A Numerical Study of the Voter Model

Lillian Makhoul

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Abstract

A voter model is a directed graph of nodes which hold one of two possible opinions. This graph can be represented as many different ways topologically, however the main shape I look at in this report is a square lattice graph, which holds "boundary" nodes, and each node may hold one of two opinions.

In this report, I will give an introductory discussed on some previously studied models, put forward a possibly new model and how it mirrors real life situations, as well as give a brief overview of some other literature that was reviewed in the development of this paper.

1 Introduction

The goal of this report is to study alternative versions of the model discussed in Sidney Redner's review [7] through numerical results and from there adjust these models based on questions brought up during the process of developing the simulations. I will also note several smaller ideas that were brainstormed throughout this project, but didn't receive the opportunity to be expanded on.

While this project is heavily focused on an elementary stochastic process, it mainly highlights results found through numerical analysis.

Assumptions

Throughout the report, these assumptions will hold unless stated otherwise:

1. **Possible interactions**: The "neighbor" can only be to the left, right, above, or below the selected particle, no diagonal interactions are accounted for.
2. **Boundaries**: In some versions of the voter model, interactions are formed so that the edge and corner particles have the same amount of possible interactions as the middle particles. That is, these outer particles are connected to the other side. If a particle in the leftmost column were selected, it can interact with its neighbors above, below, and to the right, but can also interact with the rightmost particle (in the respective row), given a "left" interaction were selected. The analogous statements hold for other locations on the perimeter of a "neighborhood". For this report, we will always assume **there are boundaries**, meaning these reach-across interactions do not occur.

Structure of Paper

This report will begin with section 2 giving an empirical study on the classic voter model presented by Redner. Following this mini-study will be the bulk of the paper: section 3, a proposal and numerical analysis of my alternative take on the voter model. Section 4 will discuss the confidence voter model as well as a few modifications I made to it. I find that while my results are not extensive, they are still worth noting. Section 5 will give a brief discussion on the process of generating the results, and section 6 will discuss the unanswered questions that have remained throughout the process of generating this report as well as questions that have sprung up towards the end of the project.

2 The Classic Voter Model

Before discussing the alternative models, we will discuss the functionality of a classic voter model.

The process of one observation of the classic voter model follows:

1. We begin by initializing our model of N particles, by giving each particle a 50/50 probability of being initialized as opinion 1 or opinion 2.
2. A particle is uniformly selected. We will name this the "individual".
3. A particle adjacent from the individual is uniformly selected. We will name this secondary particle the "neighbor".
4. The opinion of the neighbor will influence the individual to change to the neighbor's opinion. If the individual already holds the same opinion as the neighbor, no changes occur [1].
5. Repeat steps 2-4 until we reach a consensus. A consensus is defined as the model reaching a state where all of the particles have the same opinion. This will be labeled as the absorbing state.

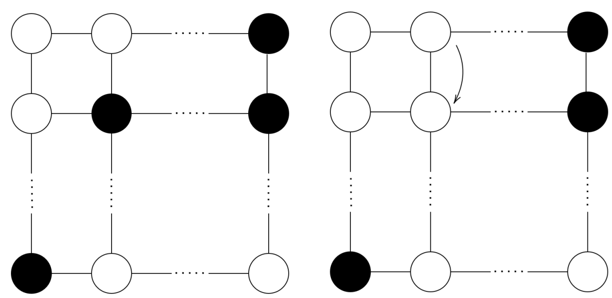


Figure 1: Example of a step in the classic voter model. The former black dot was the uniformly selected individual, and the white dot that influenced the black dot was the uniformly selected neighbor.

To visualize the results of the average time to absorption, one may consider the amount of particles to be the independent variable, however an important consideration in my definition is the shape of the model. While a 3x4 model and a 6x2 model both have 12 individuals in their population, and since boundaries are taken into consideration, the proportion of "boundary" individuals to "interior" individuals causes a variation in the average time to absorption. To plot the average time to absorption, it will be modeled as a function over the number of columns, and keeping the number of rows fixed 2.

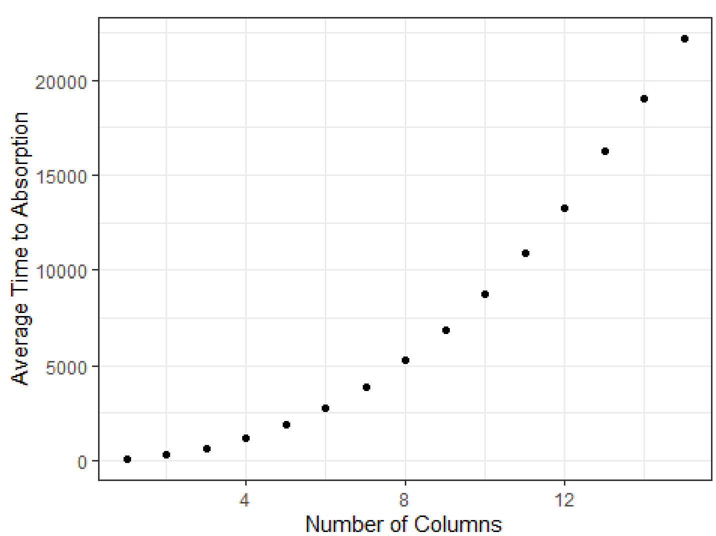


Figure 2: The average time to absorption plotted as a function of the number of columns when we fix the number of rows to 8. Each point was found with 10,000 observations, 8 rows

In addition to studying the empirical results, I manually calculated the expected time to absorption of a 3x2 classic voter model 3. The purpose of this calculation is to confirm the validity of the empirical results the algorithm produces and to better understand where the empirical results are coming from.

Theorem ([3])**.** *For an absorbing Markov chain with all states either transient or absorbing, let $F = (I - Q)^{-1}$. The expected number of steps from transient state i util the chain is absorbed in some absorbing state is $(F1)_i$.*

Firstly, I put together the transition matrix and extracted the Q-matrix:

$$Q = \begin{bmatrix}
\frac{25}{36} & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{11}{18} & 0 & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{7}{18} & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & \frac{13}{18} & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{7}{18} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & \frac{11}{36} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{9} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{9} & 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{9} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{7}{12} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{18} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{11}{18} & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{5}{12} & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{12} & \frac{1}{18} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{12} & \frac{1}{9} & 0 & 0 & 0 & 0 & \frac{5}{12} & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{12} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{5}{18} & \frac{1}{6} & 0 & \frac{1}{12} & \frac{1}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{18} & 0 & 0 & 0 & \frac{7}{12} & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{1}{9} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 & 0 & 0 & \frac{2}{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{9} & 0 & 0 & \frac{1}{12} & 0 & 0 & \frac{11}{36} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{9} & 0 & 0 & 0 & \frac{7}{18} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{18} & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{18} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{18} & 0 & 0 & 0 & 0 & \frac{11}{18} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & \frac{25}{36}
\end{bmatrix}$$

and calculated the probability that we begin at states 1 through 22:

$$P(X_0 = i) = \begin{bmatrix} \frac{1}{16} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{16} & \frac{1}{64} & \frac{1}{16} & \frac{1}{32} & \frac{1}{16} & \frac{1}{16} & \frac{1}{32} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{64} & \frac{1}{16} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{16} \end{bmatrix}$$

To see how each state is labeled, see the Appendix. The expected time to absorption of a 3x2 classic voter model is 18.31765 steps 2. Setting the seed to 1, the simulation generates an average of 18.1296 steps to absorption with 10,000 observations.

While I have not come across a formula for this specific graph, a group of mathematicians [2] proposed an order for a formula of any connected graph.

$$O\left(\frac{n}{\nu(1-\lambda)}\right) \tag{1}$$

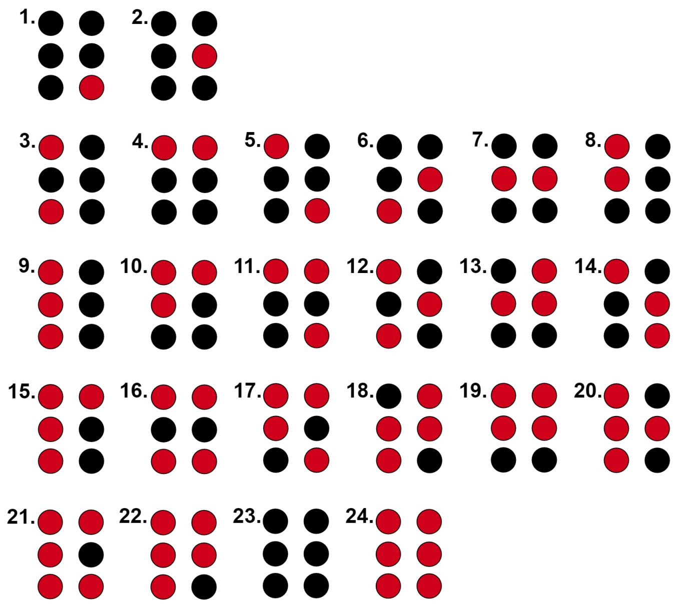


Figure 3: The possible states that a 3x2 voter model can take, ordered for symmetry.

where λ is the second largest eigenvalue of the absolute value of the transition matrix of a random walk on a general connected graph, and ν is a value dependent on m degrees in the graph. To calculate nu, each row will be multiplied by it's respective common denominator (so the transition matrix becomes a matrix of zeros and ones). We find ν to be equal to the sum of the squared row sums divided by d^2N where $d^2N = 2m^2/N$. Using this formula, I found the order for a 3x2 graph to be $O(6/(51/46)(0.5)) \approx O(10.83)$, which is not far off from our expected time.

Redner's Definition

Redner proves an already-existing formula for the time to absorption of the classic voter model. Let ρ represent the density of individuals with a given opinion, and $T(\rho)$ be the time to absorption when beginning an observation with any ρ. Then,

$$T(\rho) = -N[(1-\rho)\ln(1-\rho) + \rho \ln \rho] \tag{2}$$

where $T \sim N^2$ for $d = 1$ dimension, $T \sim N \ln N$ for $d = 2$, and $T \sim N$ for $d = 3$.

I used the following algorithm to match Redner's definition of the classic voter model:

Let N be the size of the popualtion and i be the number of individuals of a given opinion.

1. We begin by initializing our model of N particles, by giving each particle an i/N probability of being initialized as opinion 1 or opinion 2.
2. A particle is uniformly selected. We will name this the "individual".
3. A second particle is uniformly selected, but does not necessarily have to be adjacent from the particle, i.e. we will be looking at a complete graph. We will name this particle the "neighbor".
4. The opinion of the neighbor will influence the individual to change into the neighbor's opinion. If the individual already holds the same opinion as the neighbor, no changes occur.
5. Repeat steps 2-4 until we reach a consensus.
6. Repeat steps 1-5 for $i = 0, 1, \ldots N$

I chose to include Redner's definition in my report to show how the voter model can hold an alternative structure to the square lattice graph, and how it affects our time to consensus. It is difficult to compare the time to consensus of the complete graph to the square lattice graph, because the complete graph can only hold one shape for each number of individuals, unlike for the square lattice graph. Still, I used 1 again to compare the order of a complete graph with 6 nodes, to the order from the 3x2 square lattice graph. I found the order to be O(7.5) and the empirical average returned was 15.5. We can hypothesize that the number of degrees within the system is inversely related to the expected and empirical time to consensus.

3 My Model

While developing simulations for this study, I was thinking back to a previous project on the coupon collector problem [5] and how we managed to create a version in which absorption was never reached; I wondered how I might be able to implement that in the voter model. What if I created an alternative model, using the all same properties as my definition of the classical model, but the probability of an individual being convinced is dependent on the proportion of individuals which share its opinion? That is, if there were 3 blue particles, and 7 red particles, the probability that a blue particle is willing to be convinced is 3/10. One notices immediately that as the model comes closer to consensus it becomes more difficult to reach an absorbing state, but it turns out is not impossible. After all, one of the basic properties of a voter model is it eventually reaches a consensus.

$$
Q = \begin{bmatrix}
\frac{185}{216} & 0 & 0 & \frac{5}{72} & 0 & 0 & 0 & \frac{5}{108} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{85}{108} & 0 & 0 & 0 & 0 & \frac{5}{108} & \frac{5}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{9} & 0 & \frac{19}{27} & 0 & 0 & 0 & 0 & 0 & \frac{4}{54} & 0 & \frac{4}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{18} & 0 & 0 & \frac{47}{54} & 0 & 0 & 0 & 0 & 0 & \frac{2}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{9} & 0 & 0 & 0 & \frac{19}{27} & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & \frac{2}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{18} & \frac{1}{18} & 0 & 0 & 0 & \frac{35}{54} & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & \frac{2}{27} & \frac{1}{18} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{27} & 0 & 0 & 0 & 0 & \frac{19}{27} & 0 & 0 & 0 & 0 & 0 & \frac{2}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{27} & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 & \frac{85}{108} & \frac{1}{18} & \frac{1}{18} & 0 & 0 & \frac{1}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{36} & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{7}{9} & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & \frac{1}{24} & 0 & \frac{29}{36} & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{24} & \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & \frac{17}{24} & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{24} & \frac{1}{36} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{1}{24} & \frac{1}{18} & 0 & 0 & 0 & 0 & \frac{17}{24} & 0 & 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} & \frac{1}{24} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{24} & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{23}{36} & \frac{1}{12} & 0 & \frac{1}{24} & \frac{1}{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{18} & \frac{1}{27} & 0 & 0 & 0 & \frac{85}{108} & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{1}{27} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{9} & 0 & 0 & 0 & 0 & \frac{19}{27} & 0 & 0 & 0 & 0 & \frac{2}{27} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{2}{27} & 0 & 0 & \frac{1}{18} & 0 & 0 & \frac{35}{54} & 0 & 0 & 0 & \frac{1}{18} & \frac{1}{18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{2}{27} & 0 & 0 & 0 & \frac{19}{27} & 0 & 0 & 0 & \frac{1}{9} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{47}{54} & 0 & 0 & \frac{1}{18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{27} & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{27} & 0 & \frac{1}{9} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{36} & \frac{5}{108} & 0 & 0 & 0 & 0 & \frac{85}{108} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{108} & 0 & 0 & 0 & \frac{5}{72} & 0 & 0 & \frac{185}{216}
\end{bmatrix}
$$

The matrix labels each state the same way as the transition matrix of the classical model. Using the previously stated theorem and probabilities vector, the expected time to absorption of a 3x2 model is 160.3006.

Setting the seed to 1, the simulation generates and empirical result of 159.7298 with 10,000 observations. From 4 and 5 we see that it is in fact possible to reach a consensus under these conditions, however as I

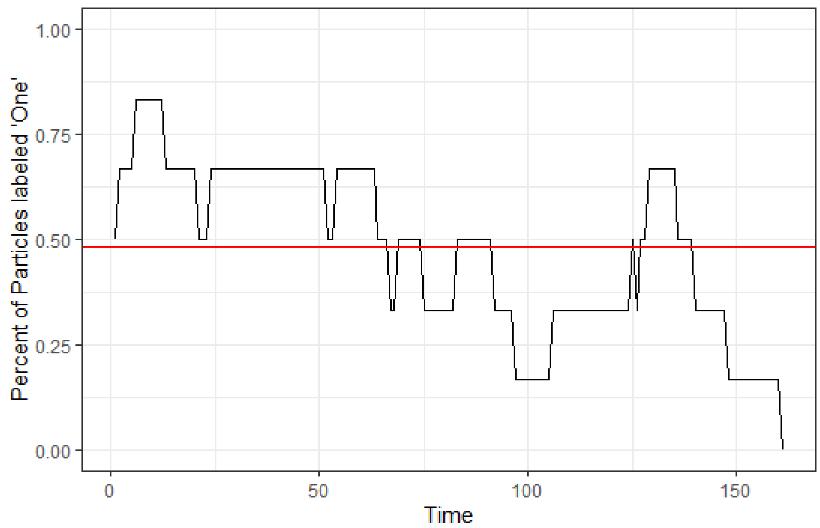


Figure 4: 3x2 voter model with 2 possible states.

increase the size of population it becomes beyond realistic. In the classic voter model, I found the average time to absorption to be approximately 18 steps, but with this new model, it takes about 9 times more steps to reach a consensus.

Using Sys.time, I found that the average time to run one observation of a 4x4 classic model is 0.004 seconds whereas the modified model had an average time of 5.078 seconds per observation. To emphasize the extremety of this discrepency, say I would like to run 1,000 observations of the 4x4 graph to obtain its average time to consensus. The classical model would talk about 4 seconds to return the result, but the modified model will take 5000 seconds, or a little less than an hour and a half, to return a result.

This dramatic change shows us that as we increase the size of the population, while reaching a consensus is possible, it is not plausible. For example, if a person were to canvas until every household in a city agrees on a subject, it may be possible, but considering the large population, it will take an incredible amount of time, and may even surpass the lifespan of the canvasser.

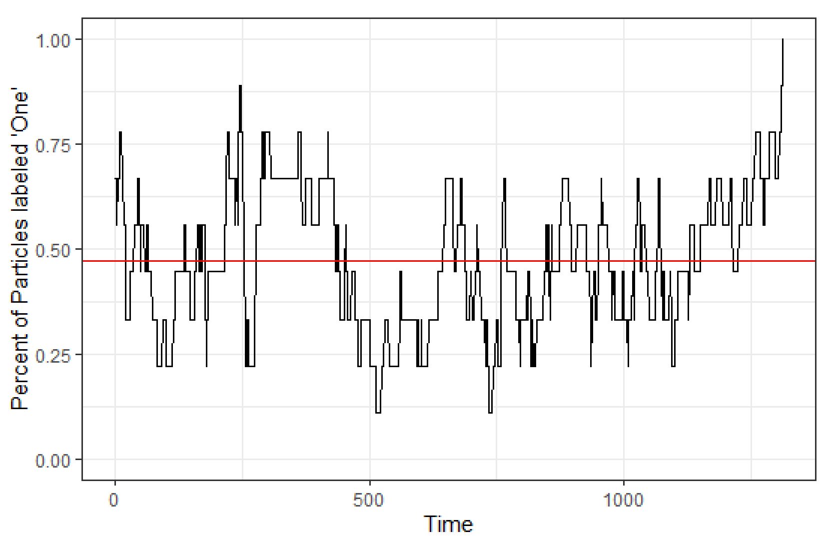


Figure 5: 3x3 modified voter model with 2 possible states.

Real-life Examples

We can compare the concept of this model to a smaller scale: a jury. When a unanimous vote is needed, most of the time is spent convincing a stubborn juror to vote with the majority. It is this interaction that causes a delay towards reaching a verdict.

An additional example would be on the Senate filibuster. With this modified model, we can see a parallel to a "loud minority". This group can hold power over the majority. Many resolutions in the Senate are passed with a simple majority, that is more than half of the sitting members vote to pass the resolution, but any singular Senator can launch a filibuster. Three-fifths of the sitting members must vote to end a filibuster. If the majority does not have enough votes, then the minority party continues to hold power over the majority for as long as the filibuster lasts, or until the majority party manages to flip enough votes. Based on historical context, it is difficult to get senators to vote against their party to end a filibuster. This is because as the party's opinion becomes more "unpopular" within the chamber (i.e. the closer the majority is to obtaining enough votes to end the filibuster), the more difficult it is to get a supporting member to turn against their cause. It has gone so far to where Ted Cruz passed time in his filibuster by reading Dr. Suess's Green Eggs and Ham, and yet his filibuster was still not put to an end.

The First Steps Towards Finding a Formula of $E[T_M]$

Size of Model	T_C	T_M	Observations (Modified)
2x4	42.4	795	5,000
2x5	78.4233	3,681.06	5,000
2x6	133.4	16,897.84	300
3x2	18.1	160	10,000
3x3	51	1,263.61	1,000
3x4	106.1	9,666.416	500
3x5	186.56	83,756.84	300
3x6	302.4223	722,845.9	500
3x7	454.0221	6,100,133	750
4x4	206.976	160,617	1,000
4x5	364.2978	2,272,649	1,000

Before many data points were drawn, a few guesses were made about the the formula of the time to consensus of the modified model.

Conjecture. *Let I be the number of "interior" individuals in an nxp model, D be the dimensions of the matrix, T_M be the time to absorption of the modified model, and T_C be the time to absorption of the classic model. Then the expected time to absorption of a model with a minimum dimension of 2 is*

$$\mathbb{E}[T_M] = \frac{\Gamma(I+1)}{2}\mathbb{E}[T_C]^2 \tag{3}$$

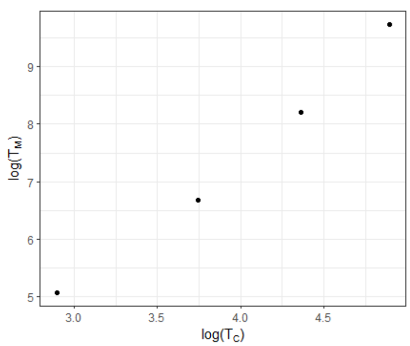
and the expected time to absorption of a model with a minimum dimension of 3 is

$$\mathbb{E}[T_M] = \frac{\Gamma(I+1)}{2}\mathbb{E}[T_C]^2 - \mathbb{E}[T_C](\max(D)^I) \tag{4}$$

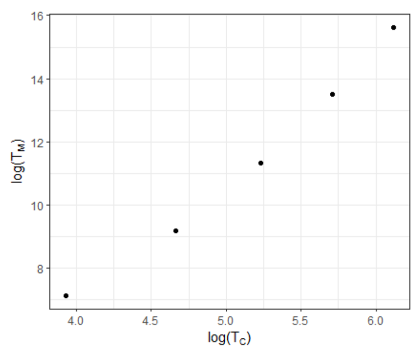
Size of Model	T_M	Eq.1	Eq.1 Residuals	Eq.2	Eq.2 Residuals
2x4	795	898.88	-103.88	856.48	-61.48
2x5	3,681.06	3,073.28	607.78	2,994.88	686.18
2x6	16,897.84	17,795.6	-897.76	17,662.16	-764.32
3x2	1,160	163.805	-3.805	145.705	14.295
3x3	1,263.61	1,300.5	-36.89	1,249.5	14.11
3x4	9,666.416	11,236	-1,569.584	9,540	126.416
3x5	83,756.84	104,458.7	-20,701.86	81,138.68	2,618.16
3x6	722,845.9	1,097,494	-374,648.1	706,102.3	16,743.6
3x7	6,100,133	12,366,960	-11,647,358.3	4,736,582	-4,016,980.3
4x4	160,617	514,188	-353,571	461,196	-300,579
4x5	2,272,649	46,785,690	-44,513,041	41,152,878	-38,880,229

While these formulas suit their respective models well, there is no real basis for supporting them. It is also likely that the formulas will continue to vary as the minimum dimension increases. As seen in the table, the residuals for both equations 3 and 4 of the models with a minimum dimension of 4 is far off.

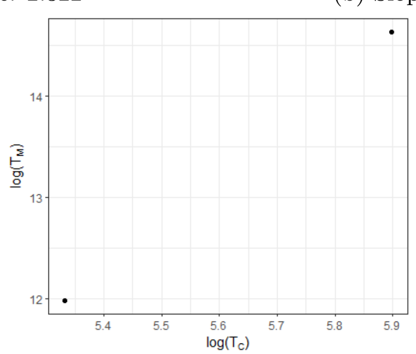
The data was split by the minimum dimension and each set was plotted on a log-log graph 6. The slopes produced for minimum dimensions of 2,3, and 4 were, respectively, 2.322, 3.870, and 4.225. This leads me to believe that either the orders may be equal to the minimum dimension, or more likely the minimum dimension plus 1/2.



(a) Slope: 2.322



(b) Slope: 3.870



(c) Slope: 4.775

Figure 6: Log-log plots of the time to consensus of models with a minimum dimension of (a) 2 rows/columns, (b) 3 rows/columns, and (c) 4 rows/columns.

The result of 6c remains unreliable since there are only two points to base the slope of the log-log plot off of. It is difficult to increase the number of observations for this modified model or to find data points for larger populations due to the length of time it takes to run the simulations. For a simulation of a 4x6 model, it took over 3 days to run 250 observations, and the results of this simulation returned a value which was far off from the path followed in 7, and had a very high variance.

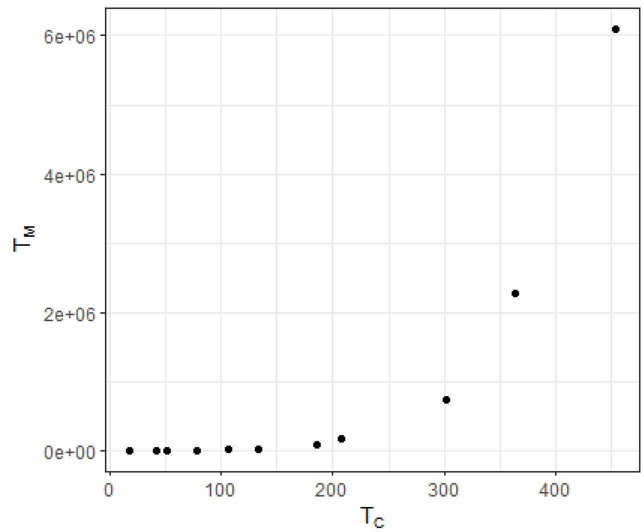


Figure 7: The time to consensus of each size as a classical model plotted against the time to consensus of it's corresponding modified model.

4 Additional Work

The Confidence Voter Model

There are two different versions of the confidence voter model— marginal and extremal. We define a confident individual as an individual who will need to have a minimum of two interactions with a disagreeing neighbor to switch states, and an unsure individual who will need to have only one interaction with a disagreeing neighbor to switch states 8. The process of the confidence voter model runs as follows:

1. Initialize our model of N particles, by giving each particle a 50/50 probability of being initialized as opinion 1 or opinion 2, and a 50/50 probability of being initialized as confident or unsure. As Redner outlines, in the extremal case, the proportion of unsure to confident will be close to 1-1, but not exactly symmetric, otherwise the extremal model will never reach a consensus.
2. A particle is uniformly selected. We will name this the "individual".
3. A particle adjacent from the individual is uniformly selected. We will name this particle the "neighbor".
4. The opinion of the neighbor will influence the individual to change opinion, and/or its level of confidence 8.

   MARGINAL: If an unsure individual of a given opinion interacts with an agreeing neighbor, then that individual becomes confident in it's current opinion. If the unsure individual interacts with a disagreeing neighbor, then that individual switches opinions yet remains unsure. If a confident individual of a given opinion interacts with an agreeing neighbor, nothing happens. If the confident individual interacts with a disagreeing neighbor, then the individual keeps its current opinion, but becomes unsure.

   EXTREMAL: Follows a similar process to marginal except if an unsure individual of a given opinion interacts with a disagreeing neighbor, then that individual switches opinions and immediately becomes confident, rather than remaining unsure. In this case the individual has "seen the light" [9].
5. Repeat steps 1-4 until a consensus is reached (the individuals do not necessarily have to be all confident or all unsure).

Applications

Damon Centola [1] executed the confidence voter model in an intriguing experiment. Controlling for as many means of bias as possible, Centola created an online environment, where each time an individual's online connections adopted a behavior, they would receive an automated message that encouraged them to join

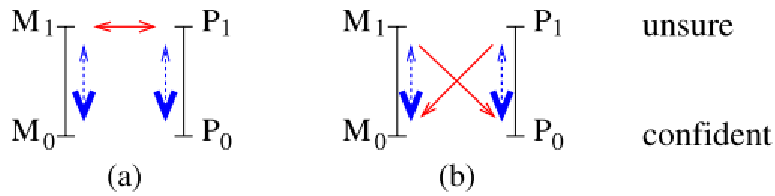


Figure 8: Demonstration of how the (a) marginal and (b) extremal confidence voter models function [9].

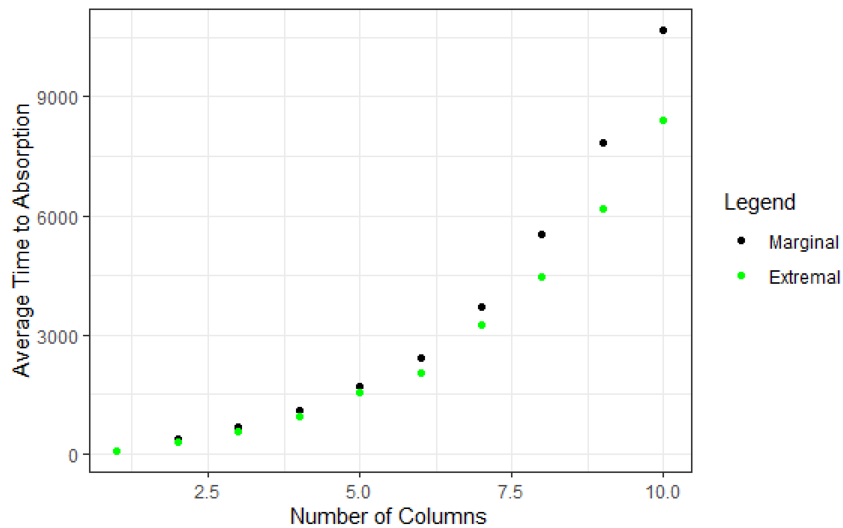


Figure 9: Comparing the times to consensus of the extremal and marginal models. The models are fixed to 6 rows, and each data point was generated with 1000 observations.

as well. The more connections that joined, the more messages the individual would receive. The results found by this experiment reinforced the main idea of a confident/stubborn voter model: the more times an individual is presented with media which reinforces a given behavior, the more likely they are to adopt that behavior.

While the numerical study I did on the confidence voter model was minimal, it is still important to introduce its functionality to understand where my modifications are brought in.

Modifying the Confidence Voter Model

Version 1

The process of version 1 follows:

1. Initialize our model of N particles, by giving each particle a 50/50 probability of being initialized as opinion 1 or opinion 2, and a 50/50 probability of being initialized as confident or unsure.

2. A particle is uniformly selected. We will name this the "individual".

3. A particle adjacent from the individual is uniformly selected. We will name this particle the "neighbor".

4. If the neighbor selected is labeled unsure, we skip step 5. Otherwise, we move on as usual.

5. The opinion of the neighbor will influence the individual to change opinion, and/or its level of confidence.

   MARGINAL: If an unsure individual of a given opinion interacts with an agreeing neighbor, then that individual becomes confident in it's current opinion. If the unsure individual interacts with a disagreeing neighbor, then that individual switches opinions and remains unsure. If a confident individual of a given opinion interacts with an agreeing neighbor, nothing happens. If the confident individual interacts with a disagreeing neighbor, then the individual keeps its current opinion, but becomes unsure.

EXTREMAL: Follows a similar process to marginal except if an unsure individual of a given opinion interacts with a disagreeing neighbor, then that individual switches opinions and immedietly becomes confident, rather than remaining unsure.

6. Repeat steps 1-5 until a consensus is reached (the individuals do not necessarily have to be all confident or all unsure).

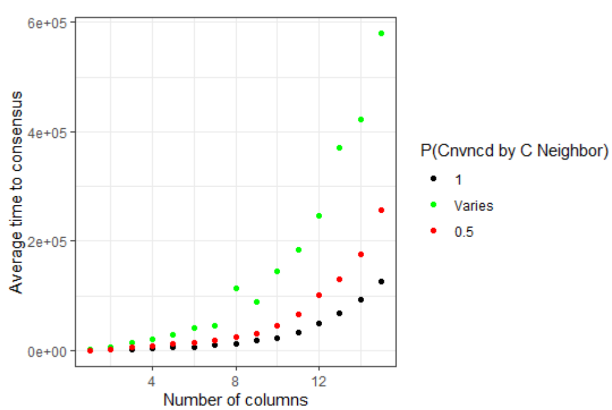
The purpose of adding step four is to account for the idea that an unsure individual cannot typically make an argument which is compelling enough for an individual, especially a confident one, to have a change of heart. Picture having a political discussion with a centrist. Often times they say they sit on the fence, not leaning towards any extreme: “Both sides make good arguments”. In political discussion, it is often difficult to convince one to agree with their opinion if they also relate to the opposite end.

Version 2

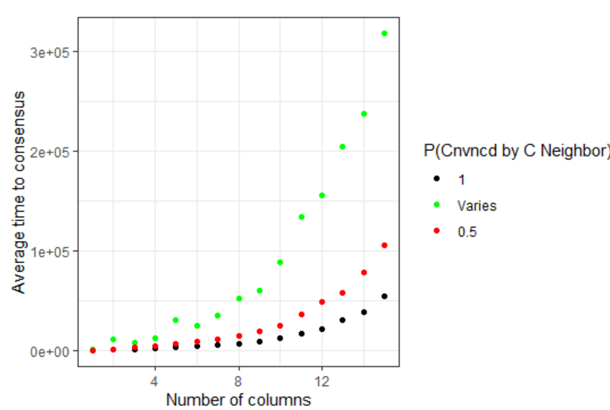
Version 2 of my model follows the same process as what is stated under version 1, however there is an additional detail added. From here, we bring in the same characteristic as what is shown in the individual heterogeneity model [6] and implement it into the version 1 model. That is, each particle has its own probability of being convinced by a neighbor, conditioned on that neighbor being confident. This will be made a random variable which is re-sampled throughout the simulation, so in each observation a new set of probabilities are generated.

Numerical Results

What is most intriguing about the results of these models is, regardless of the probability of being convinced, the shape of the curve begins linear, however after the number of columns exceeds the number of rows 10 (in this example, the number of columns exceeds 8), the shape becomes non-linear. This is likely due to the condition that the individual cannot be convinced by an unsure neighbor.



(a) Marginal



(b) Extremal

Figure 10: Each version fixes the number of rows to 8, and runs 1000 observations for each dot, except for the data points associated with a varying probability, which were found with 5000 observations. This model is much more prone to a high variance. Thus a larger number of observations are required to keep the points consistent with their actual average time.

Comparing Each Version

A possible question we had was is there really a difference in the time to consensus between the version-2 model and a model which has a fixed probability of being convinced at 0.5? In a population which an individuals probability of being convinced is uniformly sampled, the average probability approaches 0.5 as the number of individuals increases.

Marginal		
$P(convinced)$ (a)	$P(convinced)$ (b)	P-Value
1	Varies	0.0188
1	0.5	0.19
Varies	0.5	0.07

Extremal		
$P(convinced)$ (a)	$P(convinced)$ (b)	P-Value
1	Varies	0.008
1	0.5	0.13
Varies	0.5	0.03

Looking at the last line of each table, we see that with $\alpha = 0.1$ a significance in the difference between the average time to consensus of the model in which the probability of being convinced varies between the individuals, and the average time to consensus of the model in which the probability of being convinced is fixed at 0.5 throughout the population.

5 Remaining Questions and Literature Review

Seeing that this is a one-semester project, there was only so much time I've been able to dedicate to this project, and lack of access to a high performance computing limited me to the results my laptop was able to generate. Still, there are a number of questions left I am anxious to answer.

First and most importantly is a conjecture to my model. As discussed in an earlier section, Redner provided a formula for the time to consensus of a classic voter model, which may serve as a basis for my modified model. If I were to continue exploring this project, I would adjust my model to meet the same properties as his definition of a classic model, being on a complex graph, and use the density of agreeing voters as the input, rather than the size and shape of the population.

Another measurement I was hoping to explore was the number of interactions that each individual in the population has until a consensus is reached. This does not necessarily have to be in one of my models, but I would be interested to study and attempt to make an observation on this behavior.

Towards the end of my project I stumbled across a paper on discordant voting [2]. This model includes a random chance that instead of an interaction occurring, the edge connecting two individuals is broken and a new edge is formed with a different individual within the population, in other words the model "restructures based on the acceptance or rejection of differing opinions among social groups". It is not necessarily a property I would have implemented into an alternative model, but I would have liked to study numerically, however due to time constraints will have to keep in the "Remaining Questions" portion of my report. I also found it intriguing how the authors explored graphs outside of a grid or complete graph, such as stars, double stars, and even a cycle, and considers cases where the individual "pushes" their opinion to their neighbor, the opposite case which has been considered all through this report.

I would have also liked to look into these models I had discussed when they are presented with a quenched disorder, rather than a random walk in a random environment [4], especially for the second version of my confidence voter model. I wonder how the results may compare if the probabilities which are generated in the first observation stays the same for the remaining observations in the simulation.

In addition to Jedrzejewski's paper, an additional paper on a related model also piqued my interest. With this model, one of the conditions is that over time, nodes may be added or removed, introducing yet another random variable into the mix [8]. The authors introduce an algorithm of their generative network automata involving extracting and replacing regions of the population 11. An important note is this replacement stage described in 11 is also memoryless, bringing in more questions on how we may make adjustments or what we can study in this process as well. In cases where a node is added, how can the parent nodes affect the initialized state of the new node, for example?

Lastly I am still curious on the results of my versions of the confidence voter model. What can we say about this linear-to-nonlinear relationship the graphs are displaying?

Overall, as I've understood more about this topic, I have found that the literature I had previously and continue to read to bring about more questions towards the end of this project then I've had begin with at the start of the semester.

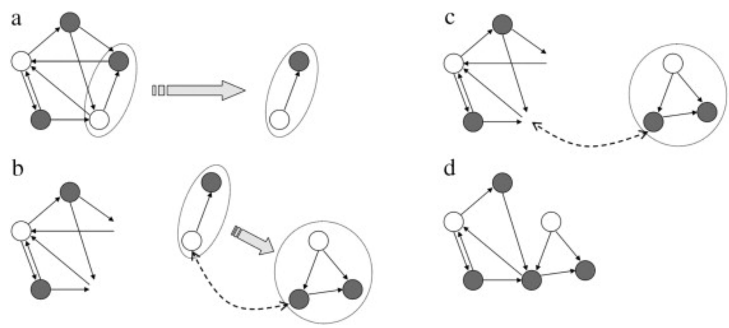


Figure 11: GNA writing process as described in associated literature [8]. (a) A portion of the population is selected and extracted from the model. (b) The extracted figure is replaced with a new figure, which has either an additional or one less node than the extracted figure from (a). (c) The new figure is placed back in the place of the former figure. (d) The new model after the GNA writing process.

Appendix

R Code

All of the code used to generate the results is uploaded to a repository in GitHub[1]. This repository also includes code for calculating the theoretical results referenced in this report, an R package of some of the functions used to generate results for this report, as well as the version control of each file. The code is extensive, but a summary is provided in the README file.

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[1] https://github.com/lem224/Math-374