Revisiting ‘penetration depth’ in falling film mass transfer

Mohammad Ali Aroon1†\*, Milad Asgarpour Khansary2‡, Saeed Shirazian2⸸

1-Membrane Research Lab., Caspian Faculty of Engineering, College of Engineering, University of Tehran, Tehran, Iran; 2-Department of Chemical Sciences, Bernal Institute, University of Limerick, Limerick, Ireland. †[maaroon@ut.ac.ir](mailto:maaroon@ut.ac.ir), ‡[Milad.Asgarpour@ul.ie](mailto:Milad.Asgarpour@ul.ie), ⸸[Saeed.Shirazian@ul.ie](mailto:Saeed.Shirazian@ul.ie). \*correspondences: M.A. Aroon ([maaroon@ut.ac.ir](mailto:maaroon@ut.ac.ir))

**Abstract**

For the analysis of falling film mass transfer such as gas absorption in liquid phase flowing over a vertical wall, conventionally an infinite penetration depth for penetrating component has been assumed. This assumption is originated from the fact that magnitude of diffusion depth is too small relative to the film thickness as if the mass transfer into the liquid film is taking place in a semi-infinite geometry. Despite its validity in the most engineering applications, it is still an unanswered question that ‘to what physically meaningful extent the diffusion may occur or the mass transfer boundary layer into the liquid film could progress alongside the wall, and consequently how it might affect the mass transfer characteristics’. We addressed such concern here by considering a finite depth of penetration into the film, derived governing equations for the continuity of mass and solved it analytically and numerically. The progress of penetration depth over the vertical coordinate i.e. flowing direction of falling film is observable as it approaches the film thickness far down the wall. The method gives the opportunity to examine the progress of the penetration depth with mass transfer properties such as diffusivity, average liquid velocity and etc. which have not been touched before.

**Keywords**: falling film; mass transfer; penetration depth; gas absorption

1. **Problem definition**

Gas absorption is one of the most common processes where a penetrating component transfers/diffuses into a falling film [[1](#_ENREF_1), [2](#_ENREF_2)]. Fig. 1 illustrates a falling film mass transfer [[3](#_ENREF_3)] in gas absorption operation where the free surface of the liquid falling film on a vertically aligned flat solid surface is exposed to a gas phase containing an unwanted component (*A)*. Therefore, component *A* is transferred and dissolved into the falling liquid film of thickness *δ*. The falling film is falling down with an average velocity of . At the top of the falling liquid film (e.g. *y*=*0*), a uniform concentration profile *cA0* could be assumed for transferring component *A*. The interfacial concentration *cAi* through the gas-liquid interface (from *y*=*0* to *y*=*L*) can be assumed constant [[1](#_ENREF_1), [2](#_ENREF_2)].

|  |
| --- |
| **Finite penetration depth**  ***ξ*(*y*)** |
| Fig. 1. Schematic illustration of falling film mass transfer in gas absorption operation |

The concentration profile of dissolved gas in liquid phase over the vertical coordinate (length of wall), is conventionally derived by utilizing an *infinite penetration depth* ***as boundary condition*** (). This condition is originated from the assumption that the magnitude of diffusion depth is too small relative to the liquid film thickness, *δ*. So one may consider the diffusive mass transfer normal to the liquid film as a process taking place in a semi-infinite geometry [[2-4](#_ENREF_2)]. The governing partial differential equation for this process is obtained (see appendix) as  where *D* is the diffusivity of *A* in liquid film, is concentration of “*A*” in liquid phase and is average velocity of falling film fluid in *y* direction [[3](#_ENREF_3)]. Considering the boundary conditions (BCs) – and noting to the assumptions: steady state, no chemical reaction, no diffusion in flow direction, no convection in parallel to the diffusion direction (), constant diffusivity, constant density and two dimensional mass transfer) – and using combination of variable (similarity) technique, the solution for the governing equations can be obtained as given in Table 1 [[3](#_ENREF_3)].

Table 1. Concentration profile for gas absorption in falling liquid film using *infinite* penetration depth (conventional solution)

|  |  |  |  |
| --- | --- | --- | --- |
| **Penetration depth** | **BCs** | **Profile of *cA*** | **Eq.** |
| Infinite [[3](#_ENREF_3)] |  |  | 1 |
| *assumptions* [[2](#_ENREF_2), [4](#_ENREF_4)]: short contact time, steady state, no chemical reaction, no diffusion in flow direction, no convection in parallel to the diffusion direction (), constant diffusivity, constant density and two dimensional mass transfer | | | |

While for the most of engineering applications, such *infinite* penetration depth assumption is valid and a concentration profile as given in Table 1 is useful. However, the extend of mass transfer boundary layer and the depth of penetration – up to which the gas diffusion may occur – cannot be analyzed. It is impossible to examine the effect of the penetration depth on the mass transfer characteristics such as mass transfer coefficient. Therefore, finding the *physically meaningful* extent of penetration depth is of great concern and interest, which is addressed and discussed in following sections.

1. Discussion on penetration depth

To relax the assumption of an infinite penetration depth, consider that a mass transfer boundary layer of thicknessis fully formed and established (as illustrated in Fig. 1). This in turn means that the conventional infinite penetration depth (which was given as, see Table 1) is refined as a *finite penetration depth*. This finite penetration depths is translated and written as a new boundary condition  (or alternatively) as listed in Table 2. By using the separation of variable technique, the solution to the governing continuity of mass equation was derived, and the concentration profile of the penetrating component “*A*” is given in Table 2 (see appendix). From Eq. 2, it can be seen that the concentration profile is implicit with respect to the finite penetration depth, . By applying the finite penetration depth boundary condition i.e., one may obtain. This can be rewritten as, where. This relationship cannot be solved analytically to derive an explicit equation for and requires a numerical treatment as follows by using an example.

Table 2. Concentration profile for gas absorption in falling liquid film with finite penetration depth

|  |  |  |  |
| --- | --- | --- | --- |
| **Penetration depth** | **BCs** | **Profile of *cA*** | **Eq.** |
| Finite |  |  | 2 |
| *assumptions* [[2](#_ENREF_2), [4](#_ENREF_4)]: steady state, no chemical reaction, no diffusion in flow direction, no convection in parallel to the diffusion direction (), constant diffusivity, constant density and two dimensional mass transfer | | | |

***Example***: For numerical evaluation of proposed approach and discussion of improvement over the conventional approach, the mass transfer of carbon dioxide (*CO2*) from gas phase as the penetrating component into the water as falling liquid film on a vertical wetted wall column of unit length (*L* = 1 *m*) has been considered. The data is retrieved from Ref. [[3](#_ENREF_3)], where initial concentration of CO2 gas in falling water film is *cA0* = 0 kmol/m3; the concentration of CO2 gas at the interface of falling water film and gas stream (at any vertical position) is *cAi* = 0.0366 kmol/m3; the density of water is *ρ*water = 998 kg/m3; the viscosity of water is *µ* = 8.94×10-4 kg/m.s; and *g* = 9.8 m/s2. Two scenarios have been considered for numerical experiments (the python script used for these calculations are given in the appendix):

1. In order to examine the effect of the magnitude of penetrating gas diffusivity on the finite penetration depth, at fixed film flow rate of = 0.05 kg/m.s, the diffusivity (*D* × 109) is varied as 1.0, 2.0, 2.5, 5.0, 7.0, 8.0, and 9.0 m2/s.
2. For a fixed diffusivity of *D* = 1.96 × 10-9 m2/s, the film flow rate (*Γ*) is varied as 0.01, 0.02, 0.05, 0.08, 0.10, and 0.20 kg/m.s to analyze the effect of the film flow rate variation on the finite penetration depth.

For varying diffusivity where the film flow rate is fixed at *Γ* = 0.05 kg/m.s (which corresponds to = 0.21 *m*/*s*) of above-mentioned example, the film thickness *δ* and the progress of penetration depth () over vertical coordinate (*y*) are shown in Fig. 2. The calculated film thickness is 0.0002394 m. It can be seen that at any vertical position, say *y* = 0.01 *m* for better visual presentation, the penetrating gas, i.e. CO2, penetrates deeper into the falling film. This concludes that the higher diffusivity, the higher penetration depth. It is evidence that at any specific diffusivity value the penetration depth gradually increases alongside the vertical position and finally reaches/converges to the film thickness *δ* which is an upper limit for penetration depth.

Comparing the penetration depth profiles, one may conclude that to reach a specific depth within the falling film, the lower the diffusivity, the penetrating gas must travel more alongside the vertical coordinate. This means that when the diffusivity of the penetrant increases in the liquid film, it can diffuse rapidly and travel deeper into the falling film in short contact time. So the penetration depth growth will be shorter alongside the vertical wall (i.e. *y* coordinates) and it faster reaches the film thickness value. For instance, when *D* = 1×10-9 m2/s, the penetrating gas is obliged to travel longer alongside the vertical coordination i.e. *y* = 0.25 *m*, as the diffusivity is too slow to penetrate deeper at very short contact times, therefore it will travel and eventually would get sufficient time to penetrate deeper. In contrast, the fastest diffusivity studied here i.e. *D* = 9×10-9 m2/s, requires lower travels alongside the vertical coordinate to penetrate deep into falling film. It is evidence from the corresponding penetration depth which reaches the film thickness around *y* < 0.05 *m*. It is very interesting that the penetration depth growth alongside the vertical coordinate increases as the diffusivity decreases in liquid film. It is worthy to mention that current approach brings the great opportunity of observing the penetration depth growth and progress which has not been touched before.

|  |
| --- |
|  |
| Fig. 2. Progress of penetration depth () over vertical coordinate (*y*) at varying diffusivity (*D*) *m*2/*s* (as indicated by legends) for above-mentioned example where *Γ* = 0.05 kg/m.s (= 0.21 *m*/*s*). dashed-line shows film thickness, *δ*. |

For varying film flow rate where the diffusivity is fixed at *D* =1.96×10-9 *m*2/*s* of above-mentioned example, the film thickness *δ* and the progress of penetration depth () over the vertical coordinate (*y*) are calculated as shown in Fig. 3. As the film flow rate (*Γ*) increases, the film thickness *δ* (dashed-line) increases as well. Increasing the film flow rate, the penetrating gas must travel longer distance alongside the vertical coordinate (to reach/converge to the film thickness *δ* which is an upper limit for penetration depth). For instance comparing *Γ* = 0.08 and *Γ* = 0.2 *kg*/*m*.*s*, the penetrating gas travels to *y* = 0.25 *m* and *y* = 0.5 *m* respectively. This is because when the film flow rate increases, the fluid flows faster alongside the vertical direction and the penetrating gas would have shorter contact time. This in turn results in less deep penetration depth for the penetrating gas, and consequently the gas will travel longer distance alongside the vertical coordinate until it reached the film boundary. At any given vertical position, strictly speaking where all penetration depth profiles are still evolving as seen in zoomed section of Fig. 3, it can be concluded that the film flow rate has a reciprocal relationship with the depth of penetration. This is because higher film flow rates imply higher average fluid velocity, which carries the penetrated gas faster, and then the penetrating gas has shorter times to diffuse deeper into the falling film. For instance at the vertical position *y* = 0.001 *m*, comparing *Γ* = 0.2 and comparing *Γ* = 0.01 *kg*/*m*.*s*, it can be seen that the gas penetrates deeper when *Γ* = 0.01 *kg*/*m*.*s*.

|  |
| --- |
|  |
| Fig. 3. Progress of penetration depth () over vertical coordinate (*y*) at varying flow rate *kg*/*m.s* (as indicated by legends) for above-mentioned example where *D* =1.96×10-9 *m*2/*s*; dashed lines show the corresponding film thickness *δ* at each film flow rate. |

The calculated concentration profiles of *CO2* in the falling film for the two conditions of finite and infinite penetration depths where *D* =1×10-9 *m*2/*s* and *Γ*=0.05 kg/m.s are shown in Fig. 4. As it is obvious, the concentration of the penetrant (“*A*”) increases alongside the vertical coordinate in each constant cross sectional *z*, and it also decreases inside the liquid film in each constant *y* (because of mass transfer resistance). It is necessary to note that new approach results in the same concentration profiles as obtained by the conventional models.

|  |
| --- |
|  |
| Fig. 4. Calculated concertation profile over vertical coordinate (*y*) at varying depths (z-coordinate as indicated by legends) for above-mentioned example where *D* =1×10-9 *m*2/*s* and *Γ*=0.05 kg/m.s; solid lines show the finite penetration depth corresponding concentrations and bullets show the infinite penetration depth corresponding concentrations |

1. Concluding remarks

For the first time, considering a finite penetration depth of penetrant in a liquid falling film, the governing equations were derived and then solved analytically and numerically using the separation of variable technique. A numeric example was used to check the consistency and accuracy of obtained concentration profiles to those of literature and conventionally used ones which use an infinite penetration depth. While the same concentration profiles were obtained, the new approach here leads to opportunities on examination of the penetration growth and its progress alongside the film direction which has not been touched before.

1. Data availability

The associated data of this article can be found in online version available at [www.sciencedirect.com](http://www.sciencedirect.com) or can be retrieved from <https://github.com/makhansary/pub.dPen>.

1. Conflict of interest

There’s no conflict of interest to declare.

1. Appendix

# Detailed derivation of governing equation and corresponding solution

The gas penetration into a falling film is focus of present modeling work. Fig. 1 illustrates the coordinates and the mass transfer properties. Assuming two dimensional, steady state mass transfer, no chemical reaction between gas (*A*) and falling liquid (*B*), no diffusion in flow direction (*y*), no convective mass transfer in diffusion direction (*z*), constant diffusivity (*D*) and liquid density, then the continuity of mass results in Eq. S. 1, where is the concentration profile of the penetrating gas.

|  |  |
| --- | --- |
|  | S. |

The velocity profile in y-direction is given as in Eq. S. 2, where *ρ* is density, *g* is gravity acceleration, *µ* is the viscosity of falling liquid film and *δ* is the liquid film thickness.

|  |  |
| --- | --- |
|  | S. 2 |

Short contact time assumption reads *z*<<*δ*, therefore, Eq. S. 2 read Eq. S. 3, where  is the maximum velocity of falling liquid film, which is related to the average velocity () as.

|  |  |
| --- | --- |
|  | S. 3 |

Therefore, the final form of governing equation reads Eq. S. 4 where 

|  |  |
| --- | --- |
|  | S. |

For the infinite penetration depth, the solution of partial differential equation (PDE) can be found in a previous paper [[3](#_ENREF_3)]. For the finite penetration depth, using the separation of variables method by setting  and , the PDE in Eq. S. 4 can be transformed into Eq. S. 5 where  is the separation factor.

|  |  |
| --- | --- |
|  | S. |

The general solutions to each parts in Eq. S. 5 can be obtained as given in Eqs. S. 6 and S. 7.

|  |  |
| --- | --- |
|  | S. |
|  | S. |

By applying boundary conditions, one would obtain

|  |  |
| --- | --- |
|  | S. |
|  | S. |
|  | S. |

Finally, the concentration profile is given as in Eq. S. 11 or similarly as in Eq. S. 12.

|  |  |
| --- | --- |
|  | S. |
|  | S. |

To find *ξ*, the introduced finite penetration depth, one could use the boundary condition, which reads Eq. S. 13.

|  |  |
| --- | --- |
|  | S. |

For engineering applications, the mass transfer characteristics are of the interest including the gas flux at the interface of falling film and gas stream (), the mass transfer coefficient (), and the average mass transfer coefficient alongside the vertical coordinate i.e. falling film wall from *y*=*0* to *y*=*L* () as given in Eq. S. 14-S. 16.

|  |  |
| --- | --- |
|  | S. |
|  | S. |
|  | S. |

# Python scripts for computational experiments

**Code for case A:**

##############################

# MAK - python3 code

# file name: varD.py

# (in)finite penetration depth : varying D

##############################

import numpy as np

import math

##############################

# system data

L=1 # wall lenght

gamma=0.05 # film flow rate kg/m.s

cA0=0 # initial concentration

cAi=0.0366 # interfacial concentration

var=[1.0, 2.0, 2.5, 5.0, 7.0, 8.0, 9.0]

varD=[vard/1000000000 for vard in var]

##############################

Rho=998 # kg/m3

Mue=0.000894 # kg/m.s at STP

g=9.807 # m/s2

##############################

# y-coordinate

YY=[0.00000000001, 0.0000000001, 0.000000001, 0.00000001, 0.0000001, 0.000001, 0.00001, 0.0001, 0.001, 0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 1]

Y=[L\*y for y in YY]

# film thickness

delta=((3\*Mue\*gamma)/((Rho\*\*2)\*g))\*\*(1/3)

u=gamma/(Rho\*delta)

# z-coordinate

Z=np.linspace(0, delta, num=50)

# mass transfer properties

############################

for D in varD:

############################

b=(2\*D)/(3\*u)

#

infinity=100

for y in Y:

# finding kisi from BC2

guesslist=np.logspace(0, 10, num=10000)

guesslist=delta/guesslist

guesslist=np.fliplr([guesslist])[0]

KICI=[delta]

for kisi in guesslist:

sigma=0

for n in range(infinity):

Sin=((-1)\*\*n)/(2\*n+1)

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=y\*Expot

Exp=np.exp(Expt)

An=Sin\*Exp

sigma += An

Err=sigma - (math.pi/4)

if abs(Err)==0:

KICI.append(kisi)

kisi=min(KICI)

# calculating kc and NA

Sum=0

for n in range(infinity):

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=y\*Expot

Exp=np.exp(Expt)

An=Exp

Sum += An

kc\_infY=(2\*D/kisi)\*Sum

NA\_infY=kc\_infY\*(cAi - cA0)

kc\_infN=((3\*u\*D)/(2\*math.pi\*y))\*\*0.5

NA\_infN=(cAi-cA0)\*kc\_infN

# kcbar calculation

Sum=0

for n in range(infinity):

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=L\*Expot

Exp=np.exp(Expt)

An=Exp/((2\*n+1)\*\*2)

Sum += An

kcbar\_infY=(12\*u\*kisi/(L\*(math.pi\*\*2)))\*((math.pi\*\*2)/8 - Sum)

kcbar\_infN=((6\*u\*D)/(math.pi\*L))\*\*0.5

# calculating cA

for z in Z:

Sum=0

for n in range(infinity):

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=y\*Expot

Exp=np.exp(Expt)

Sin=math.sin(z\*lmbda)

An=Sin\*Exp/(2\*n+1)

Sum += An

cA\_infY=cA0+(cAi-cA0)\*(1-(4/math.pi)\*Sum)

Exp\_infN=0.5\*(((3\*u\*(z\*\*2))/(2\*D\*y))\*\*0.5)

cA\_infN=cA0+(cAi-cA0)\*(1-math.erf(Exp\_infN))

print (D, y, z, delta, kisi, cA\_infY, cA\_infN, NA\_infY, NA\_infN, kc\_infY, kc\_infN, kcbar\_infY, kcbar\_infN, gamma, u)

**Code for case B:**

##############################

# MAK - python3

# varG.py

# (in)finite penetration depth : varying Gamma

##############################

import numpy as np

import math

##############################

# system data

L=1 # wall length

varG=[0.01, 0.02, 0.05, 0.08, 0.10, 0.20]

cA0=0 # initial concentration

cAi=0.0366 # interfacial concentration

D=1.96/1000000000 # diffusivity

##############################

Rho=998 # kg/m3

Mue=0.000894 # kg/m.s at STP

g=9.807 # m/s2

##############################

# y-coordinate

YY=[0.00000000001, 0.0000000001, 0.000000001, 0.00000001, 0.0000001, 0.000001, 0.00001, 0.0001, 0.001, 0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 1]

Y=[L\*y for y in YY]

############################

for gamma in varG:

############################

# mass transfer properties

delta=((3\*Mue\*gamma)/((Rho\*\*2)\*g))\*\*(1/3)

u=gamma/(Rho\*delta)

# z-coordinate

Z=np.linspace(0, delta, num=50)

b=(2\*D)/(3\*u)

#

infinity=100

for y in Y:

# finding kisi from BC2

guesslist=np.logspace(0, 10, num=10000)

guesslist=delta/guesslist

guesslist=np.fliplr([guesslist])[0]

KICI=[delta]

for kisi in guesslist:

sigma=0

for n in range(infinity):

Sin=((-1)\*\*n)/(2\*n+1)

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=y\*Expot

Exp=np.exp(Expt)

An=Sin\*Exp

sigma += An

Err=sigma - (math.pi/4)

if abs(Err)==0:

KICI.append(kisi)

kisi=min(KICI)

# calculating kc and NA

Sum=0

for n in range(infinity):

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=y\*Expot

Exp=np.exp(Expt)

An=Exp

Sum += An

kc\_infY=(2\*D/kisi)\*Sum

NA\_infY=kc\_infY\*(cAi - cA0)

kc\_infN=((3\*u\*D)/(2\*math.pi\*y))\*\*0.5

NA\_infN=(cAi-cA0)\*kc\_infN

# kcbar calculation

Sum=0

for n in range(infinity):

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=L\*Expot

Exp=np.exp(Expt)

An=Exp/((2\*n+1)\*\*2)

Sum += An

kcbar\_infY=(12\*u\*kisi/(L\*(math.pi\*\*2)))\*((math.pi\*\*2)/8 - Sum) kcbar\_infN=((6\*u\*D)/(math.pi\*L))\*\*0.5

# calculating cA

for z in Z:

Sum=0

for n in range(infinity):

lmbda=((2\*n+1)\*math.pi)/(2\*kisi)

lmbda2=lmbda\*\*2

Expot=(-1)\*lmbda2\*b

Expt=y\*Expot

Exp=np.exp(Expt)

Sin=math.sin(z\*lmbda)

An=Sin\*Exp/(2\*n+1)

Sum += An

cA\_infY=cA0+(cAi-cA0)\*(1-(4/math.pi)\*Sum)

Exp\_infN=0.5\*(((3\*u\*(z\*\*2))/(2\*D\*y))\*\*0.5)

cA\_infN=cA0+(cAi-cA0)\*(1-math.erf(Exp\_infN))

print (gamma, u, y, z, delta, kisi, cA\_infY, cA\_infN, NA\_infY, NA\_infN, kc\_infY, kc\_infN, kcbar\_infY, kcbar\_infN, D)

References

1. McCabe, W.L. and J.C. Smith, *Unit Operations of Chemical Engineering*. 2 ed. 1967, New York: McGraw-Hill.

2. Treybal, R.E., *Mass Transfer Operations*. 3rd ed. McGraw-Hill Chemical Engineering Series. 1980: McGraw-Hill.

3. Ali Aroon, M. and M.A. Khansary, *Generalized similarity transformation method applied to partial differential equations (PDEs) in falling film mass transfer.* Computers & Chemical Engineering, 2017. **101**: p. 73-80.

4. Bird, R.B., W.E. Stewart, and E.N. Lightfoot, *Transport Phenomena*. 2006: John Wiley & Sons, Inc.