Chapter 2 Finite Automata

Elements Of The Theory Of Computation
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Goal:

— to define increasingly powerful models of computation, more and more sophisticated devices for

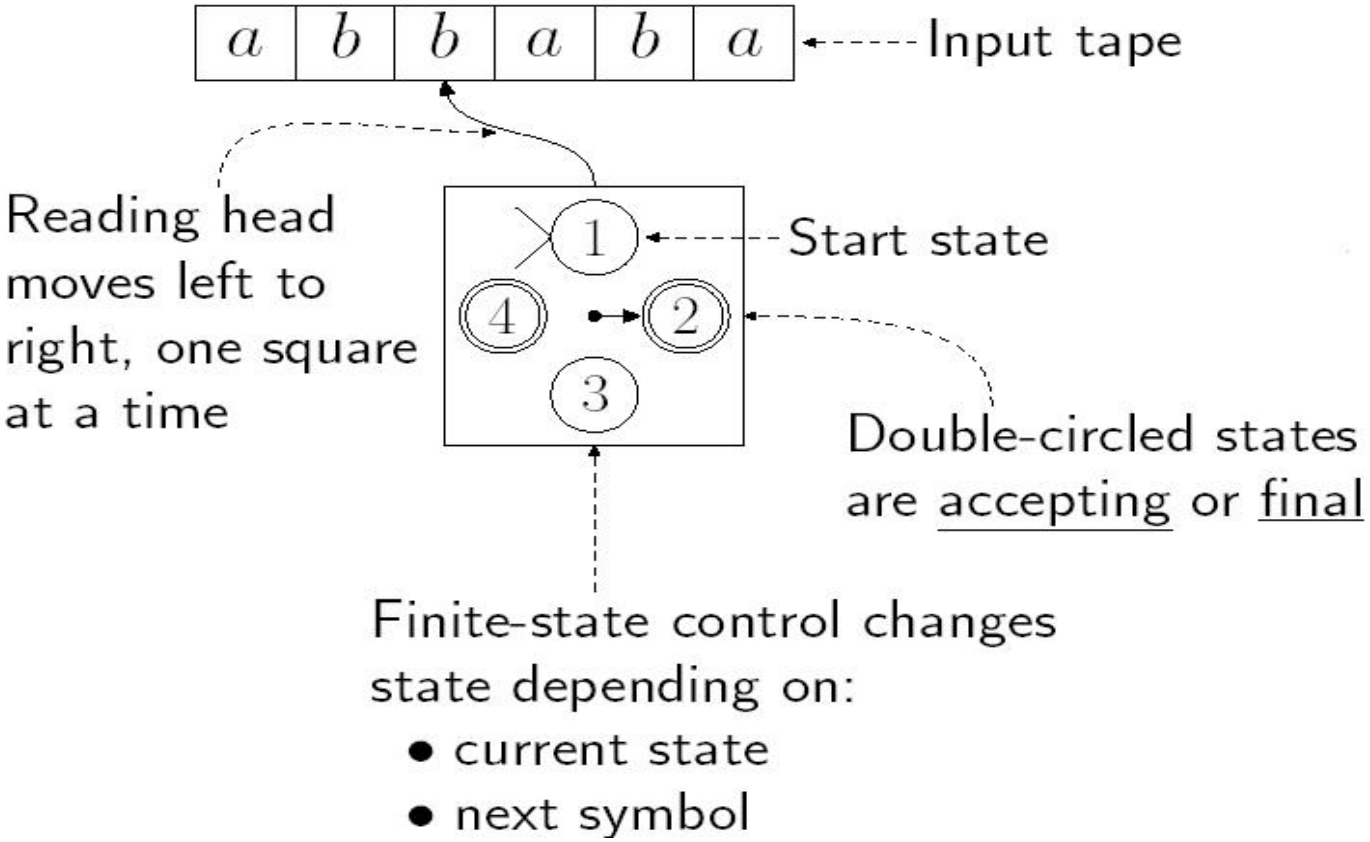
- accepting languages
- generating languages

The Chomsky hierarchy

Language type	Automata type
regular	finite
context-free	pushdown
context-sensitive	linear bounded
unrestricted	Turing Machine

2.1 Deterministic Finite Automata

Introduction to DFA



Definition: A **deterministic finite automata(DFA)** is a quintuple $(K, \sum, \delta, s, F)$, where

- K is a finite set of states
- $\sum$ is an alphabet
- $s \in K$ is the initial state
- $F \subseteq K$ is the set of final states
- δ: transition function, $K \times \sum \rightarrow K$.

Remarks:

Transition function will determine **unique** next state based on current input and state.

Remark:

1) A **configuration** of a DFA $(K, \sum, \delta, s, F)$ is any element of $K \times \sum^*$.

2) The binary relation $\vdash_M$ between two configurations of M:

$(q, w) \vdash_M (q', w') \Leftrightarrow \exists a \in \sum, w = aw'$, and $\delta(q, a) = q'$.

— say (q, w) yields (q', w') in one step.

3) The reflexive, transitive closure of $\vdash_M$:$\vdash_M^*$
$(q, w) \vdash_M^* (q', w') \Leftrightarrow (q, w)$ yields (q', w') after some number, possibly zero, of steps.

4) A **string** $w \in \sum^*$ **is said to be accepted by** M iff there is a state $q \in F$ such that $(s, w) \vdash_M^* (q, e)$.
The language accepted by M, $L(M)$ is the set of all strings accepted by M.

Example: $\sum = \{a, b\}$

$K = \{q_0, q_1\}$

$s = q_0$

$F = \{q_0\}$

Transition function

q	σ	$\delta(q, \sigma)$
q_0	a	q_0
q_0	b	q_1
q_1	a	q_1
q_1	b	q_0

Consider the string $aabba$:

$$q_0 \xrightarrow[a]{} q_0 \xrightarrow[a]{} q_0 \xrightarrow[b]{} q_1 \xrightarrow[b]{} q_0 \xrightarrow[a]{} q_0$$

— q_0 is final, $aabba$ is accepted.

Graphical representation

— State Diagram

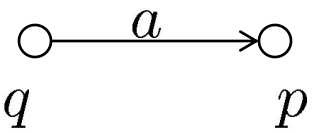
State ○

Initial state >○

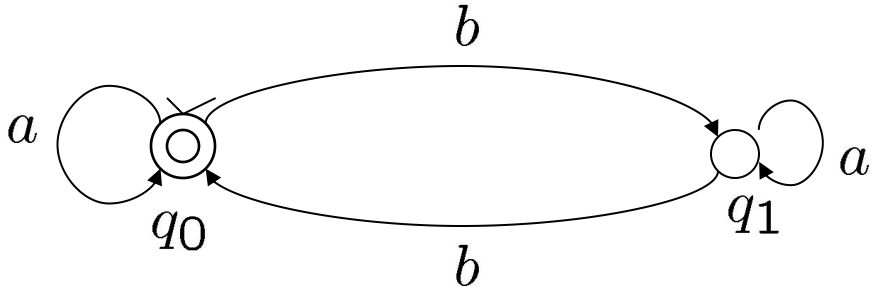
Final state ◎

Transition function:

$\delta(q, a) = p$



The DFA in example can be represented:



Example:

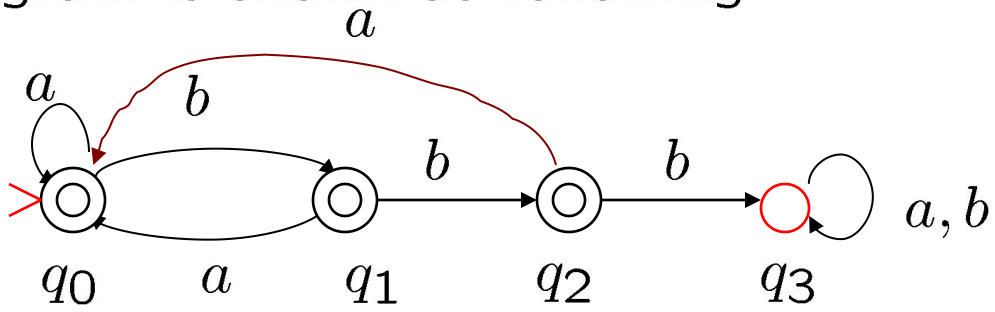
Let us design a DFA M that accepts the language

$L_1 = \{w \in \{a, b\}^* : w$ does not contain three consecutive b's$\}$.

Solution: Let DFA $M = (K, \sum, \delta, s, F)$ where

$$K = \{q_0, q_1, q_2, q_3\}$$
$$\sum = \{a, b\}$$
$$s = q_0$$
$$F = \{q_0, q_1, q_2\}$$

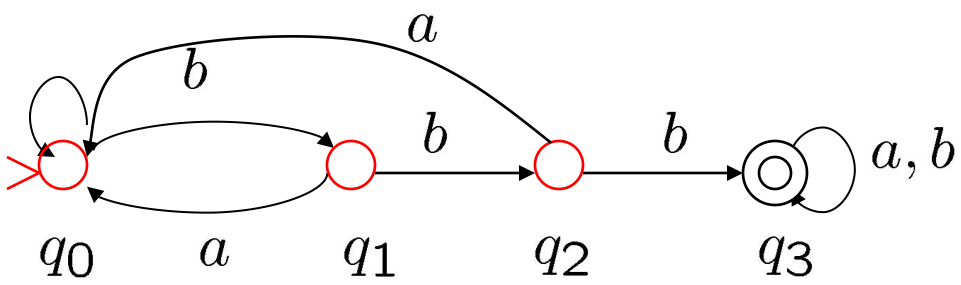
The state diagram is shown as following:



Question:

$L_2 = \{w \in \{a,b\}^* : w$ contains three consecutive b's$\}$.

The DFA that accept language L_2?



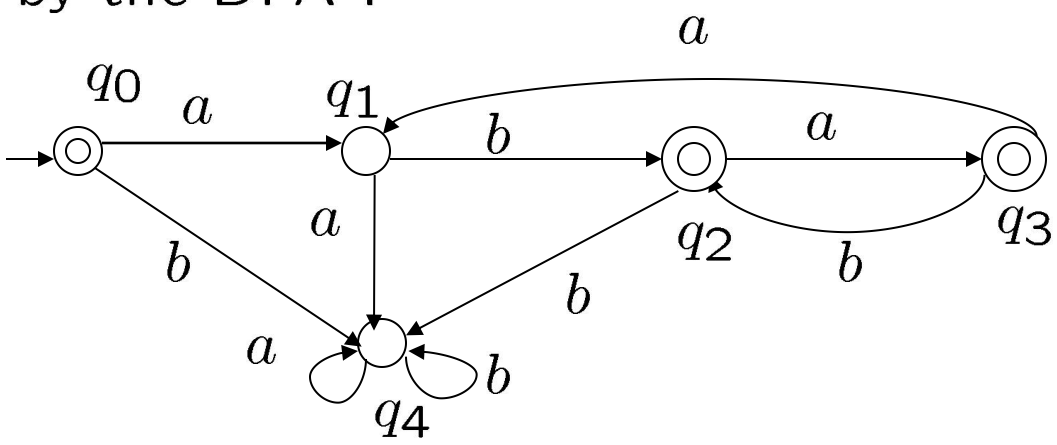
Note: $L_2 = \sum^* - L_1$.

2.2 Nondeterministic Finite Automata

Generalization of determinism:

- Many "next-states"
- Computation is a " tree "
- Acceptance: $\exists$ a path to accepting leaf

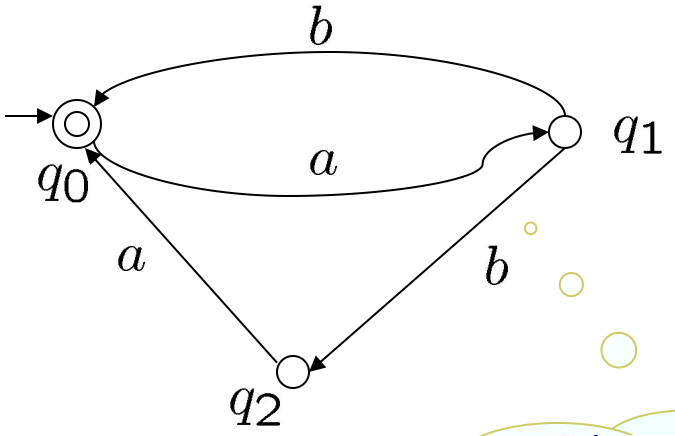
Example: Consider the language $L = (ab \cup aba)^*$ which is accepted by the DFA :



Note:

- No DFA with fewer than **five** states can accept this language.
- It is not easy to ascertain that a DFA is shown.
- L is accepted by the simple Nondeterministic Finite Automata:

$$M = (K, \sum, \delta, s, F), \text{ where}$$
$$K = \{q_0, q_1, q_2\},$$
$$S = \{a, b\},$$
$$s = q_0,$$
$$F = \{q_0\}$$



$$\delta(q_1, b) = \{q_0, q_2\}$$
$$\delta(q_0, b) = \emptyset$$

- Consider input aba
- In state diagram of a NFA arrows that are labeled by the empty string e.

Definition: A **nondeterministic finite automata(NFA)** is a quintuple $(K, \sum, \Delta, s, F)$, where

- K is a finite set of states
- $\sum$ is an alphabet
- $s \in K$ is the initial state
- $F \subseteq K$ is the set of final states
- Δ, transition relation, is a subset of $K \times (\sum \cup \{e\}) \times K$.

Remark:

For DFA, the transition function δ is a function, but for NFA, Δ only a relation.

- triple $(q, u, p) \in \Delta$ — transition of M:

  $u \neq e$

  $u = e$: no input symbol

- configuration — an element of $K \times \sum^*$.

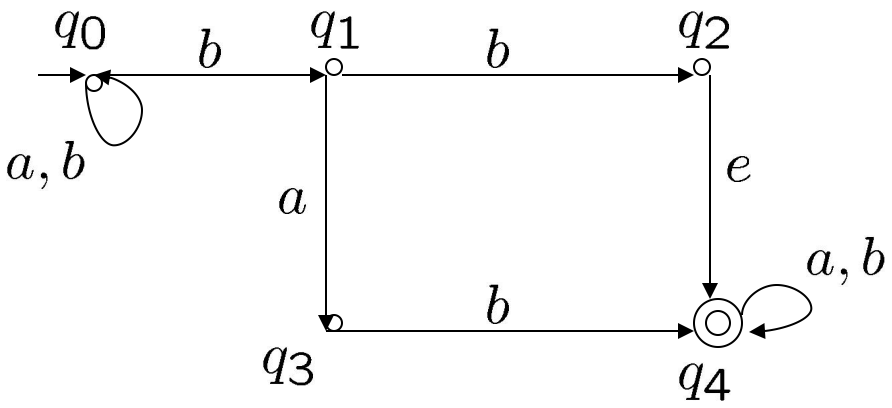
- binary relation $\vdash_M$ and its reflexive transitive closure $\vdash_M^*$.

- A **string** $w \in \sum^*$ **is accepted by** M iff there is a state $q \in F$ such that $(s, w) \vdash_M^* (q, e)$.

  The **language accepted by** M, $L(M)$ is the set of all strings accepted by M.

Example:

$L = \{w \in \{a,b\}^* : w$ contains the pattern bb or $bab\}$.

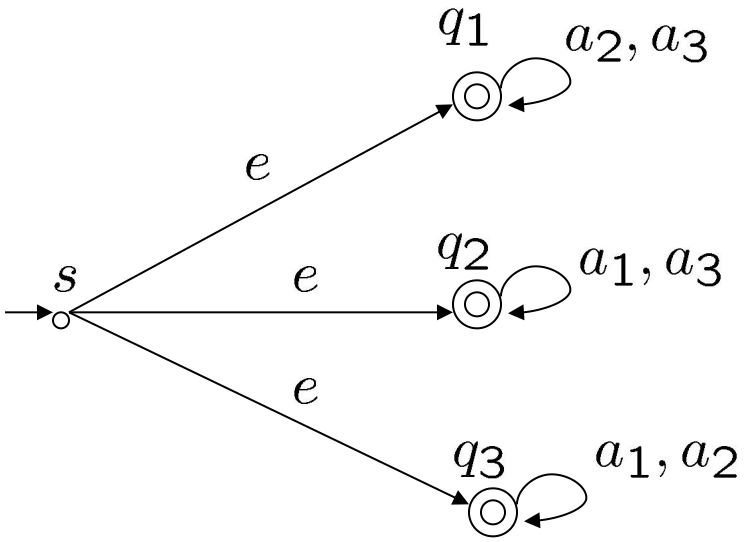


- Consider the string $bababab \in L$?

Example:

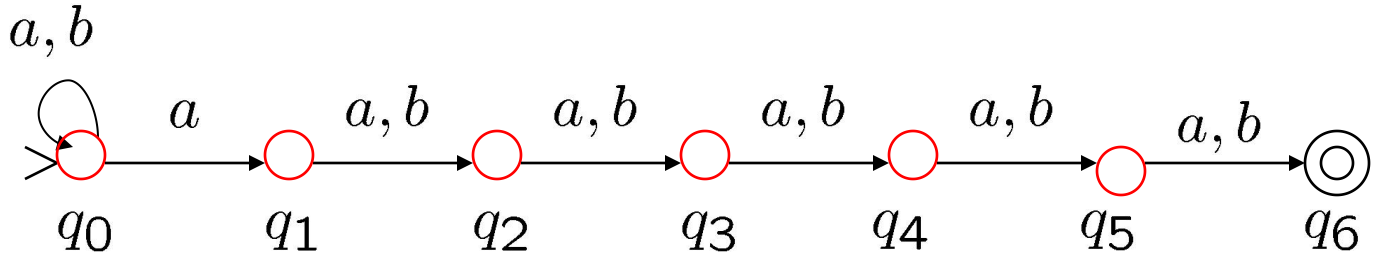
$\sum = \{a_1, \cdots, a_n\}$, where $n \geq 2$.

$L = \{w \in \sum^* : \exists a_i$ not appearing in $w\}$.



Example:

$L = \{w \in \{a,b\}^* : w \text{ } 6^{th} \text{ symbol from end is an } \text{“}a\text{”}\}$.

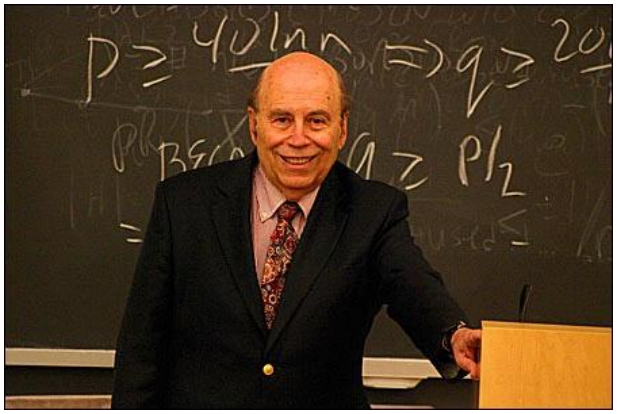


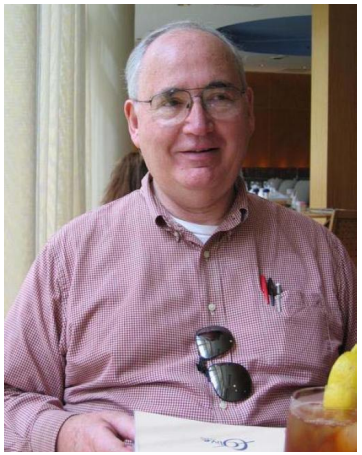
Expressive power of NFAs vs. DFAs

- Every DFA is an NFA
- NFAs allow more transitions

Is the language more expressive?

– *Is there a language* $L = L(M)$ *for some NFA* M*, where* $L = L(M')$ *for some DFA* M'*?*





Michael Oser Rabin (1931–) **Dana Stewart Scott** (1932—)

Turing Award Citation (1976): For their joint paper "Finite Automata and Their Decision Problem," which introduced the idea of nondeterministic machines, which has proved to be an enormously valuable concept. Their (Scott & Rabin) classic paper has been a continuous source of inspiration for subsequent work in this field.

NFA/DFA Equivalence

Definition: Two FA M_1 and M_2 (deterministic or non deterministic) are **equivalent** iff $L(M_1) = L(M_2)$.

- To prove equivalence of DFAs and NFAs we must do two things:

> *I: For each DFA, produce an NFA that accepts the same language*
>
> *II: For each NFA, produce a DFA that accepts the same language*

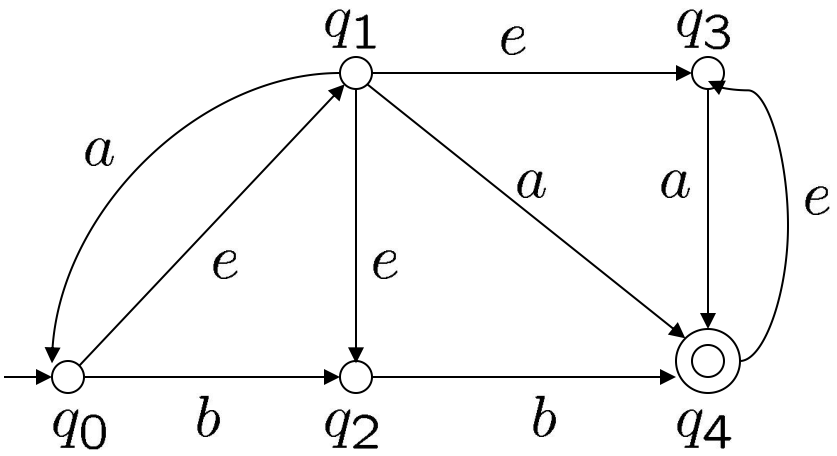
We'll do part II; part I is immediate.

Theorem: For each NFA, there is an equivalent DFA.

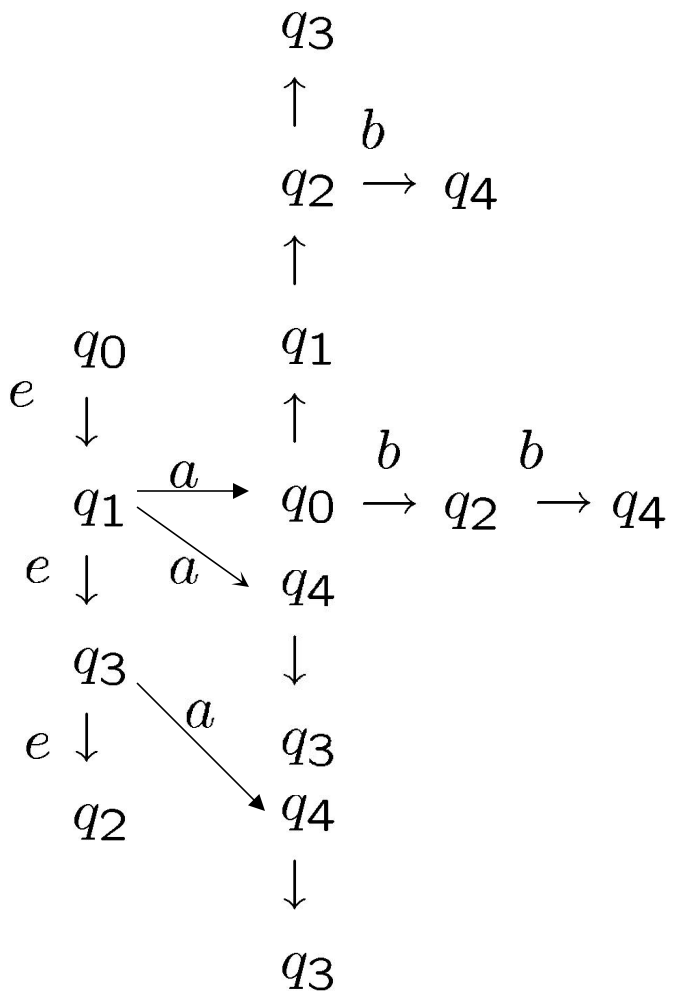
Proof:

Key idea: Every subset of K becomes a single state in our new machine!

For Example:
NFA M, $K = \{q_0, q_1, q_2, q_3, q_4\}$, after reading a certain input string, M could be in $\{q_0, q_2, q_3\}$, not in $\{q_1, q_4\}$, then DFA M' is in state $\{q_0, q_2, q_3\}$.



$$w = abb$$



NFA $\Rightarrow$ DFA construction

Consider an arbitrary NFA $M = (K, \sum, \Delta, s, F)$.
To construct an equivalent DFA $M' = (K', \sum, \delta, s', F')$.

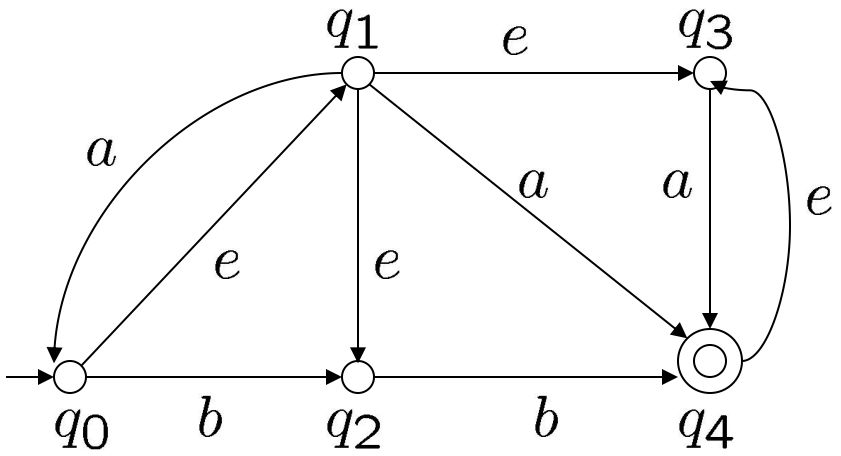
Definition:

$$E(q) = \{p \in K : (q, e) \vdash_M^* (p, e)\}$$

Remark:

$E(q)$— the set of all states of M that are reachable from state q without reading any input.

Example:



$$E(q_0) = \{q_0, q_1, q_2, q_3\} \qquad E(q_1) = \{q_1, q_2, q_3\}$$

$$E(q_2) = \{q_2\} \qquad E(q_3) = \{q_3\}$$

$$E(q_4) = \{q_3, q_4\}$$

I. Formal Definition of DFA

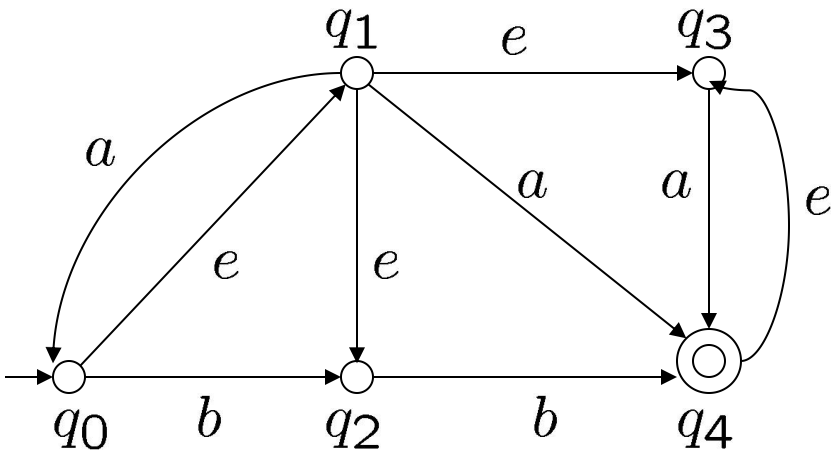
NFA $M = (K, \sum, \Delta, s, F)$.

To construct an equivalent DFA $M' = (K', \sum, \delta, s', F')$.

- Let $K' = 2^K$
- Let $s' = E(s)$
- Let $F' = \{Q \mid Q \subseteq K, Q \cap F \neq \emptyset\}$
- For each $Q \subseteq K$ and $\forall a \in \sum$,

  Let $\delta(Q, a) = \cup\{E(p) \mid p \in K$ and $(q, a, p) \in \Delta$ for some $q \in Q\}$

Example: (continued)



$$E(q_0) = \{q_0, q_1, q_2, q_3\}$$
$$E(q_1) = \{q_1, q_2, q_3\}$$
$$E(q_2) = \{q_2\}$$
$$E(q_3) = \{q_3\}$$
$$E(q_4) = \{q_3, q_4\},$$

$$s' = E(q_0) = \{q_0, q_1, q_2, q_3\}$$

since $(q_1, a, q_4), (q_1, a, q_0) \in \Delta$

$$\delta(\{q_1\}, a) = E(q_0) \cup E(q_4) = \{q_0, q_1, q_2, q_3, q_4\}$$

II. **NFA $\Rightarrow$ DFA construction:** verification

We next show:

- M' is deterministic
- M' is equivalent to M.

- **M' is deterministic:**

□ δ is a single-valued and well defined on all $Q \in K'$ and $a \in \sum$ by the way it is constructed.

□ For some $Q \in K'$ and $a \in \sum$, $\delta(Q, a) = \emptyset$ does not mean δ is not well defined; $\emptyset$ is a member of K'.

- M' **is equivalent to** M**.**

Claim: For any string $w \in \sum^*$ and any states $p, q \in K$,

$$(q, w) \vdash_M^* (p, e) \Leftrightarrow (E(q), w) \vdash_{M'}^* (P, e)$$

for some set P containing p.

Consider any string $w \in \sum^*$.

definition

claim

$$\begin{aligned} w \in L(M) &\Leftrightarrow (s, w) \vdash_M^* (f, e) \text{ for some } f \in F \\ &\Leftrightarrow (E(s), w) \vdash_{M'}^* (Q, e) \text{ for some } f \in Q \\ &\Leftrightarrow (s', w) \vdash_{M'}^* (Q, e) \text{ with } Q \in F' \\ &\Leftrightarrow w \in L(M') \end{aligned}$$

From the claim, M' is equivalent to M.

- **Proof of Claim:**

  Basis step:

  For $|w| = 0$, i.e. $w = e$, we must show that

  $$(q, e) \vdash_M^* (p, e) \Leftrightarrow (E(q), e) \vdash_{M'}^* (P, e)$$

  for some set P containing p.

  $$(q, e) \vdash_M^* (p, e) \Leftrightarrow p \in E(q)$$

  M' is deterministic
  $(E(q), e) \vdash_{M'}^* (P, e)$ $\Leftrightarrow P = E(q)$

  $$p \in P \Rightarrow p \in E(q)$$

Induction hypothesis:

Suppose the claim is true for all string w of length k or less for some $k \geq 0$.

Induction step:

Let $|w| = k + 1$. $w = va$, where $a \in \sum, v \in \sum^*$.

$\Rightarrow$

$$(q, w) \vdash_M^* (p, e) \Leftrightarrow (E(q), w) \vdash_{M'}^* (P, e)$$

$(q, w) \vdash_M^* (p, e) \Leftrightarrow \exists$ states r_1 and r_2 such that

$$(q, va) \vdash_M^* (r_1, a) \vdash_M (r_2, e) \vdash_M^* (p, e)$$

I $(q, va) \vdash_M^* (r_1, a) \Leftrightarrow (q, v) \vdash_M^* (r_1, e)$

$\Leftrightarrow (E(q), v) \vdash_{M'}^* (R_1, e)$ for some $R_1, r_1 \in R_1$

Induction hypothesis

II $(r_1, a) \vdash_M (r_2, e) \Leftrightarrow (r_1, a, r_2) \in \Delta$

$\Rightarrow E(r_2) \subseteq \delta(R_1, a)$

Definition of $\delta(R_1, a)$

$p \in \delta(R_1, a)$

III $(r_2, e) \vdash_M^* (p, e) \Rightarrow p \in E(r_2)$

$\Rightarrow (E(q), va) \vdash_{M'}^* (R_1, a) \vdash_{M'}^* (P, e).$

$P = \delta(R_1, a)$

$\Leftarrow$

- $(E(q), va) \vdash^*_{M'} (R_1, e) \vdash^*_{M'} (P, e)$ for some $p \in P$
- $(R_1, a) \vdash^*_{M'} (P, e) \Rightarrow P = \delta(R_1, a)$
- $\delta(R_1, a) = \cup\{E(r_2) : \exists r_2 \in K$ and $(r_1, a, r_2) \in \Delta$ for some $r_1 \in R_1\}$

$p \in P \Rightarrow \exists r_2$ such that $p \in E(r_2)$ and for smoe $r_1 \in R_1, (r_1, a, r_2) \in \Delta$

- By IH, $(q, va) \vdash^*_M (r_1, a)$

Therefore, $(q, va) \vdash^*_M (r_1, a) \vdash^*_M (r_2, e) \vdash^*_M (p, e)$.

Remark:

1) Size of new DFA can be exponential in size of old NFA!

2) The proof of theorem provides an actual algorithm for constructing an equivalent DFA from any NFA.

Example:

Convert the NFA in example(this section) to a DFA.

$|K| = 5 \Rightarrow |K'| = 2^5 = 32.$

— *Only a few of these states will be relevant to the operation of M.*

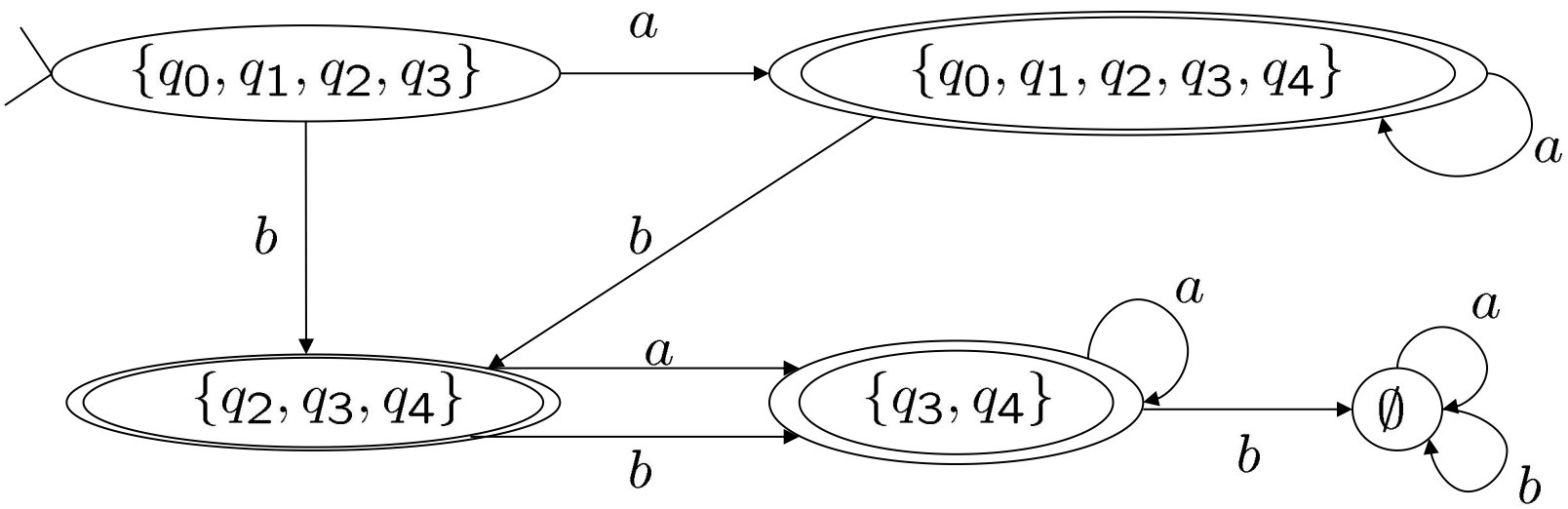
(q_1, a, q_0),(q_1, a, q_4),(q_3, a, q_4) are all the transitions for some $q \in s'$.

$s' = E(q_0) = \{q_0, q_1, q_2, q_3\}$

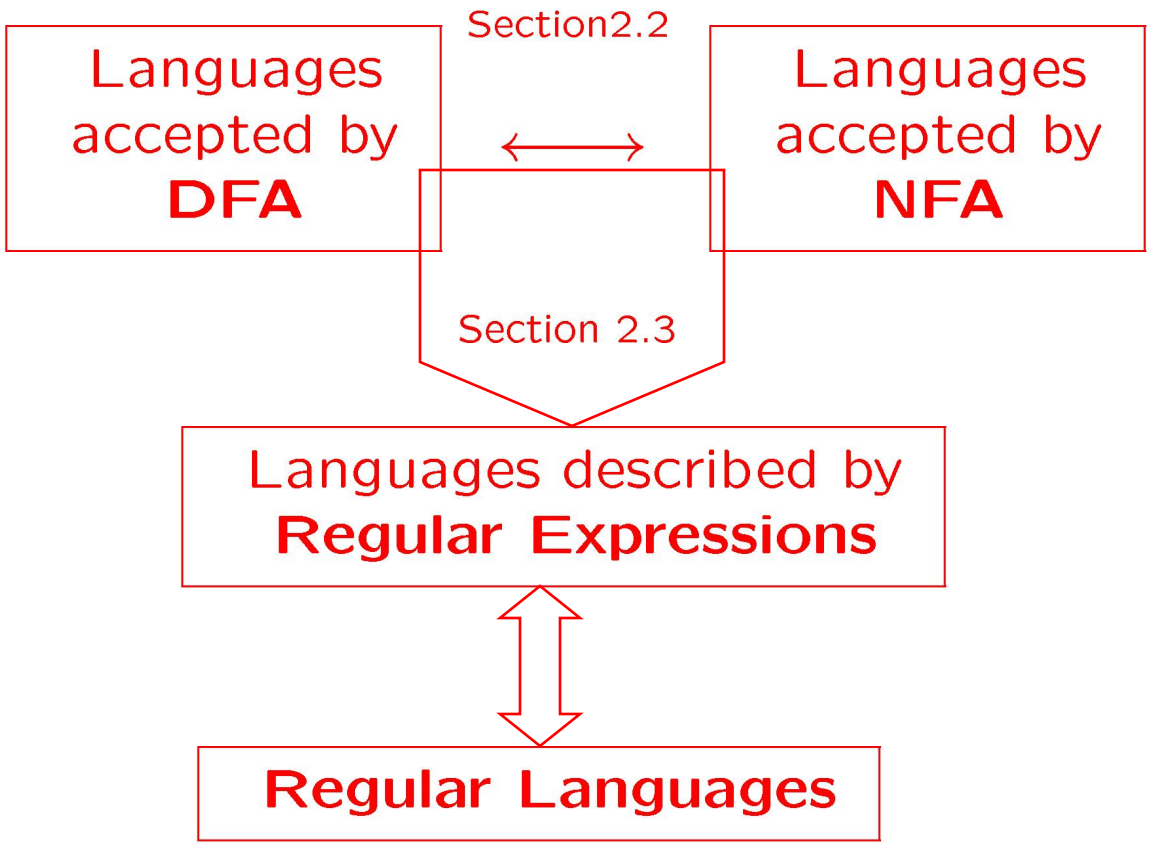
$\delta(s', a) = E(q_0) \cup E(q_4) = \{q_0, q_1, q_2, q_3, q_4\}$

$\delta(s, b) = E(q_2) \cup E(q_4) = \{q_2, q_3, q_4\}$

$$\delta(\{q_0,q_1,q_2,q_3,q_4\},a)=\{q_0,q_1,q_2,q_3,q_4\}$$
$$\delta(\{q_0,q_1,q_2,q_3,q_4\},b)=\{q_2,q_3,q_4\}$$
$$\delta(\{q_2,q_3,q_4\},a)=\{q_3,q_4\}$$
$$\delta(\{q_2,q_3,q_4\},b)=\{q_3,q_4\}$$
$$\delta(\{q_3,q_4\},a)=\{q_3,q_4\},\ \delta(\{q_3,q_4\},b)=\emptyset$$
$$\delta(\emptyset,a)=\delta(\emptyset,b)=\emptyset$$



2.3 Finite Automata & Regular Expressions



Closure Properties of Regular Languages

Theorem: The class of languages accepted by FA is **closed** under

(a) Union

(b) Concatenation

(c) Kleene star

(d) Complementation

(e) Intersection

Proof:

Main idea To construct an FA that accept the appropriate language in each case.

(a) Union:

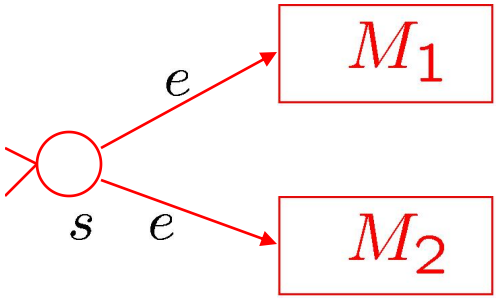
Let $M_i = (K_i, \sum, \Delta_i, s_i, F_i)$ $(i = 1, 2)$ be two NFA.

$\Rightarrow$ construct a NFA $M = (K, \sum, \Delta, s, F)$, such that

$$L(M) = L(M_1) \cup L(M_2).$$

Where

- $K = K_1 \cup K_2 \cup \{s\}$
- $F = F_1 \cup F_2$
- $\Delta = \Delta_1 \cup \Delta_2 \cup \{(s, e, s_1), (s, e, s_2)\}$



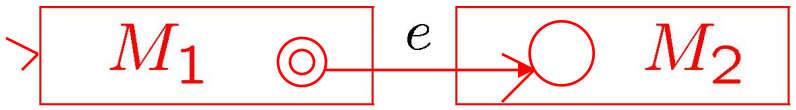
Note: Any finite union of regular sets is regular. Inifinite unions may not be regular.

(b) Concatenation:

Let $M_1 = (K_1, \sum, \Delta_1, s_1, F_1)$ and $M_2 = (K_2, \sum, \Delta_2, s_2, F_2)$ be two NFA.

$\Rightarrow$ construct a NFA $M = (K, \sum, \Delta, s, F)$, such that

$$L(M) = L(M_1) \circ L(M_2).$$



Where

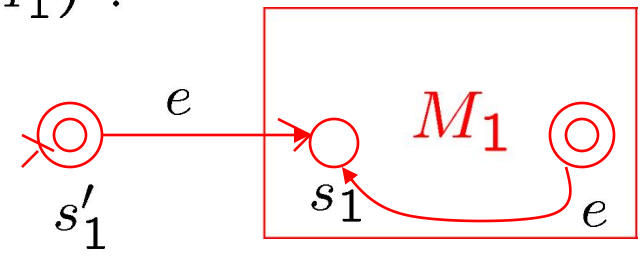
- $K = K_1 \cup K_2$
- $F = F_2$
- $\Delta = \Delta_1 \cup \Delta_2 \cup \{(q_i, e, s_2) : q_i \in F_1\}$

(c) Kleene star:

Let $M_1 = (K_1, \sum, \Delta_1, s_1, F_1)$ be a NFA.

$\Rightarrow$ construct a NFA $M = (K, \sum, \Delta, s, F)$, such that

$$L(M) = L(M_1)^*.$$



Where

- $K = K_1 \cup \{s_1'\}$
- $F = F_1 \cup \{s_1'\}$
- $\Delta = \Delta_1 \cup \{(q_i, e, s_1) : q_i \in F_1\} \cup \{(s_1', e, s_1)\}$

(d) Complementation:

Let $M_1 = (K_1, \sum, \delta_1, s_1, F_1)$ be a DFA.

$\Rightarrow$ construct a DFA $M = (K, \sum, \delta, s, F)$, such that

$$L(M) = \overline{L(M_1)} = \sum^* - L(M_1).$$

Where

- $K = K_1$
- $s = s_1$
- $F = K - F_1$
- $\delta = \delta_1$

(e) Intersection:

$$L(M_1) \cap L(M_2) = \sum^* - (\sum^* - L(M_1)) \cup (\sum^* - L(M_2))$$

Let $M_i = (K_i, \sum, \delta_i, s_i, F_i)$ $(i = 1, 2)$ be two DFA.

$\Rightarrow$ construct a DFA $M = (K, \sum, \delta, s, F)$, such that

$$L(M) = L(M_1) \cap L(M_2).$$

Where

- $K = K_1 \times K_2$
- $s = (s_1, s_2)$
- $F = F_1 \times F_2$
- $\delta : K \times \sum \to K$

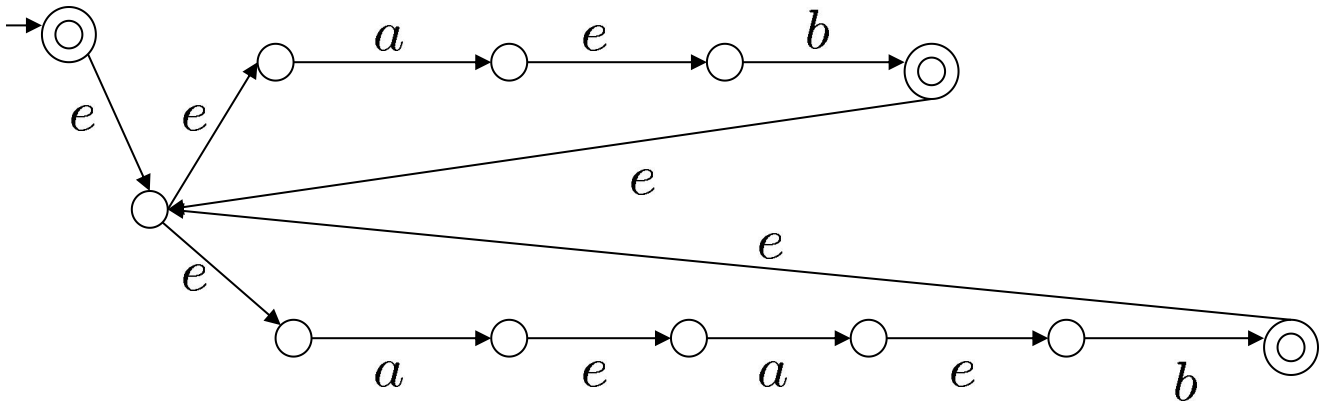
$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$$

"parallel" simulation!

How about $L(M_1) \cup L(M_2)$?

Example:

Consider the regular expression $(ab \cup aab)^*$. A NFA accepting the language denoted by this regular expression can be shown as following:



□ FA and Regular Express are equivalent

Theorem: A language is regular iff it is accepted by a FA.

Proof: $\Rightarrow$

The class of regular languages is the smallest class of languages containing $\emptyset$ and the singletons $\{a|a \in \sum\}$, and closed under **union**, **concatenation**, and **Kleene star**.

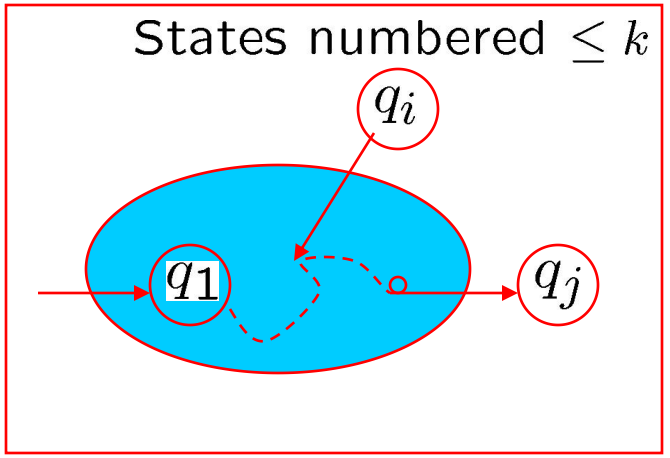
$\emptyset$ and the singletons $\{a|a \in \sum\}$ are accepted by FA.

$\Rightarrow$ every regular language is accepted by some FA.

Let $M = (K, \sum, \Delta, s, F)$ be a FA.
To construct a regular expression R, s. t. $L(M) = L(R)$.

Let $K = \{q_1, \cdots, q_n\}$ and $s = q_1$.

We'll do induction over k for executions that use only states in $\{q_1, \cdots, q_k\}$



Definition: For $i, j = 1, \cdots, n$ and $k = 0, \cdots, n$, define

$$R(i,j,k) = \{w \mid w \in \sum^*, \delta(q_i, w) = q_j \text{ and for any prefix } x \text{ of } w,\ x \neq e,\ \delta(q_i, x) = q_l \wedge (l \leq k)\}$$

$$R(i,j,0) = \begin{cases} \{a \mid \delta(q_i, a) = q_j\} & \text{if } i \neq j \\ \{a \mid \delta(q_i, a) = q_j\} \cup \{e\} & \text{if } i = j \end{cases}$$

Note:

$$R(i,j,n) = \{\ w \mid w \in \sum^*, (q_i, w) \vdash_M^* (q_j, e)\ \}$$

$$L(M) = \cup\{\ R(1,j,n) \mid q_j \in F\ \}$$

Theorem: $R(i,j,k)$ are regular languages.

Proof: By induction on k.

Basis step: $k = 0$

$$R(i,j,0) = \begin{cases} \{a|\ \delta(q_i,a) = q_j\} & \text{if } i \neq j \\ \{a|\ \delta(q_i,a) = q_j\} \cup \{e\} & \text{if } i = j \end{cases}$$

Induction step:

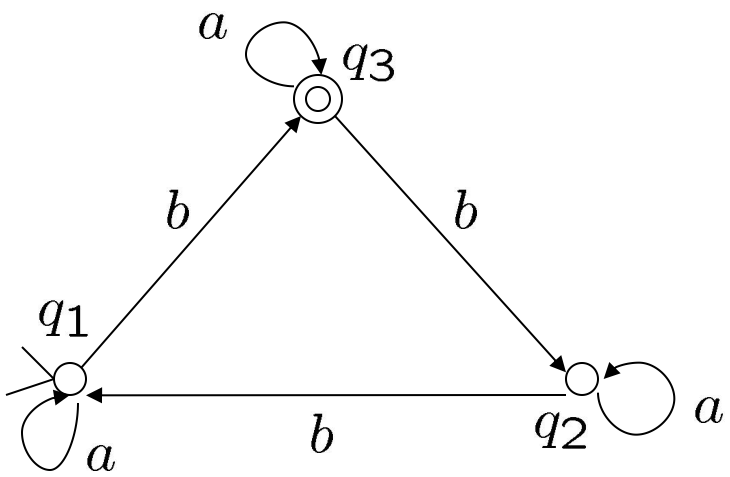
$$\begin{aligned} R(i,j,k) =& R(i,j,k-1) \cup R(i,k,k-1)R(k,k,k-1)^* \\ & R(k,j,k-1) \end{aligned}$$

$\Rightarrow$ $R(i,j,k)$ is regular for all i,j,k.

Example: Construct a regular expression for the language

$$L = \{w \in \{a,b\}^* : w \text{ has } 3k+1 \ b\text{'s for some } k \in \mathbb{N}\}$$

accepted by the following DFA.



Assume that the DFA M has two simple properties:

1) It has a single final state, $F = \{f\}$.

2) If $(q, u, p) \in \Delta$, then $q \neq f$, and $p \neq s$.

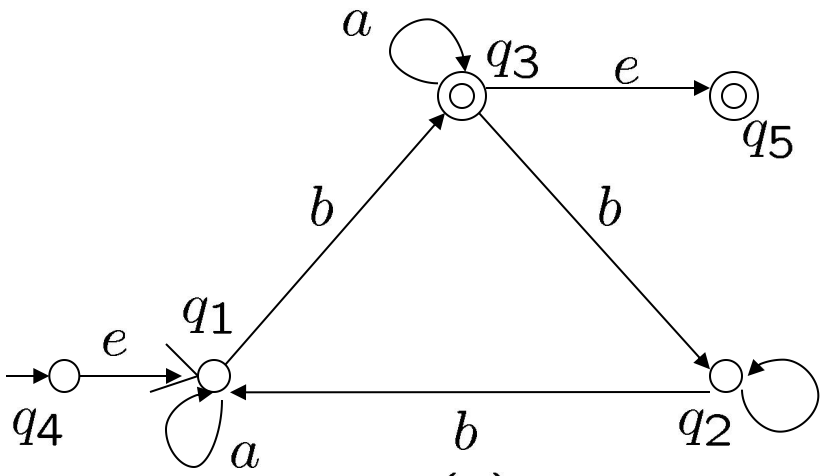
Any DFA $\Rightarrow$ An equivalent FA with properties 1)2).

Let $K = \{q_1, q_2, \cdots, q_n\}$, so that $s = q_{n-1}, f = q_n$.

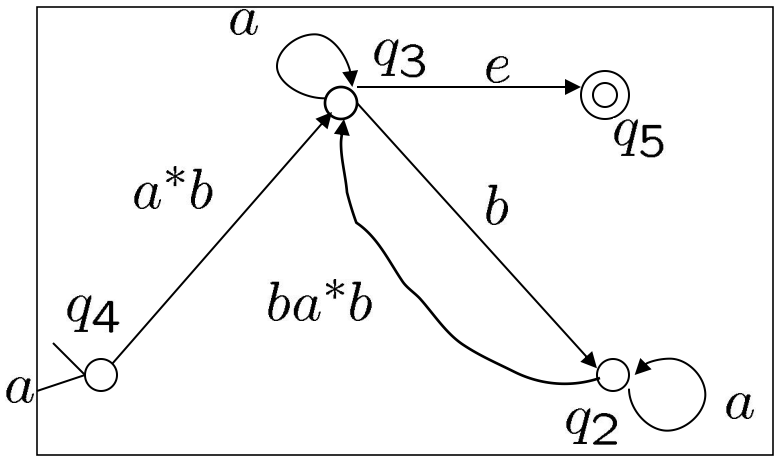
$\Rightarrow$ The regular expression $R(n-1, n, n) = L(M)$.

We compute

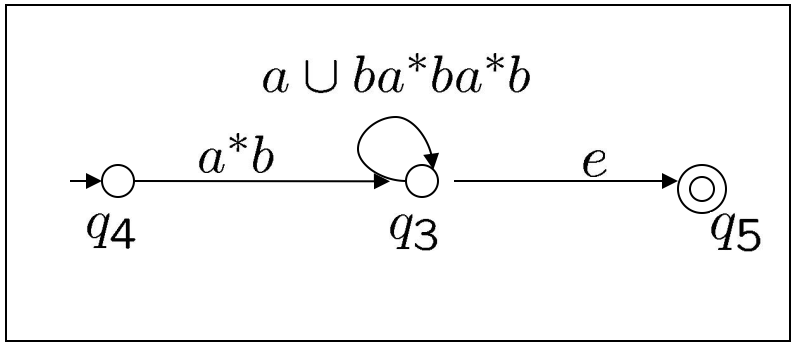
$$R(i, j, 0) \rightarrow R(i, j, 1) \rightarrow \cdots \rightarrow R(i, j, n)$$


a
q3
e
q5
b
b
q1
e
q4
a
b
q2

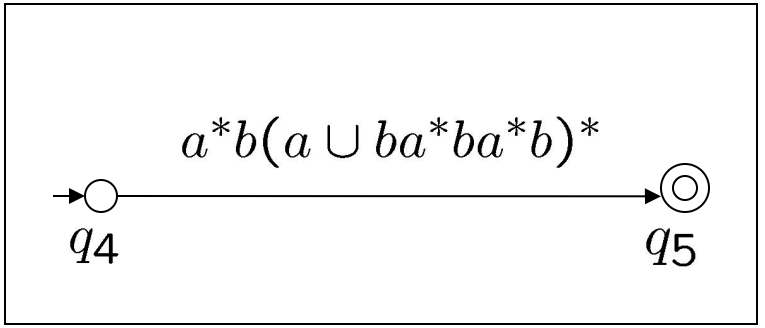
(a)


a
q3
e
q5
a*b
b
q4
ba*b
a
q2
a

(b)


a ∪ ba*ba*b
a*b
e
q4
q3
q5

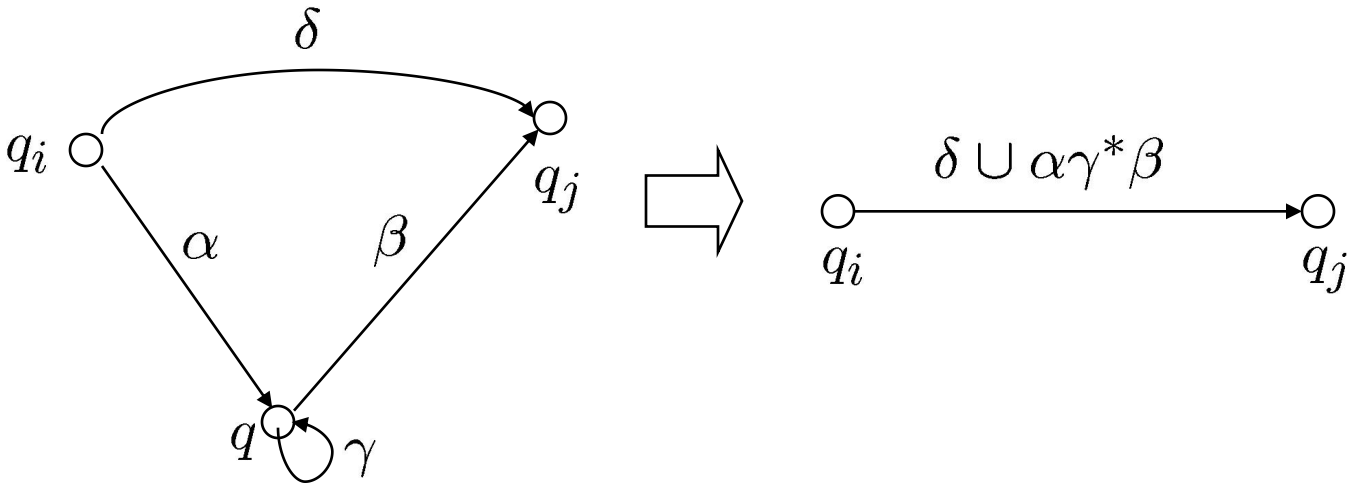
(c)


a*b(a ∪ ba*ba*b)*
q4
q5

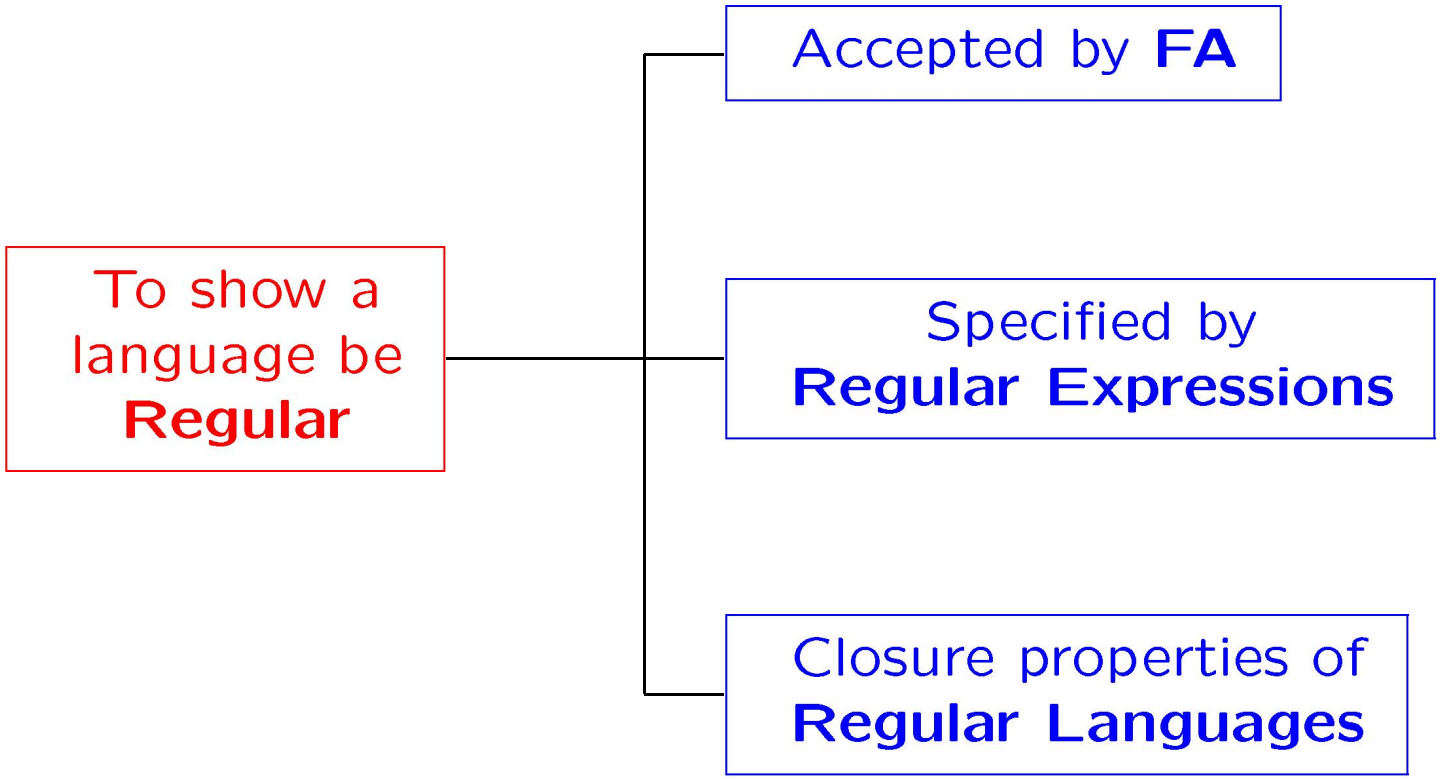
(d)

Remark:

In general, for each pair of states $q_i \neq q$ and $q_j \neq q$, eliminate state q as following:



2.4 Languages that are and are not Regular



Example: Let $\sum = \{0, 1, \cdots, 9\}$.
$L = \{w : w \in \sum^*$ be the decimal representation for non-negative integers(without redundant leading 0's) divisible by 2 and 3$\}$. Then L is regular.

Solution:

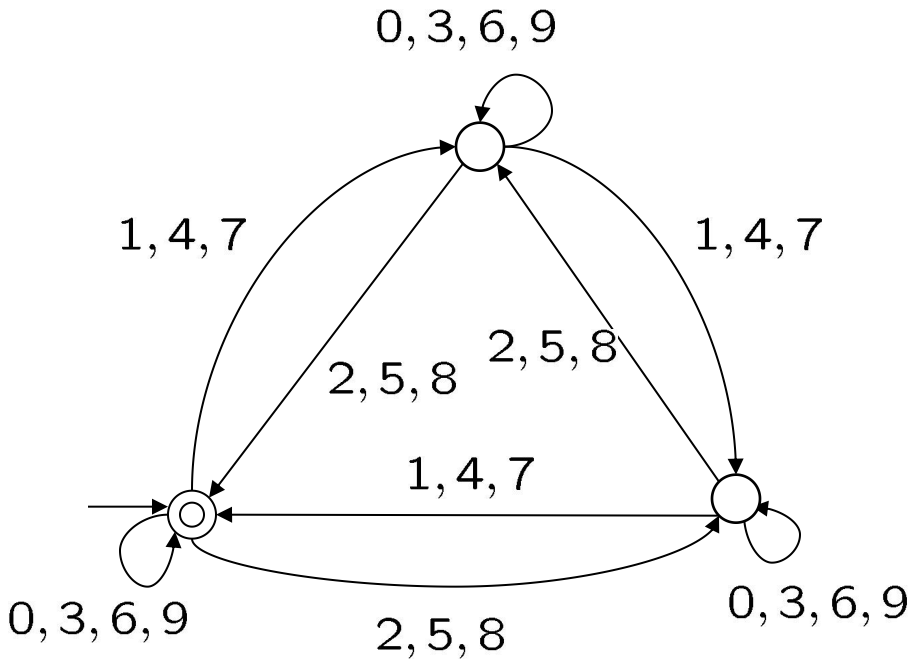
- Let L_1 be the set of decimal representation of nonnegative integers.

$$L_1 = 0 \cup \{1, 2, \cdots, 9\} \sum^*.$$

- Let L_2 be the set of decimal representation of nonnegative integers divisible by 2.

$$L_2 = L_1 \cap \sum^* \{0, 2, 4, 6, 8\}.$$

• Let L_3 be the set of decimal representation of nonnegative integers divisible by 3. L_3 can be accepted by the following FA.



$$L = L_2 \cap L_3$$

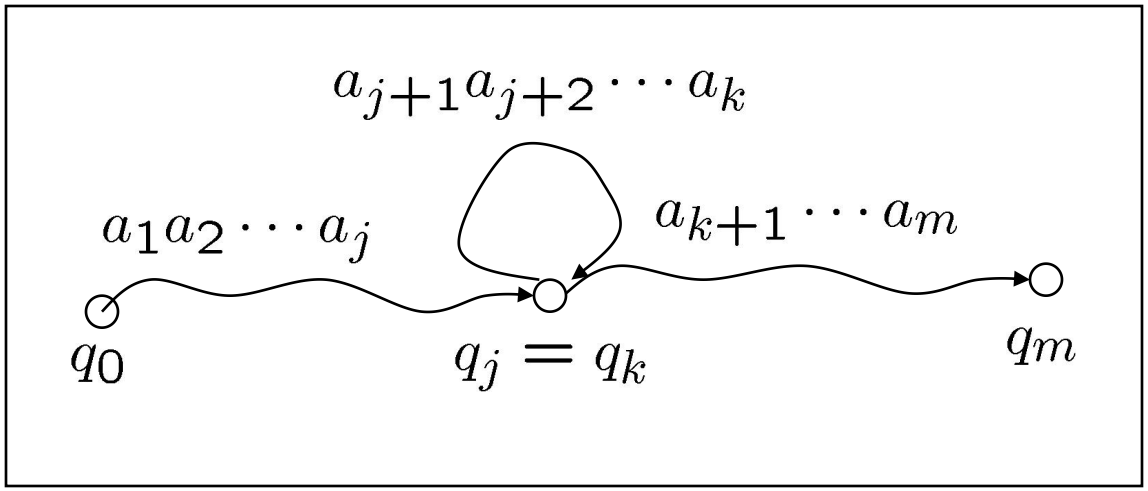
– L is regualar.

What languages are not regular?

Intuition:
Since a FA has only finite state, it can "remember" only a finite number of things

- Some things we would expect are not regular
  - *Balanced parentheses (have to remember an arbitrary nesting depth)*
  - $\{a^n b^n : n \geq 0\}$

If a FA accepts a string that is "long enough", it must repeat a state



– $\{a^n : n \geq 1$ *is a prime*$\}$ *is not regular.*

Theorem: (Pumping Theorem) Let L be a regular language. $\exists$ an integer $n \geq 1$ such that any string $w \in L$ with $|w| \geq n$ can be written as $w = xyz$ such that

- $y \neq e$
- $|xy| \leq n$
- for each $i \geq 0$ $xy^iz \in L$

Proof:

Let L be a regular language, accepted by a DFA M with n states, and string w, $|w| \geq n$.

Consider the first n steps of computation of M:

$$(q_0, a_1 \cdots a_n) \vdash_M (q_1, a_2 \cdots a_n) \vdash_M \cdots \vdash_M (q_n, e)$$

Pigeonhole Principle

$\Rightarrow \exists i, j,\ 0 \leq i < j \leq n$, such that $q_i = q_j$

Let $y = a_i \cdots a_j, x = a_1 \cdots a_{i-1}, z = a_{j+1} \cdots a_m$.

Example: Show $L = \{a^i b^i | i \geq 0\}$ is not regular.

Proof:

Assume L regular. By Pumping Theorem, $\exists$ integer n.

Consider the string $w = a^n b^n \in L$.

It can be written as $w = xyz$ such that

- $|xy| \leq n$
- $y \neq e$

$\Rightarrow y = a^i$ for some $i > 0$

But $xz = a^{n-i} b^n \notin L$

contradiction!!

Remark: Proving that a language is not regular:

- Let L be the proposed regular language
- There is some n, by the pumping lemma
- Choose a string s, longer than n symbols, in the language L
- Using the pumping lemma, construct a new string s' that is not in the language

Example: Show $L = \{a^n | n \text{ is prime } \}$ is not regular.

Proof:

Assume L regular. $\exists w = xyz$, and $x = a^p$, $y = a^q$ and $z = a^r$, where $p, r \geq 0$ and $q > 0$.

By Pumping theorem, $xy^n z \in L$ for each $n \geq 0$; that is, $p + nq + r$ is prime for each $n \geq 0$.

But it is impossible; for let $n = p + 2q + r + 2$; then $p + nq + r = (q + 1) \cdot (p + 2q + r)$.

contradiction!!

Example: Show

$L = \{w \in \{a, b\}^* : w$ has an equal number of a's and b's$\}$

is not regular.

Proof:

If L is regular, then so would be $L \cap a^*b^*$.

Closure under intersection of regular language

But $L \cap a^*b^* = \{a^n b^n : n \geq 0\}$

— not regular language.

L is not regular.

Reprise on FA and Regular Languages

Which of the following are necessarily regular?

- A finite language
- A union of a finite number of regular languages
- A union of a countable number of regular languages
- An intersection of a countable number of regular languages
- $\{x : x \in L_1 \text{ and } x \notin L_2\}$, L_1 and L_2 are both regular
- A subset of a regular language

$$\bigcup_{i=0}^{\infty} L_i = \overline{\bigcap_{i=0}^{\infty} \overline{L_i}}$$

2.5 State Minimization*

Objective:

To find the smallest DFA equivalent to a given DFA

- *Smallest: Fewest states*
- *Equivalent: Accepts the same language*

Our method:

We will show that for any regular language L, there is a unique, "standard" DFA associated with L (up to renaming the states)

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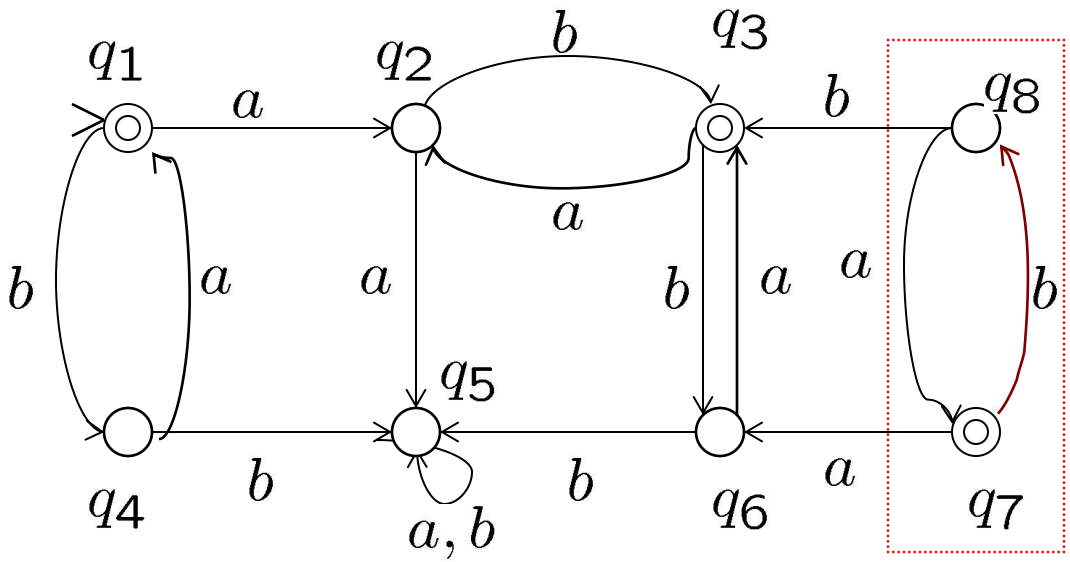
* Graduated course

Question: Minimization the following DFA

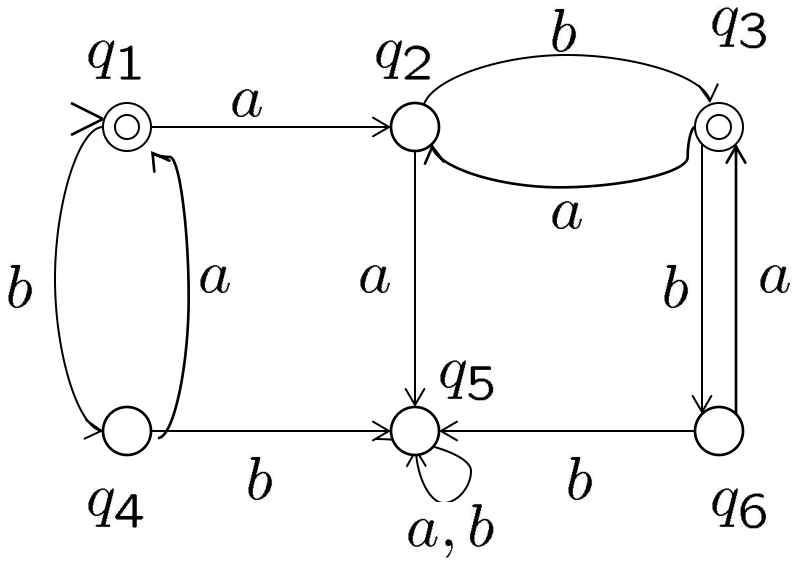


- The DFA accepts language

$$L = (ab \cup ba)^*$$

- States q_7 and q_8: — *unreachable.*

Step 1. Remove all unreachable states



Identifying the reachable states is easy to do in polynomial time.

- Compute the set of reachable states

— the transitive closure of $\{s\}$ under the relation $\{(p, q) : \delta(p, a) = q$ for some $a \in \sum\}$.

Algorithm: Computing the set of reachable states

$R := \{s\}$
While $\exists$ a state $p \in R$ and $a \in \sum$ such that $\delta(p, a) \notin R$ **do**
    add $\delta(p, a)$ to R

Step 2. Merge the equivalent states

- Consider states q_4 and q_6.
  — equivalent

- Intuitively, the states are equivalent:

– from either state, precisely the same string lead the automaton to acceptance.

□ Decomposition of $\sum^*$ into Equivalence Classes with respect to Language

---

Definition Let $L \subseteq \sum^*$ be a language, and let $x, y \in \sum^*$. We say x and y are **equivalent with respect to** L, denoted $\boxed{x \approx_L y}$, if for all $z \in \sum^*$, $xz \in L$ iff $yz \in L$.

$\boxed{z = e}$

Remark:

- $x \approx_L y$ if either both strings belong to L or neither is in L
- $x \approx_L y$: appending any fixed string to both x and y results in two strings are either both in L or both not in L

Example: If $x \in \sum^*$, denote by $[x]$ the equivalence class with respect to L, where $L = (ab \cup ba)^*$.

It is not hard to see that $\approx_L$ has four equivalence classes:

1) $[e] = L$
2) $[a] = La$
3) $[b] = Lb$
4) $[aa] = L(aa \cup bb) \sum^*$.

Decomposition via a DFA

Definition Let $M = (K, \sum, \delta, s, F)$ be a DFA. We say two strings $x, y \in \sum^*$ are **equivalent with respect to** M, denoted $\boxed{x \sim_M y}$, if, intuitively, they both drive M from s to tha same state. Formally,

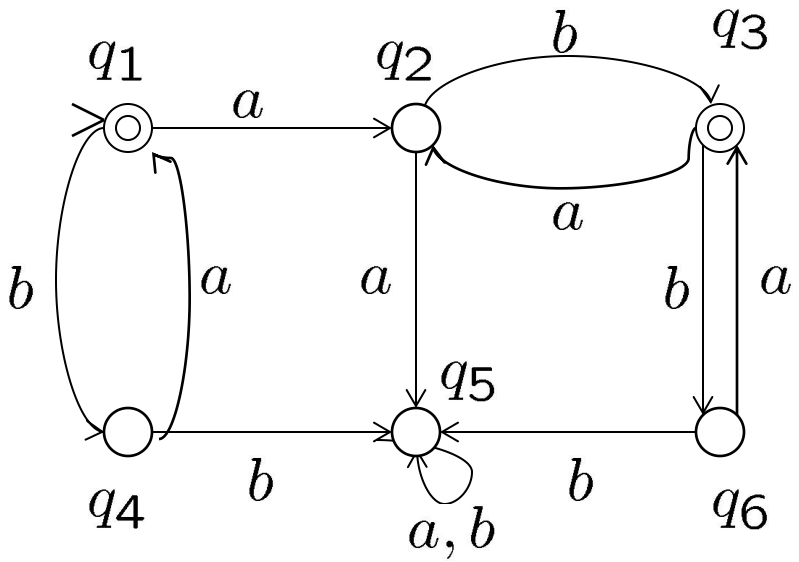
$$x \sim_M y \Leftrightarrow \exists q \in K \text{ s.t.}((s,x) \vdash_M^* (q,e)) \wedge ((s,y) \vdash_M^* (q,e))$$

Remark:

The equivalence classes can be identified by the states of M.

— denoted by E_q

Example: $L = (ab \cup ba)^*$ is the language accepted by M.



The equivalence class as following:

(1) $E_{q_1} = (ba)^*$

(2) $E_{q_2} = La$

(3) $E_{q_3} = abL$

(4) $E_{q_4} = (ba)^* b$

(5) $E_{q_5} = L(bb \cup aa)\sum^*$

(6) $E_{q_6} = abLb$

Relationship Between Two Equivalent Relations

Theorem For any DFA $M = (K, \sum, \delta, s, F)$ and any strings $x, y \in \sum^*$. If $x \sim_M y$, then $x \approx_{L(M)} y$.

Proof:

- For any string $x \in \sum^*$, let $q(x) \in K$ be unique state such that $(s, x) \vdash_M^* (q(x), e)$.
- Notice that, for any $x, z \in \sum^*$,
$$xz \in L(M) \Leftrightarrow (q(x), z) \vdash_M^* (f, e) \text{ for some } f \in F$$
- If $x \sim_M y$ then, by the definition of $\sim_M$, $q(x) = q(y)$.

- $x \sim_M y$ implies that the following holds:

$$xz \in L(M) \Leftrightarrow yz \in L(M) \text{ for all } z \in \sum^*$$

which is the same as $x \approx_{L(M)} y$.

Definition: An equivalence relation $\sim$ is a refinment of another $\approx$, if

$$\forall x, y \; x \sim y \Rightarrow x \approx y.$$

- If $\sim$ is a refinment of $\approx$, then each equivalence class of $\sim$ is contained in some eqivalence class of $\approx$.

Remark:

Equivalence relation $\sim_M$ is a **refinement** of $\approx_{L_M}$.

Example: (continued)

$\approx_{L(M)}$ equivalence classes:

(1) $[e] = L$

(2) $[a] = La$

(3) $[b] = Lb$

(4) $[aa] = L(aa \cup bb)\sum^*$.

(1) $E_{q_1} = (ba)^*$

(2) $E_{q_2} = La$

(3) $E_{q_3} = abL$

(4) $E_{q_4} = (ba)^*b$

(5) $E_{q_5} = L(bb \cup aa)\sum^*$

(6) $E_{q_6} = abLb$

$\sim_M$ is the refinement of $\approx_{L($

$E_{q_1} \cup E_{q_3} = [e]$, $E_{q_5} = [aa$ $q_6} = [b]$

Can this lower bound be archived?

Remark: The numbers of states of any FA equivalent to M must be *at least as large as* the number of equivalence classes of $L(M)$.— **a natural lower bound.**

Lemma Let $L \subseteq \sum^*$ be a regular language. Then there is a DFA with precisely states as there are equivalence classes in $\approx_L$ that accepts L.

Proof: Define $[x]$ to be the equivalence class containing x. Construct DFA $M = (K, \sum, \delta, s, F)$ from $\approx_L$

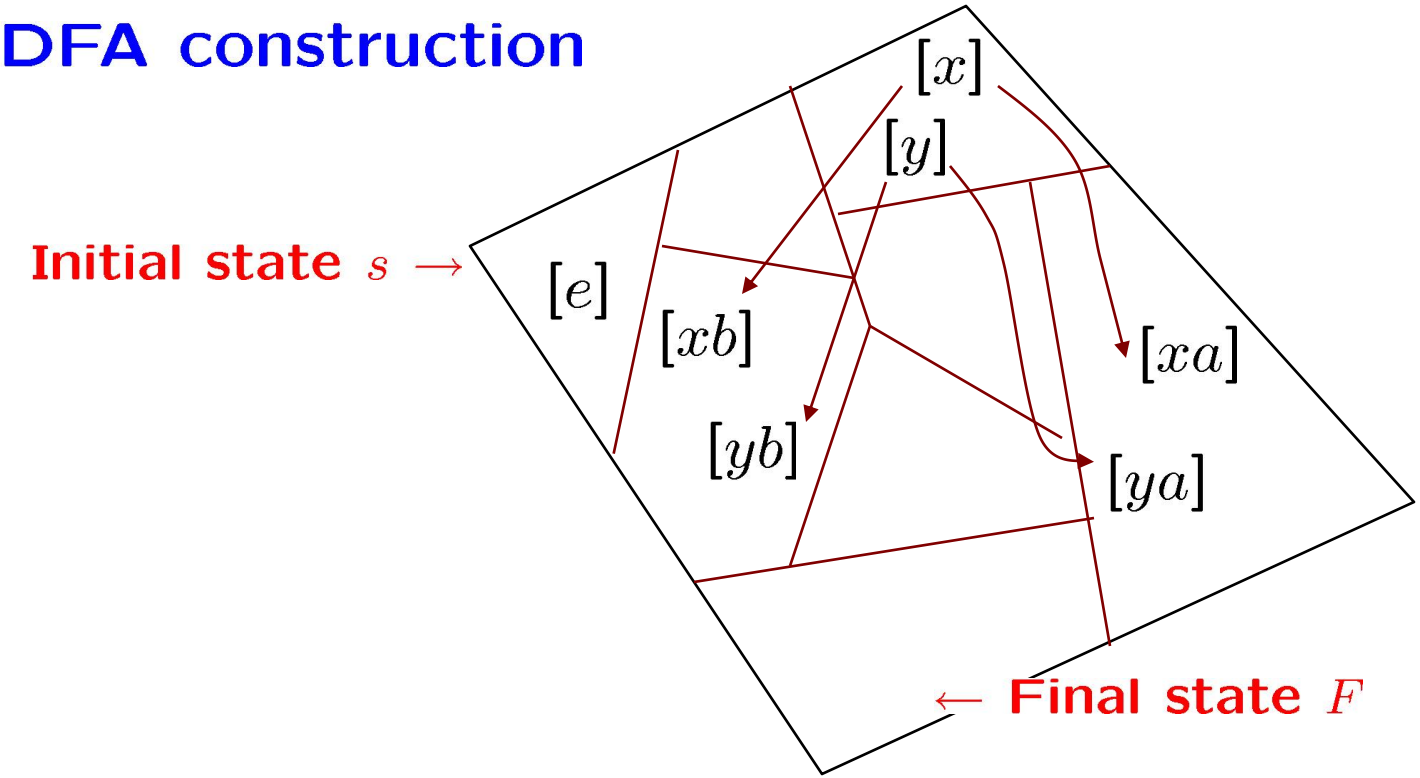
— Let $K = \{[x] : x \in \sum^*$, the set of equivalence classes under $\approx_L$

— Let $s = [e]$

— Let $F = \{[x] \mid x \in L\}$

— For any $[x] \in K$ and any $a \in \sum$, define $\delta([x], a) = [xa]$

DFA construction



$k =$ Equivalence classes of $\approx_L$

We next show:

- K is finite
- $\delta([x], a)$ is a function
- $L = L(M)$

Part 1. K is finite

Let L be a regular language, and so it is accepted by some DFA M'.

$\Rightarrow \sim_{M'}$ is a refinement of $\approx_L$.

$\Rightarrow$ The number of equivalence classes of $\approx_L$

$\leq$ The number of equivalence classes of $\sim_{M'}$

$=$ The number of states of M'

$\Rightarrow K$ is finite.

Part 2. $\delta([x], a)$ is well-defined

$\delta([x], a) = [xa]$ is independent of the string of $x \in [x]$, because

$$x \approx_L x' \Leftrightarrow xa \approx_L x'a.$$

Part 3. $L = L(M)$

$\square$ $\forall x, y \in \sum^*$ $([x], y) \vdash_M^* ([xy], e)$

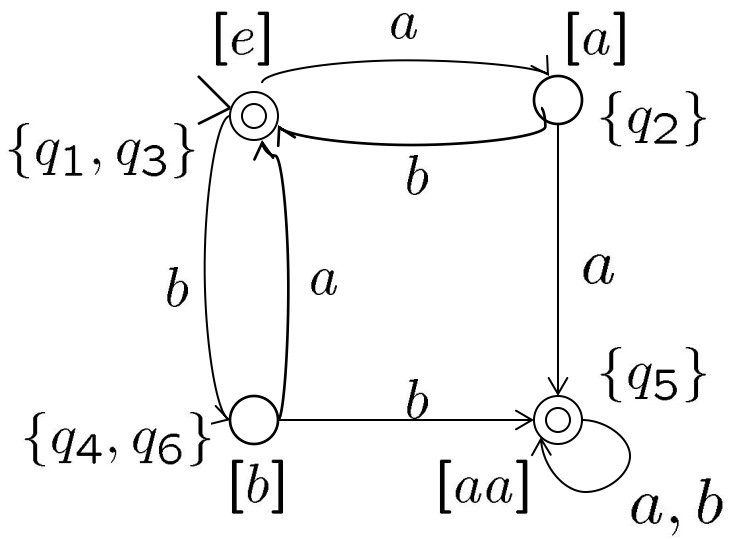
$\square$ $\forall x \in \sum^*$ $x \in L(M) \Leftrightarrow ([e], x) \vdash_M^* (q, e)$ for some $q \in F$

$\Leftrightarrow q = [x] \in F$

$\Leftrightarrow x \in L$ By definition of DFA M

Example: (continued)

The standard FA corresponing to the language $L = (ab \cup ba)^*$ is shown following:



Theorem (The Myhill-Nerode theorem) A language L is regular iff $\approx_L$ has *finitely* many quivalence classes.

Proof: $\Rightarrow$

L is regular

$\Rightarrow L = L(M)$ for some DFA M and M has at least as many states as $\approx_L$ has equivalence classes

$\Rightarrow$ There are finitely many equivalece classes in $\approx_L$

$\Leftarrow$

If $\approx_L$ has finitely many equivalece classes

$\Rightarrow$ The standard DFA M_L accpets L.

Application of Myhill-Nerode Theorem

Alternate proof that $L = \{a^n b^n : n \geq 0\}$ is not regular:

$$a^i \approx_L a^j \Leftrightarrow i = j$$

($\Rightarrow$) If $i \neq j$ then $a^i b^i \in L$, but $a^j b^i \notin L$

($\Leftarrow$) Obvious.

Thus, $\approx_L$ has infinitely many equivalent classes. By Myhill-Nerode theorem, L must not be regular.

Construction of the Standard DFA Equivalent to a Given DFA M

- The minimum automaton accepting a language L is unique (up to renaming of states), and is given by the construction in the Myhill-Nerode theorem.

- An algorithm is a procedure that always terminates (with the right answer)

Minimization algorithm

- Let $M = (K, \sum, \delta, s, F)$ be a DFA.
- Define $\boxed{p \equiv q}$ iff for each $z \in \sum^*$,

$$\delta(p, z) \in F \Leftrightarrow \delta(q, z) \in F \qquad (*)$$

Basic idea:

- The standard DFA is constructed by clumping together the equivalence classes under $\equiv$

- To find $\equiv$: form a sequence of equivalence relations: $\equiv_0, \equiv_1, \cdots, \equiv_n$, where:

$p \equiv_n q$ iff $(*)$ holds for all z of length $|z| \leq n$

Finding $\equiv$

- $\equiv_0$ has just two equivalence classes: F and $K - F$

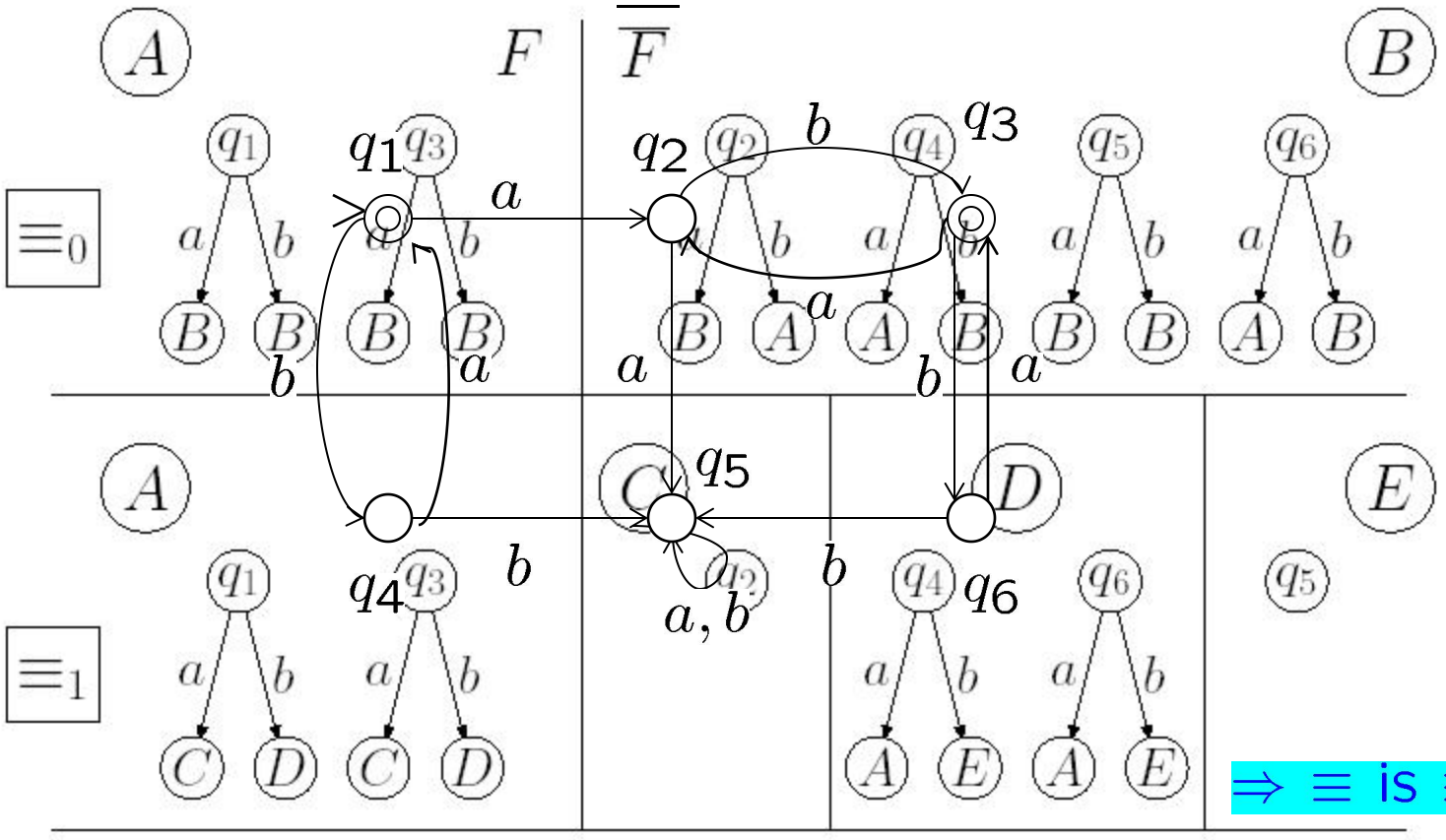
- $p \equiv_{n+1} q$ iff

  1. $p \equiv_n q$
  2. For all $a \in \sum$, $\delta(q, a) \equiv_n \delta(p, a)$

$\equiv_{n+1}$ is a refinement of $\equiv_n$.

Since K is finite, eventually there is an n such that $\equiv_n$ and $\equiv_{n+1}$ are the same: this is $\equiv$.

Example: (continued)



$\Rightarrow \equiv$ is $\equiv_1$.

Example: The language $L \subseteq \{a_1, \cdots, a_n\}^*$ of all strings that do not contain occurrences of all n symbols, and corresponding DFA with 2^n states.

- $\approx_L$ has 2^n equivalecce classes.
- Let $A \subseteq \sum$, $L_A = \{w : w \in \sum^*, w$ contains occurrences of all symbols in A, and of no symbols in $\sum - A\}$.

The 2^n sets L_A are precisely the equivalence classes of $\approx_L$.

2.6 Algorithms for Finite Automata*

Question: Convert a NFA $(K, \sum, \Delta, s, F)$ to a DFA.

– Computation of all $E(q)$ $\cdots\cdots$ $\mathscr{O}(|K|^3)$

For each $Q \subseteq K$, $\forall a \in \sum$,

$\delta(Q, a) = \cup\{E(p) \mid p \in K$ and $(q, a, p) \in \Delta$ for some $q \in Q\}$

– Computation of all $\delta(Q, a)$ $\cdots\cdots$ $\mathscr{O}(|\Delta||K|^2)$

The total complexity: $\mathscr{O}(2^{|K|}|K|^3|\sum||\Delta||K|^2)$.

- Convert a regular expression R into an equivalent NFA. No more than $4|R|^2$ transitions.

- Turning a given FA $M = (K, \sum, \Delta, s, F)$. The resulting regular expressions may have length as large as $3^{|K|}$.

- The total complexity of the minimization algorithm is $\mathcal{O}(|\sum||K|^3)$.

Theorem:

(a) There is an exponential algorithm which, given a NFA, constrcuts an equivalent DFA.

(b) There is a polynomial algorithm which, given a regular expressions, constrcuts an equivalent NFA.

(c) There is an exponential algorithm which, given a NFA, constrcuts an equivalent regular expression.

(d) There is a polynomial algorithm which, given a DFA, constrcuts an equivalent DFA with the smallest possible number of states.

Theorem: (Continued)

(e) There is a polynomial algorithm which, given two DFA, decides whether they are equivalent.

(f) There is an exponential algorithm which, given two NFA, decides whether they are equivalent; similarly for the equivalence of two regular expressions.

FA as Algorithms

A DFA M is an efficient algorithm for deciding wether a given string is in $L(M)$.

Theorem If L is a regular language, then there is an algorithm which, given $w \in \sum^*$, tests wether it is in L in $\mathscr{O}(|w|)$ time.

Theorem If $M = (K, \sum, \Delta, s, F)$ is a NFA, then there is an algorithm which, given $w \in \sum^*$, tests wether it is in L in $\mathscr{O}(|K|^2|w|)$ time.

Homework 2:	
P60	2.1.1 2.1.2 (c)(d) 2.1.3 (c)(e)
P74	2.2.2 (a)(b) 2.2.6 (a)(b) 2.2.10
P83	2.3.4 (b) 2.3.7 (a)(b)(c)(d)
P90	2.3.5 (a) 2.4.8