A Hicksian Multiplier-Accelerator Model with Floor Determined by Capital Stock

Tongues of Periodicity in a Family of Two-dimensional Discontinuous Maps of Real Möbius Type

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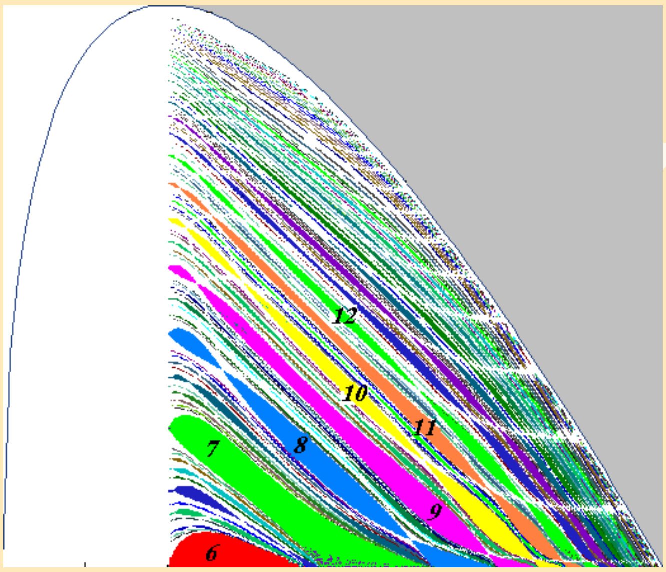

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A Hicksian Multiplier-Accelerator Model with Floor Determined by Capital Stock

Tönu Puu, Laura Gardini, and Irina Sushko

Abstract: This article reconsiders the Hicksian multiplier-accelerator model with the "floor" related to the the depreciation on actual capital stock. Through the introduction of the capital variable a growth trend is created endogenously by the model itself, along with growth rate oscillations around it. The "ceiling" can be dispensed with altogether. As everything is growing in such a model, a variable transformation is introduced to the end of focusing relative dynamics of the income growth rate and the actual capital output ratio.

Introduction

The Samuelson-Hicks Model

The objective of the present article is to reconsider the Hicksian multiplier-accelerator model of business cycles. This model was introduced by Samuelson [12] in 1939, and was based on two interacting principles: Consumers spending a fraction c of past income, $C_t = cY_{t-1}$, and investors aiming at maintaining a stock of capital K_t in given proportion a to the income Y_t to be produced. With an additional time lag for the construction period for capital equipment, net investments, by definition the change in capital stock, $I_t = K_t - K_{t-1}$, become $I_t = a\left(Y_{t-1} - Y_{t-2}\right)$. As income is generated by consumption and investments, i.e. $Y_t = C_t + I_t$, a simple feed back mechanism $Y_t = (c+a)Y_{t-1} - aY_{t-2}$ was derived. It could generate growth or oscillations in income.

The consumption component was referred to in terms of the "multiplier" and the investment component in terms of the "principle of acceleration". To be quite true to history, two remarks should be added: First, Samuelson applied the accelerator to consumption expenditures only, the above described application to all expenditures is due to Hicks [7] 1950, discussed in more detail below. The difference in terms of substance is marginal. Second, all these models contain an additional term called "autonomous expenditures", i.e. government expenditures, any consumption independent of income, and investments not dependent on the business cycles generated by the model. The multiplier, $1/(1-c)$, i.e. the multiplicative factor applied to such autonomous expenditures resulted in an equilibrium income, or a particular solution to the difference equation. Income could then be redefined as a deviation from this equilibrium income, and the original equation regained, though now in income deviations from the equilibrium. This makes sense of negative values of income which inevitably result from the above difference equation.

However, there was more to the Hicksian reformulation. He realized that there must be limits to the accelerator-based investment function. In a depression phase $I_t = a\left(Y_{t-1} - Y_{t-2}\right) < 0$, and it can even happen that income (=production) decreases at a pace so fast that more capital can be dispensed with than disappears through natural wear. As nobody actively destroys capital to such an end, there is a lower limit to disinvestment, the so called "floor", fixed at the (negative) net investment which occurs when no worn out capital is replaced at all.

At the same time Hicks suggested that there be a "ceiling" at full employment, when income could not be expanded any more. Hicks [7] never assembled the pieces to a complete formal model. It is clear that the floor constraint is applied to the investment function, so it becomes something like $I_t = Max\{a\left(Y_{t-1} - Y_{t-2}\right), -I^f\}$, where I^f is the absolute value of the floor disinvestment. On the other hand it is not quite clear what the ceiling is applied to. Most likely Hicks thinks of it as applied to income, so that the income formation equation is changed to $Y_t = Min\{cY_{t-1} + I_t, Y^c\}$, where Y^c is the full employment capacity income. Gandolfo [4] interpreted the model this way, and Hommes [8] gave a more or less full analysis of this model. The above symbols conform to Hommes's notation.

It is not obvious in such a formulation which agents cut their expenditures when the ceiling is reached. As an alternative one might incorporate it in the investment function, along with the floor, thereby implying that it is the investors who abstain from further investment when they realize that full employment is reached. This was the choice of Goodwin [5] and many other students of the Hicksian business cycle machine, even the present authors. See [10], [11].

To the complete model also belong the autonomous expenditures which were already mentioned. These can be constant, or growing. In his verbal description Hicks seems to have been in favour of exponentially growing autonomous expenditures, as he obviously wanted to model both growth and cycles around a growing trend. Growth, however, was not created endogenously by the model, as the cycles were, but introduced ad hoc. To make this type of model suitable for analysis, the floor and ceiling must be assumed to be growing too, at the same rate as the autonomous expenditures, and this seems to have been Hicks's own tacit assumption. The assumption of equal growth rates is fairly arbitrary. Gandolfo [4] modelled it this way, though Hommes [8] preferred to analyze the stationary case where autonomous expenditures, floor, and ceiling were all constant. A recent mathematical analysis of Gandolfo's case may be found in [3].

A Suggested Reformulation

The assumed growth of the floor along with the autonomous expenditures is particularly problematic. A growing capital stock, as a result of growing autonomous expenditures, should *increase* the absolute value of maximum disinvestment, so it is not only arbitrary to assume the floor to grow at the same rate as the autonomous expenditures, but the change even *goes in the wrong direction*. The floor would rather be *decreasing* with capital accumulation. For this reason it seems to be important to make capital an explicit variable in the model, and to relate the floor directly to capital stock, i.e. put $I_t^f = rK_t$, where K_t is capital stock and r is the rate of depreciation. Just to avoid misunderstanding it should be understood that, though the income variable, as we have seen, can be negative in the sense of a negative *deviation* from equilibrium, capital by necessity always is nonnegative.

Making this change to the model, we get the extra benefit that the growing trend need not be introduced exogenously. It would result *within* the model through capital accumulation. The model hence *explains* both the growth trend and the business cycles that take place around it.

As for the ceiling, at least for a start, we dispense with it. It was noted by Duesenberry [2] in his review of Hicks's book in 1950, that both floor and ceiling were not always needed for bounded motion, and that in particular the ceiling could be dispensed with. Allen [1] gives a very clear account of the argument: "*On pursuing this point, as Duesenberry does, it is seen that the explosive nature of the oscillations is largely irrelevant, and no ceiling is needed. A first intrinsic oscillation occurs, the accelerator goes out in the downswing, and a second oscillation starts up when the accelerator comes back with new initial conditions. The explosive element never has time to be effective - and the oscillations do not necessarily hit a ceiling*".

Of course, with accumulating capital, the floor is no longer fixed, and the growth of capital stock allows increasing amplitude swings around the growth trend for which it is also responsible. Growth is something that economists regard as a good feature for a model, but it is no good for the use of mathematical methods which favour the study of fixed points and their destabilization, stationary cycles, quasiperiodicity, and chaos. To make the model suitable for standard analysis, we focus on relative dynamics, the rate of growth of income, and of the actual capital/output ratio.

Before this reduction to relative dynamics, we, however, have to state the complete model with the explicit inclusion of the stock of capital.

The Model

The Absolute Growth Dynamics

Let us first just restate the consumption function:

$$C_t = cY_{t-1} \tag{1}$$

and the investment function:

$$I_t = Max\{a\,(Y_{t-1} - Y_{t-2})\,, -rK_{t-1}\} \tag{2}$$

where c, a and r are real parameters such that $0 < c < 1$, $a > 0$, $0 < r < 1$.

We now also need a relation for capital stock updating:

$$K_t = K_{t-1} + I_t \tag{3}$$

which just says that capital stock changes with net investments according to (2), accelerator generated as $K_t = K_{t-1} + a\,(Y_{t-1} - Y_{t-2})$, or, in the case when the floor is activated, just decays, like a radioactive substance as $K_t = (1-r)\,K_{t-1}$. As there in no ceiling, the income generation equation just reads:

$$Y_t = C_t + I_t \tag{4}$$

Equations (1)-(4) now define the complete system. It is easy to see through numerical studies that the model can create a process of accumulating capital, along with a growth trend in income, and this without any growing autonomous expenditures at all. Further, the model creates growth cycles around these secular trends.

Fixed Points and their Stability

Let us first note that if

$$a\,(Y_{t-1} - Y_{t-2}) + rK_{t-1} \geq 0 \tag{5}$$

then the first alternative in (2) applies. Let us call the part of phase space where the inequality (5) is satisfied **Region I**. Then, eliminating the consumption and investment variables through substitution from (1) and (2) into (3) and (4), we see that in **Region I** the system is defined by:

$$K_t = K_{t-1} + a\,(Y_{t-1} - Y_{t-2}) \tag{6}$$

$$Y_t = cY_{t-1} + a\,(Y_{t-1} - Y_{t-2}) \tag{7}$$

Let us now look at the other alternative, where the second branch of (2) is activated, i.e. $I_t = -rK_{t-1}$. This occurs when

$$a\,(Y_{t-1} - Y_{t-2}) + rK_{t-1} < 0 \tag{8}$$

Let us call this **Region II**. From (1)-(4) we get:

$$K_t = (1-r)\,K_{t-1} \tag{9}$$

$$Y_t = cY_{t-1} - rK_{t-1} \tag{10}$$

It is easy to find the fixed points for (6)-(7). From (7), there is just one fixed point for income, $Y_t = Y_{t-1} = Y_{t-2} = 0$. Next, putting $Y_{t-1} = Y_{t-2}$ in (6), we conclude that $K_t = K_{t-1}$, i.e. *any* (positive) capital stock may be an equilibrium stock. Simulation experiments indicate that, depending on the dynamical process, i.e. on the initial conditions, the stock of capital may end up at different equilibrium values. As for stability, this also implies that, if there is some perturbation of the capital stock, then the process will again end up at a new equilibrium stock. However, income always goes to the single zero equilibrium. This, of course, is true, only if the equilibrium is *stable*.

It is also obvious that the system (9)-(10) has only one fixed point located at the origin of phase space.

Let us first investigate the stability of the system (6)-(7). As (7) is independent of the capital stock, we can study this single equation alone. However, we have to observe that (7) is a linear second order difference equation. Writing down the Jacobian matrix of (7) and the corresponding characteristic equation, one can easily get its roots, or eigenvalues:

$$\lambda_{1,2} = \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a+c)^2 - 4a} \tag{11}$$

From (11) we can immediately see that the zero fixed point is a node if $(a+c)^2 - 4a > 0$ and a focus if $(a+c)^2 - 4a < 0$. It is stable iff $|\lambda_{1,2}| < 1$. The latter condition holds for the parameter ranges: $c < 1$, $a < 1$, $a > -(1+c)/2$. Taking into account the feasible parameter range, we conclude that the *stability region* for the fixed point of (7) is

$$0 < c < 1, 0 < a < 1$$

Because of the linearity, the system (7) is a contraction for the above parameter range not only at the fixed point but in the whole **Region I**.

The system (6)-(7) becomes an expansion if $|\lambda_{1,2}| > 1$, that is true for $a > 1$.

Now let us check the stability of the system (9)-(10). Its eigenvalues are $\mu_1 = c$ and $\mu_2 = 1 - r$. As $0 < r < 1$, and $0 < c < 1$, we note that both eigenvalues are positive and less than unity. Accordingly, the system (9)-(10), defined in the **Region II,** is a contraction.

The economics of this is that in **Region II** the system tends to equilibrium, with zero capital and zero income. The stability of the map (9)-(10) would be a big problem, if the process were not easily mapped *back* into **Region I**, where the fixed point may be unstable. If $c < 1 - r$ holds, which seems to be the factually most likely case, then the jumping between regions (in a finite number of iterations) occurs for *any* initial conditions we may care to choose. If not, then there exist initial conditions, such that the process goes to equilibrium in **Region II**, even when $a > 1$ (see Appendix 1 for the details).

The Fixed Point Bifurcation

As we have seen, only the fixed point of (7) may become unstable. Indeed, at

$$a = 1 \tag{12}$$

the eigenvalues (11) are complex conjugate and have unitary modulus. Thus, the fixed point has a bifurcation analogous to the Neimark bifurcation. At the bifurcation we can write the eigenvalues (11) as $\lambda_{1,2} = \cos\theta \pm i\sin\theta$, where

$$\cos\theta = \frac{a+c}{2} \tag{13}$$

and

$$\sin\theta = \frac{1}{2}\sqrt{4a - (a+c)^2}$$

In case there is a rational rotation m/n around the fixed point, the solution $\theta = 2\pi m/n$ holds. Then, using (13) and (12), we get the exact value of the parameter c which corresponds to the rotation number m/n:

$$c = 2\cos\left(\frac{2\pi m}{n}\right) - 1 \tag{14}$$

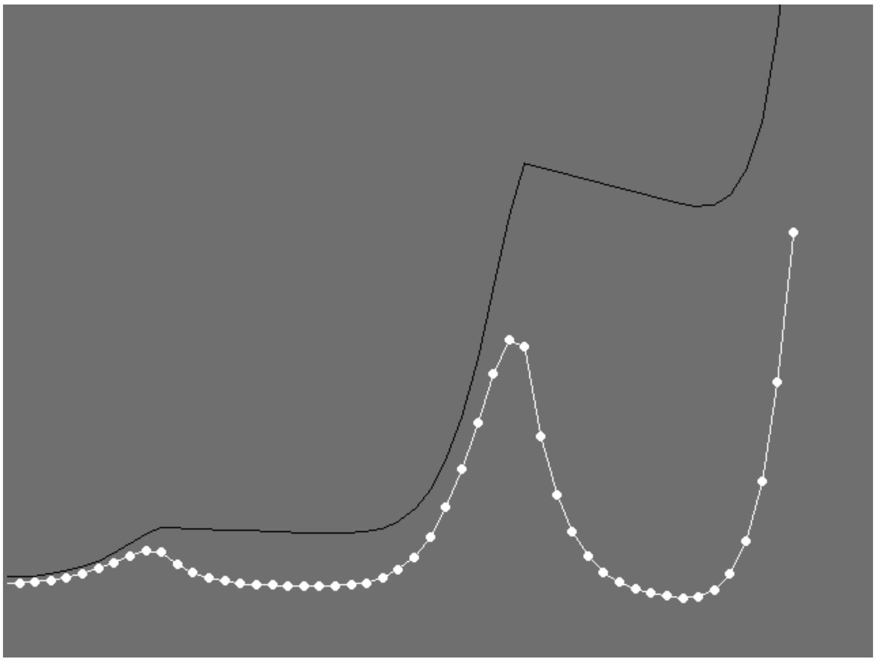

Figure 1: Time series of oscillating income around a growth trend (white curve), and growing capital stock with recessions (black).

For instance, $m = 1$ and $n = 1, 2, 3, 4, 5$ and 6 yield $c = 1, -3, -2, -1, \left(\sqrt{5} - 3\right)/2$ and 0, respectively, so $1 : 7$ is the lowest basic resonance that falls into the admissible parameter range. The global dynamics of the system (1)-(4) at $a = 1$ is briefly described in Appendix 2.

Our study of this bifurcation was confined to the fixed point of (6)-(7), and the reader may note that then (7) is no different from the original Samuelson-Hicks model. However, globally, the moving floor component, which is related to capital stock, is important, and eventually it is the growing capital stock that is responsible for the secular growth created by the proposed model for $a > 1$.

In Figure 1 we show typical growth paths for capital and income. Income (or rather its deviation from equilibrium) is the white curve, oscillating around the zero value, with increasing height of the peaks and increasing amplitude. The picture was calculated for parameters $a = 2.25, c = 0.65, r = 0.01$. As we will see below this case results in a 23-period growth cycle around a growing trend. As the growth rate makes a constant amplitude oscillation, income itself oscillates with increasing amplitude. As for capital, the black curve shows an ever increasing trend with periodic recessions. The falling segments occur where the floor is activated. As we see, the slope of these increases. This is a reflection of the fact that the floor restriction slackens with increasing capital, and larger disinvestment is allowed with growing capital stock.

Stationary Relative Dynamics

It is convenient to study the phenomena in the above way only when $a \leq 1$, i.e. when the zero fixed point for income is stable. If it is not, then there is exponential growth in the model, so all variables eventually explode. If we try to display time series such as Figure 1 over prolonged periods, we just get horizontal lines first and then oscillations growing so fast that they break any frame for the figure. In the same way, in the phase diagrams we only see spiralling motions that move out from the frame, even if there is something more to see, like growth cycles, or quasiperiodic trajectory around a trend. However, we cannot catch these visually, neither can we use standard mathematical analyses for such growing systems.

We would need to find some variable transformations, which make the oscillations around the exploding trends stationary periodic, quasiperiodic, or chaotic, whatever they are, but such that they can be studied by standard methods. As the growth trend is not given by any exogenous term growing at a *given* rate, unlike the Gandolfo [4] version, we have to *define some transformed variables within the model* such that they undergo stationary cyclic or other motion.

As a pedagogical device, one of the present authors 40 years ago [9] suggested to study the evolution of new relative variables for the original Samuelson-Hicks model, i.e. $Y_t = (c+a)Y_{t-1} - aY_{t-2}$ through defining $y_t := Y_t/Y_{t-1}$. The objective was to avoid complex numbers in the study of second order difference equations through making the iteration one dimensional, though non-linear. In fact the original model becomes $y_t = (c+a) - a/y_{t-1}$. This strategy, however, makes for instance cyclic variations in the income growth variable really become cyclic. We now have a system with capital as an additional variable, but never mind, we can still use the same strategy.

So, let us again eliminate investments and consumption in (1)-(4), through substitution in (3) and (4), and then define new variables. Let:

$$x_t := K_t/Y_{t-1} \tag{15}$$

and

$$y_t := Y_t/Y_{t-1} \tag{16}$$

These new variables are the actual capital/output ratio, as distinguished from the optimal ratio a, and the relative change of income from one period to the next, quite as in the framework suggested in [9]. Again there is a reduction of dimension for the system, now from 3 to 2. Using these new variables defined in (15)-(16), we can restate the dynamical system as follows. Suppose we have:

$$x_{t-1}\left(a\left(y_{t-1}-1\right)+rx_{t-1}\right) \geq 0$$

As we see this corresponds to **Region I** in the original model. The system can then be written:

$$x_t = \frac{x_{t-1}}{y_{t-1}} + a\left(1 - \frac{1}{y_{t-1}}\right) \tag{17}$$

$$y_t = c + a\left(1 - \frac{1}{y_{t-1}}\right) \tag{18}$$

Suppose that, on the contrary,

$$x_{t-1}\left(a\left(y_{t-1}-1\right)+rx_{t-1}\right) < 0$$

This obviously corresponds to **Region II** in the original model. Then (17)-(18) are replaced by:

$$x_t = (1-r)\frac{x_{t-1}}{y_{t-1}} \tag{19}$$

$$y_t = c - r\frac{x_{t-1}}{y_{t-1}} \tag{20}$$

Fixed Growth Points

Written as relative dynamics (17)-(20), the new system has fixed points as well. In terms of economics they represent equilibrium growth rates. Consider first **Region I**. Putting $x_t = x_{t-1} = x$, $y_t = y_{t-1} = y$ in (17)-(18) we obtain:

$$x = a \tag{21}$$

and

$$y^2 - (a+c)y + a = 0 \tag{22}$$

According to (21) the equilibrium capital/output ratio x equals the optimal one as indicated by the accelerator a, which intuitively seems most reasonable. As for (22), it determines either two real, or two complex conjugate equilibrium values for the relative income growth rate:

$$y_{1,2} = \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a+c)^2 - 4a} \tag{23}$$

We may observe that (22) is equal in form to the characteristic equation for the original Samuelson-Hicks model, so the growth rate equilibria exist whenever the original multiplier-accelerator model has solutions with two *real* roots.

A saddle-node bifurcation resulting in the appearance of the fixed points (x, y_1) and (x, y_2) obviously occurs when

$$(a+c)^2 = 4a$$

For $(a+c)^2 < 4a$ no fixed point exists for the income growth rate, whereas for $(a+c)^2 > 4a$ there are two fixed points, one stable and one saddle. This is easy to see from (18), as the derivative (putting $y_{t-1} = y$ at equilibrium) is:

$$\frac{\partial y_t}{\partial y_{t-1}} = \frac{a}{y^2}$$

Using the larger root according to (23), we get:

$$\frac{a}{y_1^2} = \frac{a+c-\sqrt{(a+c)^2 - 4a}}{a+c+\sqrt{(a+c)^2 - 4a}} < 1$$

and, using the smaller,

$$\frac{a}{y_2^2} = \frac{a+c+\sqrt{(a+c)^2 - 4a}}{a+c-\sqrt{(a+c)^2 - 4a}} > 1$$

So far we only checked the stability of (23). We should complete the discussion by differentiating (17), and again deleting the index in the right hand side:

$$\frac{\partial x_t}{\partial x_{t-1}} = \frac{1}{y}$$

so stability for the capital variable at the fixed point depends on the equilibrium value of y. From (23) we see that if the roots are real, then both are positive (due to the minus term under the root sign). Further, whenever $c < 1$, which it must be to make any sense, we have $1 < y_2 < y_1$ according to (23). Hence:

$$\frac{1}{y_1} < 1$$

and

$$\frac{1}{y_2} < 1$$

So, the fixed point (x, y_1) is a stable node while (x, y_2) is a saddle. This is a little pedestrian way of checking stability, for income and capital separately. But, quite as in the original model, the Jacobian matrix of the system (17)-(18) is triangular, so the main diagonal derivatives actually are the eigenvalues.

Fixed Decline Points

As was the case before the reduction to relative dynamics, the model also has fixed points in **Region II**, i.e., where (19)-(20) hold. Put again $x_t = x_{t-1} = x, y_t = y_{t-1} = y$ in (19)-(20). The system obviously has the fixed point:

$$x = \frac{1-r}{r}(c+r-1)$$

and

$$y = 1-r$$

If we recall that the new y-variable is the income ratio for one period to the previous, then we realize that the fixed point means income change at the constant rate $(1-r)$, which is the rate of capital depreciation. The accelerator then gives the same investment as the floor condition, and capital is just depreciating at the same rate. When the system is in this new fixed point, corresponding to a negative constant growth rate $-r$, then the system is on its way towards the zero equilibrium for capital and income according to the original setup in **Region II**. The eigenvalues in the fixed point are 0 and $c/(1-r)$. It is stable point if $c < 1-r$, but for the same condition this fixed point does not belong to **Region II.** Thus, the system switches between the two definition regions, so the attractivity of the new fixed point does not really matter.

There is an additional fixed point in **Region II**:

$$x = 0$$

and

$$y = c$$

The eigenvalues are 0 and $(1-r)/c$, so, as the second eigenvalue is the reciprocal of the corresponding one in the previous case, it exchanges stability with the aforementioned fixed point. The economics of this fixed point is a classical multiplier process converging to equilibrium, with zero capital. In a sense the two fixed points are the same. They both eventually lead to zero income (deviation from equilibrium due to autonomous expenditures) and zero capital. However, the paths of approach are different, and we now focus growth rates, not the income and capital variables themselves.

Periodicity

So far we studied the fixed points of the relative dynamics system. Numerical experiment indicates cycles in the growth rates. For instance, as we see in Figure 2, the parameter combination $a = 2.25,\ c = 0.65,\ r = 0.01$ results in a 23-period growth cycle. The case was illustrated in Figure 1, though it was difficult to see any regular pattern of oscillation in that picture. On the white income growth trace in Figure 2 we marked the successive iterates through little circles, so it is easy to identify the periodicity. The black trace is for the capital to income ratio. It is worth noting that the long horizontal sections coincide with the level of the accelerator coefficient, which seems to catch the trajectory for considerable periods of time.

To see some more possibilities, we show a bifurcation diagram in Figure 3. It represents the parameter plane a (horizontal), c (vertical). The third parameter r is fixed at 0.01. Changing this parameter, the rate of depreciation, changes little in the picture. The changes mainly concern the structure of white streaks that seem to run through the points where the periodicity tongues look twisted.

The tongues were computed for the relative dynamics model (17)-(20), for periodicity 1-45. However, the starting points for the periodic tongues were computed in the original model in (14). On the bifurcation line, the original system *is* periodic (for rational rotation numbers), so this is not surprising. To the left of the vertical line at $a = 1$, the zero fixed point of the original system is stable.

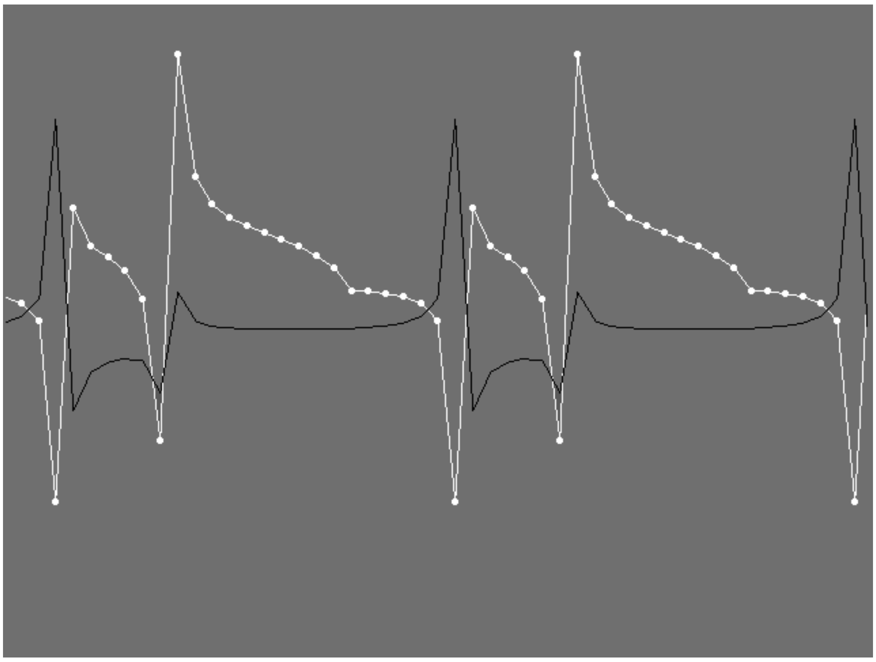

Figure 2: 23-period cycle in the relative dynamics system. (Same case as in Fig. 1.)

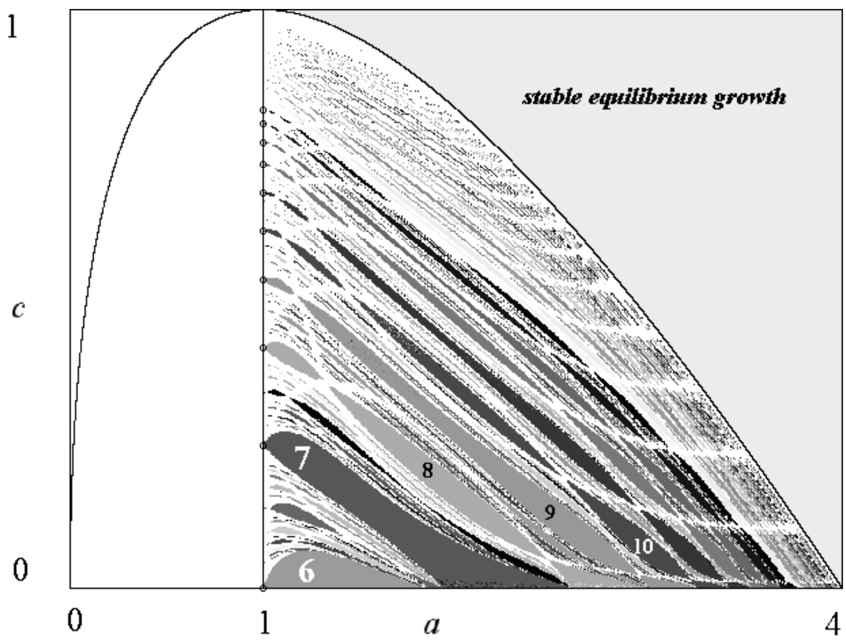

Figure 3: Bifurcation diagram. Tongues of periodicity and stable equilibrium region.

In Figure 3, we also see a parabola turned upside down. It is the locus of points $(a+c)^2 = 4a$, i.e. the borderline between complex (below) and real (above) roots to the characteristic equation. To the left of the line $a = 1$, the attracting zero fixed point is a focus below the parabola and a node above it. To the right of the line $a = 1$, we have the tongues of periodicity drawn below the parabola. Above it there is the area where the roots (23) are real, and there is an attractive fixed point for the relative dynamics system. The original system then goes to a stable growth rate at the larger value of (23).

Conclusion

Above we suggested a business cycle model, consisting of about half of the bits and pieces proposed by Hicks in his classical work [7]. In particular, the "floor" was retained, but the "ceiling" omitted, in concordance with Duesenberry's argument [2].

As a new element, the floor was tied to actual capital stock through a fixed depreciation factor. Hence, in the process of growth with capital accumulation, the level of the "floor" changes, thus allowing increasing amplitude oscillations around the growing trends for income and capital. Further, the secular growth trends are created within the model, and need not be introduced in terms of exogenous growing expenditures.

In order to analyze the oscillating growth rates around the rising trends, a transformed system of relative dynamics, in terms of the income growth rate and the capital/income ratio, was proposed. This reduced the system from three to two dimensions, though it also introduced new complexity through transforming linear relations to nonlinear with vanishing denominators.

It should be said that the original system for the evolution of capital and income can always be retrieved from the relative system. One could arbitrarily put income in the first time period equal to unity, and obtain its evolution as a continued product of the growth factors. A different initial first period income would then just scale the entire time series obtained (up or down) in proportion.

The relative dynamics system and the detailed structure we see in Fig. 3 have some intricacy of a mathematical nature, which the authors intend to study more closely in a coming publication.

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Appendix 1

Let us rewrite the system of equations (1)-(4) in the form of an iterated map. We introduce the following variables:

$$\begin{aligned} x_t &= Y_{t-1} \\ y_t &= Y_t \\ z_t &= K_t \end{aligned}$$

For convenience we skip the index and use a dash to denote the one period advancement operator. Then the system (1)-(4) can be written as a three-dimensional piecewise linear continuous map F given by two linear maps F_1 and F_2 which are defined, respectively, in **Region I** (denoted R_1) and **Region II** (denoted R_2):

$$F : \begin{cases} (x', y',' z') = F_1(x,y,z), \text{ if } (x,y,z) \in R_1, \\ (x', y',' z') = F_2(x,y,z), \text{ if } (x,y,z) \in R_2, \end{cases} \tag{24}$$

where

$$F_1 : \begin{cases} x' = y \\ y' = -ax + (c+a)y \\ z' = a(y-x) + z \end{cases}, \; R_1 = \left\{ (x,y,z) : z \geq \frac{a}{r}(x-y),\ z > 0 \right\}$$

$$F_2 : \begin{cases} x' = y \\ y' = cy - rz \\ z' = z(1-r) \end{cases}, \; R_2 = \left\{ (x,y,z) : 0 < z < \frac{a}{r}(x-y) \right\}$$

Here a, c and r are real parameters: $0 < c < 1$, $a > 0$, $0 < r < 1$; x, y and z are real variables: $z > 0$ (one can check that if an initial value of z is positive, then it remains positive under the iterations).

The purpose of this consideration is to give conditions and explain a mechanism of constant shifting between the two regions which keeps the process going for $a > 1$ without converging to zero equilibrium.

To proceed we need to reformulate some results described in Sec. 2.2.

The fixed point (x^*, y^*, z^*) of the map F_1 is any point of the z-axes, i.e. $(x^*, y^*, z^*) = (0, 0, z^*)$ where $z^* \geq 0$. The eigenvalues of F_1 are

$$\lambda_1 = 1,\ \lambda_{2,3} = \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a+c)^2 - 4a}$$

and the corresponding eigenvectors are

$$v_1 = (0,0,1),\ v_{2,3} = (1, \lambda_{2,3}, a)$$

The fixed point $(0,0,z^*)$ is stable iff $|\lambda_{2,3}| < 1$, which holds for $a < 1$. It is a stable node for $(a+c)^2 > 4a$ and a stable focus for $(a+c)^2 < 4a$. For $a > 1$ the fixed point $(0,0,z^*)$ is unstable.

The fixed point (x^*, y^*, z^*) of the map F_2 is at the origin, i.e. $(x^*, y^*, z^*) = (0,0,0)$. The eigenvalues of F_2 are

$$\mu_1 = 0,\ \mu_2 = c,\ \mu_3 = 1 - r$$

and the corresponding eigenvectors are

$$w_1 = (1,0,0),\ w_2 = (1,c,0),\ w_3 = (-1, r-1, (c-1+r)(r-1)/r)$$

The fixed point of F_2 is always stable for the parameter range considered.

Thus, for $a < 1$ the whole system is stable.

Let us denote the plane, which separates the two regions R_1 and R_2, by LC_{-1} :

$$LC_{-1} = \left\{ (x,y,z) : z = \frac{a}{r}(x-y) \right\}$$

Its consecutive images by F are

$$\begin{aligned} LC_0 &= F(LC_{-1}) = \left\{ (x,y,z) : z = \frac{1-r}{r}(cx-y) \right\}, \\ LC_i &= F(LC_{i-1}),\ i = 1,\dots \end{aligned}$$

We call these sets *critical planes*, emphasizing the fact that they play an important role, as critical lines, or critical surfaces, for noninvertible maps [6].

Proposition. *Let $a > 1$, $(a+c)^2 < 4a$, $c < 1-r$. Then any point $(x_0,y_0,z_0) \in R_1$ is mapped to R_2 in a finite number of iterations while any $(x_0,y_0,z_0) \in R_2$ is mapped to R_1 also in a finite number of iterations.*

The first part of the Proposition is obvious: For $(a+c)^2 < 4a$, $a > 1$ any point of the z-axes is an unstable focus, thus, any point $(x_0,y_0,z_0) \in R_1$ rotates under iterations by F_1 around the z-axes away from it. Note that in the direction of the eigenvector $v_1 = (0,0,1)$ we have $\lambda_1 = 1$, thus, all points $(x_i,y_i,z_i) = F_1(x_{i-1},y_{i-1},z_{i-1})$, $i = 1,\dots,j$, belong to a rotation plane $z = ax+z^*$ corresponding to the eigenvectors v_2 and v_3 and passing through the fixed point $(0,0,z^*)$. Any such a rotation plane obviously intersects the critical plane LC_{-1}. Thus, after a finite number of iterations the trajectory enters the region R_2, i.e., there exists $j > 0$ such that $(x_j,y_j,z_j) \in R_2$.

Let now $(x_0,y_0,z_0) \in R_2$. First note, that, as the map F_2 does not depend on x (a fact that causes one eigenvalue to be zero), any point $(x_0,y_0,z_0) \in R_2$ is mapped by F_2 on the plane LC_0 in one step, i.e., $(x_1,y_1,z_1) = F_2(x_0,y_0,z_0) \in LC_0$. Obviously, as long as iterated points are in R_2, they all belong to LC_0 and approach the stable zero fixed point of the map F_2 in the eigendirections w_2 and w_3.

For $c < 1-r$ the eigenvalues μ_2, μ_3 of the map F_2 are such that $\mu_2 < \mu_3$, that is the iterated points move more quickly to the zero fixed point in the w_2 direction and asymptotically tangent to w_3. But for the same condition $c < 1-r$ we have $w_3 \in R_1$. Thus, approaching the zero fixed point, the trajectory necessarily enters the region R_1 : There exists $k > 0$ such that $(x_k,y_k,z_k) = F_2(x_{k-1},y_{k-1},z_{k-1}) \in R_1$. $\square$

Appendix 2

It is also interesting to describe the dynamics of the map F given in (24) at $a = 1$. It is a bifurcation value for the map F_1 when two of its eigenvalues, being complex-conjugate, are on the unit circle: $|\lambda_{2,3}| = 1$. It means that the fixed point $(0,0,z^*)$ is a center. Any point in its neighborhood, denoted P, is either periodic with rotation number m/n, or quasiperiodic, depending on the parameters. The value of the parameter c which corresponds to the rotation number m/n is given in (14). What is this neighborhood P? Obviously, $P \in R_1$ and it belongs to the rotation plane $z = ax+z^*$ passing through $(0,0,z^*)$ corresponding to the eigenvectors v_2 and v_3. Without going into details we just say that in the case of a periodic rotation the set P is a polygon whose boundary is made up by segments of critical lines which are intersections of LC_i, $i = 0,\dots,m-1$, with the rotation plane (see [11] where an analogous consideration is provided in detail for a two-dimensional piecewise linear map). In the case of quasiperiodic rotation the set P is an ellipse, each point of which is tangent to some critical line.

We can consider a union U of such sets P constructed for each fixed point $(0,0,z^*)$. This union is either a polygonal cone with m sides (in the case of m-period rotation), or a cone (quasiperiodic case), which issue from the origin. Any point $(x,y,z) \in U$ is either m-periodic, or quasiperiodic, while any point $(x,y,z) \notin U$ is mapped to the boundary of U in a finite number of iterations.

Tongues of Periodicity in a Family of Two-dimensional Discontinuous Maps of Real Möbius Type

Iryna Sushko, Laura Gardini and Tönu Puu

Abstract: In this paper we consider a two-dimensional piecewise-smooth discontinuous map representing the so-called "relative dynamics" of an Hicksian business cycle model. The main features of the dynamics occur in the parameter region in which no fixed points at finite distance exist, but we may have attracting cycles of any periods. The bifurcations associated with the periodicity tongues of the map are studied making use of the first return map on a suitable segment of the phase plane. The bifurcation curves bounding the periodicity tongues in the parameter plane are related with saddle-node and border-collision bifurcations of the first-return map. Moreover, the particular "sausages structure" of the bifurcation tongues is also explained.

Introduction

The present article reconsiders the Hicksian multiplier-accelerator model of business cycles. It was introduced by Samuelson [8] in 1939, and was based on two interacting principles: Consumers that spend a fraction c of past income, $C_t = cY_{t-1}$, and investors that maintain a stock of capital K_t in given proportion a to the income Y_t. With an additional time lag for the construction period of capital equipment, net investment, by definition the change in capital stock, $I_t = K_t - K_{t-1}$, becomes $I_t = a\left(Y_{t-1} - Y_{t-2}\right)$. As income is generated by consumption and investments, i.e. $Y_t = C_t + I_t$, a simple feed back mechanism $Y_t = (c+a)\,Y_{t-1} - aY_{t-2}$ was derived.

Hicks [3] in 1950 further developed this model. As it stands, it just produces damped or explosive oscillatory motion. Hicks preferred to model a system producing sustained limited amplitude oscillations and gave an economic explanation for how the model should be changed to give this result: In a depression phase of the business cycle we have $I_t = a\left(Y_{t-1} - Y_{t-2}\right) < 0$, and it can even happen that income decreases at a pace so fast that more capital can be dispensed with than disappears through natural wear. As nobody actively destroys capital, there is a lower limit to disinvestment, called the "floor", and fixed at the (negative) net investment when no worn out capital is replaced at all. So the investment function is changed to $I_t = \max\left(a\left(Y_{t-1} - Y_{t-2}\right), -I^f\right)$, where I^f is the absolute value of the floor disinvestment. (Hicks also suggested that there be a "ceiling" at full employment, when income could not be expanded any further, but we do not consider this at present.)

To the complete model, which Hicks never formulated mathematically, also belong exponentially growing "autonomous expenditures", which produce a growth trend, around which the business cycles, produced by the model, provide the fluctuations . To make this type of model suitable for analysis, the floor and the ceiling must then be assumed to be growing too, and even at the same rate as the autonomous expenditures. This seems to have been Hicks's own tacit assumption, however, all these equal growth rates look fairly arbitrary.

In a previous paper [7], the authors tried their hands at a slight reformulation, through actually relating the floor to the stock of capital, putting $I_t^f = rK_t$, where K_t is capital stock and r is the rate of depreciation. Making this change to the model, we get the benefit that the growing trend need not be exogenously introduced. It results *within* the model through capital accumulation and hence *explains* both the growth trend and the fluctuations around it. Growth is something economists regard as a good feature, but it is no good for the use of standard mathematical methods, as all variables explode to infinity at an exponential rate. To make the model suitable for analysis, we focus on relative dynamics, the rate of growth of income, and the actual capital/output ratio.

Before this reduction to relative dynamics, we, however, have to state the complete model. As noted, the consumption function is $C_t = cY_{t-1}$, and the investment function is $I_t = \max\left(a\left(Y_{t-1} - Y_{t-2}\right), -rK_{t-1}\right)$, where c, a and r are real parameters. As the fraction of income spent is positive but less than unity, $0 < c < 1$. Further, the capital output ratio obviously is a positive number, so $a > 0$. Finally, the capital depreciation is a small positive number, so we definitely have $0 < r < 1$. As before, the income formation equation reads $Y_t = C_t + I_t$, and we now also add an updating equation for capital stock $K_t = K_{t-1} + I_t$. This completes the model. We can eliminate C_t and I_t, and thus obtain a recurrence map in the income and capital variables alone: $Y_t = cY_{t-1} + \max\left(a\left(Y_{t-1} - Y_{t-2}\right), -rK_{t-1}\right)$, and $K_t = K_{t-1} + \max\left(a\left(Y_{t-1} - Y_{t-2}\right), -rK_{t-1}\right)$.

In order to obtain the relative dynamics, define: $x_t := K_t / Y_{t-1}$ and $y_t := Y_t / Y_{t-1}$ These new variables are the actual capital/output ratio, and the relative change of income from one period to the next, i.e. the growth factor. Using these variables results in the following iterated map:

$$x_t = \frac{x_{t-1}}{y_{t-1}} + a\left(1 - \frac{1}{y_{t-1}}\right), \quad y_t = c + a\left(1 - \frac{1}{y_{t-1}}\right)$$

if

$$x_{t-1}\left(a\left(y_{t-1} - 1\right) + rx_{t-1}\right) \geq 0,$$

and

$$x_t = (1-r)\frac{x_{t-1}}{y_{t-1}}, \quad y_t = c - r\frac{x_{t-1}}{y_{t-1}}$$

if

$$x_{t-1}(a(y_{t-1} - 1) + rx_{t-1}) < 0.$$

It is obvious that the domain of definition for this map is not the entire phase plane (x, y), but this plane with exclusion of the line of non-definition, $y = 0$, as well as of all its preimages of any rank.

In the next sections we shall investigate the dynamic properties of this map. We shall see that an attracting fixed point may exist. However, the more interesting features occur in a parameter region in which no attracting fixed point exists, whereas we can have attracting cycles of any period. The bifurcation diagram in the (a, c)-parameter plane shows a structure qualitatively similar to that occurring at a Neimark bifurcation. However, there is no Neimark bifurcation in our model. The main purpose of the present paper is to explain the bifurcation structure associated with such tongues of periodicity. To perform this study we construct a one-dimensional "first-return map" on a suitable segment. I.e., we reduce the degree of our map by using a suitable Poincaré section on a well defined segment which is necessarily visited by the trajectories.

The plan of the work is as follows. After this introduction, section 2 describes the main characteristics of the two maps which are involved in our model, showing a two-dimensional bifurcation diagram at a fixed value of r. We only use one value of r in this paper as any other value of this parameter in its admitted range gives bifurcation diagrams having a qualitatively similar structure. In section 3 we introduce the Poincaré section and show how the bifurcation curves may be detected by using this "first-return map". Section 4 illustrates some more properties of the bifurcation curves, related to the "sausages structure", and section 5 is the conclusion.

Description of the Model

As introduced in the previous section, we are interested in a family of two-dimensional nonlinear discontinuous maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by two maps F_1

and F_2 defined in the regions R_1 and R_2, respectively:

$$F:(x,y)\longmapsto\begin{cases}F_1(x,y), \text{ if } (x,y)\in R_1;\\ F_2(x,y), \text{ if } (x,y)\in R_2;\end{cases}\tag{1}$$

where

$$F_1:\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}x/y+a(1-1/y)\\c+a(1-1/y)\end{pmatrix},\ R_1=\{(x,y):x(a(y-1)+rx)\geq 0\};$$

$$F_2:\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}x(1-r)/y\\c-x/y\end{pmatrix},\ R_2=\{(x,y):x(a(y-1)+rx)<0\}.$$

As we recall, a,c and r are real parameters such that $a>0$, $0<c<1$, $0<r<1$.

One can see that with this definition of the map F the (x,y)-phase plane is separated into four regions by the straight lines $x=0$ and $y=1-rx/a$, so that the map F_1 is defined in $R_1=\{x\geq 0,\ y\geq 1-rx/a\}\cup\{x\leq 0,\ y\leq 1-rx/a\}$ and F_2 is defined in $R_2=\{x<0,\ y>1-rx/a\}\cup\{x>0,\ y<1-rx/a\}$. We call these straight lines *critical lines* and denote them LC_{-1} and LC'_{-1} :

$$LC_{-1}=\{(x,y):y=1-rx/a\};$$

$$LC'_{-1}=\{(x,y):x=0\}.$$

The map F is continuous on LC_{-1}. Its image by F is the straight line

$$LC=\{(x,y):y=c-rx/(1-r)\}.$$

The map F is discontinuous on LC'_{-1}, the image of which, using the map F_1, is the straight line

$$LC'=\{(x,y):y=x+c\},$$

while by using F_2 we get just a point $(0,c)$.

Studying the map F numerically we get an interesting two-dimensional bifurcation diagram in the (a,c)-parameter plane (see Figure 1) with tongues of periodicity which look like the Arnol'd tongues that usually appear due to the Neimark bifurcation. But for the map F this is not the case, as there are no fixed point with complex eigenvalues on the bifurcation curve. So, the main purpose of the present consideration is to explain the origin and structure of these tongues.

Let us first show that the maps F_1 and F_2 alone have rather simple dynamics.

The map F_1 is triangular: The variable y is mapped, independently of x, by a one-dimensional map f:

$$f:y\mapsto\frac{(c+a)y-a}{y}\tag{2}$$

which is a so-called real Möbius map[1]. It has two fixed points denoted y_+ and y_-

$$y_\pm=\frac{c+a\pm\sqrt{(c+a)^2-4a}}{2}\tag{3}$$

which have real values for

$$c\geq c^*\overset{def}{=}2\sqrt{a}-a.\tag{4}$$

At $c=c^*$, these fixed points appear due to a saddle-node bifurcation. For $0<c<c^*$ the map f has neither fixed points, nor cycles of any period (due to the fact that for Möbius maps the solutions to the equation $f^k(y)=y$, $k>1$, are the same as those to $f(y)=y$). Thus, for $0<c<c^*$ any trajectory of f, and of F_1 as well, is diverging.

1 A real rational map $f:\mathbb{R}\to\mathbb{R}$ of the form

$$f:x\mapsto\frac{ax+b}{cx+d},\ \textit{where } ad-cb\neq 0,$$

is called real Möbius map.

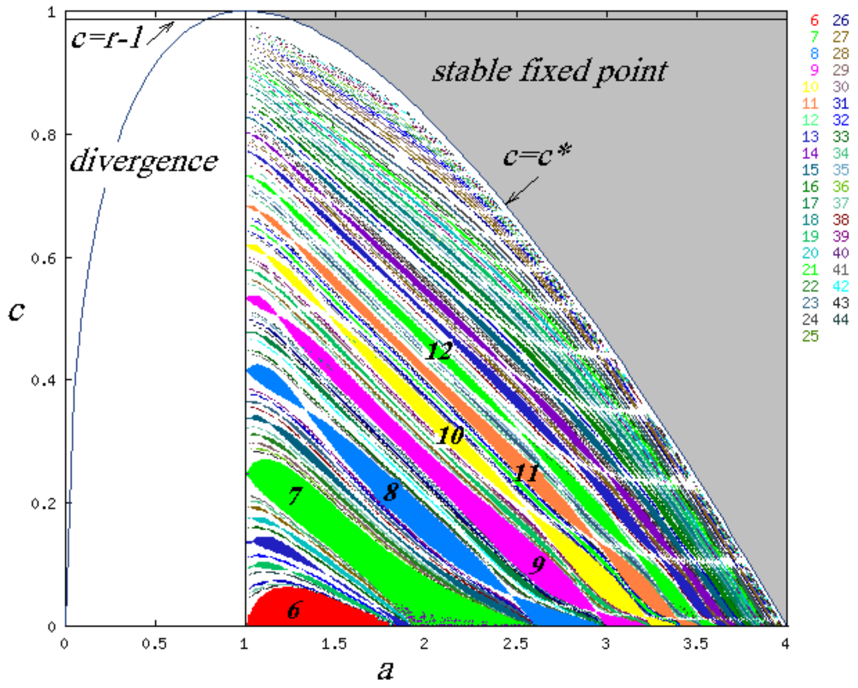

Figure 1: Bifurcation diagram of the map F at $r = 0.01$.

Let us check the stability of y_+ and y_- when they exist. An eigenvalue of f can be written $\lambda_1(y) = a/y^2$. For the parameter range considered $0 < \lambda_1(y_+) < 1$ and $\lambda_1(y_-) > 1$. Thus, the fixed point y_+ is attracting and y_- is repelling.

The corresponding fixed points of the map F_1 are (a, y_-) and (a, y_+). The second eigenvalue of F_1 is $\lambda_2(y) = 1/y$. For $c < 2 - a$ we have $\lambda_2(y_+) > 1$ and $\lambda_2(y_-) < 1$, while for $c > 2 - a$ the inequalities $\lambda_2(y_+) < 1$ and $\lambda_2(y_-) < 1$ hold. Thus, taking (4) into account, we conclude that for $a > 1$, $c^* < c < 1$ the fixed point (a, y_+) is an attracting node and (a, y_-) a saddle, whereas, for $0 < a < 1$, $c^* < c < 1$, (a, y_+) is a saddle and (a, y_-) is a repelling node. To summarize, we can state the following

Proposition 1. *For $0 < c < c^*$ any trajectory of F_1 is diverging. For $0 < a < 1$, $c^* < c < 1$ any trajectory (except those with initial points at (a, y_+) and (a, y_-), or their preimages) is diverging as well.*

Consider now the map F_2. Any straight line (t, mt), $t \in R$, of slope m is mapped by F_2 into one point $(x', y') = ((1 - r)/m, c - r/m)$ belonging to the critical line LC. The dynamic behavior of F_2 is thus reduced to a one-dimensional map on the straight line LC. If x is the first coordinate of a point $(x, y) \in LC$, then its image by F_2 on LC is given by a one-dimensional map g:

$$g : x \mapsto g(x) = \frac{x(1 - r)^2}{c(1 - r) - rx}, \tag{5}$$

which again is a Möbius map. It has two fixed points: $x_1 = 0$ and $x_2 = (1 - r)(c - 1 + r)/r$. If $c < 1 - r$, then x_1 is repelling and x_2 is attracting, while if $c > 1 - r$, then x_2 is repelling and x_1 is attracting. At $c = 1 - r$ these fixed points merge, i.e., $x_1 = x_2 = 0$. The corresponding fixed points of the map F_2 are $p_1(x_1, c)$ and $p_2(x_2, (1 - r))$.

We come back to the map F defined in (1). It is now obvious that only if both maps F_1 and F_2 are applied, we can get attracting cycles corresponding to the tongues of periodicity shown in Figure 1. Numerical analysis shows that for $a < 1$ a generic trajectory of the map F, after some transient, belongs only to R_1 where the

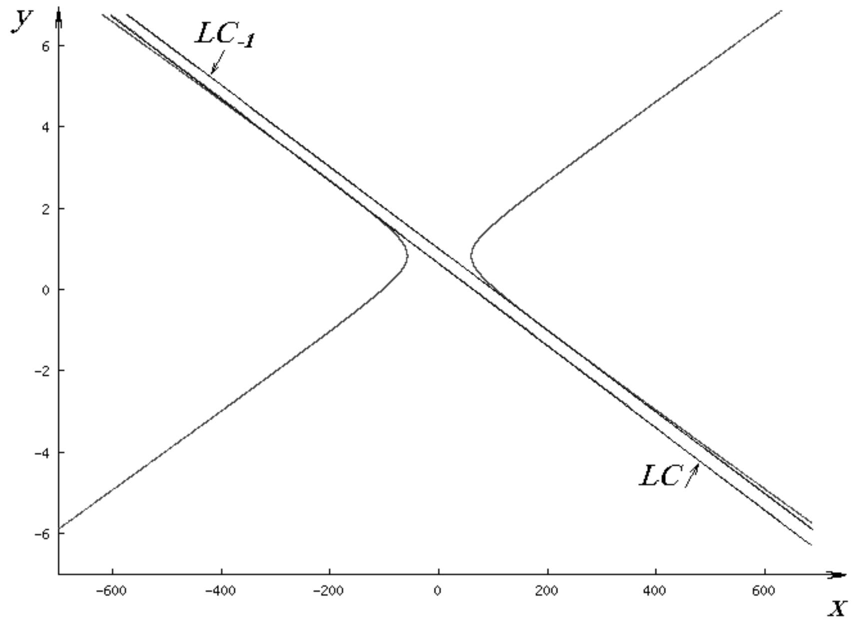

Figure 2: A trajectory of the map F for $r = 0.01$, $a = 1$ and $c = 0.65$. The trajectory is tangent to the boundary of R_1.

map F_1 is applied and, according to Proposition 1, is diverging. (See Figure 2 where a transient part of a trajectory of the map F is shown for $a = 1$ when this trajectory is just tangent to the boundary of R_1). Thus, we restrict our considerations to the case $a > 1$.

First return map

Let the following inequalities hold: $a > 1$, $c < c^*$ and $c < 1 - r$.

Let $[AB]$ denote a segment of LC which belongs to R_1, that is $[AB] = LC \cap R_1$, where $A = LC \cap LC_{-1}$ and $B = LC \cap LC'_{-1}$ (see Figure 3a). Assume that a trajectory has a point $(x_k, y_k) \in R_2$. Then, as it was shown, its image by F_2 belongs to LC, i.e., $(x_{k+1}, y_{k+1}) \in LC$. We have that either $(x_{k+1}, y_{k+1}) \in [AB]$, where the map F_1 applies, or we have to apply the one-dimensional map g given in (5). One can easily check, that, for the parameter range considered, the attracting fixed point x_2 of the map g belongs to $[AB]$ while the repelling fixed point of g is just the point B (see Figure 3b). Thus, approaching x_1, the trajectory must enter the segment $[AB]$, i.e., there exists an integer $s > 0$ such that $(x_{k+1+s}, y_{k+1+s}) \in [AB]$. It follows that we can describe the essential features of the two-dimensional map F by studying a one-dimensional return map on the segment $[AB]$.

Definition 1. *The first return map $\varphi : [AB] \to [AB]$ is a map φ through which the x-coordinate of a given point $(x, y) \in [AB]$ is mapped to the point $\varphi(x)$ which is the x-coordinate of the "first" point satisfying $F^k(x, y) \in [AB]$.*

Clearly, the "second" return map of a point of $[AB]$ to the segment $[AB]$ is given by the map φ^2, and generally, the k-th return on $[AB]$ is given by φ^k. This will be used to characterize the dynamics of the map F completely. Moreover, the following proposition is immediate, and its proof is left as an exercise:

Proposition 2. *Let C be an attracting (repelling) cycle of the map F of period $k > 1$ having p (≥ 1) periodic points in the segment $[AB]$. Then it corresponds to an attracting (repelling) fixed point of the p-return map φ^p.*

We recall that the map F is discontinuous on the y-axis and thus also the map φ

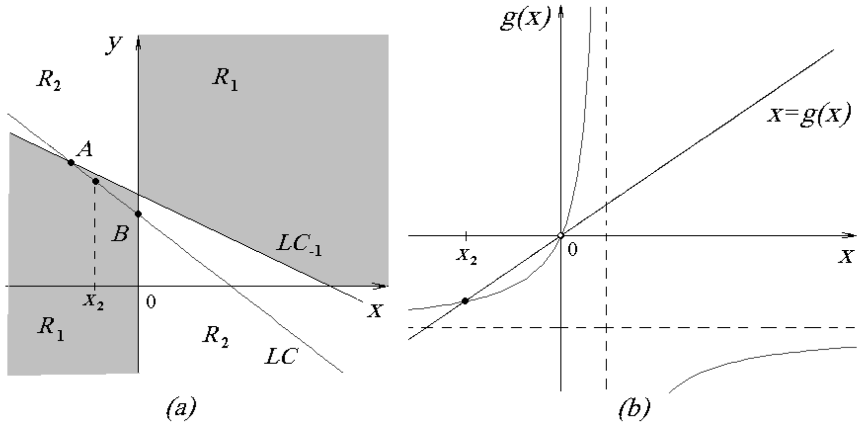

Figure 3: s

uitable for the first return map. In (b) it is shown the one-dimensional map $g(x)$, restriction of F_2 to LC.]In (a) it is shown the segment $[AB]$ suitable for the first return map. In (b) it is shown the one-dimensional map $g(x)$, restriction of F_2 to LC.

is discontinuous. As a matter of fact, we cannot write down the analytical expression for φ, but it is easy to compute it numerically. In the following examples the map φ is used to explain the structure of the periodicity tongues which are shown in Figure 1.

Consider the point $(a, c) = (1.5, 0.32)$, which in Figure 1 is above the 8-tongue, and then decrease c. Our objective is to describe the bifurcations which characterize the upper and lower boundary of the tongue.

Let c_1^* denote the value of c corresponding to the upper boundary of the 8-tongue, and c_2^* denote the value corresponding to the lower boundary. The first return map φ for $c = c_1^* \approx 0.3174$ is shown in Figure 4, from which we can see that at this parameter value a *saddle-node bifurcation* occurs, giving rise to two fixed points of φ existing for $c_2^* < c < c_1^*$ (see Figure 5 where $c = 0.3$). This bifurcation is also a border-collision bifurcation for piecewise-smooth maps, because the saddle-node bifurcation is not a tangent bifurcation as it is defined for different maps on the right and on the left of the contact point.

Clearly, just knowing the shape of φ we cannot deduce the period of the corresponding orbit of F. We can get this information only through iterating F. In this example the period of the two cycles (one attracting and one saddle) is 8.

As c decreases, the graph of φ is modified so that the distance between the two fixed points increases and they both approach the discontinuity points of φ. At the bifurcation value $c = c_2^* \approx 0.2868$ (the point of the lower boundary of the 8-tongue), the fixed points of φ merge with the discontinuity points (see Figure 6) and disappear (for $c < c_2^*$). Thus, the lower bifurcation curve corresponds to the *border-collision bifurcation*.

Similar bifurcations occur in all other "main" tongues in which the rotation number of the q-periodic orbit of F is $1/q$.

The order of the tongues with different periodicity in the bifurcation diagram of Figure 1 follows the usual Farey rule which applies in the similar bifurcation diagram associated with the Neimark bifurcation. This means that between two tongues corresponding to the cycles with rotation numbers p_1/q_1 and p_2/q_2 a tongue exists which corresponds to a cycle with rotation number $(p_1 + p_2)/(q_1 + q_2)$ (see [4], [6], [1], [5]).

In our case a tongue associated with a rotation number p/q corresponds to a q-periodic orbit of the map F having p points in the segment $[AB]$, and thus the cycles are represented as fixed points of the p-return map φ^p. As an example, Figure 7 shows the graph of φ at $a = 1.26, c = 0.149$, which has no fixed points. But

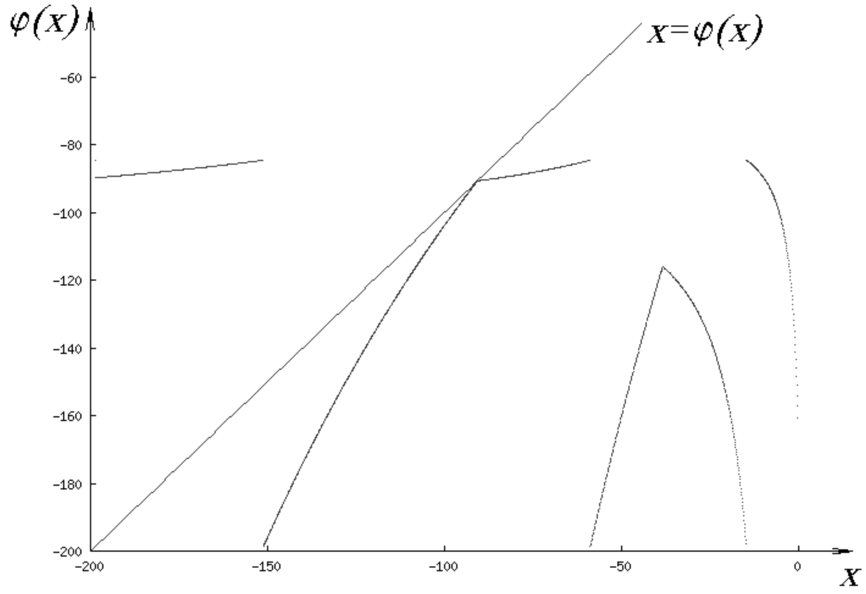

Figure 4: Saddle-node bifurcation bifurcation of the first return map $\varphi(x)$ at $r = 0.01$, $a = 1.5$, $c = 0.3174$.

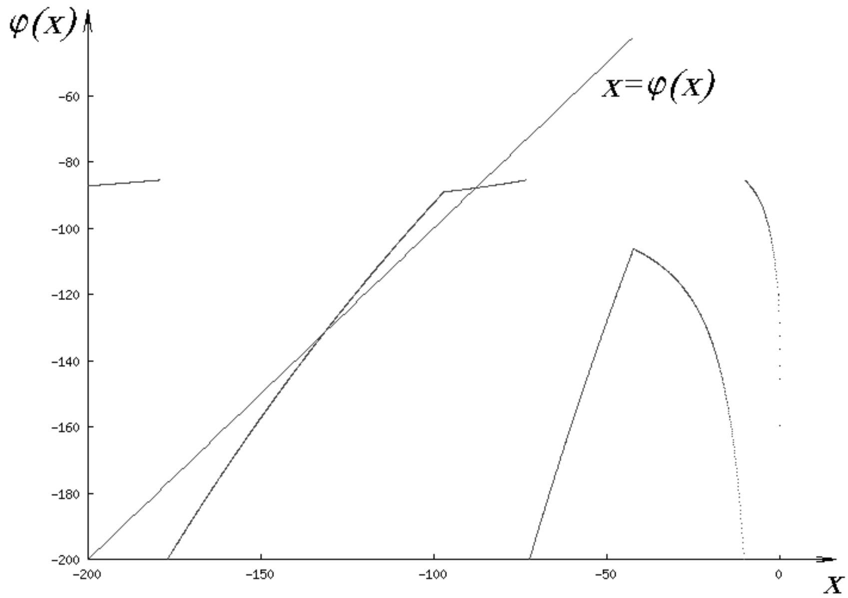

Figure 5: First return map $\varphi(x)$ at $r = 0.01$, $a = 1.5$, $c = 0.3$. Two fixed points are clearly visible, one stable and one unstable.

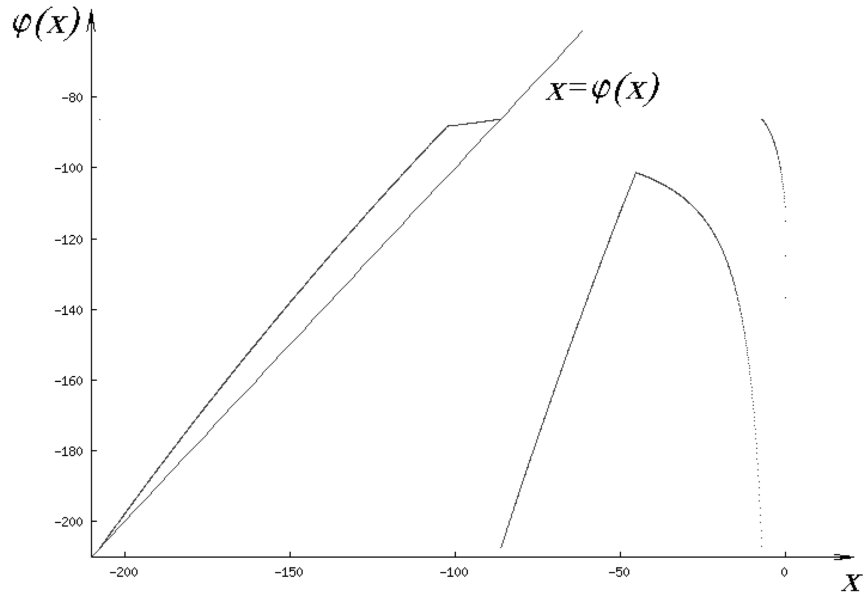

Figure 6: First return map $\varphi(x)$ at $r = 0.01$, $a = 1.5$, $c = 0.2868$. Border-collision bifurcation.

the corresponding map F for such parameter values has a cycle of period $q = 20$ with $p = 3$ points in the segment $[AB]$. The corresponding 3-return map φ^3 is shown in Figure 8.

>From the properties described above we know that the dynamics of the points belonging to the segment $[AB]$ are representative of the dynamics of the map F, and we can state the following

Proposition 3. *Let $a > 1$, $c < c^*$ and $c < 1 - r$. Then an invariant attracting set of the map F belongs to the closure of the set $A = \cup_{n\geq 0} F^n([AB])$*

Sausages structure of the tongues of periodicity

In this section we give reason of the particular shape of the tongues of periodicity which can clearly be seen in Figure 1. This shape is due to the piecewise definition of F, and a similar shape in a two-dimensional parameter plane was already described in [1], [10], [9], [2]. We recall here the main features. Let us consider a point $(a, c) = (1.25, 0.365)$ which belongs to the first area of the sausages structure associated with the tongue of periodicity 8 in Figure 1. At this parameter value (as for any other value in this area), of the periodic points of the $8-$cycle of F two belong to region R_2 and six belong to R_1 (see Figure 9).

At the transition point between the first and the second area of the same tongue, one of the periodic points belongs to the critical line LC_{-1}. For $(a, c) = (1.5, 0.3)$, belonging to the second area (as well as for any other point in this second area) of the tongue of period 8 the 8-cycle of F has three points in region R_2 and five in region R_1 (see Figure 10). For (a, c), belonging to the third area of the same tongue the 8-cycle has four points in region R_1 and four in region R_2.

It follows that inside the different "sausage" portions of the areas associated with a tongue, the period is the same, whereas changes occur in the sequence in which the maps (F_1 and F_2) are applied to give the cycle. The "waist" points of such a structure correspond to a border collision bifurcation of the periodic orbit, which changes only the structure of the cycle (but not its period).

We close this section with a final remark on the dynamics of F associated with points outside the periodicity tongues shown in Figure 1. It is well known that qua-

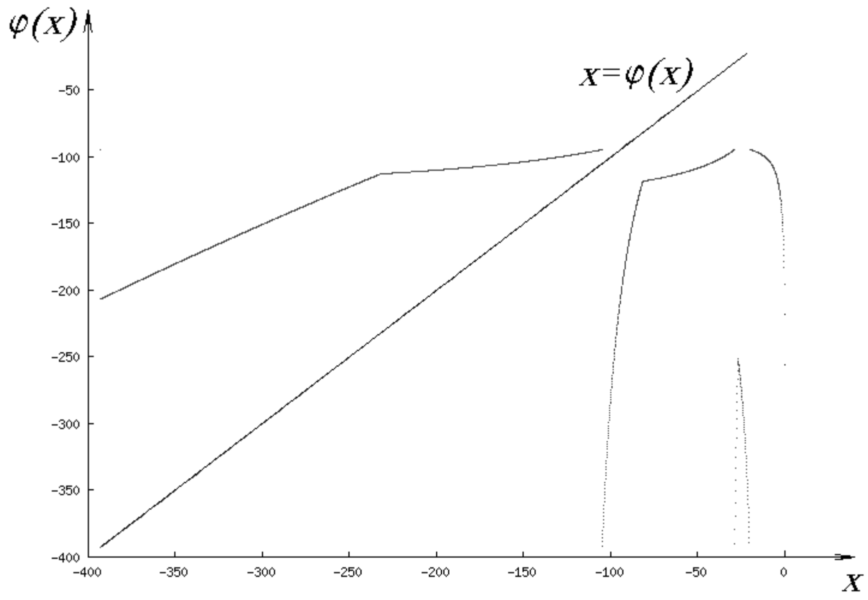

Figure 7: First return map $\varphi(x)$ at $r = 0.01$, $a = 1.26$, $c = 0.149$.

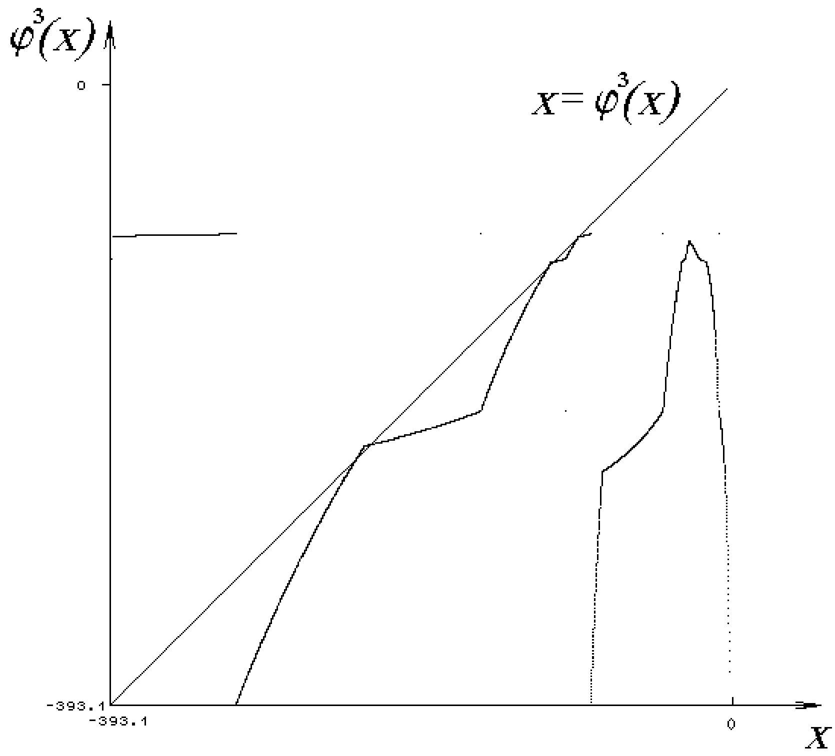

Figure 8: 3-return map $\varphi^3(x)$ at $r = 0.01$, $a = 1.26$, $c = 0.149$. Three stable and three unstable fixed points are clearly visible.

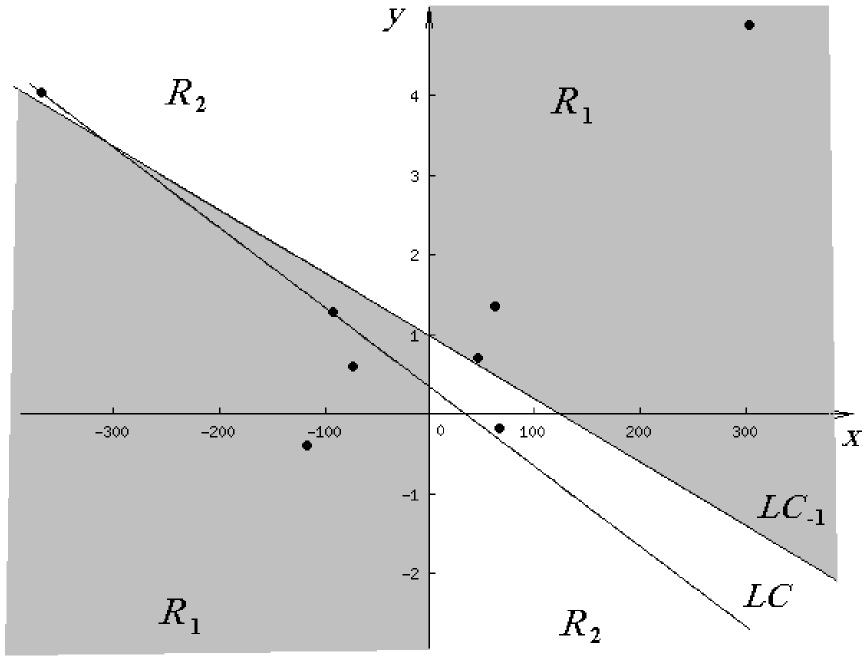

Figure 9: Attracting 8-cycle of F in the phase plane at $r = 0.01$ and $(a, c) = (1.25, 0.365)$ belonging to the first area of the sausage structure of the tongue of periodiciy 8. Two periodc points belong to the region R_2 and six belong to R_1.

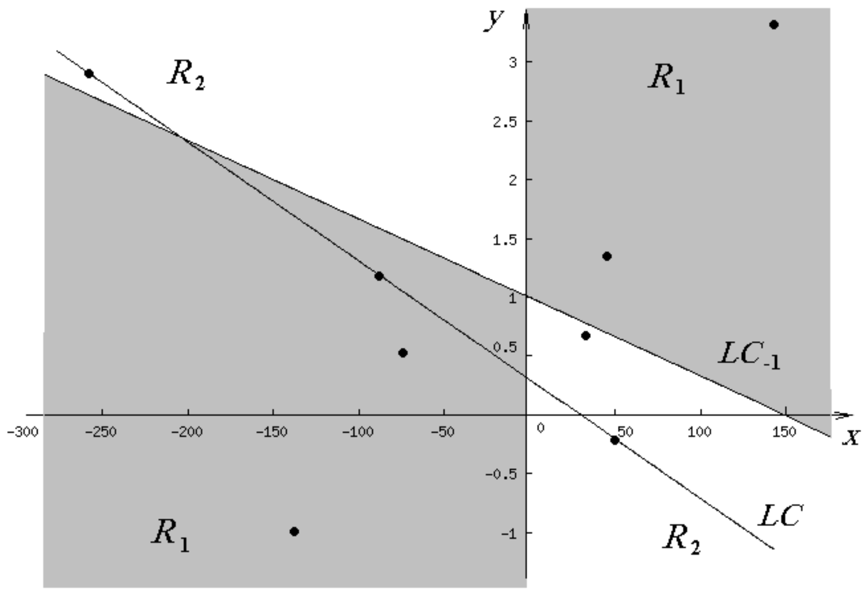

Figure 10: Attracting 8-cycle of F in the phase plane at $r = 0.01$ and $(a, c) = (1.5, 0.3)$ belonging to the second area of the sausage structure of the tongue of periodiciy 8. Three periodc points belong to the region R_2 and five belong to R_1.

siperiodic orbits, associated with the Neimark bifurcation, correspond to irrational rotation numbers, and also that chaotic regimes may exist between the tongues. For the map F it is not easy to give a definition of an "irrational rotation number". However, in such a case we numerically observe an invariant attracting set which consists of two curves (crossing the line of nondefinition $y = 0$), approaching infinity, whose shape is similar to the one shown in Figure 2. Moreover, we conjecture that chaotic dynamics cannot occur for the map F. A numerical computation of the Lyapunov exponents gives one exponent equal to 0 and the second one with a negative value.

Conclusion

In this paper we have considered a two-dimensional piecewise-smooth discontinuous map F representing the so-called "relative dynamics" of the Hicksian business cycle model proposed in [7]. The main features of the dynamics related to this map occur in the parameter region in which no fixed points at finite distance exist, but we may have attracting cycles of any periods. The bifurcations associated with the periodicity tongues of the map have been studied, making use of the first return map on a suitable segment of the phase plane, belonging to an invariant attracting set of the map. We have thus explained the bifurcation curves bounding the periodicity tongues shown in Figure 1 and the related "sausages" structure.

The peculiarity of this bifurcation diagram is that it looks like (and possesses the same properties as) a bifurcation diagram associated with the Neimark bifurcation of a fixed point. However, no fixed point at finite distance is involved on the bifurcation line $a = 1$ of that picture, so that it is not a Neimark bifurcation. Thus, the starting points of the tongues, issuing from the line $a = 1$, are still an open problem for our map F, even if we can explicitly obtain their values (described in [7]). The bifurcation points of a tongue of periodicity of p/q are given by $a = 1$ and $c = 2\cos(2\pi p/q) - 1$. But their relation to the two-dimensional map F is still an open question.

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