Contents

1 Introduction

This text is aimed at readers who are familiar with manifolds, differential forms and the outer derivative covered in a first course on differential geometry. Such a course at its end oftentimes touches on the topic of the de Rham cohomology and maybe even shows the homotopy invariance of the cohomology in order to explicitly compute some examples. Naturally, the question arises, what the de Rham cohomology of a given manifold actually depends on – the differential structure or the topology? Through the study of other (co-)homology theories like simplicial and singular homology, or Čech cohomology, all which analyze only topological aspects of a space, we see that the de Rham cohomology of a manifold is actually isomorphic to these and thus we can say that it only depends on the topology of the manifold.

These types of cohomologies however will not be discussed in this thesis. Rather, I want to present a more generalized approach to cohomology through derived functors, which are a tool to measure the "non-exactness" of functors between abelian categories. A basic understanding of what a categories and functors are is assumed for this part, but any more specific definitions will be explicitly stated in the text. The construction of the derived functors and their application to sheaf cohomology follows closely to the book *Hodge Theory and Complex Algebraic Geometry I* by Claire Voisin, which in this sense was the main source for my thesis.

The main part of my work for was spent on defining abelian categories and working through their properties that are used in the Voisin's book. For my own understanding and for that of any other reader coming to this topic from the world of differential geometry, I tried to prove as many statements as I could, which are most often assumed to be familiar to the reader in standard textbooks on cohomology theory in abelian categories.

Specifically, I wanted to prove all of the required results without invoking the *Freyd-Mitchell embedding theorem*, which states that any small abelian category admits a full, faithful and exact functor into the category $\mathsf{k\text{-}Mod}$ of modules over some unital ring. This is a standard tool for studying abelian categories, as it allows one to prove desired properties in $\mathsf{k\text{-}Mod}$, making them automatically true in the original abelian category. Towards the end of the thesis, I did not have time to rigorously prove everything "from first principles" and at some points referred to the Freyd-Mitchell theorem or the literature.

The most important results in this thesis will be the theorem on the existence of derived functors and a general version of the *de Rham-Weil theorem* in abelian categories, showing that the derived functors can be computed through an acyclic resolution of an object. These theorems are then applied to the abelian category $\mathsf{Sh}(X, \mathsf{Ab})$ of sheaves of abelian groups over a topological space X, or a manifold in particular. Thus, we define the sheaf cohomology as the right derived functors of the global sections functor, and show that the de Rham cohomology is isomorphic to the sheaf cohomology of the sheaf $\underline{\mathbb{R}}$ of locally constant functions, since the sheaves Ω^r of differential forms constitute an acyclic resolution of $\underline{\mathbb{R}}$.

2 Manifolds and Sheaves

2.1 Smooth Manifolds

We begin with a recapitulation of the standard definitions concerning smooth manifolds. The approach chosen here is to start with a topological space X, and to define what it means for X to be locally euclidean, a topological manifold, etc. Some authors (c.f. Hitchin [5]) instead choose to take a set X, and to define coordinate charts simply as bijective maps onto open subsets of $\mathbb{R}^d$. We then do not a priori have a topology on X, which is instead in the end defined to be the initial topology generated by the coordinate charts of a maximal atlas. Such an approach not only is very elegant, but it already shows in its definitions that the topology and differential structure of a manifold are closely intertwined. The reason I chose not to define manifolds this way is that later on I will present another alternative definition of smooth manifolds in terms of sheaves, which requires a topological space to start with.

Definition 2.1.1. A topological space X is ***locally euclidean*** if for every point $x \in X$ there exists an open neighborhood U of x and a homeomorphism $\varphi : U \to V$ to an open subset $V \subseteq \mathbb{R}^d$. Such a homeomorphism is called a ***coordinate chart*** of U. We will oftentimes omit the range V of coordinate charts and just denote them as $\varphi : U \to \mathbb{R}^d$, with the understanding that a chart is a homeomorphism not necessarily to $\mathbb{R}^d$, but to its image $\varphi(U) \subseteq R^d$.

Remark. Coming from differential geometry, one expects that the dimension of a connected locally euclidean space is globally constant, as is the case for smooth manifolds. This is in fact true, but much less obvious than when dealing with diffeomorphisms, where you can invoke arguments about the rank of the derivatives. In the homeomorphic case, it follows from Brouwer's *invariance of domain theorem*:

Theorem 2.1.2 (Invariance of domain)**.** *If $f : U \subseteq \mathbb{R}^n \to f(U) \subseteq \mathbb{R}^n$ is an injective, continuous map, then $f(U)$ is open and hence f is indeed a homeomorphism to its image.*

Proof. The original proof due to Brouwer [2], as well as most contemporary sources (c.f. Hatcher [4]) make use of singular homology, which already turns this problem into a motivating example for the study of homological algebra. It should be mentioned though, that there also is an analytical proof of the theorem, based on Brouwer's *fixed point theorem.* This version of the proof, which is possibly due to Kulpa [7] can be found in an online article by Tao [10]. □

Corollary 2.1.3. *For $n > m$, there is no continuous injective map from an open subset $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$.*

Proof. Suppose, there was such a continuous injective map $f : U \to \mathbb{R}^m$. Then the map $\overline{f} : U \to \mathbb{R}^m \times \mathbb{R}^{n-m} \cong \mathbb{R}^n : x \to (f(x), 0)$ would also be continuous and injective, hence $\overline{f}(U) \subseteq \mathbb{R}^n$ would be open. But $\overline{f}(U) = f(U) \times \{0\}^{n-m}$ clearly cannot be open in $\mathbb{R}^n$, contradicting our assumption that such an f exists. □

Proposition 2.1.4. *The dimension d of a connected topological manifold X is globally well-defined.*

Proof. Take two coordinate charts $\varphi_1 : U \to \mathbb{R}^d, \psi : V \to \mathbb{R}^e$ around points x, y of X that have intersecting domains, $U \cap V \neq \emptyset$. Consider the crossover map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. This is an injective, continuous map between open subsets of $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively. By the previous corollary, this is only possible for $d = e$. Now, since X is connected and covered by coordinate charts, we see that the dimension is constant globally. $\square$

Definition 2.1.5. A ***topological manifold*** is a locally euclidean space that is Hausdorff and second countable.

Definition 2.1.6. A ***smooth*** or ***differential*** or ***$\mathscr{C}^\infty$-atlas*** on a locally euclidean space X is a covering of X by a family of coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^d$, i.e. $\bigcup_{\alpha \in I} U_\alpha = X$, that fulfill the following compatibility condition: For all $\alpha, \beta \in I$, the ***crossover map*** $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a $\mathscr{C}^\infty$-diffeomorphism of open sets in $\mathbb{R}^d$.

Remark. Under the alternative approach referred to in the beginning, we would have to also assume that for all $\alpha, \beta \in I : \varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^d$, because we would not yet have equipped X with a topology. However, here the crossover maps automatically are homeomorphisms.

As another remark, we can switch out the condition on the crossover maps to define different types of atlases: We could impose that the crossover maps should be $\mathscr{C}^k$-diffeomorphisms for some $k \in \mathbb{N}$, or that they should be analytic, etc. But from now on, by "atlas", we always mean a smooth atlas, and in general, the terms "smooth", "differential" and "$\mathscr{C}^\infty$" will be used interchangeably.

Definition 2.1.7. Two atlases $\mathcal{A} = \{\varphi_\alpha\}_{\alpha \in I}$ and $\mathcal{B} = \{\psi_\beta\}_{\beta \in J}$ are ***compatible***, if their union $\mathcal{A} \cup \mathcal{B}$ also is an atlas, i.e., fulfills the compatibility condition. This defines an equivalence relation on the set of atlases on a locally euclidean space and a ***differential structure*** is an equivalence class of compatible atlases.

Definition 2.1.8. A ***smooth manifold*** is a topological manifold together with a differential structure. With the exception of this definition, we will just use the term manifold for smooth manifolds.

Remark. A differential structure is also called a ***maximal atlas***, as we can identify each equivalence class with the union of all atlases contained in it. This gives us the maximal atlas of that structure to work with, so to speak.

It should be noted that there can generally be more than one distinct differential structures on a given locally euclidean space or topological manifold. In that sense, there can be multiple distinct smooth manifolds on the same topological manifold and these then are not diffeomorphic in the sense of differential geometry.

Example 2.1.9. Perhaps the most famous examples of this are given by the ***exotic spheres***, i.e. the spaces $S^n \subseteq \mathbb{R}^{n+1}$, which for some n admit multiple incompatible differential structures. This was first demonstrated by Milnor, who in 1956 proved that S^7 admits 28 incompatible differential structures ([8]). It, as of December 2024, still an open question whether S^7 is the lowest-dimensional sphere exhibiting this property. For $n \in \{1, 2, 3, 5, 6\}$ it has been shown that only one differential structure on S^n exists, whilst the number is still unknown for S^4, with the *smooth Poincaré conjecture* stating that there is only one.

Example 2.1.10. Of course, $\mathbb{R}^n$, $n \geq 1$ is a manifold, with the atlas generated by the identity $\mathsf{id}_{\mathbb{R}^n}$, as well as any open subset U of $\mathbb{R}^n$, with the atlas generated by the inclusion $U \hookrightarrow \mathbb{R}^n$. Generally, every open subset $U \subseteq M$ of a manifold M is itself a manifold with the differential structure given by the restriction of all coordinate charts of M to U.

Definition 2.1.11. Let M and N be manifolds of dimension m and n, respectively. A continuous map $f : U \to N$ defined on an open subset $U \subseteq M$ is a ***smooth map***, if for all points $x \in U$, all coordinate charts $\varphi : V \to \mathbb{R}^m$ around x, and all charts $\psi : W \to \mathbb{R}^n$ around $f(x)$ the function

$$\psi \circ f \circ \varphi^{-1} : \varphi\big(f^{-1}(W) \cap V\big) \to \psi(W)$$

is a smooth, i.e. $\mathscr{C}^\infty$, function.

Remark. It suffices to check these differentiability condition for one atlas of M and N respectively, as it then follows for the whole differential structures by the compatibility conditions on atlases.

Example 2.1.12. The most important example of smooth maps are the smooth functions on any manifold M, defined as smooth maps from M to $\mathbb{R}$. Here, we can simplify the definition, as the differential structure on $\mathbb{R}$ is generated by $\mathsf{id}_{\mathbb{R}}$, and $f^{-1}(\mathbb{R}) = M$, so we just need the maps $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha) \to \mathbb{R}$ to be smooth for some atlas $\{\varphi_\alpha\}_{\alpha \in I}$ of M.

Definition 2.1.13. For a smooth manifold M, we denote by $\mathscr{C}^\infty(U)$ the smooth functions on $U \subseteq M$.

Remark. Since every open set U is again a manifold, we should note that $\mathscr{C}^\infty(U)$, viewing U as a manifold itself, is the same set of functions as when viewing U as subset of M. Also, for open sets $V \subseteq U \subseteq M$, $\mathscr{C}^\infty(V)$ again gives the same set when viewing V as an open subset of the manifold M or the manifold U, respectively. So there is no potential for confusion in this notation.

Also in the last described case, we see that any smooth function in $\mathscr{C}^\infty(U)$ can be restricted to a smooth function in $\mathscr{C}^\infty(V)$ by the standard restriction of functions. So we have restriction maps $\mathsf{res}_{VU} : \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(V) : f \mapsto f|_V$ for any pair $V \subseteq U$ of open sets in M. Obviously, $\mathscr{C}^\infty(U)$ is not only a set, but also carries the structure of a commutative $\mathbb{R}$-algebra with the standard linear combination and multiplication of functions, and the restriction morphisms respect this structure, i.e. are homomorphisms of commutative algebras.

Therefore, the assignment $\mathscr{C}^\infty : \tau \to \mathsf{Comm}_{\mathbb{R}}$ actually becomes a contravariant functor from the topology τ of M to the category of commutative $\mathbb{R}$-algebras. Such a functor is called a presheaf which will be discussed in the next chapter.

2.2 Sheaves

Let (X, τ) be a topological space, where we regard τ as a poset category. Let C be an arbitrary category.

Definition 2.2.1. The category $\mathsf{Presh}_{\mathsf{C}}(X)$ of C-valued ***presheaves*** on X is defined as the category $\mathsf{C}^{\tau^{\mathrm{op}}}$ of contravariant functors from τ to C.

Remark. Less categorically speaking, F assigns to each open set an object $F(U)$ in C and to each inclusion $V \subseteq U$ of open sets (which are the morphisms in τ) a ***restriction morphism*** $\mathsf{res}_{VU} : F(U) \to F(V)$ such that the following diagram commutes:

$$\begin{tikzcd}
\tau : & V \arrow[r, "\subseteq"] \arrow[d, mapsto] & U \arrow[d, mapsto] \\
\mathsf{C} : & F(V) & F(U) \arrow[l, "\mathsf{res}_{VU}"]
\end{tikzcd}$$

A ***presheaf morphism*** $\varphi : F \to G$ between two presheaves F and G then is a morphism in the category $\mathsf{C}^{\tau^{\mathrm{op}}}$, i.e. a natural transformation between the functors F and G. Broken down, this is a collection of morphisms $\varphi_U : F(U) \to G(U)$ in C for all $U \in \tau$, such that the following diagram commutes:

$$\begin{tikzcd}
F(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathsf{res}^F_{VU}"] & G(U) \arrow[d, "\mathsf{res}^G_{VU}"] \\
F(V) \arrow[r, "\varphi_V"'] & G(V)
\end{tikzcd}$$

Remark. Notation: As usual, we suppress the topology of a space in our notation and denote presheaves (valued in any category C) as $F : X \to \mathsf{C}$. When talking about a presheaf where the objects $F(U)$ are sets, we use the usual notation and write $s|_V := \mathsf{res}_{VU}(s)$ for elements $s \in F(U)$.

As already discussed, $\mathscr{C}^\infty$ on a manifold is a presheaf valued in the category $\mathsf{Comm}_{\mathbb{R}}$. In the following, we will mainly be interested in presheaves valued in the category Set of sets. If we forget about the commutative algebra structure of the smooth functions, they of course form such a presheaf. There will be many examples of this type, where a (pre-)sheaf is valued not only in Set, but some specific subcategory (meaning that both the objects mapped to by the (pre-)sheaf and the restriction morphism belong to that subcategory). Still, all of the following definitions apply and can sometimes also be generalized for presheaves valued in other categories. But from now on, unless stated otherwise, we will assume all (pre-)sheaves to be valued in Set.

Definition 2.2.2. A ***sheaf*** is a presheaf $F : X \to \mathsf{Set}$ satisfying two additional conditions:

- **Locality:** For $U \in \tau$ and any open cover $\{U_\alpha\}_{\alpha \in I}$ of U: If $s, t \in F(U)$ such that $s_{|U_\alpha} = t_{|U_\alpha}\ \forall \alpha \in I$, then $s = t$.

- **Gluing**: For $U \in \tau$ and any open cover $\{U_\alpha\}_{\alpha\in I}$ of U: If $\{s_\alpha\}_{\alpha\in I}$, $s_\alpha \in F(U_\alpha)$ such that $s_{\alpha|U_\alpha\cap U_\beta} = s_{\beta|U\alpha\cap U_\beta}\,\forall\alpha,\beta \in I$, then there exists $s \in F(U)$ such that $s_{|U_\alpha} = s_\alpha\,\forall\alpha \in I$.

The gist of the definition of a sheaf is that the elements of $F(U)$ are more well behaved and admit arguments that we would expect from classical examples, like the smooth functions on a manifold, which indeed form a sheaf. Other examples include the r-times continuously differentiable functions or the analytic functions on $\mathbb{R}^n$, or also the holomorphic functions on $\mathbb{C}$. But, as a counterexample, the continuous bounded functions do not in general form a sheaf, as they can violate the gluing axiom: If $X = \mathbb{R}$ with some open cover $\{U_\alpha\}$ consisting of only bounded sets and we choose the elements to be $s_\alpha = \mathsf{id}_{U_\alpha}$, then these are indeed bounded, but the resulting function of the gluing process would have to be $\mathsf{id}_\mathbb{R}$, which clearly is not bounded.

Definition 2.2.3. We define a ***morphism of sheaves*** $\varphi : F \to G$ to be a morphism of the underlying presheaves. The category of all C-valued sheaves on a space X is denoted as $\mathsf{Sh}(X, \mathsf{C})$. (Here C is understood as some subcategory of Set)

Definition 2.2.4. A ***fibre bundle***, denoted as $F \to E \xrightarrow{\pi} B$, consists of topological spaces F, E, B and a continuous surjective map $\pi : E \to B$ such that: For all $x \in B$ there is an open neighborhood U of x and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ that makes the following diagram commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathsf{proj}_U"] \\ & U & \end{tikzcd}$$

In this case, we call F the ***fibre space***, E the ***total space***, B the ***base space*** and π the ***bundle projection***.

Example 2.2.5. The ***trivial bundle*** bundle is given by $E = B \times F$.

Remark. The local homeomorphisms in the definition of a general fibre bundle mean that E is locally homeomorphic to the trivial bundle $U \times F$, and the pairs (U, φ) are called the ***local trivializations*** of the fibre bundle. It is furthermore easy to see that $\pi^{-1}(\{x\}) \subseteq E$ with the subspace topology of E, is homeomorphic to F for all $x \in B$.

Definition 2.2.6. For a fibre bundle $F \to E \xrightarrow{\pi} B$ and an open subset $U \subseteq B$, we call a map $s : U \to E$ a ***section*** of π over U, if $\pi \circ s = \mathsf{id}_U$, and we set

$$\mathsf{sec}_\pi(U) := \{s : U \to \ | s \text{ section of } \pi \text{ over } U\}$$

Example 2.2.7. Given a fibre bundle $F \to E \xrightarrow{\pi} X$, we see that $\mathsf{sec}_\pi : B \to \mathsf{Set}$ is a sheaf with the restriction morphism being standard restrictions of functions. The restriction of a section over U to an open subset V will clearly still be a section of π over V, and the locality and gluing properties are also easily verified.

Remark. Actually, this type of construction yields sheaves even in more general settings: A ***bundle*** could be defined to simply consist of a two sets E, B and a (not necessarily surjective) map $f : E \to B$. For any $x \in B$, we call $f^{-1}(\{x\}) \subseteq E$ the ***fibre*** of f at x. Thus a fibre bundle has its name due to the fact that all the fibres are homeomorphic to the same space F, although this condition alone does not fully characterize fibre bundles.

For a subset $U \subseteq B$, we define a section of f over U to be a map $s : U \to E$ such that $f \circ s = \mathsf{id}_U$, and define $\mathsf{sec}_f(U)$ in the same way as before. Of course, if f is not surjective, then $\mathsf{sec}_f(U)$ will be empty for some $U \subseteq B$, but this is not a problem. If B is then equipped with a topology, e.g. $\mathcal{P}(B)$, the assignment $\mathsf{sec}_f : B \to \mathsf{Set}$ is a sheaf. In light of this, we will now construct for any presheaf $F : X \to \mathsf{Set}$ a bundle $E(F) \xrightarrow{\pi} X$ that will yield a sheaf $\overline{F}$ associated to F.

Definition 2.2.8. Let $x \in X$ and $F \in \mathsf{Presh}(X, \mathsf{C})$. Define

$$\Sigma_x := \Big\{ (U, s) \,|\, U \text{ open neighborhood of } x, s \in F(U) \Big\}$$

Define on this set the equivalence relation $(U, s) \sim (V, t) :\Leftrightarrow \exists W \subseteq U \cap V$ open such that $s|_W = t|_W$. The quotient

$$F_x := {}^{\Sigma_x}\!/\!_{\sim}$$

is called the ***stalk*** of F at x. An element in F_x is called the ***germ*** of its representatives and for a section $s \in F(U)$ and $x \in U$ we write the germs as $s_x := [(U, s)]$. It is easy to see that for C being the abelian groups, rings, modules, algebras etc., the stalks will also be objects in that respective category. E.g., for abelian groups we can define an abelian group structure on F_x by $s_x + t_x := (s + t)_x$, with the neutral element given by 0_x.

Remark. The stalk can be defined without the need to talk about elements as a category-theoretic limit, if the category in which F is valued allows for such limits. Take $\nu_x := \{U \in \tau \,|\, x \in U\}$. This set of open neighborhoods becomes again a poset category with the inclusion of open sets, and we define

$$F_x := \varprojlim \big(F : \nu_x \to \mathsf{C} \big)$$

With this definition, it is clear that the stalks are again objects of C.

Definition 2.2.9. The ***étale space*** of a presheaf F on X is the set

$$E(F) := \bigsqcup_{x \in X} F_x$$

For an element $s \in F(U)$ we define $\overline{s} : U \to E(F) : x \mapsto (x, s_x)$ and equip $E(F)$ with the topology τ_F generated by the subbasis $\sigma := \big\{ \overline{s}(U) \,|\, U \in \tau, s \in F(U) \big\}$.

Proposition 2.2.10. *Let U be an open neighborhood of a point $x \in X$. Then for any $s \in F(U)$, the map $\overline{s} : U \to E(F)$ is continuous.*

Proof. It suffices to show that $\overline{s}^{-1}(W)$ is open for all sets W in the subbasis σ of τ_F. So let $V \subseteq X$ be open, $t \in F(V)$ and $W = \overline{t}(V)$. We then have the preimage

$$P := \overline{s}^{-1}(W) = \overline{s}^{-1}(\overline{t}(V)) = \{y \in U \cap V \mid s_y = t_y\}$$

If this set is empty, it is of course open. Otherwise we have for any $y \in \overline{s}^{-1}(W)$ that $s_y = t_y$, i.e., there is an open neighborhood $O \subseteq U \cap V$ such that $s|_O = t|_O$. But this in turn also means that $s_z = t_z$ for all $z \in O$. Hence $O \subseteq P$. Thus we have shown that any point in P has an open neighborhood in P and thus P is open, concluding the proof. $\square$

Remark. Let $\pi : E(F) \to X : (x, s_x) \mapsto x$ be the natural projection, which is a continuous surjective map. This turns $E(F) \xrightarrow{\pi} X$ into a bundle whose fibres are the stalks F_x. Also $\pi \circ \overline{s} = \mathsf{id}_U$ for any $s \in F(U)$ and hence $\overline{s}$ is a continuous section of π over U.

Definition 2.2.11. By the previous remark, the assignment

$$\overline{F} : X \to \mathsf{Set} : U \mapsto \{\overline{s} \mid \overline{s} \text{ continuous section of } \pi \text{ over } U\}$$

is a sheaf, the sheaf ***associated*** to F, also called the ***sheafification*** of F.

Remark. Again, if $F \in \mathsf{Sh}(X, \mathsf{C})$ with C being the abelian groups etc., then we know that the stalks also carry the respective structure, and so do the sets $\overline{F}(U)$ by point-wise definition: E.g., for abelian groups, set $(\overline{s} + \overline{t})(x) := (x, s_x + t_x)$. Thereby, the sheafification of F is an element in $\mathsf{Sh}(X, \mathsf{C})$.

For any presheaf F, we get an induced presheaf morphism $\iota : F \to \overline{F}$ as follows:

$$\iota_U : F(U) \to \overline{F}(U) : s \mapsto \overline{s}$$

Evidently, for $V \subseteq U, s \in U$, the germs $[(U, s)]$ and $[V, \mathsf{res}_{VU}(s)]$ are equal by the definition of the stalks, and hence $\mathsf{res}^{\overline{F}}_{VU}(\iota_U(s)) = \iota_V(\mathsf{res}^F_{VU}(s))$, so that ι really is a morphism of set-valued presheaves. It is furthermore again easy to see that if F is valued in the abelian groups etc., ι will respect this structure as well, i.e. $\iota \in \mathsf{Hom}_{\mathsf{Presh}(X,\mathsf{C})}(F, \overline{F})$.

Proposition 2.2.12. $F \in \mathsf{Presh}(X, \mathsf{Set})$ *fulfills the locality axiom of sheaves if and only if* $\iota_U : F(U) \to \overline{F}(U)$ *is injective for all open sets* $U \subseteq X$.

Proof. Ramanan [9], p. 7 $\square$

Proposition 2.2.13. $F \in \mathsf{Presh}(X, \mathsf{Set})$ *is a sheaf if and only if* $\iota_U : F(U) \to \overline{F}(U)$ *is bijective for all open sets* $U \subseteq X$.

Proof. Ramanan [9], p. 7 $\square$

Remark. If F is a sheaf, we can identify it with its sheafification $\overline{F}$. In that sense, a presheaf F is a sheaf if and only if it is its own sheafification.

Definition 2.2.14. Let F be a sheaf. In light of the above result, we call an element $s \in F(U)$ a ***section*** of F over U, as it can be identified with an actual section in $\overline{F}(U)$. We define the ***section functors*** $\Gamma(U, -) : \mathsf{Sh}(X, \mathsf{C}) \to \mathsf{C} : F \mapsto F(U)$, that takes sheaves to their sections over open sets U. The special case $\Gamma(X, -)$ is called the ***global section functor*** and is often denoted just as $\Gamma : \mathsf{Sh}(X, \mathsf{C}) \to \mathsf{C}$. The section functors $\Gamma(U, -)$ take morphisms of sheaves to morphisms in C by taking $\varphi : F \to G$ to $\varphi_U : F(U) \to G(U)$.

The global section functor will later on be the focal point of our study, as it turns out to be a left exact functor of abelian categories, for which we can compute the right derived functors, that yield the sheaf cohomology.

Definition 2.2.15. For a continuous map $f : X \to Y$ of topological spaces and a sheaf F on Y, we define the ***inverse image sheaf*** $f^{-1}F$ on X by taking the fibre product $X \times_{f,Y,\pi} E(F)$, short $X \times_Y E(F)$, which is the subspace $\{(x, (y, s_y)) \in X \times E(F) \mid f(x) = \pi((y, s_y))\} \subseteq X \times E(F)$, that has a continuous projection $\rho : X \times_Y E(F) \to X : (x, (y, s_y)) \mapsto x$. The pullback sheaf f^*F then is defined as the sheaf of continuous sections of ρ.

For a subspace $X \subseteq Y$, we define the ***restriction*** $F|_X$ of F to X to be the pullback of F via the inclusion map $X \hookrightarrow Y$.

Definition 2.2.16. A sheaf H is a ***subsheaf*** of another sheaf F, if there is a sheaf morphism $\varphi : H \to F$ that is injective for all open $U \subseteq X$, i.e. $\varphi_U : H(U) \to F(U)$ is an injective morphism.

With the inverse image sheaf we can give the previously alluded to alternative definition of a smooth manifold:

Definition 2.2.17. A ***smooth manifold*** is a topological manifold M together with a subsheaf $\mathscr{D}$ of the sheaf $\mathscr{C}^0 : X \to \mathsf{Comm}_{\mathbb{R}} : U \mapsto \mathscr{C}^0(U)$ of continuous $\mathbb{R}$-valued functions, where $\mathscr{D}$ is required to fulfill the following condition: For any coordinate chart $\varphi : U \to \mathbb{R}^d$ on X, we have that $\mathscr{D}|_U \cong \varphi^{-1}\mathscr{C}^\infty(\varphi(U))$ in $\mathsf{Sh}(X, \mathsf{Comm}_{\mathbb{R}})$.

This captures the essence of what a smooth manifold should be: A topological manifold on which we have a way to talk about smooth functions. All the definitions in the upcoming section concerning the tangent space, vector fields etc., can analogously be made for a smooth manifold in the sense of the definition above.

2.3 Sheaves on Manifolds

Definition 2.3.1. Let R be a ring (commutative, with 1) and A a R-algebra. A ***derivation*** on A is an R-linear map $D : A \to A$, such that $D(ab) = D(a)b + aD(b)$ for all $a, b\ inA$. The set of all R-linear derivations on A is denoted as $\mathsf{Der}_R(A)$. It is a left R-module by

Definition 2.3.2. Let M be a manifold. A ***tangent vector*** to M at a point $p \in M$ is a derivation on the stalk $\mathscr{C}^\infty_p$ of the smooth functions. The ***tangent space*** to M at p is then $T_pM := \mathsf{Der}_{\mathbb{R}}(\mathscr{C}^\infty_p)$, and the ***tangent bundle*** of M is the topological space $TM := \bigsqcup_{p \in M} T_pM$.

Remark. From the definition via the stalk it is clear that for any open subset $U \subseteq M$ and $p \in U$ we have $T_pU \cong T_pM$.

Lemma 2.3.3. *The tangent space of a n-dimensional manifold is a n-dimensional real vector space, which is to $\mathbb{R}^n$ as a vector space and thereby also a n-dimensional manifold. Therefore, the tangent bundle is a manifold of dimension $2n$.*

This lemma demonstrates that the tangent bundle TM is indeed a fibre bundle over M and since it is also a manifold, we can talk about smooth maps between M and TM. In fact, the projection $\pi : TM \to M : (p, v) \mapsto p$ is smooth.

Definition 2.3.4. The sheaf $\Gamma(T-)$ that takes an open subset $U \subseteq M$ to the set $\Gamma(TU)$ of smooth sections of the tangent bundle TU is called the sheaf of ***smooth vector fields***.

Remark. Every element $X \in \Gamma(TU)$ induces a derivation $D_X \in \mathsf{Der}_{\mathbb{R}}(\mathscr{C}^\infty(U))$ with $D_X(f)(p) = X(p)(f_p)$. On the other hand, every derivation $D \in \mathsf{Der}_{\mathbb{R}}(\mathscr{C}^\infty(U))$ induces a smooth vector field $X_D \in \Gamma(TU)$ with $X_D(p)(f_p) = D(f)_p$. So one can equivalently define a smooth vector field to be a derivation on $\mathscr{C}^\infty(U)$ and the set of all smooth vector fields in this sense is denoted by $\mathfrak{X}(U)$. Both $\Gamma(TU)$ and $\mathfrak{X}(U)$ are $\mathscr{C}^\infty(U)$-modules.

Definition 2.3.5. Given a module V over some ring k, we define an ***alternating r-form*** ω on V to be an element of $\mathsf{Hom}(V^{\otimes r}, \mathsf{k})$ (i.e. a multilinear map that takes r elements of V to a value in k) that satisfies:

$$\forall v_1, ..., v_r \in V \ : \ v_i = v_j \text{ for some } i \neq j \Rightarrow \omega(v_1, ..., v_r) = 0$$

As a convention, an alternating 0-form is a constant $\lambda \in \mathsf{k}$. The set of all alternating r-forms on V is denoted by $\Lambda^r(V)$ and is itself a module over k.

Lemma 2.3.6 (Properties of alternating forms)**.** *Given $\omega \in \Lambda^r(V)$ and elements $v_1, ..., v_r \in V$ we have:*

- ***Shear invariance:*** *For $\lambda \in \mathsf{k} : \omega(v_1, ..., v_i + \lambda v_j, ..., v_r) = \omega(v_1, ..., v_r)$*
- ***Antisymmetry:*** *For any permutation $\sigma \in S_r : \omega(v_{\sigma(1)}, ..., v_{\sigma(r)}) = \mathsf{sgn}(\sigma) \cdot \omega(v_1, ..., v_r)$*

Definition 2.3.7. The ***wedge product*** of two alternating forms $\alpha \in \Lambda^r(V), \beta \in \Lambda^s(V)$ is defined by

$$(\alpha \wedge \beta)(v_1, ..., v_{r+s}) := \sum_{\sigma \in \mathcal{S}_{r,s}} \alpha(v_{\sigma(1)}, ..., v_{\sigma(r)})\beta(v_{\sigma(r+1)}, ..., v_{\sigma(r+s)})$$

Where $\mathcal{S}_{r,s} = \{\sigma \in S_{r+s} \,|\, \sigma(1) < ... < \sigma(r), \sigma(r+1) < ... < \sigma(r+s)\}$ are called the (r, s)-shuffles. The result of the wedge product is an alternating $(r+s)$-form. Moreover, the wedge product is bilinear and it has the following antisymmetry property:

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha$$

Lemma 2.3.8. *If* k *is a field and* V *a* n*-dimensional vector space over* k*, then* $\Lambda^r(V)$ *is a* $\binom{n}{r}$*-dimensional vector space over* k*, with the convention that* $\binom{n}{r} = 0$ *for* $r > n$*. We can also explicitly give a basis for* $\Lambda^r(V)$*: If* $e_1, ..., e_n$ *is a basis of* V*, then the*

$$e_j^* : V \to \mathsf{k} : \sum_{i=1}^{n} \lambda_i e_i \mapsto \lambda_j$$

for $j = 1, ..., n$ *form a basis of* $\Lambda^1(V)$*, and the family given by* $e_{j_1}^* \wedge ... \wedge e_{j_r}^*$ *for* $j_1 < ... < j_r \in \{1, ..., n\}$ *forms a basis of* $\Lambda^r(V)$*.*

Definition 2.3.9. If k is a field and and V a k-vector-space, the direct sum

$$\Lambda^*(V) := \bigoplus_{r=0}^{\infty} \Lambda^r(V)$$

together with the wedge product becomes what is called a ***graded*** k***-algebra***. An element $\alpha \in \Lambda^*(V)$ that is actually an element of some $\Lambda^r(V)$ (which is then uniquely determined) is called a ***homogeneous*** element. Every element in $\Lambda^*(V)$ is uniquely determined as a finite sum homogeneous elements and the ***degree*** $\mathsf{deg}(\omega)$ of an element $\omega \in \Lambda^*(V)$ is defined to be the highest degree of the homogeneous elements making up its sum. The wedge product then has the property that taking the product of elements α, β, of degree r and s respectively, yields an element $\alpha \wedge \beta$ of degree $r + s$.

Definition 2.3.10. An ***antiderivation*** on a graded k-algebra A^* is an k-linear map $D : A^* \to A^*$ such that for all homogeneous elements α, β there holds

$$D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{\mathsf{deg}(\alpha)} \alpha \wedge D(\beta)$$

If the antiderivation furthermore has the special property that for all homogeneous elements α: $\mathsf{deg}(D(\alpha)) = \mathsf{deg}(\alpha) + m$ for some $m > 0$, then m is called the ***degree*** of the antiderivation.

Definition 2.3.11. A ***differential*** r***-form*** on an open subset U of a manifold M is an element $\omega \in \Lambda^r(\Gamma(TU))$, viewing $\Gamma(TU)$ as a $\mathscr{C}^\infty(U)$-module. The set of all differential r-forms on U is denoted by $\Omega^r(U)$ and is a $\mathscr{C}^\infty(U)$-module. Note that a differential 0-form is simply a smooth function on U, i.e. $\Omega^0(U) = \mathscr{C}^\infty(U)$.

Remark. Most authors first define an ***alternating*** r***-form*** on U to be a family $\omega = \{\omega_p\}_{p \in U}$ of alternating r-forms $\omega_p : T_pU \to \mathbb{R}$. Such a form through its basis representation in each point yields $\binom{n}{r}$ functions $\omega_{j_1,...,j_r} : U \to \mathbb{R}$, i.e.

$$\omega_p = \sum_{j_1 < ... < j_r} \omega_{j_1,...,j_r}(p) e_{j_1}^* \wedge ... \wedge e_{j_r}^*$$

A differential r-form ω then is one such that all the functions $\omega_{j_1,...,j_r}$ are smooth. Alternatively, without the need to talk about bases, an alternating r-form ω is a differential form if $\omega(X_1, ..., X_r)$ is a smooth function for all $X_1, ..., X_r \in \Gamma(TU)$.

Definition 2.3.12. Let $\mathcal{A}$ be a sheaf of commutative algebras and $\mathcal{M}$ a sheaf of abelian groups, both on the same topological space X. We then say that $\mathcal{M}$ is an ***$\mathcal{A}$-module*** if for all open $U \subseteq X$, the group $\mathcal{M}(U)$ has the structure of a left $\mathcal{A}(U)$-module, and these module-structures are compatible with the restrictions: $\mathsf{res}^{\mathcal{M}}_{VU}(f \cdot m) = \mathsf{res}^{\mathcal{A}}_{VU}(f) \cdot \mathsf{res}^{\mathcal{M}}_{VU}(m)$.

Example 2.3.13. $\Gamma(T-), \mathfrak{X}(-)$ and $\Omega^r(-)$ are a $\mathscr{C}^\infty(-)$-modules.

Definition 2.3.14. The direct sum

$$\Omega^*(M) := \bigoplus_{k=0}^{\mathsf{dim}\, M} \Omega^k(M)$$

is a graded $\mathbb{R}$-algebra with the wedge product given by

$$\alpha \wedge \beta : TM \to \mathbb{R} : v \in T_p(M) \mapsto (\alpha_p \wedge \beta_p)(v)$$

for all homogeneous differential forms α, β. It can be easily checked that this $\alpha \wedge \beta$ is a differential form of degree $\mathsf{deg}(\alpha) + \mathsf{deg}(\beta)$.

Definition 2.3.15. An ***outer derivative*** on a manifold M is an antiderivation D of degree 1 on $\Omega^*(M)$ such that $D^2 := D \circ D = 0$ and for all $f \in \mathscr{C}^\infty(M) \subseteq \Omega^*(M)$ and smooth vector fields X on M we have $D(f)(X) = X(f)$.

Theorem 2.3.16. *On any manifold M there exists exactly one outer derivative, and it is denoted as $d : \Omega^*(M) \to \Omega^*(M)$ to express that it is agrees with the standard derivative on $\mathscr{C}^\infty$.*

Proof. Tu [11], p. 212 □

Since we know that the outer derivative is of degree 1, we can not only view it as a singular map on the graded algebra $\Omega^*(M)$, but also as multiple maps between the different degrees of homogeneous differential forms: $d^k : \Omega^k(M) \to \Omega^{k+1}(M) : \omega \mapsto d\omega$. This turns the modules $\Omega^k(M)$ into what is called a cochain complex:

Definition 2.3.17. A ***cochain complex*** of modules is a sequence of modules V^k with maps $\delta^k : A^k \to A^{k+1}$ such that $\delta^{k+1} \circ \delta^k = 0$. The cochain complex of as a whole is denoted as $A^\bullet$, and the maps $\delta^\bullet$ are called the ***coboundary operators*** of the complex.

If there is a k_0 such that $A^k = 0$ for all $k \geq k_0$, we say that the cochain complex is ***bounded above***. Similarly, we define complexes that are ***bounded below***, and a complex that is both bounded above and below is of course said to be ***bounded***.

Remark. As that name suggests, cochain complexes are dual to chain complexes, where the boundary operators are of degree -1, but these will not be of further interest for this thesis.

The terminology of (co-)boundary operators historically stems from the simplicial homology in algebraic topology, where the boundary operator is based on the actual geometric notion of the boundary of a shape. Therefore, we call an element in the image $\mathcal{B}^k(A^\bullet) := \mathsf{im}\, d^{k-1} \subseteq A^k$ a ***coboundary of degree*** k. On the contrary, elements of the kernel $\mathcal{Z}^k(A^\bullet) := \mathsf{ker}\, d^k \subseteq A^k$ are called ***cocycles of degree*** k, because in the geometric setting, a shape that has 0 boundary is called a cycle. For a cochain complex, we evidently have $\mathcal{B}^k \subseteq \mathcal{Z}^k$, i.e. every boundary is a cycle. Therefore the quotient $H^k(A^\bullet) := \mathcal{Z}^k/\mathcal{B}^k$ exists and is called the k-th **cohomology group** of the complex, though it is of course a module itself.

Example 2.3.18. For a n-dimensional manifold M, we have the bounded complex $\Omega^\bullet$ of $\mathscr{C}^\infty$-modules:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \Omega^n(M) \longrightarrow 0$$

and the cohomology groups of this complex are called the ***de-Rham-cohomology*** of the manifold M with the special notation

$$\mathcal{H}^k_{\mathrm{dR}}(M) := H^k(\Omega^\bullet(M))$$

Lemma 2.3.19. *Let M be a manifold and $U \subseteq M$ an open subset, viewed as a manifold itself. We denote by d_M the outer derivative on M, and by d_U that on U. For $\omega \in \Omega^r(M)$:*

$$(d_M\omega)_{|U} = d_U\omega_{|U}$$

Proof. Tu [11], p. 215 □

Corollary 2.3.20. *Since it commutes with the restriction of differential forms, the outer derivative induces morphisms of sheaves $d^r : \Omega^r(-) \to \Omega^{r+1}(-)$ and we obtain and the cochain complex*

$$0 \longrightarrow \Omega^0(-) \xrightarrow{d^0} \Omega^1(-) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \Omega^n(-) \longrightarrow 0$$

of sheaves, which for now remains an informal notion, but will be made exact through the discussion of abelian categories, which are the setting to study general (co-)homology.

3 Abelian Categories

In this section we want to generalize the concept of a chain complex to objects that are not necessarily modules over a ring, e.g. sheaves whose values are groups, rings, modules etc. The correct setting to study generalized chain complexes will be abelian categories, which are modeled after the category of abelian groups and offer all the necessary language to talk about homological algebra. We will build up to the definition of an abelian category via preadditive and additive categories.

3.1 Preadditive Categories

Definition 3.1.1. A category C is said to be ***preadditive***, if

- **(PA1)** For any two objects $A, B \in \mathsf{C}$ the hom-set $\mathsf{Hom}(A, B)$ is endowed with a binary operation $+_{AB}$, so that $(\mathsf{Hom}(A, B), +_{AB})$ is an abelian group.
- **(PA2)** Composition of morphisms is bilinear, i.e. for $f, g \in \mathsf{Hom}(A, B)$ and $u, v \in \mathsf{Hom}(B, C)$ we have:

$$u \circ (f +_{AB} g) = u \circ f +_{BC} u \circ g$$

$$(u +_{BC} v) \circ f = u \circ f +_{BC} v \circ f$$

Remark. With the exception of this definition, we will omit the subscripts on the abelian operations and just use the $+$ symbol for all hom-sets.

A first observation is that since the hom-sets are groups, there must be a unique zero-morphism from any object to another and from bilinearity it follows that composing any morphism on either side with a zero-morphism results in a zero-morphism: $0 \circ f = 0$ and$f \circ 0 = 0$. Note that we are omitting subscripts here as well, as domain and codomain of a zero-morphism technically matter.

In the following, let C be a preadditive category, although it should be noted that the following definitions can also be generalized to categories with a notion of a zero-object and zero-morphisms.

Definition 3.1.2. A ***kernel*** of a morphism $f : X \to Y$ is an object K together with a morphism $k : K \to X$ such that the following diagram commutes:

$$\begin{tikzcd} & K \arrow[dl, "k"'] \arrow[dr, "0"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Furthermore, a kernel is required to fulfill the following universal property: For any object K' and morphism $k' : K' \to X$ that also makes the defining diagram commute, there is a unique morphism $u : K' \to K$ such that the following diagram commutes:

$$
\begin{tikzcd}
& K' \arrow[d, dotted, "u"] \arrow[ddl, "k'"'] \arrow[ddr, "0"] & \\
& K \arrow[dl, "k"] \arrow[dr, "0"'] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

Lemma 3.1.3. *All kernels are monomorphisms.*

Proof. Let $k : K \to X$ be a kernel of $f : X \to Y$ and suppose there are some parallel morphisms $a, b : M \to K$ such that $ka = kb = h$. Then $fh = 0$ since $fk = 0$, so we have the commuting diagram:

$$
\begin{tikzcd}
& M \arrow[ddl, "h"'] \arrow[ddr, "0"] & \\
& K \arrow[dl, "k"] \arrow[dr, "0"'] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

By the universal property of K, this means that there is a unique $u : M \to K$ making the diagram commute. But a and b make it commute as well, hence $a = u = b$. So k is indeed monic. $\square$

Although we in general for a given morphism in a preadditive category cannot be sure that a kernel exists, we have the following nice property if it does:

Lemma 3.1.4. *If $k : K \to X, l : L \to X$ are both kernels of $f : X \to Y$, then there exists a unique isomorphism $\varphi : L \to K$ such that $l\varphi = k$.*

Proof. The uniqueness and existence of φ such that $l\varphi = k$ follows from the universal property of K:

$$
\begin{tikzcd}
& L \arrow[d, dotted, "\varphi"] \arrow[ddl, "l"'] \arrow[ddr, "0"] & \\
& K \arrow[dl, "k"] \arrow[dr, "0"'] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

Since we can make the same argument using the universal property of L, we also get a unique morphism $\psi : K \to L$ such that $k\psi = l$. Hence $k\psi\varphi = k = k\mathsf{id}_K$ and by k being monic it follows that $\psi\varphi = \mathsf{id}_K$. Analogously, $\varphi\psi = \mathsf{id}_L$, so φ and ψ constitute an inverse pair of isomorphisms. $\square$

Remark. With this knowledge, we call the (up to isomorphism unique) kernel object of $\mathsf{ker}\, f$, if it exists.

Definition 3.1.5. A ***cokernel*** of a morphism $f : X \to Y$ is an object Q together with a morphism $q : Y \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "0"'] & & Y \arrow[dl, "q"] \\ & Q & \end{tikzcd}$$

Additionally, a cokernel is required to fulfill the universal property that for any other $q' : Y \to Q$ that makes the diagram commute, there exists a unique morphism $v : Q \to Q'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "0"] \arrow[ddr, "0"'] & & Y \arrow[dl, "q"'] \arrow[ddl, "q'"] \\ & Q \arrow[d, dashed, "v"] & \\ & Q' & \end{tikzcd}$$

Lemma 3.1.6. *All cokernels are epimorphisms and any two cokernels of the same morphism are isomorphic through a uniquely determined isomorphism.*

Proof. The proof is entirely dual to that for the kernel. $\square$

Remark. We call the (up to isomorphism unique) cokernel object of $\mathsf{coker}\, f$, if it exists.

Example 3.1.7. In the category $\mathsf{k\text{-}Mod}$ of modules over some ring k one can easily check that for a module-homomorphism $f : X \to Y$ the categorical kernel indeed agrees with the standard algebraic definition: Let K be the algebraic kernel, then K together with its canonical inclusion $k : K \hookrightarrow X$ obviously makes the required diagram commute. If $k' : K' \to X$ is some other module-morphism with this property, this means that $f \circ k' = 0$, i.e. $k'(K') \subseteq K$, so the unique morphism $u : K' \to K$ is just the right restriction of $k' : K' \to X$ to $K' \to K$.

Analogously, the categorical cokernel matches its algebraic counterpart: The object $Y/_{\mathsf{im}\, f}$ together with the canonical projection $\pi : Y \to Y/_{\mathsf{im}\, f}$ fits the definition.

This shows that $\mathsf{k\text{-}Mod}$ has all kernels and cokernels.

Before moving on to the next section, we want to show the following lemma, which gives useful properties for computing kernels and cokernels:

Lemma 3.1.8. *Let $f : A \to B$ be a morphism in some preadditive category. Additionally, let $m : A \to M$ be a monomorphism and $e : E \to A$ be an epimorphism.*

(1) $\mathsf{ker}\, f = 0 \Leftrightarrow f$ *is monic.*

(2) $\mathsf{coker}\, f = 0 \Leftrightarrow f$ *is epic.*

(3) $\mathsf{ker}\,(mf) \cong \mathsf{ker}\, f$

(4) $\mathsf{coker}\,(fe) \cong \mathsf{coker}\,f$

Proof. We show the properties (1) and (3), as proving (2) and (4) is entirely dual. Let $K = \mathsf{ker}\,f$ and $L = \mathsf{ker}\,(mf)$.

(i): If $K = 0$, then for any two morphism $x, y : X \to A$ such that $fx = fy$, we have $f(x - y) = 0$ and thus by the universal property of $k : K \to A$, there is a unique morphism $u : X \to K$ such that $ku = x - y$. But since $K = 0$, we have $k = 0$ and $u = 0$, so $x - y = 0$, hence $x = y$. Therefore, f is monic. On the other hand, assume that f is monic. Then $fk = 0$ immediately implies $k = 0$ for the kernel morphism, and by the universal property of K we see that it must in fact be the 0-object itself.

(iii): The kernel morphism $\ell : L \to B$ has the property $mf\ell = 0$, but since m is monic, we have $f\ell = 0$. Therefore, by the universal property of $k : K \to A$, there exists a unique morphism $u : L \to K$ such that $ku = \ell$. On the other hand, since $mfk = 0$, by the universal property of ℓ, there exists a unique morphism $v : K \to L$ such that $k = \ell v$. Thus we have $\ell = ku = \ell vu$ and $kuv = \ell v = k$. Since ℓ and k are monic, we have $vu = \mathsf{id}_L$ and $uv = \mathsf{id}_K$, and thus $K \cong L$. ☐

3.2 Additive Categories

Let C be a preadditive category.

Definition 3.2.1. A ***product*** of a family $(A_i)_{i\in I}$ of objects in C is an object P together with morphisms $(\pi_i : P \to A_i)_{i\in I}$ such that for any object X and morphisms $(f_i : X \to A_i)_{i\in I}$ there exists a unique morphism $f : X \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & X \arrow[ddl, "f_i"'] \arrow[d, dotted, "f"] \arrow[ddr, "f_j"] & & \\
 & & P \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & & \\
\cdots & A_i & \cdots & A_j & \cdots
\end{tikzcd}$$

Definition 3.2.2. A ***coproduct*** of a family $(A_i)_{i\in I}$ of objects in C is an object S together with morphisms $(\iota_i : A_i \to S)_{i\in I}$ such that for any object Y and morphisms $(g_i : A_i \to Y)_{i\in I}$ there exists a unique morphism $g : S \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & Y & & \\
 & & S \arrow[u, dotted, "g"] & & \\
\cdots & A_i \arrow[uur, "g_i"] \arrow[ur, "\iota_i"'] & \cdots & A_j \arrow[uul, "g_j"'] \arrow[ul, "\iota_j"] & \cdots
\end{tikzcd}$$

Lemma 3.2.3. *Both products and coproducts are determined up to a unique isomorphism. Hence, we denote the product by $\prod A_i$ and the coproduct by $\coprod A_i$.*

Proof. The proof is analogous to the one for the kernel in leveraging the respective universal property of the (co-)product. $\square$

Example 3.2.4. In the category $\mathsf{k}\text{-}\mathsf{Mod}$, the product of a family of modules is simply the Cartesian product with component-wise operations and the coproduct is the direct sum. Therefore, if the family of modules is finite, product and coproduct coincide.

Lemma 3.2.5. *For a finite family $A_1, ..., A_n$ of objects in a preadditive category, the the following are equivalent:*

(i) The product $\prod A_i$ exists.

(ii) The coproduct $\coprod A_i$ exists.

And, as the proof will show, $\prod A_i \cong \coprod A_i$, if they exist.

Proof. We will only show "(i)⇒(ii)", as the other direction is entirely analogous.

Let $P = \coprod A_i$ together with the morphisms $\pi_i : P \to A_i$. Let further $\delta_{ij} : A_i \to A_j$ be given by $\delta_{ii} = \mathsf{id}_{A_i}$ and $\delta_{ij} = 0$ for $i \neq j$. Then, by the universal property of the product, there exist unique induced morphisms $\iota_i : A_i \to P$ such that $\pi_j \circ \iota_i = \delta_{ij}$. Furthermore $s := \sum \iota_i \pi_i : P \to P$ is the identity id_P: We have

$$\pi_j \sum \iota_i \pi_i = \sum \pi_j \iota_i \pi_i = \sum \delta_{ji} \pi_i = \delta_{jj} = \mathsf{id}_{A_j}$$

And therefore the morphism s makes the following diagram commute:

$$\begin{tikzcd}
& P \arrow[d, dotted, "s"] \arrow[ddl, "\pi_j"'] \arrow[ddr, "\pi_k"] & \\
& P \arrow[dl, "\pi_j"] \arrow[dr, "\pi_k"'] & \\
A_j & & A_k
\end{tikzcd}$$

But by the universal property of P there is a uniquely determined morphism $P \to P$ that makes this diagram commute and since id_P makes the diagram commute as well, we have $s = \mathsf{id}_P$.

We now want to show that P together with the ι_i is a coproduct of the A_i. Let therefore $g_i : A_i \to Y$ be morphisms to some object Y. We then have the morphism $g = \sum g_i \pi_i : P \to Y$ and $g\iota_j = \sum g_i \pi_i \iota_j = \sum g_i \delta_{ij} = g_i$. Assume that $h : P \to Y$ is a morphism with the property $h\iota_i = g_i$ for all i. Then

$$h = h\mathsf{id}_P = h \sum \iota_i \pi_i = \sum h\iota_i \pi_i = g_i \pi_i = g$$

Hence, the morphism g is uniquely determined and P fulfills the universal property of the coproduct.

$\square$

Remark. For a finite family of objects A_i, an object B that is both a product and a coproduct of the A_i is called a ***biproduct*** if the morphism π_i, ι_i fulfill

$$\pi_i \iota_j = \begin{cases} \mathsf{id}_{A_i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

As the proof above shows, any finite product or coproduct can be turned into a biproduct.

Definition 3.2.6. A preadditive category is ***additive*** if all finite biproducts (i.e. all finite products or all finite coproducts) exist.

Example 3.2.7. Of course, $\mathsf{k\text{-}Mod}$ is additive.

3.3 Abelian Categories

We have previously seen that all kernels are monic and all cokernels are epic. The question then becomes: Is the inverse statement true as well?

Definition 3.3.1. We call a monomorphism ***normal*** if it is the kernel of some morphism. A category, in which all the monomorphisms are normal is in turn also called ***normal***. Analogously, we define ***conormal*** epimorphisms and categories, although the word normal is often times used as well when talking about a given epimorphism, since we understand that conormality is the property in question. A category that is both normal and conormal is called ***binormal***.

Example 3.3.2. It is now also easy to see that $\mathsf{k\text{-}Mod}$ is binormal: If $f : X \to Y$ is a monomorphism, we can view $X \cong \mathsf{im}\, f$ as a submodule of Y and hence f is the kernel of its cokernel $\mathsf{coker}\, f$. Analogously, every epimorphism in $\mathsf{k\text{-}Mod}$ is the cokernel of its kernel.

Definition 3.3.3. Finally, a preadditive category C is ***abelian*** if

- **(AB1)** C has an initial object which we will call the 0-object.
- **(AB2)** C has all finite biproducts.
- **(AB3)** C has all kernels and cokernels.
- **(AB4)** C is binormal.

Example 3.3.4. Throughout this chapter we have seen that $\mathsf{k\text{-}Mod}$ is abelian.

We have seen that in $\mathsf{k\text{-}Mod}$, that $\mathsf{coker}\, f = {}^{Y}\!/_{\mathsf{im}\, f}$ and hence $\mathsf{im}\, f = \mathsf{ker}\,\mathsf{coker}\, f$. To conclude this chapter on abelian categories, we want to discuss in the following how one can generally define the image of a morphism in abelian categories.

Lemma 3.3.5. *For any morphism $f : X \to Y$ in an abelian category we have* $\mathsf{ker}\,\mathsf{coker}\, f \cong \mathsf{coker}\,\mathsf{ker}\, f$.

Proof. Let $K = \ker f, Q = \operatorname{coker} f$. We obtain the commutative diagram below, where $u : X \to \ker q$ is induced by the universal property of $\ker q$ since $qf = 0$. Similarly, $v : \operatorname{coker} k \to Y$ is induced by the universal property of $\operatorname{coker} k$, as $fk = 0$.

$$
\begin{tikzcd}
\operatorname{coker} k \arrow[rrdd, dotted, bend left, "v"] & & \\
& K \arrow[lu, "0"'] \arrow[ld, "k"] \arrow[rd, "0"] & \\
X \arrow[uu, "r"] \arrow[rr, "f"] \arrow[rd, "0"] \arrow[rrdd, dotted, bend right, "u"'] & & Y \arrow[ld, "q"] \\
& Q & \\
& & \ker q \arrow[uu, "\ell"'] \arrow[lu, "0"]
\end{tikzcd}
$$

From the diagram we read that $qvr = 0$, and as r is epic, we have $qv = 0$. Therefore, by the universal property of $\ker q$, there is a unique induced morphism $\alpha : \operatorname{coker} k \to \ker q$ that keeps the diagram commutative.

Analogously, we obtain a unique induced morphism $\beta : \ker q \to \operatorname{coker} k$ that keeps the diagram commutative. In the usual fashion, we then see that these constitute a pair of inverse isomorphisms, proving the lemma. $\square$

Definition 3.3.6. In any category, an ***image*** of a morphism $f : X \to Y$ is an object I together with a monomorphism $m : I \to Y$ such that there exists a morphism $e : X \to I$ that makes the following diagram commute:

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rd, "e"'] & & Y \\
& I \arrow[ru, "m"'] &
\end{tikzcd}
$$

Additionally, the image is required to fulfill the following universal property: For any other object I' together with a monomorphism $m' : I' \to Y$ that admits a morphism $e' : X \to I'$ making the required diagram commute, there is a unique morphism $v : I \to I'$ keeping the diagram commutative:

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rd, "e"] \arrow[rdd, "e'"'] & & Y \\
& I \arrow[ru, "m"] \arrow[d, dotted, "v"] & \\
& I' \arrow[ruu, "m'"'] &
\end{tikzcd}
$$

Remark. Again, the image is determined up to a unique isomorphism and evidently, the image of a monomorphism $f : X \to Y$ is given by the morphism itself. Also, the morphism e such that $me = f$ is unique, since m is monic.

Lemma 3.3.7. *In a preadditive category, the morphism e in the definition of the image is an epimorphism.*

Proof. Suppose there are morphisms $a, b : I \to M$ such that $ae = be$, i.e. $(a-b)e = 0$. We want to show that $a - b = 0$, which is to say, $K := \mathsf{ker}\,(a-b) \cong I$. Since $(a-b)e = 0$, we get a unique induced morphism $u : X \to K$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"'] \arrow[dd, dotted, "u"'] & & Y \\
& I \arrow[ur, "m"'] \arrow[dr, "a-b"] & \\
K \arrow[ur, "k"] & & M
\end{tikzcd}$$

Since mk is monic, there is a unique morphism $v : I \to K$ keeping the diagram commutative, by the universal property of the image I. Now, $m = mkv$ and as m is monic, we have $kv = \mathsf{id}_I$. Also, $kvk = k$ and since k is monic, we $vk = \mathsf{id}_K$. Therefore, we see that $K \cong I$ through the inverse isomorphisms v, k. Hence, $a-b = 0$ and e is an epimorphism. ☐

Lemma 3.3.8. *For a morphism $f : X \to Y$ in a preadditive category that has all kernels and cokernels, the image is given by* $\mathsf{im}\, f \cong \mathsf{coker}\,\mathsf{ker}\, f \cong \mathsf{ker}\,\mathsf{coker}\, f$.

Proof. Firstly, $\mathsf{ker}\,\mathsf{coker}\, f$ comes with a monomorphism k into Y, and since $qf = 0$, we also get a unique morphism $u : X \to \mathsf{ker}\, q$ by the universal property of $\ker q$, such that the diagram below is commutative:

$$\begin{tikzcd}
& X \arrow[ddl, "f"'] \arrow[d, dotted, "u"] \arrow[ddr, "0=qf"] & \\
& \mathsf{ker}\, q \arrow[dl, "k"] \arrow[dr, "0"'] & \\
Y \arrow[rr, "q"'] & & \mathsf{coker}\, f
\end{tikzcd}$$

Because of the isomorphism $\mathsf{ker}\,\mathsf{coker}\, f \cong \mathsf{coker}\,\mathsf{ker}\, f$, we also have a monomorphism $\ell : \mathsf{ker}\,\mathsf{coker}\, f \to Y$ and a morphism $v : X \to \mathsf{coker}\,\mathsf{ker}\, f$ such that $\ell v = f$. Now let $m : I \to Y, e : X \to I$ be the image of f. By the universal property of the image, there exists a unique induced morphism $\alpha : I \to \mathsf{coker}\,\mathsf{ker}\, f$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, "v"'] & & Y \\
& I \arrow[ur, "m"] \arrow[d, dotted, "\alpha"] & \\
& \mathsf{coker}\,\mathsf{ker}\, f \arrow[uur, "\ell"'] &
\end{tikzcd}$$

Let now $i : \mathsf{ker}\, f \to X$ be the canonical map of the kernel, such that $\mathsf{coker}\,\mathsf{ker}\, f = \mathsf{coker}\, i$. Then we have $mei = fi = 0$ and since m is monic, we have $ei = 0$ and therefore, by the universal property of $\operatorname{coker} i$, we get a unique induced morphism $\beta : \operatorname{coker} i \to I$ that keeps the diagram commutative. In the usual fashion, we can now argue that α and β form an inverse pair of isomorphisms, proving the lemma. $\square$

Remark. In abelian categories, the image is often defined as $\ker \operatorname{coker} f$, motivated by the example from k-Mod and the lemma above. The object $\operatorname{coker} \ker f$ then is called the ***coimage***.

4 Homological Algebra

Throughout this chapter, let C be an abelian category.

4.1 The Category of Cochain Complexes

Definition 4.1.1. A ***cochain complex*** in C is a sequence of objects $(A^j)_{j\in\mathbb{Z}}$ together with morphisms

$$\dots \xrightarrow{d^{j-2}} A^{j-1} \xrightarrow{d^{j-1}} A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \dots$$

such that $d^{j+1}\circ d^j = 0$. We denote this cochain complex as $A^\bullet$. A ***morphism of cochain complexes*** $A^\bullet$, $B^\bullet$ is a sequence of morphisms $(\varphi^j)_{j\in\mathbb{Z}}$ with $\varphi^j : A^j \to B^j$ (denoted $\varphi^\bullet$) such that the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & A^j \arrow[r, "d_A^j"] \arrow[d, "\varphi^j"] & A^{j+1} \arrow[r] \arrow[d, "\varphi^{j+1}"] & \dots \\
\dots \arrow[r] & B^j \arrow[r, "d_B^j"] & B^{j+1} \arrow[r] & \dots
\end{tikzcd}$$

In the following we show that the class of all cochains in C becomes an abelian category itself, denoted $\mathsf{Ch}^\bullet(\mathsf{C})$. First, it is clear that we can concatenate morphisms of cochains by concatenating in each degree and that every cochain has its unique identity morphism given by the identity in each degree. Thus, $\mathsf{Ch}^\bullet(\mathsf{C})$ in fact becomes a category. Furthermore, $\mathsf{Ch}^\bullet(\mathsf{C})$ inherits preadditivity from C: Given two morphisms $\varphi^\bullet, \psi^\bullet : A^\bullet \to B^\bullet$, their sum is defined by taking the sum in each degree. The result is indeed a morphism of cochains, as the required diagram commutes:

$$\begin{aligned}
d_B^j \circ (\varphi^j + \psi^j) &\overset{\textbf{(PA2)}\text{ in }\mathsf{C}}{=} d_B^j \circ \varphi^j + d_B^j \circ \psi^j \\
&= \varphi^{j+1}\circ d_A^j + \psi^{j+1}\circ d_A^j \\
&\overset{\textbf{(PA2)}\text{ in }\mathsf{C}}{=} (\varphi^{j+1} + \psi^{j+1})\circ d_A^j
\end{aligned}$$

It is also easy to show from the preadditivity of C that the addition of cochain morphisms is commutative and bilinear, i.e. that $\mathsf{Ch}^\bullet(\mathsf{C})$ is a preadditive category.

Now we move on to show that $\mathsf{Ch}^\bullet(\mathsf{C})$ is indeed abelian. First, $\mathsf{Ch}^\bullet(\mathsf{C})$ fulfills **(AB1)**: The 0-object in $\mathsf{Ch}^\bullet(\mathsf{C})$ is the 0-cochain consisting of the 0-object of C and the 0-morphism in every degree.

Lemma 4.1.2. $\mathsf{Ch}^\bullet(\mathsf{C})$ *fulfills* ***(AB2)****, i.e. it admits all finite biproducts.*

Proof. We define the biproduct $B^\bullet$ of $A_1^\bullet, \dots, A_n^\bullet$ by taking $B^j := \bigoplus_i A_i^j$. Then, by the universal property of B^j, there is a unique morphism δ^j such that the following diagram commutes:

$$\begin{tikzcd}
 & & A_1^{j+1} \\
B^j \arrow[urr, "d^j_{A_1}\pi^j_1"] \arrow[drr, "d^j_{A_n}\pi^j_n"'] \arrow[r, dotted, "\delta^j"] & B^{j+1} \arrow[ur, "\pi_1^{j+1}"'] \arrow[dr, "\pi_n^{j+1}"] & \dots \\
 & & A_n^{j+1}
\end{tikzcd}$$

It remains to see that $\delta^{j+1}\delta^j = 0$. By the universal property of B^{j+2} there is a unique morphism $u : B^j \to B^{j+2}$ making the diagram below commute:

$$\begin{tikzcd}
 & & A_1^{j+2} \\
B^j \arrow[urr, "0=d^{j+1}_{A_1}d^j_{A_1}\pi^j_1"] \arrow[drr, "0=d^{j+1}_{A_n}d^j_{A_n}\pi^j_n"'] \arrow[r, dotted, "u"] & B^{j+2} \arrow[ur, "\pi_1^{j+2}"'] \arrow[dr, "\pi_n^{j+2}"] & \dots \\
 & & A_n^{j+2}
\end{tikzcd}$$

The 0-morphism $B^j \to B^{j+2}$ makes this diagram commute, so $u = 0$. But we have

$$\pi_i^{j+2}\delta^{j+1}\delta^j = d^{j+1}_{A_i}\pi_i^{j+1}\delta^j = d^{j+1}_{A_i}d^j_{A_i}\pi^j_i$$

Therefore $\delta^{j+1}\delta^j$ also makes the diagram commute, hence by the uniqueness of u, we have $\delta^{j+1}\delta^j = 0$. $\square$

Lemma 4.1.3. $\mathsf{Ch}^\bullet(\mathsf{C})$ *fulfills **(AB3)**, i.e. it admits all kernels and cokernels.*

Proof. Let $\varphi^\bullet : A^\bullet \to B^\bullet$ be a morphism of cochains. Let $K^j := \ker\varphi^j, Q^j := \operatorname{coker}\varphi^j$. We want to construct morphisms $\delta^j : K^j \to K^{j+1}, \partial^j : Q^j \to Q^{j+1}$ such that $K^\bullet, Q^\bullet$ become cochains themselves.

$$\begin{tikzcd}
\dots & K^j \arrow[d, "k^j"] & K^{j+1} \arrow[d, "k^{j+1}"] & \dots \\
\dots \arrow[r] & A^j \arrow[r, "d^j_A"] \arrow[d, "\varphi^j"] & A^{j+1} \arrow[r] \arrow[d, "\varphi^{j+1}"] & \dots \\
\dots \arrow[r] & B^j \arrow[r, "d^j_B"] \arrow[d, "q^j"] & B^{j+1} \arrow[r] \arrow[d, "q^{j+1}"] & \dots \\
\dots & Q^j & Q^{j+1} & \dots
\end{tikzcd}$$

From the diagram above we find the commutative diagram

$$\begin{tikzcd}
& K^j \arrow[ddl, "d_A^j k^j"'] \arrow[ddr, "\varphi^{j+1} d_A^j k^j"] & \\
& K^{j+1} \arrow[dl, "k^{j+1}"] \arrow[dr, "0"'] & \\
A^{j+1} \arrow[rr, "\varphi^{j+1}"'] & & B^{j+1}
\end{tikzcd}$$

But we also know that $\varphi^{j+1} d_A^j k^j = d_B^j \varphi^j k^j = d_B^j 0 = 0$. So by the universal property of K^{j+1} there exists a unique morphism $K^j \to K^{j+1}$ keeping the diagram commutative, and we will call it δ^j.

$$\begin{tikzcd}
& K^j \arrow[ddl, "d_A^j k^j"'] \arrow[ddr, "0"] \arrow[d, dashed, "\delta^j"] & \\
& K^{j+1} \arrow[dl, "k^{j+1}"] \arrow[dr, "0"'] & \\
A^{j+1} \arrow[rr, "\varphi^{j+1}"'] & & B^{j+1}
\end{tikzcd}$$

We can use a similar observation to construct ∂^j and thus end up with:

$$\begin{tikzcd}
\dots \arrow[r] & K^j \arrow[r, "\delta^j"] \arrow[d, "k^j"] & K^{j+1} \arrow[r] \arrow[d, "k^{j+1}"] & \dots \\
\dots \arrow[r] & A^j \arrow[r, "d_A^j"] \arrow[d, "\varphi^j"] & A^{j+1} \arrow[r] \arrow[d, "\varphi^{j+1}"] & \dots \\
\dots \arrow[r] & B^j \arrow[r, "d_B^j"] \arrow[d, "q^j"] & B^{j+1} \arrow[r] \arrow[d, "q^{j+1}"] & \dots \\
\dots \arrow[r] & Q^j \arrow[r, "\partial^j"] & Q^{j+1} \arrow[r] & \dots
\end{tikzcd}$$

It is easy to see that $K^\bullet$ actually is a cochain: $k^{j+1}\delta^{j+1}\delta^j = d_A^{j+1} d_A^j k^j = 0$. But since k^{j+2} is monic, it follows that $\delta^{j+1}\delta^j = 0$. The argument for $Q^\bullet$ is analogous.

It remains to show that $K^\bullet, Q^\bullet$ fulfill the respective universal properties of the kernel, respectively cokernel, in $\mathsf{Ch}^\bullet(\mathsf{C})$. Suppose we have another cochain morphism $\ell^\bullet : L^\bullet \to A^\bullet$ such that $\varphi^\bullet \ell^\bullet = 0$, which translates to $\varphi^j \ell^j = 0$ for all j. Then by the universal property of the K^j, there exist unique morphism $u^j : L^j \to K^j$ such that $\ell^j = k^j u^j$. We want to show $\delta^j u^j = u^{j+1} d_L^j$, i.e. that $u^\bullet$ is a morphism of cochains.

$$\begin{aligned}
d_A^j \ell^j &= \ell^{j+1} d_L^j \\
d_A^j k^j u^j &= k^{j+1} u^{j+1} d_L^j \\
k^{j+1} \delta^j u^j &= k^{j+1} u^{j+1} d_L^j
\end{aligned}$$

Since k^{j+1} is monic, we have $\delta^j u^j = u^{j+1} d_L^j$. So $K^\bullet$ indeed fulfills the universal property of the kernel in $\mathsf{Ch}^\bullet(\mathsf{C})$. Analogously, one can show that $Q^\bullet$ is the cokernel in $\mathsf{Ch}^\bullet(\mathsf{C})$, concluding our proof.

□

Lemma 4.1.4. $\mathsf{Ch}^\bullet(\mathsf{C})$ *fulfills **(AB4)**, i.e. it is binormal.*

Corollary 4.1.5. *The category* $\mathsf{Ch}^\bullet(\mathsf{C})$ *is abelian.*

4.2 The Cohomology Functors

Let $A^\bullet$ be a cochain complex. By the universal property of the kernel $\mathsf{ker}\, d^j$, the coboundary operator d^{j-1} factors through k^j, yielding a unique morphism $e^{j-1} : A^{j-1} \to \mathsf{ker}\, d^j$ as in the diagram below:

$$\begin{tikzcd}
 & A^{j-1} \arrow[dl, "d^{j-1}"'] \arrow[d, dotted, "e^{j-1}"] \arrow[dr, "d^j d^{j-1}=0"] & \\
 & \mathsf{ker}\, d^j \arrow[dl, "k^j"] \arrow[dr, "0"] & \\
A^j \arrow[rr, "d^j"'] & & A^{j+1}
\end{tikzcd}$$

Definition 4.2.1. The ***cohomology in degree*** j of a cochain complex is the object

$$H^j(A^\bullet) := \mathsf{coker}\,(e^{j-1} : A^{j-1} \to \mathsf{ker}\, d^j)$$

Example 4.2.2. Let k be a ring. In $\mathsf{k}\text{-}\mathsf{Mod}$, we can explicitly compute the cohomology:

$$\begin{aligned} H^j(A^\bullet) &= \mathsf{coker}\, e^{j-1} \\ &= {}^{\mathsf{ker}\, d^j}\!\big/_{\mathsf{im}\, e^{j-1}} \\ &= {}^{\mathsf{ker}\, d^j}\!\big/_{\mathsf{im}\, d^{j-1}} \end{aligned}$$

This agrees with our definition of the cohomology in the first chapter.

Lemma 4.2.3. *The assignments* $H^j : \mathsf{Ch}^\bullet(\mathsf{C}) \to \mathsf{C}$ *are additive functors.*

Proof. Let $\varphi^\bullet : A^\bullet \to B^\bullet$ be a cochain morphism. From the commuting diagram

$$\begin{tikzcd}
\mathsf{ker}\, d^j \arrow[d, "k^j"] \arrow[ddd, dotted, bend right, "\alpha^j"'] & \\
A^j \arrow[r, "d^j"] \arrow[d, "f^j"] & A^{j+1} \arrow[d, "f^{j+1}"] \\
B^j \arrow[r, "\delta^j"] & B^{j+1} \\
\mathsf{ker}\, \delta^j \arrow[u, "\kappa^j"'] &
\end{tikzcd}$$

we get a unique induced morphism $\alpha^j : \ker d^j \to \ker\delta^j$ by the universal property of $\ker\delta^j$, since $\delta^j f^j k^j = f^{j+1} d^j k^j = f^{j+1}0 = 0$. Furthermore, by the universal property of $H^j(B) = \operatorname{coker} e^{j-1}$, we get the morphism $H^j(f) : H^j(A) \to H^j(B)$:

$$\begin{tikzcd}
& \ker d^j \arrow[r, "\ell^j"] \arrow[d, "k^j"] \arrow[ddd, dotted, bend right, "\alpha^j"'] & H^j(A) \arrow[ddd, dotted, bend left, "H^j(f)"] \\
A^{j-1} \arrow[ur, "e^{j-1}"] \arrow[r, "d^{j-1}"] \arrow[d, "f^{j-1}"] & A^j \arrow[r, "d^j"] \arrow[d, "f^j"] & A^{j+1} \arrow[d, "f^{j+1}"'] \\
B^{j-1} \arrow[r, "\delta^{j-1}"'] \arrow[dr, "\varepsilon^{j-1}"'] & B^j \arrow[r, "\delta^j"'] & B^{j+1} \\
& \ker\delta^j \arrow[u, "\kappa^j"'] \arrow[r, "\lambda^j"] & H^j(B)
\end{tikzcd}$$

because there is the commuting subdiagram

$$\begin{tikzcd}
A^{j-1} \arrow[rr, "e^{j-1}"] \arrow[dr, "0"] \arrow[ddr, "\lambda^j\varepsilon^{j-1}f^{j-1}"'] & & \ker d^j \arrow[dl, "\ell^j"'] \arrow[ddl, "\lambda^j\alpha^j"] \\
& H^j(A) & \\
& H^j(B) &
\end{tikzcd}$$

where $\lambda^j\varepsilon^{j-1}f^{j-1} = 0f^{j-1} = 0$. Hence, H^j also acts on morphisms. To further check that it is a functor, we need to show: $H^j(\mathrm{id}_A) = \mathrm{id}_{H^j(A)}$ and that for $f : A \to B, g : B \to C : H^j(g \circ f) = H^j(g) \circ H^j(f)$. The first property is evident if one follows the construction of $H^j(\mathrm{id})$. The second one involves a very large diagram to verify, so we will not present the whole proof here, but in short: With the same construction as before, we get $\alpha^j : \ker d^j \to \ker\delta^j$ induced by f and $\beta^j : \ker\delta^j \to \ker\partial^j$ induced by g, but also $\gamma : \ker d^j \to \ker\partial^j$ induced by $g \circ f$. And by the universal property of $\ker\partial^j$ we know that γ^j is the unique morphism making the involved diagram commute. Because $\beta^j \circ \alpha^j$ also does so, $\beta^j \circ \alpha^j = \gamma^j$ and the hence also $H^j(g \circ f) = H^j(g) \circ H^j(f)$.

Next, we show the additivity of H^j: Let $f_1, f_2 : A \to B$ be two cochain morphisms. We get $\alpha_1^j, \alpha_2^j : \ker d^j \to \ker\delta^j$ and $H^j(f_1), H^j(f_2) : H^j(A) \to H^j(B)$. We also get σ^j induced by $(f_1 + f_2)^j = f_1^j + f_2^j$, but the diagram

$$\begin{tikzcd}
& \ker d^j \arrow[d, dotted] \arrow[ddl, "(f_1+f_2)^j \circ k^j"'] \arrow[ddr, "0"] & \\
& \ker\delta^j \arrow[dl, "\kappa^j"] \arrow[dr, "0"'] & \\
B^j \arrow[rr, "\delta^j"'] & & B^{j+1}
\end{tikzcd}$$

is also made commutative by $\alpha_1^j + \alpha_2^j$ instead of σ^j, so by the uniqueness part of the universal property, $\sigma^j = \alpha_1^j + \alpha_2^j$. The analogous argument goes for the universal property of $H^j(B)$, and so $H^j(f_1 + f_2) = H^j(f_1) + H^j(f_2)$.

$\square$

Definition 4.2.4. Let $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ be two morphisms of cochain complexes, then a ***homotopy*** $\eta^\bullet$ between them is a family of morphisms $\eta^j : A^j \to B^{j-1}$ such that

$$\eta^{j+1}d^j + \delta^{j-1}\eta^j = f^j - g^j$$

The order of $f^\bullet$ and $g^\bullet$ does not matter, in the sense that if we take $\zeta^\bullet := -\eta^\bullet$, this clearly fulfills $\zeta^{j+1}d^j + \delta^{j-1}\zeta^j = g^j - f^j$ and hence we call two cochain morphisms ***homotopic*** if there exists a homotopy between them.

A morphism $h^\bullet : A^\bullet \to B^\bullet$ is called ***null-homotopic*** if it is homotopic to the 0-morphism, i.e. $\eta^{j+1}d^j + \delta^{j-1}\eta^j = h^j$. Thus, another way of saying that $f^\bullet$ and $g^\bullet$ are homotopic is to say that their difference is null-homotopic.

Remark. For $A^\bullet, B^\bullet \in \mathsf{Ch}^\bullet(\mathsf{C})$, the relation $f^\bullet \sim g^\bullet \Leftrightarrow f - g$ is null-homotopic is an equivalence relation on $\mathsf{Hom}(A^\bullet, B^\bullet)$, and the sum of two null-homotopic morphisms also is null-homotopic. Thus, $\mathsf{Hom}(A^\bullet, B^\bullet)/\sim$ is an abelian group and we can define the ***homotopy category of cochains*** of C to be the category $\mathsf{K}^\bullet(\mathsf{C})$ that has the objects of $\mathsf{Ch}^\bullet\mathsf{C}$ as objects and the morphisms

$$\mathsf{Hom}_{\mathsf{K}^\bullet(\mathsf{C})}(A^\bullet, B^\bullet) = \mathsf{Hom}(A^\bullet, B^\bullet)/\sim$$

This also is an abelian category.

Proposition 4.2.5. *If f, g are homotopic, then $H^j(f) = H^j(g)$*

Proof. We know that $f^j - g^j = \eta^{j+1}d^j + \delta^{j-1}\eta^j$. In the construction of $H^j(f-g)$, the induced morphism α^j is the unique morphism that makes the diagram

$$\begin{tikzcd}
& \ker d^j \arrow[d, dotted] \arrow[ddl, "(f^j-g^j)k^j"'] \arrow[ddr, "0"] & \\
& \ker \delta^j \arrow[dl, "\kappa^j"] \arrow[dr, "0"'] & \\
B^j \arrow[rr, "\delta^j"'] & & B^{j+1}
\end{tikzcd}$$

commute. Hence

$$\begin{aligned}
\kappa^j\alpha^j &= (f^j - g^j)k^j \\
&= (\eta^{j+1}d^j + \delta^{j-1}\eta^j)k^j \\
&= \eta^{j+1}d^jk^j + \delta^{j-1}\eta^jk^j \\
&= \eta^{j+1}0 + \delta^{j-1}\eta^jk^j \\
&= \delta^{j-1}\eta^jk^j = \kappa^j\varepsilon^{j-1}\eta^jk^j
\end{aligned}$$

Since κ^j is monic, we get $\alpha^j = \varepsilon^{j-1}\eta^j k^j$. Taking a look at the diagram

$$\begin{tikzcd}
A^{j-1} \arrow[rr, "e^{j-1}"] \arrow[dr, "0"] \arrow[ddr, "0"'] & & \ker d^j \arrow[dl, "\ell^j"'] \arrow[ddl, "\lambda^j\alpha^j"] \\
& H^j(A) & \\
& H^j(B) &
\end{tikzcd}$$

which induces $H^j(f-g)$, we notice that $\lambda^j\alpha^j = \lambda^j\varepsilon^{j-1}\eta^j k^j = 0\eta^j k^j$, by the definition of λ^j as the cokernel morphism of ε^j. Hence, the unique induced morphism that makes the above diagram commutative must be 0, so $H^j(f-g) = 0$ and therefore, by the additivity of H^j, $H^j(f) = H^j(g)$. $\square$

Corollary 4.2.6. *H^j also is well-defined as an additive functor $\mathsf{K}^\bullet(\mathsf{C}) \to \mathsf{C}$.*

Definition 4.2.7. A pair of morphisms $f^\bullet : A^\bullet \to B^\bullet, g^\bullet : B^\bullet \to A^\bullet$ is a ***homotopy equivalence*** between $A^\bullet$ and $B^\bullet$ if $g^\bullet f^\bullet$ is homotopic to $\mathsf{id}_{A^\bullet}$, and $f^\bullet g^\bullet$ is homotopic to $\mathsf{id}_{B^\bullet}$. Note that this is the same as saying that $[f^\bullet], [g^\bullet]$ are inverse isomorphisms in $\mathsf{K}^\bullet(\mathsf{C})$.

4.3 Exact Sequences

Definition 4.3.1. A cochain complex is called ***exact*** or an ***exact sequence*** if its cohomology is 0 in every degree.

Lemma 4.3.2 (Characterization of Exactness)**.** *Consider the sequence $U \xrightarrow{f} V \xrightarrow{g} W$, where $gf = 0$. This induces the diagram*

$$\begin{tikzcd}
& \ker g \arrow[d, "k"] & \\
U \arrow[ur, "e"] \arrow[r, "f"] & V \arrow[r, "g"] \arrow[d, "q"'] & W \\
& \operatorname{coker} f \arrow[ur, "m"'] &
\end{tikzcd}$$

where e and m are induced by the universal properties of $\ker g$ and $\operatorname{coker} f$, respectively. The following are equivalent:

(i) $\ker m = 0$, *i.e. m is monic.*

(ii) $\operatorname{coker} e = 0$, *i.e. e is epic.*

(iii) $\operatorname{im} f \cong \ker g$.

(iv) $\operatorname{coker} f \cong \operatorname{im} g$

Proof. "(i) $\Rightarrow$ (iii)": Let $L := \mathsf{ker}\,\mathsf{coker}\, f \cong \mathsf{im}\, f$, $K := \mathsf{ker}\, g$ and $Q := \mathsf{coker}\, f$. We have the commutative diagram

$$
\begin{tikzcd}
 & & K \arrow[d, "k"] & \\
U \arrow[urr, "e"] \arrow[rr, "f"] & & V \arrow[r, "g"] \arrow[d, "q"] & W \\
 & L \arrow[ur, dashed, "\ell"] \arrow[r, "0"'] & Q \arrow[ur, "m"'] &
\end{tikzcd}
$$

We see that $g\ell = mq\ell = 0$ and therefore, by the universal property of K, there is a unique morphism $u : L \to K$ that leaves the diagram commutative. Next, we see that $mqk = gk = 0$, but we know that m is monic, so $qk = 0$. Hence, by the universal property of L, there also is a morphism $v : K \to L$ that keeps the diagram commutative. But then $vu : L \to L$ and $uv : K \to K$ keep the diagram commutative. By the uniqueness part of the universal properties of L and K, these must be the respective identities, yielding an isomorphism $L \cong K$.

"(ii)$\Rightarrow$ (iii)": We start with the same commutative diagram as above, but this time notice that $qke = qf = 0$ and because e is epic, we have again have $qk = 0$. The rest follows as before.

"(iii)$\Rightarrow$(i)": See the remark below.

"(iii)$\Rightarrow$(ii)": Since $K \cong \mathsf{im}\, f$, the morphism $e : U \to K \cong \mathsf{im}\, f$ has to be an epimorphism by the properties of the image.

"(ii)$\Rightarrow$(iv)": Since e is epic, we have

$$\mathsf{im}\, g \cong \mathsf{coker}\,\mathsf{ker}\, g = \mathsf{coker}\, k \cong \mathsf{coker}\,(ke) = \mathsf{coker}\, f$$

"(iv)$\Rightarrow$(iii)": See the remark below.

□

Remark. I was not able to prove the steps"(iii)$\Rightarrow$(i)" and "(iv)$\Rightarrow$(iii)" in all generality, hence I will only present a proof in $\mathsf{k\text{-}Mod}$ and refer to the Freyd-Mitchell theorem:

"(iii)$\Rightarrow$(i)": When talking about modules, we not only have $\mathsf{im}\, f \cong \mathsf{ker}\, g$, but in fact $\mathsf{im}\, f = \mathsf{ker}\, g$. Furthermore $\mathsf{coker}\, f = {}^{V}\!/_{\mathsf{im}\, f}$, with $q : V \to \mathsf{coker}\, f : v \mapsto v + \mathsf{ker}\, g$. Thus, the map $m : \mathsf{coker}\, f \to W$ is given by $m(v + \mathsf{ker}\, g) = g(v)$, and hence is monic, since $m(v + \mathsf{ker}\, g) = 0$ means $v \in \mathsf{ker}\, g$, i.e. $v + \mathsf{ker}\, g = 0$.

"(iv)$\Rightarrow$(iii)": By the first isomorphism theorem, $\mathsf{im}\, g \cong {}^{V}\!/_{\mathsf{ker}\, g}$. Again, $\mathsf{coker}\, f = {}^{V}\!/_{\mathsf{im}\, f}$, thus we have ${}^{V}\!/_{\mathsf{ker}\, g} \cong {}^{V}\!/_{\mathsf{im}\, f}$, and hence $\mathsf{ker}\, g \cong \mathsf{im}\, f$.

Lemma 4.3.3 (Weaving Lemma)**.** *Given a long exact sequence*

$$\ldots \longrightarrow A^{j-1} \xrightarrow{d^{j-1}} A^{j} \xrightarrow{d^{j}} A^{j+1} \xrightarrow{d^{j+1}} \ldots$$

we have $\mathsf{im}\, d^{j-1} \cong \mathsf{ker}\, d^{j} =: K^{j}$ *and thus we obtain short exact sequences*

$$0 \longrightarrow K^{j} \xrightarrow{k^{j}} A^{j} \xrightarrow{e^{j+1}} K^{j+1} \longrightarrow 0$$

where k^j is the natural kernel morphism and e^{j+1} is the epimorphism such that the monomorphism $k^{j+1} : K^{j+1} \to A^{j+1}$ together with e^{j+1} constitutes the image of d^j. These short exact sequences can be displayed in the following commutative diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & & & K^j \arrow[ur] \arrow[dr, "k^j"] & & & \\
\cdots \arrow[rr, "d^{j-2}"] \arrow[dr, "e^{j-1}"'] & & A^{j-1} \arrow[rr, "d^{j-1}"] \arrow[ur, "e^j"] & & A^j \arrow[rr, "d^j"] \arrow[dr, "e^{j+1}"'] & & \cdots \\
 & K^{j-1} \arrow[ur, "k^{j-1}"'] \arrow[dr] & & & & K^{j+1} \arrow[ur, "k^{j+1}"'] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}
$$

In this sense, every long exact sequence can be obtained by "weaving together" short exact sequences, which are the simpler objects to study.

Lemma 4.3.4 (Splitting Lemma)**.** *For a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following are equivalent:*

*(i) There exists a **left split** or **retraction** of f, i.e. a morphism $r : B \to A$ such that $rf = \mathsf{id}_A$.*

*(ii) There exists a **right split** or **section** of g, i.e. a morphism $s : C \to B$ such that $gs = \mathsf{id}_C$.*

(iii) There exists an isomorphism $\varphi : B \to A \oplus C$ such that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mathsf{id}_A"] & B \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "\mathsf{id}_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota_A"'] & A \oplus C \arrow[r, "\pi_C"'] & C \arrow[r] & 0
\end{tikzcd}
$$

*If any of these equivalent conditions is met, the sequence is called **split exact**.*

Lemma 4.3.5. *An additive functor $F : \mathsf{C} \to \mathsf{D}$ between abelian categories takes split exact sequences to split exact sequences.*

Proof. Consider a spit exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mathsf{id}_A"] & B \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "\mathsf{id}_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota_A"'] & A \oplus C \arrow[r, "\pi_C"'] & C \arrow[r] & 0
\end{tikzcd}
$$

Since F is additive, it takes this diagram to

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r, "F(f)"] \arrow[d, "\mathsf{id}_{F(A)}"] & F(B) \arrow[r, "F(g)"] \arrow[d, "F(\varphi)"] & F(C) \arrow[r] \arrow[d, "\mathsf{id}_{F(C)}"] & 0 \\
0 \arrow[r] & F(A) \arrow[r, "\iota_{F(A)}"'] & F(A) \oplus F(C) \arrow[r, "\pi_{F(C)}"'] & F(C) \arrow[r] & 0
\end{tikzcd}$$

Since $F(\varphi)$ is an isomorphism and the second row is exact, it follows that the sequence in the first row is split exact. $\square$

4.4 Injective Resolutions

Definition 4.4.1. An ***injective object*** I in an abelian category is one such that for all monomorphisms $\iota : A \to B$ and all other morphisms $\varphi : A \to I$ there is a morphism $\psi : B \to I$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "\iota"] \arrow[dr, "\varphi"'] & & B \arrow[dl, dashed, "\psi"] \\
& I &
\end{tikzcd}$$

We say that C has ***sufficiently many*** or ***enough injectives*** if for all $A \in \mathsf{C}$ there exists a monomorphism $m : A \to I$ for some injective object $I \in \mathsf{C}$.

Remark. This generalizes the notion of an injective module: Let k be some ring and $\iota : A \to B$ a monomorphism in k-Mod, i.e. an injective module homomorphism. Then we can view A as a submodule of B. By the property of an injective module I, we can extend any given homomorphism $\varphi : A \to I$ to some (not necessarily unique) homomorphism ψ from the entire module B to I.

It has been shown, first by Baer ([1]) in 1940, and then in a simplified proof by Eckmann and Schopf ([3]) in 1953, that for any unital ring k, the category k-Mod has enough injectives.

Definition 4.4.2. A ***resolution*** $A^\bullet$ of an object A is an exact cochain complex

$$0 \longrightarrow A \xrightarrow{\alpha} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \ldots$$

An ***injective resolution*** of A is a resolution where all the objects A^j are injective.

Lemma 4.4.3. *If C has enough injectives, then every object A has an injective resolution:*

Proof. Since C has enough injectives, we can choose a monomorphism $\alpha : A \to I^0$ into some injective object I^0. We have the natural map $q^\alpha : I^0 \to \operatorname{\mathsf{coker}} \alpha$ and can choose another monomorphism $\iota^0 : \operatorname{\mathsf{coker}} \alpha \to I^1$ into some injective object I^1. With this we construct $d^0 = \iota^0 q^\alpha : I^0 \to I^1$.

Having constructed d^j, we continue inductively by choosing a monomorphism $\iota^{j+1} : \operatorname{\mathsf{coker}} d^j \to I^{j+1}$ into some injective object I^{j+1} and then setting

$$d^{j+1} = \iota^{j+1} q^j : I^j \to I^{j+1}$$

We thus obtain the sequence

$$\begin{tikzcd}
 & & \mathsf{coker}\,\alpha \arrow[dr, dotted, "\iota^0"] & & \mathsf{coker}\,d^0 \arrow[dr, dotted, "\iota^1"] & \\
A \arrow[r, dotted, "\alpha"] & I^0 \arrow[ur, "q^\alpha"] \arrow[rr, "d^0"] & & I^1 \arrow[ur, "q^0"] \arrow[rr, "d^1"] & & \ldots
\end{tikzcd}$$

where $d^{j+1}d^j = \iota^{j+1}q^jd^j = 0$, and it remains to see the exactness of this complex:

$$\mathsf{ker}\,d^{k+1} \cong \mathsf{ker}\left(\iota^{k+2}q^{k+1}\right) \cong \mathsf{ker}\,q^{k+1} = \mathsf{ker}\,\mathsf{coker}\,d^k \cong \mathsf{im}\,d^k$$

$\square$

Lemma 4.4.4. *Let $M^j, j \geq 0$ and $I^j, j \geq 0$ be resolutions of objects A and B, respectively, with the second one being injective. Then for any morphism $f : A \to B$ there exists an extension of f to a morphism of complexes $f : M \to I$.*

Proof.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"] & M^0 \arrow[r, "d^0"] \arrow[d, dotted, "f^0"] & M^1 \arrow[r, "d^1"] & M^2 \arrow[r, "d^2"] & \ldots \\
0 \arrow[r] & B \arrow[r, "\beta"'] & I^0 \arrow[r, "\delta^0"'] & I^1 \arrow[r, "\delta^1"'] & I^2 \arrow[r, "\delta^2"'] & \ldots
\end{tikzcd}$$

First, we can extend the morphisms $\beta \circ f : A \to I^0$ to $f^0 : M^0 \to I^0$, since $\alpha : A \to M^0$ is monic and I^0 is injective. The morphism $f^1 : M^1 \to I^1$ then is constructed as follows: First, we notice that $d^0 : M^0 \to M^1$ factors through a morphism $\overline{d^0} : \mathsf{coker}\,\alpha \to M^1$:

$$\begin{tikzcd}
 & \mathsf{coker}\,\alpha \arrow[dr, dotted, "\overline{d^0}"] & \\
A \arrow[r, "\alpha"'] & M^0 \arrow[u, "q^\alpha"] \arrow[r, "d^0"'] & M^1
\end{tikzcd}$$

This follows from $d^0 \circ \alpha = 0$ and the universal property of $\mathsf{coker}\,\alpha$. By the exactness of the extension at M^0 and Proposition 4.26, the morphism $\overline{d^0}$ is monic. Furthermore, the natural property of $\mathsf{coker}\,\alpha$ induces a morphism $v^0 : \mathsf{coker}\,\alpha \to I^1$, since $\delta^0 \circ f^0 \circ \alpha = \delta^0 \circ \beta \circ f = 0$. Because I^1 is injective and $\overline{d^0}$ is monic, there exists an extension of $\overline{d^0}$ to a morphism $f^1 : M^1 \to I^1$.

$$\begin{tikzcd}
 & & \mathsf{coker}\,\alpha \arrow[dr, dotted, "\overline{d^0}"] \arrow[ddr, dotted, "v^0"'] & & & \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"] & M^0 \arrow[u, "q^\alpha"] \arrow[r, "d^0"] \arrow[d, dotted, "f^0"] & M^1 \arrow[r, "d^1"] \arrow[d, dotted, "f^1"] & M^2 \arrow[r, "d^2"] & \ldots \\
0 \arrow[r] & B \arrow[r, "\beta"'] & I^0 \arrow[r, "\delta^0"'] & I^1 \arrow[r, "\delta^1"'] & I^2 \arrow[r, "\delta^2"'] & \ldots
\end{tikzcd}$$

Continuing inductively, we suppose that we have already constructed $f^j : M^j \to I^j$. We then factor d^k through $\overline{d^k} : \mathsf{coker}\,d^{k-1} \to M^{k+1}$ which by the exactness of the extension and Proposition 4.26 is monic. Additionally, we obtain $v^k : \mathsf{coker}\,d^{k-1} \to I^{k+1}$, as before, and construct $f^{k+1} : M^{k+1} \to I^{k+1}$ as its extension by the injectivity of I^{k+1}. $\square$

Corollary 4.4.5. *In the situation of lemma 4.5.4., given two extension $f^\bullet, g^\bullet$ of the same morphism $f : A \to B$, there exists a homotopy between them, meaning that such an extension is uniquely determined up to homotopy.*

Proof. Since $f^0 \circ \alpha = \beta \circ f = g^0 \circ \alpha$, we follow that $(f^0 - g^0) \circ \alpha = 0$ as a morphism $A \to I^0$. Thus, by the universal property of $\operatorname{coker} \alpha$, there exists an induced morphism $v^0 : \operatorname{coker} \alpha \to I^0$. Again, we can extend this morphism to M^1, since $\overline{d^0}$ is monic and I^0 is injective. Hence, we obtain a morphism $\eta^1 : M^1 \to I^0$, and by setting $\eta^0 : M^0 \to B$ to be the 0-morphism, this fulfills $f^0 - g^0 = \eta^1 d^0 + \beta\eta^0$.

In order to construct the homotopy in higher degrees, we continue inductively: Let $\delta^{-1} := \beta$. Then, having constructed η^j, we consider the morphism $h^j = f^j - g^j - \delta^{j-1}\eta^j$, for which we have

$$\begin{aligned} h^j d^{j-1} &= (f^j - g^j)d^{j-1} - \delta^{j-1}\eta^j d^{j-1} \\ &= \delta^{j-1}(f^{j-1} - g^{j-1}) - \delta^{j-1}\eta^j d^{j-1} \\ &= \delta^{j-1}(\delta^{j-2}\eta^{j-1} + \eta^j d^{j-1}) - \delta^{j-1}\eta^j d^{j-1} \\ &= \delta^{j-1}\delta^{j-2}\eta^{j-1} + \delta^{j-1}\eta^j d^{j-1} - \delta^{j-1}\eta^j d^{j-1} \\ &= 0 \end{aligned}$$

Thus, by the universal property of $q^{j-1} : M^j \to \operatorname{coker} d^{j-1}$, there exists a unique morphism $v^j : \operatorname{coker} d^{j-1} \to I^j$ such that $v^j q^{j-1} = h^j$. Again, we have the monic morphism $\overline{d^j} : \operatorname{coker} d^{j-1} \to M^{j+1}$ and by the injectivity of I^j, we can extend v^j to a morphism $\eta^{j+1} : M^{j+1} \to I^j$ such that $\eta^{j+1}\overline{d^j} = v^j$ and thus $\eta^{j+1}d^j = v^j q^{j-1} = h^j = f^j - g^j - \delta^{j-1}\eta^j$. Therefore, $f^j - g^j = \eta^{j+1}d^j + \delta^{j-1}\eta^j$, as required for $\eta^\bullet$ to be a homotopy. □

Corollary 4.4.6. *Injective resolutions are unique up to homotopy equivalence.*

Proof. Let $I^\bullet, J^\bullet$ both be injective resolutions of A. By lemma 4.5.4, the identity $\mathrm{id}_A : A \to A$ induces two extensions $f^\bullet : I^\bullet \to J^\bullet$ and $g^\bullet : J^\bullet \to I^\bullet$. Composition yields $g^\bullet f^\bullet : I^\bullet \to I^\bullet$ and $f^\bullet g^\bullet : J^\bullet \to J^\bullet$, which are both extensions of id_A to morphisms $I^\bullet \to I^\bullet$ and $J^\bullet \to J^\bullet$, respectively. By corollary 4.5.5, these are homotopic to the identity of $I^\bullet$, respectively $J^\bullet$, and thus constitute a homotopy equivalence between $I^\bullet$ and $J^\bullet$. □

Lemma 4.4.7. *For a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ there exist injective resolutions $0 \to A \to I^\bullet, 0 \to B \to J^\bullet, 0 \to C \to K^\bullet$ and extensions $f^\bullet, g^\bullet$ such that $0 \to I^\bullet \xrightarrow{f^\bullet} J^\bullet \xrightarrow{g^\bullet} K^\bullet \to 0$ is a short exact sequence*

Proof. VOISIN [12], p. 100 □

Lemma 4.4.8. *A short exact sequence $0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$, where I is injective, is split exact.*

Proof. Since the morphism $f : I \to B$ is monic and I is injective, we can extend the identity id_I to a morphism $r : J \to I$. We see that, by the definition of this extension, $rf = \mathsf{id}_I$, so there exists a retraction of f and hence the sequence is split exact.

$$
\begin{tikzcd}
I \arrow[rr, "f"] \arrow[dr, "\mathsf{id}_I"'] & & B \arrow[dl, dotted, "r"] \\
& I &
\end{tikzcd}
$$

$\square$

4.5 Derived Functors

Theorem 4.5.1 (Snake Lemma). *In an abelian category C, consider the following commuting diagram where both rows are exact:*

$$
\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}
$$

Then there is a unique morphism $\partial : \mathsf{ker}\, c \to \mathsf{coker}\, a$ such that the sequence

$$\mathsf{ker}\, a \to \mathsf{ker}\, b \to \mathsf{ker}\, c \xrightarrow{\partial} \mathsf{coker}\, a \to \mathsf{coker}\, b \to \mathsf{coker}\, c$$

is exact and the following diagram commutes

$$
\begin{tikzcd}
& \mathsf{ker}\, a \arrow[r, "u_f"] \arrow[d, "k_a"] & \mathsf{ker}\, b \arrow[r, "u_g"] \arrow[d, "k_b"] & \mathsf{ker}\, c \arrow[d, "k_c"] \arrow[dddll, dotted, "\partial"] & \\
& A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d, "q_a"] & B' \arrow[r, "g'"] \arrow[d, "q_b"] & C' \arrow[d, "q_c"] & \\
& \mathsf{coker}\, a \arrow[r, "v_f"] & \mathsf{coker}\, b \arrow[r, "v_g"] & \mathsf{coker}\, c &
\end{tikzcd}
$$

Proof. We will only prove this lemma for the simpler case of the category k-Mod for some ring k. Actually, this also implies the statement of the lemma in any abelian category, by the Freyd-Mitchell embedding theorem. A general proof without invoking the Frey-Mitchell theorem can be found in KASHIWARA, SHAPIRA [6], lemma 12.1.1.

First, we extend the diagram by the kernels and cokernels of a, b, c, yielding exact columns. Next, we show the existence and uniqueness of the morphisms u_f, u_g, v_f, v_g: We have the morphism $fk_a : \mathsf{ker}\, a \to B$, for which $bfk_a = f'ak_a = 0$. So by the universal property of $\mathsf{ker}\, b$, there exists a unique morphism $u_f : \mathsf{ker}\, a \to \mathsf{ker}\, b$ that keeps the diagram commutative. The morphisms u_g, v_f, v_g are constructed analogously. In k-Mod, these morphism are actually just the restrictions of f, g, f', g', respectively, and thus $\mathsf{im}\, u_f \subseteq \mathsf{ker}\, u_g$, along with the other respective inclusions, is evident.

Moreover, the exactness can then be seen as follows: Suppose $y \in \mathsf{ker}\, b$ such that $u_g(y) = 0$, thus $g(y) = 0$ as well. Therefore, by the exactness of the first row, $y \in \mathsf{im}\, f$, i.e. here is some $x \in A$ such that $y = f(x)$. Since $y \in \mathsf{ker}\, b$, we have $0 = b(y) = b(f(x))$, and since the diagram is commutative, this means $f'(a(x)) = 0$. But by the exactness of the second row, f' is injective, and therefor $a(x) = 0$, i.e. $x \in \mathsf{ker}\, a$. Thus, there is an element $x \in \mathsf{ker}\, a$ such that $u_f(x) = f(x) = y$, hence $y \in \mathsf{im}\, u_f$ and the sequence is exact at $\mathsf{ker}\, b$. The exactness at $\mathsf{coker}\, b$ is shown similarly.

Now, it remains to construct the connecting morphism $\partial : \mathsf{ker}\, c \to \mathsf{coker}\, a$. Pick some $z \in \mathsf{ker}\, c \subseteq C$. By the exactness of the first row, g is surjective, so there exists $y \in B$ such that $g(y) = z$. By the commutativity of the diagram, $g'(b(y)) = c(g(y)) = c(z) = 0$, i.e. $b(y) \in \mathsf{ker}\, g' = \mathsf{im}\, f'$, so there is some $u \in A'$ such that $f'(u) = b(y)$. This element u furthermore is unique, since f' is injective, hence we can define

$$\partial(z) = u + \mathsf{im}\, a$$

This is well-defined, since if $y' \in B$ was some other element such that $g(y') = z$, we would have $g(y - y') = 0$, i.e. $y - y' \in \mathsf{ker}\, g = \mathsf{im}\, f$, thus there is some $x \in A$ such that $y - y' = f(x)$. Hence $f'(a(x)) = b(f(x)) = b(y - y') = b(y) - b(y') = f'(u) - f'(u') = f'(u - u')$, where u' is the element corresponding to y' in the same way as u to y. Since f' is injective, $a(x) = u - u'$, and therefore

$$u + \mathsf{im}\, a = u' + a(x) + \mathsf{im}\, a = u' + \mathsf{im}\, a$$

and the map ∂ is well-defined. It is furthermore easy to see that it is a homomorphism by its construction.

We now want to show the exactness at $\mathsf{ker}\, c$: Suppose $z \in \mathsf{im}\, u_g$, i.e. there is some $y \in \mathsf{ker}\, b$ such that $u_g(y) = z$. Then $g(y) = u_g(y) = z$, hence we may choose y for the construction of $\partial(z)$. But then $b(y) = y$ and hence, by the injectivity of f', the unique $u \in A'$ such that $f'(u) = b(y)$ must be 0, hence $\partial(z) = 0 + \mathsf{im}\, a = 0$.

On the other hand, suppose that $z \in \mathsf{ker}\, \partial$. Let $y \in B$ such that $g(y) = z$ and $u \in A'$ such that $f'(u) = b(y)$. But then $\partial(u) = u + \mathsf{im}\, a$, so $u \in \mathsf{im}\, a$, i.e. there exists $x \in A$ such that $u = a(x)$. We have $b(y) = f'(u) = f'(a(x)) = b(f(x))$, hence $b(y - f(x)) = 0$ and so $y - f(x) \in \mathsf{ker}\, b$. Therefore

$$u_g(y - f(x)) = g(y - f(x)) = g(y) - g(f(x)) = g(y) = z$$

so $z \in \mathsf{im}\, u_g$ and the sequence is exact at $\mathsf{ker}\, c$. The exactness at $\mathsf{coker}\, a$ is shown analogously.

$\square$

Theorem 4.5.2 (Zig-zag lemma). *Given a short exact sequence of cochain complexes $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ in an abelian category, there exist morphisms $\partial^n : H^n(C^\bullet) \to H^{n+1}(A^\bullet)$ such that the long sequence*

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H^{n-1}(C^\bullet) \arrow[dll, "\partial^{n-1}"] \\
H^n(A^\bullet) \arrow[r] & H^n(B^\bullet) \arrow[r] & H^n(C^\bullet) \arrow[dll, "\partial^{n}"] \\
H^{n+1}(A^\bullet) \arrow[r] & \cdots &
\end{tikzcd}
$$

is exact.

Proof. The proof in terms of element is analogous to that of the snake lemma. An element-free proof can be found in KASHIWARA, SHAPIRA [6], theorem 12.3.3. $\square$

Definition 4.5.3. A functor $F : \mathsf{C} \to \mathsf{D}$ is ***left-exact*** if for any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in C, the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact in D.

Theorem 4.5.4 (Derived Functors)**.** *Let $F : \mathsf{C} \to \mathsf{D}$ be an additive left exact functor of abelian categories. Then for $M \in \mathsf{C}$ and $j \geq 0$, there exist objects $R^jF(M)$ in D, called the **right derived functors** of F, such that:*

(1) $R^0F(M) = F(M)$

(2) For any injective object $I \in \mathsf{C}$, the right derived functors are trivial: $R^jF(I) = 0$, for $j > 0$.

(3) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence in C, there exist morphisms $\partial^j : R^jF(C) \to R^{j+1}F(A)$ such that the following sequence is exact in D.

$$
\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[dll, "\partial^0"] \\
 & R^1F(A) \arrow[r] & \cdots \arrow[r] & R^jF(C) \arrow[dll, "\partial^j"] \\
 & R^{j+1}F(A) \arrow[r] & \cdots &
\end{tikzcd}
$$

(4) The objects $R^jF(M)$ determined by the previous properties are unique up to isomorphism.

Proof. We proof the theorem by constructing the objects $R^jF(M)$. For M in C, we choose an injective resolution

$$0 \longrightarrow M \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \ldots$$

which by the additivity of F in turn yields a complex

$$0 \longrightarrow F(M) \xrightarrow{\beta} F(I^0) \xrightarrow{\delta^0} F(I^1) \xrightarrow{\delta^1} \ldots$$

with $\beta = F(\alpha), \delta^j = F(d^j)$, which is not necessarily exact. We truncate the complex on the left to obtain the complex

$$0 \longrightarrow F(I^0) \xrightarrow{\delta^0} F(I^1) \xrightarrow{\delta^1} \ldots$$

which we call $F(I^\bullet)$ and use it to define the derived functors:

$$R^jF(M) := H^j(F(I^\bullet))$$

These objects do not, up to isomorphism, depend on the choice of the injective resolution: Let $I^\bullet, J^\bullet$ be two injective resolutions of M. Then, by corollary 4.5.6, there exists a homotopy equivalence between them. That is, morphisms $\varphi : I^\bullet \to J^\bullet, \psi^\bullet : J^\bullet \to I^\bullet$ and homotopies $\eta^\bullet$ between $\psi^\bullet\varphi^\bullet$ and $\mathsf{id}_{I^\bullet}$, and $\zeta^\bullet$ between $\varphi^\bullet\psi^\bullet$ and $\mathsf{id}_{J^\bullet}$. Because F is additive, it takes the homotopies $\eta^\bullet, \zeta^\bullet$ to homotopies $F(\eta^\bullet), F(\zeta^\bullet)$ between $F(\psi^\bullet\varphi^\bullet)$ and $\mathsf{id}_{F(I^\bullet)}$, and between $F(\varphi^\bullet\psi^\bullet)$ and $\mathsf{id}_{JF(\bullet)}$, respectively. This means that $F(\varphi^\bullet)$ and $F(\psi^\bullet)$ constitute a homotopy equivalence in $\mathsf{Ch}^\bullet(\mathsf{D})$. The morphisms $F(\psi^\bullet\varphi^\bullet), F(\varphi^\bullet\psi^\bullet)$ induce morphisms on the cohomologies $H^j(F(I^\bullet))$ and $H^j(F(J^\bullet))$, respectively. Because of the homotopy equivalence, these the same as the morphisms induced by $\mathsf{id}_{I^\bullet}$ and $\mathsf{id}_{J^\bullet}$, in other words, they are the respective identities on the cohomologies. Thus, the morphisms induced by $F(\varphi^\bullet)$ and $F(\psi^\bullet)$ are inverse isomorphisms between the cohomologies of $F(I^\bullet)$ and $F(J^\bullet)$, showing that the objects $H^j(F(I^\bullet)) \cong H^j(F(J^\bullet))$ do not depend on the choice of the injective resolution.

(1) We apply the weaving lemma and get the short exact sequences

$$0 \longrightarrow K^0 \xrightarrow{k^0} I^0 \xrightarrow{e^1} K^1 \longrightarrow 0$$

$$0 \longrightarrow K^1 \xrightarrow{k^1} I^1 \xrightarrow{e^2} K^2 \longrightarrow 0$$

Since F is left exact, this yields exact sequences

$$0 \longrightarrow F(K^0) \xrightarrow{F(k^0)} F(I^0) \xrightarrow{F(e^1)} F(K^1)$$

$$0 \longrightarrow F(K^1) \xrightarrow{F(k^1)} F(I^1) \xrightarrow{F(e^2)} F(K^2)$$

From the second line we see that $F(k^1)$ is monic. But we also know

$$\delta^0 = F(d^0) = F(k^1 e^1) = F(k^1)F(e^1)$$

and hence $\operatorname{ker}\delta^0 \cong \operatorname{ker}F(e^1)$. Now we can read form the first line that $\operatorname{ker}F(e^1) \cong \operatorname{im}F(k^0) \cong F(K^0)$, since $F(k^1)$ is monic. Since $0 \to M \to I^\bullet$ is exact, we also know that $K^0 \cong \operatorname{im}\alpha \cong M$, since α is monic. Thus, we have $\operatorname{ker}\delta^0 \cong F(M)$, and in the complex

$$0 \longrightarrow F(I^0) \xrightarrow{\delta^0} F(I^1) \xrightarrow{\delta^1} ...$$

the induced morphism $0 \to \operatorname{ker}\delta^0$ used to define the cohomology at $F(I^0)$ must be the 0-morphism, hence $H^0(F(I^\bullet)) = \operatorname{coker}(0 \to \operatorname{ker}\delta^0) \cong \operatorname{ker}\delta^0 \cong F(M)$.

(2) For an injective object I we can choose the resolution $0 \to I \to I \to 0 \to ...$, for which $R^jF(I) = 0$ for all $j > 0$. Thus, this also holds for any other injective resolution of I.

(3) By lemma 4.5.7, there are injective resolutions $0 \to A \to I^\bullet, 0 \to B \to J^\bullet, 0 \to C \to K^\bullet$ and extensions $f^\bullet, g^\bullet$ such that $0 \to I^\bullet \xrightarrow{f^\bullet} J^\bullet \xrightarrow{g^\bullet} K^\bullet \to 0$ is a short exact sequence. By the zig-zag lemma, this induces the long exact sequence in question.

(4) Suppose that there are objects $R^jF(M)$ satisfying the conditions (1)-(3). Then again, we can split up the injective resolution $I^\bullet$ of M by the weaving lemma, yielding short exact sequences

$$0 \longrightarrow K^n \xrightarrow{k^n} I^n \xrightarrow{e^{n+1}} K^{n+1} \longrightarrow 0$$

By (3), these induce long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F(K^n) \arrow[r] & F(I^n) \arrow[r] & F(K^{n+1}) \arrow[dll, rounded corners, "\partial_n^0"] \\
& R^1F(K^n) \arrow[r] & 0 \arrow[r] & R^1F(K^{n+1}) \arrow[dll, rounded corners, "\partial_n^1"] \\
& R^2F(K^i) \arrow[r] & ... \arrow[r] & R^jF(K^{n+1}) \arrow[dll, rounded corners, "\partial_n^j"] \\
& R^{j+1}F(K^n) \arrow[r] & ... &
\end{tikzcd}$$

As seen before, $\operatorname{ker}\delta^n = \operatorname{ker}(k^{n+1}e^n) \cong \operatorname{ker}F(e^n) \cong \operatorname{im}F(k^n)$. Thus, the morphism δ^{n-1} factors through $F(k^n)$ via a unique ε^{n-1} such that $\delta^{n-1} = F(k^n)\varepsilon^{n-1}$. At the same time, $\delta^{n-1} = F(k^n)F(e^{n-1})$, thus $\varepsilon^{n-1} = F(e^{n-1})$. Therefore, the cohomology $H^n(I^\bullet)$ is given by $\operatorname{coker}\varepsilon^{n-1} = \operatorname{coker}F(e^{n-1})$. On the other hand, since the sequence

$$0 \longrightarrow F(K^n) \xrightarrow{F(k^n)} I^n \xrightarrow{F(e^{n+1})} F(K^{n+1}) \xrightarrow{\partial_n^0} R^1F(K^n) \longrightarrow 0$$

is exact, we have $\mathsf{coker}\, F(e^{n-1}) \cong \mathsf{im}\, \partial_n^0 \cong R^1F(K^n)$ by the lemma on the characterization of exactness.

Furthermore, we can read from the long exact zig-zag sequence above that $R^jF(K^{n+1}) \cong R^{j+1}F(K^n)$ for all $j \geq 1, n \geq 0$. Thus, inductively, we see $H^n(I^\bullet) \cong R^1F(K^n) \cong R^2F(K^{n-1}) \cong ... \cong R^{n+1}F(K^0) = R^{n+1}(M)$.

□

Lemma 4.5.5. *For a morphism $f^\bullet : A^\bullet \to B^\bullet$ and injective resolutions $0 \to A \to I^\bullet, 0 \to B \to J^\bullet$ there exist induced morphisms $R^j(f) : R^jF(A) \to R^jF(B)$*

Proof. VOISIN [12], p. 101 □

Remark. This is the functorial property due to which the objects $R^jF(M)$ are called derived functors, even though they are not well-defined functors $\mathsf{C} \to \mathsf{D}$, as they technically depend on the choice of the injective resolution. Still, the right derived functors measure the "non-exactness" to the right of an additive left exact functor, as they extend the sequence $0 \to F(A) \to F(B) \to F(C)$ to a long exact sequence in a natural way.

Definition 4.5.6. An ***acyclic*** object for a left-exact additive functor F is an object A such that $R^jF(A) = 0$ for all $j > 0$.

Theorem 4.5.7 (de Rham-Weil theorem)**.** *Given an acyclic resolution $0 \to M \to A^\bullet$ of an object M, we have isomorphisms*

$$R^jF(M) \cong H^j(F(A^\bullet))$$

Proof. The proof is similar to step (4) in Theorem 4.5.4, in that we use the weaving lemma to split up the acyclic resolution into short exact sequences. These then induce long exact sequences through the right derived functors, where $R^jF(A^n) = 0$ for all $j \geq 1, n \geq 0$. This is carried out in VOISIN [12], p. 102. □

5 Sheaf Cohomology

5.1 The Category $\mathsf{Sh}(X, \mathsf{Ab})$

Lemma 5.1.1. *The category* $\mathsf{Sh}(X, \mathsf{Ab})$ *is an additive category.*

Proof. The abelian group structure on the Hom-sets in $\mathsf{Sh}(X, \mathsf{C})$ is given by $(\varphi + \psi)_U := \varphi_U + \psi_U$ for sheaf morphisms $\varphi, \psi : F \to G$. It is immediate that this structure is bilinear with respect to composition, thus $\mathsf{Sh}(X, \mathsf{Ab})$ is preadditive.

The biproducts are given by $(F \oplus G)(U) := F(U) \oplus G(U)$. This assignment clearly is a presheaf and it is also a sheaf, as we can check the sheaf-axioms "component-wise". ☐

Definition 5.1.2. For presheaf morphism $\varphi : F \to G$ we define the ***presheaf kernel***, ***presheaf cokernel*** and ***presheaf image*** of φ as follows:

$$(\ker \varphi)(U) := \ker \varphi_U, \quad (\operatorname{coker} \varphi)(U) := \operatorname{coker} \varphi_U, \quad (\operatorname{im} \varphi)(U) := \operatorname{im} \varphi_U$$

With the restriction morphisms

$$\begin{aligned}
\mathsf{res}^{\ker}_{VU} &: \ker \varphi_U \to \ker \varphi_V : s \mapsto \mathsf{res}_{VU}(s) \\
\mathsf{res}^{\operatorname{coker}}_{VU} &: \operatorname{coker} \varphi_U \to \operatorname{coker} \varphi_V : s \mapsto \mathsf{res}_{VU}(s) \\
\mathsf{res}^{\operatorname{im}}_{VU} &: \operatorname{im} \varphi_U \to \operatorname{im} \varphi_V : s \mapsto \mathsf{res}_{VU}(s)
\end{aligned}$$

these actually become presheaves themselves.

Lemma 5.1.3. *If* $\varphi : F \to G$ *is a morphism of sheaves, then* $\ker \varphi$ *is a sheaf as well.*

Proof. ☐

The same however is not true in general for the cokernel and image of a morphism of sheaves, hence the following definition:

Definition 5.1.4. The ***cokernel*** and ***image*** of a morphism of sheaves $\varphi : F \to G$ are defined as the respective sheafifications $\overline{\operatorname{coker} \varphi}, \overline{\operatorname{im} \varphi}$ of the presheaf cokernel and image. In abuse of notation we will still denote these by as $\operatorname{coker} \varphi, \operatorname{im} \varphi$ when talking about morphisms of sheaves.

Lemma 5.1.5. *The kernel, cokernel and image defined above fit the category theoretic definition of the kernel, cokernel and image in* $\mathsf{Sh}(X, \mathsf{Ab})$.

Proof. ☐

Definition 5.1.6. A morphism $\varphi : F \to G$ of sheaves is ***injective*** if its kernel is the 0-sheaf. Likewise, it is ***surjective*** if its image is G. It is an ***isomorphism*** if it is both injective and surjective.

In general, a morphism of presheaves $\varphi : F \to G$ induces morphisms on the stalks of F and G in the following way: $\varphi_x : F_x \to G_x : (x, s_x) \mapsto (x, \varphi_U(x)_s)$, where U is any open neighborhood of x. In the case that both F and G are sheaves, this leads to a particularly nice result:

Lemma 5.1.7. *For a morphism $\varphi : F \to G$ of sheaves, we have the following results:*

(1) $\ker \varphi = 0$ *if and only if* $\ker \varphi_x = 0$ *for all* $x \in X$

(2) $\operatorname{im} \varphi = G$ *if and only if* $\operatorname{im} \varphi_x = G_x$ *for all* $x \in X$

Proof. Ramanan [9], pp. 9, 93 □

Remark. In the preceding lemma, neither (1) nor (2) are generally true for morphisms of presheaves. c.f. Tu

Corollary 5.1.8. *The category* $\mathsf{Sh}(X, \mathsf{Ab})$ *is abelian.*

Proof. The only things remaining to be seen are that $\mathsf{Sh}(X, \mathsf{Ab})$ has an initial 0-object and that it is binormal. Firstly, the 0-object is given by the constant 0-sheaf. Secondly, the binormality follows from that of Ab:

Let $\varphi : F \to G$ be a monomorphism, i.e. an injective morphism of sheaves. This means that $\varphi_x : F_x \to G_x$ is injective for all $x \in X$. Thus, all the φ_x are kernels of some group-morphisms $\psi_x : G_x \to H_x$. The family H_x of groups actually induces a sheaf H on X by equipping each H_x with the discrete topology and constructing the space $E := \bigsqcup_{x \in X} H_x$, which automatically makes the projection $\pi : E \to X$ continuous. Then, we define H to be the sheaf of continuous sections of π. This lifts the family ψ_x of group-morphisms to a morphism of sheaves $\psi : G \to H$ by taking a section $s \in G(U)$ to the function $\psi_U(s) : U \to E : x \mapsto \psi_x(s_x)$. Finally, the monomorphism φ then is the kernel of ψ.

Showing that every epimorphism, i.e. surjective sheaf morphism, is the cokernel of some sheaf morphism, is similar to this construction. □

5.2 Sheaf Cohomology

Lemma 5.2.1. *The global section functor* $\Gamma : \mathsf{Sh}(X, \mathsf{Ab}) \to \mathsf{Ab}$ *is additive and left-exact.*

Proof. The additivity of Γ is clear by how the addition of sheaf morphisms is defined.

To show the second statement, we take a short exact sequence of sheaves

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 0$$

We want to show that

$$0 \longrightarrow F(X) \xrightarrow{\varphi_X} G(X) \xrightarrow{\psi_X} H(X)$$

is exact, i.e. $0 = \ker \varphi_X$ and $\operatorname{im} \varphi_X = \ker \psi_X$, where the inclusions "$\subseteq$" are clear.

Suppose that $s \in \ker \varphi_X$, so $\varphi_X(s) = 0$. This implies $\varphi_x(s_x) = \varphi_X(s)_x = 0$ for all $x \in X$. Since the sequence of sheaves is assumed to be short exact, the morphism φ is injective, i.e. injective at the stalk-level, we must have $s_x = 0$ for all $x \in X$, and therefore $s = 0$. This proves the exactness at $F(X)$.

Suppose that $t \in \ker \psi_X$, so $\psi_X(t) = 0$. Then $\psi_x(t_x) = \psi_X(t)_x = 0$ for all $x \in X$, i.e. $t_x \in \ker \psi_x = \operatorname{im} \varphi_x$, since the sequence is exact at the stalk level. But this means that there are elements $s_x \in F_x$ for all $x \in X$ such that $\varphi_x(s_x) = t_x$. These induce a section $s \in F(X)$ given by $s(x) = (x, s_x)$, and $\varphi_X(s)_x = t_x$, hence $\varphi_X(s) = t$ and thus $t \in \operatorname{im} \varphi_X$. This shows that the sequence is exact at $G(X)$. □

Lemma 5.2.2. *The category* $\mathsf{Sh}(X, \mathsf{Ab})$ *has enough injectives.*

Proof. It is a well-known fact that the category Ab has enough injectives and this result was, as already mentioned, generalized by BAER ([1]) to the category $\mathsf{k}\text{-}\mathsf{Mod}$ for any unital ring k.

Knowing this, we can find for any sheaf $F \in \mathsf{Sh}(X, \mathsf{Ab})$ inclusions of the stalks F_x into injective objects I_x. Then, F can be injected into the sheaf I given by $I(U) := \bigoplus_{x \in X} I_x$. □

Corollary 5.2.3. *Combining the two previous lemmas, we see that the left derived functors of* Γ *exist and we call the objects* $\mathcal{H}^j(X, F) := R^j\Gamma(F)$ *the* ***sheaf cohomology*** *of the sheaf* F *on* X*.*

Definition 5.2.4. A sheaf F is ***flasque*** or ***flabby*** if for all open sets $U \subseteq X$ the map res_{UX} is surjective. In other words: Any section of a flasque sheaf over a subset of X has at least one extension to a section over all of X.

Lemma 5.2.5. *Flasque sheaves are acyclic for* Γ*.*

Proof. VOISIN [12], p. 103 □

Definition 5.2.6. A sheaf of $F \in \mathsf{Sh}(X, \mathsf{Ab})$ is called ***fine*** if for any locally finite open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X there exists a partition of the identity id_F subordinate to $\mathcal{U}$. That is, a family of sheaf endomoprhisms $h_i : F \to F$ such that $(h_i)_x = 0$ for all $x \notin U_i$ (also written as $\mathsf{supp}\, h_i \subseteq U_i$), and $\sum_{i \in I} h_i = \mathsf{id}_F$. This infinite formal sum is defined in the following sense: Since $\mathcal{U}$ is locally finite, any point $x \in X$ has an open neighborhood U such that the set $I_U = \{i \in I \,|\, U_i \cap U \neq \emptyset\}$ is finite. Thus, we can take the sum $\sum_{i \in I_U} h_i$ and the infinite formal sum is defined as the morphism which agrees with these sums everywhere.

Lemma 5.2.7. *Let* $\mathcal{A}$ *be a sheaf of unital algebras such that for any locally finite open covering* $\mathcal{U} = \{U_i\}_{i \in I}$ *of* X *there exists a partition unity subordinate to* $\mathcal{U}$*. That is, there exist sections* $f_i \in \mathcal{A}(X)$ *such that* $\mathsf{supp}\, f_i \subseteq U_i$ *and* $\sum_{i \in I} f_i = 1_{\mathcal{A}(X)}$*. Then, any sheaf* $\mathcal{M}$ *of* $\mathcal{A}$*-modules is* $\mathcal{M}$ *fine.*

Proof. A partition of unity subordinate to an open covering $\mathcal{U}$ induces a partition of $\mathsf{id}_{\mathcal{M}}$ subordinate to $\mathcal{U}$: Take $h_i : F \to F$ given by $(h_i)_U : \mathcal{M}(U) \to \mathcal{M}(U) : m \mapsto \mathsf{res}^{\mathcal{A}}_{UX}(f_i) \cdot m$. This is a partition of the identity, since the identity $\mathsf{id}_{\mathcal{M}}$ is given by the morphism

$$F(U) \to F(U) : m \mapsto \mathsf{res}^{\mathcal{A}}_{UX}(1_{\mathcal{A}}(X)) \cdot m = 1_{\mathcal{A}(U)} \cdot m = m$$

□

Example 5.2.8. It is a well-known fact (c.f. TU [11], p. 146), that for a manifold M and any locally finite open covering $\mathcal{U}$ of M there exists a partition of unity subordinate to $\mathcal{U}$. Hence, the sheaves $\Gamma(T-), \mathfrak{X}$ and Ω^r, being $\mathscr{C}^\infty$-modules, are fine.

Lemma 5.2.9. *Fine sheaves are acyclic for* Γ*.*

Proof. VOISIN [12], p. 104 $\square$

Corollary 5.2.10. *For a manifold M, the sheaves $\Omega^\bullet$ form an acyclic resolution of the sheaf $\underline{\mathbb{R}}$ of locally constant functions $M \to \mathbb{R}$:*

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \ldots$$

since they are fine sheaves. Therefore, we have

$$\mathcal{H}^j(M, \underline{\mathbb{R}}) \cong H^j(\Omega^\bullet) = \mathcal{H}^j_{dR}(M)$$

showing that the de Rham cohomology of a manifold M is the same as the sheaf cohomology of the sheaf of locally constant functions on M.

The sheaf of locally constant functions of course only depends on the topology of the manifold, showing that the de Rham cohomology also does so. As we discussed in the chapter on smooth manifolds, a topological manifold may admit a multitude of distinct differential structures. This last result however particularly demonstrates, that the de Rham cohomology is independent up to isomorphism of the differential structure. Since the de Rham cohomology in a certain sense measures the integrability of differential forms (i.e. measuring for $\omega \in \Omega^r$ whether there exists $\alpha \in \Omega^{r-1}$ such that $d\alpha = w$), this tells us that there is no "preferred" differential structure with respect to integration.

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