A Numerical Study of the Voter Model

MATH 374 Presentation

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Outline

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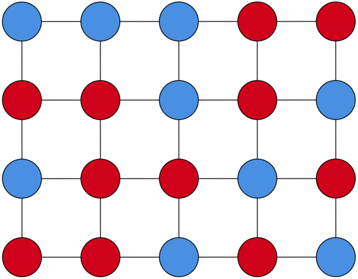
An Introduction to the Voter Model

A voter model is a directed graph of nodes which hold one of 2 possible opinions. This graph can be represented as many different ways topologically, however the main shape I look at is a square lattice graph, which holds "boundary" nodes, and each node may hold one of two opinions.



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The Classical Model, My Definition: Algorithm

The process of one observation of the classic voter model follows:

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4. The opinion of the neighbor will influence the individual to change to the neighbor's opinion. If the individual already holds the same opinion as the neighbor, no changes occur.
5. Repeat steps 2-4 until we reach a consensus. A consensus is defined as the model reaching a state where all of the particles have the same opinion. This will be labeled as the absorbing state.

The Classical Model, My Definition

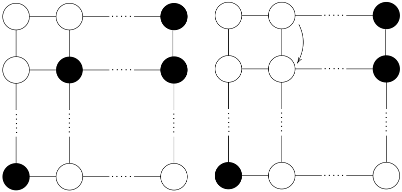


Figure: Example of a step in the classic voter model.

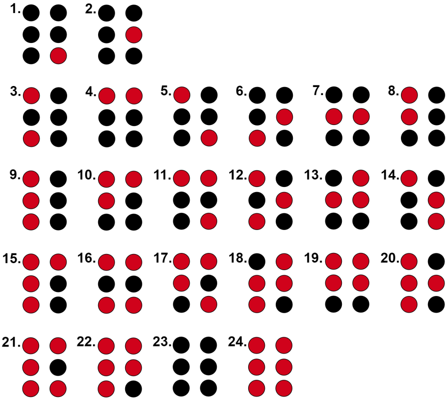
The Classical Model, My Definition: Results

In addition to studying the empirical results, I manually calculated the expected time to absorption of a 3x2 classic voter model. The purpose of this calculation is to confirm the validity of the empirical results the algorithm produces and to better understand where the empirical results are coming from.

Theorem ([Dob16])

For an absorbing Markov chain with all states either transient or absorbing, let $F = (I - Q)^{-1}$. The expected number of steps from transient state i util the chain is absorbed in some absorbing state is $(F1)_i$.

The Classical Model, My Definition: Results



A 3x2 model yielded a total of 24 possible states (accounting for symmetry) and a 24x24 transition matrix. Using 1, the expected time to consensus of a 3x2 classic voter model is 18.31765 steps. Setting the seed to 1, the simulation generates an average of 18.1296 steps to absorption with 10,000 observations.

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My Model: Introduction

What if I created an alternative model, using the all same properties as my definition of the classical model, but the probability of an individual being convinced is dependent on the proportion of individuals which share its opinion? That is, if there were 3 blue particles, and 7 red particles, the probability that a blue particle is willing to be convinced is 3/10.

My Model: Results

Using the properties I have explained, I derived another 24x24 transition matrix of the 3x2 model, and applied the theorem on expected time to absorption 1 to find the expected time to be 160.3006 Setting the seed to 1, the simulation generates and empirical result of 159.7298 with 10,000 observations.

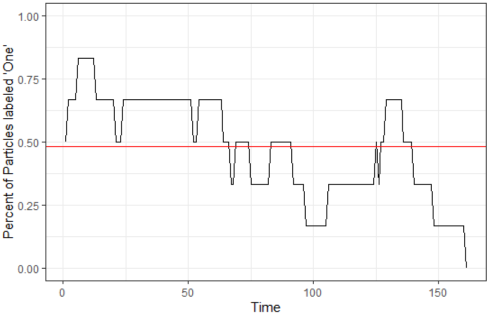


Figure: 3x2 Voter Model with 2 possible states.

My Model: Results

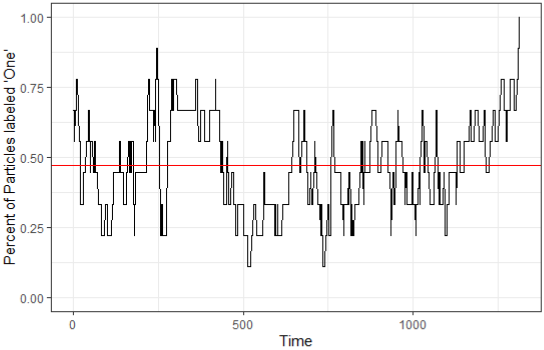


Figure: 3x3 Voter Model with 2 possible states.

Connections to Real Life

I've found that this model parallels the concept of a "loud minority" holding power over a majority, such as the minority party in the Senate using a filibuster to prevent a vote on a resolution. Based on historical context, it is difficult to get senators to vote against their party to end a filibuster. This is because as the party's opinion becomes more "unpopular" within the chamber (i.e. the closer the majority is to obtaining enough votes to end the filibuster), the more difficult it is to get a supporting member to turn against their cause.

Limitations

We see that it is in fact possible to reach a consensus under these conditions, however as we increase the size of population it becomes beyond realistic. In the classic voter model, we found the average time to absorption to be approximately 18 steps, but with this new model, it takes about 9 times more steps to reach a consensus.

Using Sys.time, I found that the average time to run one observation of a 4x4 classic model is 0.004 seconds whereas the modified model had an average time of 5.078 seconds per observation. This dramatic change shows us that as we increase the size of the population, while reaching a consensus is possible, the discrepancy between the time to consensus of the classic model and my modified model grows. We will see more extreme examples of this later.

The First Steps Towards Finding a Formula of $E[T_M]$

Size of Model	T_C	T_M	Observations (Modified)
2x4	42.4	795	5,000
2x5	78.4233	3,681.06	5,000
2x6	133.4	16,897.84	300
3x2	18.1	160	10,000
3x3	51	1,263.61	1,000
3x4	106.1	9,666.416	500
3x5	186.56	83,756.84	300
3x6	302.4223	722,845.9	500
3x7	454.0221	6,100,133	750
4x4	206.976	160,617	1,000
4x5	364.2978	2,272,649	1,000

The First Steps Towards Finding a Formula of $E[T_M]$

Conjecture

Let I be the number of "interior" individuals in an nxp model, D be the dimensions of the matrix, T_M be the time to absorption of the modified model, and T_C be the time to absorption of the classic model. Then the expected time to absorption of a model with a minimum dimension of 2 is

$$\mathbb{E}[T_M] = \frac{\Gamma(I+1)}{2}\mathbb{E}[T_C]^2 \tag{1}$$

and the expected time to absorption of a model with a minimum dimension of 3 is

$$\mathbb{E}[T_M] = \frac{\Gamma(I+1)}{2}\mathbb{E}[T_C]^2 - \mathbb{E}[T_C](\max(D)^I) \tag{2}$$

The First Steps Towards Finding a Formula of $E[T_M]$

Size	T_M	Eq.1 Residuals	Eq.2 Residuals
2x4	795	-103.88	-61.48
2x5	3,681.06	607.78	686.18
2x6	16,897.84	-897.76	-764.32
3x2	1,160	-3.805	14.295
3x3	1,263.61	-36.89	14.11
3x4	9,666.416	-1,569.584	126.416
3x5	83,756.84	-20,701.86	2,618.16
3x6	722,845.9	-374,648.1	16,743.6
3x7	6,100,133	-11,647,358.3	-4,016,980.3
4x4	160,617	-353,571	-300,579
4x5	2,272,649	-44,513,041	-38,880,229

The First Steps Towards Finding a Formula of $E[T_M]$

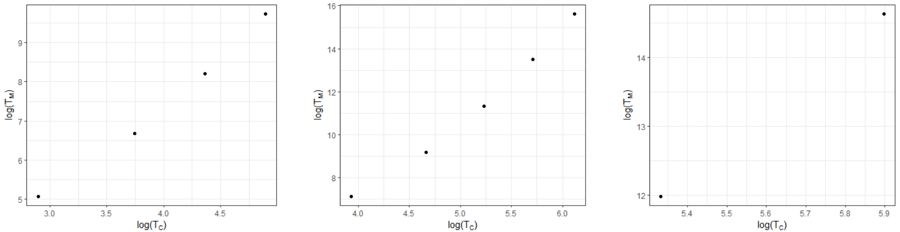
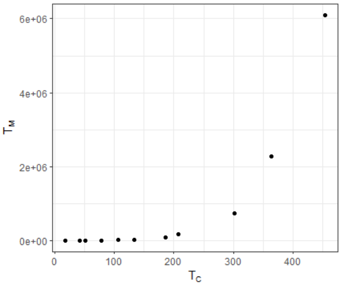


Figure: (a) 2 rows/columns, (b) 3 rows/columns, and (c) 4 rows/columns.

While these formulas suit their respective models well, there is no real basis to support them. It looks like the formulas will continue to vary as the minimum dimension increases. The data was split by the minimum dimension and each set was plotted on a log-log graph. The slopes for 2, 3, and 4 were, respectively, 2.322, 3.870, and 4.225. This leads me to believe that the exponents may be equal to the minimum dimension, or even the minimum dimension plus 1/2.

The First Steps Towards Finding a Formula of $E[T_M]$



The result of log-log plot (c) remain unreliable since there are only two points to base the slope of the log-log plot off of. It is difficult to increase the number of observations for this modified model or to find data points for larger populations due to the length of time it takes to run the simulations. For a simulation of a 4x6 model, it took over 3 days to run 250 observations, and the results of this simulation returned a value which was far off from the pattern shown in the plot of T_C v.s. T_M, and had a very high variance.

Outline

The Confidence Voter Model

1. Initialize our model of N particles, by giving each particle a 50/50 probability of being initialized as opinion 1 or opinion 2, and a 50/50 probability of being initialized as confident or unsure. As Redner outlines, in the extremal case, the proportion of unsure to confident will be close to 1-1, but not exactly symmetric, otherwise the extremal model will never reach a consensus.
2. A particle, the "individual", is uniformly selected. We will name this the "individual".
3. A particle adjacent from the individual is uniformly selected. We will name this particle the "neighbor".

The Confidence Voter Model

4. **MARGINAL**: If an unsure individual of a given opinion interacts with an agreeing neighbor, then that individual becomes confident in it's current opinion. If the unsure individual interacts with a disagreeing neighbor, then that individual switches opinions yet remains unsure. If a confident individual of a given opinion interacts with an agreeing neighbor, nothing happens. If the confident individual interacts with a disagreeing neighbor, then the individual keeps its current opinion, but becomes unsure.
   EXTREMAL: Follows a similar process to marginal except if an unsure individual of a given opinion interacts with a disagreeing neighbor, then that individual switches opinions and immediately becomes confident, rather than remaining unsure. In this case the individual has "seen the light" [VR12].
4. Repeat steps 1-4 until a consensus is reached (the individuals do not necessarily have to be all confident or all unsure).

The Confidence Voter Model

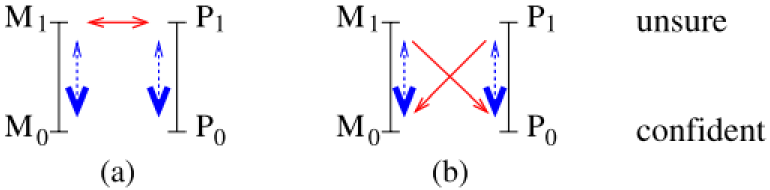


Figure: Demonstration of how the (a) marginal and (b) extremal confidence voter models function [VR12].

Modifying the Confidence Voter Model: Version 1

1. Initialize our model of N particles, by giving each particle a 50/50 probability of being initialized as opinion 1 or opinion 2, and a 50/50 probability of being initialized as confident or unsure.
2. A particle is uniformly selected. We will name this the "individual".
3. A particle adjacent from the individual is uniformly selected. We will name this particle the "neighbor".
4. If the neighbor selected is labeled unsure, we skip step 5. Otherwise, we move on as usual.
5. The opinion of the neighbor will influence the individual to change opinion, and/or its level of confidence. The extremal and marginal models follow the same as described previously.
6. Repeat steps 1-5 until a consensus is reached (the individuals do not necessarily have to be all confident or all unsure).

Modifying the Confidence Voter Model: Version 1

The purpose of adding step four is to account for the idea that an unsure individual cannot typically make an argument which is compelling enough for an individual, especially a confident one, to have a change of heart. Picture having a political discussion with a centrist. Often times they say they sit on the fence, not leaning towards any extreme: “Both sides make good arguments”. In political discussion, it is often difficult to convince one to agree with their opinion if they also relate to the opposite end.

Modifying the Confidence Voter Model: Version 2

Version 2 of my model follows the same process as what is stated under version 1, however there is an additional detail added. From here, we bring in the same characteristic as what is shown in the individual heterogeneity model [MGR10] and implement it into the version 1 model. That is, each particle has its own probability of being convinced by a neighbor, conditioned on that neighbor being confident. This will be made a random variable which is re-sampled throughout the simulation, so in each observation a new set of probabilities are generated.

Numerical Results

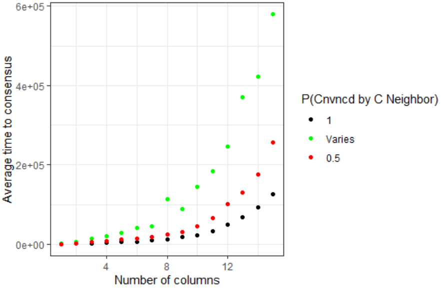


Figure: Marginal

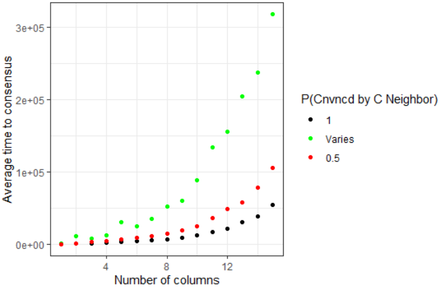


Figure: Extremal

Each version fixes the number of rows to 8, and runs 1000 observations for each dot, except for the data points associated with a varying probability, which were found with 2500 observations. This model is much more prone to a high variance.

Comparing Each Version

A possible question we had was is there really a difference in the time to consensus between the version-2 model and a model which has a fixed probability of being convinced at 0.5?

Marginal		
$P(convinced)$ (a)	$P(convinced)$ (b)	P-Value
1	Varies	0.0188
1	0.5	0.19
Varies	0.5	0.07

Extremal		
$P(convinced)$ (a)	$P(convinced)$ (b)	P-Value
1	Varies	0.008
1	0.5	0.13
Varies	0.5	0.03

Outline

Remaining Questions

Seeing that this is a one-semester project, there was only so much time I've been able to dedicate to this project, and lack of access to a high performance computing limited me to the results my laptop was able to generate. Still, there are a number of questions left I am anxious to answer.

- Conjecture to my model;
- "Push" opinions, rather than "pull" every time;
- Other quantities: the number of interactions that each individual has until a consensus is reached; and
- The modified confidence model: linear v.s. non-linear portions of the plots.

References

Robert Dobrow, *Introduction to stochastic processes with r*, John Wiley & Sons, Inc., 2016.

Naoki Masuda, N. Gibert, and S. Redner, *Heterogeneous voter models*, Phys. Rev. E **82** (2010).

D. Volovik and S. Redner, *Dynamics of confident voting*, Journal of Statistical Mechanics: Theory and Experiment **2012** (2012), no. 04.

Thank you!