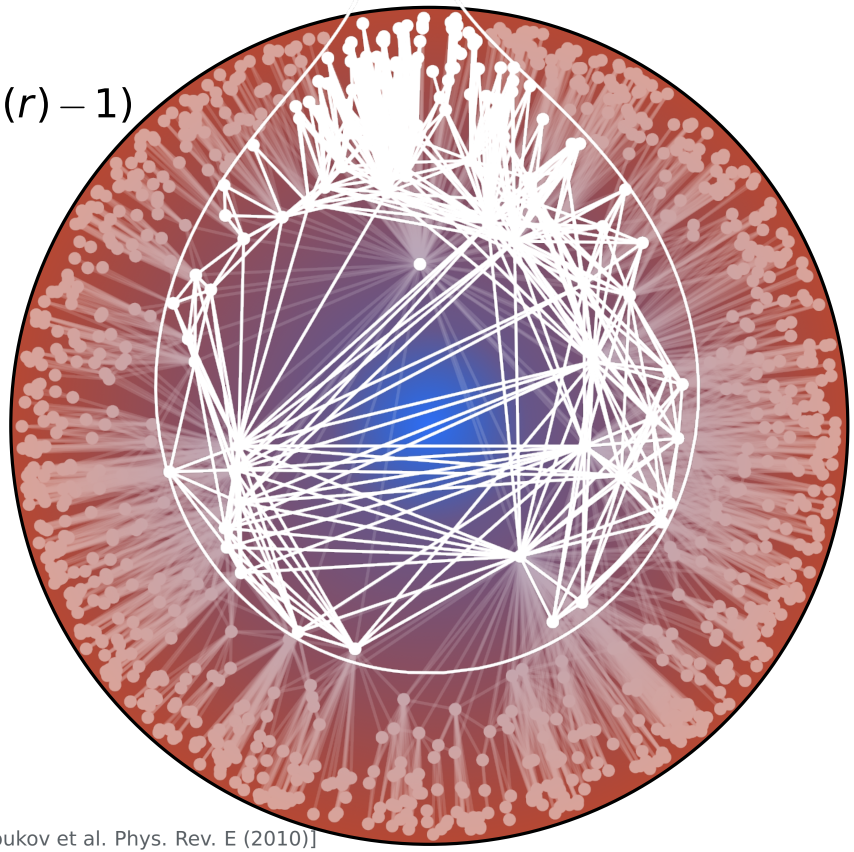

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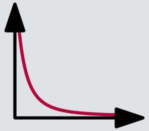
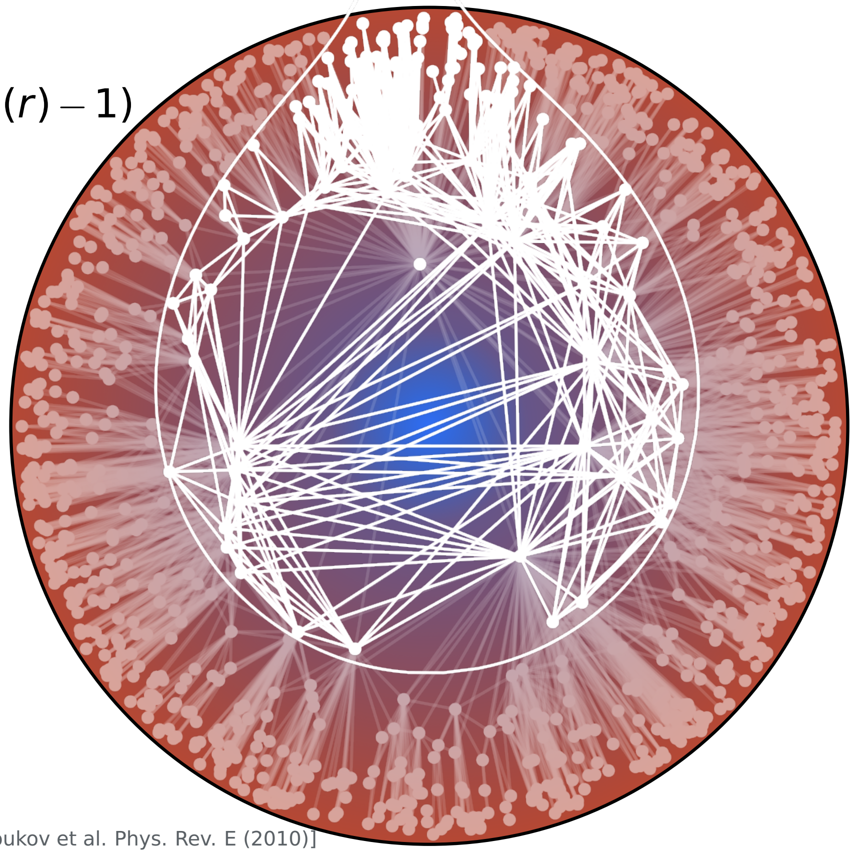
- ... a logarithmic diameter





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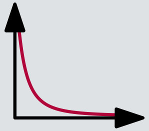
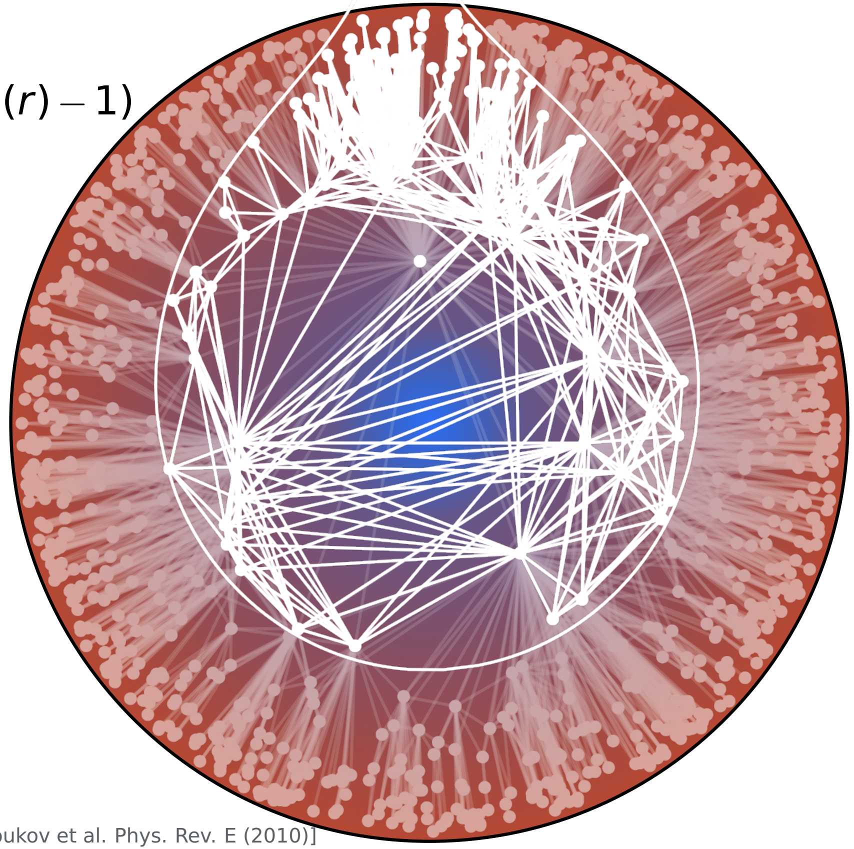
- ... a non-vanishing clustering coefficient
- ... a logarithmic diameter
- ... negative assortativity





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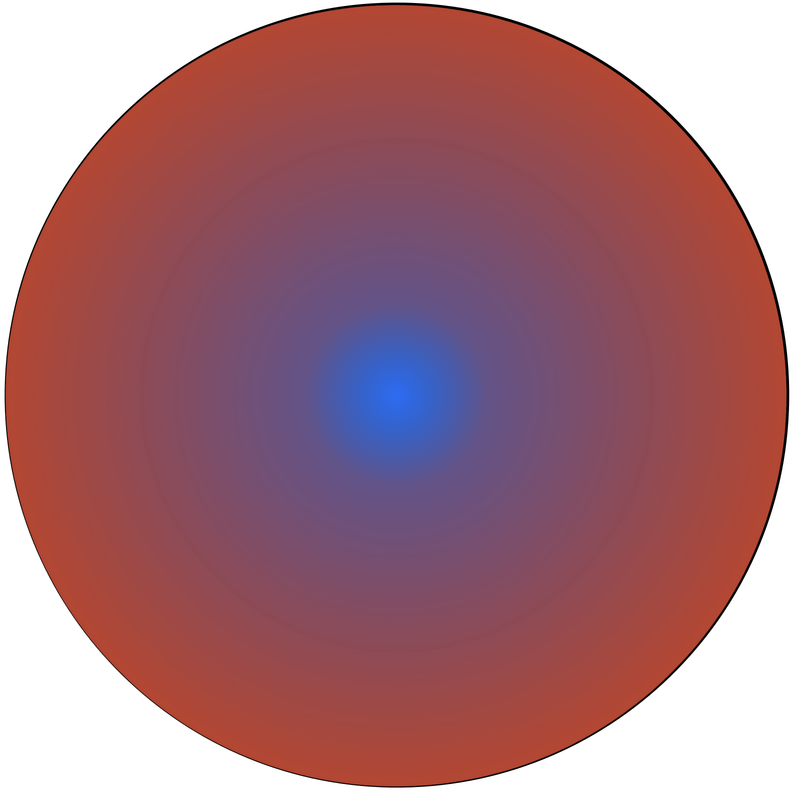

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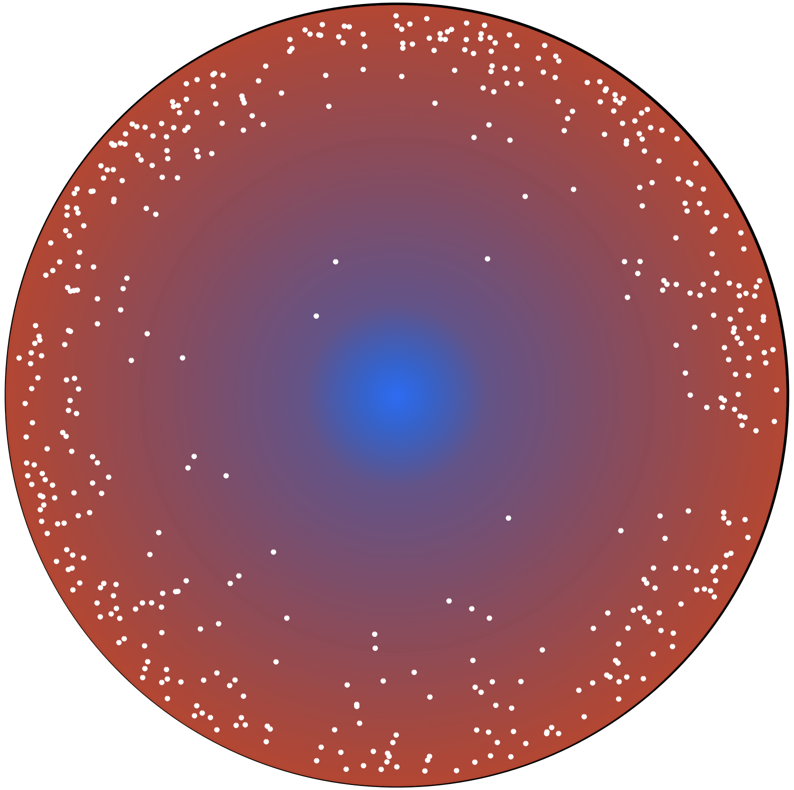

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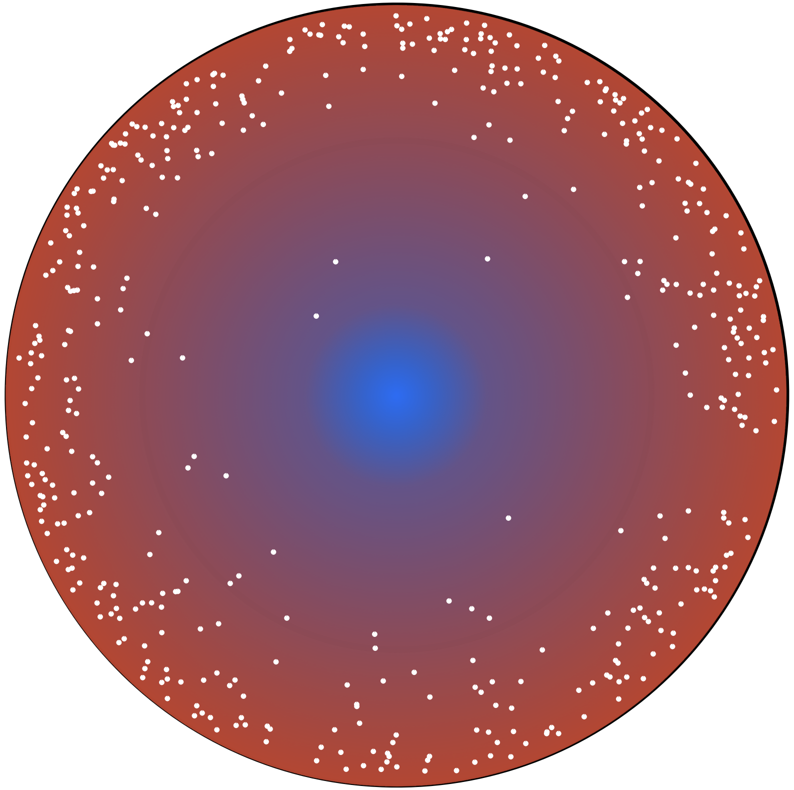

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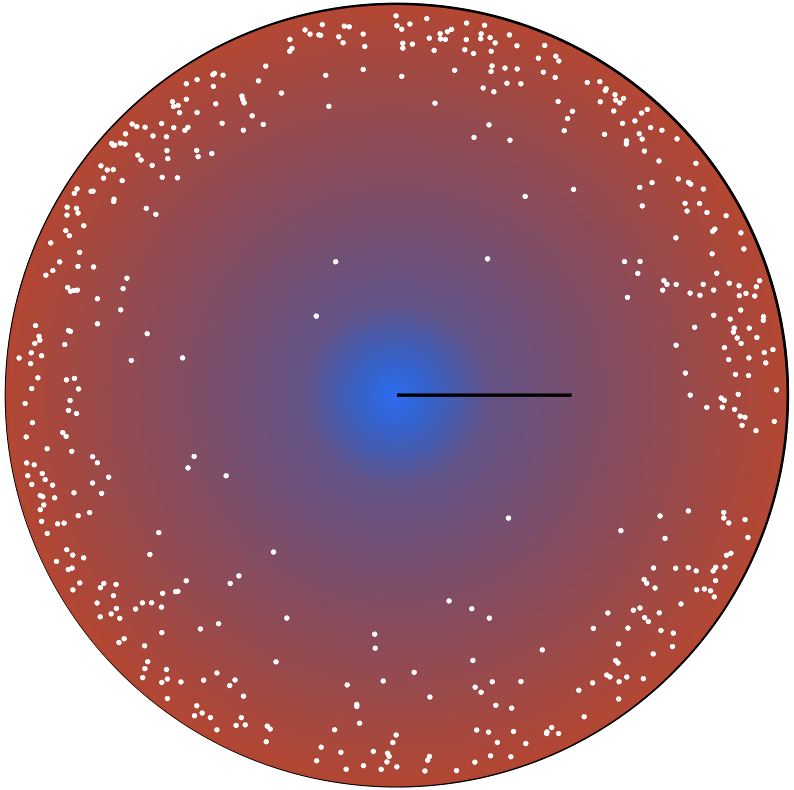

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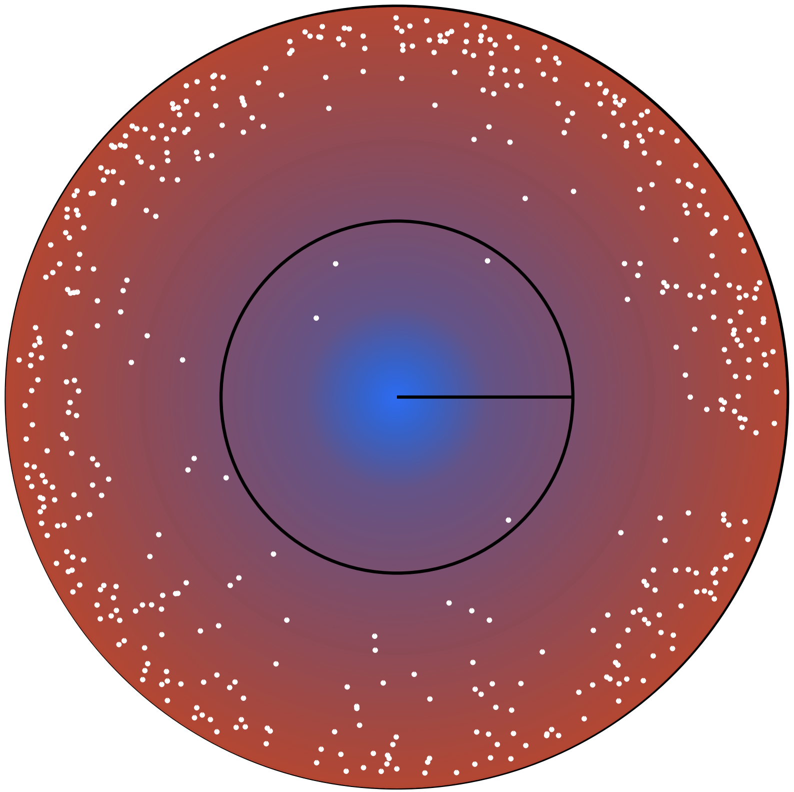

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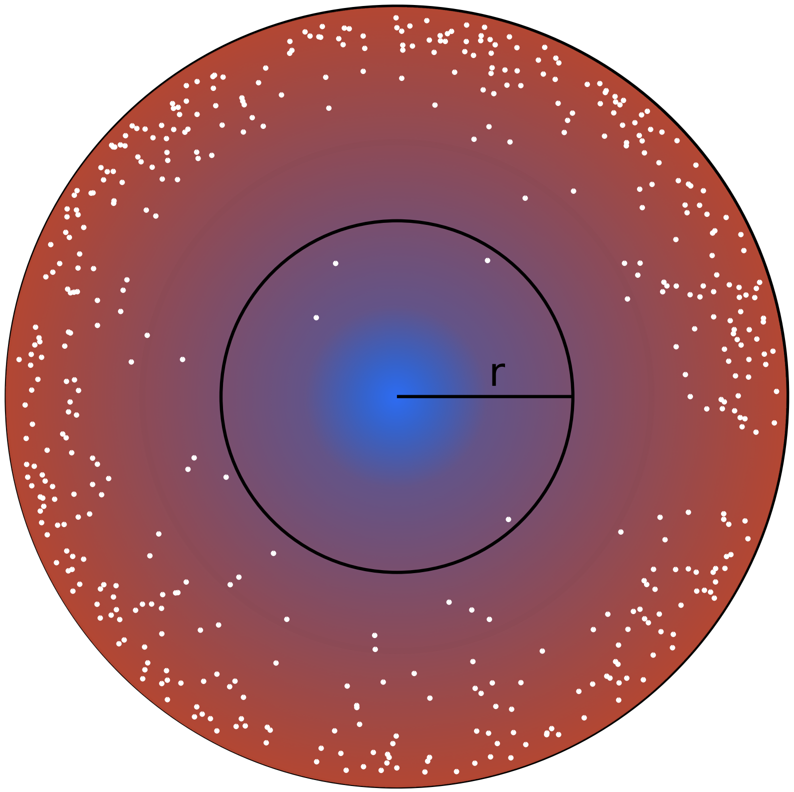

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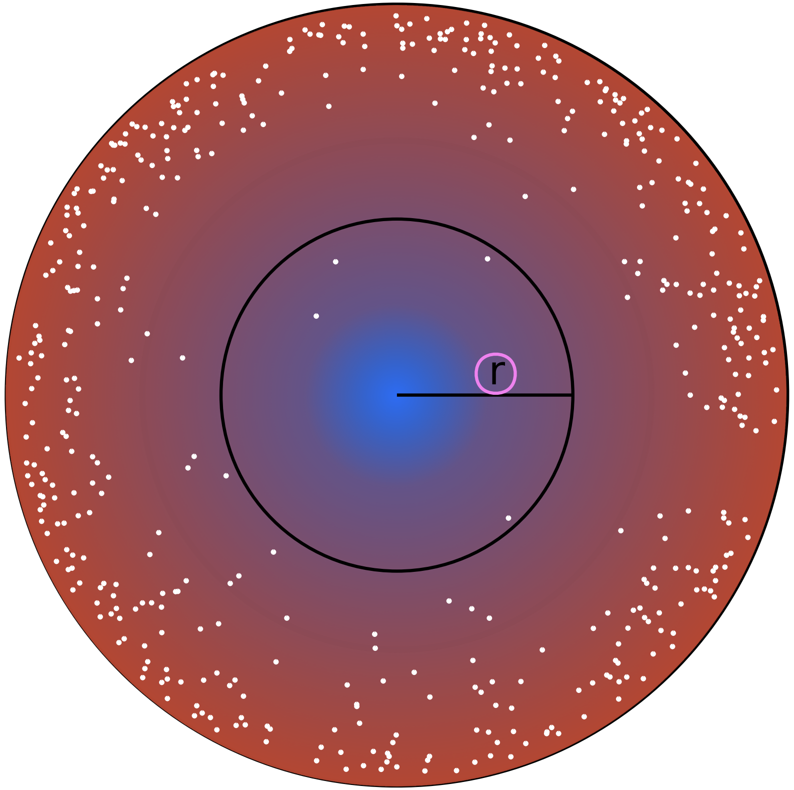

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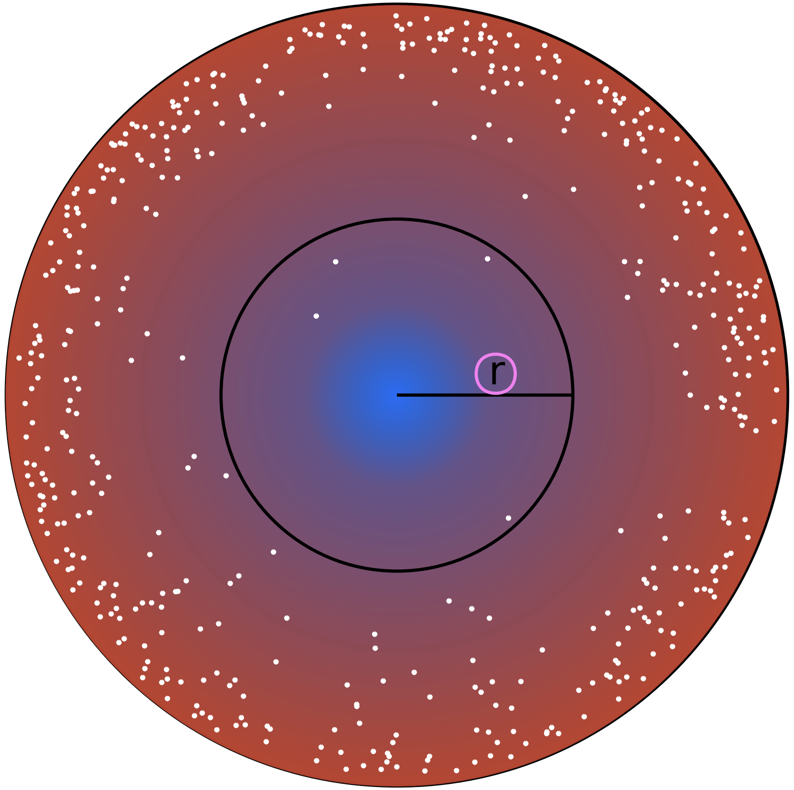

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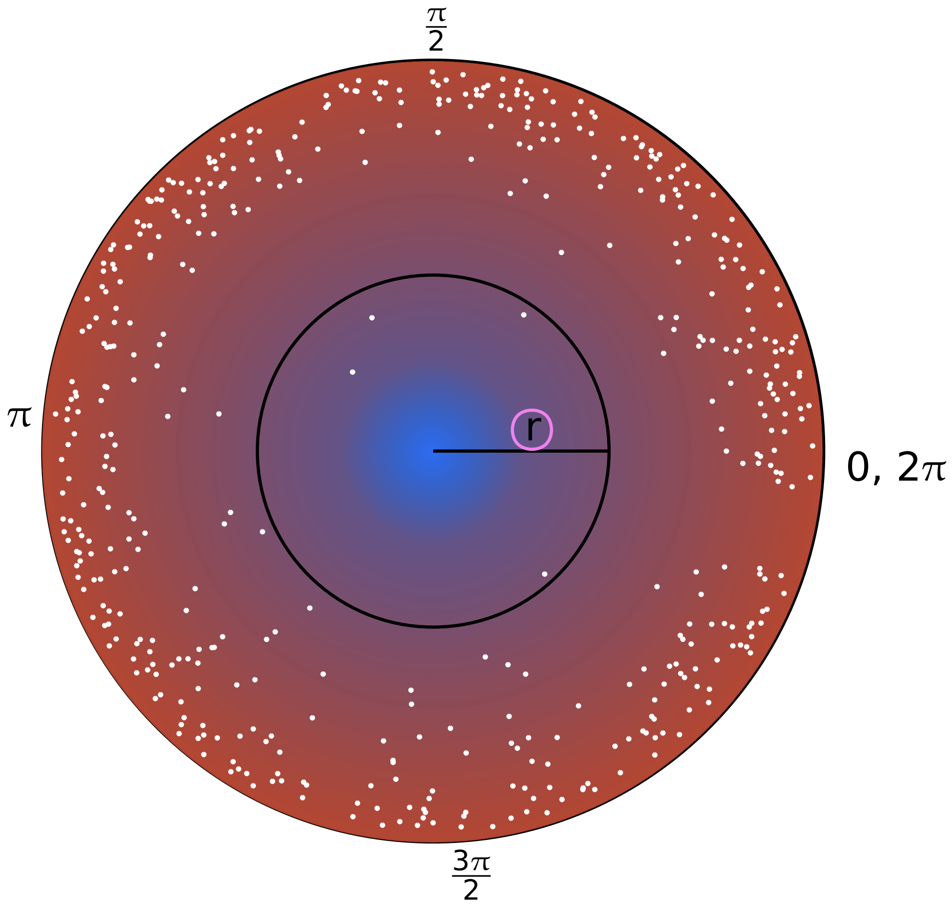

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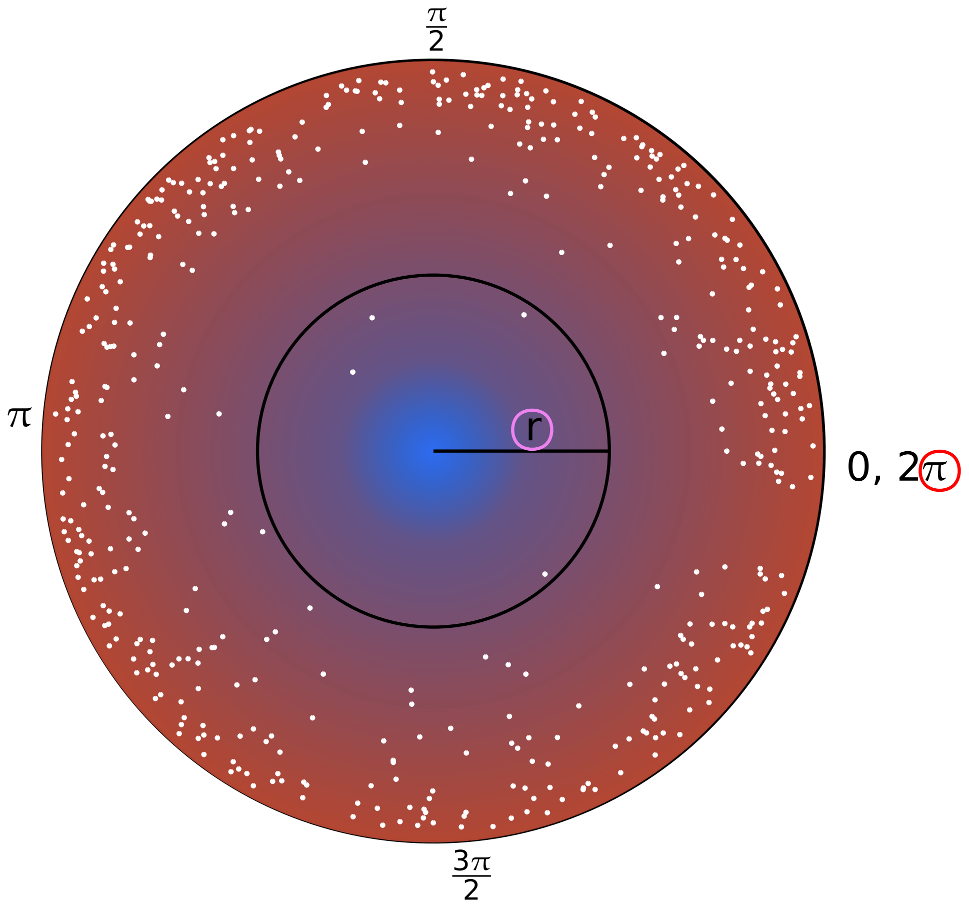

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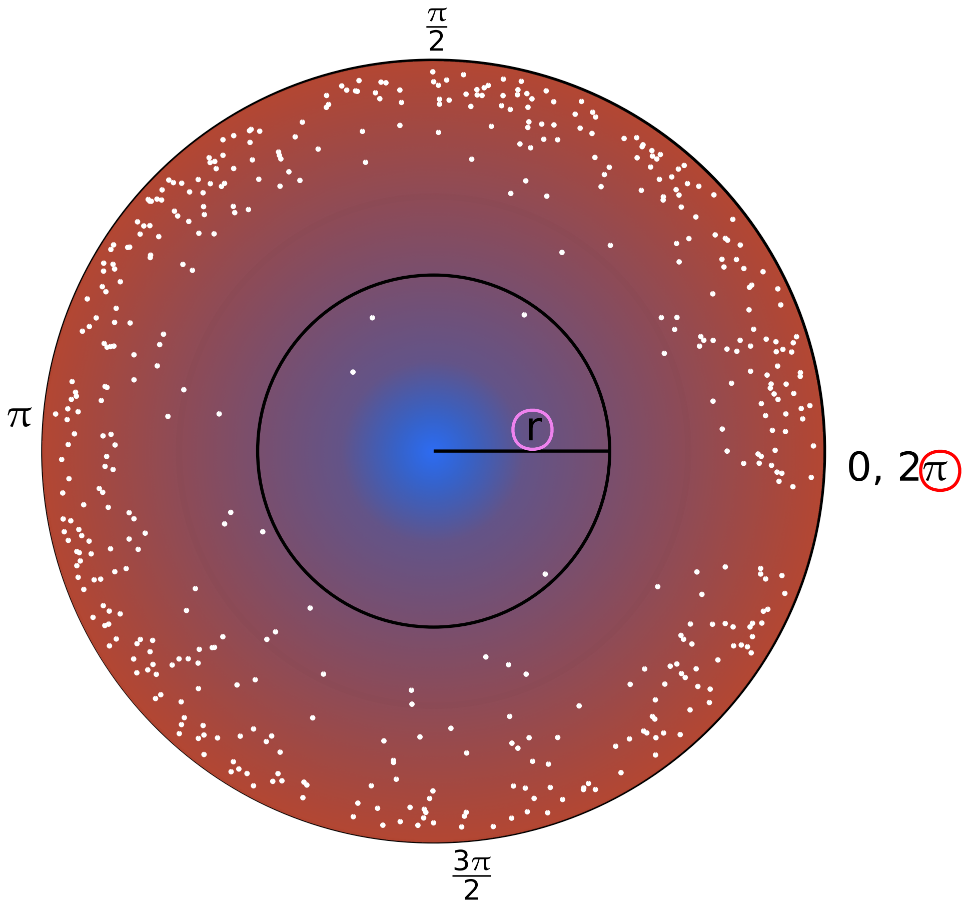

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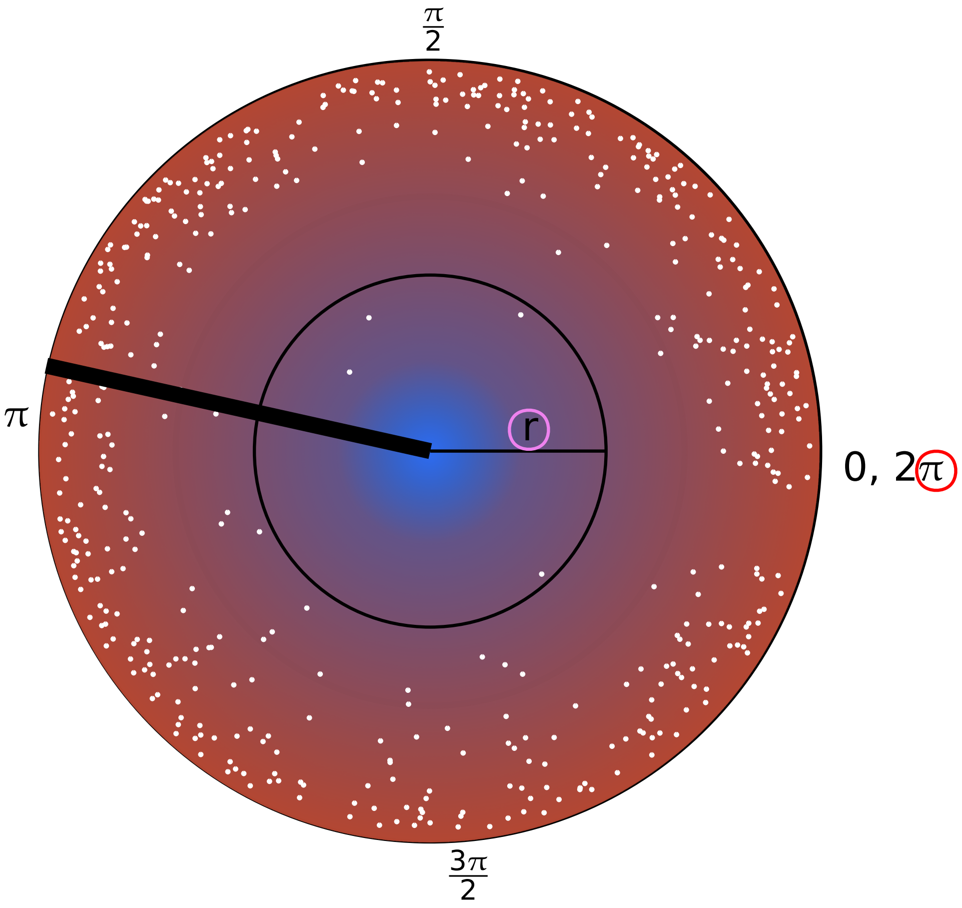

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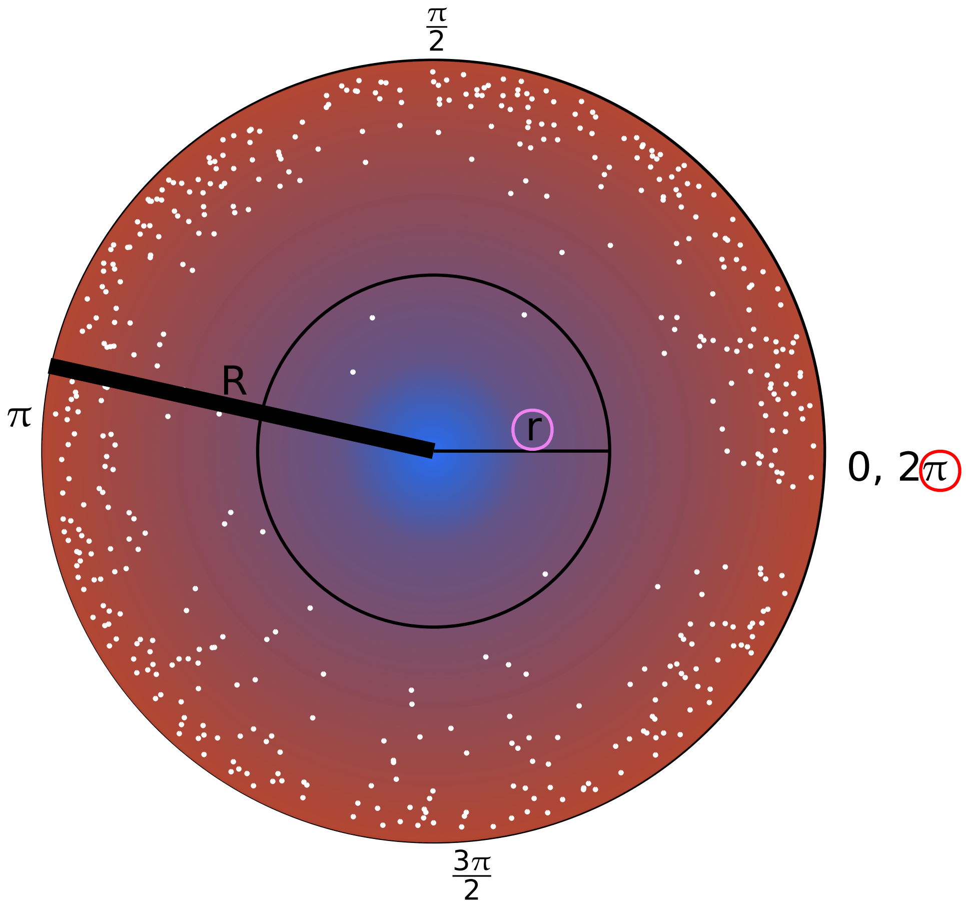

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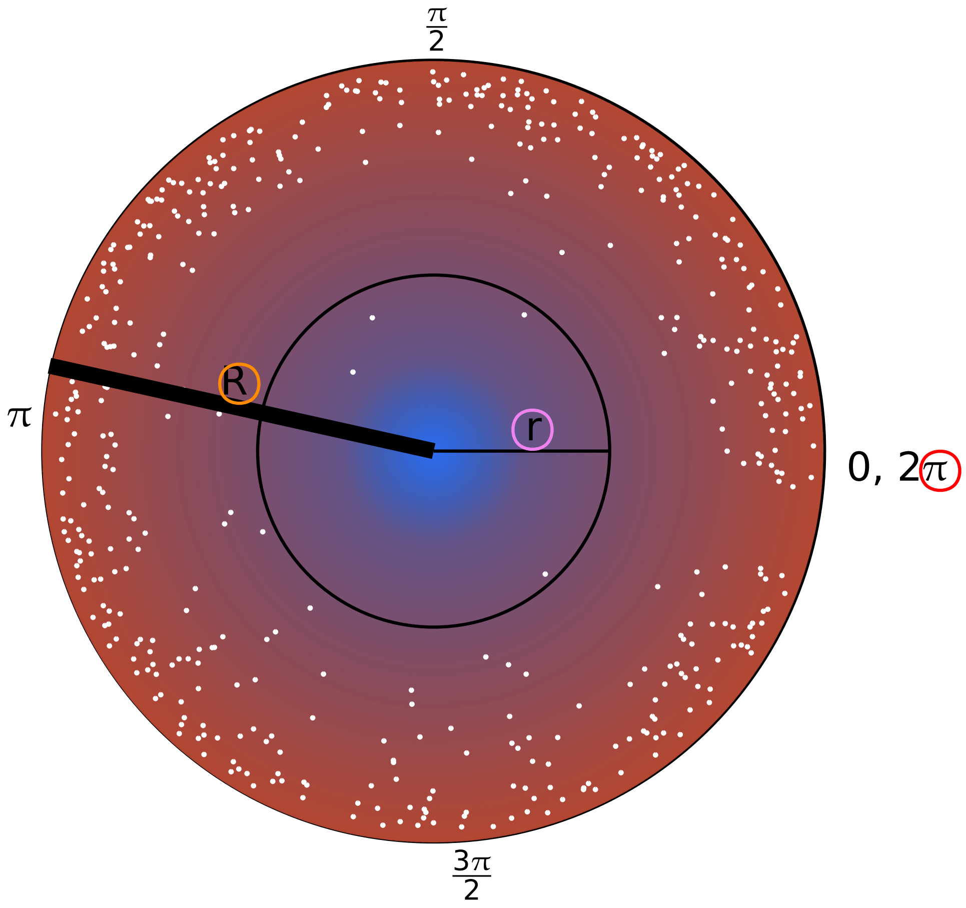

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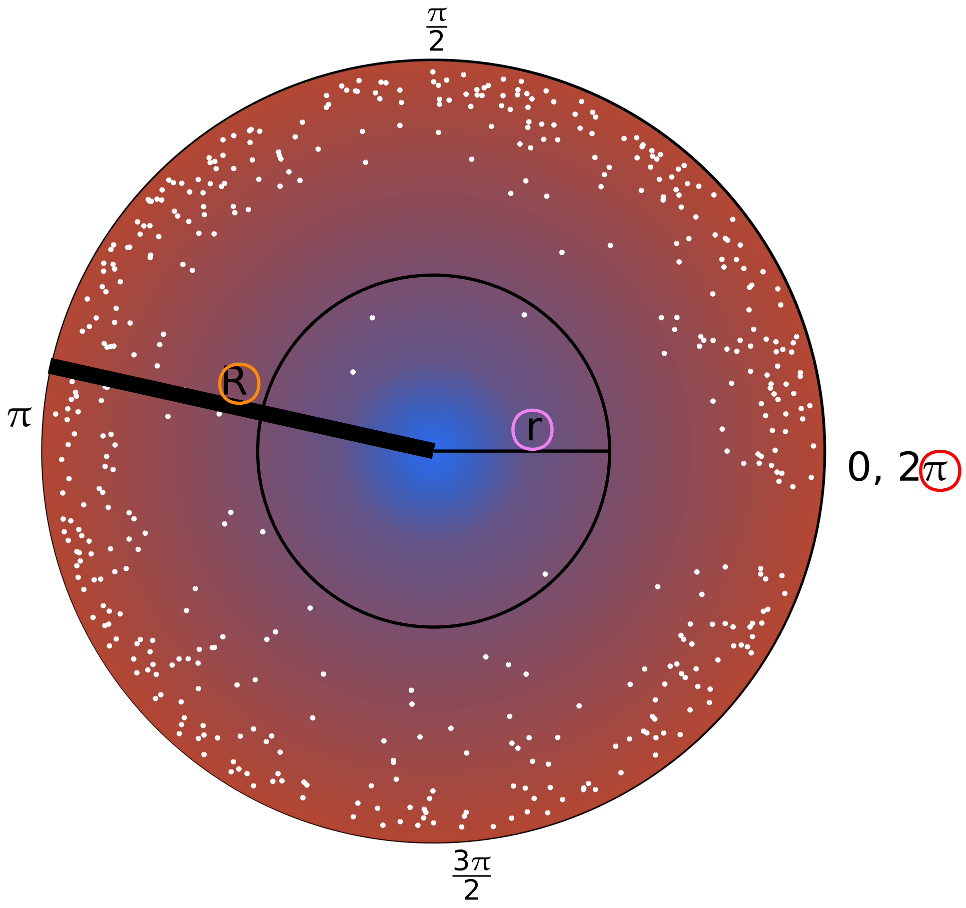

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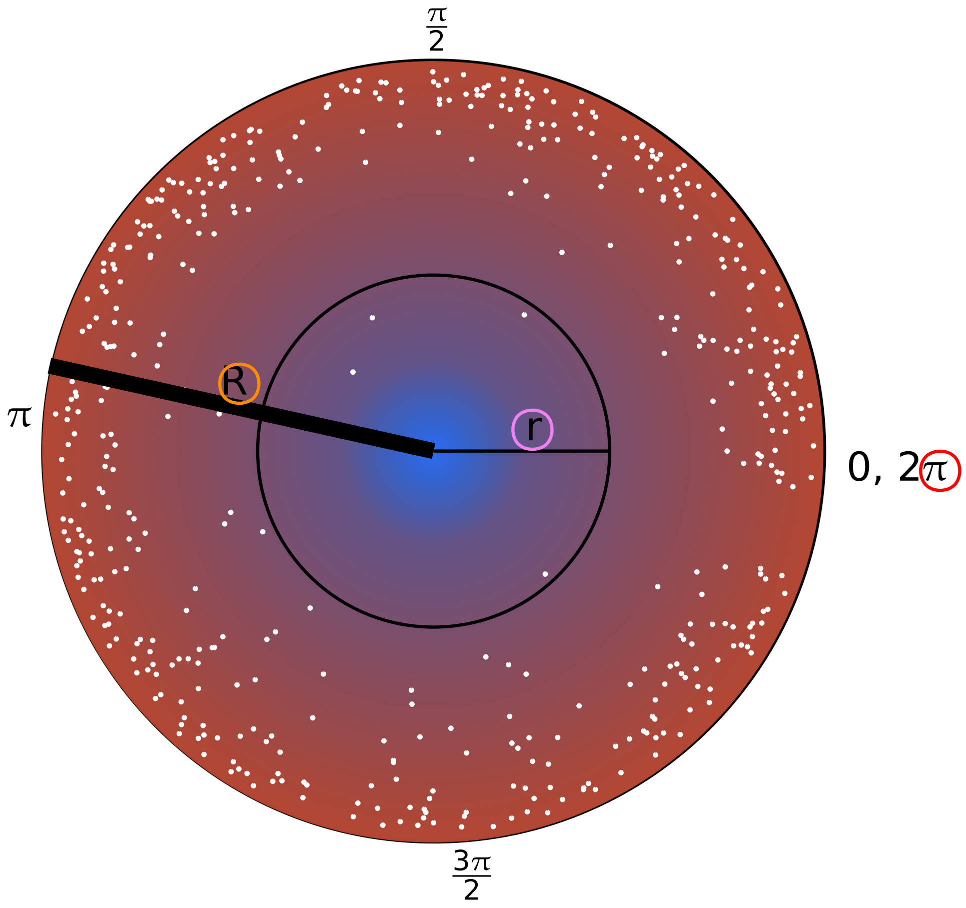

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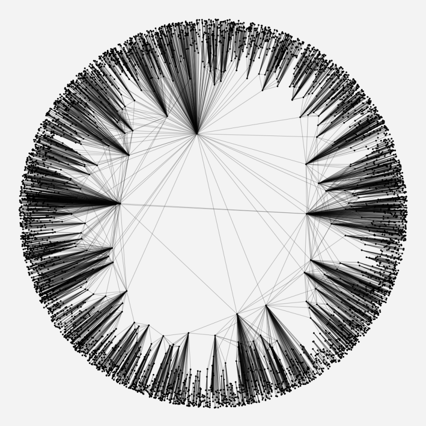

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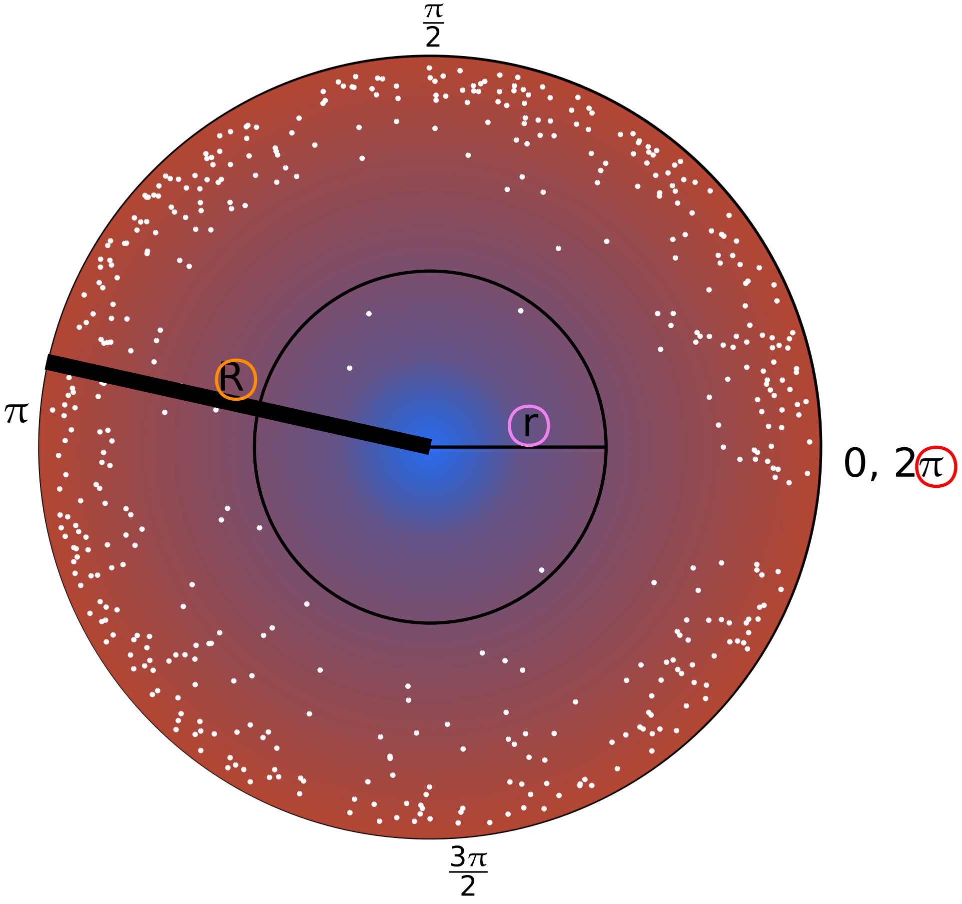

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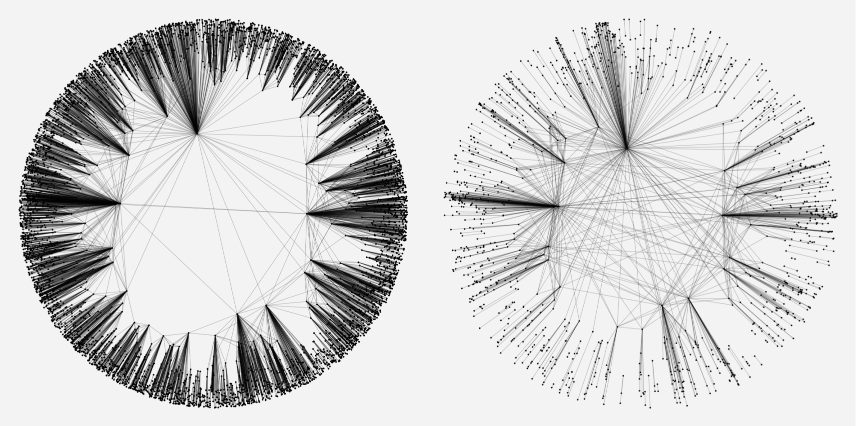

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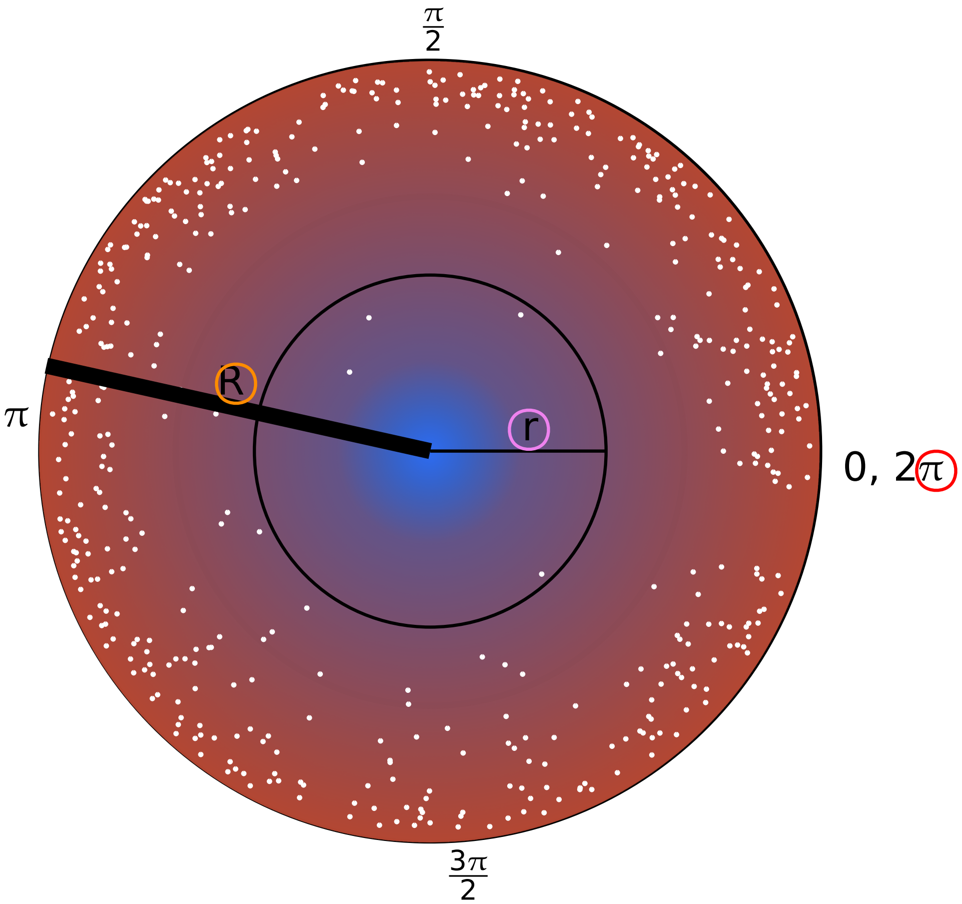

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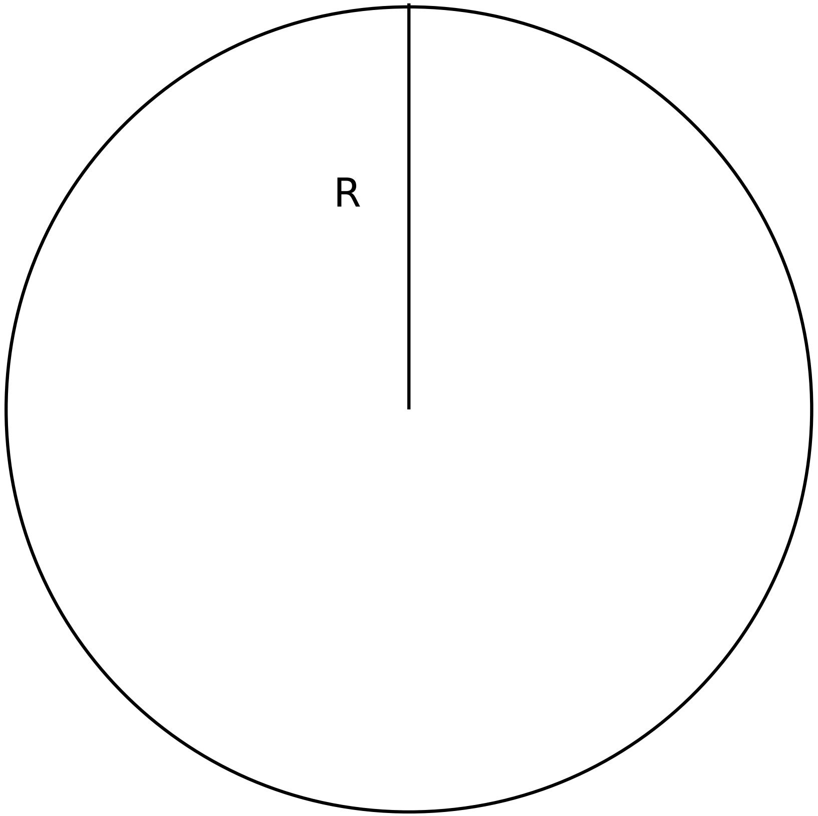

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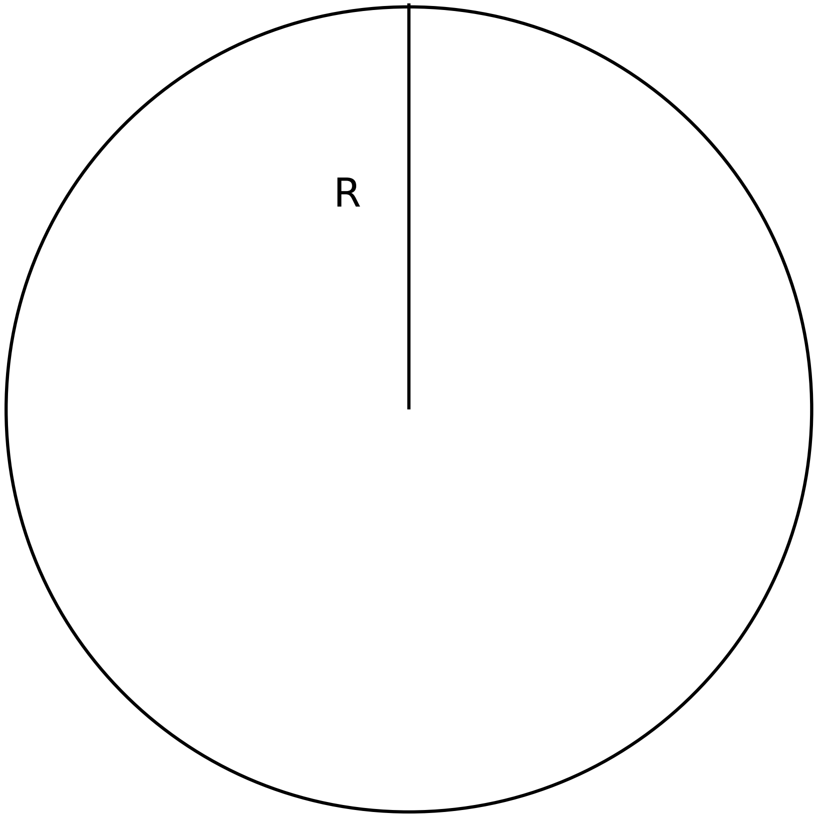

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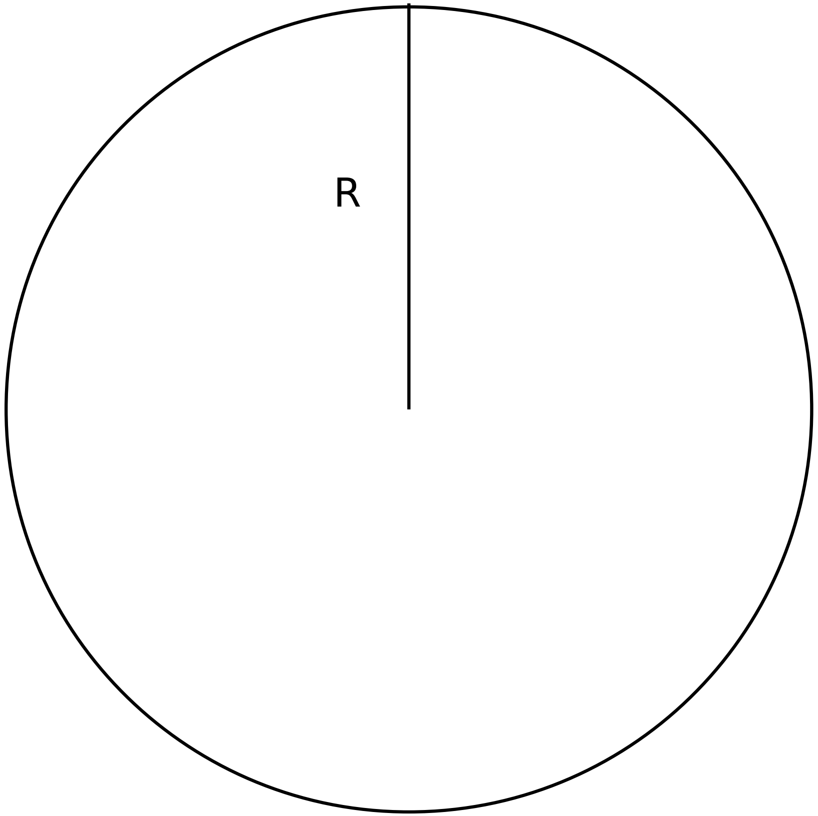

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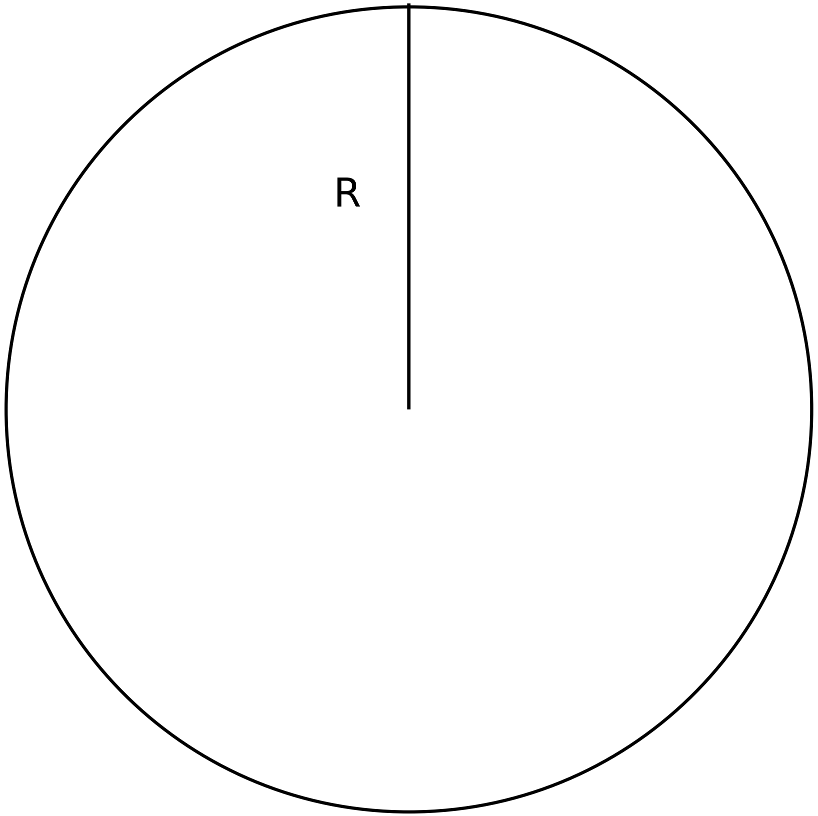

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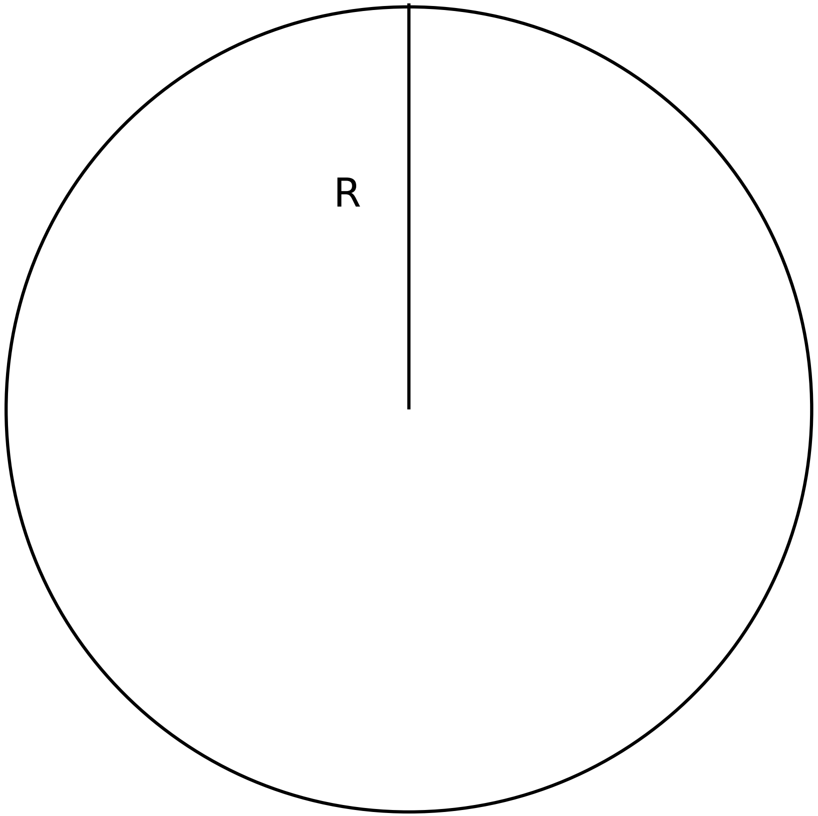

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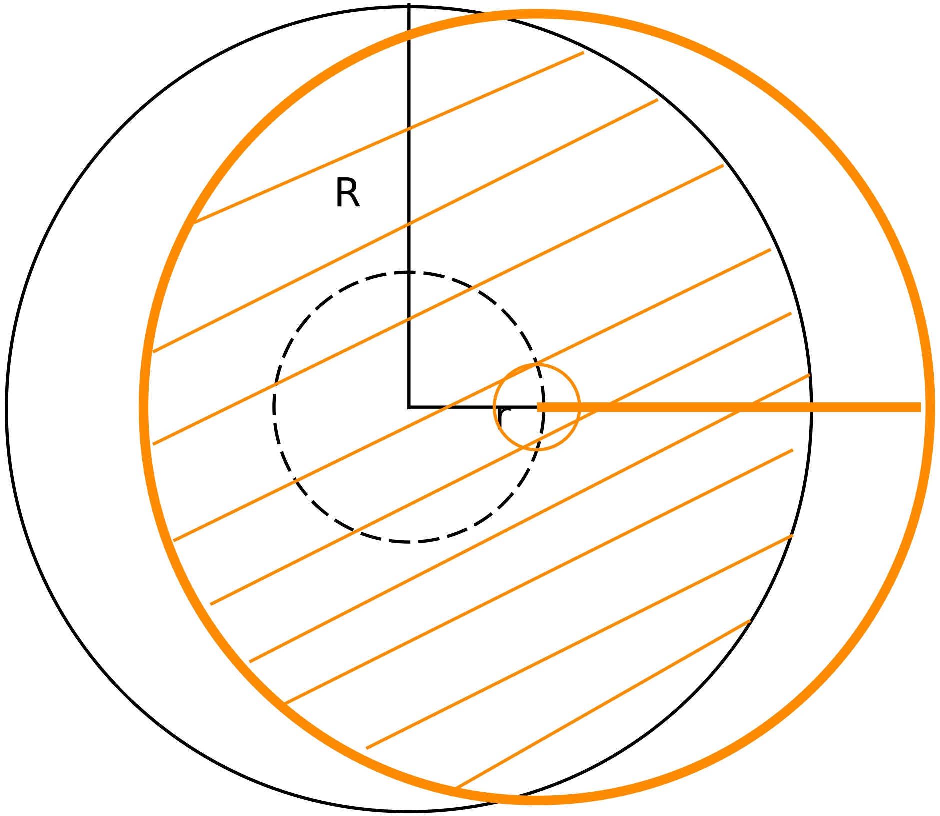

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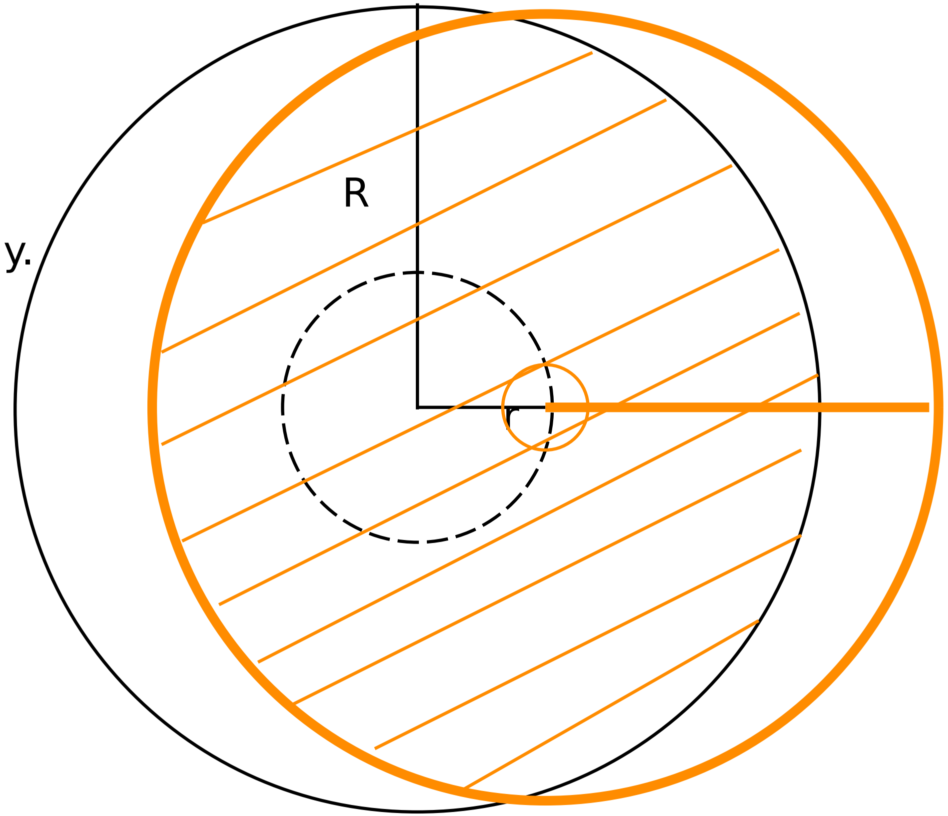

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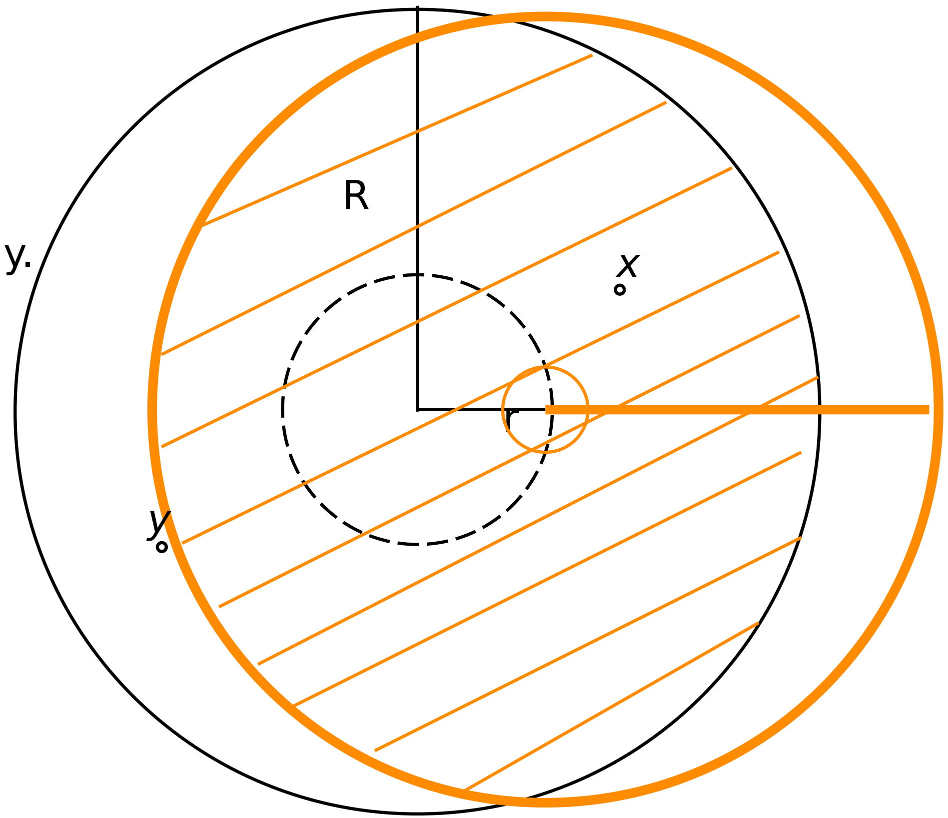

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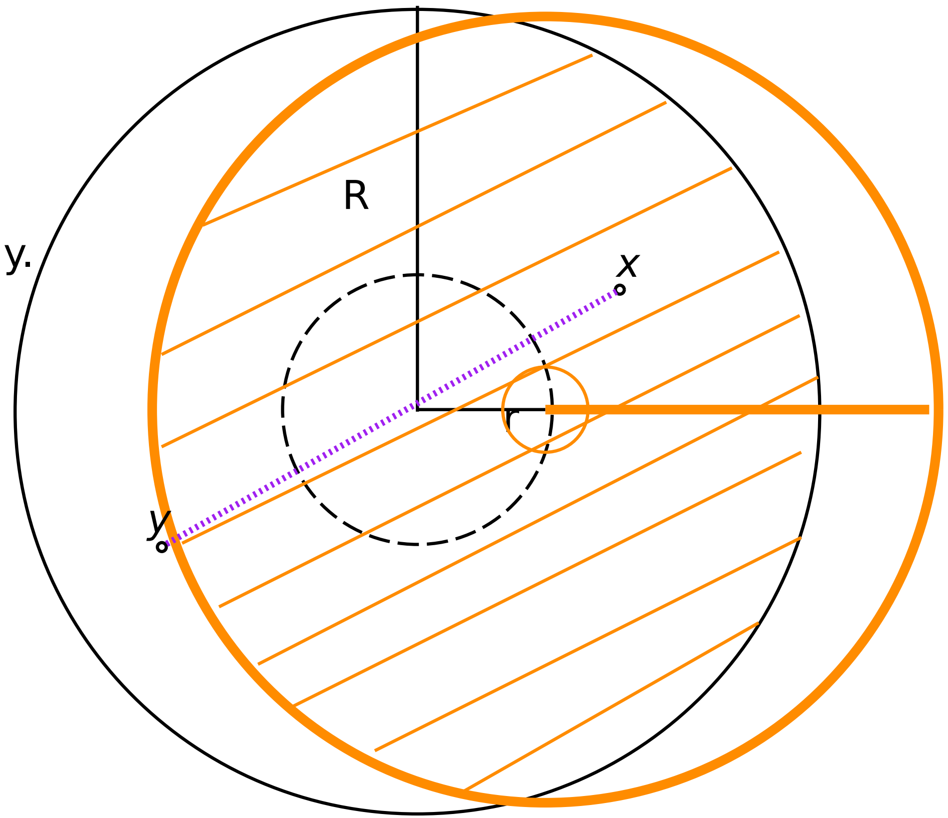

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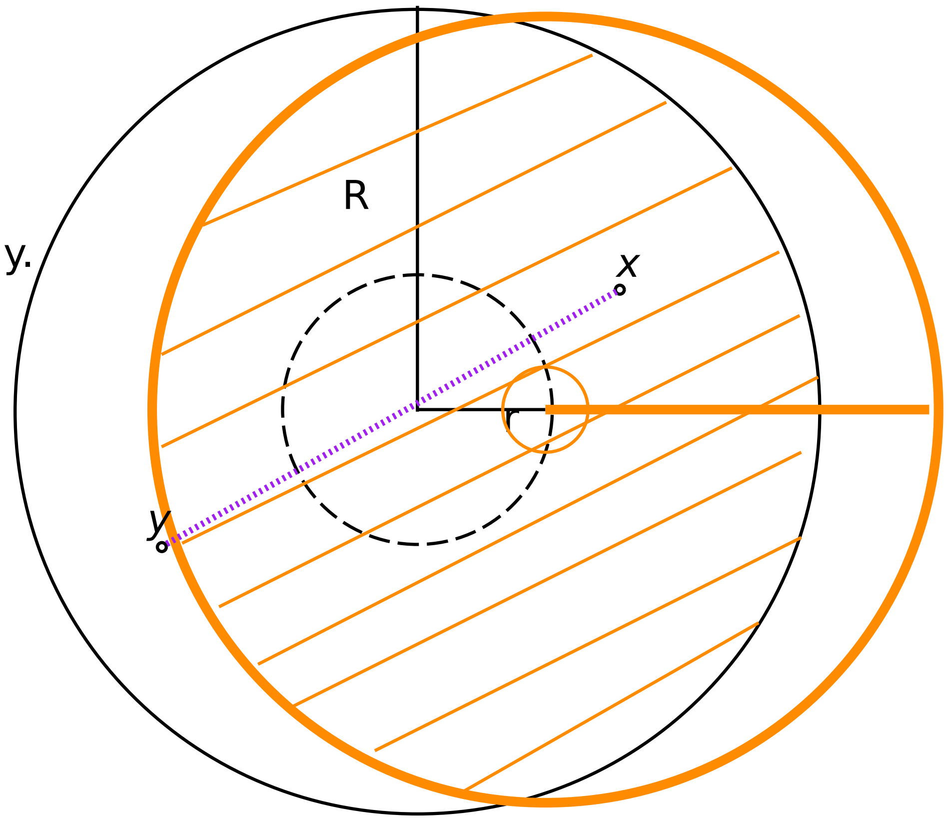

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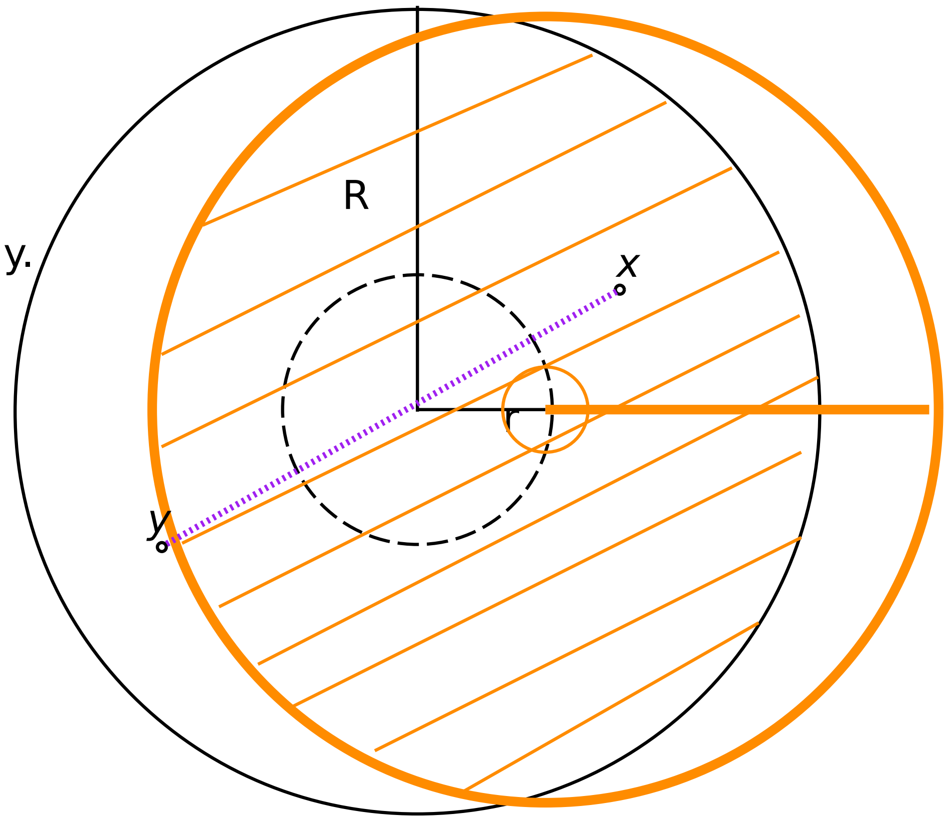

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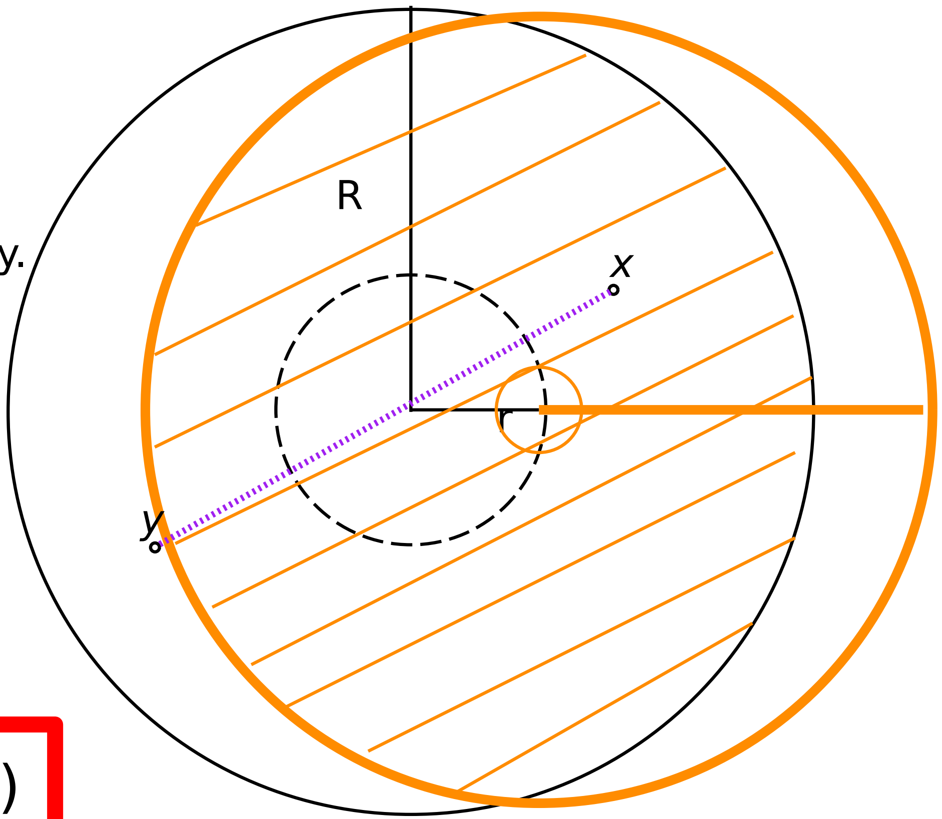

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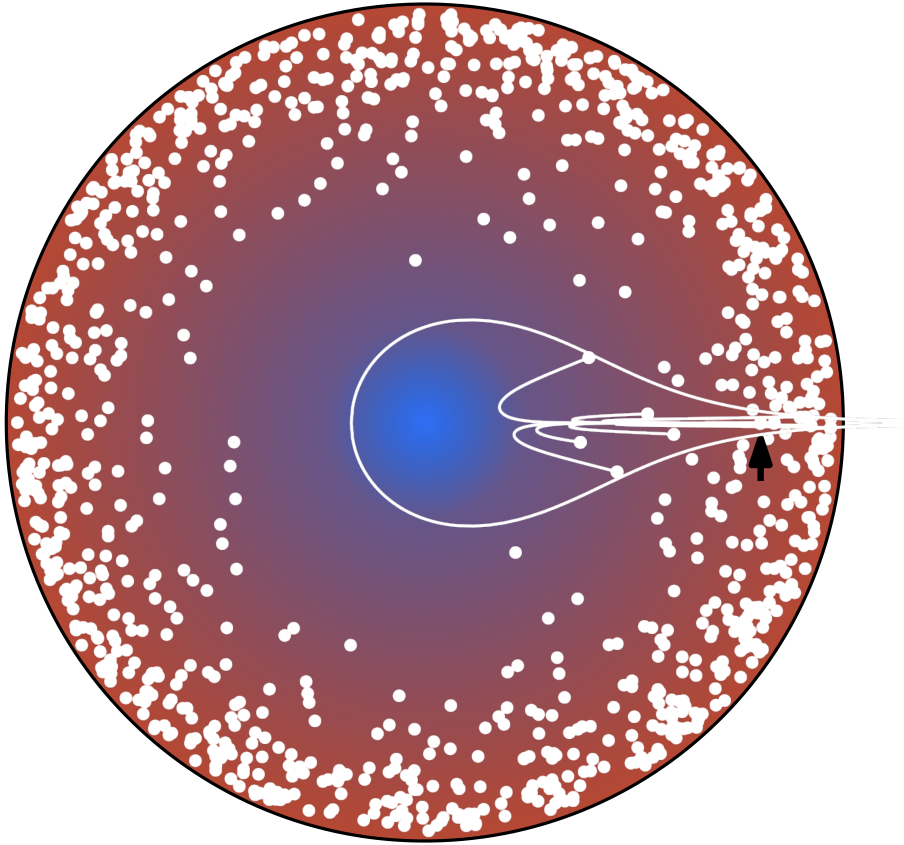

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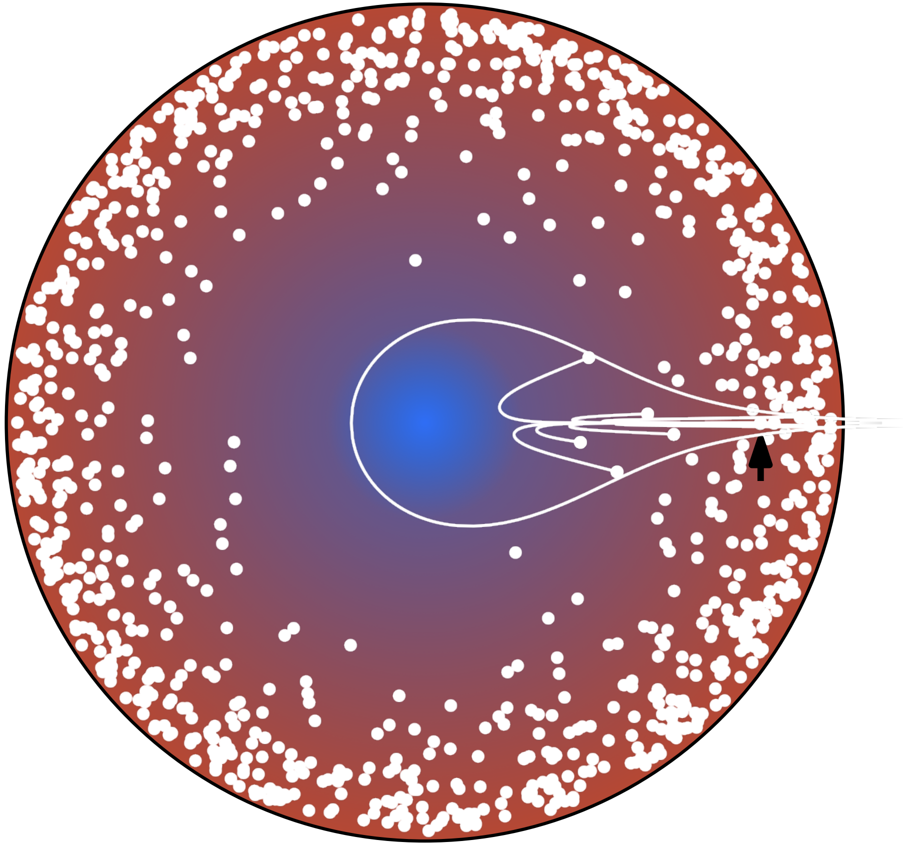

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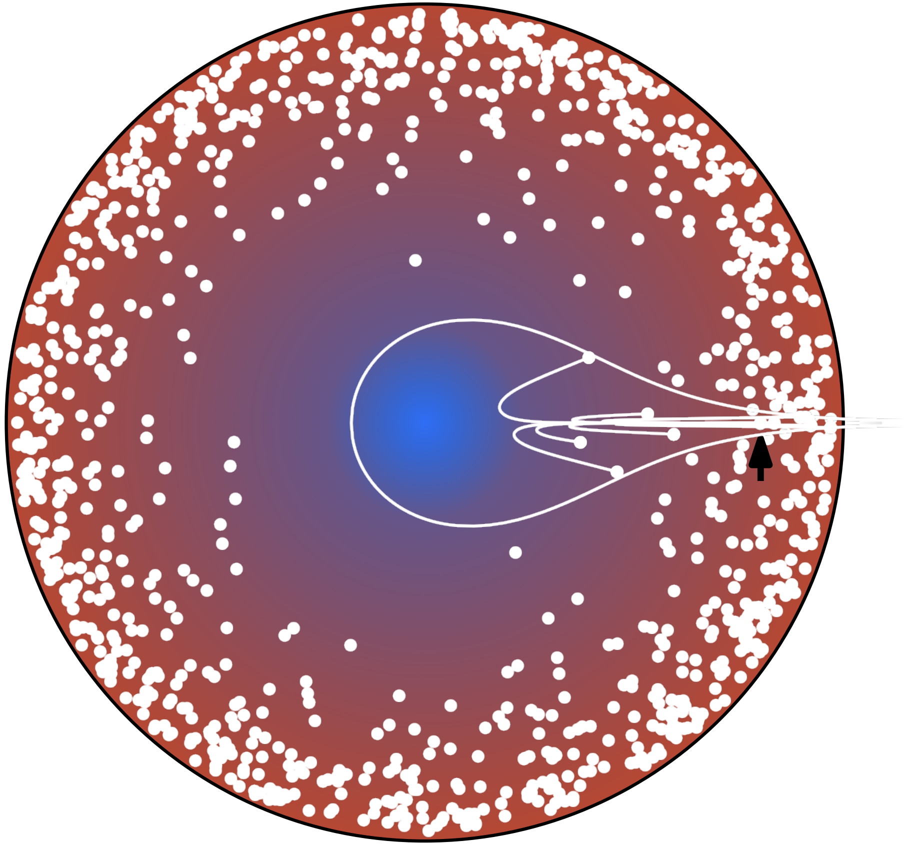

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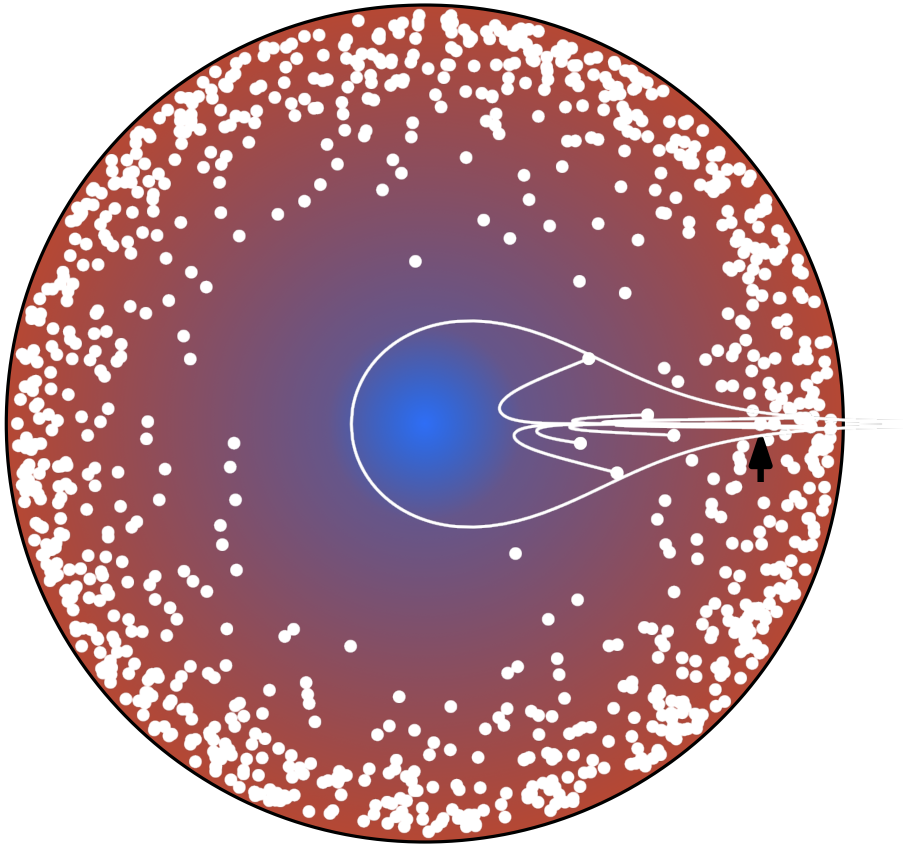

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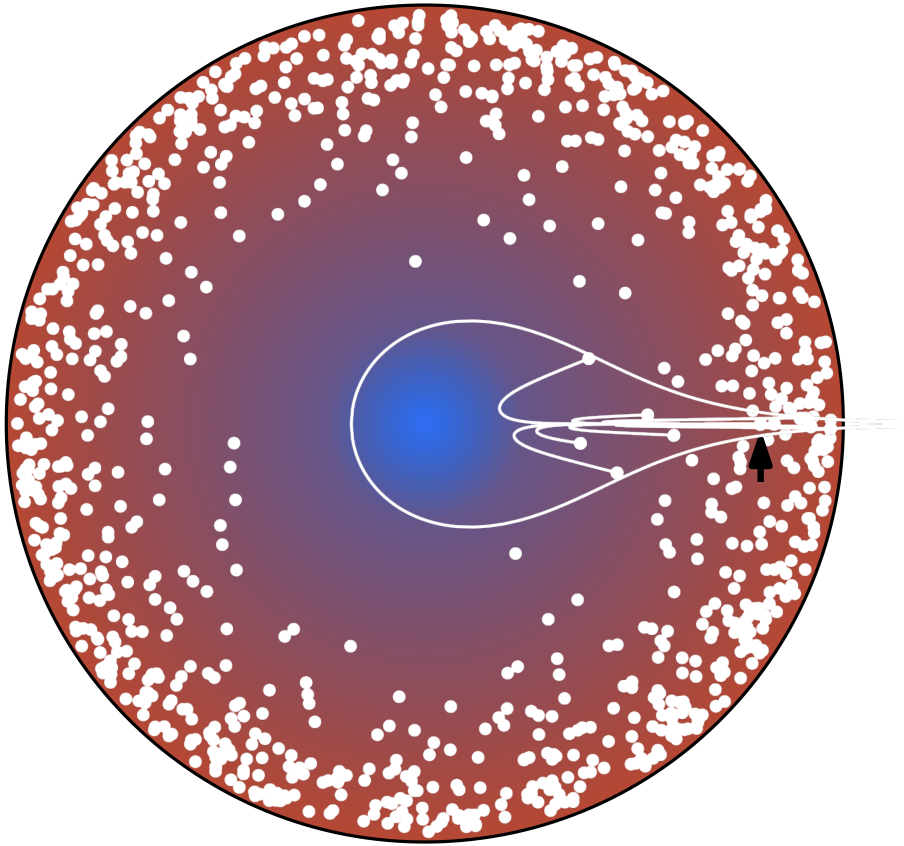

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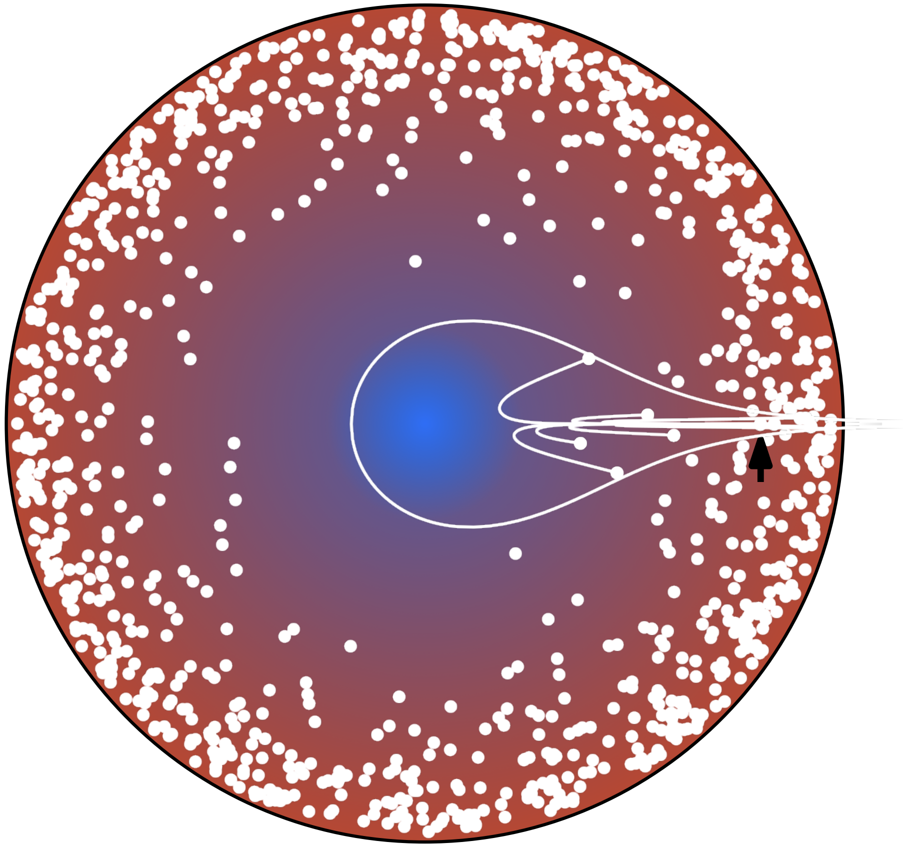

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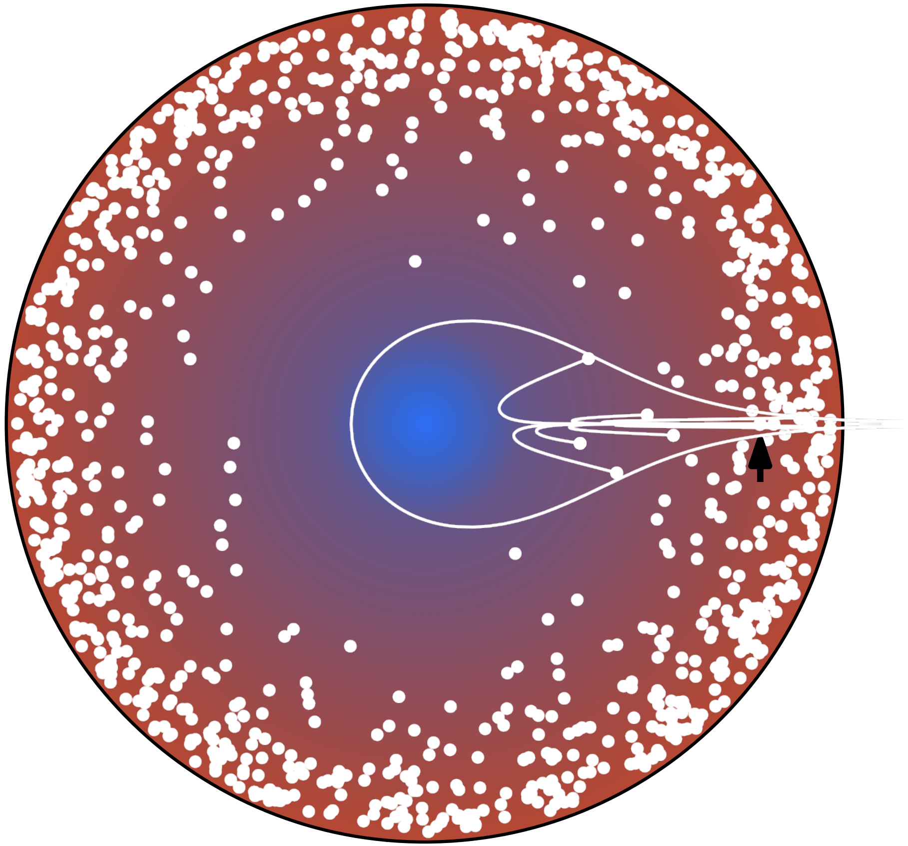

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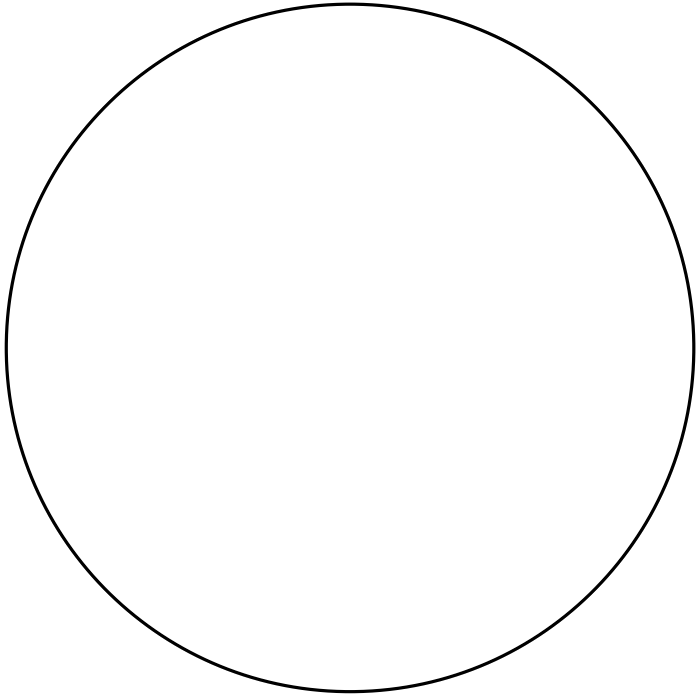

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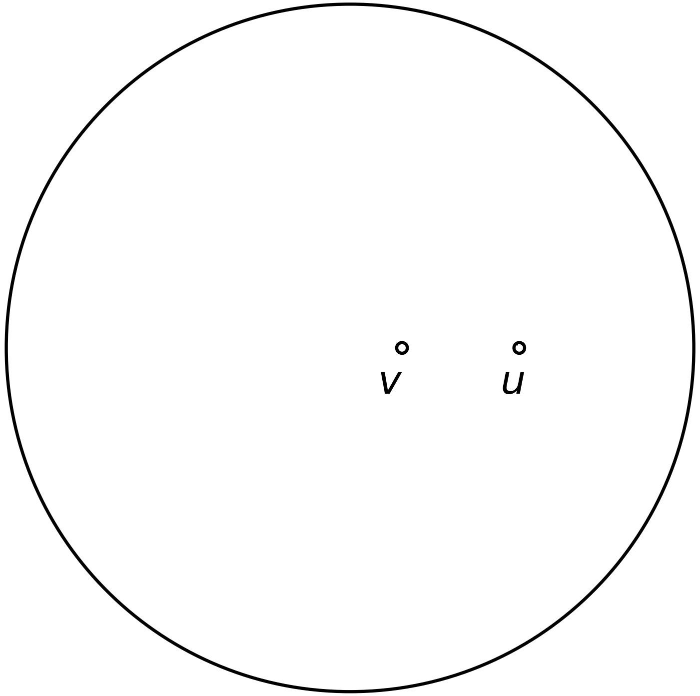

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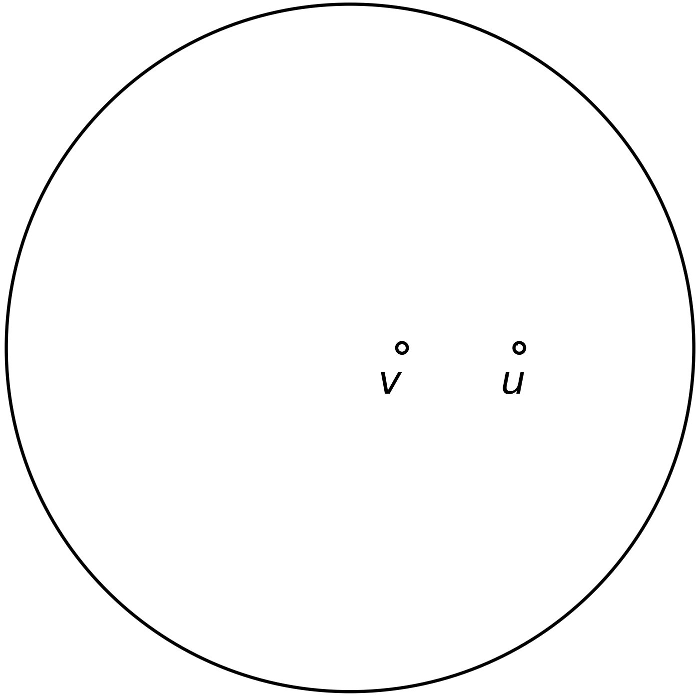

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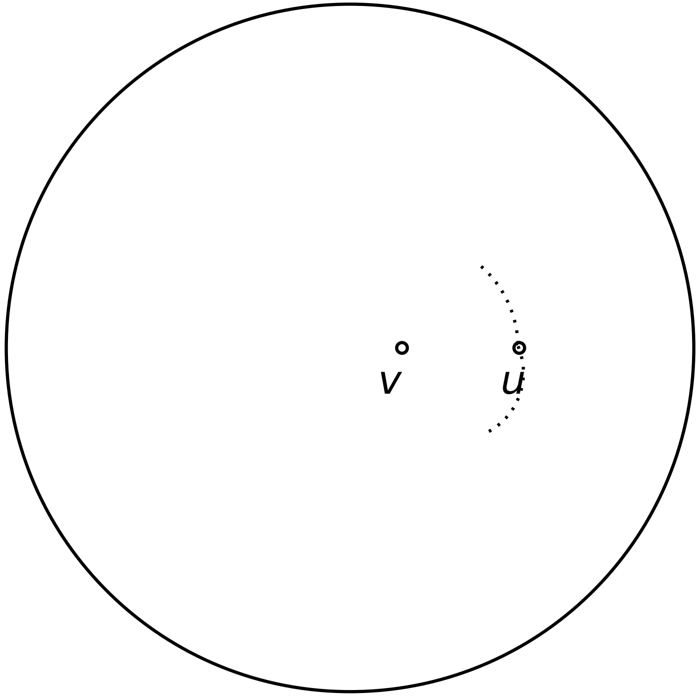

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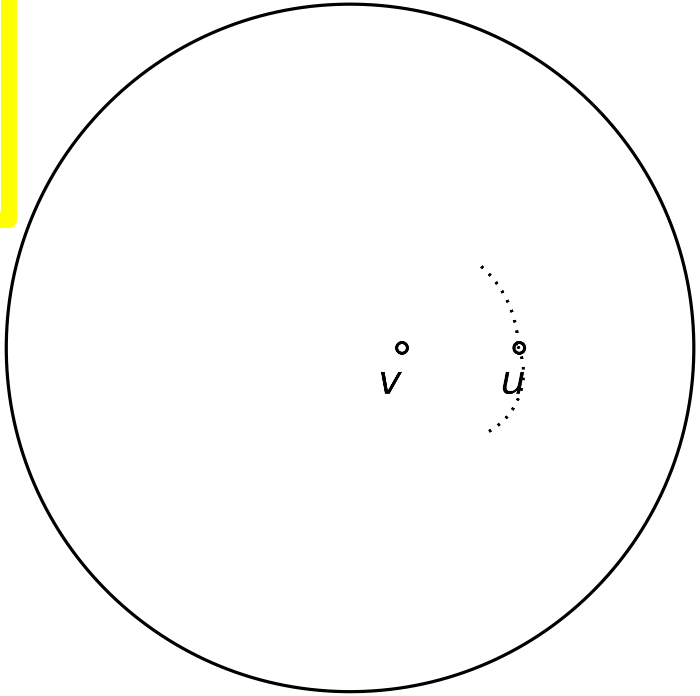

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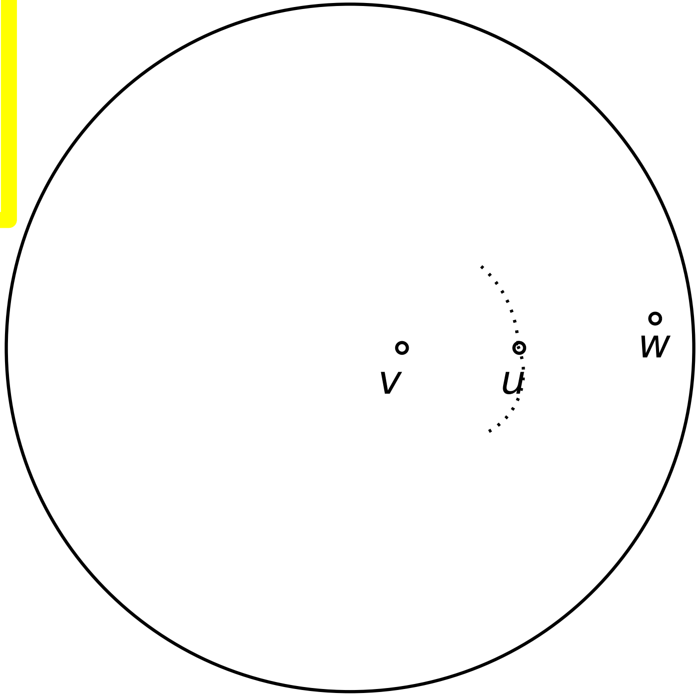

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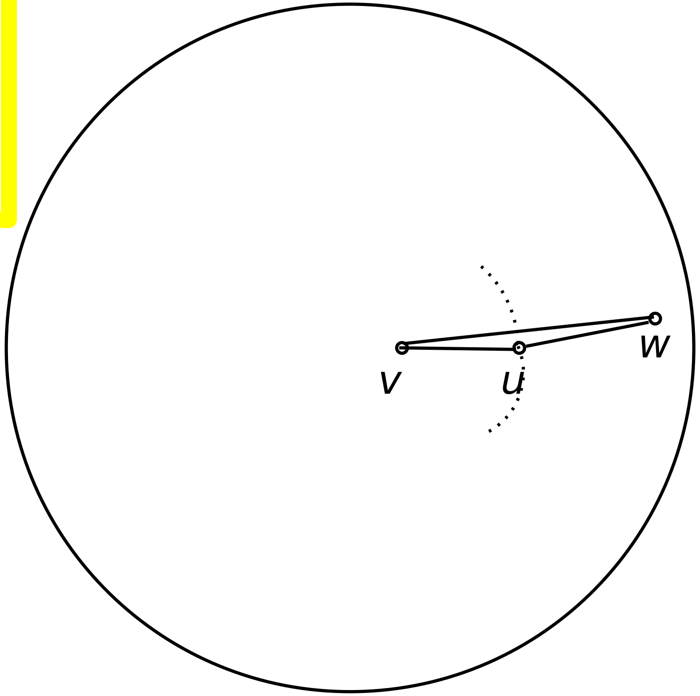

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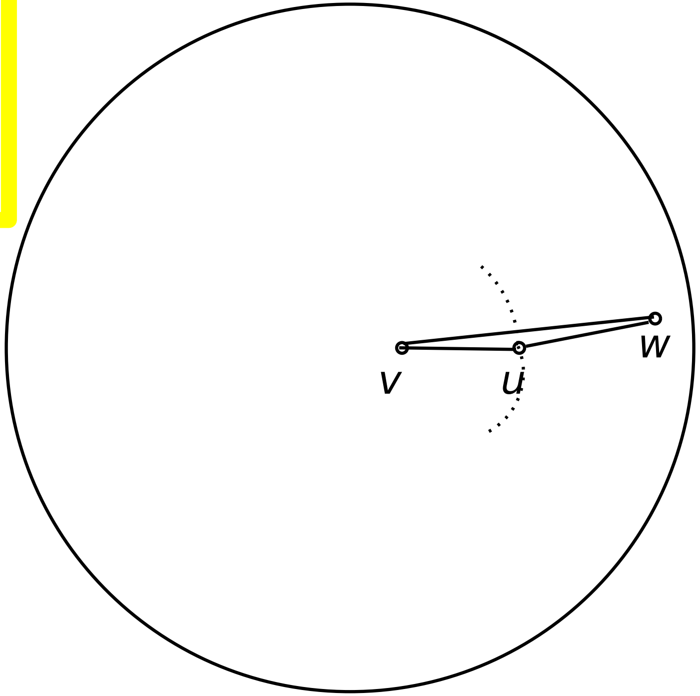

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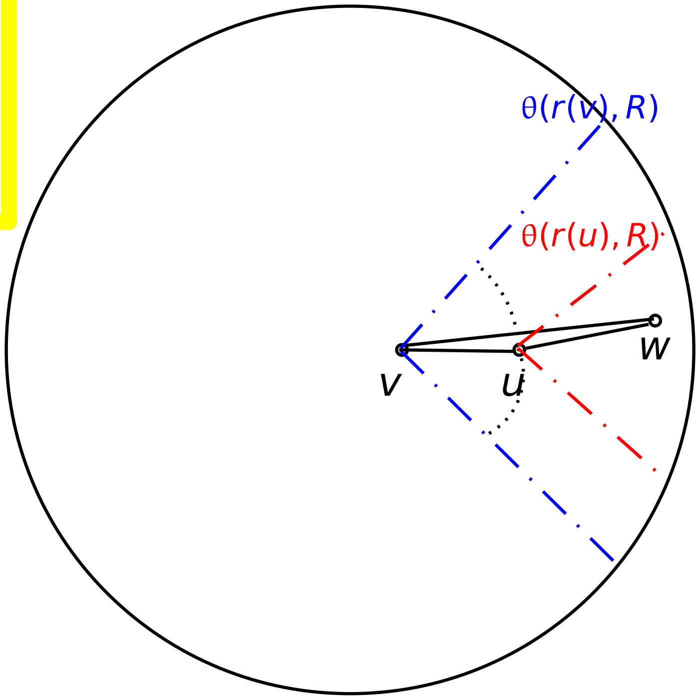

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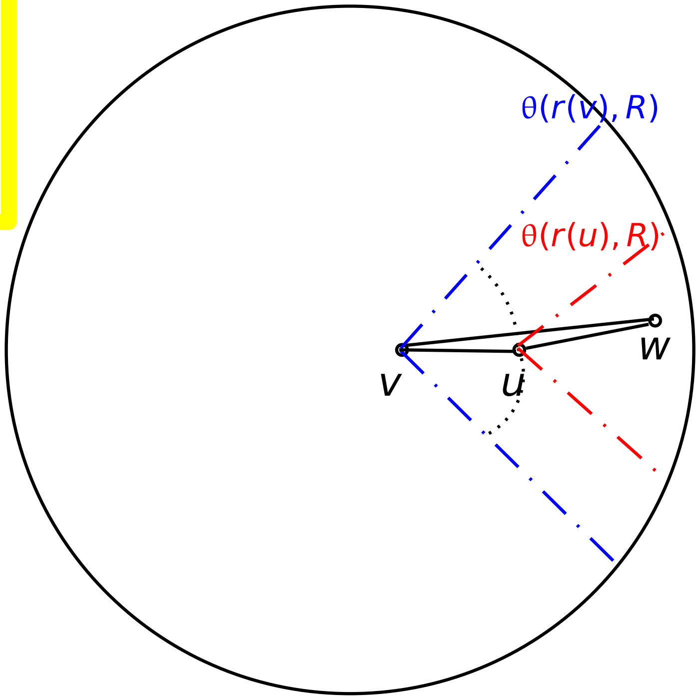

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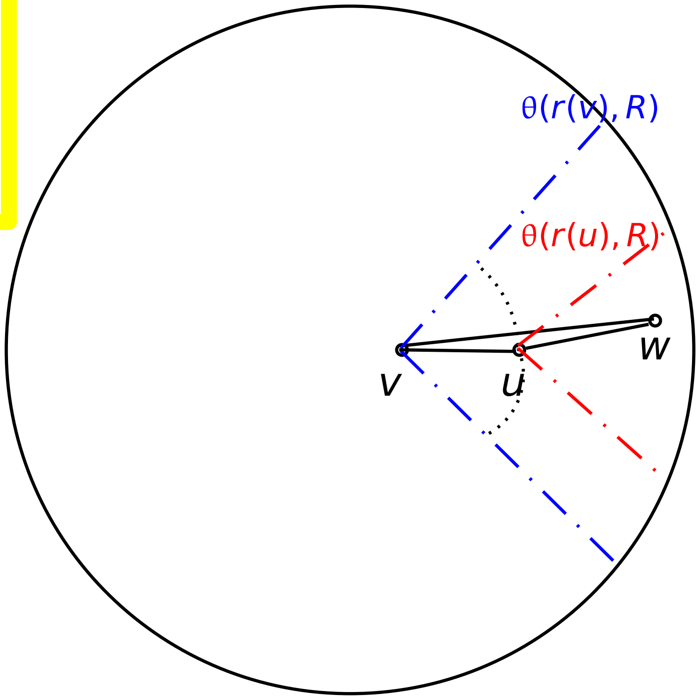

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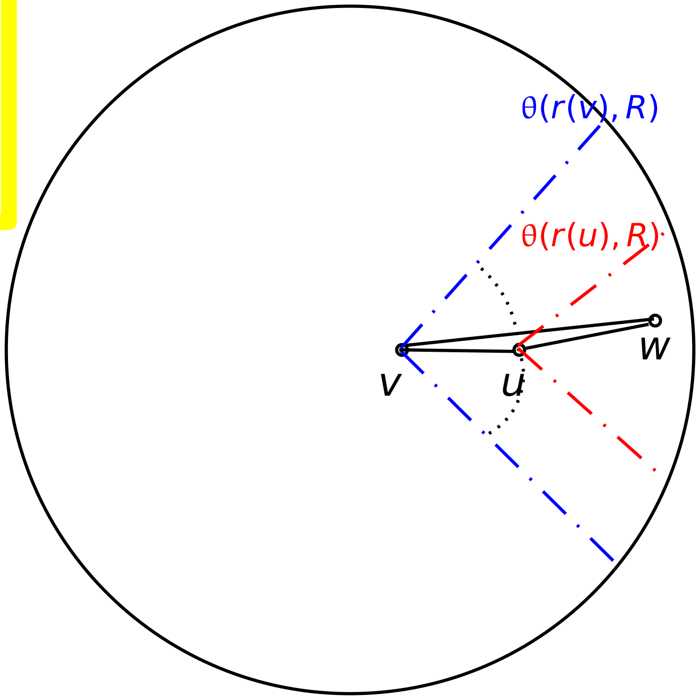

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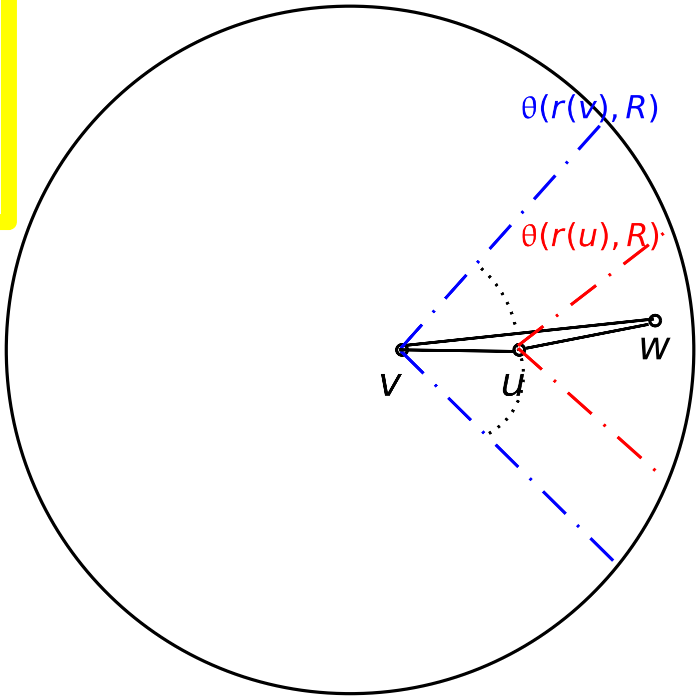

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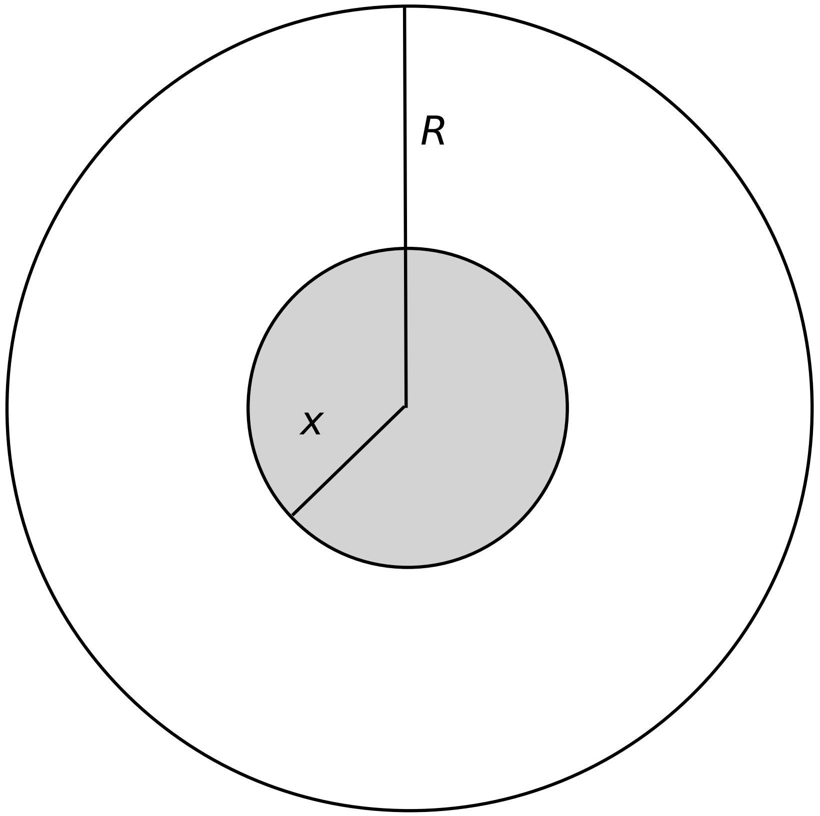

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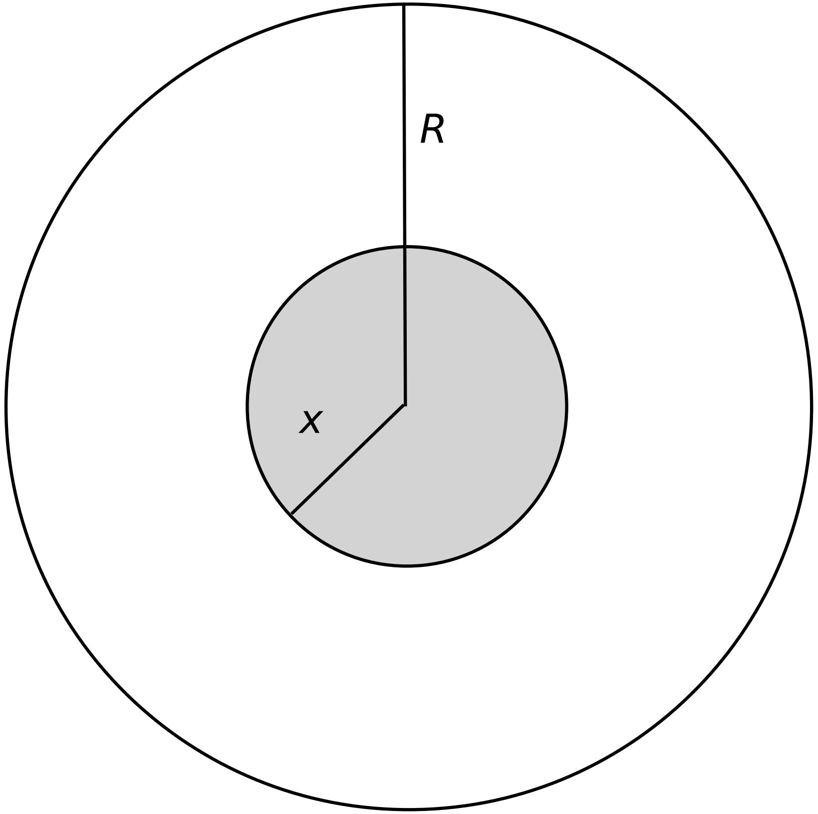

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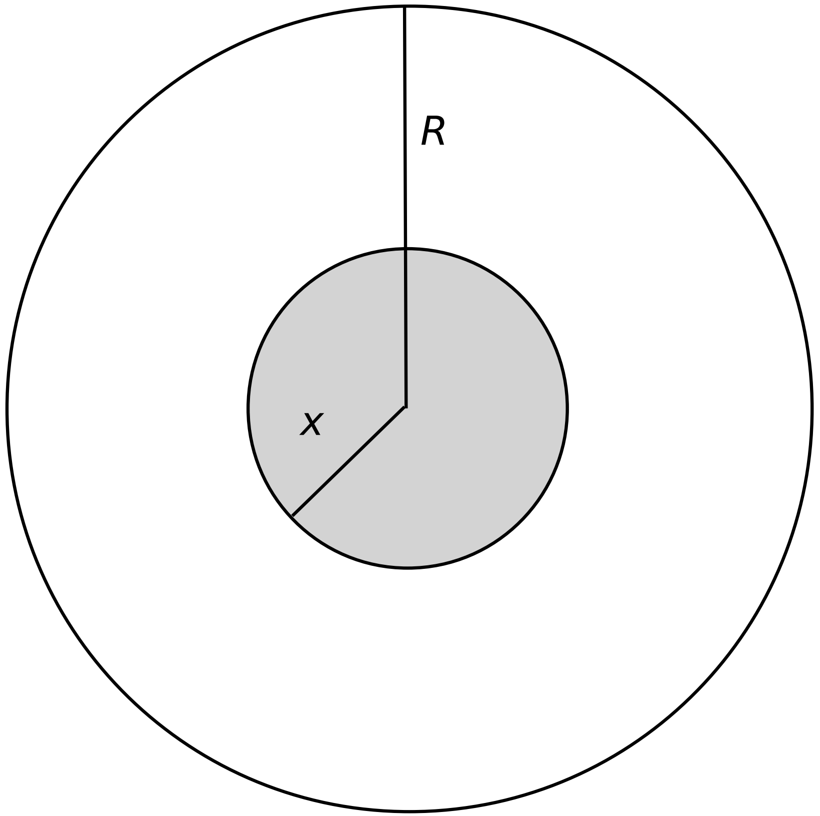

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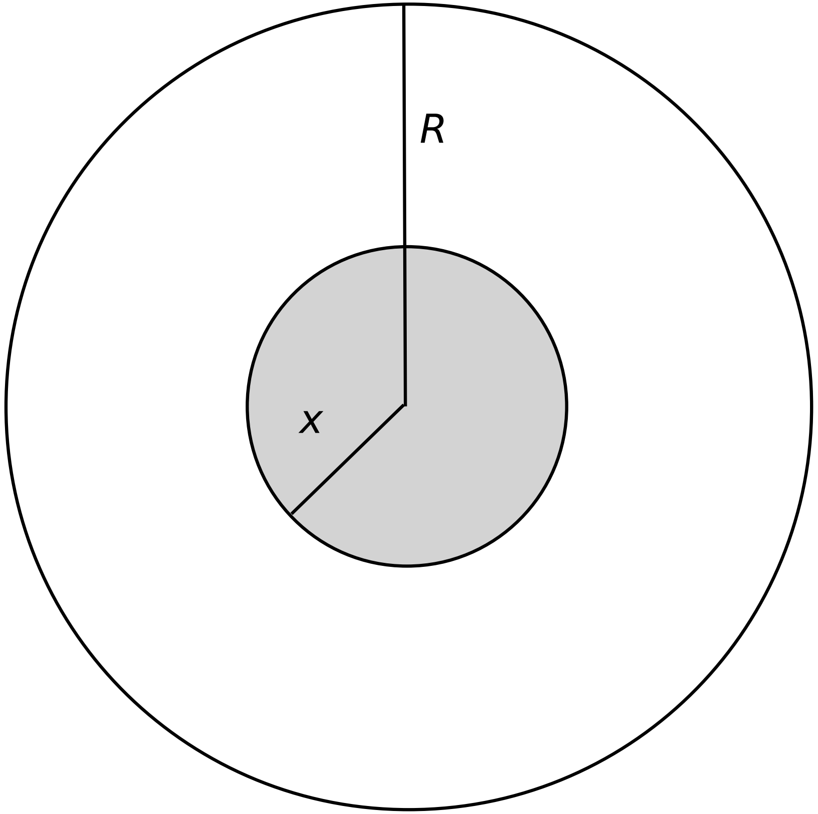

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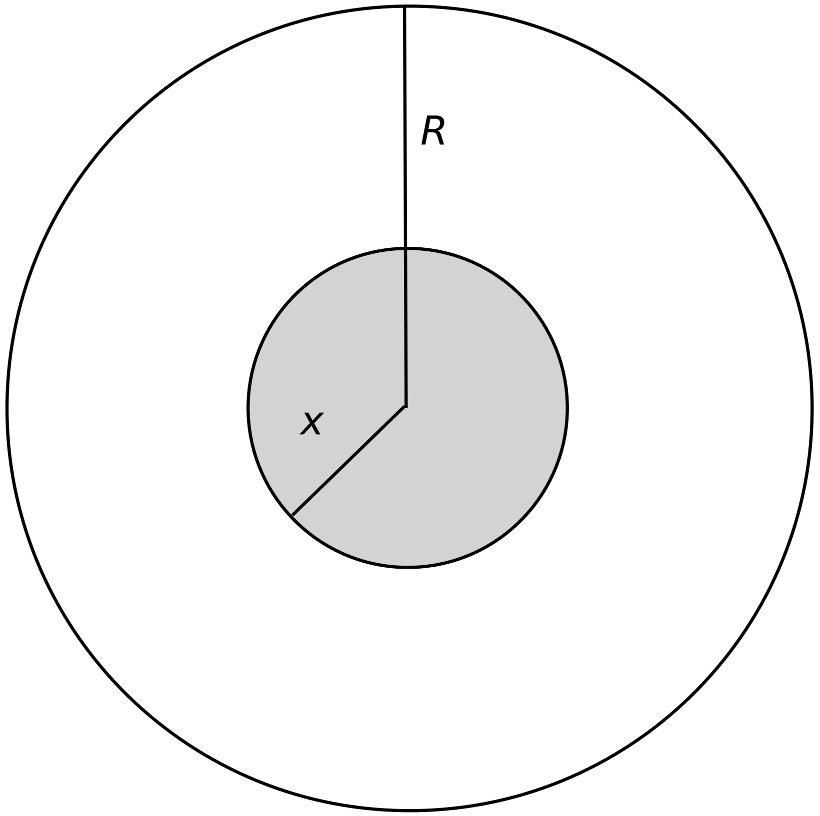

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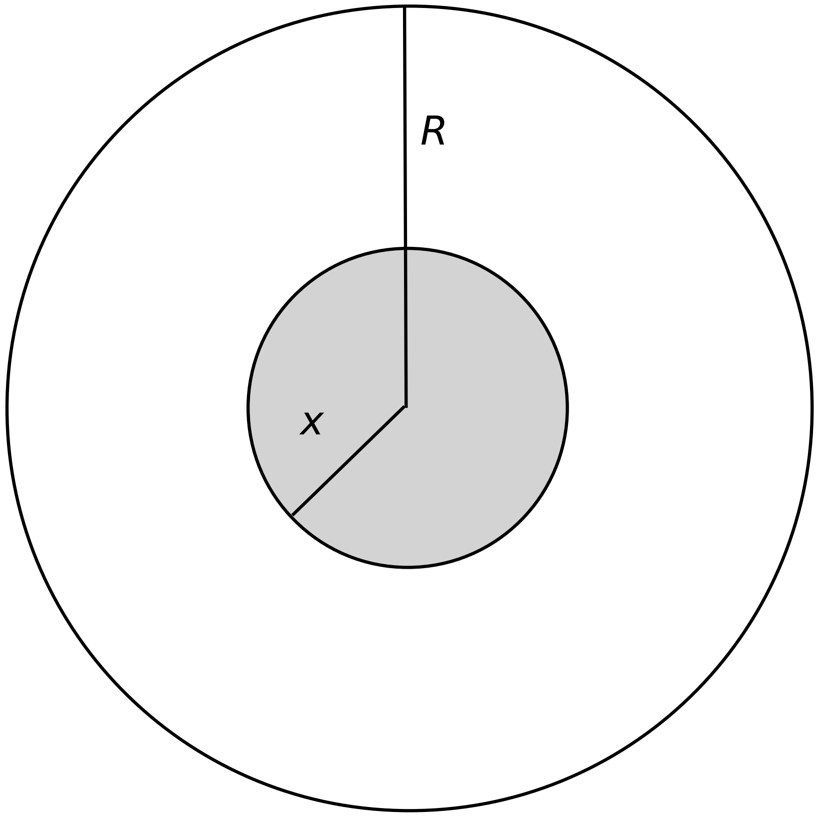

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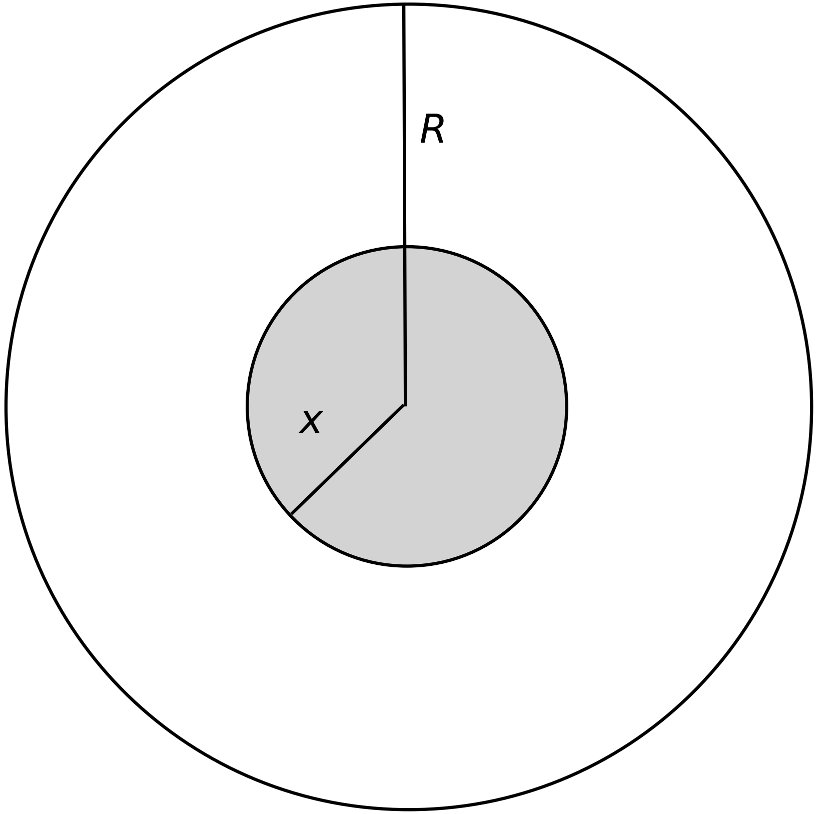

Remember: $\frac{\Delta}{\Delta x} \cosh(x) = \sinh(x)$.

Measure of the inner ball

Let $0 \leq x \leq R$.

Question: What is the measure of the grey area?

Lemma: $\mu(B_0(x)) = e^{-\alpha(R-x)}(1 + o(1))$

Proof. Just compute the probability measure for the integral from 0 to x.

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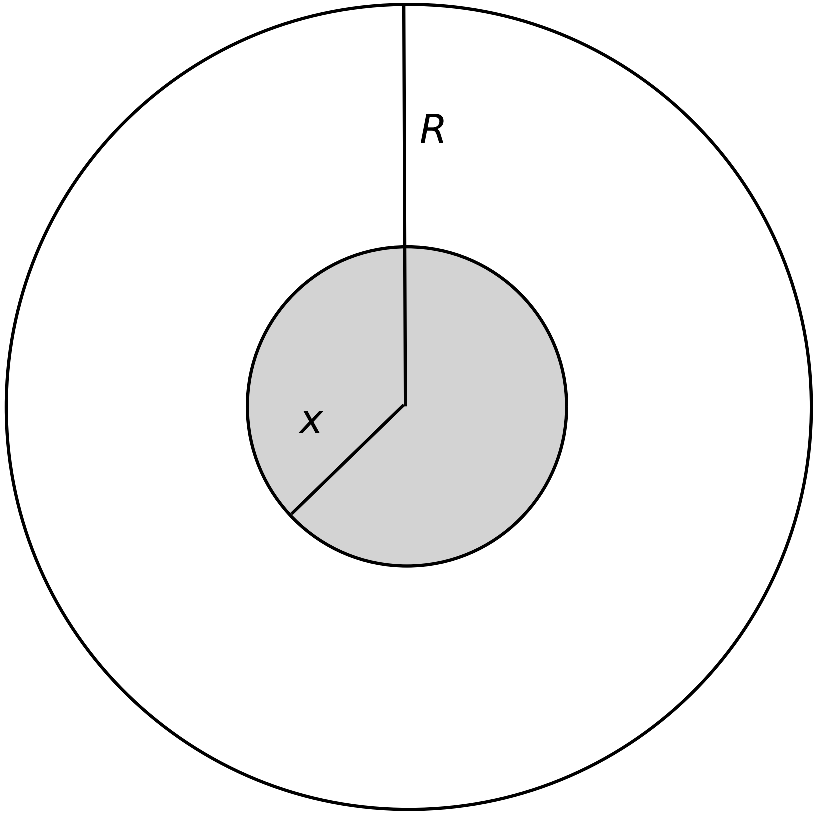

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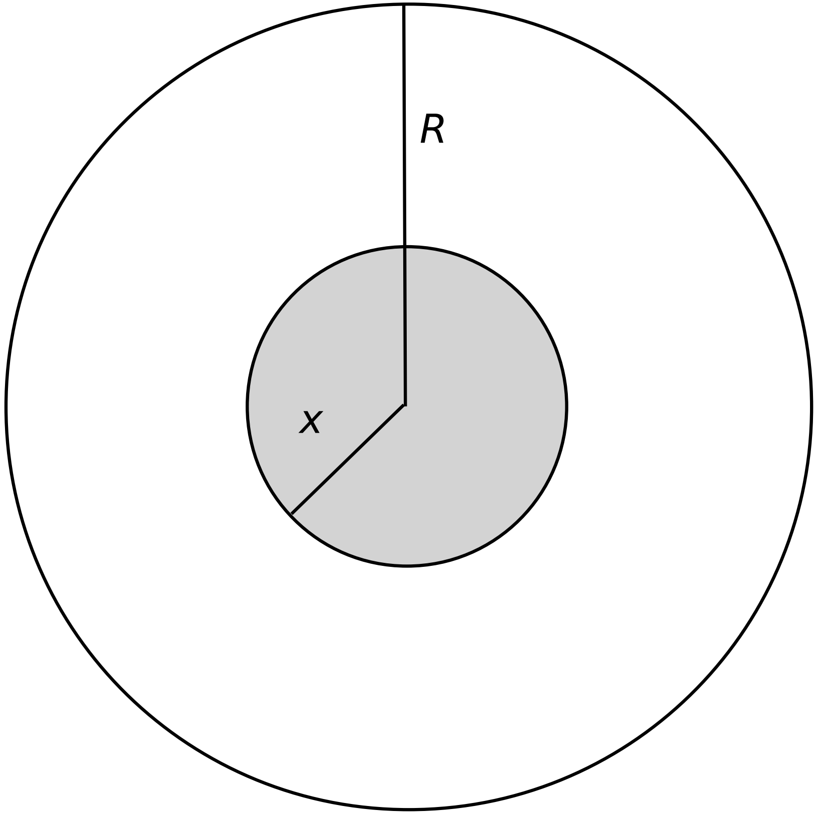

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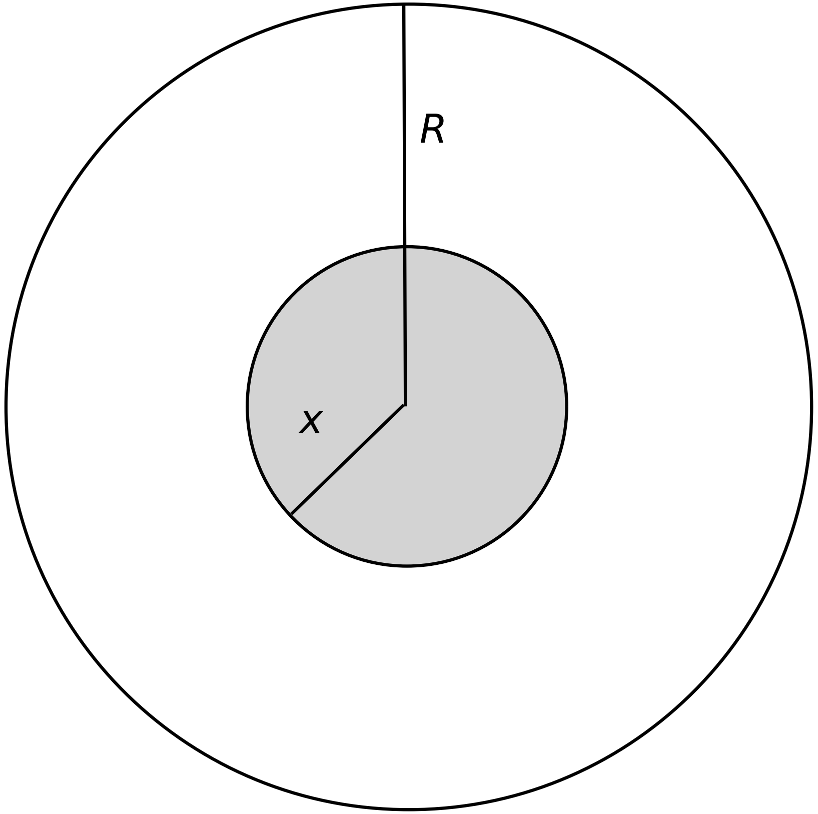

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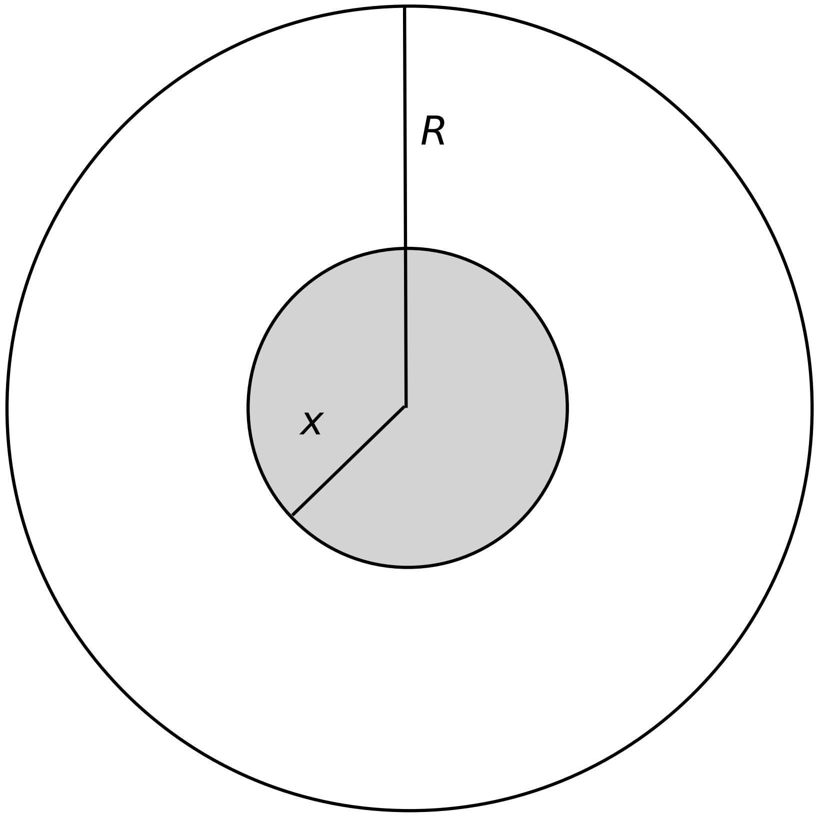

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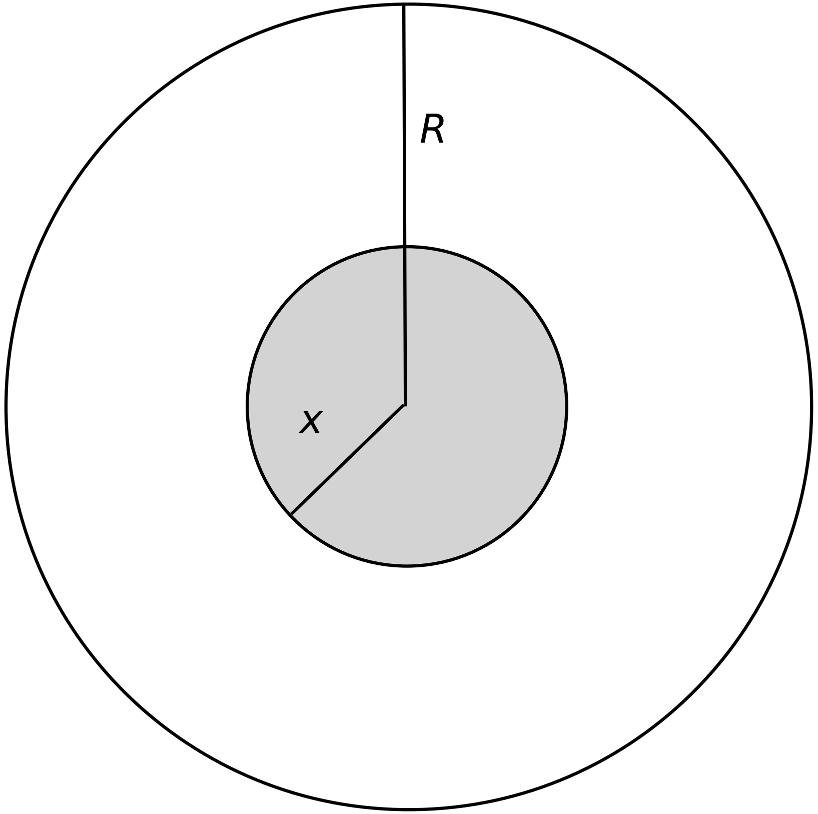

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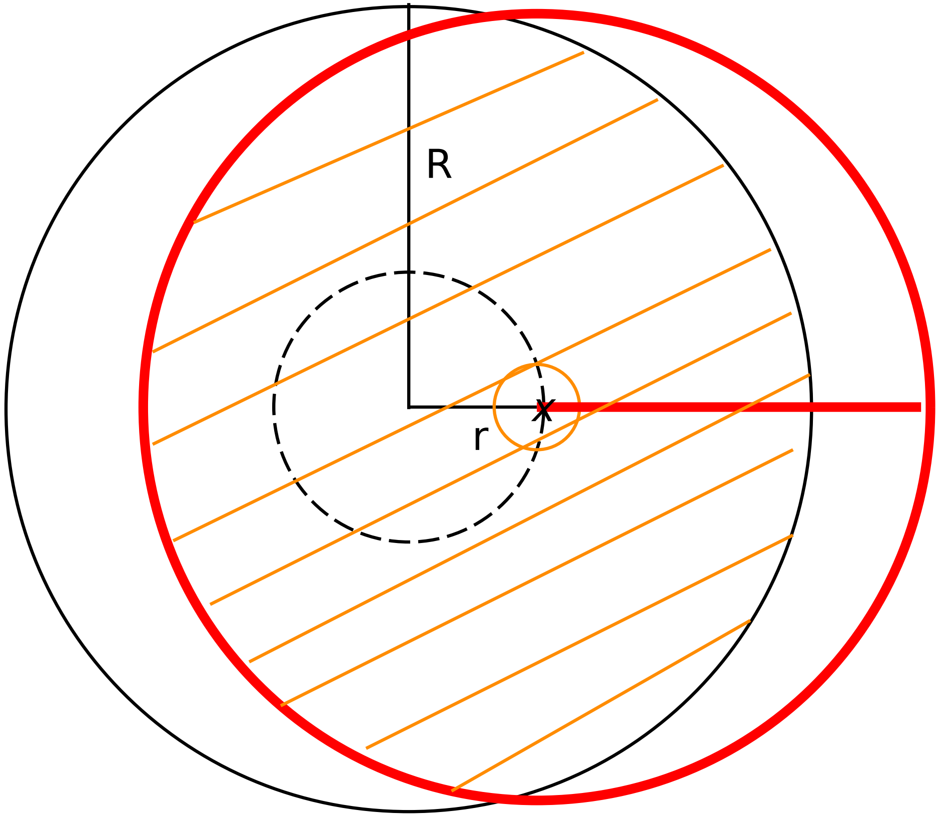

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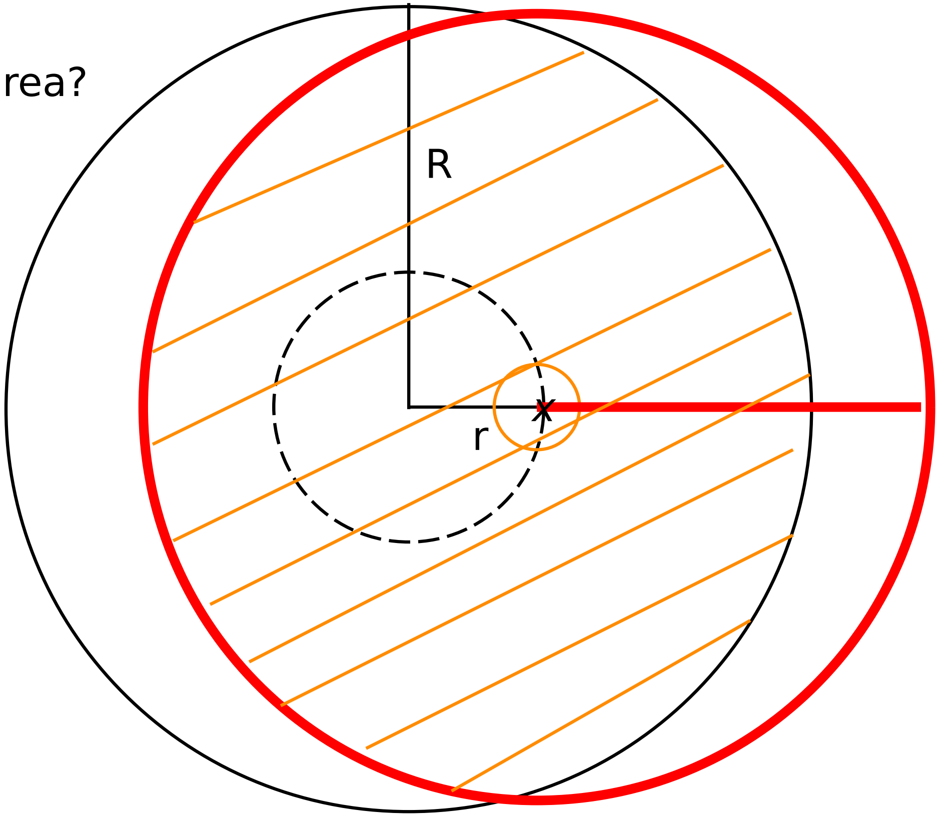

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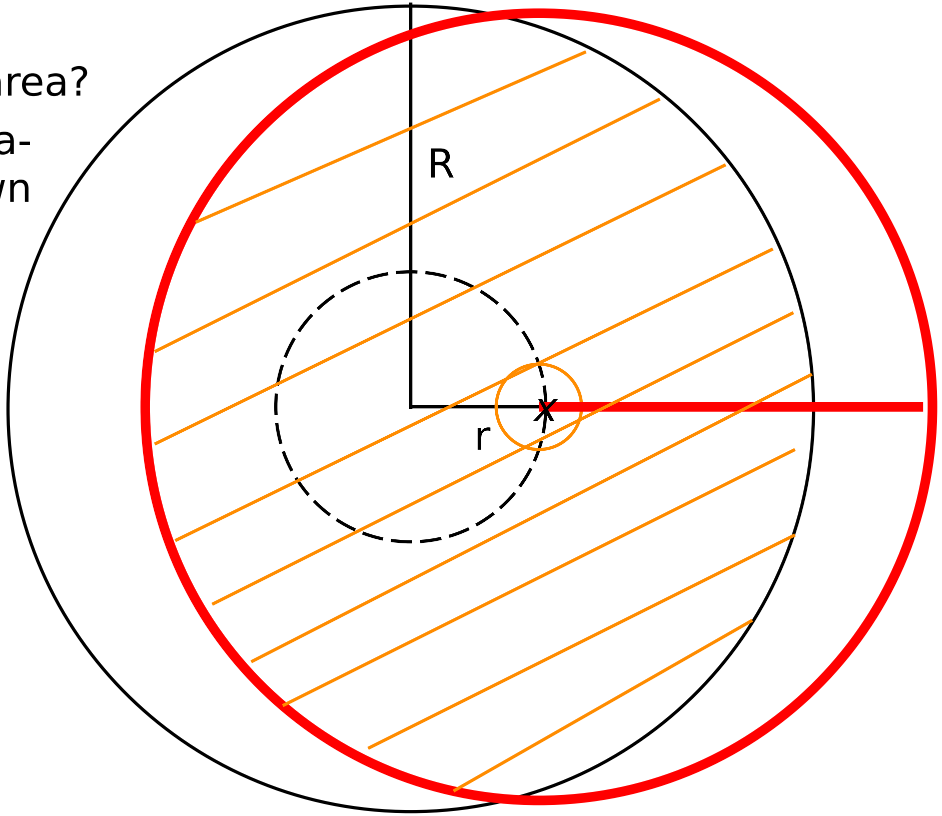

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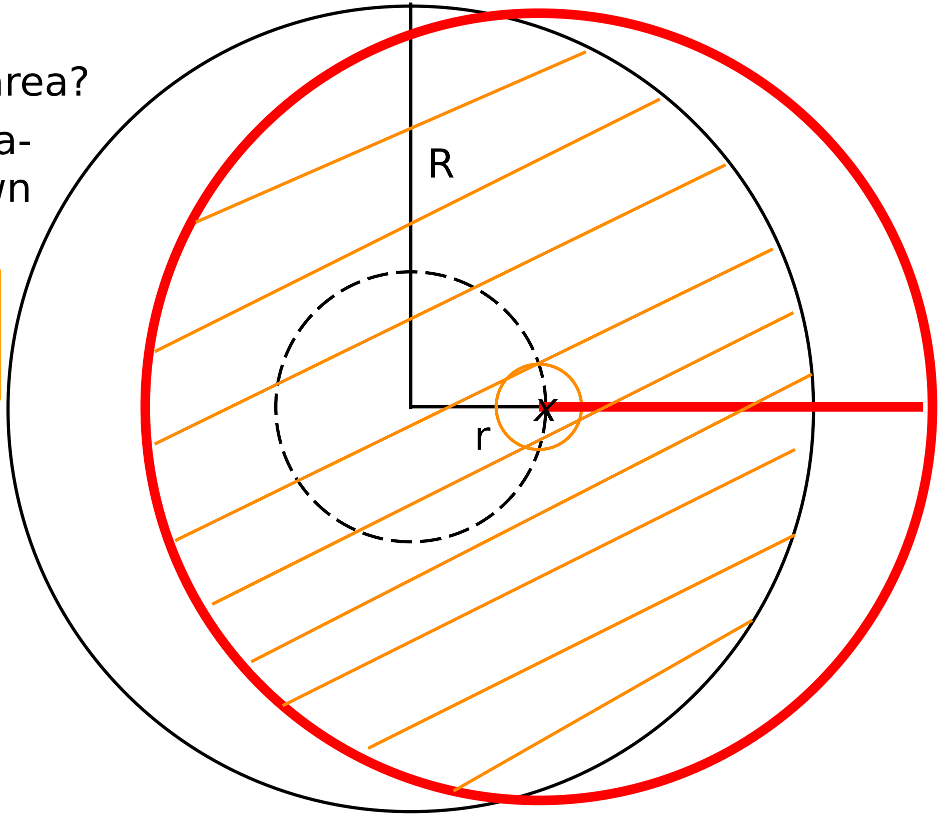

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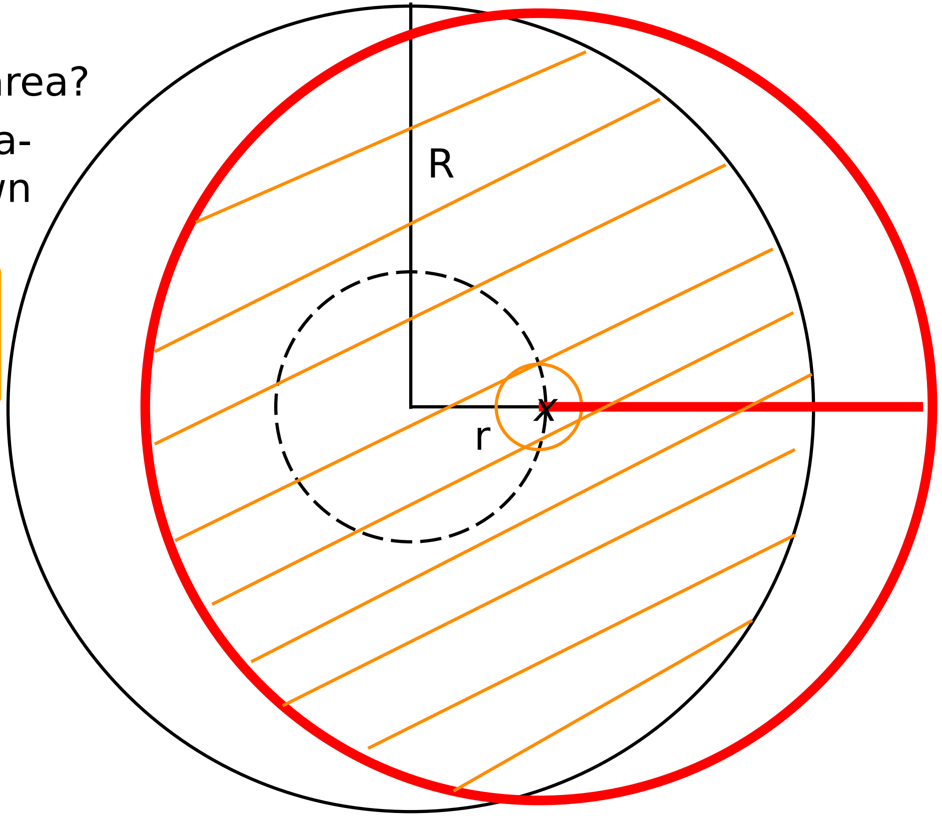

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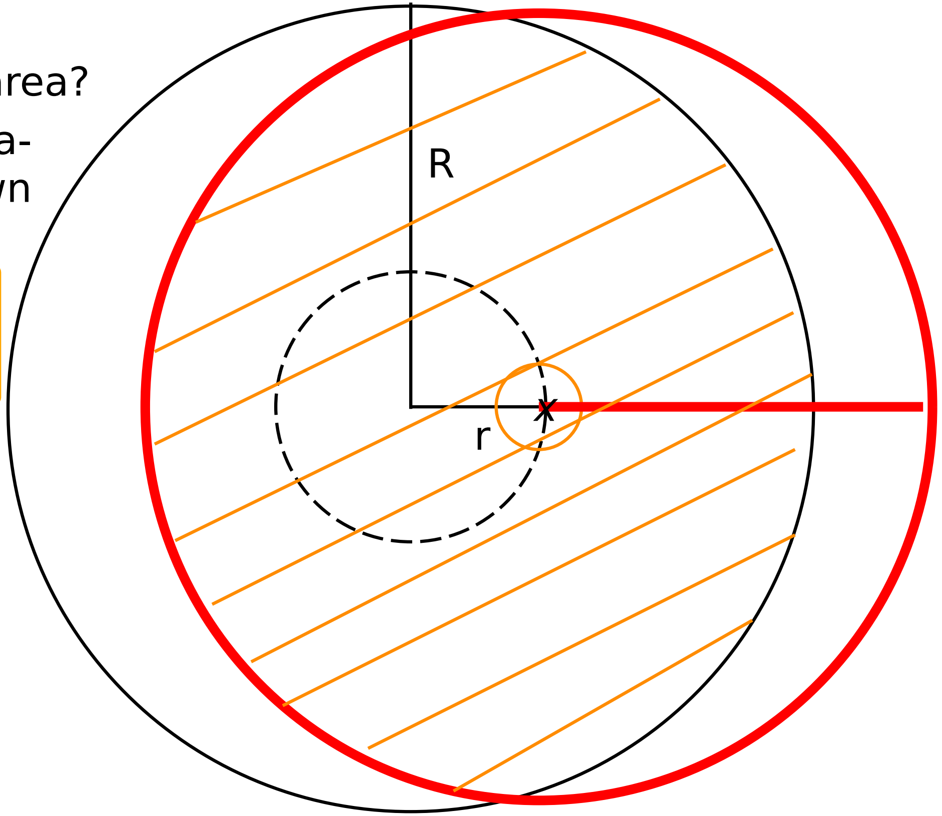

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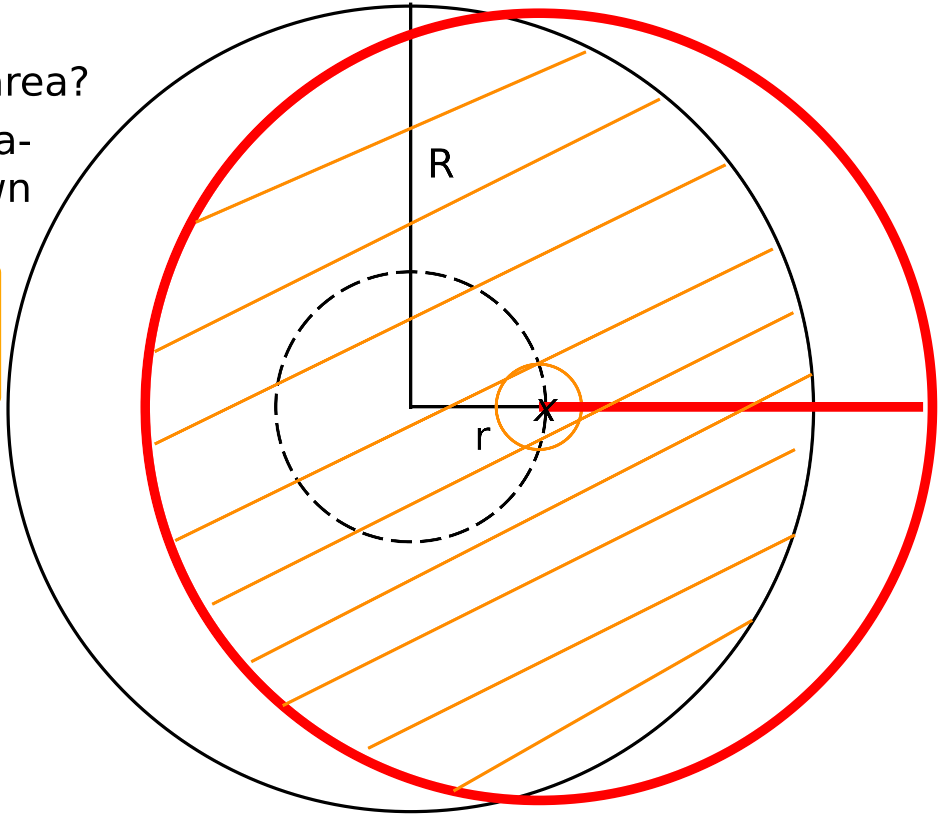

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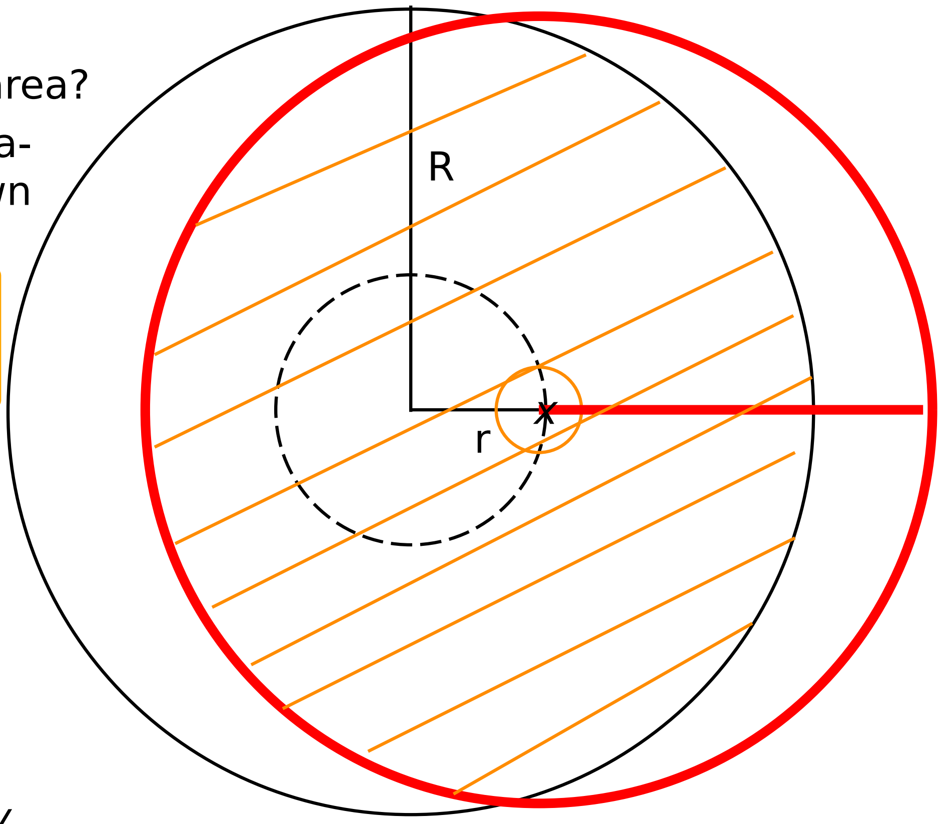

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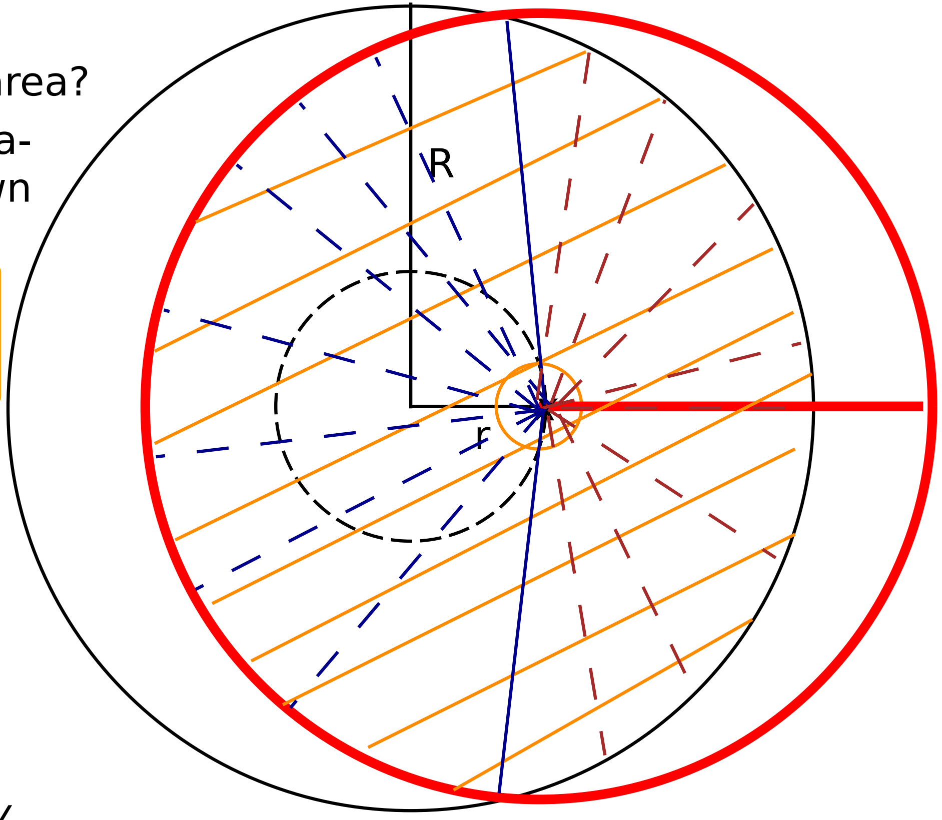

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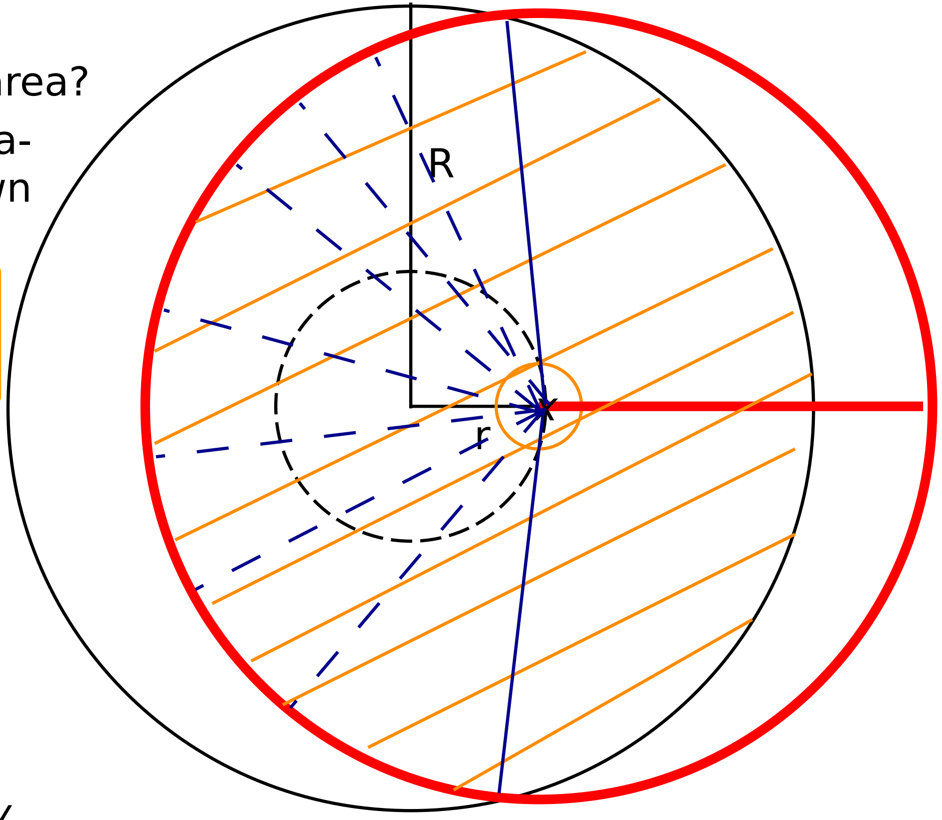

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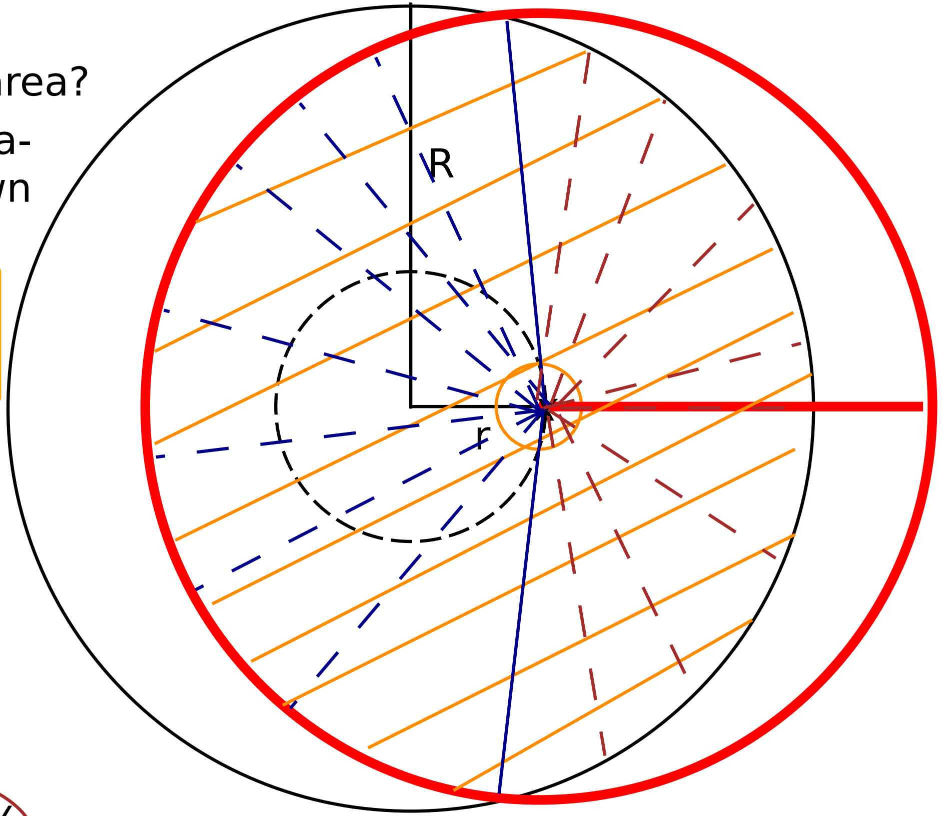

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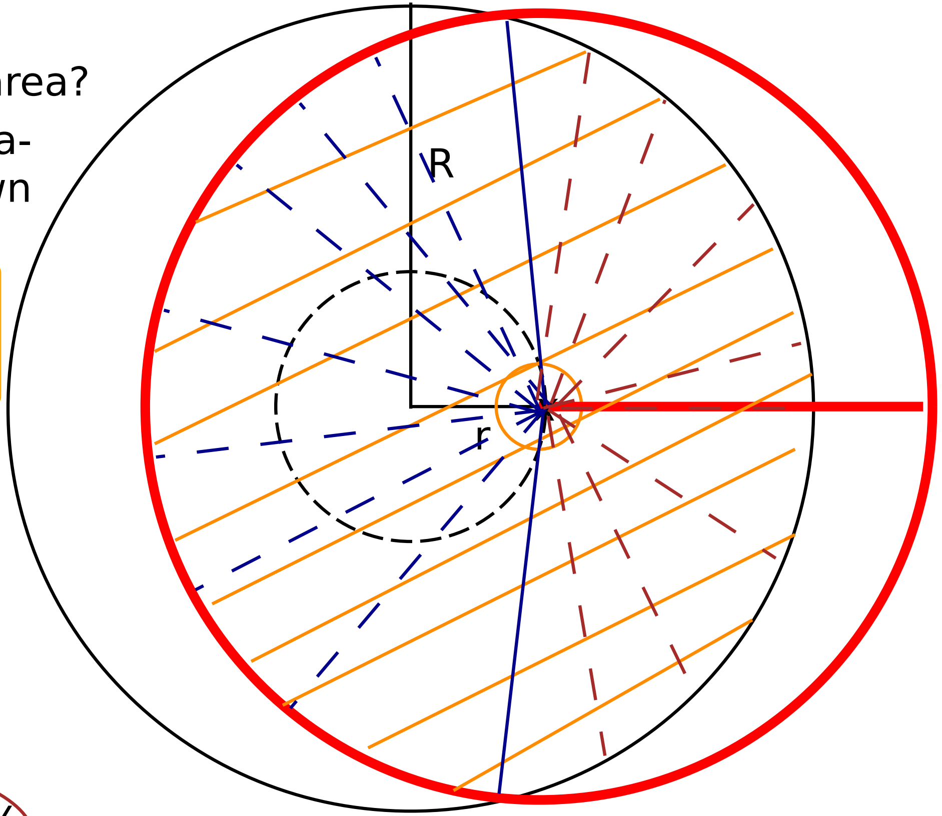

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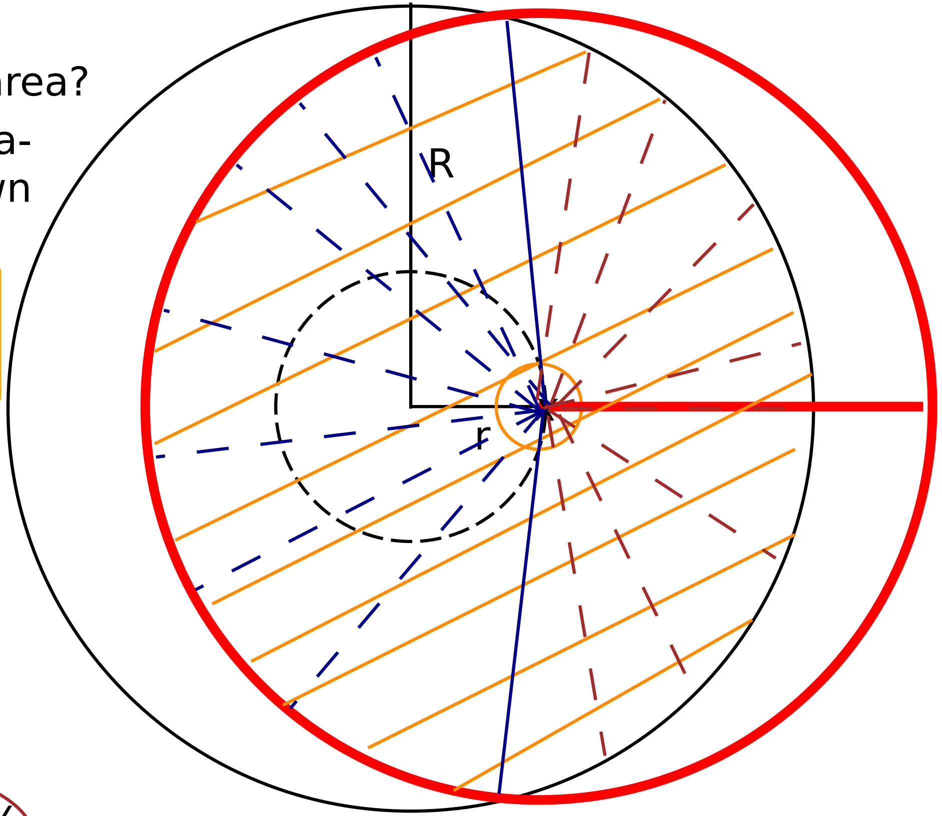

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For the details check the paper! For refinements of the lemma check out the cheat sheet!

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$$\mu((B_r(R) \cap B_o(R)) - B_O(x)) = \frac{2\alpha e^{-r/2}}{\pi(\alpha - \frac{1}{2})}(1 \pm \mathcal{O}(e^{-(\alpha-\frac{1}{2})r} + e^{-r}))$$

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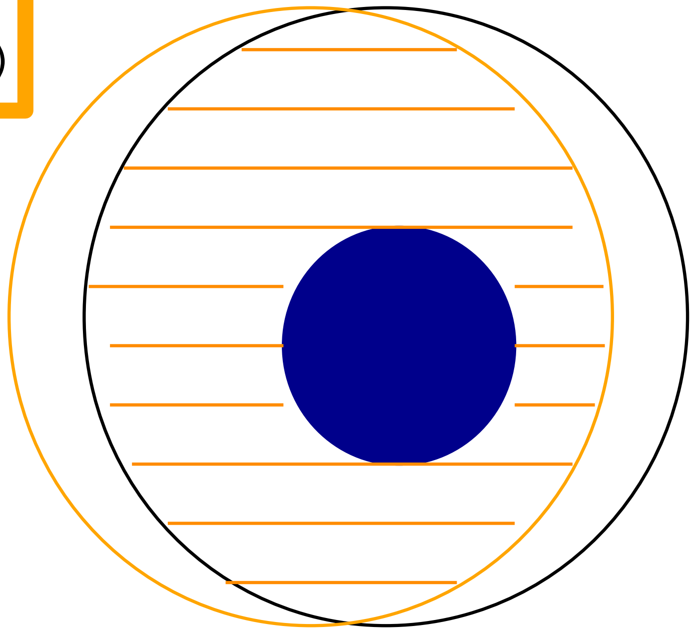

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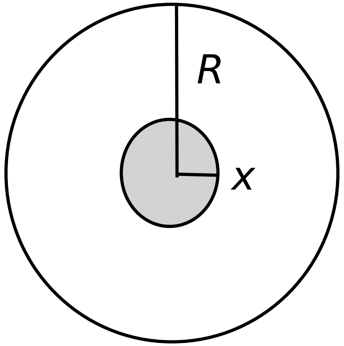

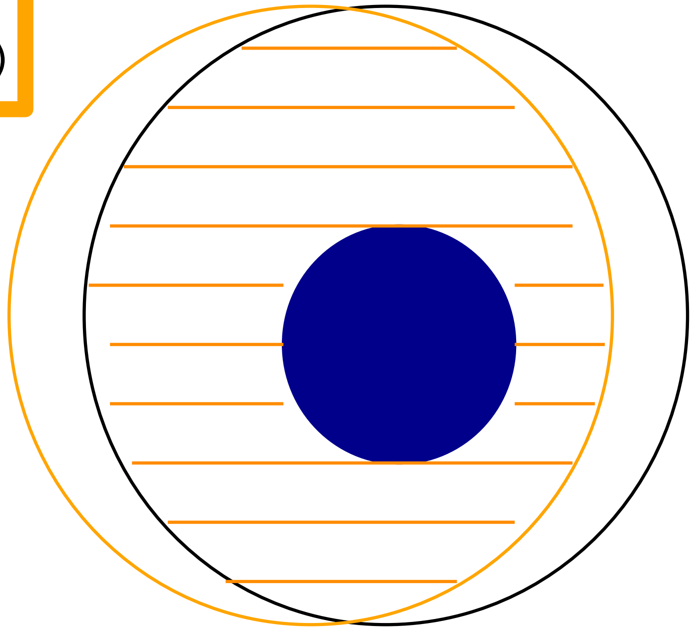

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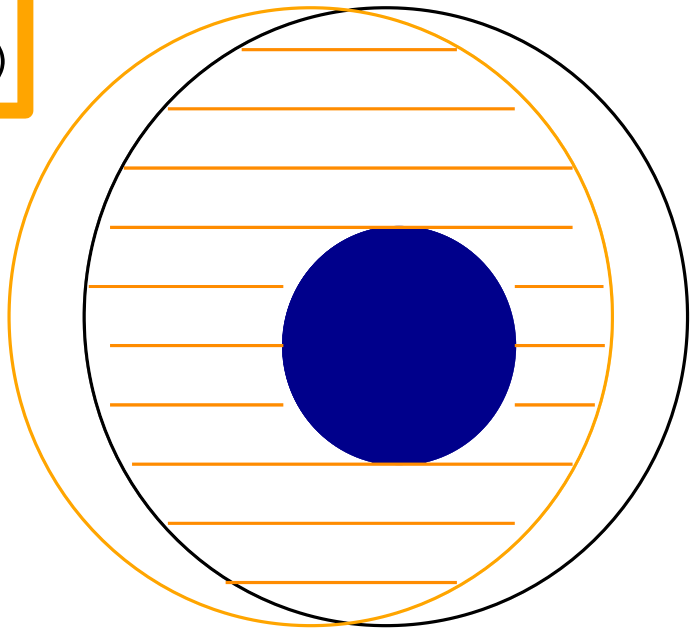

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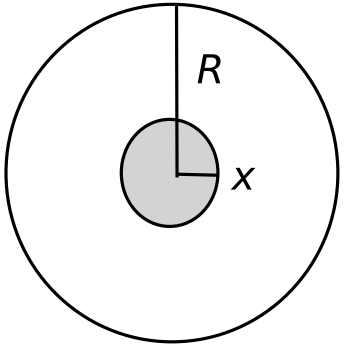

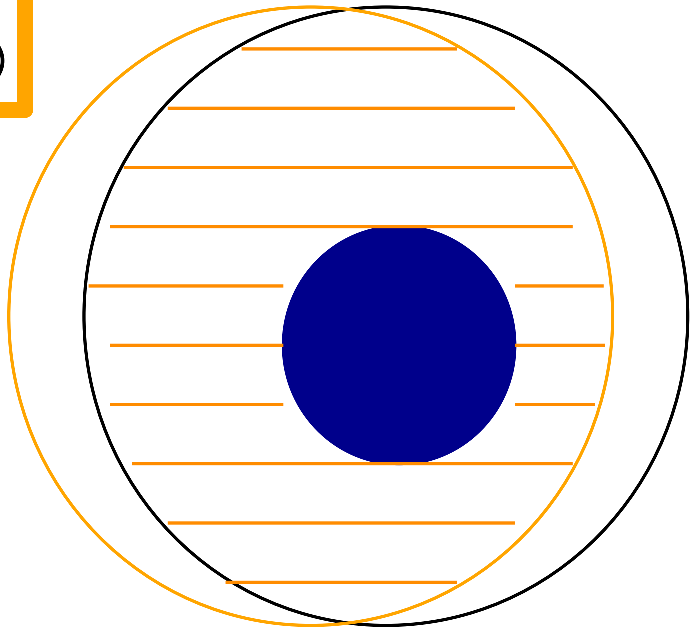

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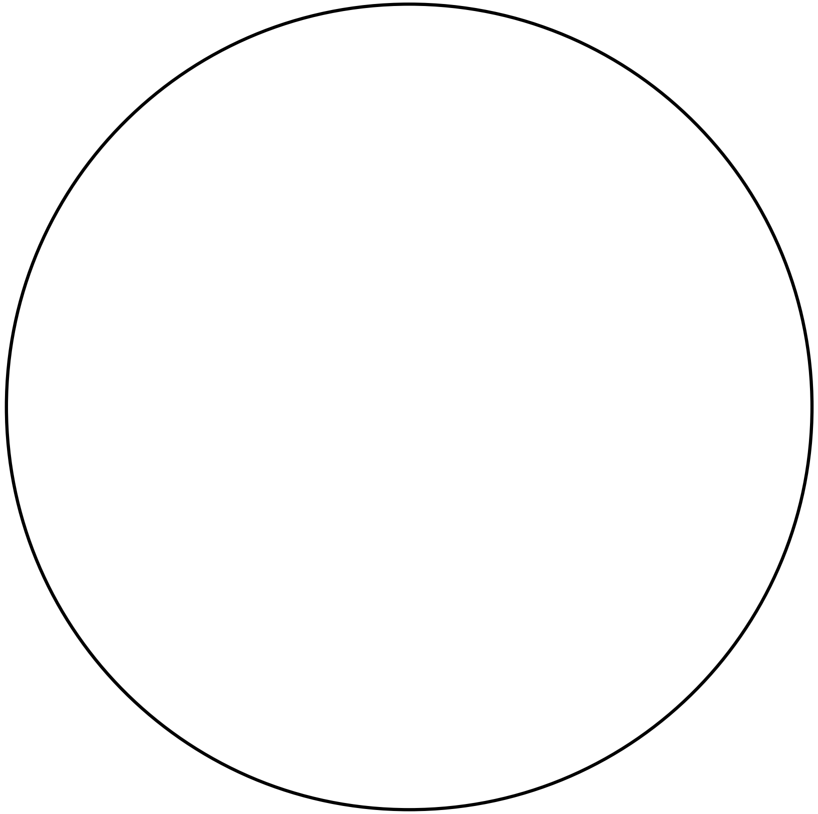

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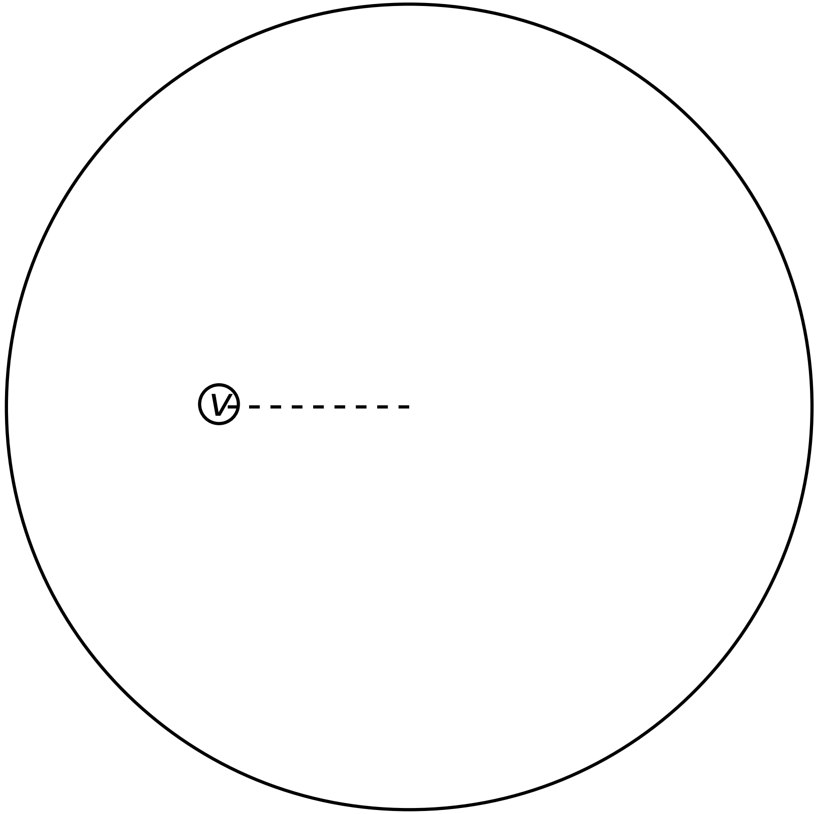

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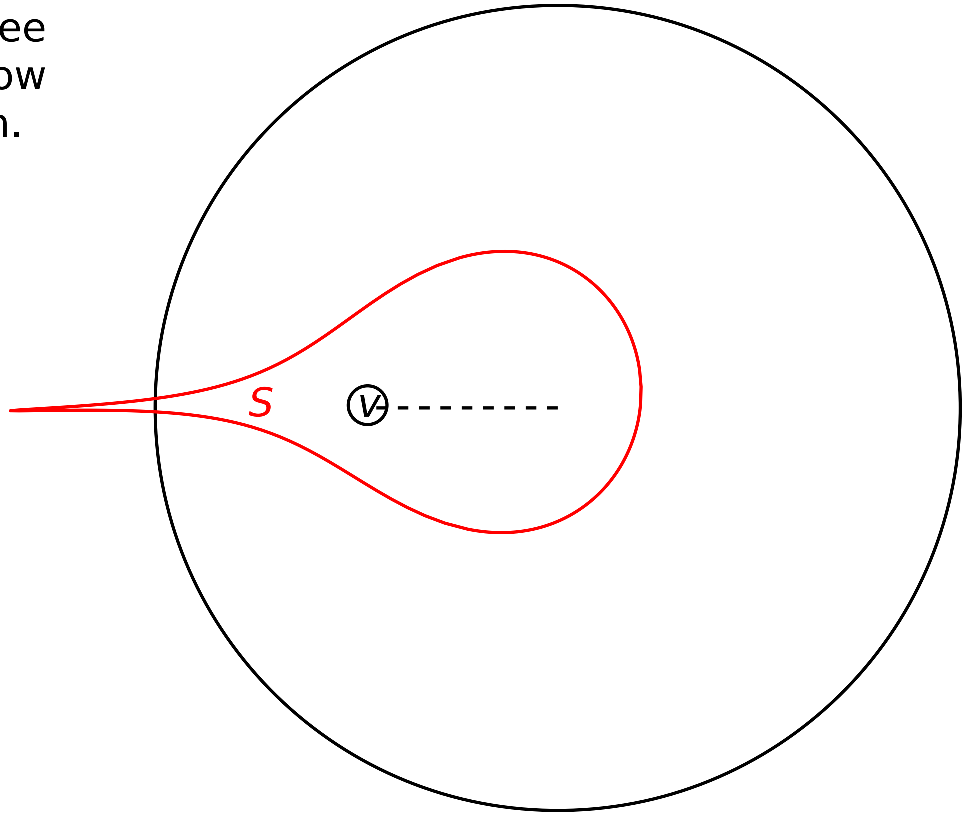
So we need to compute $\mathbb{E}[|S|] = \delta(v)$.



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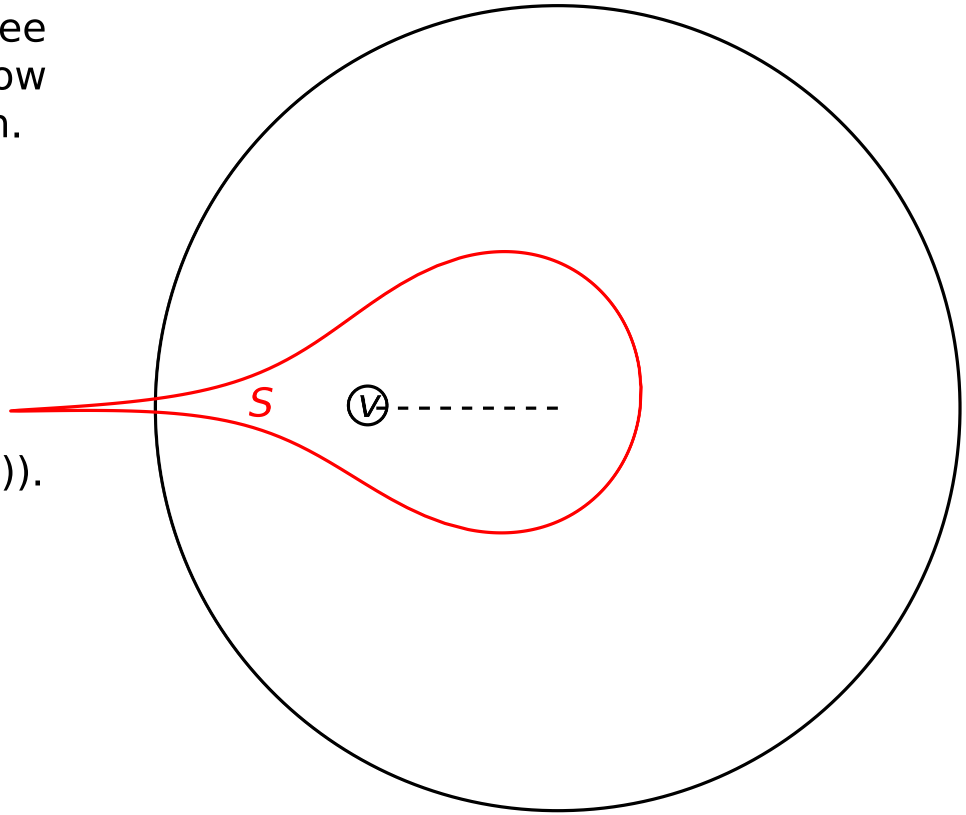

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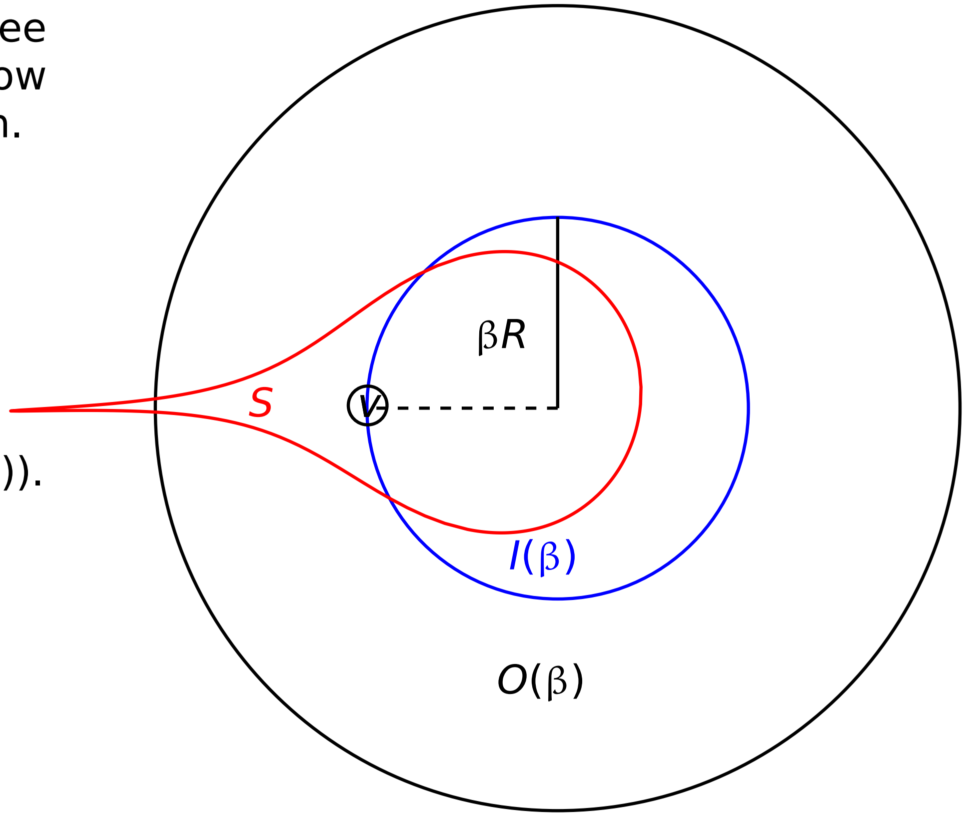

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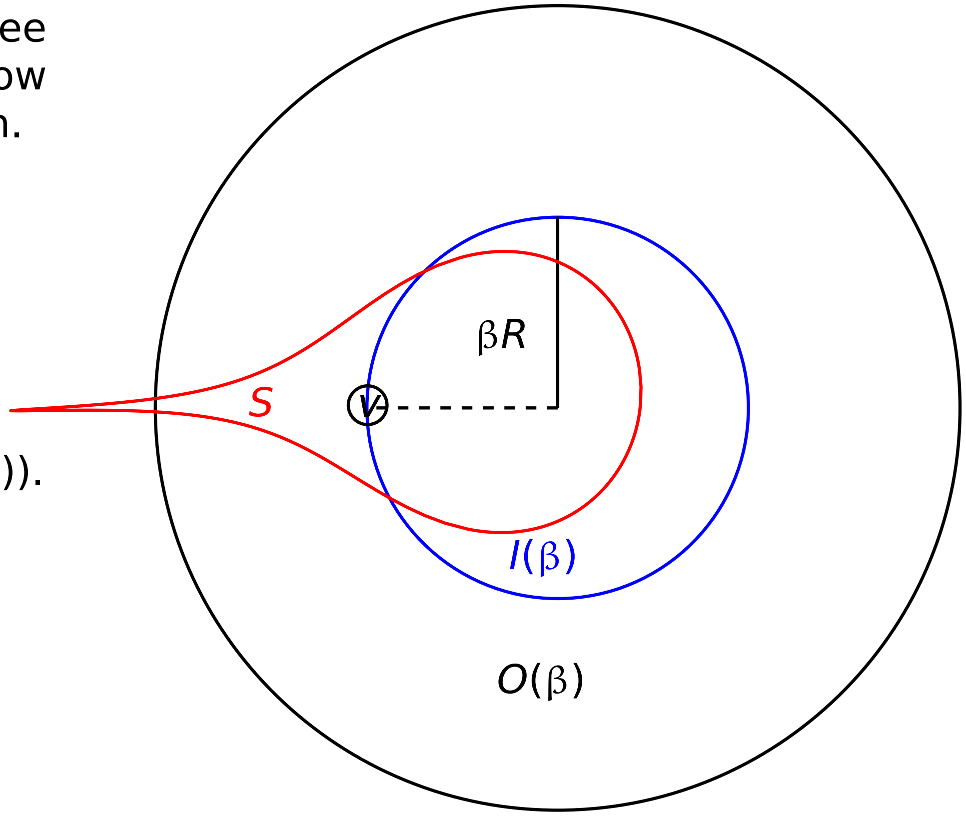

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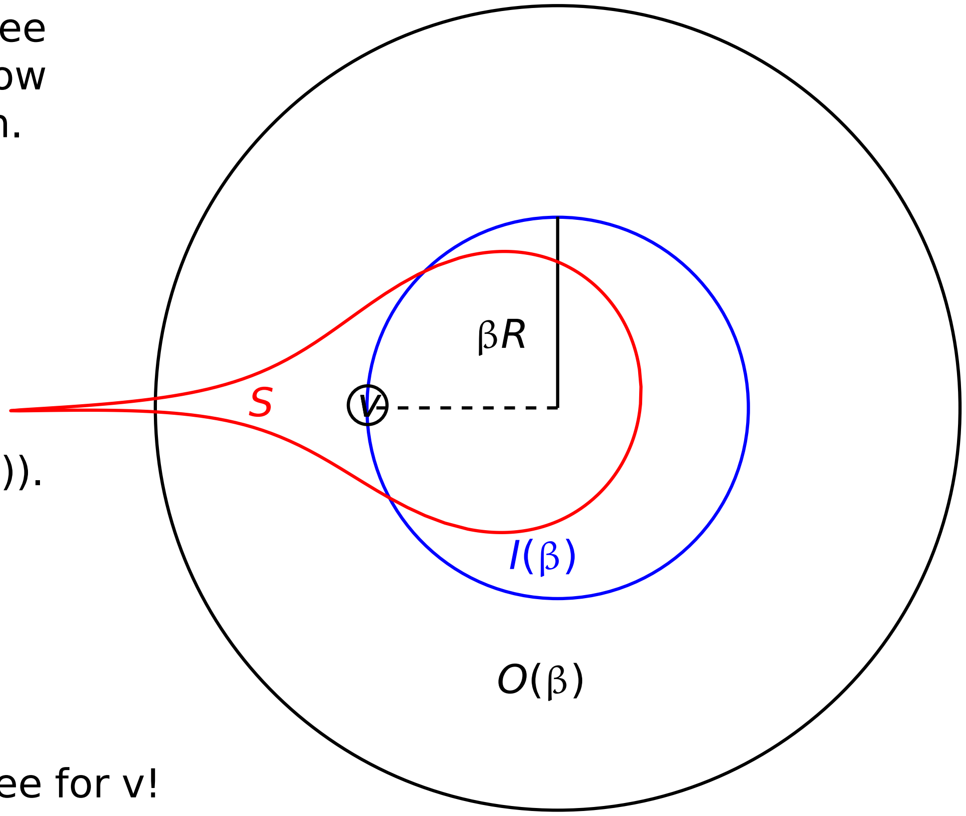

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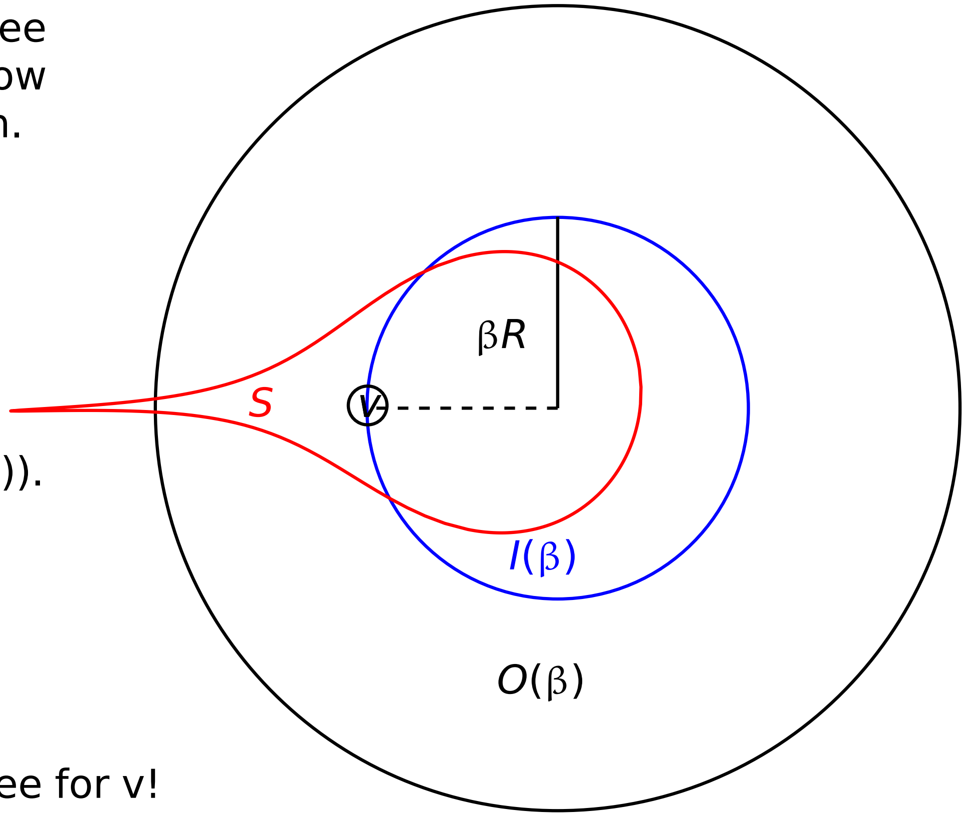

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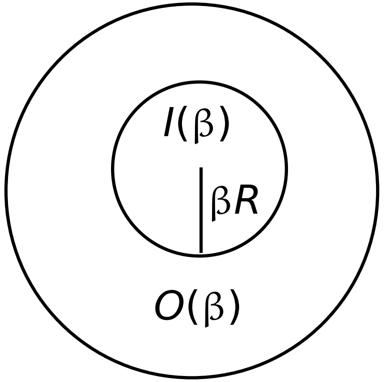

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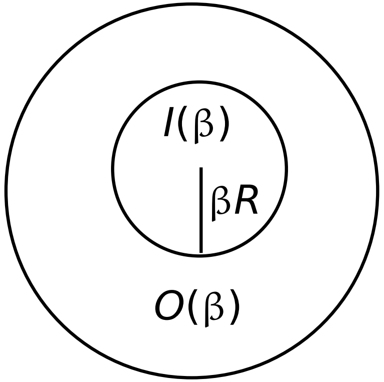

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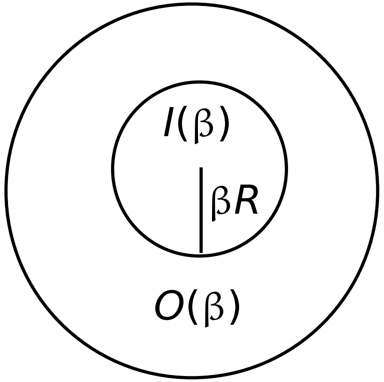

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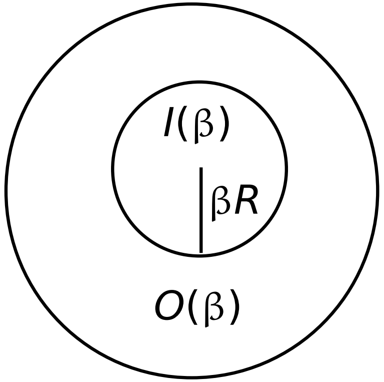

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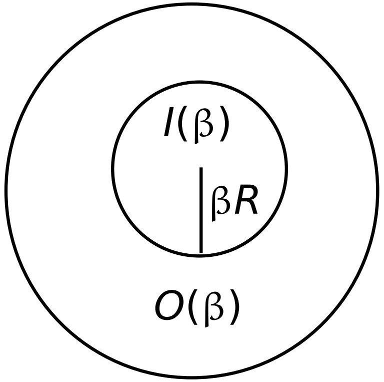

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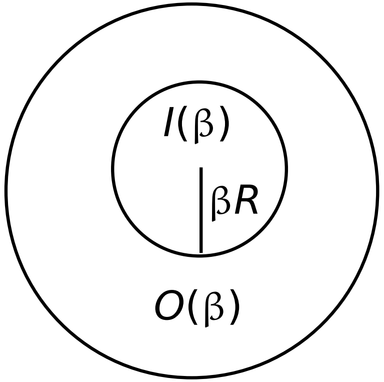

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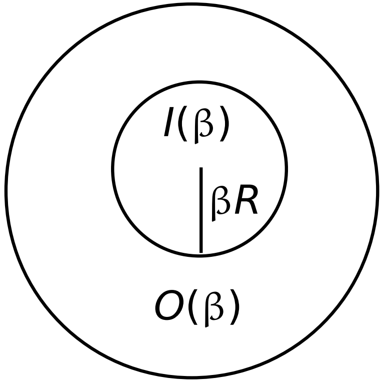

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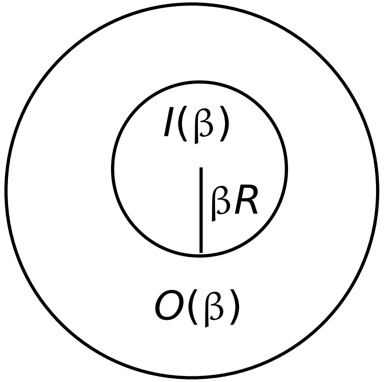

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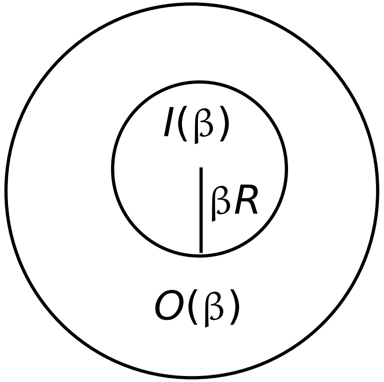

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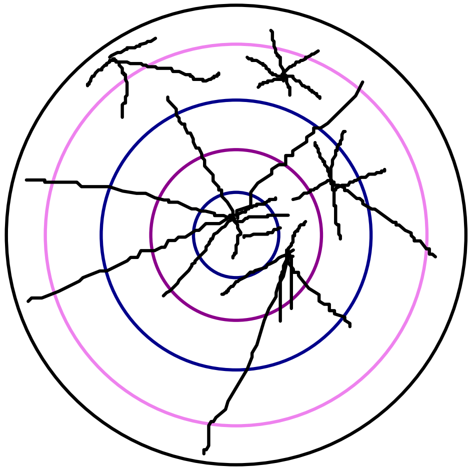

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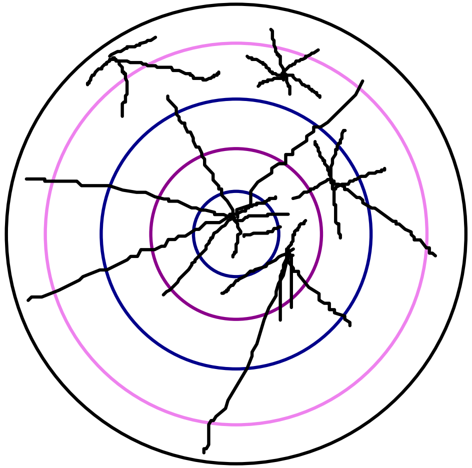

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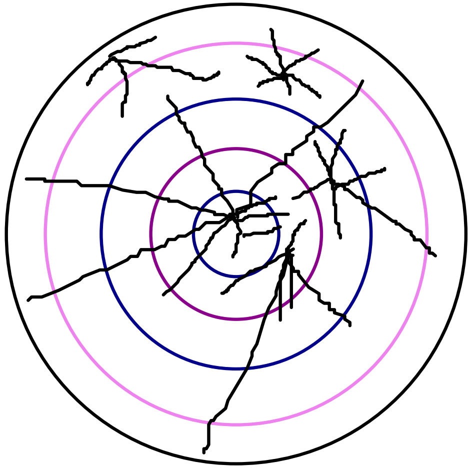

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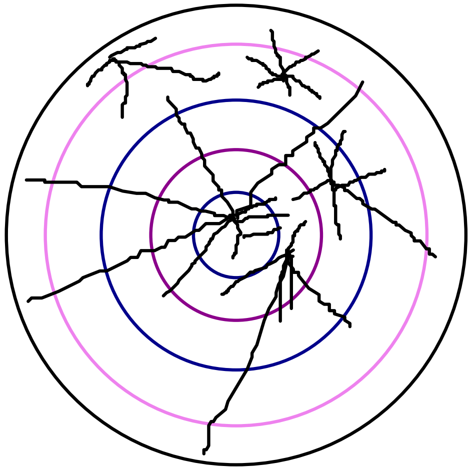

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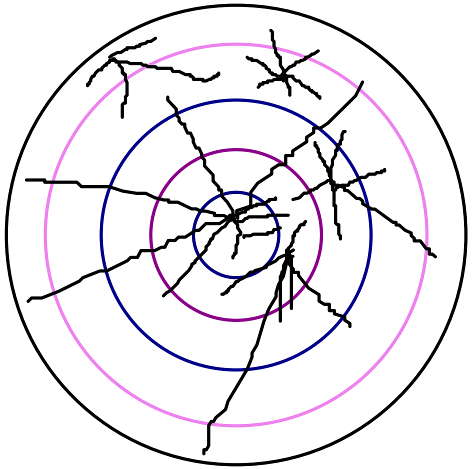

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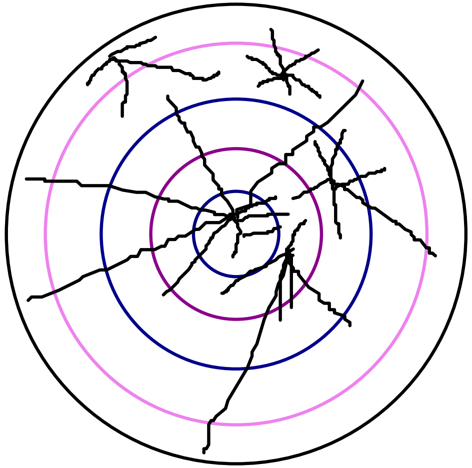

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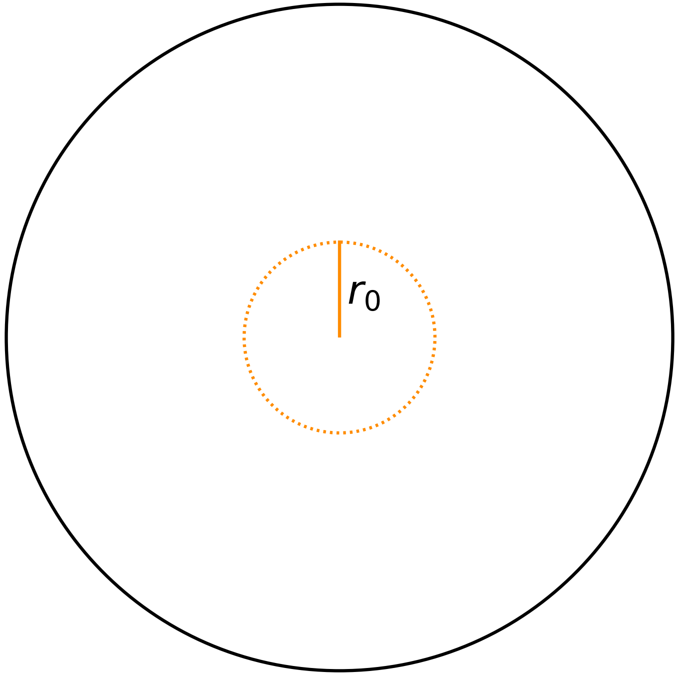

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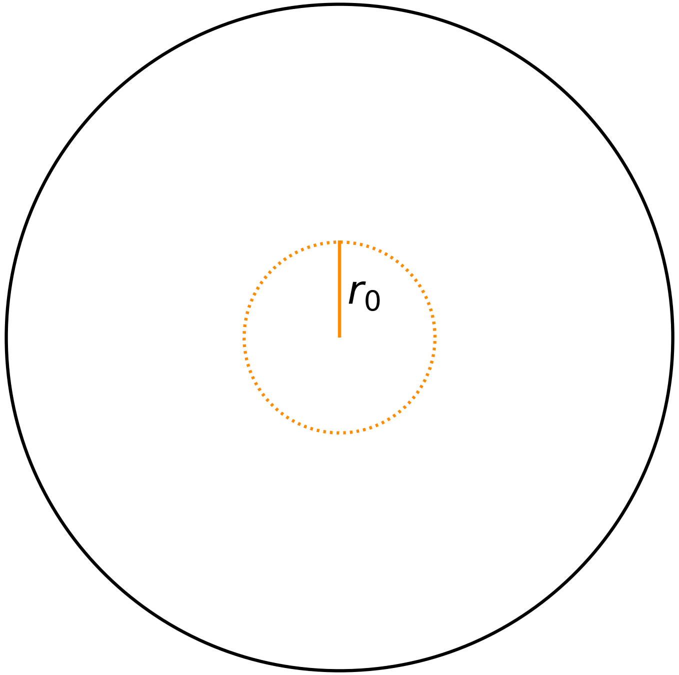

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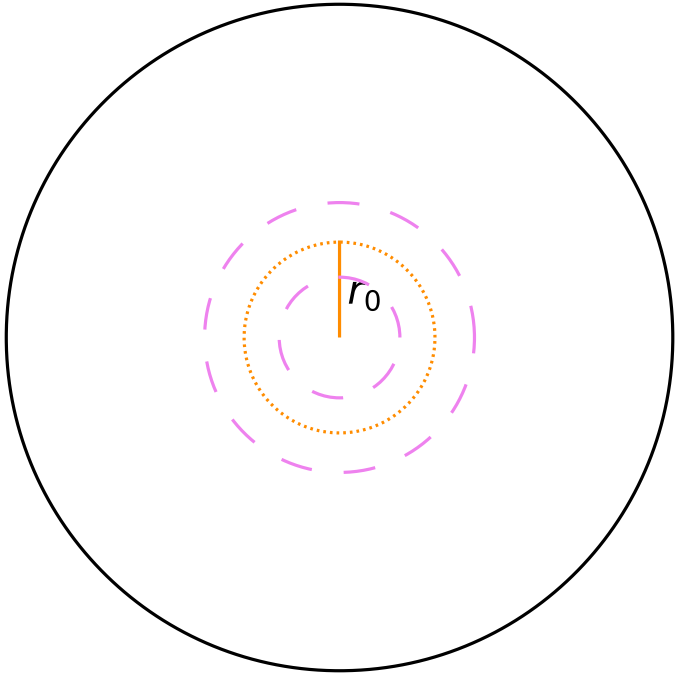

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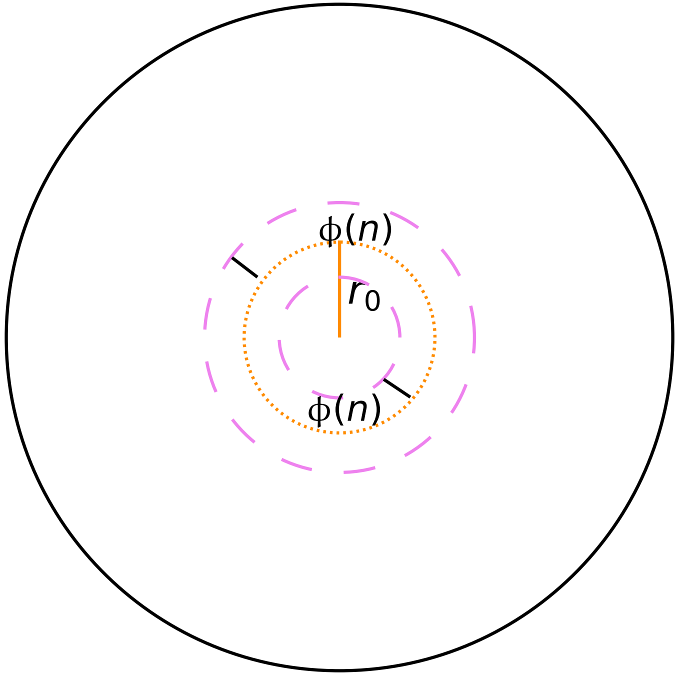

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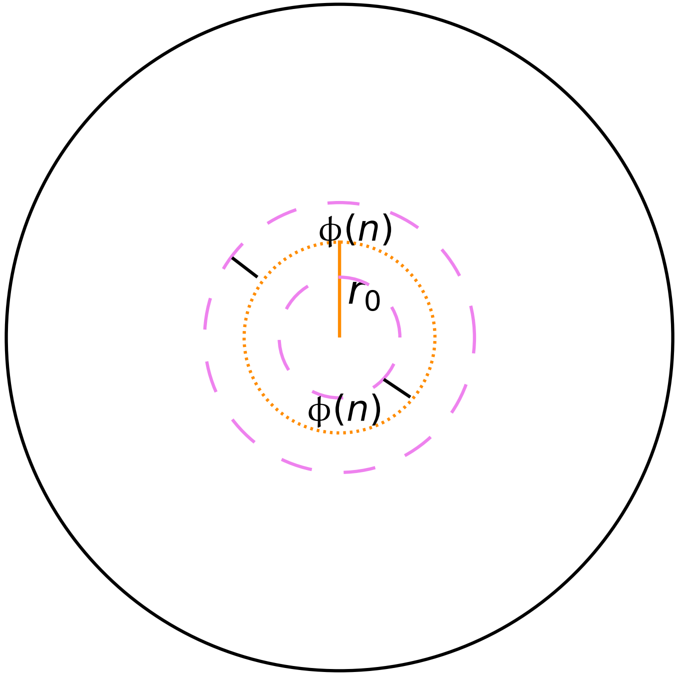

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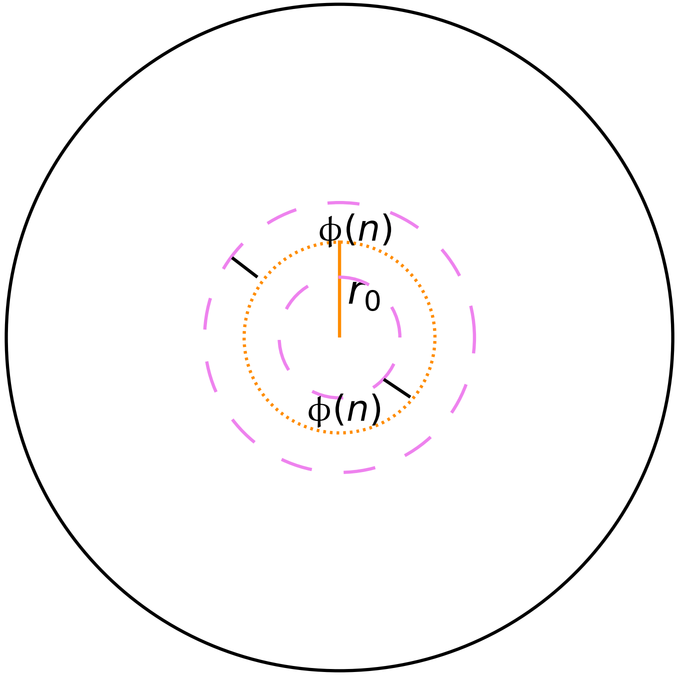

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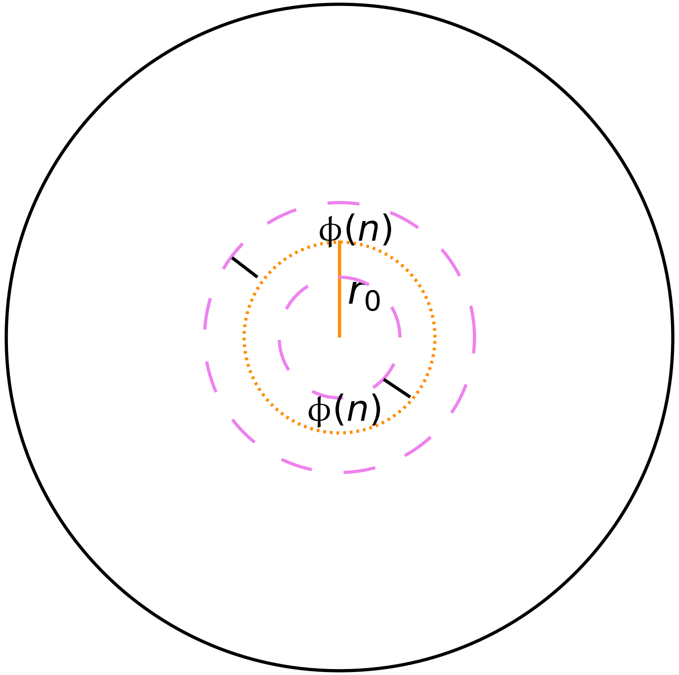

Maximum degree of a node of a HRG

Theorem: Let $\alpha > 1/2$ and $C \in \mathbb{R}$.
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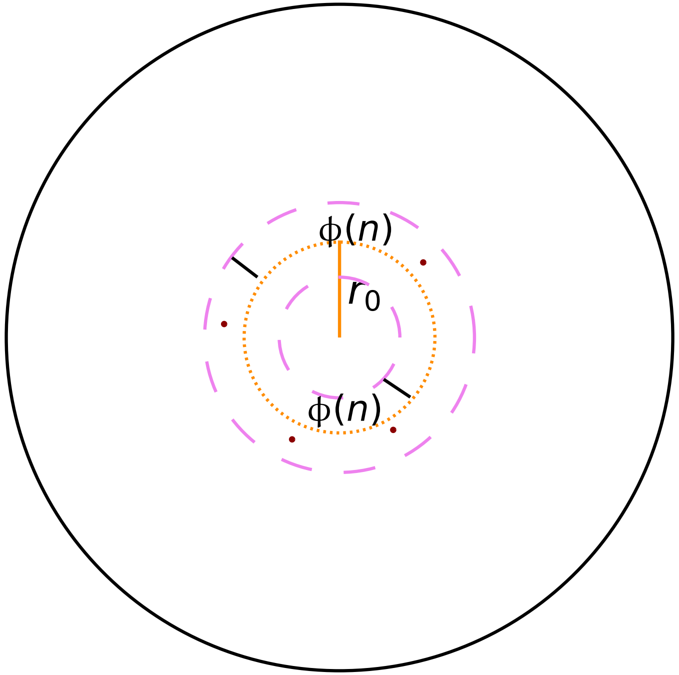

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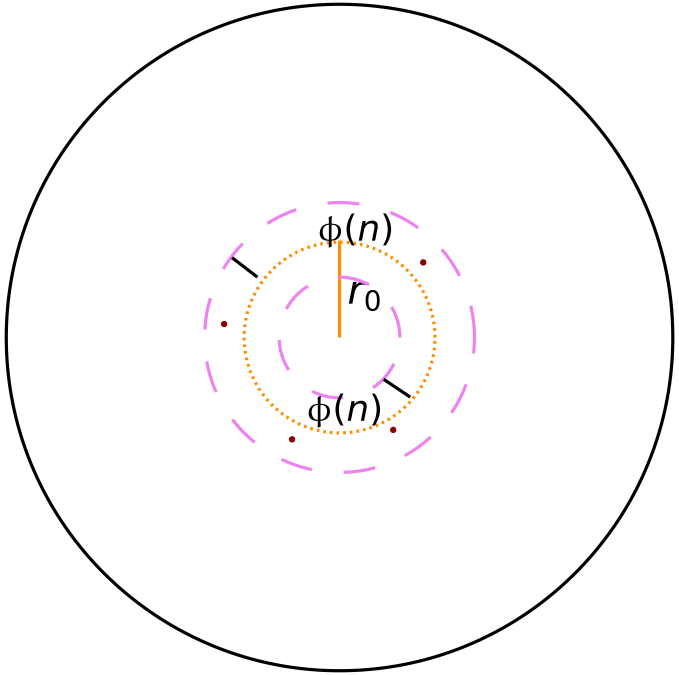

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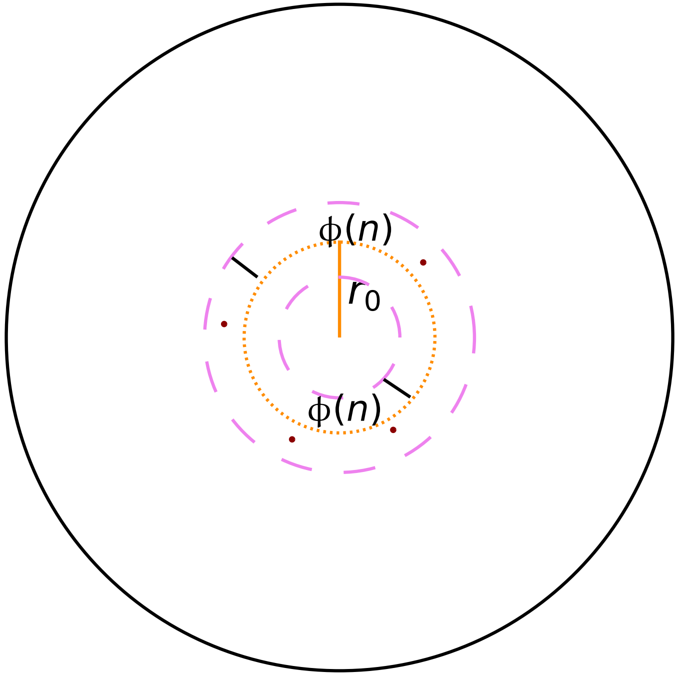

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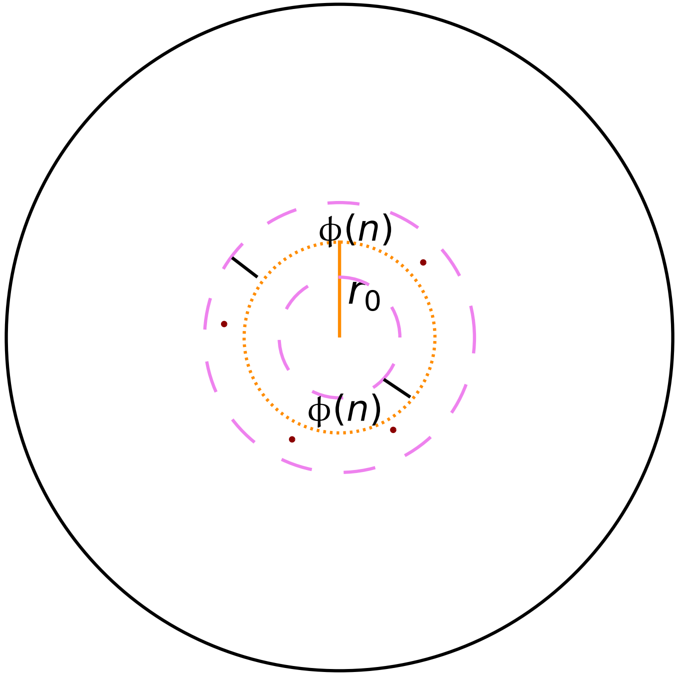

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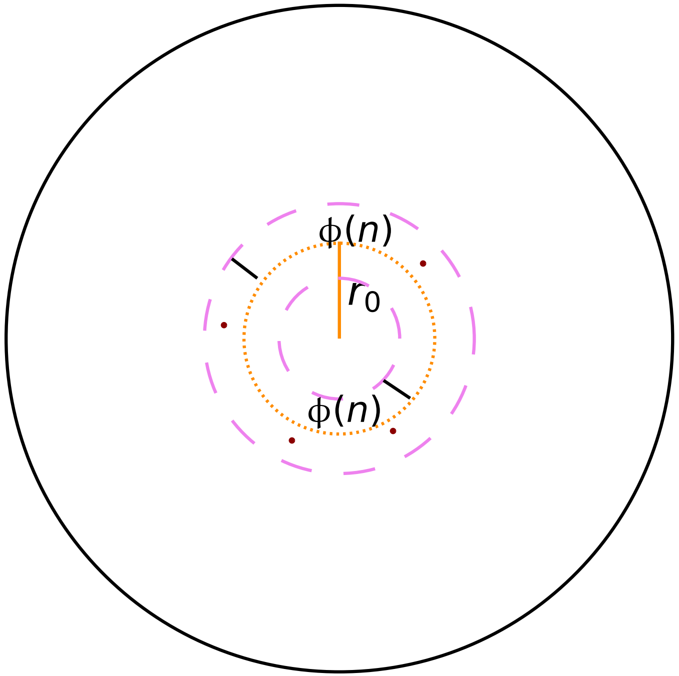

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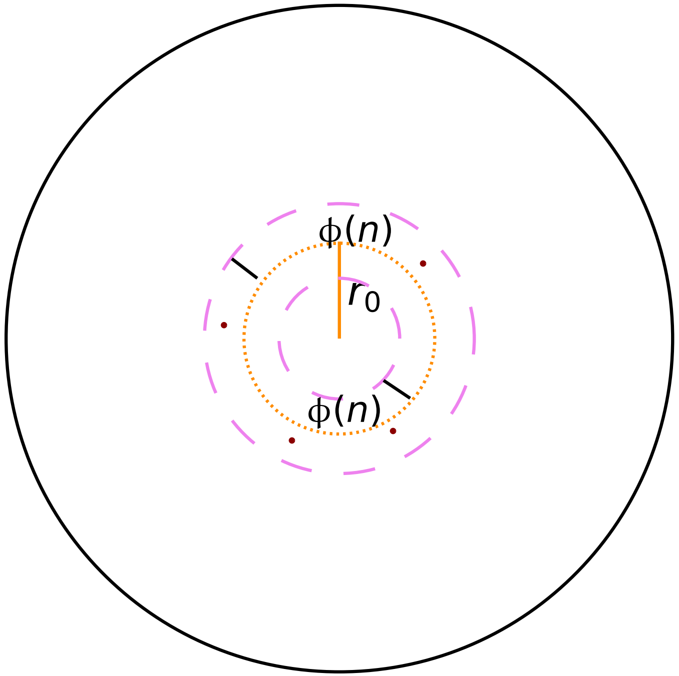

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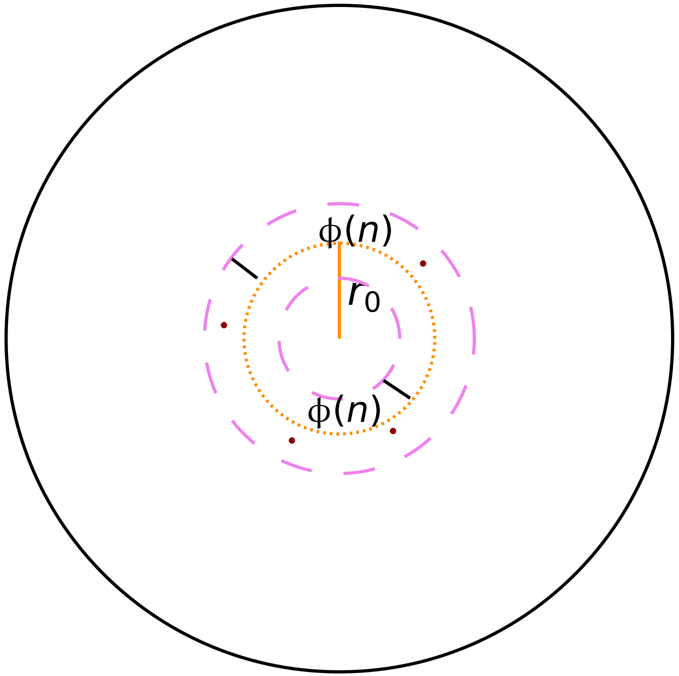

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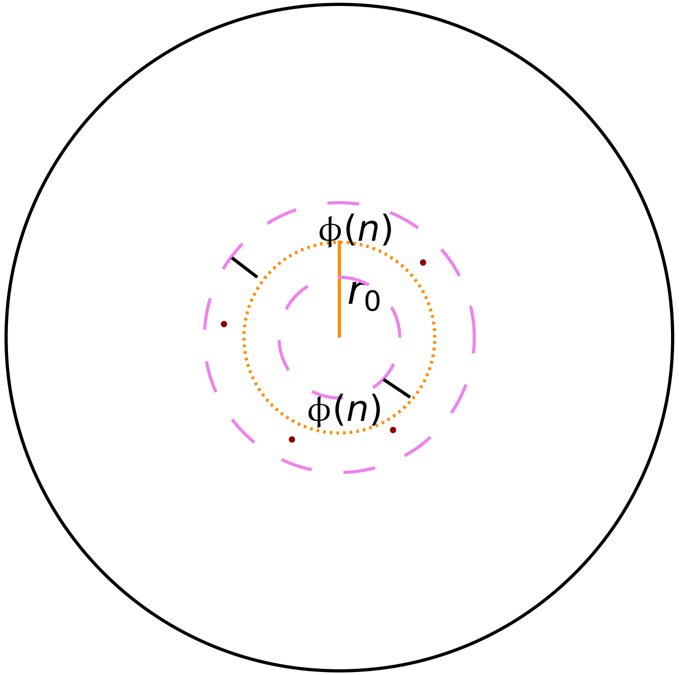

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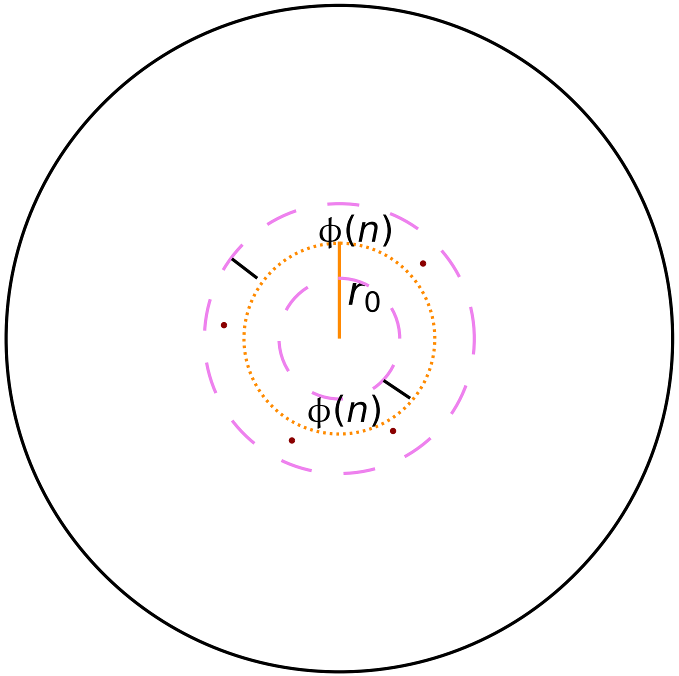

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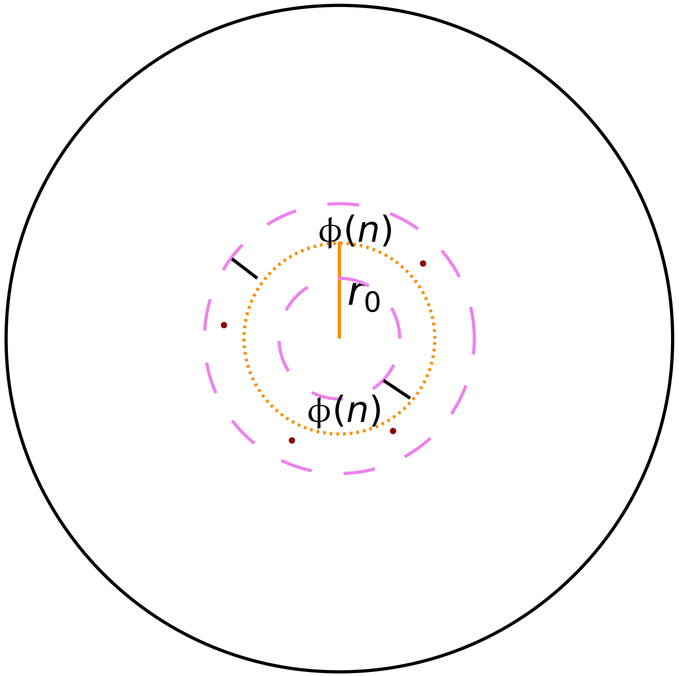

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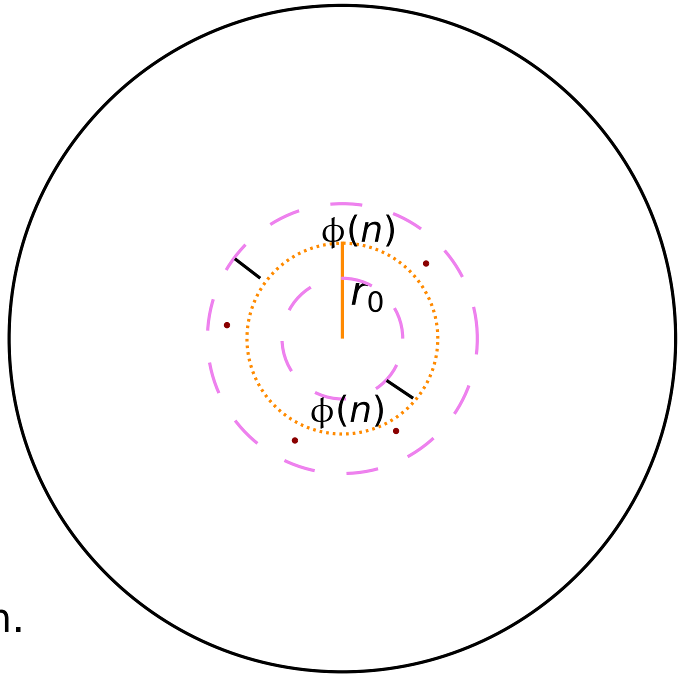

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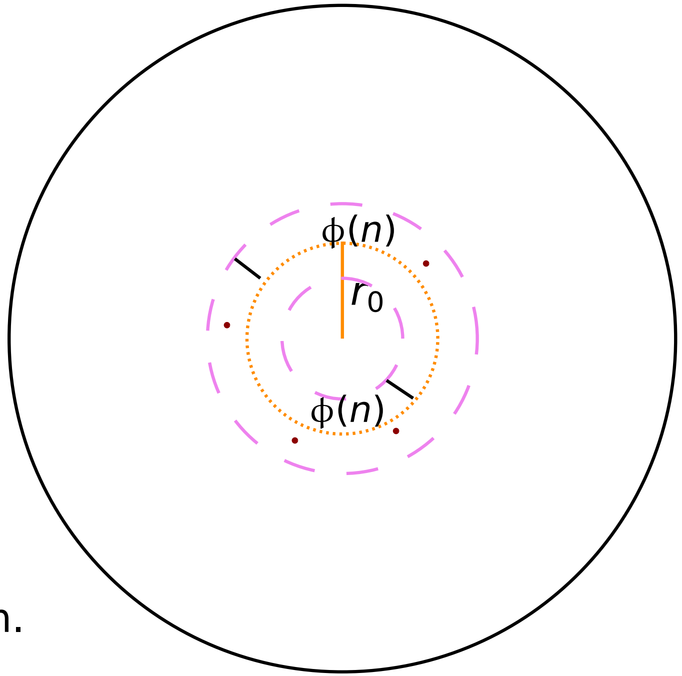

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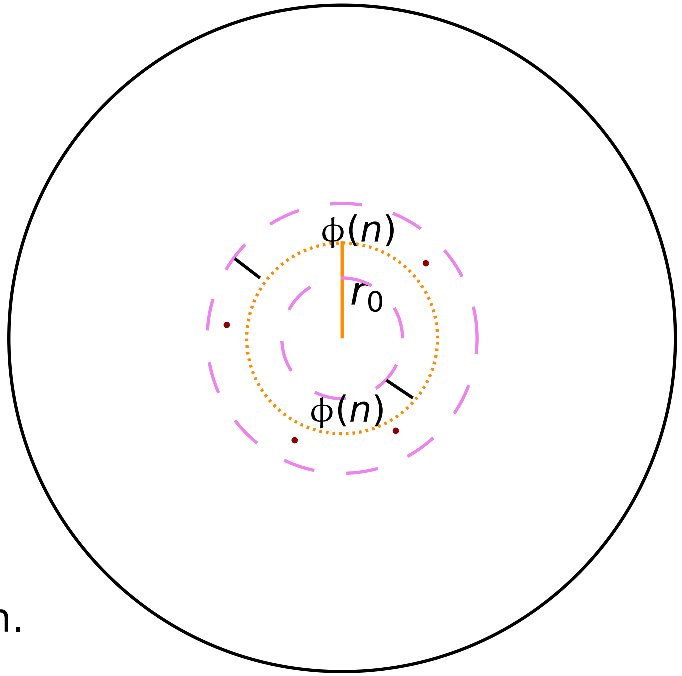

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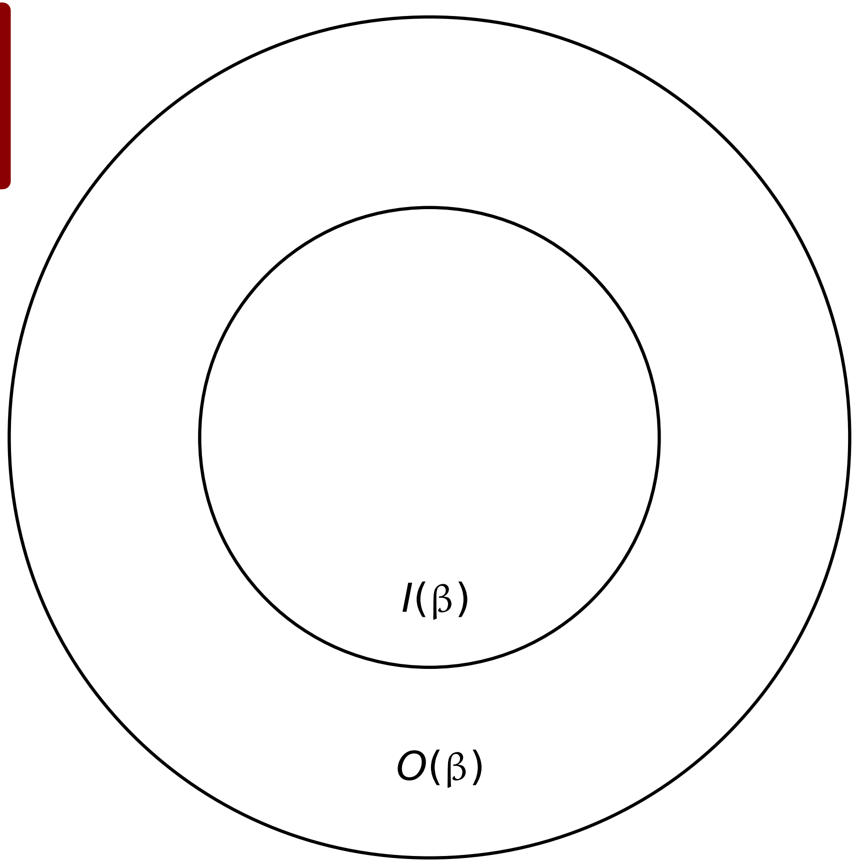

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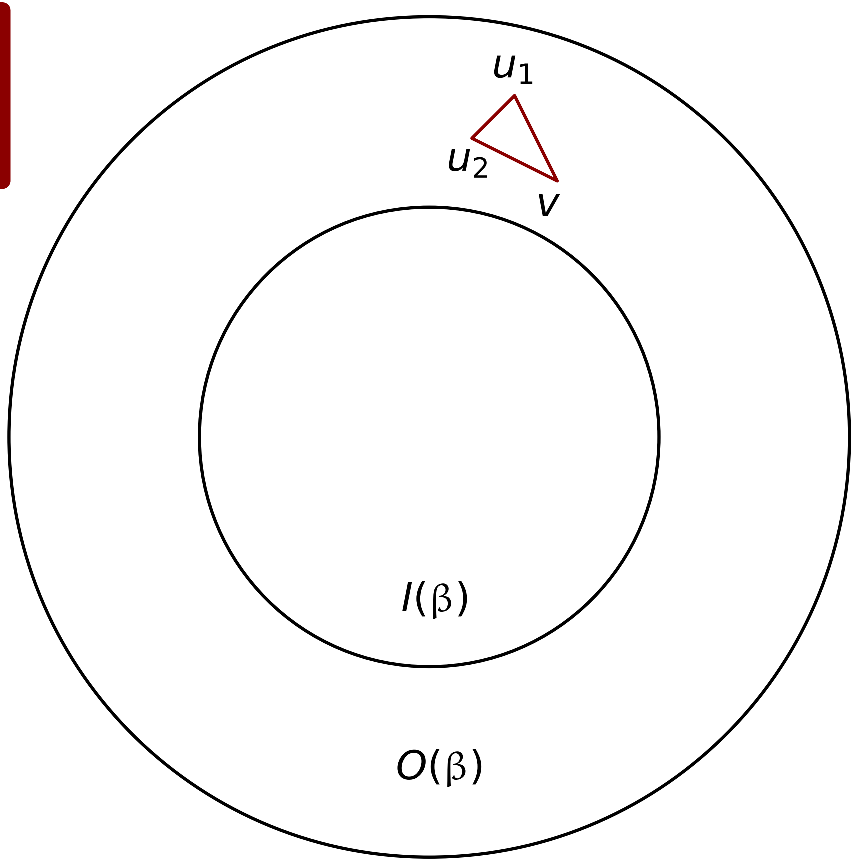

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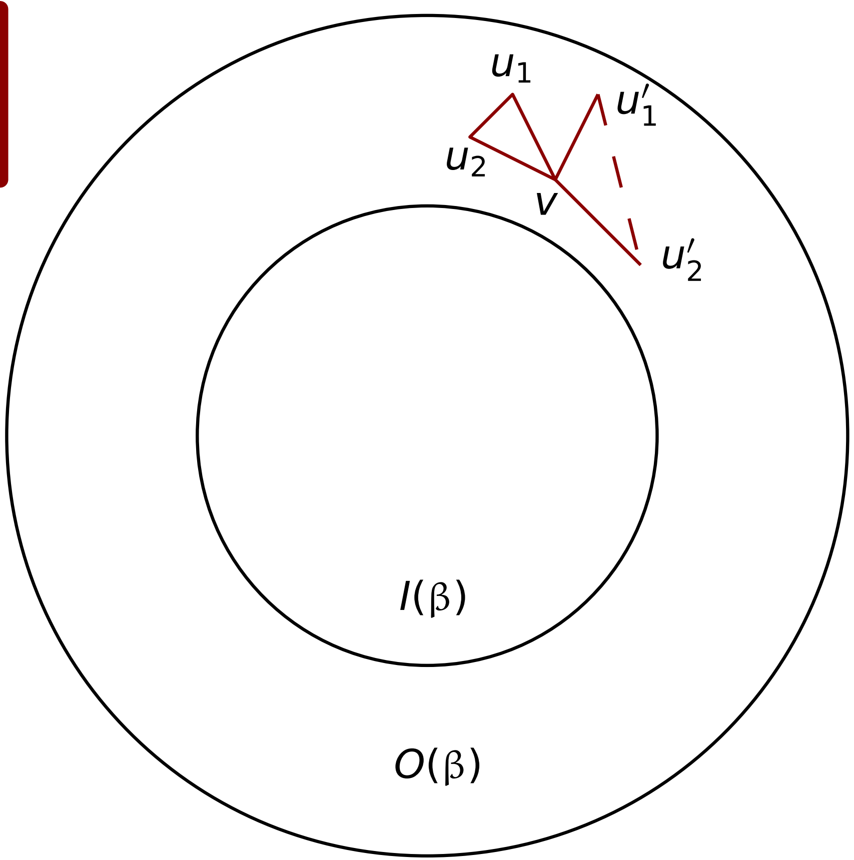

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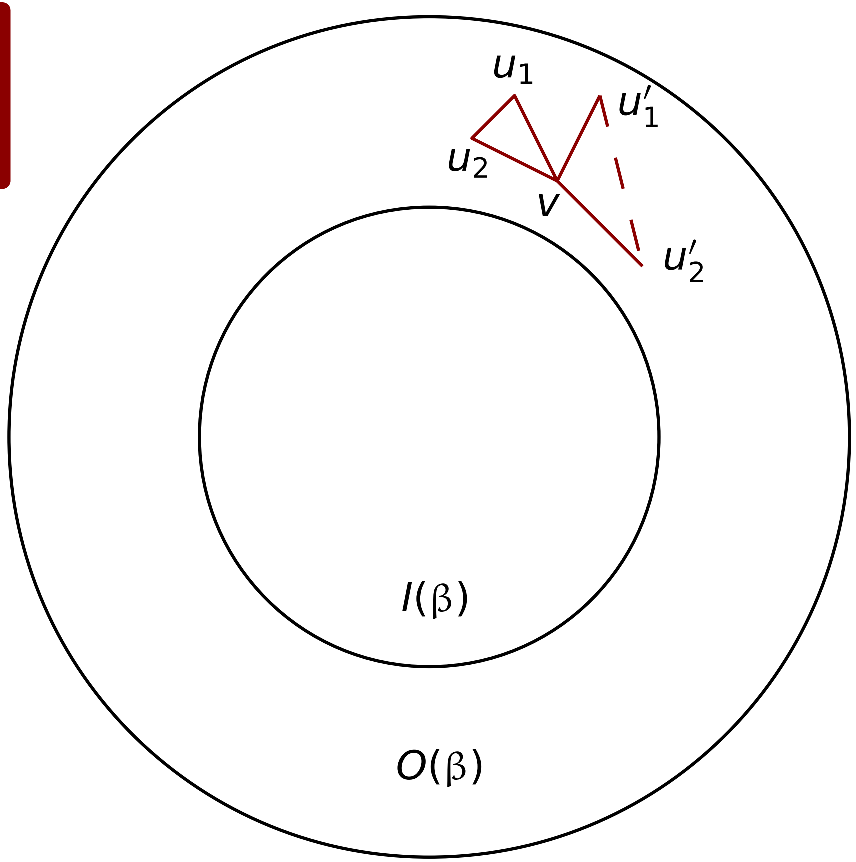

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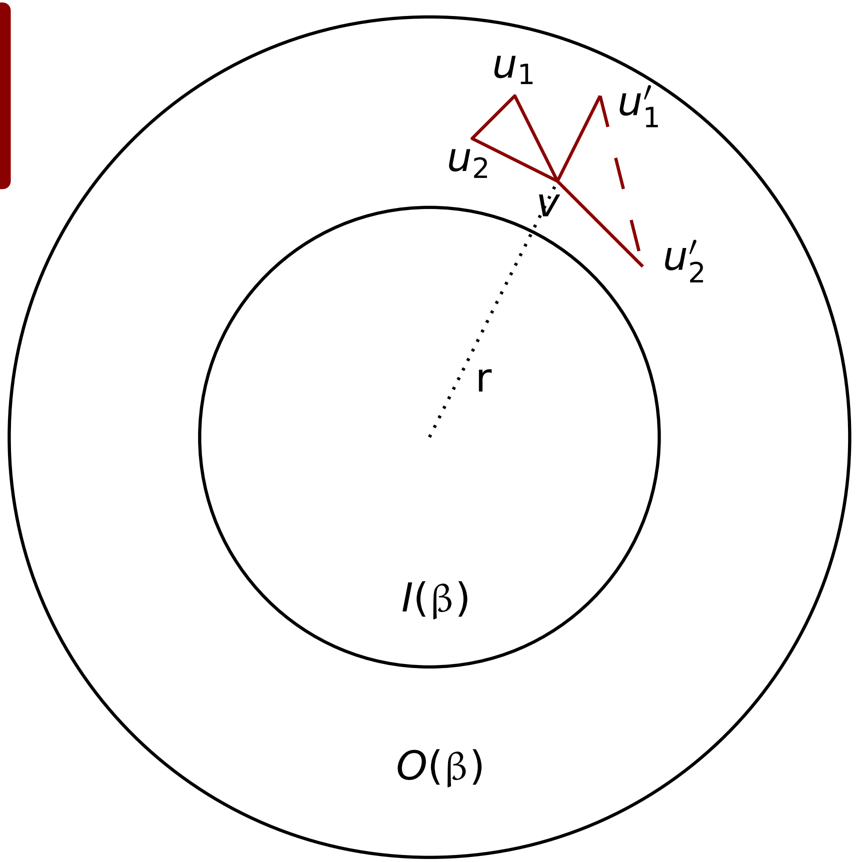

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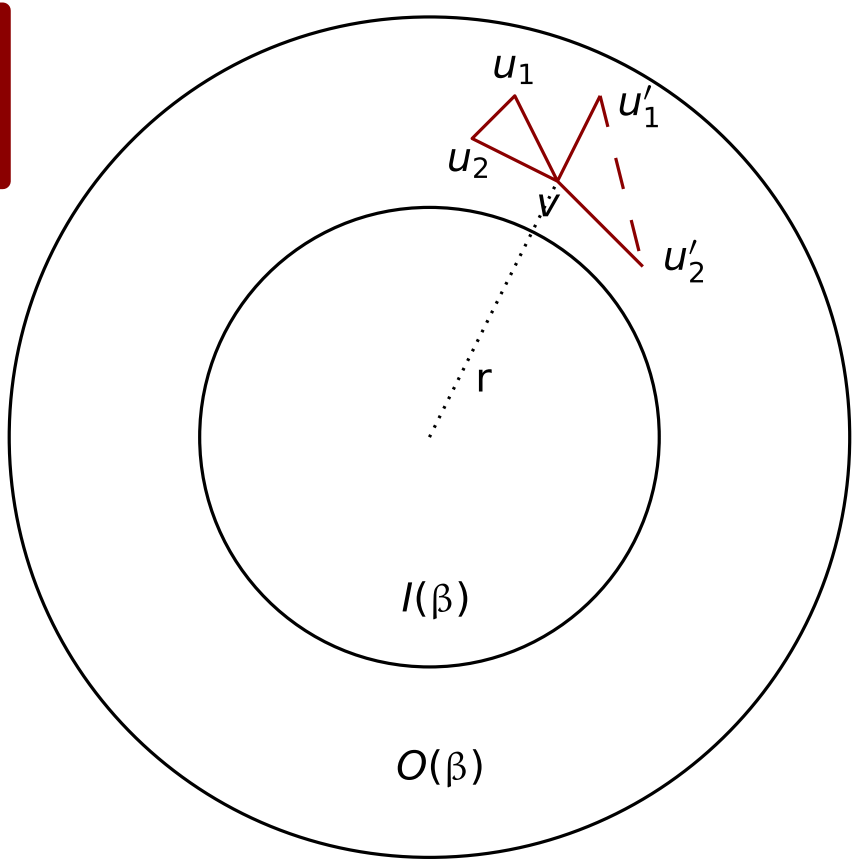
The Probability is clearly $\frac{f(y)}{\mu(B_0(R) \cap B_r(R))}$



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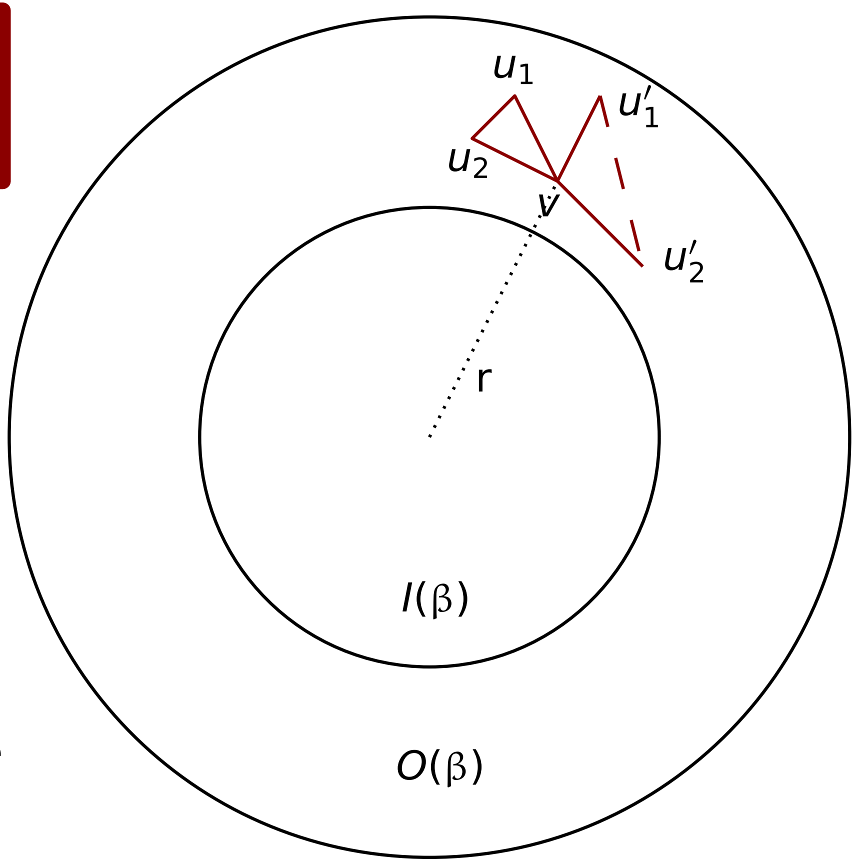
Let $\mathbb{E}[Y_r|\mathcal{E}]$ be the expected value of Y for a vertex v with radius r conditioned on the event $\mathcal{E}$ that the vertex has at least degree 2.

Observation:
$\mathbb{E}[Y_r|\mathcal{E}] = \mathbb{P}[u_1 \sim u_2 | v \sim u_1, v \sim u_2, r(v) = r]$

Fixing a point u_1 at $(y, \varphi) \in B_0(R) \cap B_r(R) \setminus B_0(\beta R)$.
The Probability is clearly $\frac{f(y)}{\mu(B_0(R) \cap B_r(R))}$

Hence, the contribution of u_2 to $\mathbb{E}[Y_r|\mathcal{E}]$, given the coordinats of u_2 is $\frac{\mu(B_r(R) \cap B_y(R) \cap B_0(R) \setminus B_0(\beta R))}{\mu(B_0(R)) \cap B_r(R))}$



*Let $X := \sum_{v \in V} \frac{|\{\{u_1,u_2\} \in E | u_1,u_2 \in \Gamma(v)\}|}{\binom{\delta(v)}{2}}$. Then $\overline{c} = \frac{X}{n}$.

Lower bound for the cluster coefficient

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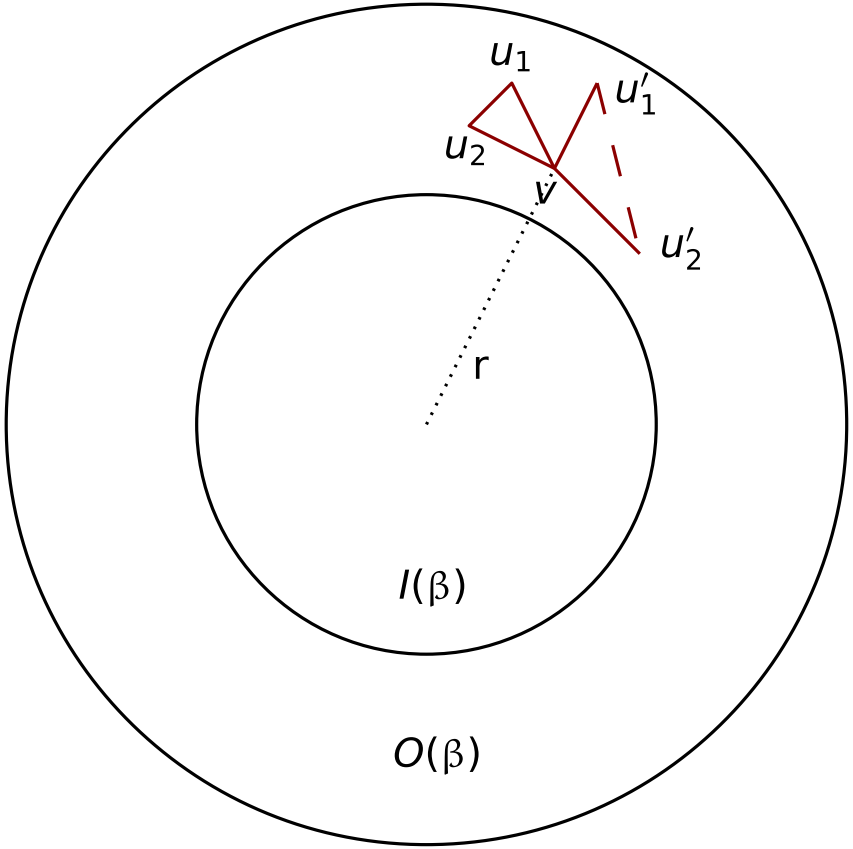

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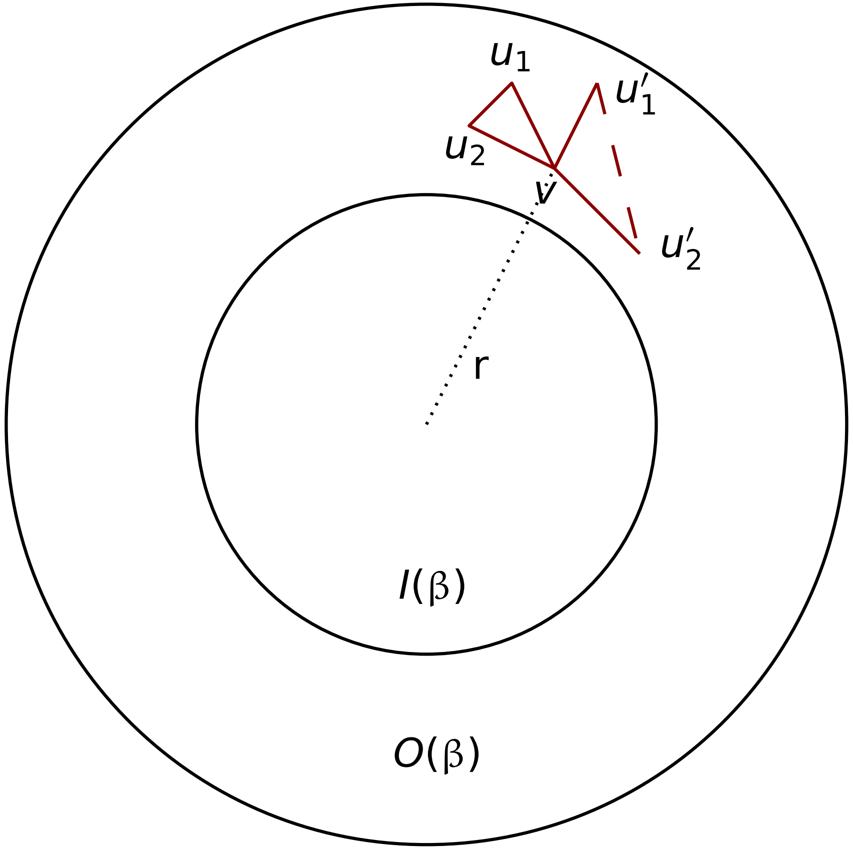

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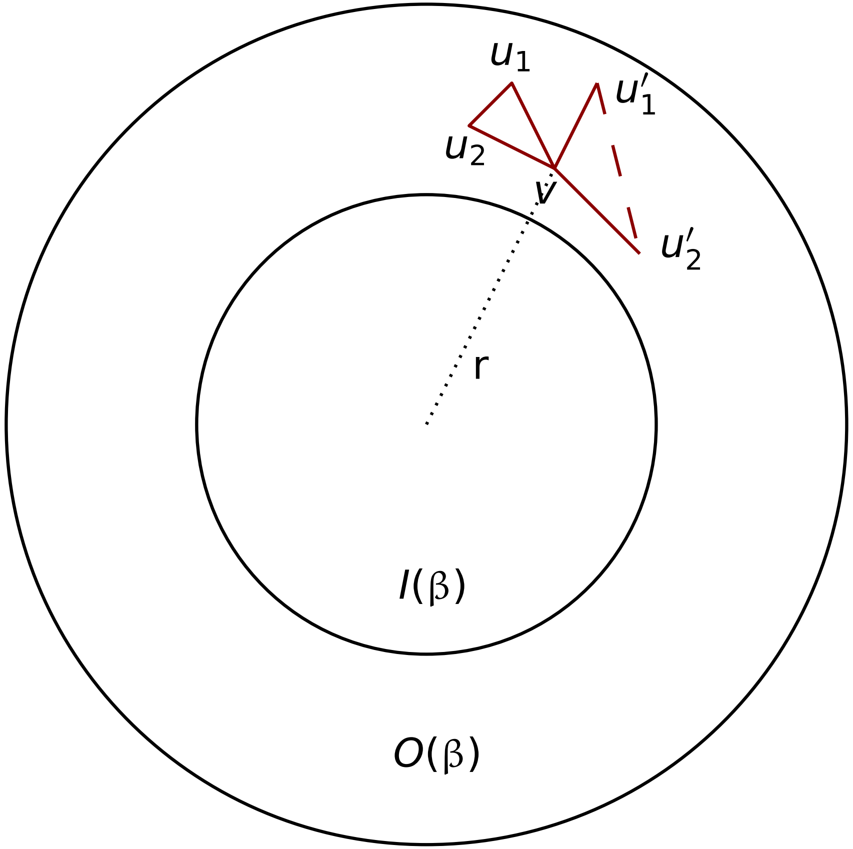

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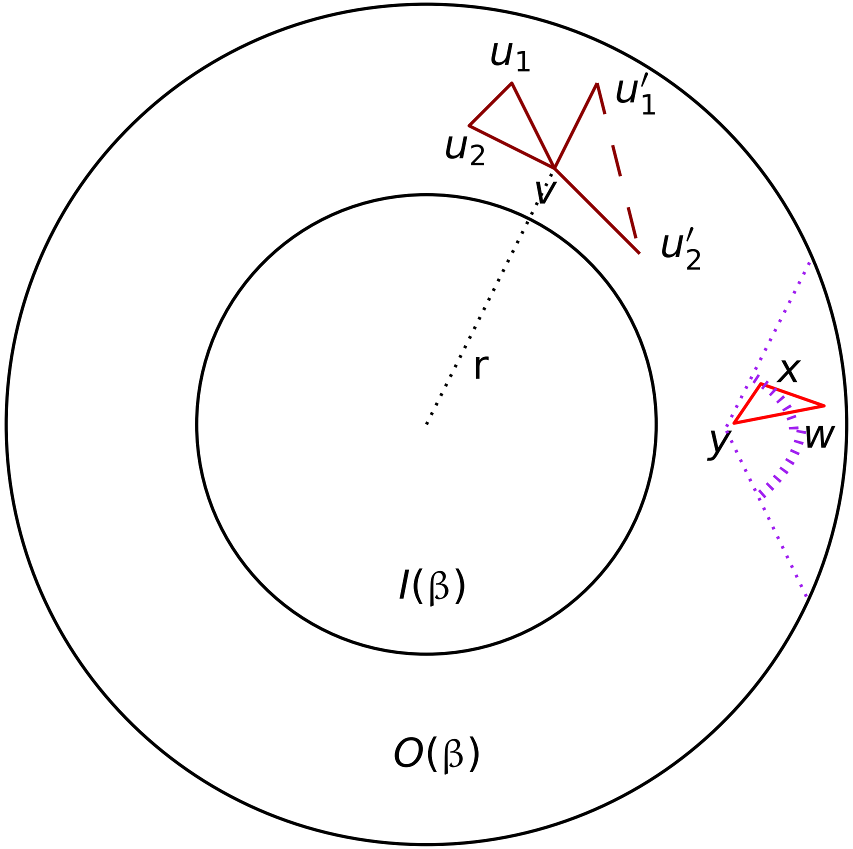

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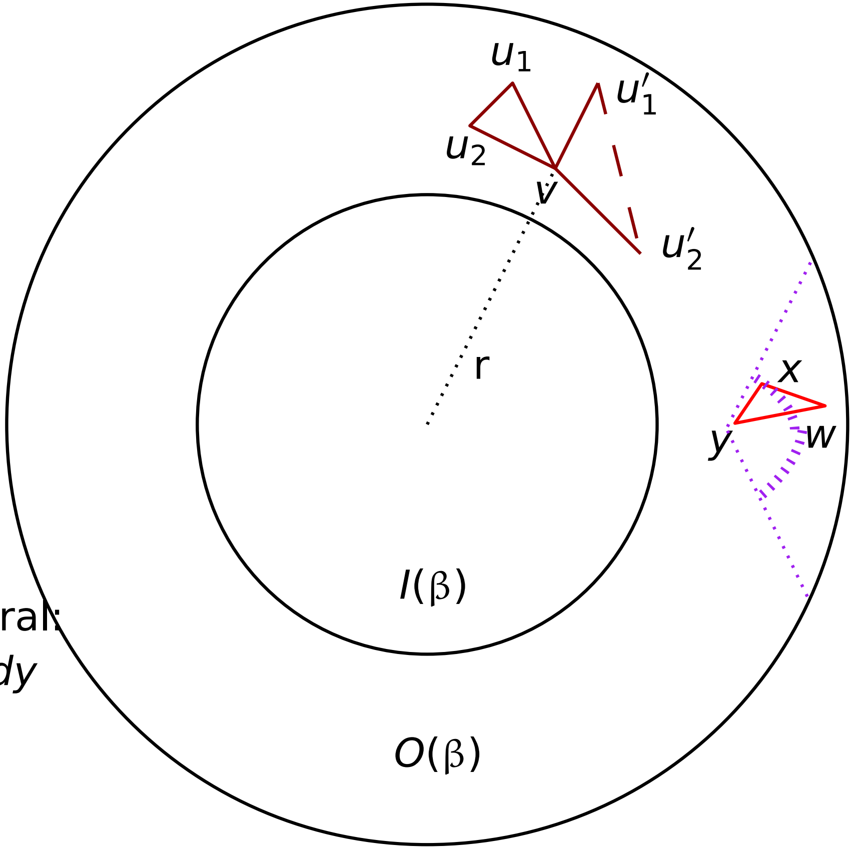

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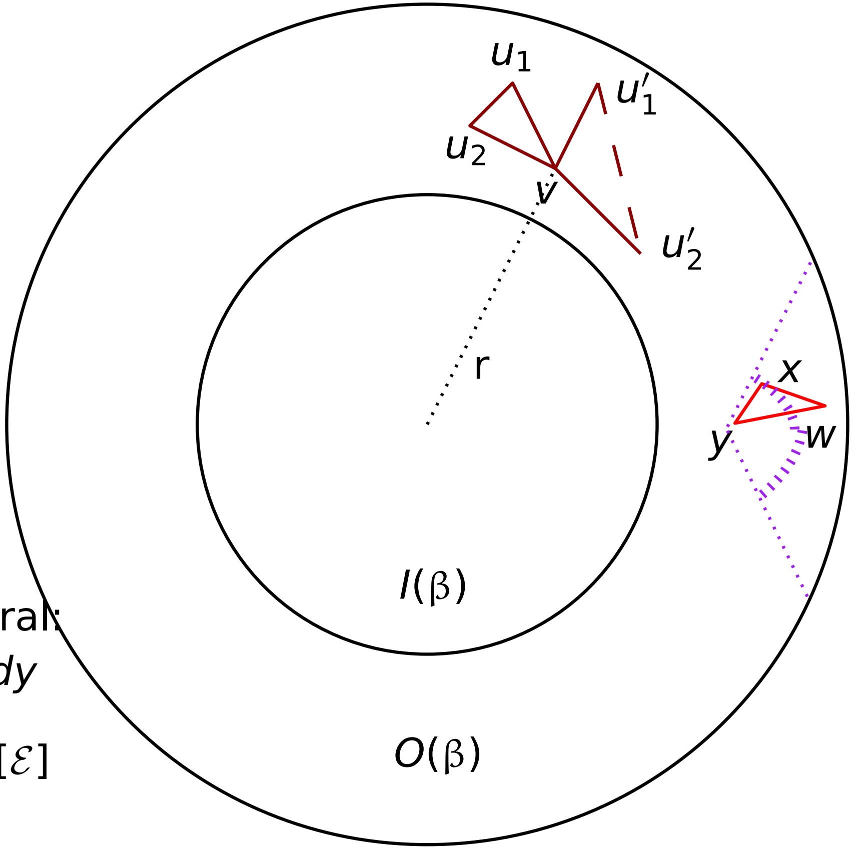
Lower bounding the integral and lower bounding $\mathbb{P}[\mathcal{E}]$ gives us with "some" calculations:



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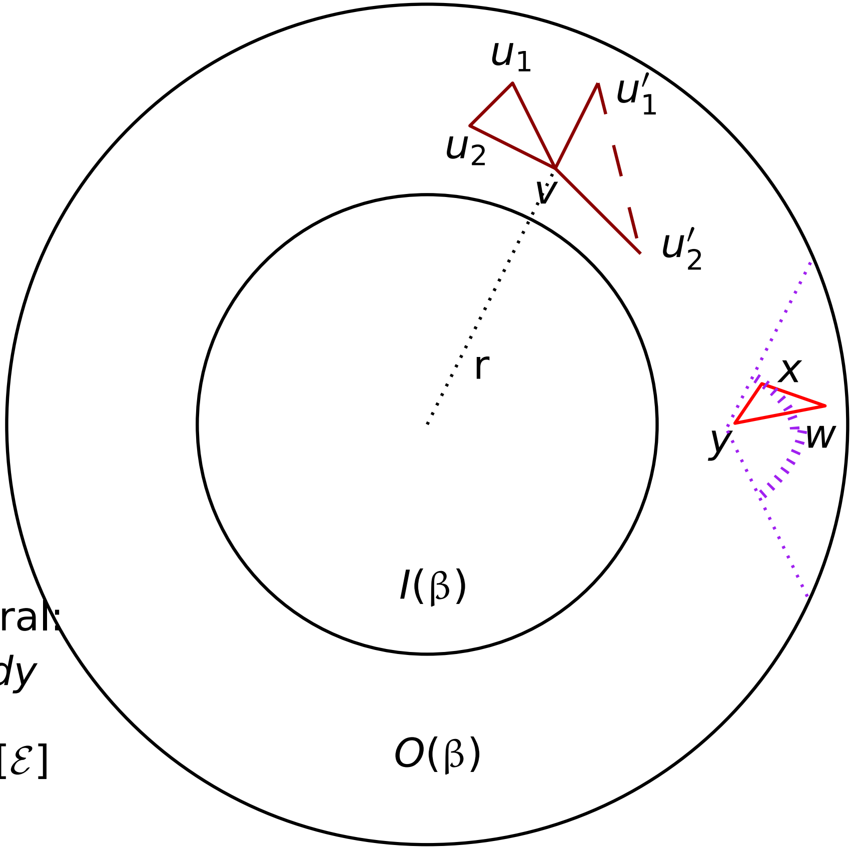

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Let $\mathcal{B} := \{\exists v \in O(\beta) : \delta(v) > 2e \underbrace{c' n^{1-\beta}}_{\mathbb{E}[\delta(v)]}\}$.

Lemma: $\mathbb{P}[\mathcal{B}] = e^{-\Omega(n^{1-\beta})}$.

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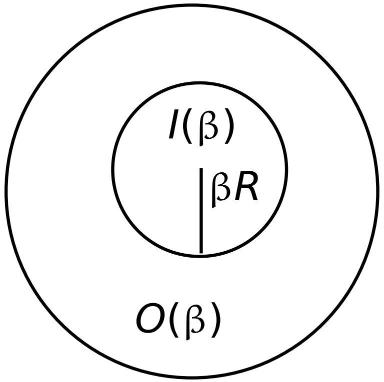

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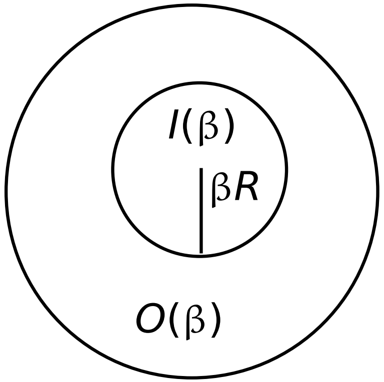

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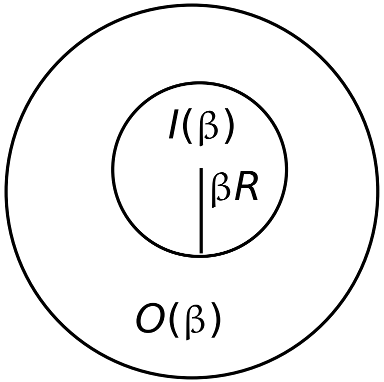

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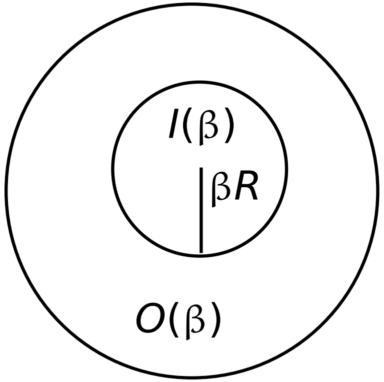

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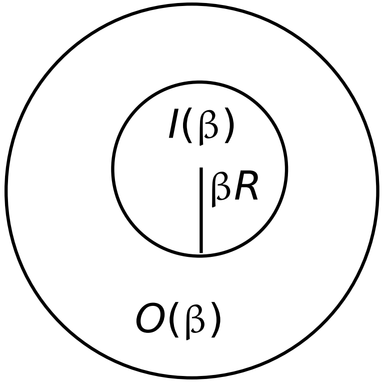

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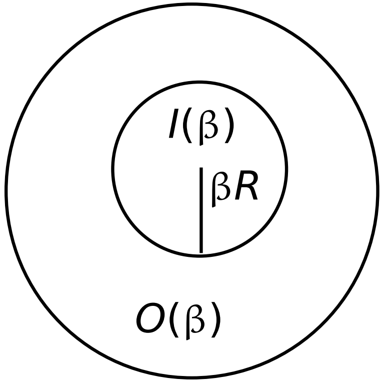

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