Coalescing Random Walks on the N-Cycle

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The Model

The **N-cycle** is a graph with the vertex set $\{0, 1, ..., N-1\}$ and the edge set $\{\{i, j\} : (i-j) \equiv 1 \mod N$ or $(j-i) \equiv 1 \mod N\}\}$. We look at a particle system on the N-cycle under the following process:

1. At time $t = 0$, all nodes on the cycle are occupied by a particle
2. At each time step, one particle is sampled uniformly and moved one node in the clockwise direction
3. If the new vertex is occupied, the particle absorbs the previous occupant
4. Repeat until only one particle remains

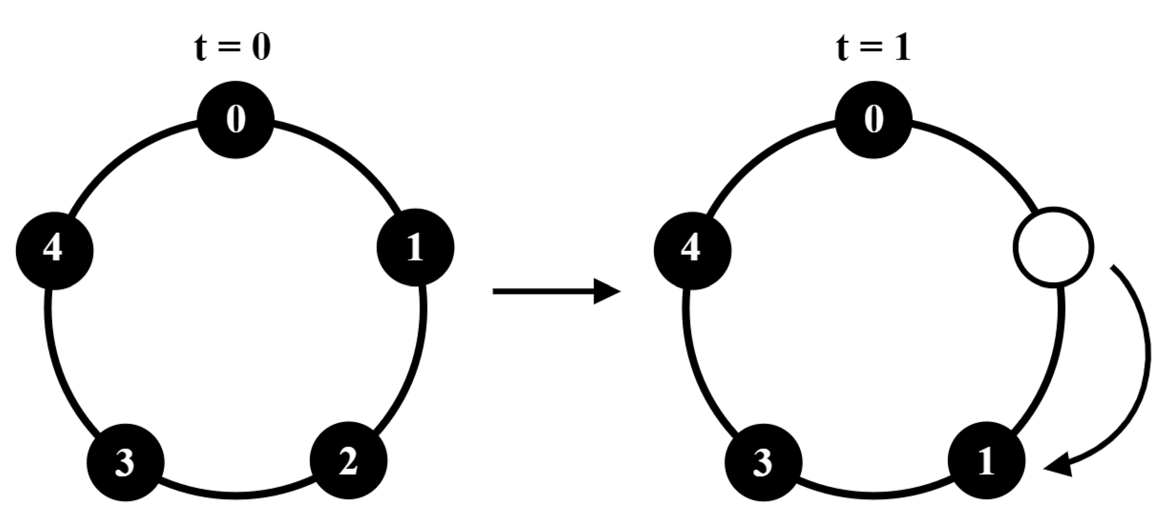


Figure 1. One step of the process on the 5-Cycle, in which particle 1 is sampled

Coalescence Model of a Line

A closely related model was studied by Russell Lyons and Michael Larsen in 1999. They focused on a coalescing random walk model on the line, where each particle moves to the left when sampled until eventually reaching an absorbing state at 0.

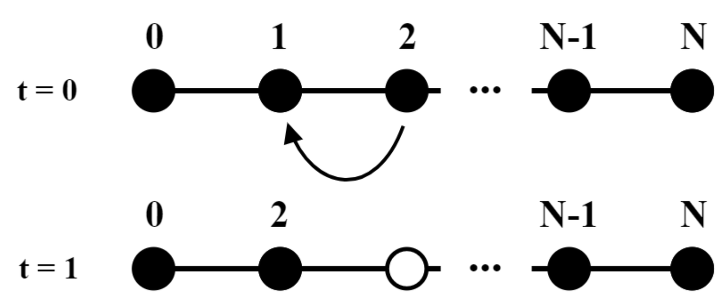


Figure 2. Coalescing random walk on the line

This linear model is essentially our model on the cycle, but with a single edge removed.

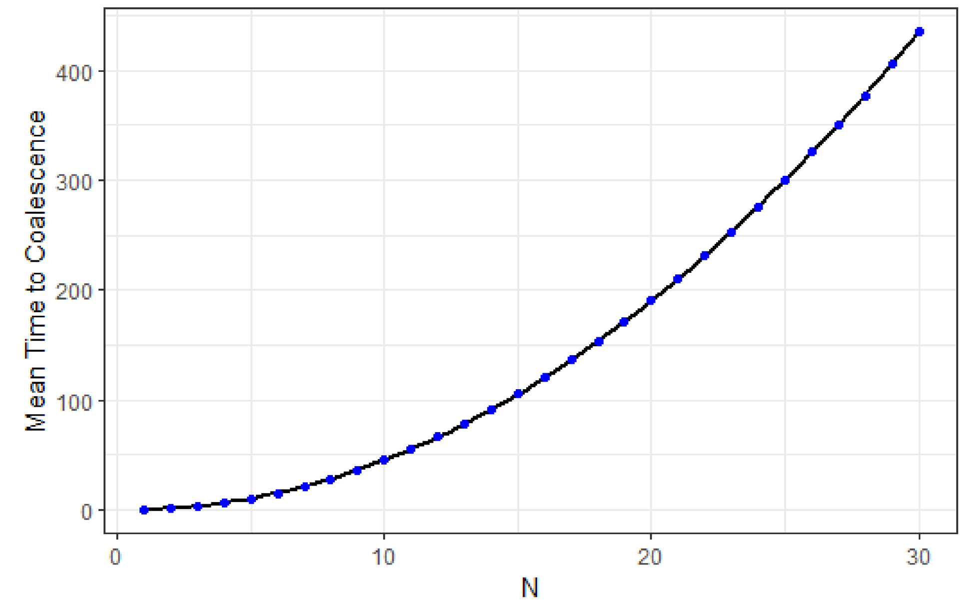
Contrasting the Models

- On the line:
  - The formula for $\mathbb{E}[T_N]$ and the upper bound for $Var[T_N]$ can be derived using a single generating function
  - The last surviving particle is always the particle starting at node N
  - The last surviving particle always takes N steps until absorption
- On the cycle:
  - The formula for $\mathbb{E}[T_N]$ is known, while the second moment and variance remain elusive
  - The identity of the last surviving particle is uniformly distributed among all particles in the system
  - The number of steps taken by the last surviving particle is random

Expectation of Time to Coalescence

Let T_N be the time until all particles have coalesced on the N-cycle.

N	Simulated $\mathbb{E}[T_N]$
1	0.0000
2	1.0000
3	2.9995
4	5.9963
5	9.9763
6	15.0168
7	20.9875
8	28.0297
9	35.9710
10	44.9907



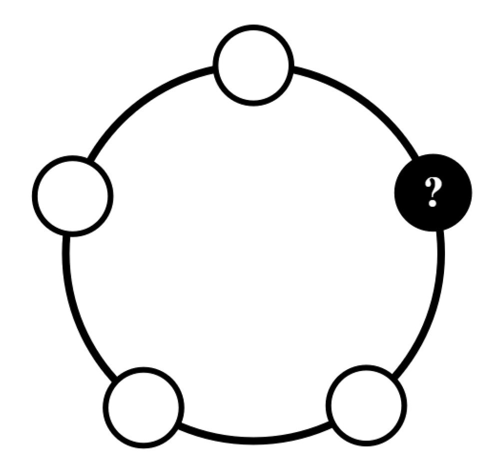
Best fit $f(N) = 0.4985N^2 - 0.4579N - 0.1658$

Theorem. Let T_N be the time at which all particles on the N-cycle coalesce. Then

$$\mathbb{E}[T_N] = \frac{N(N-1)}{2}$$

The Last Surviving Particle

At first glance, we can tell that it must take at least $N-1$ steps before the process concludes, as it must necessarily absorb its counterclockwise neighbor as its final action.



However, beyond this basic fact, it is not necessarily clear what we should expect.

Similar to the time to coalescence, multilinear regression was used to evaluate the number of steps taken by the last surviving particle. We obtained the following models for the average, variance, and second moment of the number of steps, respectively:

$$f(N) = 0.1666N^2 + 0.5058N - 0.7037$$
$$f(N) = 0.0462N^4 - 0.2260N^3 + 6.6775N^2 - 40.4179N + 58.5697$$
$$f(N) = 0.0149N^4 - 0.2059N^3 + 3.4974N^2 - 21.3330N + 31.6806$$

Theorem. Let $\mathcal{L}$ denote the number of steps taken by the last surviving particle. Then,

$$\mathbb{E}[\mathcal{L}] = \frac{N^2}{6} + \frac{N}{2} - \frac{2}{3} \tag{1}$$

$$\mathbb{E}[\mathcal{L}^2] = \frac{7N^4}{180} + \frac{N^3}{6} - \frac{N^2}{36} - \frac{2N}{3} + \frac{22}{45} \tag{2}$$

$$Var[\mathcal{L}] = \frac{(N^2-1)(N^2-4)}{90} \tag{3}$$

The Last Surviving Particle: Continued

Equations (1)-(3) were found through the following process:

1. Note that $\mathcal{L}$ is equal to the number of steps taken by b in a two particle system. If we condition on particle b absorbing its counterclockwise neighbor a.
2. Think of this two-particle system as a single particle random walk on the line. That is, it can be translated to a simple random walk of a particle p on $[0, N]$, starting at $N-1$, and with absorbing states at 0 and N.
   - If particle b moves, then p moves right.
   - If particle a moves, then p moves left.
   - Absorption corresponds to p reaching either 0 or N, however in this case we condition on absorbing when p reaches N.

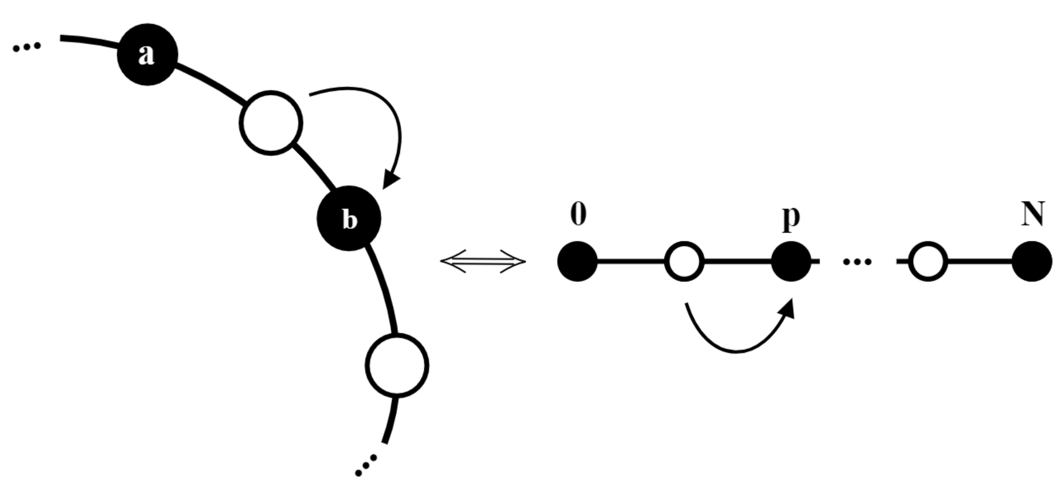


Figure 3. Walk on the cycle vs. walk on the line

3. Find a moment generating function for this conditioned random walk and take the limits of the first and second derivatives as $\lambda \to 0$ to find (1)-(3).

   Lemma. Let a and b be two consecutive particles on an N-cycle, and let T_N be the time before b absorbs a, conditioned on b absorbing a. Then,

$$\mathbb{E}[e^{-\lambda T_N}] = N \frac{\sinh\left(\frac{1}{2}\log(x)\right)}{\sinh\left(\frac{N}{2}\log(x)\right)}$$

where

$$x = \frac{1 - \sqrt{1 - e^{-2\lambda}}}{1 + \sqrt{1 - e^{-2\lambda}}}$$

Future Work

We have yet to prove a formula for the variance of the time to coalescence but through our numerical results have come up with the following conjecture:

Conjecture. Let T_N be the time at which all particles on the N-cycle coalesce. Then

$$\mathbb{E}[T_N^2] = \frac{3N^4}{10} - \frac{N^3}{2} + \frac{1}{5}$$

$$Var[T_N] = \frac{(N^2-1)(N^2-4)}{20}$$

References

Dykiel, P. (2005). Asymptotic properties of coalescing random walks.
Larson, M. and Lyons, R. (1999). Coalescing particles on an interval. *Journal of Theoretical Probability*, 12(1):201–205.
Levin, D., Peres, Y., and Wilmer, E. (2009). *Markov Chains and Mixing Times*. American Mathematical Society, second edition.