A Stochastic Approach to the Burrito Problem

An Introduction to the Two-Stage Model

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May 2, 2024

Outline

Stochastic programming

In general, a stochastic program is a type of optimization problem where **some number of parameters are uncertain**, but, we assume that the probability distributions of these parameters are known.

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Examples:

- Robust Optimization
  - optimize worst case
- Deterministic Equivalent
  - replace uncertain parameter with deterministic value (ex: mean)
- Two-Stage Model

The Two-Stage Model

General Two-Stage Model

The general two-stage model is as follows (1):

$$\begin{aligned} \text{Minimize} \quad & f(x) + \mathbb{E}_\xi Q(x, \xi(\omega)) \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where ω is an outcome such that $\xi(\omega)$ is a particular realization of the random vector ξ, and $Q(x, \xi(\omega))$ represents the optimal value of the second stage problem. Here, $\mathbb{E}_\xi Q(x, \xi(\omega))$ is known as the recourse function.

$$\begin{aligned} \text{Minimize} \quad & q(y, \xi(\omega)) \\ \text{s.t.} \quad & Wy + T(\omega)x = h(\omega) \\ & y \geq 0. \end{aligned}$$

Here, $q(y, \xi(\omega))$ represents the second stage problem, and $Q(x, \xi(\omega))$ is the optimal value of that problem.

A Farming Example (1)

$$\begin{aligned} \text{Minimize} \quad & c^T x + \mathbb{E}_\xi Q(x,s) \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

where,

$$\begin{aligned} Q(x,s) = \text{Minimize} \quad & 238y_1 + 210y_2 - 170w_1 - 150w_2 - 36w_3 - 10w_4 \\ \text{s.t.} \quad & t_1(s)x_1 + y_1 - w_1 \geq 200, \\ & t_2(s)x_2 + y_2 - w_2 \geq 240, \\ & w_3 + w_4 \leq t_3(s)x_3, \\ & w_3 \leq 6000, y, w \succeq 0, \end{aligned}$$

where $t_i(s)$ is the yield of crop i under scenario s. Suppose we have three different scenarios, then $\xi = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ and ξ can take on three different values which represent $(t_1(1), t_2(1), t_3(1))$, $(\mathrm{t}_1(2), t_2(2), t_3(2))$, and $(t_1(3), t_2(3), t_3(3))$.

Sample Average Approximation (1)

To make our model deterministic, we replace the recourse function with a Monte Carlo estimate

$$\mathbb{E}_{\xi}^{v} Q(x,\xi) = \frac{1}{V}\sum_{k=1}^{V} Q(x,\xi_k).$$

We create a fixed number of different scenarios (different realizations of ξ) and name them $\xi_1, \ldots, \xi_V$.

$$\begin{aligned} \text{Minimize } & c^T x + \frac{1}{V}\sum_{k=1}^{V} q_k^T y_k \\ \text{s.t. } & Ax = b \\ & W y_k + T_k x = d_k \\ & x,\ y_k \geq 0. \end{aligned}$$

The Burrito Problem

Original Burrito Problem

Sets

C: the set of customer locations

L: the set of potential locations for placing a food truck

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Parameters

r: the revenue (in \$) from selling one burrito

k: the cost in (in \$) of ingredients for one burrito

f_j: the fixed cost (in \$) of placing a truck at location j, $j \in L$

d_i: the demand for burritos at location i, $i \in C$

$\alpha_{i,j}$: the fraction of demand that will be captured from customer location i by placing a truck at location j, $i \in C$, $j \in L$

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Decision Variables

$$x_j: \begin{cases} 1 & \text{if we place a truck at location } j, \\ 0 & \text{otherwise} \end{cases}, \; j \in L$$

$$y_{i,j}: \begin{cases} 1 & \text{if the demand at customer location } i \text{ is assigned to location } j, \\ 0 & \text{otherwise} \end{cases}, \; i \in C, \; j \in L$$

Original Burrito Problem (Gurobi)

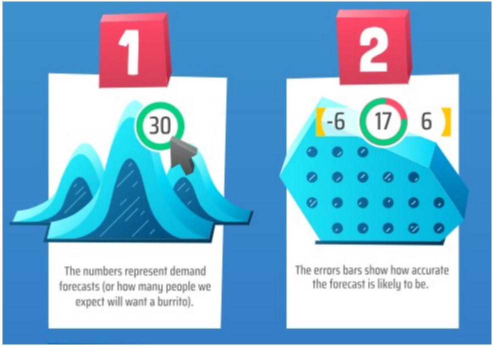
$$
\begin{aligned}
\text{Maximize} \quad & \sum_{i \in C} \sum_{j \in L} (r-k)\alpha_{i,j} d_i y_{i,j} - \sum_{j \in L} f_j x_j & \\
\text{s.t.} \quad & \sum_{j \in L} y_{i,j} \leq 1 & i \in C \\
& y_{i,j} \leq x_j & i \in C,\ j \in L \\
& x_j, y_{i,j} \in \{0,1\} & i \in C,\ j \in L
\end{aligned}
$$

Our Model

Introducing Uncertainty

What if we don't have an exact idea of what the demand at each location will be? In the burrito game we are now given demand forcasts, as well as how they may vary.

Suppose we let $d(\xi)$ be a realization of $\xi \sim \mathcal{U}(d^l, d^h)$, where d^l is the lowest our demand can be, and d^h is the highest our demand can be at a given location.



Stochastic Burrito Model

$$
\begin{aligned}
\text{Maximize} \quad & \frac{1}{B}\sum_{b=1}^{B}\sum_{i\in C}\sum_{j\in L}(r-k)\alpha_{i,j}d(\xi_{i,b})y_{i,j,b} - \sum_{j=1}^{n} f_j x_j \\
\text{s.t.} \quad & \sum_{j\in L} y_{i,j,b} \le 1 && i\in C, \quad b=1,\dots,B \\
& y_{i,j,b} \le x_j && i\in C, \quad j\in L, \quad b=1,\dots,B \\
& x_j \in \{0,1\} && j\in L \\
& y_{i,j,b} \in \{0,1\} && i\in C, \quad j\in L, \quad b=1,\dots,B
\end{aligned}
$$

Results

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<u>$|C| = 3$ AND $|L| = 8$</u>

Optimal profit (original): $\$85$

Optimal profit (SP):$\$103$

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Optimal profit (original): $\$85$

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$|C| = 25$ and $|L| = 70$

Optimal profit (original): $\$4168$

Optimal profit (SP): $\$4232$

Applying Our Model to the Game

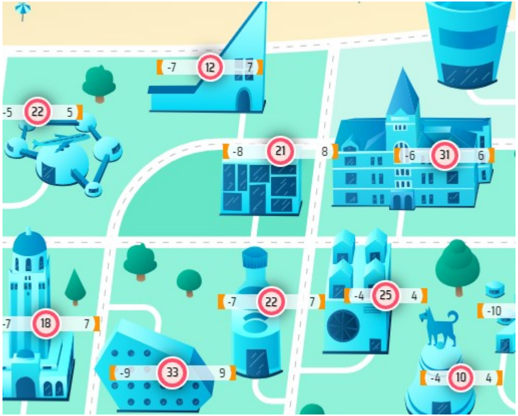

Figure: Round 2 day 1

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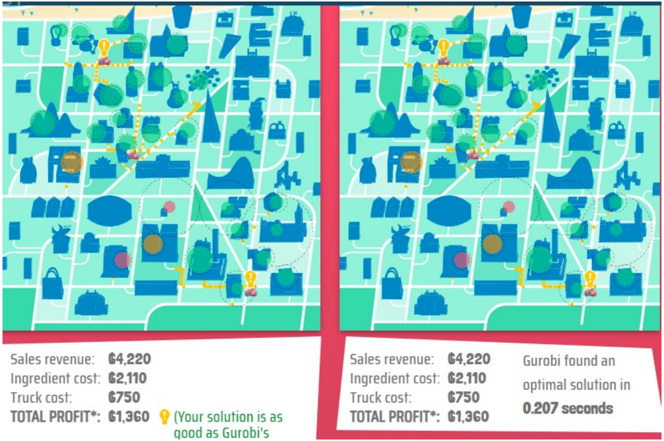

Figure: Round 2 day 1 results

Applying Our Model to the Game

Figure: Round 2 day 4

Applying Our Model to the Game

Figure: Round 2 day 4 results

Thank you!

https://github.com/makhoullillian153/A-Stochastic-Approach-on-the-Burrito-Problem

References

[1] Birge, J. R. and Louveaux, F. (2011). *Introduction to Stochastic Programming*. Springer New York, NY.

[Gurobi] Gurobi. *Burrito Optimization Game: Game Guide*.