

## VE477

### Introduction to Algorithms

#### Homework 7

Manuel — UM-JI (Fall 2019)

#### Reminders

- Write in a neat and legible handwriting or use  $\text{\LaTeX}$
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

Questions preceded by a \* are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.

#### Ex. 1 — Karger-Stein's Algorithm

In the lectures, although Karger-Stein's Algorithm was presented (5.28), only a sketch of proof was provided (5.29). In this exercise we want to prove the missing part, i.e. solve the recurrence relation

$$P(t) = 1 - \left(1 - \frac{1}{2}P\left(\frac{t}{\sqrt{2}}\right)\right)^2.$$

1. Prove that the probability of a cut to survive when  $n < 6$ , at least  $1/15$ .
2. Using an appropriate change of variable, show that

$$\begin{cases} p_{k+1} &= p_k - \frac{1}{4}p_k^2, \\ p_0 &= 1/15. \end{cases}$$

3. Let  $z_k = 4/p_k - 1$ .

a) Prove that

$$\begin{cases} z_{k+1} &= z_k + 1 + \frac{1}{z_k}, \\ z_0 &= 59. \end{cases}$$

\* b) Show that for all  $k \geq 0$ ,  $k < z_k < 59 + 2k$ .

4. Recalling that  $t = n/\sqrt{2}$  and noting that the depth of the recursion is  $2\log_2 n + \mathcal{O}(1)$ , conclude that  $P(n) = \Omega(1/\log n)$ .

#### Ex. 2 — Simplex method

Explain how the simplex method can be represented and applied from the following perspectives:

1. Tableaux (matrices);
2. Geometric;

For each case show the details of solving example 6.264 (6.16).

#### Ex. 3 — Critical thinking

Is it possible to design a stack supporting push, pop, and retrieving the minimum element in constant time? Explain.

**Ex. 4** — *Farka's lemma*

Prove the following result. Let  $M$  be an  $m \times n$  matrix and  $V$  be an  $n$ -vector. Then given an  $n$ -vectors  $x$  and an  $m$ -vector  $y$ , exactly one of the following can be true: (i)  $Mx \leq 0$  and  $V^T x > 0$ , or (ii)  $M^T y = V$  and  $y \geq 0$ .