

Pairs Trading

Prof. Daniel P. Palomar

MAFS5310 - Portfolio Optimization with R
MSc in Financial Mathematics

The Hong Kong University of Science and Technology (HKUST)
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Outline

- 1 Cointegration
- 2 Basic Idea of Pairs Trading
- 3 Design of Pairs Trading
 - Pairs selection
 - Cointegration test
 - Optimum threshold
- 4 LS Regression and Kalman for Pairs Trading
- 5 From Pairs Trading to Statistical Arbitrage (StatArb)*
 - VECM
 - Optimization of mean-reverting portfolio (MRP)
- 6 Summary

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Cointegration

- Cointegration is a very interesting property that can be exploited in finance for trading.
- Idea: While it may be difficult to predict individual stocks, it may be easier to predict relative behavior of stocks.
- Illustrative example: A drunk man is wandering the streets (random walk) with a dog. Both paths of man and dog are nonstationary and difficult to predict, but the distance between them is mean-reverting and stationary.



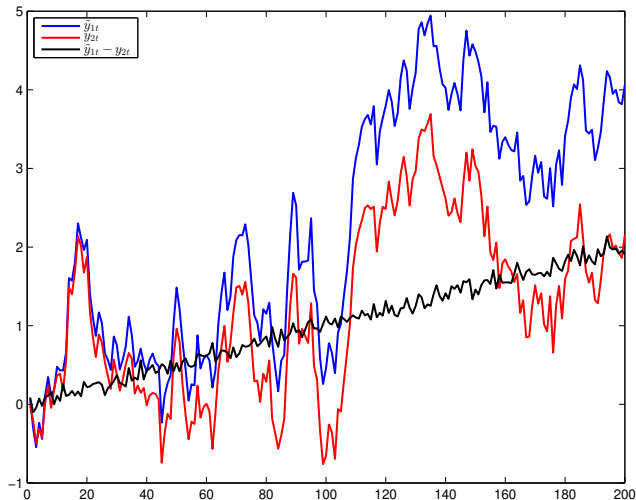
Correlation vs. cointegration

- Everybody is familiar with the concept of correlation between two random variables:
 - correlation is high when they co-move
 - correlation is zero when they move independently
- So what is cointegration?
 - cointegration is high when two quantities move together or remain close to each other
 - cointegration is inexistent if the two quantities do not stay together
- Clear? 😊 You can see why this concept may be difficult to grasp at first, but the truth is that it's easy.¹
- In the financial context:
 - Cointegration of (log-)prices y_t refers to **long-term** co-movements.
 - Correlation of (log-)returns $\Delta y_t = y_t - y_{t-1}$ characterizes **short-term** co-movements in (log-)prices y_t .

¹Y. Feng and D. P. Palomar, *A Signal Processing Perspective on Financial Engineering*. Foundations and Trends in Signal Processing, Now Publishers, 2016.

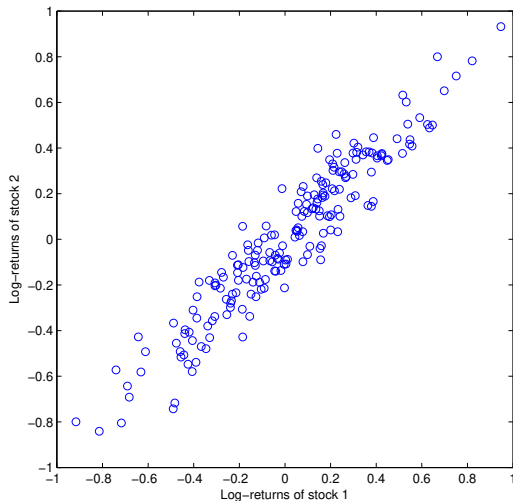
Correlation vs. cointegration

- Example of high correlation with no cointegration:



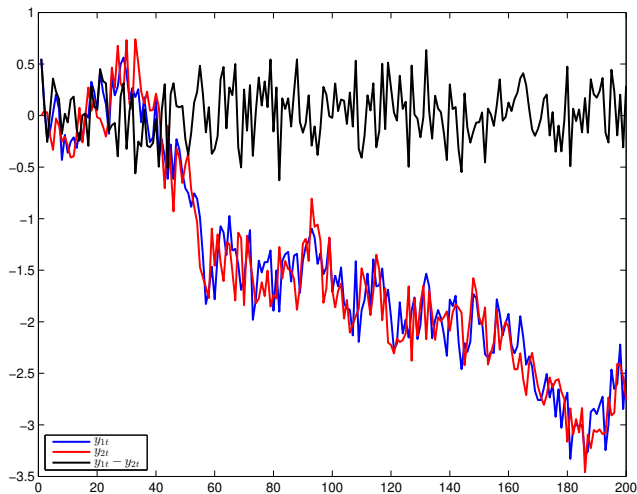
Correlation vs. cointegration

- Indeed the returns are highly correlated, see scatter plot:



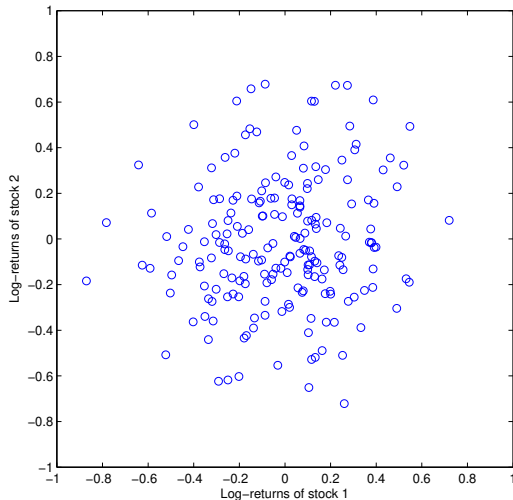
Correlation vs. cointegration

- Opposite example of high cointegration with no correlation:



Correlation vs. cointegration

- Indeed the returns are not correlated, see scatter plot:



- A time series is called integrated of order p , denoted as $I(p)$, if the time series obtained
 - by differencing the time series p times is weakly stationary,
 - while by differencing the time series $p - 1$ times is not weakly stationary.
- Example: stock log-prices y_t are integrated of order $I(1)$ because
 - log-prices are not stationary
 - but log-returns $y_t - y_{t-1}$ are stationary (at least for some period of time).
- A multivariate time series is said to be cointegrated if it has at least one linear combination being integrated of a lower order, e.g., \mathbf{y}_t is not stationary but $\mathbf{w}^T \mathbf{y}_t$ is stationary for some weights \mathbf{w} .

Cointegration

- Consider the following two nonstationary time series (e.g., log-prices of stocks):

$$y_{1t} = \gamma x_t + w_{1t}$$

$$y_{2t} = x_t + w_{2t}$$

with a stochastic common trend defined as a random walk:

$$x_t = x_{t-1} + w_t$$

where w_{1t} , w_{2t} , w_t are i.i.d. residual terms mutually independent.

- The coefficient γ is the secret ingredient here.
- If γ is known, then we can define the so-called “spread”

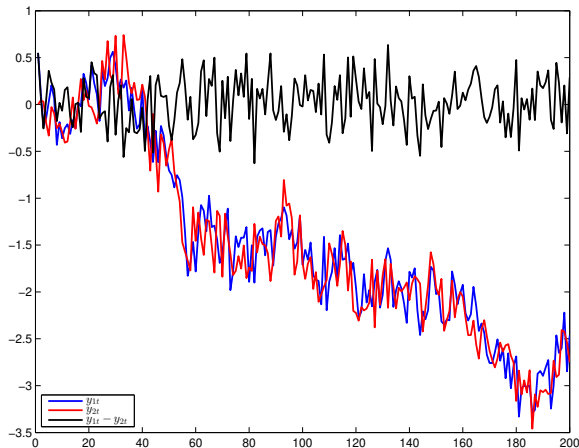
$$z_t = y_{1t} - \gamma y_{2t} = w_{1t} - \gamma w_{2t}$$

which is stationary and **mean reverting**.

- Interestingly, the differences (i.e., log-returns) Δy_{1t} and Δy_{2t} can have an arbitrarily small correlation: $\rho = 1 / \left(\sqrt{1 + 2\sigma_1^2/\sigma^2} \sqrt{1 + 2\sigma_2^2/\sigma^2} \right)$.

Cointegration

- The log-prices y_{1t} and y_{2t} are cointegrated and the spread $z_t = y_{1t} - \gamma y_{2t}$ is stationary (assume $\gamma = 1$):



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Basic Idea of Pairs Trading

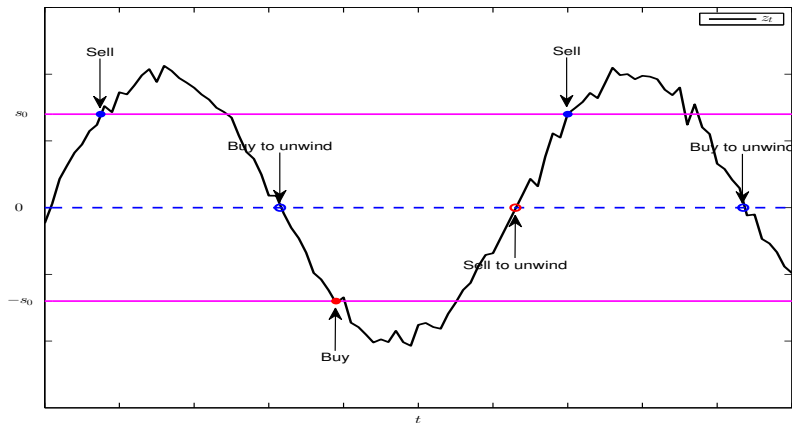
- Recall that if two time series are cointegrated, then in the long term they remain close to each other.
- In other words, the spread $z_t = y_{1t} - \gamma y_{2t}$ is **mean reverting**.
- This mean-reverting property of the spread can be exploited for trading and it is commonly referred to as “pairs trading” or “statistical arbitrage”.
- The idea behind pairs trading is to
 - short-sell the relatively overvalued stocks and buy the relatively undervalued stocks,
 - unwind the position when they are relatively fairly valued.

Trading the spread

- Suppose the spread $z_t = y_{1t} - \gamma y_{2t}$ is mean-reverting with zero mean.
- Stat-arb trading:
 - if spread is low ($z_t < -s_0$), then stock 1 is undervalued and stock 2 overvalued:
 - buy the spread (i.e., buy stock 1 and short-sell stock 2)
 - unwind the positions when it reverts to zero after i time steps $z_{t+i} = 0$
 - if spread is high ($z_t > s_0$), then stock 1 is overvalued and stock 2 undervalued:
 - short-sell the spread (i.e., short-sell stock 1 and buy stock 2)
 - unwind the positions when it reverts to zero after i time steps $z_{t+i} = 0$
- The profit, say, from buying low and unwinding at zero is $z_{t+i} - z_t = s_0$. So easy!
- Indeed $z_{t+i} - z_t = -\gamma(y_{2,t+i} - y_{2t}) + (y_{1,t+i} - y_{1t})$, so the whole process is like having used a portfolio with weights $\mathbf{w} = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}$.
- Recall that the return of a portfolio \mathbf{w} is $\mathbf{w}^T \Delta \mathbf{y}_t$.

Trading the spread

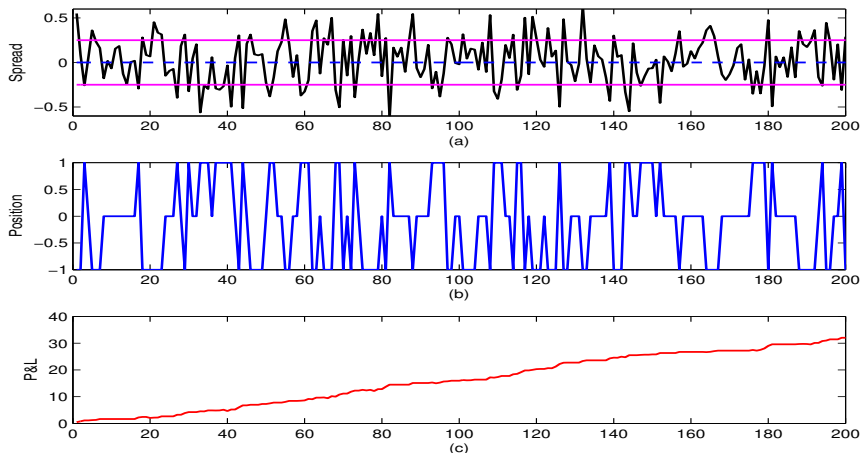
- Illustration on how to trade the spread $z_t = y_{1t} - \gamma y_{2t}$ ²



²G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*. John Wiley & Sons, 2004.

Pairs trading or statistical arbitrage

- Statistical arbitrage can be used in practice with profits:³



³M. Avellaneda and J.-H. Lee, "Statistical arbitrage in the US equities market," *Quantitative Finance*, vol. 10, no. 7, pp. 761–782, 2010.

But how to discover cointegrated pairs and γ ?

- One interesting approach is based on a VECM modeling of the universe of stocks: From the parameter β contained in the low-rank matrix $\Pi = \alpha\beta^T$ one can extract a cointegration subspace. After that, one can design some portfolio within that cointegration subspace.⁴
- A simpler approach to discover pairs is by brute force, i.e., try exhaustively different combinations of pairs of stocks and see if they are cointegrated.
- But, given a potential pair, how do we obtain the “secret” γ ?
- Easy! Just a simple LS regression!
- Recall that
 - γ is needed to form the spread to be traded (i.e., portfolio)
 - the spread mean μ is needed to determine the thresholds for entering a trade and unwind later the position.

⁴Z. Zhao and D. P. Palomar, “Mean-reverting portfolio with budget constraint,” *IEEE Trans. Signal Process.*, vol. 66, no. 9, pp. 2342–2357, 2018.

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Design of a pairs trading strategy

- We first focus on pairs trading (i.e., statistical arbitrage between two stocks) as the example to introduce the main steps of statistical arbitrage.
- In practice, pairs trading contains three main steps⁵:
 - **Pairs selection**: identify stock pairs that could potentially be cointegrated.
 - **Cointegration test**: test whether the identified stock pairs are indeed cointegrated or not.
 - **Trading strategy design**: study the spread dynamics and design proper trading rules.

⁵G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*. John Wiley & Sons, 2004.

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Pairs selection: normalized price distance

- Normalized price distance⁶ (as a rough proxy to measure cointegration):

$$\text{NPD} \triangleq \sum_{t=1}^T (\tilde{p}_{1t} - \tilde{p}_{2t})^2$$

where the normalized price \tilde{p}_{1t} of stock 1 is given by $\tilde{p}_{1t} = p_{1t}/p_{10}$. The normalized prices of stock 2 defined similarly.

- One can easily (i.e., cheaply) compute the NPD for all the possible combination of pairs and select some pairs with smallest NPD as the potentially cointegrated pairs.
- Later one can use a more refined measure of cointegration (more computationally demanding).

⁶E. Gatev, W. N. Goetzmann, and K. G. Rouwenhorst, "Pairs trading: Performance of a relative-value arbitrage rule," *Review of Financial Studies*, vol. 19, no. 3, pp. 797–827, 2006.

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Least Squares (LS) regression

- If the spread z_t is stationary, it can be written as⁷

$$z_t = y_{1t} - \gamma y_{2t} = \mu + \epsilon_t$$

where

- μ represents the equilibrium value and
- ϵ_t is a zero-mean residual.
- Equivalently, it can be written as

$$y_{1t} = \mu + \gamma y_{2t} + \epsilon_t$$

which now has the typical form of linear regression.

- Least squares (LS) regression over T observations:

$$\underset{\mu, \gamma}{\text{minimize}} \quad \sum_{t=1}^T (y_{1t} - (\mu + \gamma y_{2t}))^2$$

⁷G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*. John Wiley & Sons, 2004.

Cointegration test

- LS regression is used to estimate the parameters μ and γ , obtaining the estimates $\hat{\mu}$ and $\hat{\gamma}$.
- If y_{1t} and y_{2t} are $I(1)$ and are cointegrated, then the estimates converge to the true values as the number of observations goes to infinity⁸.
- Using the estimated parameters $\hat{\mu}$ and $\hat{\gamma}$, we can compute the residuals

$$\hat{\epsilon}_t = y_{1t} - \hat{\gamma}y_{2t} - \hat{\mu}.$$

- Then, one has to decide whether the spread is stationary, i.e., ϵ_t is stationary. In practice, the estimated residuals are used $\hat{\epsilon}_t$
- There are many well-defined mathematical tests for the stationarity of $\hat{\epsilon}_t$, e.g., augmented Dicky-Fuller (ADF) test, Johansen test, etc.

⁸R. F. Engle and C. W. J. Granger, "Co-integration and error correction: Representation, estimation, and testing," *Econometrica: Journal of the Econometric Society*, pp. 251–276, 1987.

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Optimum threshold

- Once some identified pairs have passed the cointegration test, one still needs to decide the entry and exit thresholds to open and unwind the positions, respectively.
- For the sake of concreteness, we focus on studying the entry threshold:
 - open positions when the spread diverges from its long-term mean by s_0
 - unwind the position when it reverts to its mean
- Thus, the key problem now is how to design the value of s_0 such that the total profit is maximized.
- Total profit:

profit of each trade \times number of trades

- profit of each trade is s_0
 - number of trades is related to the zero crossings, which can be analyzed theoretically as well as empirically.
- We focus now on estimating the number of trades.

Optimum threshold s_0 : Parametric approach*

- Suppose the spread follows a standard Normal distribution.
- The probability that the spread deviates above from the mean by s_0 or more is

$$1 - \Phi(s_0)$$

where $\Phi(\cdot)$ is the c.d.f. of the standard Normal distribution.

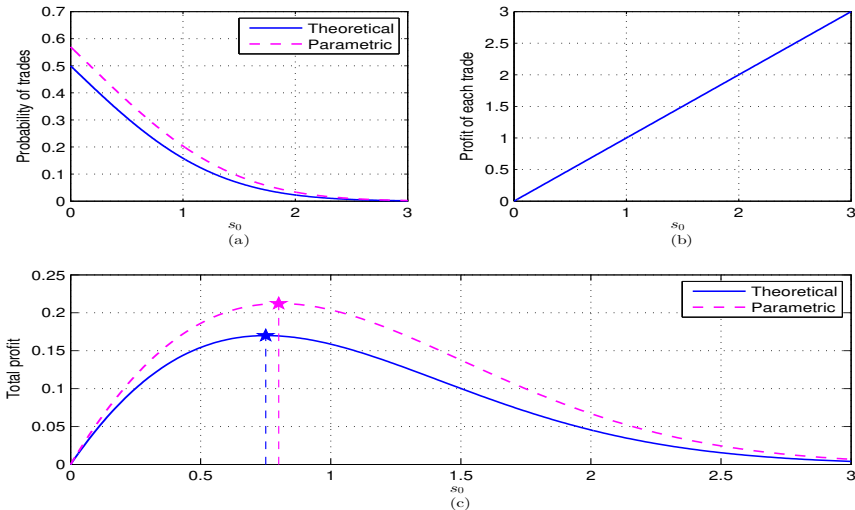
- For a path with T days, the number of tradable events is

$$T(1 - \Phi(s_0)).$$

- For each trade, the profit is s_0 and then the total profit is $s_0 T(1 - \Phi(s_0))$.
- Then the optimal threshold is $s_0^* = \arg \max_{s_0} \{s_0 T(1 - \Phi(s_0))\}$.
- In practice, one cannot know the true distribution but can estimate the distribution parameters.
- Then one can compute the total profit based on estimated distribution.

Optimum threshold s_0 : Parametric approach*

- Optimal threshold s_0^* maximizes the total profit:



Optimum threshold s_0 : Non-parametric approach*

- Suppose the observed sample path has length T : z_1, z_2, \dots, z_T .
- We consider J discretized threshold values as $s_0 \in \{s_{01}, s_{02}, \dots, s_{0J}\}$ and the empirical trading frequency for the threshold s_{0j} is

$$\bar{f}_j = \frac{\sum_{t=1}^T \mathbf{1}_{\{z_t > s_{0j}\}}}{T}.$$

- The empirical values \bar{f}_j may not be smoothed enough and the resulted profit function may not be accurate enough.
- Smooth the trading frequency function by regularization:

$$\underset{\mathbf{f}}{\text{minimize}} \quad \sum_{j=1}^J (\bar{f}_j - f_j)^2 + \lambda \sum_{j=1}^{J-1} (f_j - f_{j+1})^2$$

Optimum threshold s_0 : Non-parametric approach*

- The problem can be rewritten as

$$\underset{\mathbf{f}}{\text{minimize}} \quad \|\bar{\mathbf{f}} - \mathbf{f}\|_2^2 + \lambda \|\mathbf{D}\mathbf{f}\|_2^2$$

where

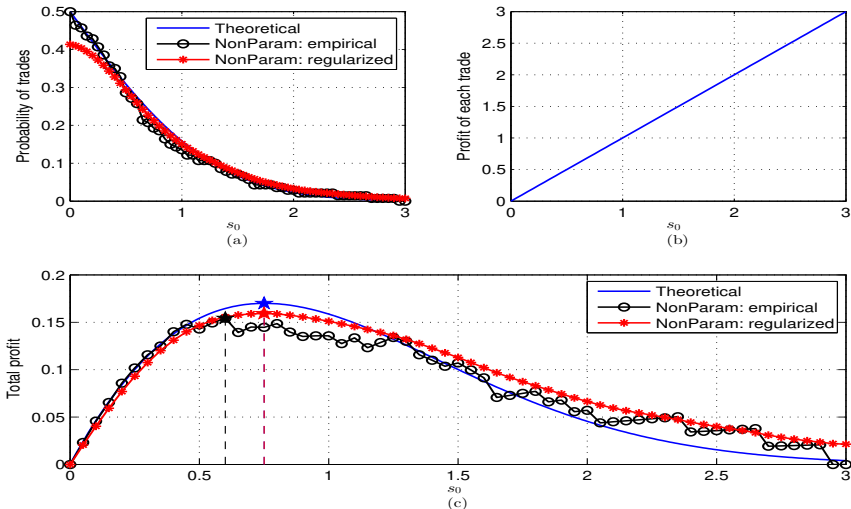
$$\mathbf{D} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(J-1) \times J}.$$

- Setting the derivative of the objective w.r.t. \mathbf{f} to zero yields the optimal solution $\mathbf{f}^* = (\mathbf{I} + \lambda \mathbf{D}^T \mathbf{D})^{-1} \bar{\mathbf{f}}$.
- The optimal threshold is the one maximizes the total profit:

$$s_0^* = \underset{s_{0j} \in \{s_{01}, s_{02}, \dots, s_{0J}\}}{\arg \max} \quad \{s_{0j} f_j\}.$$

Optimum threshold s_0 : Non-parametric approach*

- Optimal threshold s_0^* maximizes the total profit:



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LS regression for pairs trading

- If the spread z_t is stationary, it can be written as

$$z_t = y_{1t} - \gamma y_{2t} = \mu + \epsilon_t$$

where

- μ represents the equilibrium value and
- ϵ_t is a zero-mean residual.
- Equivalently, it can be written as

$$y_{1t} = \mu + \gamma y_{2t} + \epsilon_t$$

which now has the typical form for linear regression.

- Least squares (LS) regression over T observations:

$$\underset{\mu, \gamma}{\text{minimize}} \quad \sum_{t=1}^T (y_{1t} - (\mu + \gamma y_{2t}))^2$$

- By stacking the T observations in the vectors \mathbf{y}_1 and \mathbf{y}_2 , we can finally write:

$$\underset{\mu, \gamma}{\text{minimize}} \quad \|\mathbf{y}_1 - (\mu \mathbf{1} + \gamma \mathbf{y}_2)\|^2$$

LS regression for pairs trading

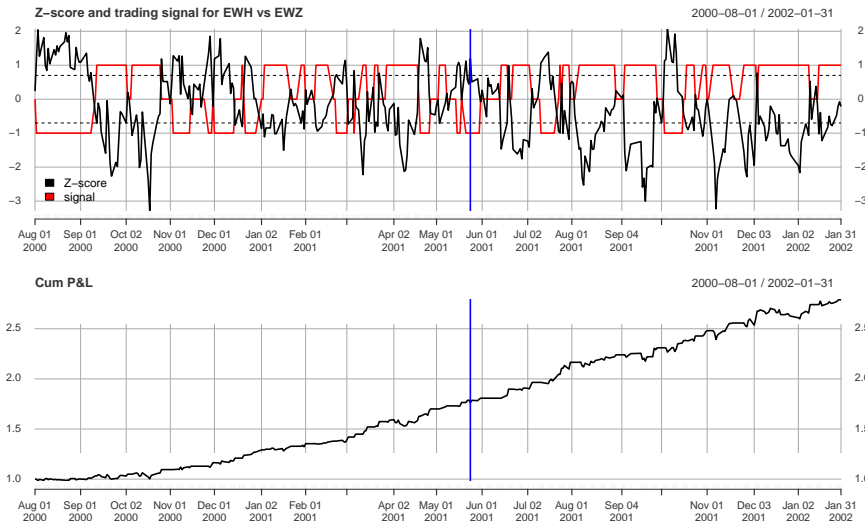
- Using the estimated parameters $\hat{\mu}$ and $\hat{\gamma}$, we can compute the residuals $\hat{\epsilon}_t = y_{1t} - \hat{\mu} - \hat{\gamma}y_{2t}$.
- Then, one has to decide whether the cointegration is acceptable or not so move to the trading part.
- There are many well-defined mathematical tests for the stationarity of $\hat{\epsilon}_t$, e.g., augmented Dicky-Fuller (ADF) test, Johansen test, etc.
- Total profit:

profit of each trade \times number of trades

- profit of each trade is s_0
 - number of trades is related to the zero crossings, which can be analyzed theoretically as well as empirically.
- Ideally, we want residuals with large amplitude (variance) as well as a strong mean reversion because they directly affect the profit.

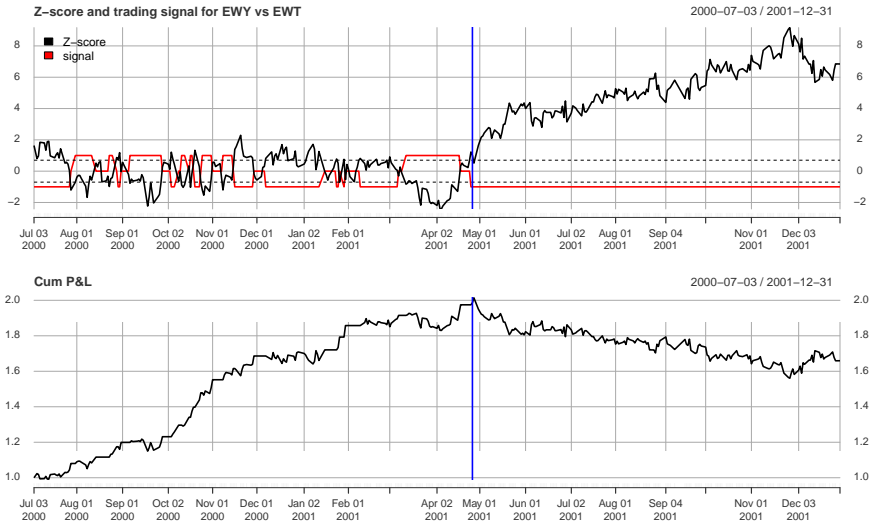
LS regression for pairs trading

- One good case: 😊



LS regression for pairs trading

- But also a bad case: 😞



LS regression for pairs trading

- The problem with the LS regression is that it assumes that μ and γ are constant.
- In practice, they can change with time, resulting in a spread that drifts from equilibrium never to revert back with huge potential losses.
- Thus, in practice, μ and γ are time-varying and have to be tracked.
- How to track time-varying parameters?
- Of course... Kalman!!!
- Well, you can also try a rolling regression or exponential smoothing, but Kalman works better.

Kalman for pairs trading

- Recall the previous static relationship for cointegrated series y_{1t} and y_{2t} :

$$y_{1t} = \mu + \gamma y_{2t} + \epsilon_t$$

- Let's make it time-varying:

$$y_{1t} = \mu_t + \gamma_t y_{2t} + \epsilon_t$$

- Let's further assume that the parameters μ_t and γ_t change slowly over time:

$$\mu_{t+1} = \mu_t + \eta_{1t}$$

$$\gamma_{t+1} = \gamma_t + \eta_{2t}$$

- Obviously, this fits nicely the Kalman framework!

Interlude: The Kalman filter

- Kalman filter consist of two equations that model the time-varying hidden state \mathbf{x}_t and the observations \mathbf{y}_t :

$$\mathbf{x}_{t+1} = \mathbf{T}_t \mathbf{x}_t + \boldsymbol{\eta}_t$$

$$\mathbf{y}_t = \mathbf{Z}_t \mathbf{x}_t + \boldsymbol{\epsilon}_t$$

- The observation equation $\mathbf{y}_t = \mathbf{Z}_t \mathbf{x}_t + \boldsymbol{\epsilon}_t$ relates the observation \mathbf{y}_t to the hidden state \mathbf{x}_t as a linear relationship, where \mathbf{Z}_t is the time-varying observation matrix and $\boldsymbol{\epsilon}_t$ is a zero-mean Gaussian error $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ with covariance matrix \mathbf{R} .
- The state transition equation $\mathbf{x}_{t+1} = \mathbf{T}_t \mathbf{x}_t + \boldsymbol{\eta}_t$ expresses the transition of the hidden state from \mathbf{x}_t to \mathbf{x}_{t+1} as a linear relationship, where \mathbf{T}_t is the time-varying transition matrix and $\boldsymbol{\eta}_t$ is a zero-mean Gaussian error $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ with covariance matrix \mathbf{Q} .
- The Kalman filter is extremely versatile in modeling a variety of real-life processes.⁹

⁹J. Durbin and S. J. Koopman, *Time Series Analysis by State Space Methods*, 2nd Ed. Oxford University Press, 2012.

Kalman for pairs trading

- Kalman filter (state transition equation and observation equation):

$$\mathbf{x}_{t+1} = \mathbf{T}\mathbf{x}_t + \boldsymbol{\eta}_t$$

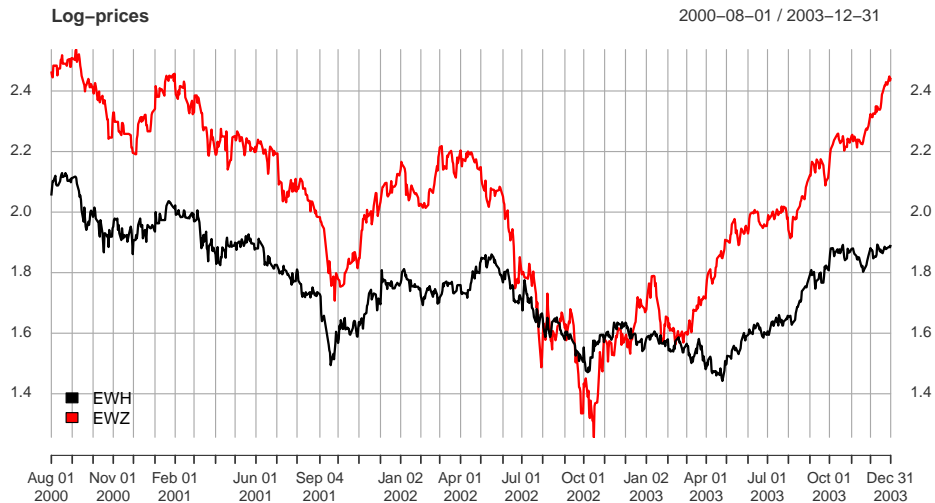
$$y_{1t} = \mathbf{Z}_t\mathbf{x}_t + \epsilon_t$$

where

- $\mathbf{x}_t \triangleq \begin{bmatrix} \mu_t \\ \gamma_t \end{bmatrix}$ is the hidden state
- $\mathbf{T} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the state transition matrix
- $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ is the i.i.d. state transition noise with $\mathbf{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$
- $\mathbf{Z}_t \triangleq \begin{bmatrix} 1 & y_{2t} \end{bmatrix}$ is the observation coefficient matrix
- $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is the i.i.d. observation noise
- Note that this is a time-varying Kalman filter since \mathbf{Z}_t is time-varying.
- Parameters σ_1^2 , σ_2^2 , σ_ϵ^2 can be estimated using the EM algorithm using historical data for calibration.
- The hidden state path \mathbf{x}_t gives the sought time-varying coefficients.

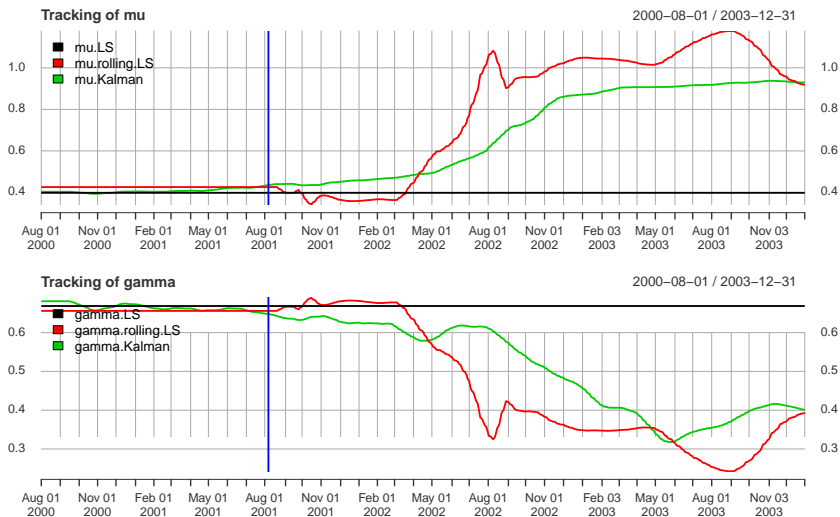
Kalman for pairs trading

- Log-prices of ETFs EWH and EWZ:



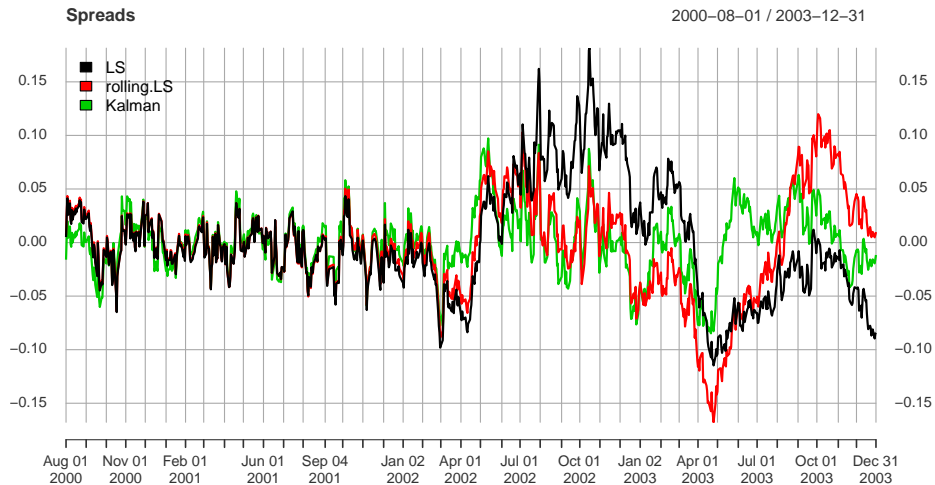
Kalman for pairs trading

- Tracking of μ and γ by LS, rolling LS, and Kalman:



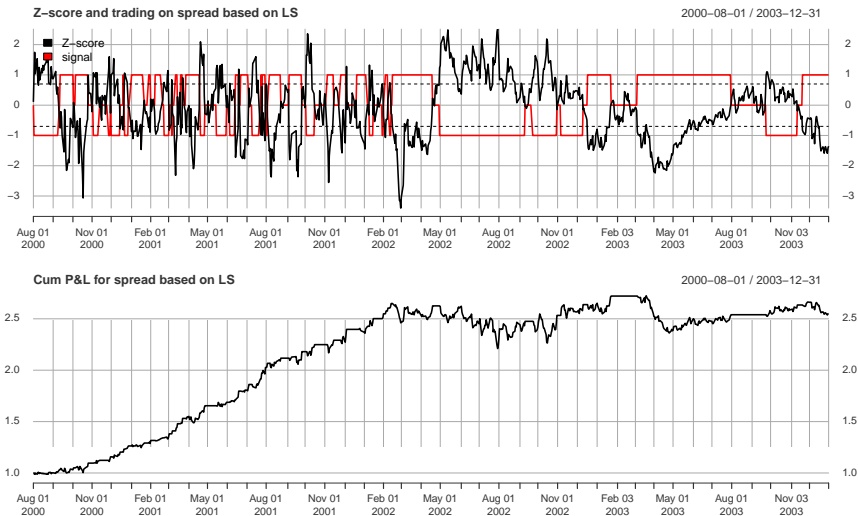
Kalman for pairs trading

- Spreads achieved by LS, rolling LS, and Kalman:



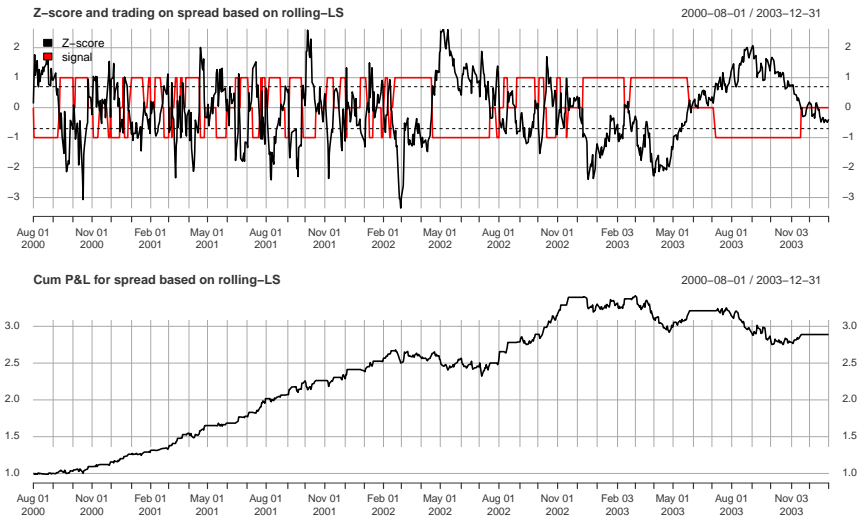
Kalman for pairs trading

- Trading of spread from LS:



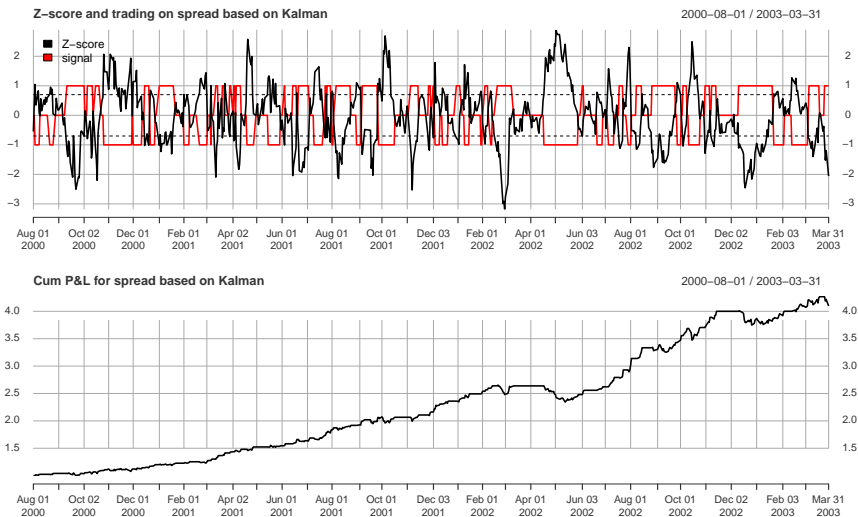
Kalman for pairs trading

- Trading of spread from rolling LS:



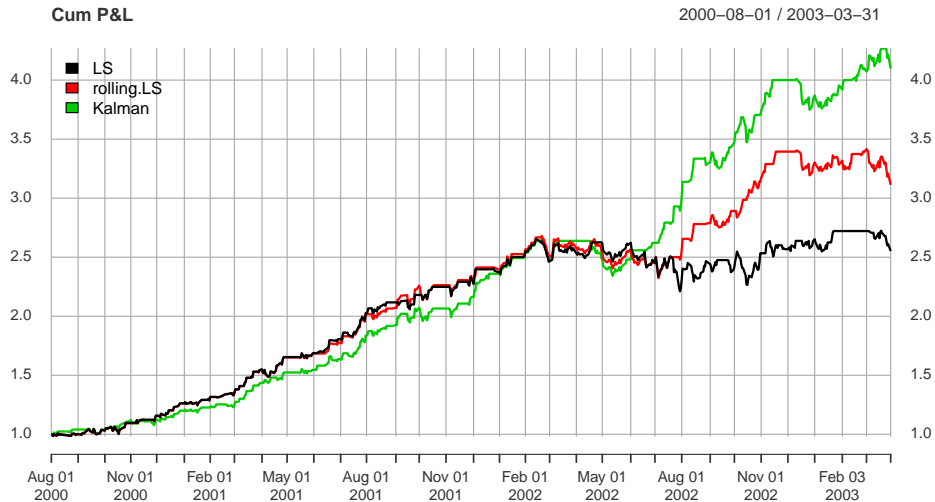
Kalman for pairs trading

- Trading of spread from Kalman:



Kalman for pairs trading

- Wealth comparison:



Kalman filter in finance

- The Kalman filter can and has been used in many aspects of financial time-series modeling as one could expect.¹⁰
- Examples of univariate time series: rate of inflation, national income, level of unemployment, etc.
- Typical models include: local model, trend-cycle decompositions, seasonality, etc.
- Examples of multivariate time series: inflation and national income.
- Multiple time series allows for more sophisticated models including common factors, cointegration, etc.
- Also data irregularities can be easily handled, e.g., missing observations, outliers, mixed frequencies.
- Plenty of applications for nonlinear and non-Gaussian models as well, e.g., GARCH modeling and stochastic volatility modeling.

¹⁰A. Harvey and S. J. Koopman, "Unobserved components models in economics and finance: The role of the Kalman filter in time series econometrics," *IEEE Control Systems Magazine*, vol. 29, no. 6, pp. 71–81, 2009.

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From pairs trading to statistical arbitrage

- Pairs trading focuses on finding cointegration between two stocks.
- A more general idea is to extend this statistical arbitrage from two stocks to more stocks.
- The idea is still based on cointegration:
*Try to construct a linear combination of the log-prices of multiple (more than two) stocks such that it is a **cointegrated mean-reversion** process.*
- In the case of two assets, the spread is $z_t = y_{1t} - \gamma y_{2t}$, which can be understood as a portfolio with weights: $\mathbf{w} = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}$.
- In the general case of many assets, one has to properly design the portfolio \mathbf{w} .

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- Denote the log-prices of multiple stocks as \mathbf{y}_t and the log-returns as $\mathbf{r}_t = \Delta \mathbf{y}_t = \mathbf{y}_t - \mathbf{y}_{t-1}$.
- Most of the multivariate time-series models attempt to model the log-returns \mathbf{r}_t (because the log-prices are nonstationary whereas the log-returns are weakly stationary, at least over some time horizon).
- However, it turns out that differencing the log-prices may destroy part of the structure.
- The VECM¹¹ tries to fix that issue by including an additional term in the model:

$$\mathbf{r}_t = \phi_0 + \boldsymbol{\Pi} \mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \tilde{\boldsymbol{\Phi}}_i \mathbf{r}_{t-i} + \mathbf{w}_t,$$

where the term $\boldsymbol{\Pi} \mathbf{y}_{t-1}$ is called error correction term.

¹¹R. F. Engle and C. W. J. Granger, "Co-integration and error correction: Representation, estimation, and testing," *Econometrica: Journal of the Econometric Society*, pp. 251–276, 1987.

- The matrix Π is of extreme importance.
- Notice that from the model $\mathbf{r}_t = \phi_0 + \Pi \mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \tilde{\Phi}_i \mathbf{r}_{t-i} + \mathbf{w}_t$ one can conclude that $\Pi \mathbf{y}_t$ must be stationary even though \mathbf{y}_t is not!!!
- If that happens, it is said that \mathbf{y}_t is cointegrated.
- There are three possibilities for Π :
 - $\text{rank}(\Pi) = 0$: This implies $\Pi = \mathbf{0}$, thus \mathbf{y}_t is not cointegrated (so no mystery here) and the VECM reduces to a VAR model on the log-returns.
 - $\text{rank}(\Pi) = N$: This implies Π is invertible and thus \mathbf{y}_t must be stationary already.
 - $0 < \text{rank}(\Pi) < N$: This is the interesting case and Π can be decomposed as $\Pi = \alpha \beta^T$ with $\alpha, \beta \in \mathbb{R}^{N \times r}$ with full column rank. This means that \mathbf{y}_t has r linearly independent cointegrated components, i.e., $\beta^T \mathbf{y}_t$, each of which can be used for pairs trading.

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Mean-reverting portfolio (MRP)

- In the case of two assets, the spread is $z_t = y_{1t} - \gamma y_{2t}$, which can be understood as a portfolio with weights: $\mathbf{w} = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}$.
- In the general case of many assets, one has to properly design the portfolio \mathbf{w} .
- One interesting approach is based on a VECM modeling of the universe of stocks:
 - From the parameter β contained in the low-rank matrix $\mathbf{\Pi} = \alpha\beta^T$ one can simply use any column of β (even all of them)
 - Even better, β defines a cointegration subspace and we can then optimize the portfolio within that cointegration subspace.^{12,13}

¹²Z. Zhao and D. P. Palomar, "Mean-reverting portfolio with budget constraint," *IEEE Trans. Signal Process.*, vol. 66, no. 9, pp. 2342–2357, 2018.

¹³Z. Zhao, R. Zhou, and D. P. Palomar, "Optimal mean-reverting portfolio with leverage constraint for statistical arbitrage in finance," *IEEE Trans. Signal Process.*, vol. 67, no. 7, pp. 1681–1695, 2019.

Mean-reverting portfolio (MRP)

- Consider the log-prices \mathbf{y}_t and use β to extract several spreads $\mathbf{s}_t = \beta^T \mathbf{y}_t$.
- Let's now use a portfolio \mathbf{w} to extract the best mean-reverting spread from \mathbf{s}_t as $z_t = \mathbf{w}^T \mathbf{s}_t$.
- To design the the portfolio \mathbf{w} we have two main objectives (recall that total profit equals: profit of each trade \times number of trades):
 - we want large variance (profit of each trade): $\mathbf{w}^T \mathbf{M}_0 \mathbf{w}$, where
$$\mathbf{M}_i = E \left[(\mathbf{s}_t - E[\mathbf{s}_t]) (\mathbf{s}_{t+i} - E[\mathbf{s}_{t+i}])^T \right]$$
 - we want strong mean reversion (number of trades): many proxies exist like the Portmanteau statistics or crossing statistics.

Mean-reverting portfolio (MRP)

- For example, if we use the Portmanteau statistics as a proxy for the mean reversion, the problem formulation becomes:

$$\begin{aligned} & \underset{\mathbf{w}}{\text{minimize}} && \sum_{i=1}^p \left(\frac{\mathbf{w}^T \mathbf{M}_i \mathbf{w}}{\mathbf{w}^T \mathbf{M}_0 \mathbf{w}} \right)^2 \\ & \text{subject to} && \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\ & && \mathbf{w} \in \mathcal{W}. \end{aligned}$$

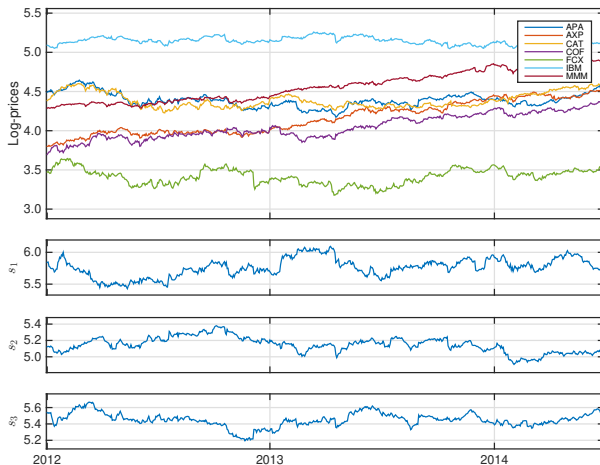
- Using other proxies, the formulation can be expressed more generally as¹⁴

$$\begin{aligned} & \underset{\mathbf{w}}{\text{minimize}} && \mathbf{w}^T \mathbf{H} \mathbf{w} + \lambda \sum_{i=1}^p \left(\mathbf{w}^T \mathbf{M}_i \mathbf{w} \right)^2 \\ & \text{subject to} && \mathbf{w}^T \mathbf{M}_0 \mathbf{w} = \nu \\ & && \mathbf{w} \in \mathcal{W}. \end{aligned}$$

¹⁴Z. Zhao and D. P. Palomar, "Mean-reverting portfolio with budget constraint," *IEEE Trans. Signal Process.*, vol. 66, no. 9, pp. 2342–2357, 2018.

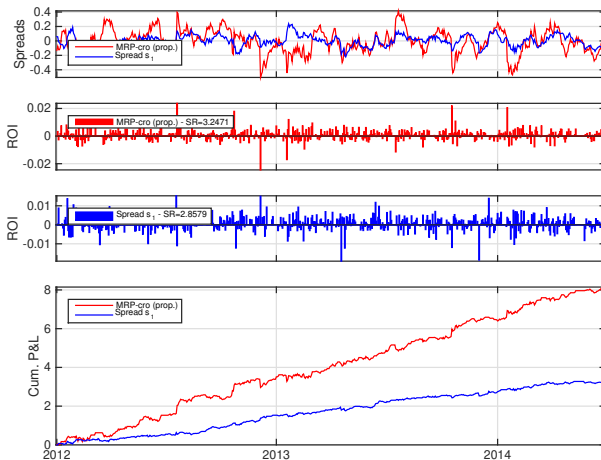
MRP in practice

- Observe several stock log-prices and the spreads obtained from β :



MRP in practice

- Observe several stock log-prices and the spreads obtained from β :



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Summary

- First of all, we have discovered the concept of cointegration.
- We have learned the basic idea of pairs trading for cointegrated assets:
 - searching for a cointegrated spread is the first step
 - making sure that the chosen spread remains cointegrated is key (cointegrated tests)
 - obtaining the cointegration ratio γ and the entering and exiting thresholds are important details.
- We have learned of the use of Kalman (initially developed for tracking missiles) filtering for improved pairs trading.
- We have briefly explored the extension of pairs trading (for two stocks) to statistical arbitrage (for more than two stocks):
 - VECM modeling is an important multivariate time-series modeling tool
 - sophisticated portfolio designs on the cointegration subspace are possible.

Thanks

For more information visit:

<https://www.danielppalomar.com>

