

Announcements

- ❖ **Project 2: Multi-agent Search**
 - ❖ Due Friday June 19 at 11:59pm.
- ❖ **Homework 4: RL**
 - ❖ Has been released! Due Wed. June 17 at 11:59pm.
- ❖ **Midterm Exam**
 - ❖ June 22, 4pm-5:40pm
 - ❖ Covers everything, CSP included.
- ❖ **Recitations**
 - ❖ Next week, answer poll on Canvas

Ve492: Introduction to Artificial Intelligence

Constraint Satisfaction Problems I



Paul Weng

UM-SJTU Joint Institute

Slides adapted from <http://ai.berkeley.edu>, AIMA, UM, CMU

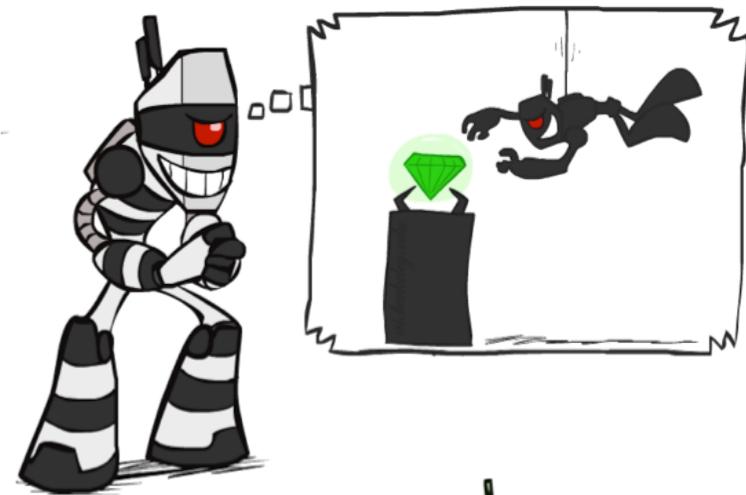
What have we learned so far?

- ❖ Search and planning
 - ❖ Define a state space, goal test; Find path from start to goal
- ❖ Game trees
 - ❖ Define utilities; Find path from start that maximizes utility
- ❖ Decision theory and game theory
 - ❖ Foundation for MEU; Basic concepts in game theory
- ❖ MDPs
 - ❖ Define rewards, utility = (discounted) sum of rewards
 - ❖ Find policy that maximizes utility
- ❖ Reinforcement learning
 - ❖ Just like MDPs, only T and / or R are not known in advance
- ❖ Today: constraint satisfaction
 - ❖ Find solution that satisfies constraints; Not just for finding a sequential plan

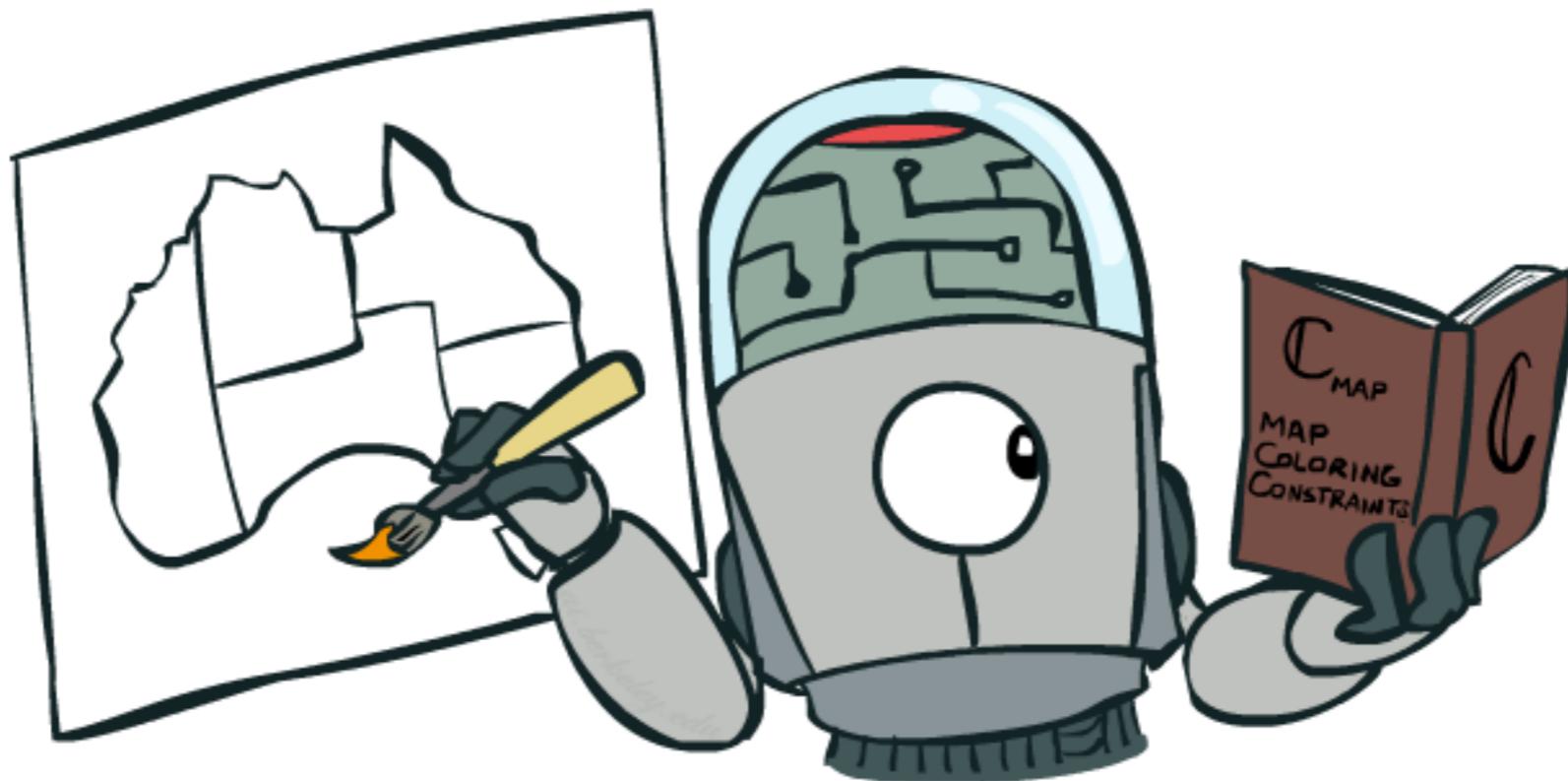


What is Search For?

- ❖ Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space
- ❖ Planning: sequences of actions
 - ❖ The path to the goal is the important thing
 - ❖ Paths have various costs, depths
 - ❖ Heuristics give problem-specific guidance
- ❖ Identification: assignments to variables
 - ❖ The goal itself is important, not the path
 - ❖ All paths at the same depth (for some formulations)
 - ❖ CSPs are specialized for identification problems



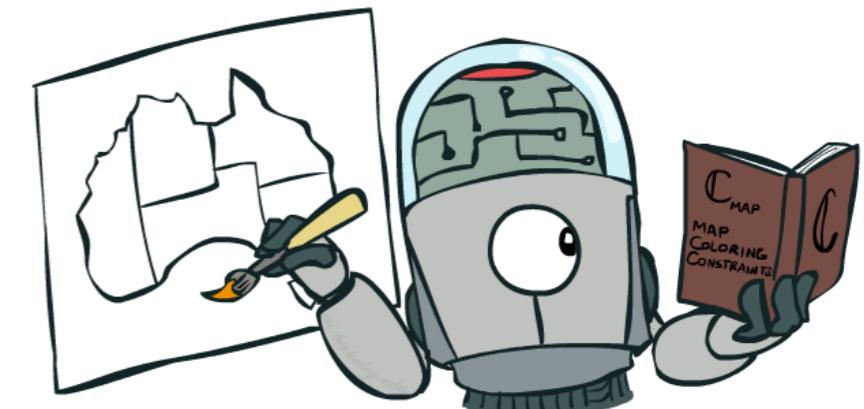
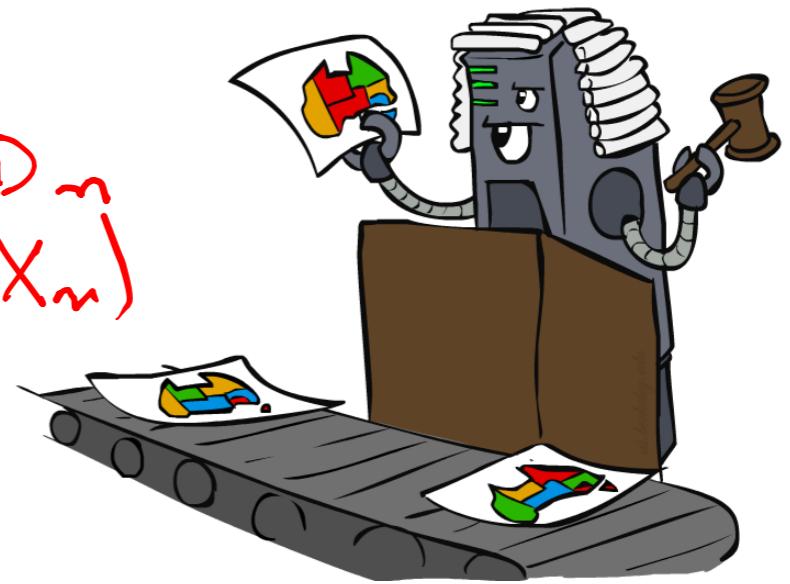
Constraint Satisfaction Problems



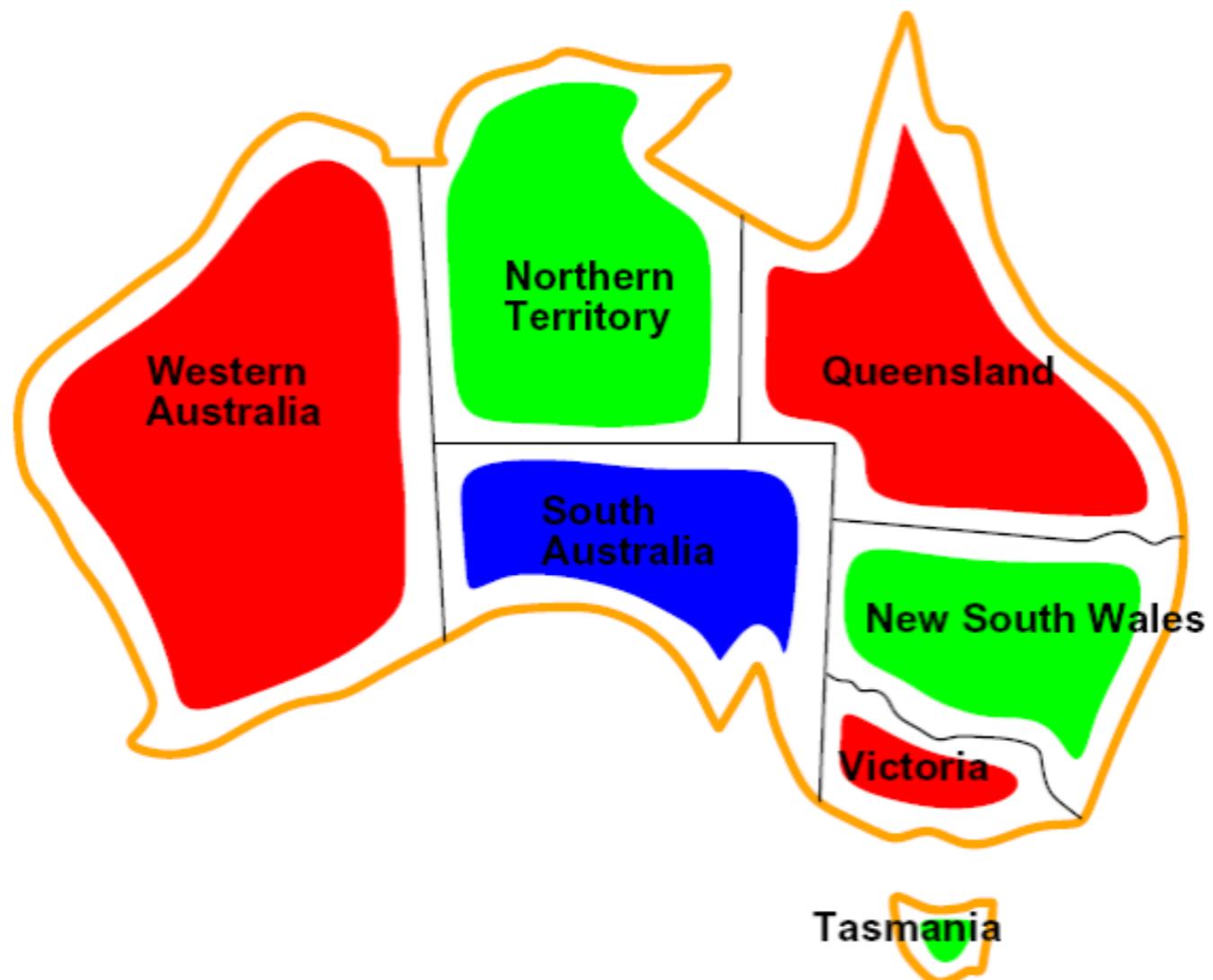
Constraint Satisfaction Problems

- ❖ Standard search problems:
 - ❖ State is a “black box”: arbitrary data structure
 - ❖ Goal test can be any function over states
 - ❖ Successor function can also be anything
- ❖ Constraint satisfaction problems (CSPs):
 - ❖ A special subset of search problems
 - ❖ State is defined by variables X_i with values from a domain D (sometimes D depends on i)
 - ❖ Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- ❖ Simple example of a *formal representation language*
- ❖ Allows useful general-purpose algorithms with more power than standard search algorithms

$$D_i \quad D_n \\ (X_1, \dots, X_n)$$



CSP Examples



Example: Map Coloring

- Variables: WA, NT, Q, NSW, V, SA, T

- Domains: $D = \{\text{red, green, blue}\}$

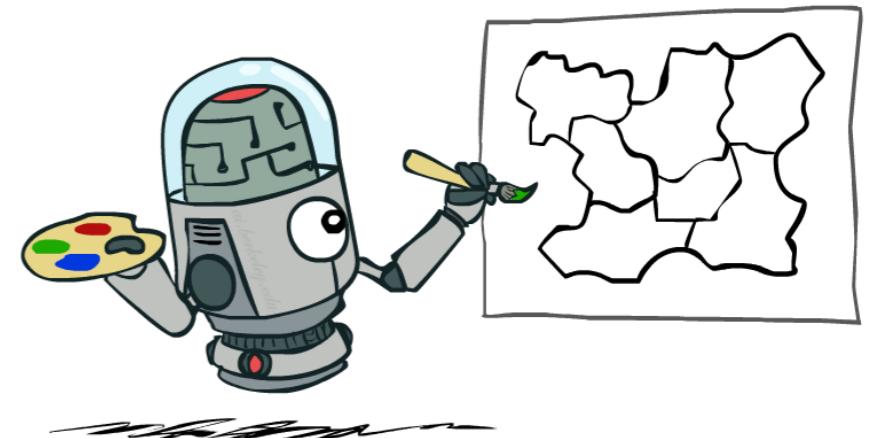
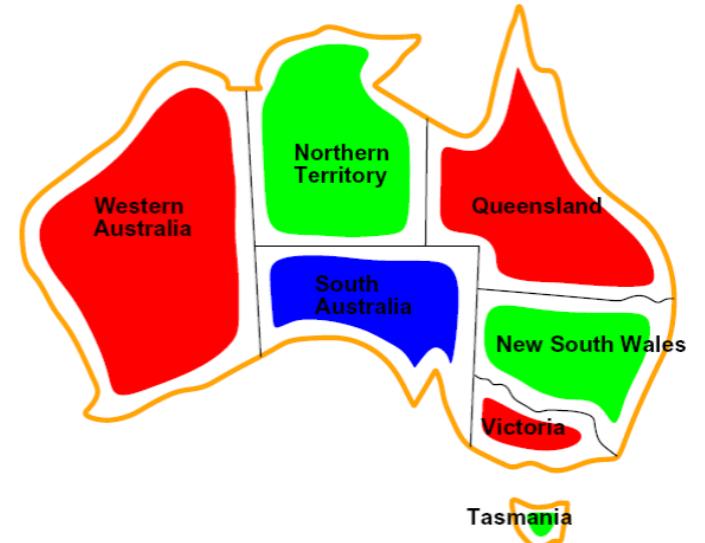
- Constraints: adjacent regions must have different colors

Implicit: $\underline{\text{WA}} \neq \text{NT}$

Explicit: $(\text{WA}, \text{NT}) \in \{(\text{red, green}), (\text{red, blue}), \dots\}$

- Solutions are assignments satisfying all constraints, e.g.:

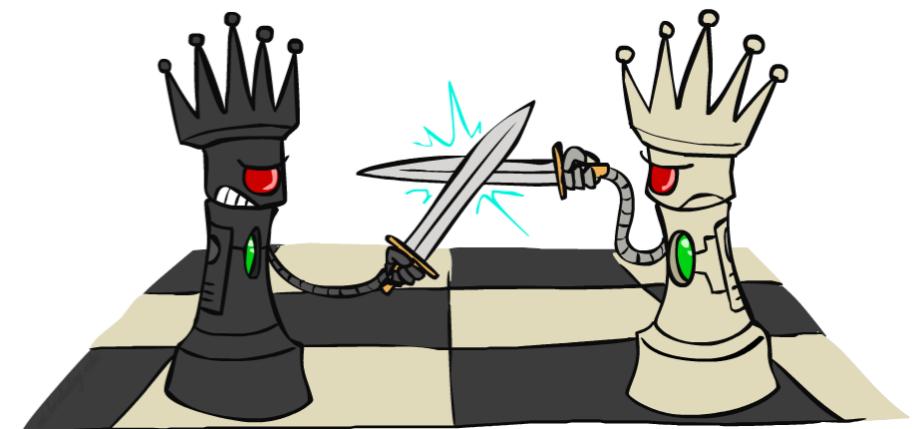
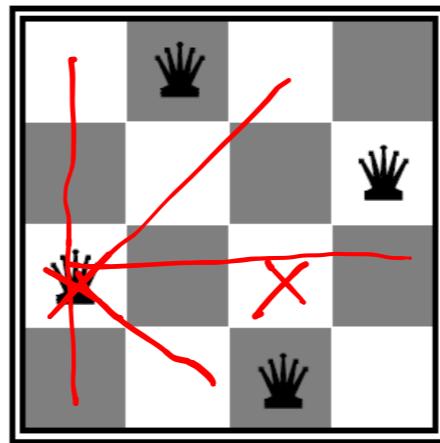
[{WA=red, NT=green, Q=red, NSW=green,
V=red, SA=blue, T=green}]



Example: N-Queens

❖ Formulation 1:

- ❖ Variables: X_{ij} N×N
- ❖ Domains: {0, 1}
- ❖ Constraints



$$\forall i, j, k \quad (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\} \quad -$$

$$\forall i, j, k \quad (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\} \quad -$$

$$\forall i, j, k \quad (X_{ij}, X_{i+k, j+k}) \in \{(0,0), (0,1), (1,0)\} \quad -$$

$$\forall i, j, k \quad (X_{ij}, X_{i+k, j-k}) \in \{(0,0), (0,1), (1,0)\} \quad -$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

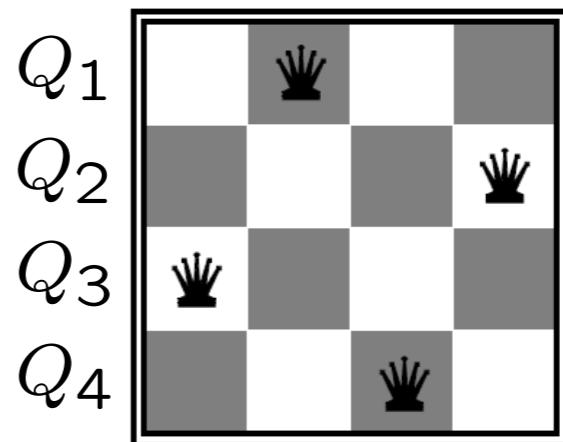
- ❖ Formulation 2:

- ❖ Variables: $Q_k \in \mathbb{N}$
- ❖ Domains: $\{1, 2, 3, \dots, N\}$
- ❖ Constraints:

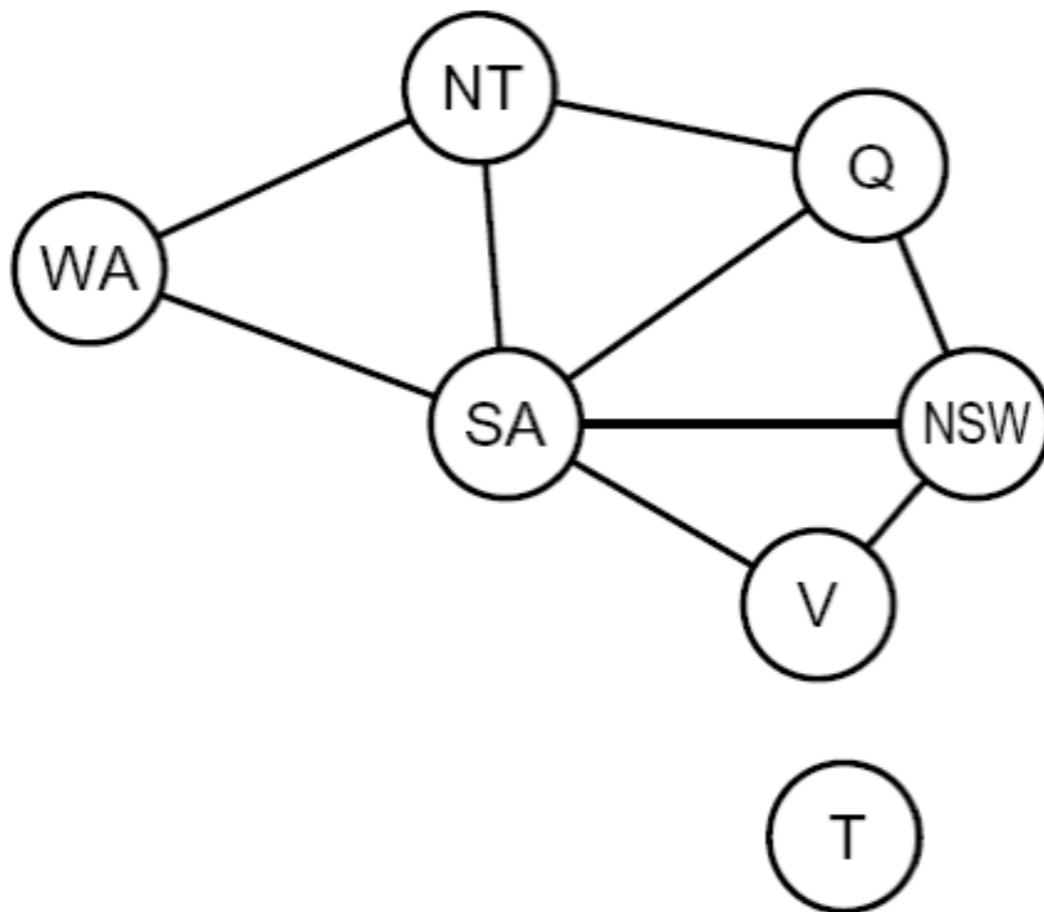
Implicit: $\forall i, j \text{ non-threatening}(Q_i, Q_j)$

Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \dots\}$

...

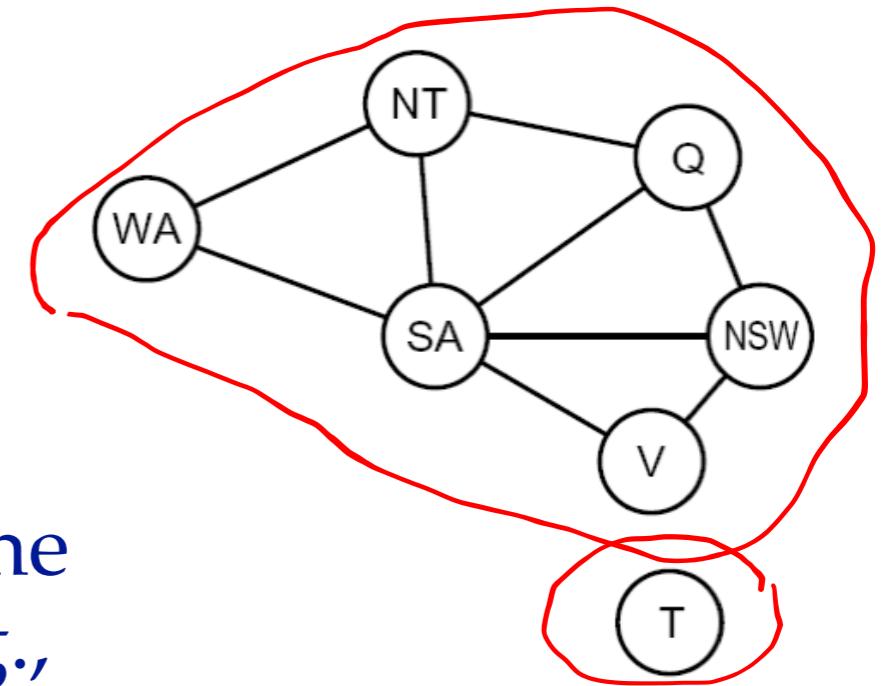


Constraint Graphs



Constraint Graphs

- ❖ Binary CSP: each constraint relates (at most) two variables
- ❖ Binary constraint graph: nodes are variables, arcs show constraints
- ❖ General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!
- ❖ Example: [CSP solver](#)



Example: Cryptarithmetic

- ❖ Variables:

$F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$

- ❖ Domains:

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

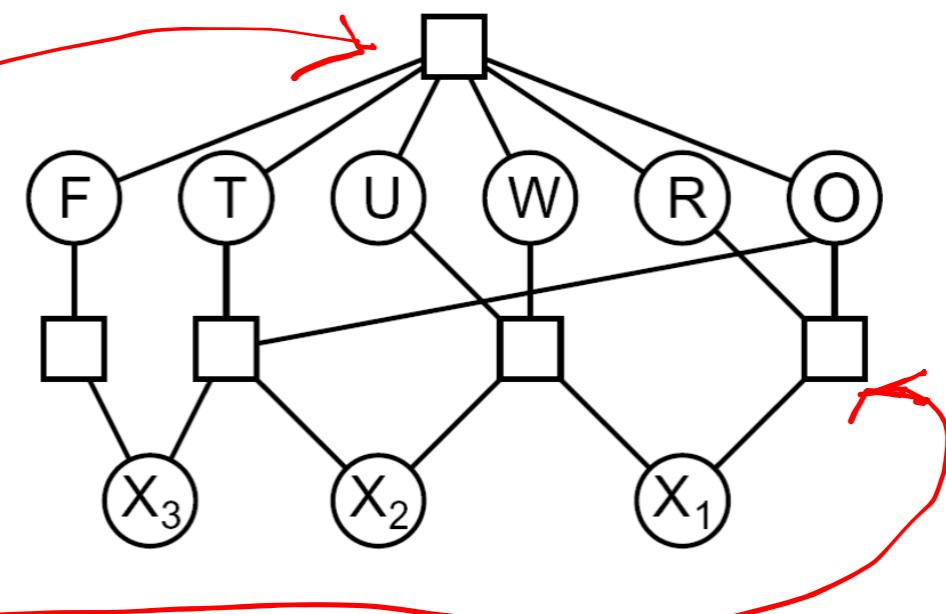
- ❖ Constraints:

alldiff(F, T, U, W, R, O)

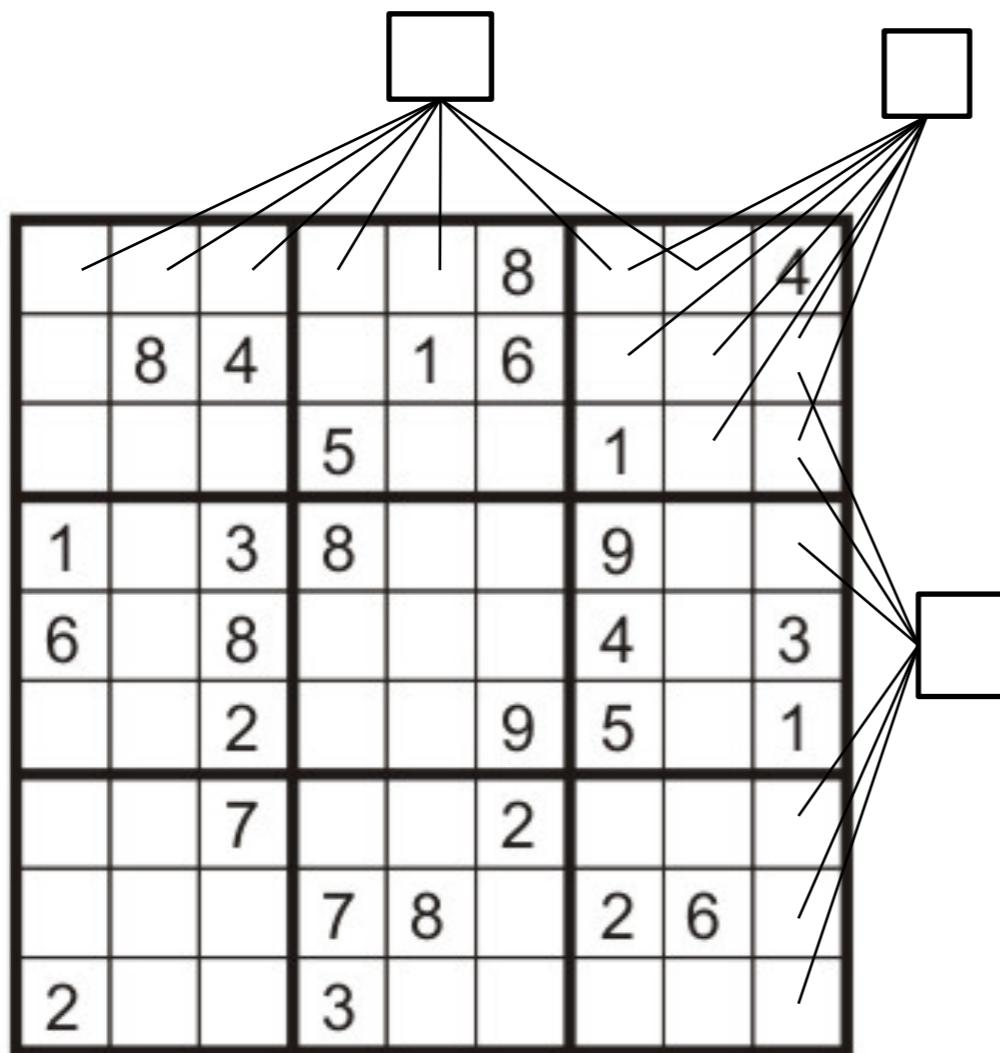
$O + O = R + 10 \cdot X_1$

...

$$\begin{array}{r} x_3 \ x_2 \ x_1 \\ \text{T} \ \text{W} \ \text{O} \\ + \ \text{T} \ \text{W} \ \text{O} \\ \hline \text{F} \ \text{O} \ \text{U} \ \text{R} \end{array}$$

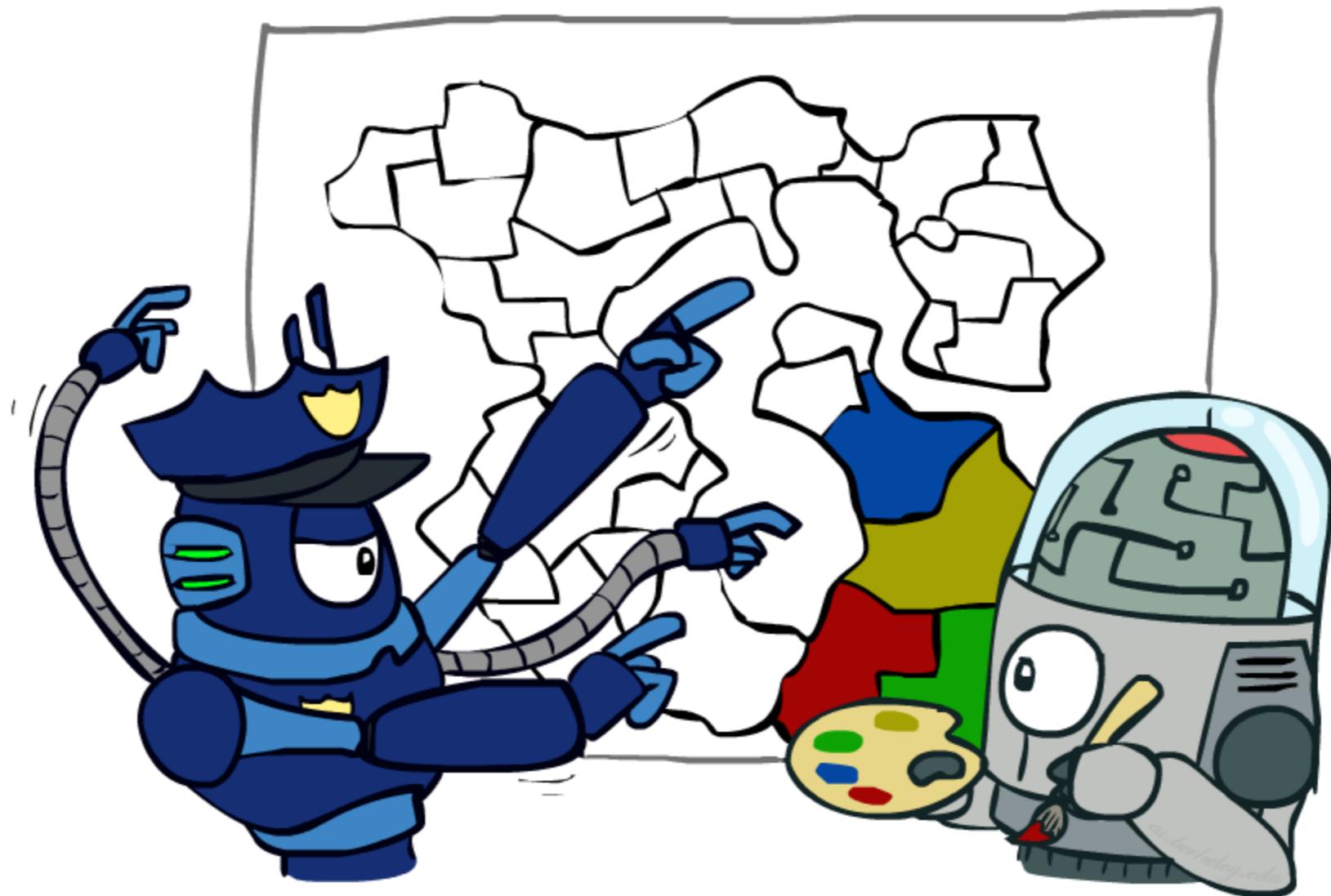


Example: Sudoku



- ❖ **Variables:**
Each (open) square
- ❖ **Domains:**
 $\{1, 2, \dots, 9\}$
- ❖ **Constraints:**
 - 9-way alldiff for each column
 - 9-way alldiff for each row
 - 9-way alldiff for each region
 - (or can have a bunch of pairwise inequality constraints)

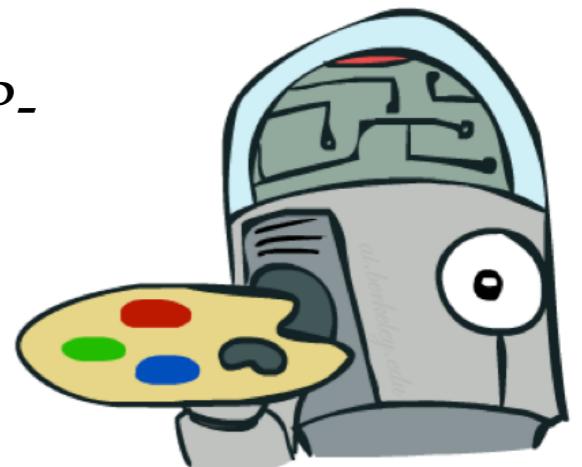
Varieties of CSPs and Constraints



Varieties of CSPs

❖ Discrete Variables

- ❖ Finite domains $D_1 \times D_2 \times \dots \times D_n$
 - ❖ Size d means $O(d^n)$ complete assignments
 - ❖ E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
- ❖ Infinite domains (integers, strings, etc.)
 - ❖ E.g., job scheduling, variables are start/end times for each job
 - ❖ Linear constraints solvable, nonlinear undecidable



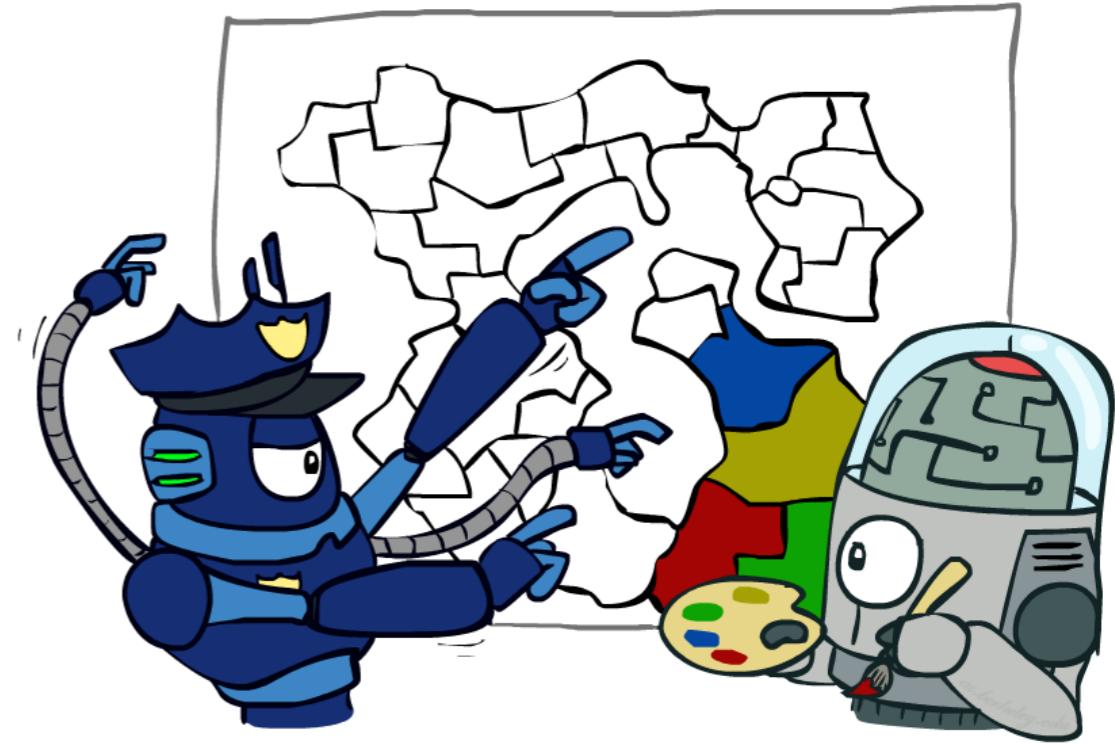
❖ Continuous variables

- ❖ E.g., start/end times for Hubble Telescope observations
- ❖ Linear constraints solvable in polynomial time by LP methods (see Ve555 for a bit of this theory)



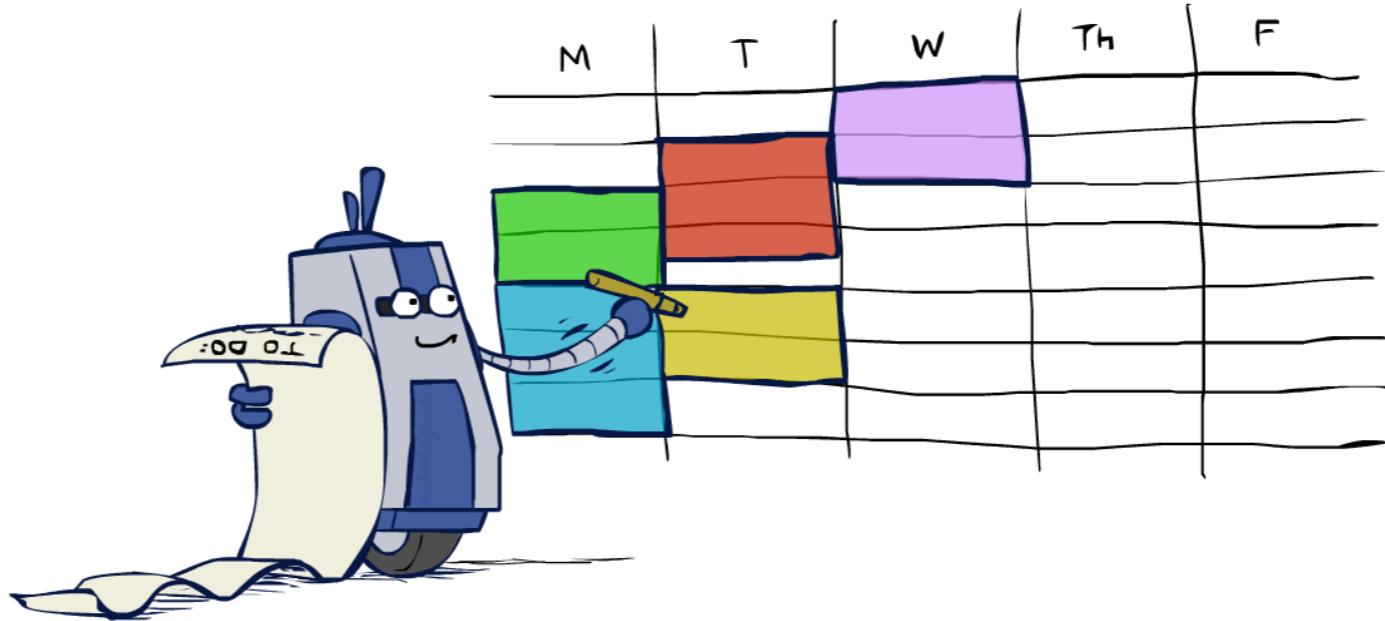
Varieties of Constraints

- ❖ Varieties of Constraints
 - ❖ Unary constraints involve a single variable (equivalent to reducing domains), e.g.:
 $SA \neq \text{green}$
 - ❖ Binary constraints involve pairs of variables, e.g.:
 $SA \neq WA$
 - ❖ Higher-order constraints involve 3 or more variables:
e.g., cryptarithmic column constraints
- ❖ Preferences (soft constraints):
 - ❖ E.g., red is better than green
 - ❖ Often representable by a cost for each variable assignment
 - ❖ Gives constrained optimization problems
 - ❖ (We'll ignore these until we get to Bayes' nets)



Real-World CSPs

- ❖ Assignment problems: e.g., who teaches what class
- ❖ Timetabling problems: e.g., which class is offered when and where?
- ❖ Hardware configuration
- ❖ Transportation scheduling
- ❖ Factory scheduling
- ❖ Circuit layout
- ❖ Fault diagnosis
- ❖ ... lots more!



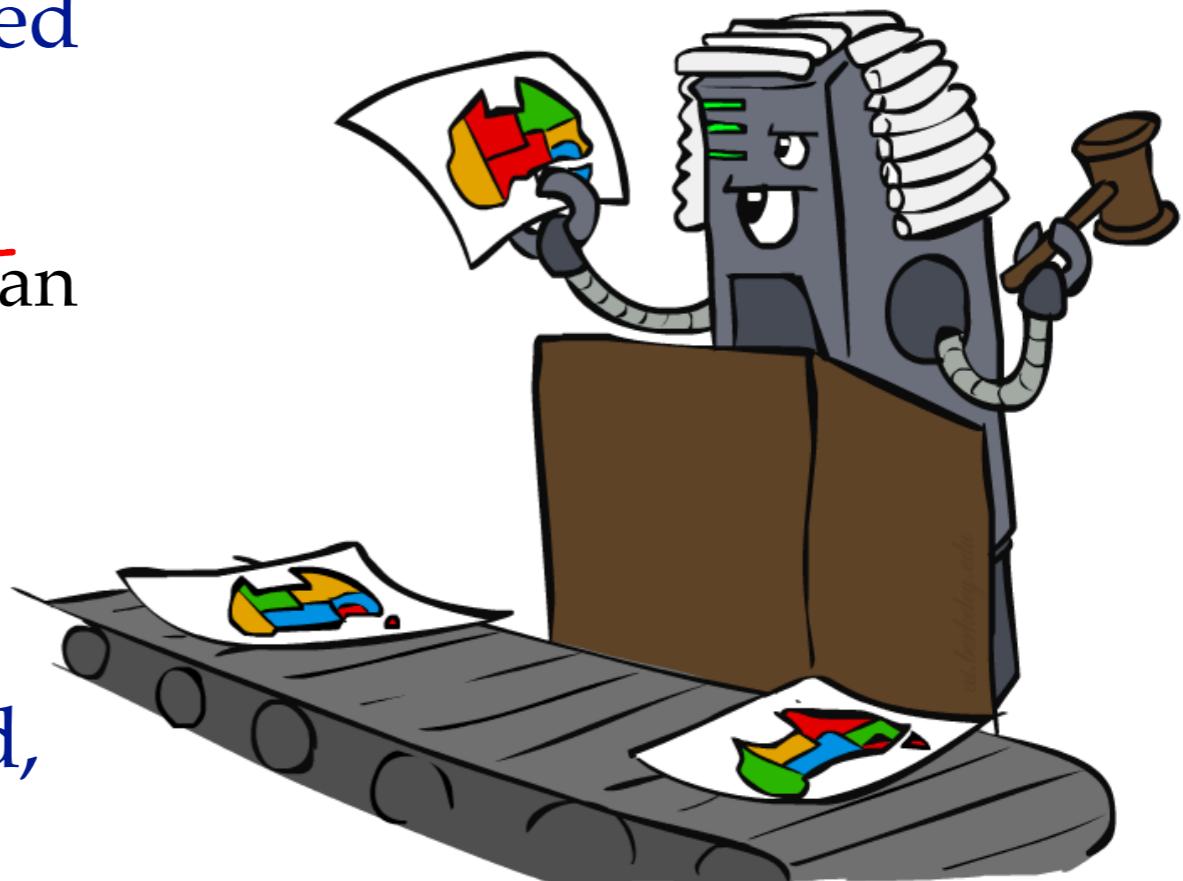
- ❖ Many real-world problems involve real-valued variables...

Solving CSPs



Standard Search Formulation

- ❖ Standard search formulation of CSPs
- ❖ States defined by the values assigned so far (partial assignments)
 - ❖ Initial state: the empty assignment, {}
 - ❖ Successor function: assign a value to an unassigned variable
 - ❖ Goal test: the current assignment is complete and satisfies all constraints
- ❖ We'll start with the straightforward, naïve approach, then improve it



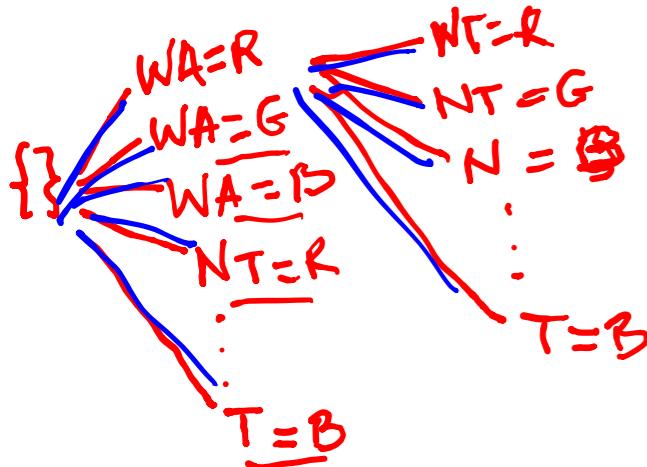
Search Methods

$m \times d$

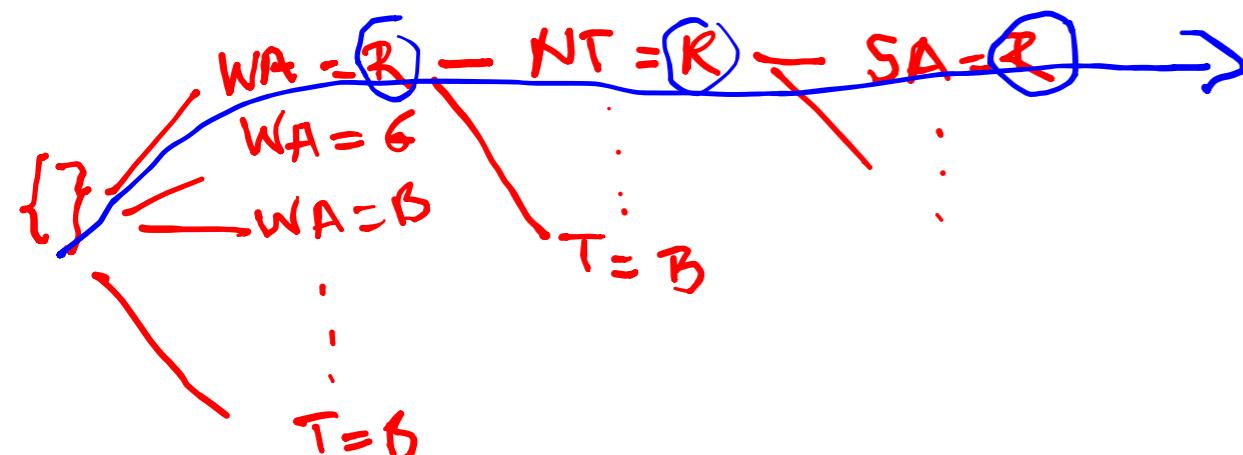
$(m-1)d$



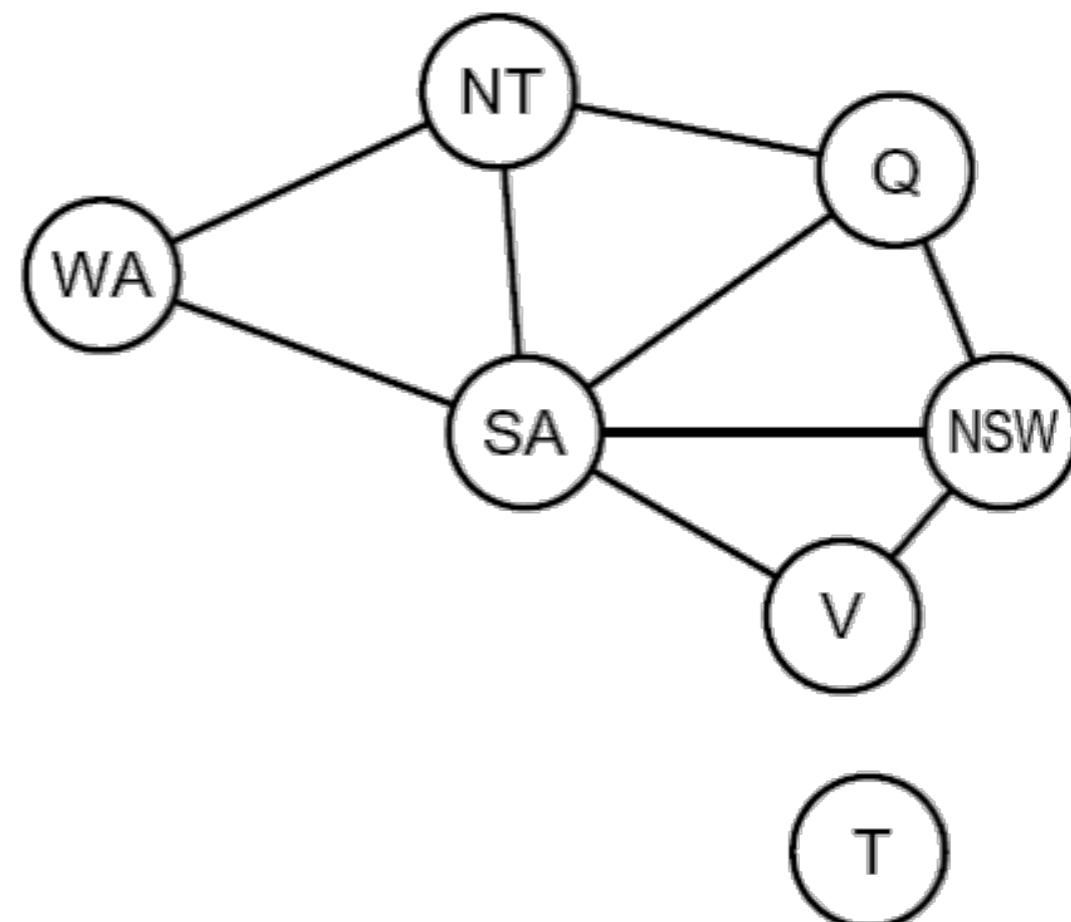
- ❖ What would BFS do?



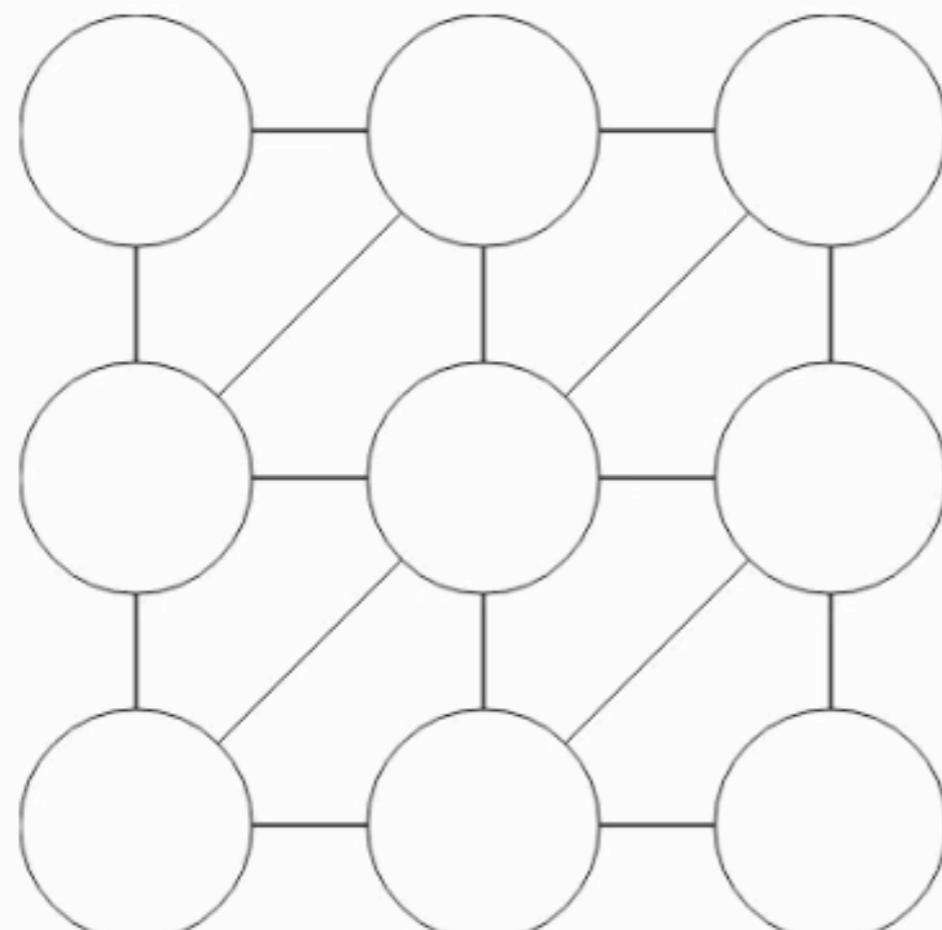
- ❖ What would DFS do?



- ❖ What problems does naïve search have?



Video of Demo Coloring -- DFS



Reset

Prev

Pause

Next

Play

Faster

Graph

Simple

Algorithm

Naive Search

Ordering

- None
- MRV
- MRV with LCV

Filtering

- None
- Forward Checking
- Arc Consistency

Speed

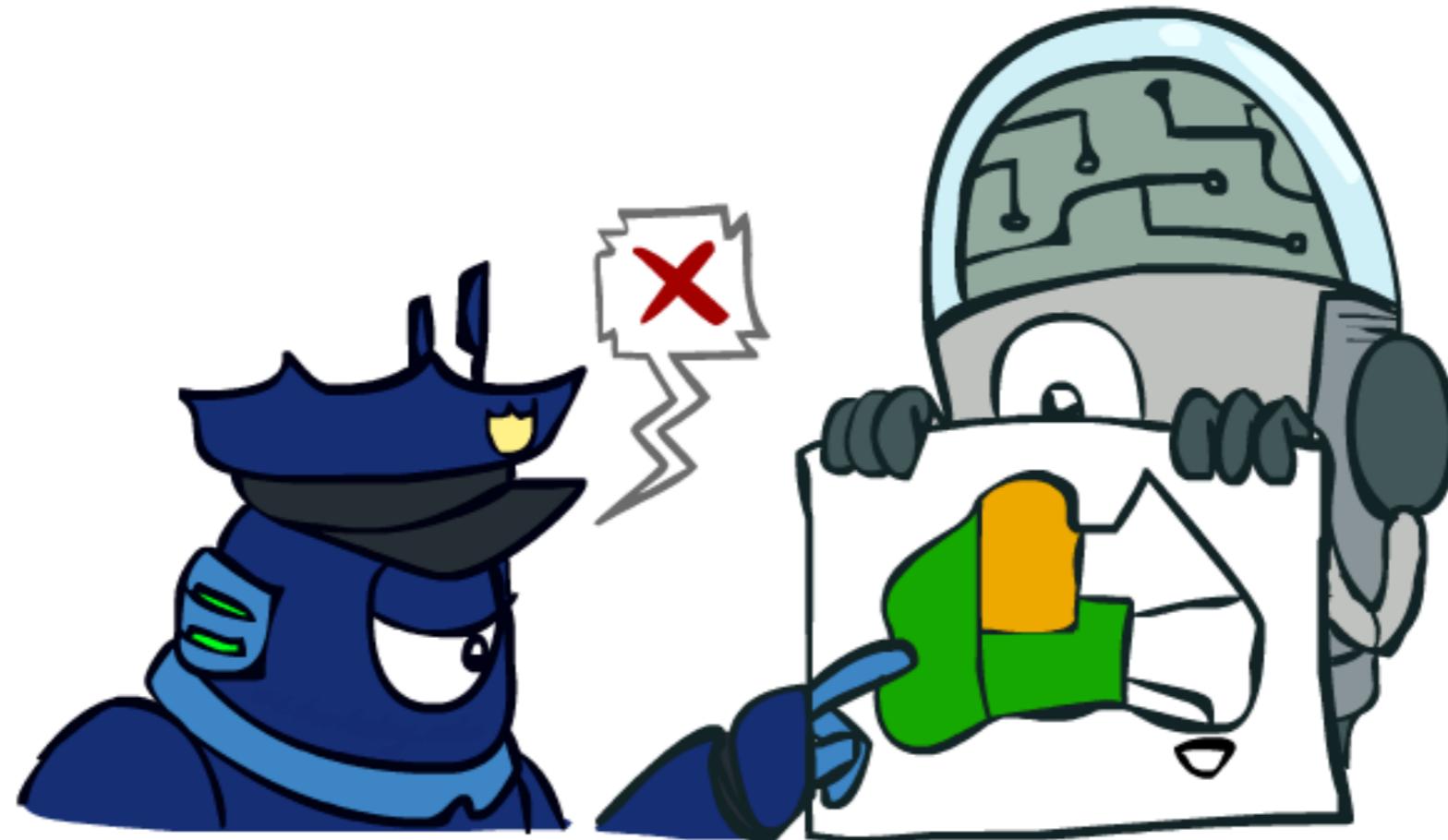
Speedup

1

Frame Delay

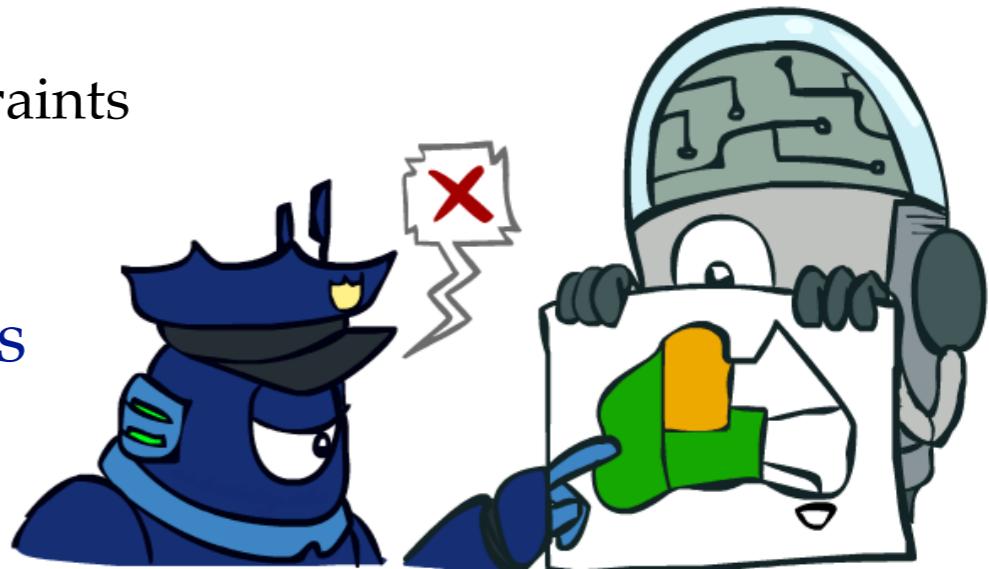
700

Backtracking Search

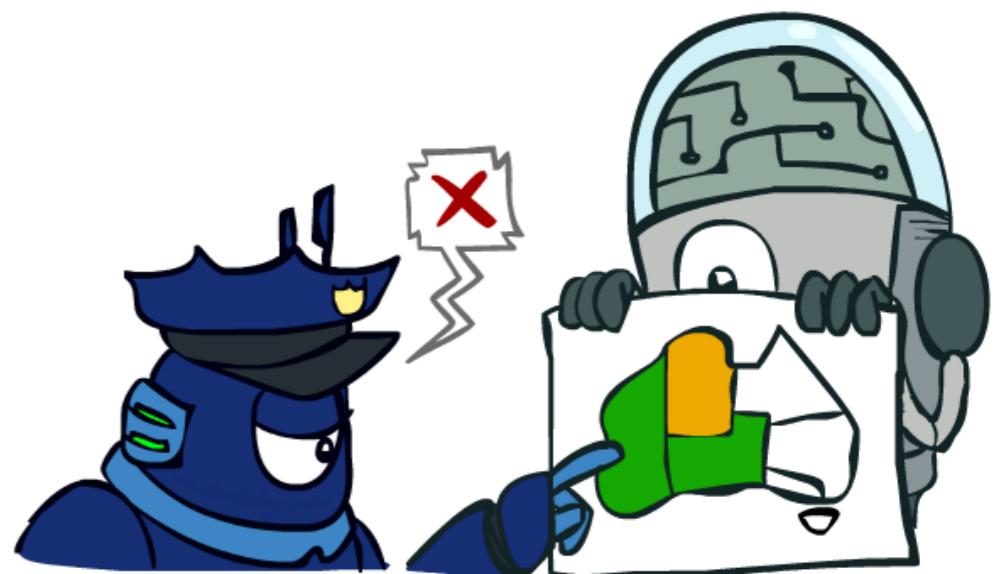
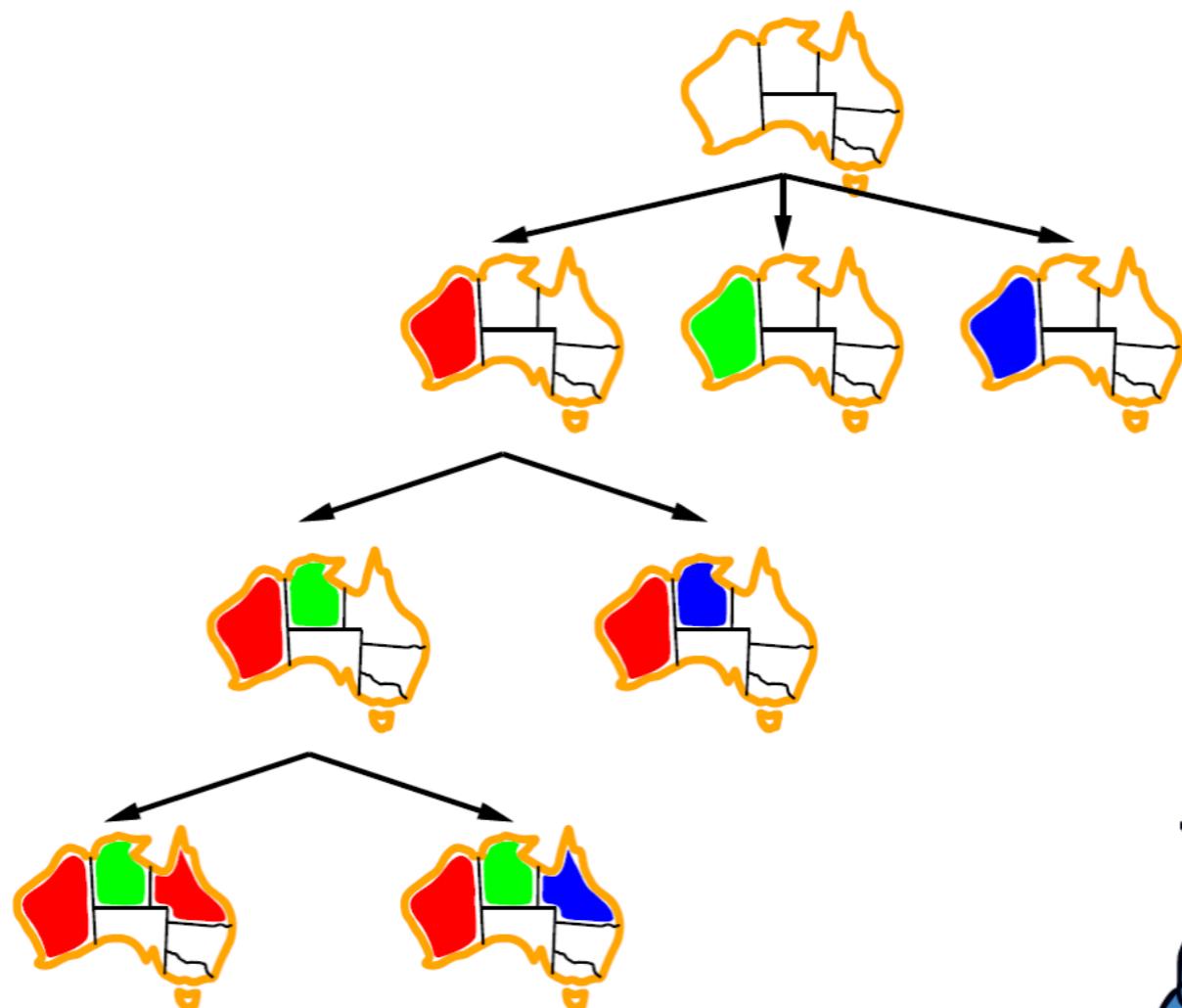


Backtracking Search

- ❖ Backtracking search is the basic uninformed algorithm for solving CSPs
- ❖ Idea 1: One variable at a time
 - ❖ Variable assignments are commutative, so fix ordering
 - ❖ I.e., [WA = red then NT = green] same as [NT = green then WA = red]
 - ❖ Only need to consider assignments to a single variable at each step
- ❖ Idea 2: Check constraints as you go
 - ❖ I.e. consider only values which do not conflict previous assignments
 - ❖ Might have to do some computation to check the constraints
 - ❖ “Incremental goal test”
- ❖ Depth-first search with these two improvements is called *backtracking search* (not the best name)
- ❖ Can solve n-queens for $n \approx 25$



Backtracking Example

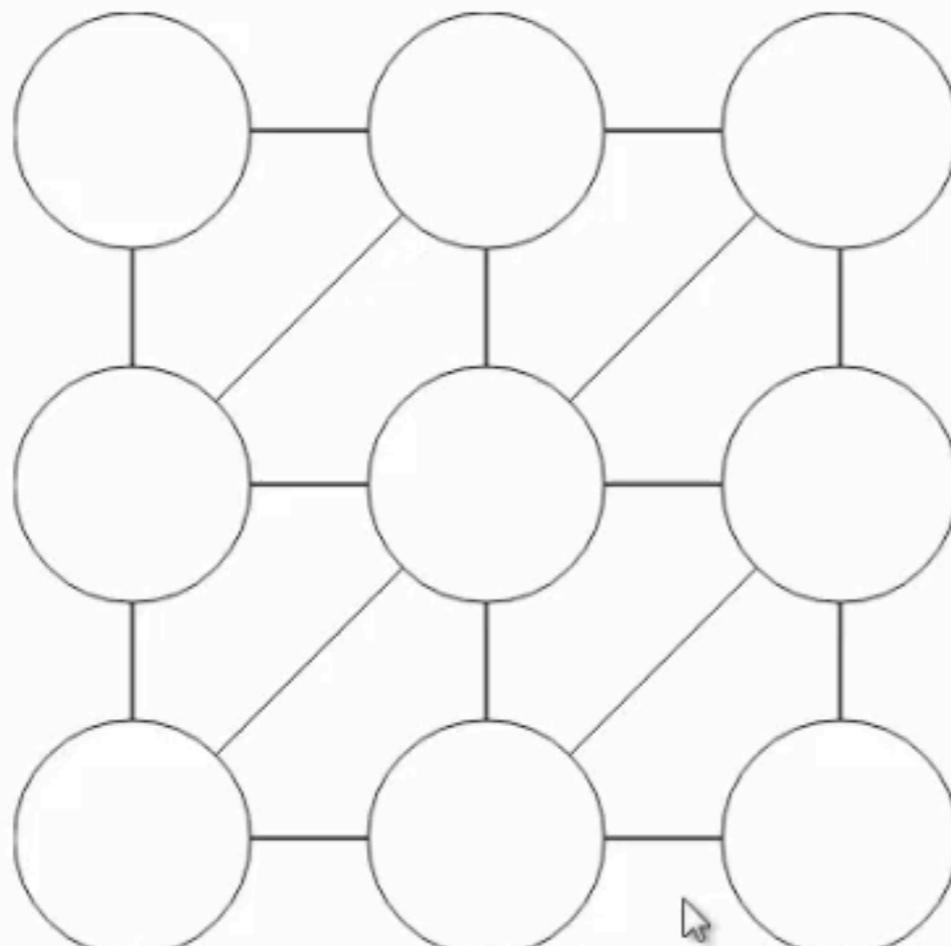


Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var  $\leftarrow$  SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add  $\{var = value\}$  to assignment
            result  $\leftarrow$  RECURSIVE-BACKTRACKING(assignment, csp)
            if result  $\neq$  failure then return result
            remove  $\{var = value\}$  from assignment
    return failure
```

- ❖ Backtracking = DFS + variable-ordering + fail-on-violation
- ❖ What are the choice points?

Video of Demo Coloring – Backtracking



Reset Prev Pause Next Play Faster

Graph

Simple ▾

Algorithm

Backtracking ▾

Ordering

- None
- MRV
- MRV with LCV

Filtering

- None
- Forward Checking
- Arc Consistency

Speed

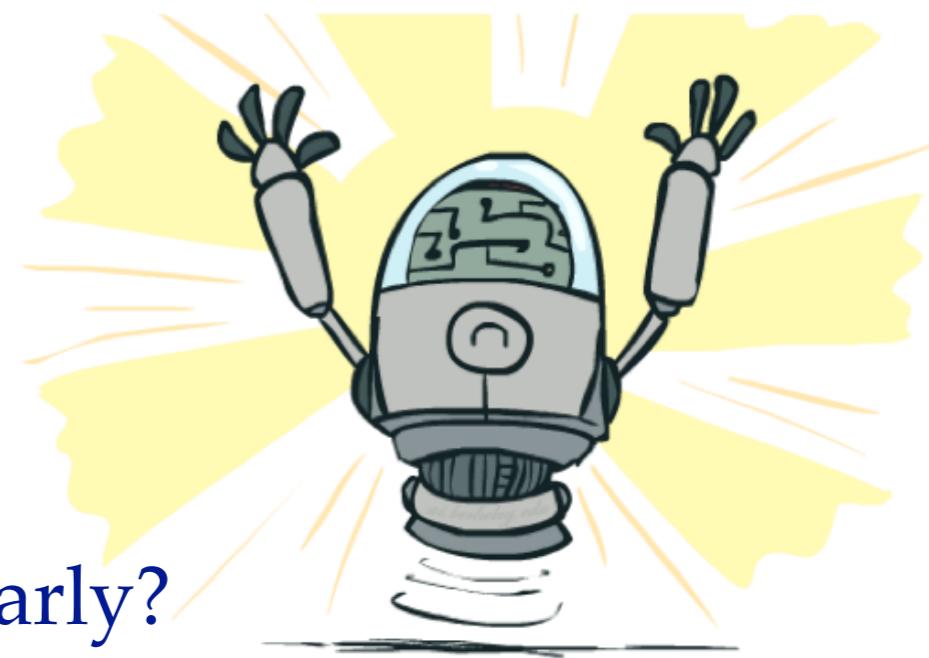
Speedup Frame Delay

1 ▾ x

700

Improving Backtracking

- ❖ General-purpose ideas give huge gains in speed
- ❖ Ordering:
 - ❖ Which variable should be assigned next?
 - ❖ In what order should its values be tried?
- ❖ Filtering: Can we detect inevitable failure early?
- ❖ Structure: Can we exploit the problem structure?

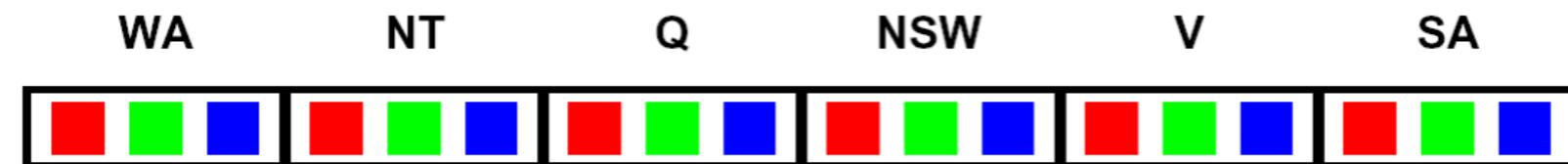


Filtering

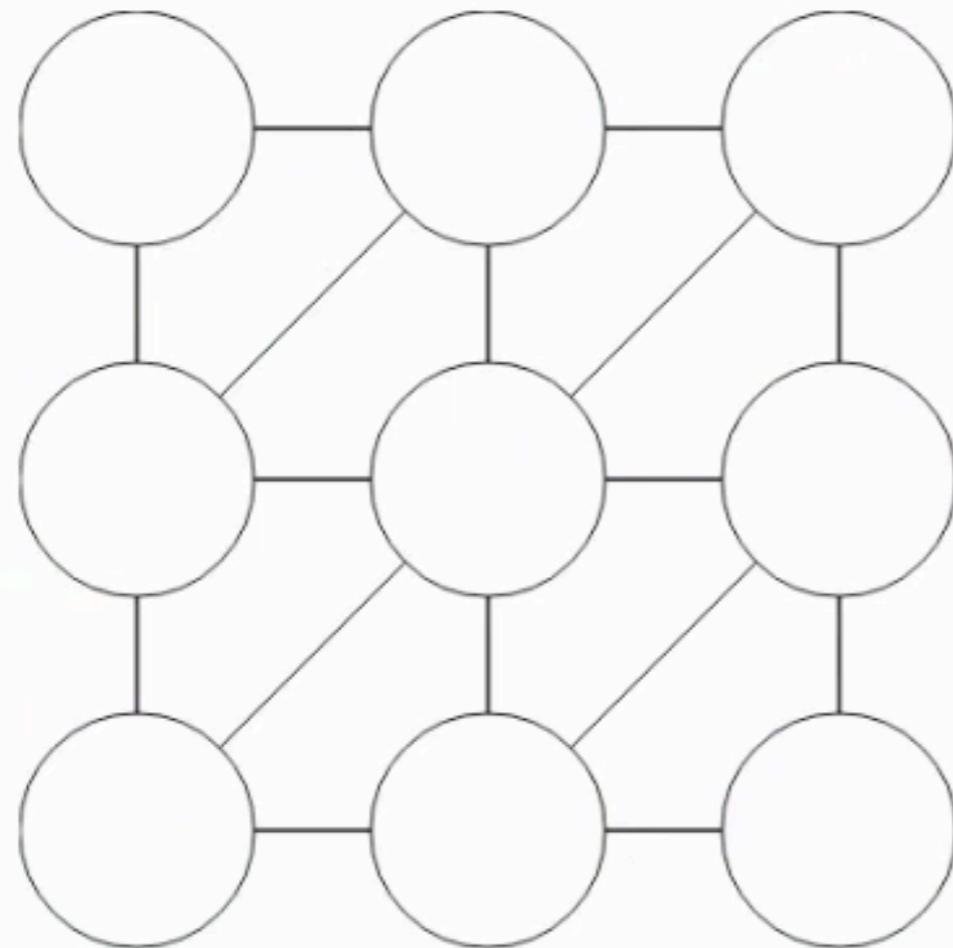


Filtering: Forward Checking

- ❖ Filtering: Keep track of domains for unassigned variables and cross off bad options
- ❖ Forward checking: Cross off values that violate a constraint when added to the existing assignment



Video of Demo Coloring – Backtracking with Forward Checking



Reset Prev Pause Next Play Faster

Graph

Simple

Algorithm

Backtracking

Ordering

- None
- MRV
- MRV with LCV

Filtering

- None
- Forward Checking
- Arc Consistency

Speed

Speedup

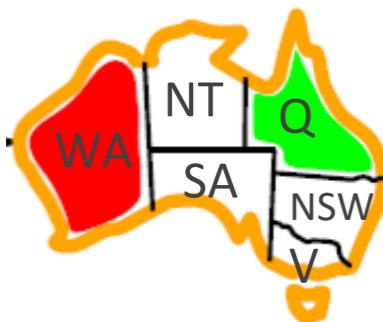
1 X

Frame Delay

700

Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



WA	NT	Q	NSW	V	SA
Red	Green	Blue	Red	Green	Blue
Red		Green	Blue	Red	Green
Red		Blue	Green	Red	Blue

- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- Constraint propagation*: reason from constraint to constraint

Consistency of A Single Arc

- An arc $X \rightarrow Y$ is **consistent** iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint

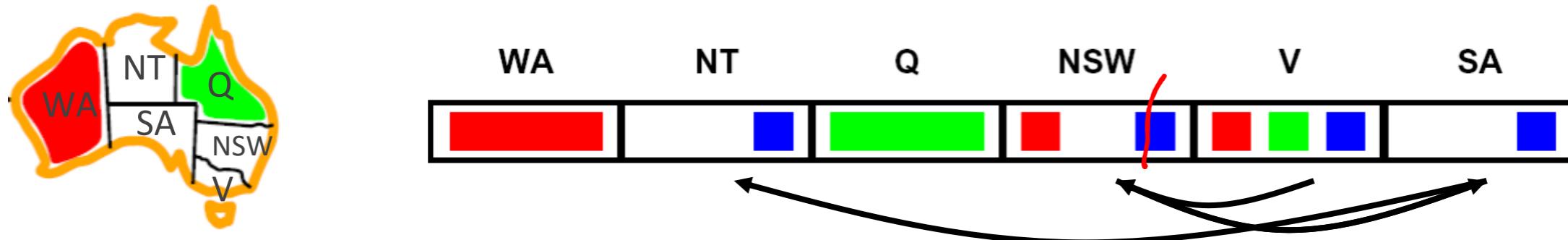


Delete from the tail!

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- ❖ A simple form of propagation makes sure all arcs are consistent:



- ❖ Important: If X loses a value, neighbors of X need to be rechecked!
- ❖ Arc consistency detects failure earlier than forward checking
- ❖ Can be run as a preprocessor or after each assignment
- ❖ What's the downside of enforcing arc consistency?

*Remember:
Delete from
the tail!*

Enforcing Arc Consistency in a CSP

function AC-3(*csp*) returns the CSP, possibly with reduced domains

inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \dots, X_n\}$

local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

while *queue* is not empty do

$(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\text{queue})$

 if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then

 for each X_k in $\text{NEIGHBORS}[X_i]$ do

 add (X_k, X_i) to *queue*

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) returns true iff succeeds

removed $\leftarrow \text{false}$

 for each x in $\text{DOMAIN}[X_i]$ do

 if no value y in $\text{DOMAIN}[X_j]$ allows (x, y) to satisfy the constraint $X_i \leftrightarrow X_j$

 then delete x from $\text{DOMAIN}[X_i]$; $\text{removed} \leftarrow \text{true}$

 return *removed*

$O(n^2)$

$O(n^2 d)$

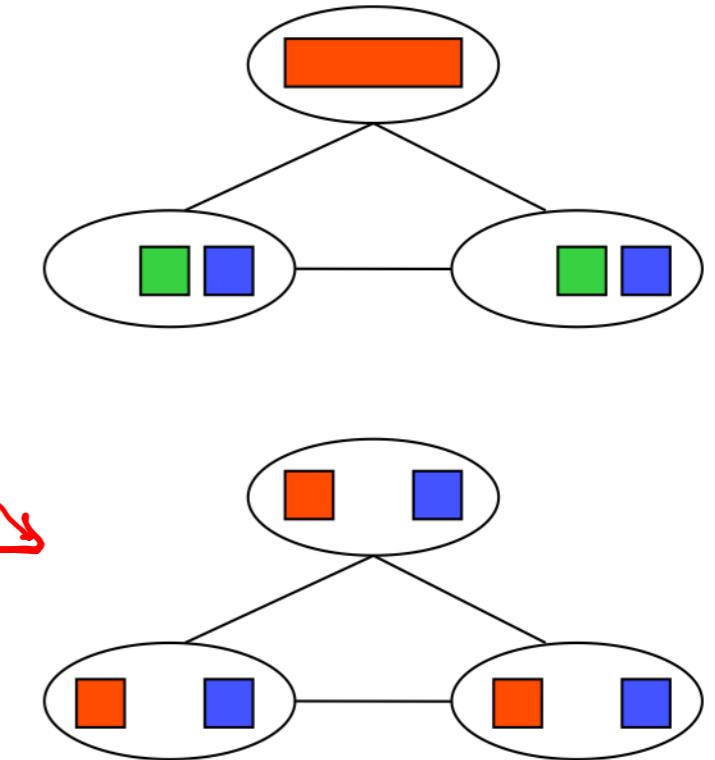
) $O(d^2)$

- ❖ Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- ❖ ... but detecting all possible future problems is NP-hard – why?

Limitations of Arc Consistency

- ❖ After enforcing arc consistency:

- ❖ Can have one solution left
- ❖ Can have multiple solutions left
- ❖ Can have no solutions left (and not know it)

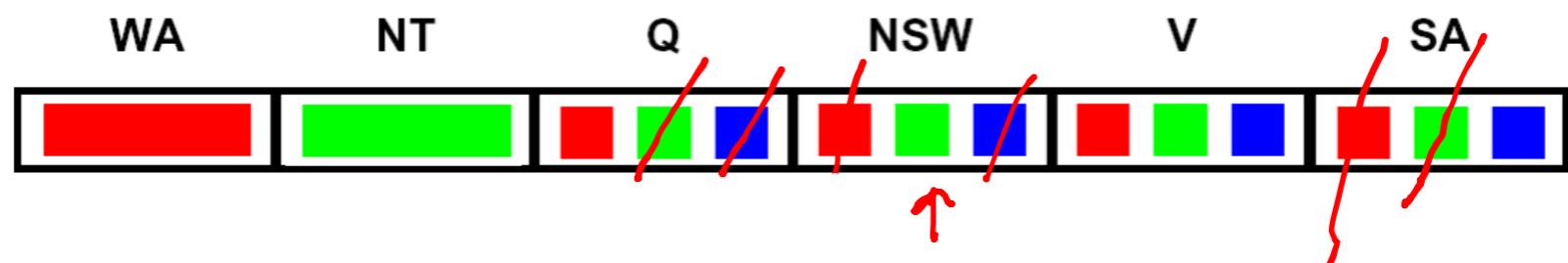


- ❖ Arc consistency still runs inside a backtracking search!

What went wrong here?

Quiz: What is the color of NSW?

- ❖ Run arc consistency on this example:



- ❖ Choose a color
 - ❖ Red
 - ❖ Green
 - ❖ Blue

Video of Demo Coloring – Backtracking with Forward Checking – Complex Graph

Graph: Complex

Algorithm: Backtracking

Ordering:

- None
- MRV
- MRV with LCV

Filtering:

- None
- Forward Checking
- Arc Consistency

Speed:

- Speedup: 1 x
- Frame Delay: 700

Buttons: Reset, Prev, Pause, **Next**, Play, Faster

Video of Demo Coloring – Backtracking with Arc Consistency – Complex Graph

A complex graph for a 3-coloring problem. The graph consists of 20 nodes arranged in four columns of five nodes each. Each node is a circle containing three colored dots (blue, red, green) representing its current state. Edges connect adjacent nodes in a zig-zag pattern across columns. Some nodes have additional edges connecting them to nodes in other columns, creating a complex web of connections.

Graph: Complex

Algorithm: Backtracking

Ordering:

- None
- MRV
- MRV with LCV

Filtering:

- None
- Forward Checking
- Arc Consistency

Speed:

Speedup: 1 x Frame Delay: 700

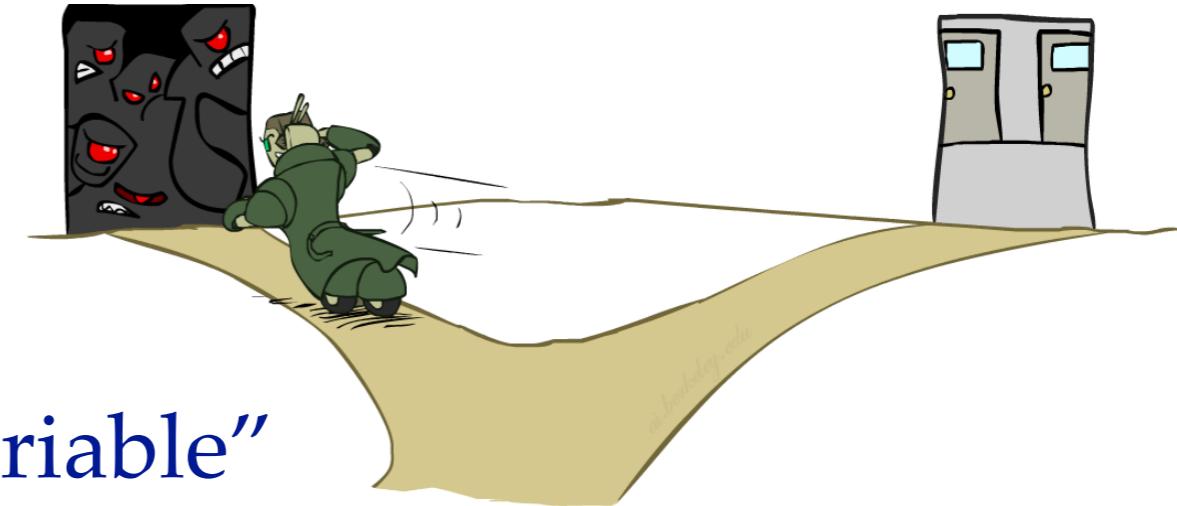
Reset Prev Pause Next Play Faster

Ordering



Ordering: Minimum Remaining Values

- ❖ Variable Ordering: Minimum remaining values (MRV):
 - ❖ Choose the variable with the fewest legal left values in its domain



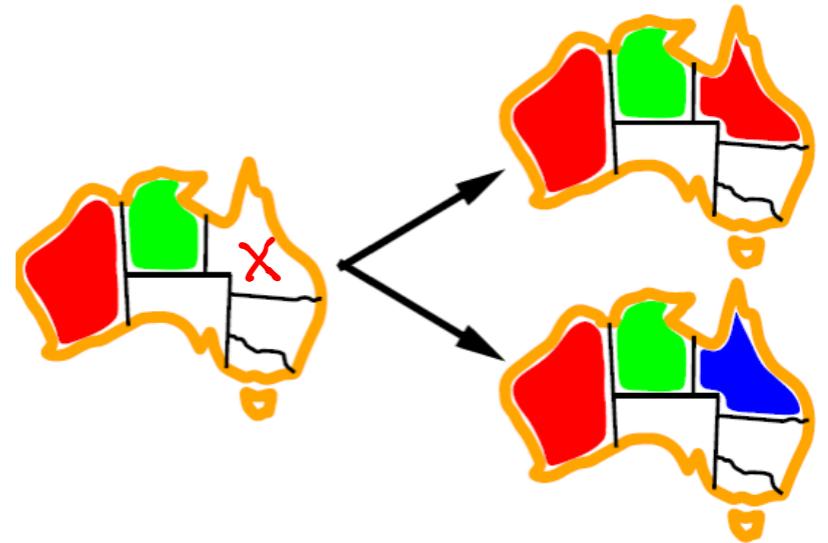
- ❖ Why min rather than max?
- ❖ Also called “most constrained variable”
- ❖ “Fail-fast” ordering

Ordering: Least Constraining Value

- ❖ Value Ordering: Least Constraining Value

- ❖ Given a choice of variable, choose the *least constraining value*
- ❖ I.e., the one that rules out the fewest values in the remaining variables
- ❖ Note that it may take some computation to determine this! (E.g., rerunning filtering)

$$x \in \{x_1, \dots, x_d\}$$



- ❖ Why least rather than most?

- ❖ Combining these ordering ideas makes 1000 queens feasible