TEAM NAMES[1] ______________________________

CSCI 5451, Professor E. Gethner, Quiz 0, Fall 2024

Instructions: Be neat, write complete sentences, and show all of your work. The way you communicate the solution to your answer is as important as the answer itself. This quiz is worth 60 points.

1. (30 points) Let $P(n)$ be the statement "$2^{2n} - 1$ is divisible by 3."

   (a) For which non-negative integers n is statement $P(n)$ true? *Be clear and coherent in your reasoning; do not leave out details and write complete sentences.*

   $P(n)$ is true for all nonnegative integers n $(n \geq 0)$.

   (b) Prove your claim in part (a) above by mathematical induction. *In order to receive any credit for this problem, you must write complete sentences, include all details, be clear in your reasoning, and use the correct format and syntax for an induction proof. Not doing so will result in massive point deductions.*

we say $2^{2n}-1$ is divisible by 3 if for some $m \in \mathbb{Z}$, $2^{2n}-1 = 3 \cdot m$. The remainder of the proof will follow by induction.

Base cases: For $n=0$,

$2^{2(0)}-1 = 1-1 = 0 = 3 \cdot 0$. Thus $P(0)$ holds.

For $n=1$,

$2^{2(1)}-1 = 4-1 = 3 = 3 \cdot 1$. Thus $P(1)$ holds.

For $n=2$,

$2^{2(2)}-1 = 16-1 = 15 = 3 \cdot 5$. Thus $P(2)$ holds.

[1] If a team member does not actively participate, let them know that their name will not be included on the quiz and let the TA know, as well.

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Induction hypothesis: Assume that $P(k)$ holds for some fixed integer $k \geq 3$.

Inductive step: We will show $P(k+1)$ is true. Since $P(k)$ holds, by our induction hypothesis, there exists $m \in \mathbb{Z}$ such that $2^{2k} - 1 = 3 \cdot m$. Then

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^2 2^k - 1 = 4 \cdot 2^k - 1 = (3+1)2^k - 1 =$$
$$= 3 \cdot 2^k + (2^k - 1) = 3 \cdot 2^k + 3 \cdot m = 3 \cdot (2^k + m)$$

Thus $2^{2(k+1)} - 1$ is divisible by 3, with a factor of $2^k + m$.

Conclusion: Thus, by induction, $2^{2n} - 1$ is divisibly by 3 for all integers $n \geq 0$.

2. (30 points) The *Lucas Numbers* are defined as $L_0 = 2$, $L_1 = 1$, and for every integer $n \geq 2$, we have $L_n = L_{n-1} + L_{n-2}$.

(a) What are the Lucas numbers $L_2, L_3, L_4, \ldots, L_8$?

$L_2 = L_1 + L_0 = 1 + 2 = 3$

$L_3 = L_2 + L_1 = 3 + 1 = 4$

$L_4 = L_3 + L_2 = 4 + 3 = 7$

$L_5 = L_4 + L_3 = 7 + 4 = 11$

$L_6 = L_5 + L_4 = 11 + 7 = 18$

$L_7 = L_6 + L_5 = 18 + 11 = 29$

$L_8 = L_7 + L_6 = 29 + 18 = 47.$

(b) Prove by induction that $\forall\, n \geq 2$ we have $L_2 + L_4 + \cdots + L_{2n} = L_{2n+1} - 1$. *In order to receive any credit for this problem, you must write complete sentences, include all details, be clear in your reasoning, and use the correct format and syntax for an induction proof. Not doing so will result in massive point deductions.*

Let $P(n)$ be the statement "$L_2 + L_4 + \ldots + L_{2n} = L_{2n+1} - 1$".
We will show $P(n)$ is true $\forall n \geq 2$ by induction.

Base cases: For $n = 2$,
$L_2 + L_4 = 10 = 11 - 1 = L_5 - 1 = L_{2(2)+1}$. Thus $P(2)$ holds.
For $n = 3$,
$L_2 + L_4 + L_6 = 28 = 29 - 1 = L_7 - 1 = L_{2(3)+1} - 1$. Thus $P(3)$ holds.

Induction hypothesis: Assume that $P(k)$ holds for some fixed integer $k \geq 4$.

Inductive step: We will show $P(k+1)$ is true. Since $P(k)$ is true, by our induction hypothesis, $L_2 + L_4 + \ldots + L_{2k} = L_{2k+1} - 1$. Then,

$$L_2 + L_4 + \ldots + L_{2k} + L_{2(k+1)} = (L_2 + L_4 + \ldots + L_{2k}) + L_{2k+2} =$$

$$= L_{2k+1} - 1 + L_{2k+2} = L_{2k+1} + L_{2k+2} - 1.$$

By the definition of the Lucas numbers,

$$(L_{2k+1} + L_{2k+2}) - 1 = L_{2k+3} - 1 = L_{2(k+1)+1} - 1$$

Thus,

$$L_2 + L_4 + \ldots + L_{2(k+1)} = L_{2(k+1)+1} - 1.$$

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conclusion: $P(n)$ is true for all integers $n \geq 2$. In other words,

$L_2 + L_4 + \ldots + L_{2n} = L_{2n+1} - 1$ for all integers $n \geq 2$.