Bayesian Causal Inference in the Convergent Cross-Mapping test and Canonical Correlation Analysis

Moscow Institute of physics and Technology

Author: Mark Ikonnikov

Expert: Vadim Strijov

December, 2025

Problem Statement

Fundamental Problem

Canonical Correlation Analysis (CCA) and Convergent Cross-Mapping (CCM) are inherently *correlational* and lack a principled causal interpretation.

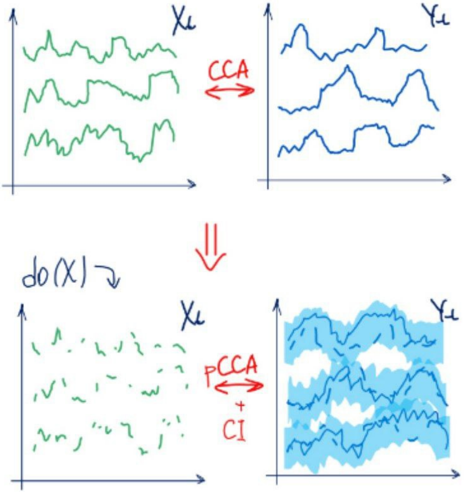
- Canonical correlations do not distinguish between **direct causal effects** and **associations induced by latent variables**.
- Existing CCM-based tests rely on predictive skill and state-space reconstruction
- In practical settings (time series, text, neural representations), **true interventions are unavailable**, making causal interpretation ambiguous.
- There is no unified Bayesian framework connecting CCA, SCMs, interventional operators, and CCM-style causal tests.

Research Objective: Bayesian Causal Inference for CCA/CCM

Objectives

1. Formulate probabilistic CCA with Bayesian interpretation.
2. Introduce `do(X)` intervention operator for causal inference for CCA.
3. Formulate the Bayesian Causal Inference with `do(X)` intervention operator to be used in the Convergent Cross-Mapping test (TO BE DONE)
4. Validate via d-variate time series, text document. (TO BE DONE)

Motivation: Why Causal Inference in Time Series?



Understanding causal influence between coupled series via CCA.

- Classical CCA: captures correlations, not causation.
- CCM: tests predictive causation via shadow manifolds.
- Bayesian CCA + `do(X)`: allows reasoning about interventions.

Background and Literature

Title	Year	Authors	Paper
A Probabilistic Interpretation of Canonical Correlation Analysis	2001	Bach et al.	JMLR
Canonical Correlation Analysis: An Overview with Application to Learning Methods	2012	Hardoon et al.	Neural Computation
Learning Relationships between Text, Audio, and Video via Deep CCA for Multimodal Language Analysis	2020	Sun et al.	AAAI

Probabilistic Model of CCA

Theorem (Probabilistic CCA)

Let $Z \sim \mathcal{N}(0, I_k)$*, then*

$$X = AZ + \varepsilon_X, \quad \varepsilon_X \sim \mathcal{N}(0, \Psi_X), \quad Y = BZ + \varepsilon_Y, \quad \varepsilon_Y \sim \mathcal{N}(0, \Psi_Y).$$

There exists a parametrization (A^*, B^*) *such that the classical canonical directions* (U, V) *satisfy*

$$A^* = C_{11}^{1/2} U \Lambda^{1/2}, \quad B^* = C_{22}^{1/2} V \Lambda^{1/2}.$$

Insight

pCCA represents CCA as a latent-variable model and relates probabilistic parameters to canonical directions.

SCM with do(X) Intervention

Consider the model

$$Z \sim \mathcal{N}(0, I_k), \quad X = AZ + U_X, \quad Y = CX + BZ + U_Y.$$

Then $\mathcal{M} = \langle \mathcal{U}, \mathcal{V}, \mathcal{F}, P(\mathcal{U}) \rangle$ defines a structural causal model for $X \to Y$ with hidden confounder Z and direct effect C.

Insight

Formalizes causal relationships and sets the basis for interventional analysis via `do(X)`.

Interventional Distribution

Theorem (Interventional Distribution)

For the SCM above,

$$P(Y \mid do(X = x)) = \mathcal{N}(Cx, BB^\top + \Psi_Y).$$

Insight

Mean encodes direct causal effect, variance accounts for latent confounders. Allows separation of causation from correlation.

Corollary (Linear Causal Effect)

$$\mathbb{E}[Y \mid do(X = x_1)] - \mathbb{E}[Y \mid do(X = x_0)] = C(x_1 - x_0)$$

Insight

Identifies linear, pure causal effect, independent of hidden confounders.

Projection onto CCA Subspace

Theorem (Projection of Causal Effect)

Let (u_i, v_i) be canonical directions normalized by $u_i^\top C_{11} u_i = v_i^\top C_{22} v_i = 1$. Then

$$v_i^\top \mathbb{E}[Y \mid do(X = x)] = (v_i^\top C u_i)(u_i^\top x).$$

Insight

Shows how interventional effect projects linearly onto canonical coordinates, facilitating interpretation in CCA space.

Theorem (Non-Causal Nature of Canonical Correlations)

For SCM with $X = AZ$, $Y = CX + BZ$ and $\Psi_X = \Psi_Y = 0$, the canonical correlation ρ_i satisfies

$$\rho_i = u_i^\top C_{11} C^\top v_i + u_i^\top A B^\top v_i.$$

Interpretation for Time Series and CCM

Time Series

- do($X_t = x$): force state at time t, observe $Y_{t+\tau}$. - SCM + pCCA isolates direct causal influence from latent dynamics.

CCM Perspective

- Project interventional response on reconstructed manifold. - Enables estimation of directed influence $X \to Y$ beyond correlation.

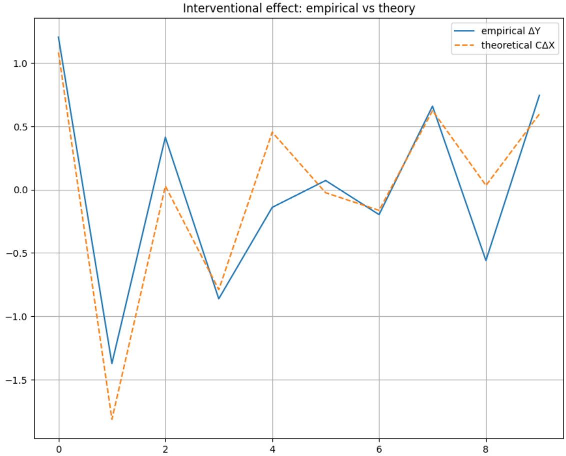
Textual Analogy

- do($X = x$): modify embeddings of linguistic features. - Z: latent semantic concepts. - Measures causal impact on related documents.

Summary of Contributions

- Formulated SCM for pCCA with $\mathrm{do}(X)$.
- Derived interventional distribution and projection onto CCA space.
- Demonstrated difference between canonical correlation and causal effect.
- Laid foundation for CCM-based causal estimation in time series and text.

Empirical Validation of the Causal pCCA Model



Comparison between the empirical response of the system and the theoretical interventional expectation predicted by the causal pCCA model.

Conclusions and Novelty

Key Findings

- Developed Probabilistic CCA with `do(X)` operator for causal inference in time series.
- Derived closed-form posterior distributions under intervention.
- Empirical results validate the theoretical formulation

Novelty / Contribution

- **Proposed** a Probabilistic causal formulation of Canonical Correlation Analysis (CCA),
- **distinct from** classical correlation-based CCA by explicitly introducing a structural causal model and the $\mathrm{do}(X)$ intervention operator,
- **enabling** identification and analysis of causal effects in latent-variable settings.