Reviving Kalecki's Business Cycle Model in a Growth Context

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Reviving Kalecki's Business Cycle Model in a Growth Context

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Second Draft, Dezember 2016

Abstract

In 1935 Kalecki formulated the first fully specified model of a macroeconomic dynamics in which he studied an endogenous and most elementary business cycle mechanism. To revive his insights, the present paper adapts his stationary economy to a growth context. Introducing a reasonable nonlinearity into the investment function, it then takes care that the model exhibits persistent cyclical behaviour that is characterized by a unique, globally attracting limit cycle. This feature is a robust property. A calibration of the numerical parameters achieves desired values for the cycle period as well as the amplitudes of the output-capital ratio and the capital growth rate. It is moreover demonstrated that a one-time shock to the dynamics can easily have long-lasting effects on the amplitudes. Despite common misgivings about delay differential equations (which here result from Kalecki's implementation lag), the analysis can be conducted with a limited mathematical effort.

JEL classification: C 13, E 12, E 30.

Keywords: Gestation period, delay differential equations, unique limit cycle, calibration, variations in the amplitude.

1 Introduction

As early as 1935, Michał Kalecki advanced the first model of a macroeconomic dynamics, formulated in a precise and rigorous way that also meets mathematical standards. On the one hand, he anticipated the idea of the multiplier-accelerator that

later became famous with the names of Samuelson (1939) and Hicks (1950). In addition, he incorporated as a phenomenon of empirical relevance an implementation lag (or "gestation period"), which is the nonnegligible amount of time that elapses from making an investment decision until the corresponding productive capacity is finally in place.

While it is true that in some versions of the discrete-time Samuelson-Hicks model family the lags may also be interpreted as taking account of such a gestation period, this is usually not an issue of any interest. By contrast, the implementation lag finds a clear expression in Kalecki's model, which is set up in continuous time with an explicit delay in capital formation. This allows Kalecki to highlight the role that this feature plays for generating endogenous cyclical behaviour. In particular, without this lag the motions would be monotonic. As a matter of fact, the model works out the most elementary foundations for a theory of the business cycle.

Kalecki even takes one step further and in an ingenious way derives from German and American statistical data concrete numerical values for the parameters in his model. With the aid of a mathematical analysis he is then able to conclude that the oscillations in his model can indeed be viewed as business cycles, with a reasonable period of ten years. Hence Kalecki also anticipated what only fifty years later, with the advent of the Real Business Cycle school, became known and appreciated as numerical calibration.

The idea of an implementation lag was later taken up by Goodwin (1951) in his nonlinear accelerator model (likewise formulated in continuous time). Here cycles are already generated by his version of an accelerator mechanism and the nonlinearity built into it; the implementation lag is added subsequently "in order to come close to reality" (Goodwin, 1951, p. 11).[1] After that, however, this line of macroeconomic research was not further developed. Thus, Kalecki's model seems today not much more than a historical footnote.

The main reason for neglecting Kalecki's approach to the business cycle is certainly that the delay differential equation which he obtains is more complex to analyse than ordinary differential equations. Many theorists might be deterred by the mathematics they expect they have to master when they turn to Kalecki's concept of an implementation lag.[2] This is a pity in two respects. First, as just

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[1]For a modern mathematical analysis of the role of the length of the implementation lag in Goodwin's model, see Matsumoto and Suzuki (2008). Because this issue of JEDC is devoted to the memory of Carl Chiarella, his alternative treatment of Goodwin's model in Chiarella (1990, Chapters 2 and 3) may be mentioned. He finds it convenient to specify the delay in investment as a distributed lag with exponential decay. This allows him to transform the dynamics into an ordinary nonlinear differential equation of second order with limit cycle behaviour. He is furthermore able to derive qualitative information about the cycle by the method of averaging.

[2]Admittedly, most contributions to differential equations with delays are fairly technical indeed.

indicated, Kalecki reveals the most elementary mechanism to generate endogenous business cycles, historically as well as logically prior to the models advanced by Samuelson/Hicks, Kaldor, Metzler, Goodwin, which proved more influential in heterodox theory. A second point is that the role of mathematics in dealing with delays in continuous-time modelling is not as crucial as it may appear at first sight. Apart from some basic facts, of course, mathematics is actually not necessarily needed—and with growing complexity soon not even available any longer. As a rule, every slightly ambitious model will have to be studied on the computer anyway. The skills required from a researcher are then different from mathematical expertise: here one has to be able to simulate the dynamics and on this basis conduct numerical experiments in systematic and meaningful ways.

Also when other themes in macroeconomics are considered it has to be said that the concept of time delays in a continuous-time framework ekes out a shadowy existence. On the whole, there are just a few specialists who are concerned with them (among them A. Matsumoto, F. Szidarovszky, M. Szydłowski, A. Krawiec). A further reason for their lacking attraction is the fact that virtually all of these models take no great interest in economic theory but focus on mathematical (plus numerical) issues. As a consequence, the models are economically rather simple. In particular, except for applications to the neoclassical Solow growth model, this work restricts itself to stationary economies.[3] This state is somewhat unsatisfactory because one feels that actually it should not be too hard to formulate economically more meaningful growth versions, which nonetheless would give rise to very similar dynamic properties.

This is the point where the present paper sets in. In reconsidering Kalecki's model, it makes a straightforward proposal to translate it into a growth context. Formally, a nonlinearity comes into being in this way, but a basic property of the originally linear dynamics persists, that is, except for a fluke, the oscillations either explode or die out. Therefore, in order to obtain robust fluctuations that are self-sustaining, a reasonable and convenient nonlinearity is subsequently introduced that smoothly bounds investment from above and below. It will be seen that over a wide range of parameters this generates a unique limit cycle that is globally attracting, which permits us to refer to *the* business cycle of the economy. We also follow Kalecki in his calibration endeavour. Numerical parameter values will be given that do not only achieve a desired cycle period, but also desired amplitudes of the output-capital ratio and the capital growth rate.

Returning to the possible fears of getting lost in too technical mathematics,

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[3]The only two exceptions the author knows of are Jarsulic (1993) and Yoshida and Asada (2007). For short it may be said, however, that they get around the problem arising from growth in a somewhat ad-hoc manner.

local stability of the equilibrium is, of course, the first issue an analysis has to deal with. In principle, one could do without mathematics and fix all numerical parameters in the model except two, lay a grid over the plane of these two, simulate the model for each such pair, and record whether the dynamics converges or diverges. In this way a stability frontier is obtained that separates the regions of stability and instability. By varying a third parameter one could furthermore study the resulting shifts of the frontier. Generally, such a procedure would indeed be the method of choice. In the present case, however, a mathematical stability theorem still exists that is not too difficult to apply. While it is a bit more involved than just checking the trace and determinant of a Jacobian matrix, the computer does not mind the arising square root and arc cosine in the stability condition. Nevertheless, only a slight extension of the present model would probably deny us this convenience and we would have to resort to the numerical brute force alternative.

The remainder of the paper is organized as follows. The next section recapitulates Kalecki's model and puts forward our growth version. Section 3 studies the conditions for local stability and cyclical behaviour. Section 4 is devoted to the calibration of the global dynamics and Section 5 discusses the long-lasting effects of a possible shock to it. Section 6 concludes.

2 Formulation of the models

2.1 The stationary economy

The characteristic feature of Kalecki's model is that productive capacities are not created instantaneously. Installing investment goods rather takes a nonnegligible amount of time. The period that elapses for the firms from making a decision to expand their capital until the corresponding additional plant and equipment is finally in place and ready for production is an implementation lag, or a "gestation period" as Kalecki calls it (his quotation marks). It is assumed to be of the same length $\theta > 0$ for any investment project. If $K(t)$ denotes the capital stock at time t and $I = I(t-\theta)$ the flow of *net* investment decided upon at $t-\theta$, we thus have in a continuous-time formulation,[4]

$$\dot{K}(t) \;=\; I(t-\theta) \tag{1}$$

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[4]A dot above a dynamic variable $x = x(t)$ denotes its derivative with respect to time, $\dot{x}(t) = dx(t)/dt$, and a caret later on its growth rate, $\hat{x}(t) = \dot{x}(t)/x(t)$. It may be noticed that Kalecki in his writings uses the symbol I to refer to gross investment.

The determination of this net investment will be set up in a moment. The other investment component, to which the same implementation lag θ applies, is the replacement of capital depreciation. Its level $U > 0$ is supposed to remain invariant over time. The capital goods $U + I(t-\theta)$ ordered at time $t-\theta$ are produced between $t-\theta$ and t at an even pace, which means $(1/\theta)[U + I(t-\theta)]$ units of them are produced per unit of time. The stock of orders still outstanding to be delivered at t or some later date amounts to $\int_{t-\theta}^{t} [U + I(\tau)]d\tau$, and the corresponding production flow $A(t)$ of investment goods at time t is given by

$$A(t) = \frac{1}{\theta} \int_{t-\theta}^{t} [U + I(\tau)] \, d\tau \tag{2}$$

The model supposes continuous market clearing in the short period. Abstracting from inventory investment, government spending and international trade, the other component of aggregate demand is consumption. It is convenient for us to postulate a constant aggregate saving propensity s $(0 < s < 1)$, so that total consumption amounts to $(1-s)Y(t)$.[5] In this way the IS condition reads $Y(t) = A(t) + (1-s)Y(t)$ and determines output as $Y(t) = A(t)/s$. Taking account of (1), the current production of capital goods can be expressed as $A(t) = (1/\theta)\int_{t-\theta}^{t}[U + \dot{K}(\tau+\theta)]d\tau = U + [K(t+\theta) - K(t)]/\theta$. Hence

$$Y(t) = \frac{1}{s}U + \frac{1}{s\theta}\left[K(t+\theta) - K(t)\right] \tag{3}$$

To close the model it remains to represent the investment decisions by a function of output and capital. The firms see their prospects improve and therefore additionally plan new investment projects when demand (and the corresponding flow of production) rises, while they become more cautious and reduce investment if *ceteris paribus* their productive capacity has increased. Specifying this principle in a linear form with two positive constant coefficients $a, b > 0$, the investment function reads,

$$I(t) = aY(t) - bK(t) \tag{4}$$

The changes in the capital stock are then described by $\dot{K}(t) = I(t-\theta) = aY(t-\theta) - bK(t-\theta)$. With the IS output from (3), everything can be reduced to a mixed differential-difference equation in just one variable, the capital stock:

$$\dot{K}(t) = \frac{a}{s}U + \frac{a}{s\theta}\left[K(t) - K(t-\theta)\right] - bK(t-\theta) \tag{5}$$

Its point of rest K^o is given by $K^o = (a/sb)U$, although it is a somewhat problematic feature that a long-run equilibrium depends on two reaction intensities, which the

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[5]Kalecki himself employs the classical saving hypothesis (workers consume all of their wages) and considers capitalist saving out of profits. The formal structure of the model is nevertheless the same.

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agents may more easily change in the short or medium term than a saving propensity. We also note in passing that in order for the economy to be meaningful, the coefficient b in the denominator of K^o must not become zero. Defining J as the deviations from this equilibrium, $J = K - K^o$, we arrive at

$$\dot{J}(t) \;=\; \frac{m}{\theta}\left[J(t) - J(t-\theta)\right] \;-\; n\,J(t-\theta) \qquad (m := a/s,\; n := b) \qquad \text{(KS)}$$

For easier reference later on, 'KS' may stand for Kalecki's stationary model. The symbols m and n are actually taken over from Kalecki (1933, 1935), who obtains the same equation for the deviations of the gross investment orders from the replacement investment U. We prefer to consider the dynamics of the capital stock in (5) or (KS) in order to point out the analogies with the growth model below.[6]

It may have been observed that the dating in the investment function (4) is not obvious. At time t the firms have already placed their orders for final delivery between t and $t+\theta$. Hence they know their capital stock at the future date $t+\theta$ and it seems reasonable that this information will not be disregarded in their decision making. Accordingly, $K(t)$ in (4) could be replaced with $K(t+\theta)$. Nevertheless, the same steps as above can be taken to derive an equation for $\dot{K}(t)$ and it is easily checked that it has the same structural form as (KS); only the coefficients are slightly different then.

It should also be mentioned that the assumption of a constant level U of capital depreciation is essential for the model's (relative) simplicity. As a matter of fact, replacing it with the nowadays usual specification of a constant rate of depreciation δ would introduce an integral $\delta \int_{t-\theta}^{t} K(\tau)\,d\tau$ in place of the constant number U in eq. (3), which determines the level of output $Y(t)$. This issue will indeed be a problem in the modelling of a growing economy, which is our next task.

2.2   Introducing growth

When in a growth framework output and capital are rising in the long run, it is not their levels but certain ratios that can remain constant in a position of balanced growth. Then, a common modelling device is to normalize output by the capital stock. In the absence of time delays, output and capital only need to be considered in the combined form of an output-capital ratio Y/K. This will be different in the present context because there are two distinct dates of the capital stock to take

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[6]The derivation of (5) has followed the reasoning in Gabisch and Lorenz (1989, pp. 72f).

into account. Hence another variable is called for that can serve the purpose of normalizing K.[7]

Our approach is that the firms themselves, independently of the short-term and medium-term fluctuations of the demand directed to them, have an idea of whether their present capacities are approximately appropriate. At the micro level they commission marketing analyses to evaluate the long-term prospects for their specific business. In a macro model we may introduce the notion of the firms' expectations about the state of long-term demand, which irrespective of the business cycle evolves in a fairly steady manner. Being a counterpart of the concept of 'potential output', it may also be referred to as potential demand, denoted D^p. Its path and the expectation formation about it will depend on a number of factors. For our present purpose, however, we are not interested in the variations that may thus come about. We will simply assume that D^p constantly grows at a 'natural' rate $g^n > 0$ (which is determined by the growth of the labour force and the rate of technical progress).

An appropriate capital stock would be a level of K that allows the firms to produce D^p at a 'normal' rate of capital utilization u^n, or a 'normal' output-capital ratio, to be precise. In this case $D^p = u^n K$ would prevail. The appropriate capital stock is thus given by $K = D^p/u^n$ and the ratio $K/(D^p/u^n)$ relates the current capital stock to the appropriate level. A ratio above (below) unity indicates that in a long-term perspective the present capacities are too high (too low). Likewise, current utilization may be different from normal utilization, so that the capital stock will be overutilized or underutilized according to whether u exceeds or falls short of u^n. In sum, the variables in intensive form that we will work with are

$$u(t) := \frac{Y(t)}{K(t)}, \quad k(t) := \frac{K(t)}{D^p(t)/u^n}; \qquad \dot{D}^p = g^n D^p \tag{6}$$

In modelling growing economies, the determination of investment refers to relative rather than absolute changes of the capital stock. Accordingly, an investment function usually models the planned (and realized) capital growth rate $\dot{K}/K$, not just the change $\dot{K}$. Furthermore, in a remark on eq. (KS) it has been noticed that when the firms plan their net investment at time $t-\theta$, they already know their future capital stock $K(t)$. Hence they can then decide on the (instantaneous) rate at which they want K to increase at time t. For the time being we designate this rate $g(t-\theta)$, where the dating indicates the information that can enter the decision bringing about $\hat{K}(t) = \dot{K}(t)/K(t)$. The net investment orders in (1) are thus written in the form

$$I(t-\theta) \;=\; g(t-\theta)\,K(t) \tag{7}$$

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[7]To be more exact, the consequence of using K for normalization in the present simple setting would be a continuum of output-capital ratios that can support a steady state growth path.

where $g(\cdot)$ is a function to be set up more precisely in a moment. Equation (7) together with (1) immediately allow us to derive the law of motion for the capital variable by way of logarithmic differentiation, $\hat{k} = \hat{K} - \hat{D}^p + 0$. Therefore,

$$\dot{k}(t) = k(t)\left[g(t-\theta) - g^n\right] \tag{8}$$

Before turning to the net investment function and its determinants, we have to re-consider replacement investment, which can no longer be treated as a fixed magnitude U. A simple way out would be to assume that it grows in step with potential demand, so that $U(t)/D^p(t) = \text{const}.$[8] This may, however, seem too artificial a device. It has moreover already been indicated above that the nowadays common supposition of a constant rate of depreciation δ, such that $U(t) = \delta K(t)$, would complicate the dynamics substantially; so this option is better discarded, too.

Instead, note that replacement investment means to maintain the existing level of plant and equipment. To a large extent, other goods are needed for this purpose than for installing an entire new investment project. Their production will tend to take less time, and their orders can be relatively well planned in advance. Let us assume that there are firms that produce these goods to stock, from which within the near future they can be delivered at any time they are demanded to restore the capital in the economy. We do not model these activities and their dating explicitly but only the prior inventory investment. A straightforward assumption in this respect is that it is proportional to current output, with some factor $v > 0$.

This idea slightly modifies the specification of the market clearing condition. First, the variable $A(t)$ does no longer include replacement investment, so that eq. (2) changes to

$$A(t) = \frac{1}{\theta} \int_{t-\theta}^{t} I(\tau)\,d\tau \tag{9}$$

Second, in addition to $A(t)$ and consumption, which with a propensity to consume c may presently be written as $cY(t)$, we have inventory investment $vY(t)$ as another component of aggregate demand. In this way the IS equation becomes $Y(t) = A(t) + vY(t) + cY(t)$. Putting $s := 1 - c - v$, the determination of output in (3) changes to

$$Y(t) = \frac{1}{s}A(t) = \frac{1}{s\theta}\left[K(t+\theta) - K(t)\right] \tag{10}$$

whereby any integral expression (alluded to at the end of previous subsection) has been avoided. Normalizing output by potential demand gives us $Y(t)/D^p(t) =$

$[K(t+\theta)/D^p(t) - K(t)/D^p(t)]/s\theta = \{ [u^n K(t+\theta)/D^p(t+\theta)] [D^p(t+\theta)/D^p(t)] - u^n K(t)/D^p(t) \}/u^n s\theta = [k(t+\theta) \exp(\theta g^n) - k(t)]/u^n s\theta$. Utilization $u(t) = Y(t)/K(t) = u^n[Y(t)/D^p(t)] [D^p(t)/u^n K(t)] = u^n Y(t)/D^p(t)/k(t)$ can thus be written as a function of two differently dated capital-to-labour ratios,

$$u(t) \;=\; \frac{1}{s\theta\, k(t)} \left[\exp(\theta g^n)\, k(t+\theta) - k(t) \right] \tag{11}$$

A direct analogy to Kalecki's investment function (4) is obtained if $Y(t)$ is replaced with the output-capital ratio $u(t)$) and the capital stock $K(t)$—or rather $K(t+\theta)$ as we argued above—with the new capital variable $k(t+\theta)$. With two reaction coefficients $\alpha, \beta > 0$ and the dating of eq. (7), we write the investment function in deviation form, where the firms have recourse to the natural rate of growth g^n, normal utilization u^n and the aforementioned capital benchmark $k = 1$,

$$g(t-\theta) \;=\; g^n \;+\; \alpha\,[u(t-\theta) - u^n] \;-\; \beta\,[k(t) - 1] \tag{12}$$

Plugging (11) with $u = u(t-\theta)$ into (12) and the latter into (8), a mixed differential-difference equation in the capital-to-labour ratio is obtained. In writing it down, 'KG' may indicate that here Kalecki's model is adapted to growth:

$$\dot{k}(t) \;=\; k(t) \left\{ \alpha \left[\frac{\exp(\theta g^n)\, k(t) - k(t-\theta)}{s\theta\, k(t-\theta)} - u^n \right] \;-\; \beta\,[k(t) - 1] \right\} \tag{KG}$$

Equation (KG) is the counterpart of Kalecki's dynamic equation as it has been formulated in (5). The main difference is that the capital variables $k(t)$ and $k(t-\theta)$ in the first part of (KG) are divided by $k(t-\theta)$, so that we no longer have a purely linear law of motion. These nonlinearities in (KG) appear, however, rather weak, that is, they will not be expected to contribute much to a stabilization of the economy. The different dating in the last capital term in (5) and (KG) should be inessential; see the remark on eq. (KS).

A long-run equilibrium is constituted by a value of k that lets the right-hand side of (KG) vanish. The firms' benchmarks g^n, u^n and $k = 1$ in their investment function are compatible with such a situation if the fraction in (KG) equals u^n. For simplicity, let us to this end invoke the following consistency condition:[9]

$$\frac{\exp(\theta g^n) - 1}{s\theta} \;=\; u^n \tag{13}$$

Clearly, $k(t+\theta) = k(t) = k^o := 1$ establishes steady state growth and (13) ensures normal utilization u^n on that path. Notice that if the consistency condition is satisfied for all θ sufficiently small, the limit of the fraction on the left-hand side is

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[9]Generally, the firms may have subjective benchmarks which they gradually revise if, in the light of the recent evolution of the economy, they do not seem appropriate any longer. Equation (13) supposes that such an adjustment process has already been successfully completed.

well-defined when the implementation lag shrinks to zero. In fact, this expression converges to g^n/s as $\theta \to 0$.[10] In any case, by virtue of the benchmark values in the investment function (12), the consistency condition as well as the steady state itself are independent of the reaction coefficients α and β. This is a conceptual advantage over an investment formulation like (4).

3   Local stability and the basic cyclical tendencies

The basic mechanism of Kalecki's theory of the business cycle is briefly summarized in the following passage of a first draft on his model (Kalecki, 1933/71, p. 9), where he describes a recovery:

> "An increase in investment orders calls forth an increase in the production of investment goods which is equal to the gross accumulation. This in turn causes a further increase in investment activity.[11] ... However, after an interval of time θ has elapsed from the time when investment orders have exceeded the level of replacement requirements, the volume of capital equipment starts to rise. Initially, this restrains the rate at which investment is increasing, and at a later stage causes a decline in investment orders."

The first stage recounts the upward tendencies that via the investment multiplier are cumulative in character. Its description is perhaps reminiscent of what today's neo-Kaleckians often call Harrodian instability. The second stage points out that the process will eventually come to a halt. Formally, the strength of the recovery is governed by the coefficient a on economic activity in the investment function (4), while the reason for the downturn to come about, or for taming Harrodian instability as we are tempted to say, is the negative feedback of the growing capital stock on investment. These insights may suggest that higher values of the reaction coefficient a tend to destabilize the economy and higher values of b have a stabilizing effect. Before turning to the cyclical potential of Kalecki's model, let us check this intuition and consider the conditions for stable and unstable behaviour.

Starting out from the numerical values for θ, $m = a/s$, $n = b$ that Kalecki proposed himself (which will be made explicit in a moment), the question could

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[10]Expanding the exponential function in an infinite series, we have $[\exp(\theta g^n) - 1]/\theta = [\theta g^n + (\theta g^n)^2/(1 \cdot 2) + (\theta g^n)^3/(1 \cdot 2 \cdot 3) + \ldots]/\theta = g^n[1 + (\theta g^n)/(1 \cdot 2) + (\theta g^n)^2/(1 \cdot 2 \cdot 3) + \ldots] \to g^n$ as $\theta \to 0$.

[11]A little later Kalecki clarifies that this is "a phase of the cycle of the length θ during which capital equipment ... has not yet begun to expand because deliveries of new equipment are as yet lower than the replacement requirements." (Kalecki, 1933/71, p. 10).

be investigated by laying a grid over the (m,n) parameter plane, simulating (KS) for each grid point, and recording whether the equilibrium $K^o = (a/sb)U$ is stable or unstable. As indicated in the Introduction, we would resort to this procedure if there is no convenient mathematics at our disposal, which would actually be the case if the model were only slightly extended.[12] Equation (KS), however, is still so elementary that we can make use of two theorems by Hayes and Burger for delay differential equations. Even simpler, we can skip the details and refer directly to the upshot of an analysis by Gandolfo (1997, p. 563). It says that asymptotic stability of (KS) is ensured if $m + \theta n \leq 1$.[13] Regarding the opposite case $m + \theta n > 1$, which is of greater significance, a sufficient as well as necessary condition for the stability of (KS) is the inequality

$$f(\theta, m, n) := \arccos\left[\frac{m}{m + \theta n}\right] - \sqrt{\theta^2 n^2 + 2m\theta n} > 0 \qquad (14)$$

By virtue of $\arccos(1) = 0$, the function f is zero when $n = 0$. Both arccos and the square root in (14) increase as n rises from zero. Laborious algebra or numerical calculation allows us to conclude that arccos increases faster, so that the expression $f(\theta, m, n)$ is positive for small positive values of n. On the other hand, for positive entries $\arccos(\cdot)$ is bounded from above by $\pi/2$, whereas the square root becomes arbitrarily large as $n \to \infty$. As a consequence, given θ and m there is a value n^H bringing about $f(\theta, m, n^H) = 0$ and stability prevails over an open interval $(0, n^H)$. It moreover turns out that once the function $n \mapsto f(\theta, m, n)$ begins to fall after its initial rise, it does not increase again. Hence such a critical value n^H is uniquely determined and the equilibrium is unstable for all $n > n^H$.

It is thus seen that already an arbitrarily weak negative influence of current capacity levels on investment, $b = n > 0$ but arbitrarily small in (4), is sufficient for stability. Perhaps somewhat surprisingly, however, the equilibrium is destabilized if these reactions are too strong. To make economic sense of this result, it has to be recognized that b is not only responsible for putting a curb on the cumulative expansionary process, it also determines the speed at which the contraction gathers momentum after the turning point has been reached. If b is high, the decline in the capital stock will overshoot its equilibrium value, and the higher b the stronger this overshooting. If b is sufficiently high, the ups and downs of K will increase in size and give rise to oscillations with an ever increasing amplitude.

Given an implementation lag θ, we can for each value of m determine the corresponding critical value $n^H = n^H(m)$. The pairs $(m, n) = (m, n^H(m))$ in the

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[12]An eigen-value analysis may still be possible, but it is quite elaborate and requires numerical methods anyway. Two examples are He and Li (2012, p. 979, Fig. 1) and Matsumoto and Szidarovszky (2015, p. 8, Figures 3–5).

[13]For simplicity, asymptotic stability will be meant when we just speak of 'stability' in the following.

(m,n) parameter plane constitute the stability frontier: (KS) is stable for all pairs below and unstable for all pairs above this line. Owing to the linearity of (KS), the economy neither converges nor diverges if exactly $n = n^H(m)$. Adopting Kalecki's length $\theta = 0.60$ of the implementation lag (see below), Figure 1 shows a numerical computation of the frontier. It is practically a straight line, even though the two functions defining $f(\cdot,\cdot,\cdot)$ are both nonlinear.

The lower line in Figure 1 is the stability frontier for another implementation lag $\theta = 0.75$. It can be viewed as being obtained from a downward rotation of the first line around the anchor $(m,n) = (1,0)$. As this reduces the stability region, we see that a longer lag θ has a destabilizing effect, too.
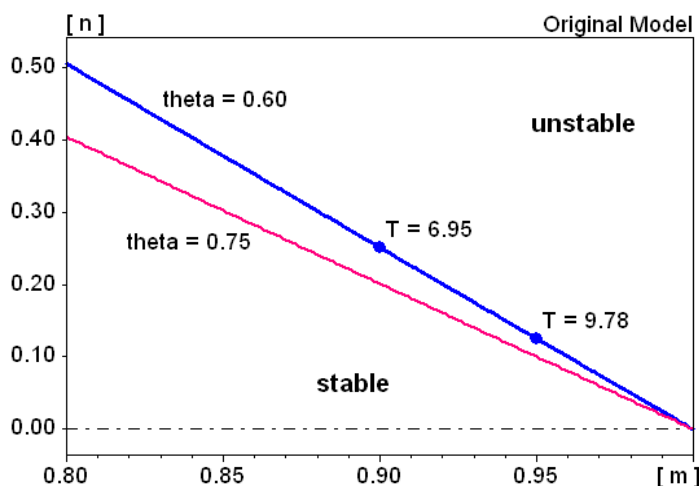
Figure 1: Stability frontiers of the original model (KS).

Kalecki in his 1935-article, where he puts forward the formal linear dynamics (KS), is not concerned with general stability issues but characterizes situations where the equation is able to generate oscillations with a constant amplitude. He even takes one step further and gets numerical information about the cycle period. He starts out from the transcendental equation that determines the 'eigen-values' $\lambda \in \mathbb{C}$ in the characteristic equation of his mixed differential-difference equation (at that time, this approach was already known among economists from Tinbergen, 1931). For later use, we may generally quote the mathematical relationship as,[14]

$$\dot{x}(t) \;=\; A\,x(t) - B\,x(t{-}\theta) \quad \overset{x(t)=\exp(\lambda t)}{\longrightarrow} \quad \lambda \;=\; A - B\exp(-\lambda\theta) \qquad (15)$$

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[14]As with ordinary differential equations, every solution can be written as a linear combination of the characteristic solutions $\exp(\lambda t)$. The difference is that (15) has a (countably) infinite number of eigen-values.

In finer detail, Kalecki considers a transformation of the characteristic equation. Its solution $z = x \pm iy\,(y > 0)$ gives rise to a sine wave with period $T = 2\pi\theta/y$. Applying the mathematics to his model, the condition for a constant amplitude (brought about by $x = m$) is given by the two relationships

$$(i) \qquad \cos y \;=\; m/(m + \theta n)$$
$$(ii) \qquad y/\tan y \;=\; m$$

Kalecki furthermore observes that his structural parameters m n and U, C_1, K_o are linked by[15]

$$(iii) \qquad n \;=\; (m-1)U/K_o + mC_1/K_o \;=\; (m-1)\cdot 0.05 + m\cdot 0.13$$

where in an ingenious reasoning he derives the numerical values for U/K_o and C_1/K_o as well as the implementation lag θ in (i) from German and American statistical records. In this way he reduces his problem to the solution of the three equations $(i) - (iii)$ for the three unknowns m, n and y. The underlying time unit being one year, his result is (Kalecki, 1935, p. 339):

$$\theta = 0.60; \quad m = 0.95; \quad n = 0.121; \quad y = 0.378 \;\to\; T = 2\pi\,\theta/y = 10.0 \qquad (16)$$

In addition, Kalecki presents a table illustrating that the period T is quite robust to wider changes in the numerical values at which U/K_o and C_1/K_o are estimated.[16]

Of course, without a pocket calculator Kaleck's solution (16) has to be an approximation.[17] In particular, the given value of n is slightly lower than $n^H = 0.125$, which we obtain from the condition $f(\theta, m, n) = 0$ and which induces a slightly lower period of $T = 9.78$ years.[18] This situation is represented by the lower point on the stability frontier in Figure 1. On this occasion it may also be mentioned that lower values of m and correspondingly higher values of $n = n^H(m)$ imply shorter cycle periods. An example is given by the second point on the same stability frontier for $m = 0.90$. On the other hand, the period T tends to infinity as m on the frontier approaches unity.

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[15] K_o is the equilibrium capital stock that we designated K^o, while C_1 is specific to Kalecki's version of the IS equation.

[16] Kalecki handed his dynamic equation over to Ragnar Frisch, who thereupon together with H. Holme provided a more general analysis on a criterion for the existence of major cycles with a period T longer than 2θ. In particular, they show that Kalecki's equation has at most one solution with a major cycle. This work was published in the same issue of *Econometrica* as Kalecki's own paper (Frisch and Holme, 1935).

[17] Actually, with $y = 0.378$ one computes $T = 9.97$. Sticking to the three values of m, n, θ, a better approximation for y in (i) and (ii) would be $y = 0.379$, which yields a period of $T = 9.94$ years.

[18] Once $n = n^H$ is known, one can directly compute $T = 2\pi/\sqrt{(m/\theta + n)^2 - (m/\theta)^2}$; see Szydłowski (2002, p. 702, Theorem 1, corrected for an obvious little misprint).

Alternatively to variations in the reaction coefficients m and n, one can fix these parameters and let the gestation period θ increase from zero. From the remark on condition (14) we know that stability prevails if θ is small, whereas sufficiently high values destabilize the economy. Again, once the function $\theta \mapsto f(\theta, m, n)$ begins to decrease, it does so in a monotonic way, so that no reswitching of stability is possible. The loss of stability occurs at two conjugate complex 'eigen-values' (rather than a real eigen-value) crossing the imaginary axis as θ increases.[19]

The phenomenon is usually referred to as a Hopf bifurcation (which is degenerate in nature if the dynamics is linear). This point of view is also taken for other models with a delay, when the analysis reveals the typical result that cycles originate from a Hopf bifurcation upon variation of the delay (some examples are Zak, 1999; Szydłowski, 2002; Szydłowski and Krawiec, 2004; Matsumoto, and Szidarovszky 2011; Ballestra et al., 2013). For Kalecki's model it is therefore a sufficiently implementation lag that can also mathematically be identified as a source of cyclical behaviour. The parsimony of his modelling framework emphasizes that it is indeed the most elementary source.

Not much has to be changed for a stability analysis of the growth version (KG) of the model. Linearizing (KG) around the equilibrium point $[k(t), k(t-\theta)] = (k^o, k^o) = (1, 1)$, it is of the same form as (KS) or (15), respectively:

$$\dot{k}(t) \;=\; A\,[k(t) - 1] \;-\; B\,[k(t-\theta) - 1] \tag{17}$$

The composite coefficients A and B are given in Table 1. They are written in such a way that they can be directly related to the stationary case (KS). Reference can thus be made to the structural coefficients a, b in the investment function (4) or the investment coefficients α, β in the firms' planned growth rate (12) on the one hand, and Kalecki's parameters m, n on the other hand. Starting out from the latter, we have a one-to-one correspondence between (KS) and (KG) if we put

$$\alpha \;=\; s\,(m + \theta\,\beta)\,/\,\exp(\theta g^n), \qquad \beta \;=\; n \tag{18}$$

The saving propensity s in (18) can be determined by solving the consistency condition (13) for s if the value of normal capital utilization u^n is considered to be given. This yields

$$s \;=\; [\exp(\theta g^o) - 1]\,/\,\theta\,u^n \tag{19}$$

While the stability discussion so far was concerned with the linearized growth model (17) and hence the local properties of the nonlinear equation (KG), the next

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[19]To see this, note that the characteristic equation $\lambda = A - B\exp(-\lambda\theta)$ in (15) cannot possibly be fulfilled for $\lambda = 0$. In fact, with $A = m/\theta$ and $B = n + m/\theta$ this would yield the contradiction $0 = A - B = -n < 0$.

	A	B
KS:	$\dfrac{m}{\theta} = \dfrac{a/s}{\theta}$	$\dfrac{m}{\theta} + n = \dfrac{a/s}{\theta} + b$
KG:	$\dfrac{m}{\theta} = \dfrac{\alpha E/s - \theta\beta}{\theta}$	$\dfrac{m}{\theta} + n = \dfrac{\alpha E/s - \theta\beta}{\theta} + \beta$

Table 1: Correspondence between (KS) and the linearized version of (KG) ($E := \exp(\theta g^n)$).

question has to be for the global behaviour of (KG). The answer is that at least within a wide and economically meaningful range, the global behaviour is a copy of the linear dynamics locally around the equilibrium point. That is, for parameter values (m, n) off the stability frontier the economy diverges or converges to the steady state, respectively. If (m, n) are exactly on the frontier, the economy converges to a periodic motion, where the amplitude increases with the size of the shock with which the steady state is initially perturbed. In terms of the discussion above, despite its formal nonlinearity the Hopf bifurcation is practically of a degenerate nature. In sum, all observations on Kalecki's model (KS) of the business cycle carry over to its growth version (KG).

4 Persistent and robust cyclical behaviour

4.1 Augmenting the growth model

Because of its quasi-linear nature, the growth version (KG) shares with Kalecki's original model the property that only very special parameter values are able to generate oscillations that neither explode nor, without shocks, die out over time. A straightforward approach to achieve persistent cyclical behaviour for a broad range of parameters is to relax the linearity assumption in the investment function (12).[20] Indeed, several reasons are conceivable why firms may not want, or are not able, to increase the growth of their capital stock in the same proportion as before when utilization is already relatively high and rises further (and symmetrically in times

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[20]In a follow-up paper, Kalecki (1937) provided a more elaborate economic reasoning than ours that suggests a nonlinear mechanism. It remained at an informal level, however.

of a continual decline in utilization). A corresponding argument may apply to the influence of the capital stock.

Such a ceiling and floor for growth need not be imposed in a crude manner. Instead, the hyperbolic tangent is a convenient specification to smooth out the upper- and lower-bounds in a functional relationship.[21] Let us therefore modify eq. (12) for the planned growth rate of the firms as

$$g(t-\theta) \;=\; g^n \;+\; \mu \tanh\left\{ \frac{1}{\mu} \left[\alpha\,[u(t-\theta)-u^n] \;-\; \beta\,[k(t)-1] \right] \right\} \quad (20)$$

where the new parameter μ represents the maximal amplitude of the growth rate, that is, the rate is restricted to the range $g^n \pm \mu$. With respect to given coefficients α and β, the reactions of $g(t-\theta)$ to changes in $u(t-\theta)$ and $k(t)$ are locally around $g^o = g^n$ approximately the same as in the linear function (12); in a wider distance of $g(t-\theta)$ from g^n they are more restrained.

As before, the coefficients α and β can serve to produce oscillations with a desired period T. However, with α, β on the stability frontier of the quasi-linear equation (KG), the nonlinearity in (20) would dampen these oscillations irrespective of their current amplitude. Hence they would eventually die out (even though extremely slowly). To obtain persistent cycles, the pair α, β should locally destabilize the economy, while further away from the steady state the hyperbolic tangent takes effect and puts an increasingly stronger curb on the fluctuations when they are building up. Ideally, all trajectories will converge to a unique periodic motion—a conjecture that will have to be checked below.

These remarks indicate that we do not only want to calibrate the cycle period of the model, as already Kalecki did, but also the amplitude of its variables. In this respect we will certainly focus on economic activity, which in our growth framework is represented by the output-capital ratio u. Considering that a certain amplitude of u is associated with a certain amplitude of the capital growth rate, an unpleasant feature of the present specification of consumption should not go unnoticed: typically condition (19) leads to very low saving propensities s. With an equilibrium growth rate $g^o = g^n = 3.00\%$, $\theta = 0.60$ and $u^n = 0.90$, for example, a value $s = 0.0336$ is obtained. As can be inferred from the denominator $s\theta$ in the output equations (10) or (11), this implies that a reasonable amplitude of utilization would go along with an extremely low amplitude for capital growth. In numerical simulations of (KG) where m, n produce a period of slightly less than ten years (cf. Figure 1), we observe that the maximal deviation of $u(t)$ from u^n is about thirty

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[21]The hyperbolic function $x \mapsto \tanh(x)$ is an everywhere increasing real function which tends to 1 as $x \to \infty$ and -1 as $x \to -\infty$. Its unique point of inflection is at $x = 0$ with $\tan(0) = 0$ and derivative $\tanh'(0) = 1$. These properties are readily seen to result from the identity $\tanh(x) = [\exp(x) - \exp(-x)] / [\exp(x) + \exp(-x)]$.

times higher than the maximal deviation of $g(t)$ from g^n. This is clearly a highly unsatisfactory ratio.

Excessive multipliers in an IS equation are a widespread phenomenon in elementary post-Keynesian modelling. It does not matter as long as only a model's qualitative properties are of interest, but it becomes a serious problem if, with higher ambitions, one proceeds to numerical experiments. The problem is at least partially defused in extensions of a model that introduce proportional tax rates on several income categories.[22] Here we prefer to take a simpler way by reconsidering the consumption function. We now posit that besides current income, consumption also takes account of the permanent income hypothesis.[23] Proxying permanent income by the level of production Y^n at normal utilization, i.e. $Y^n = u^n K$, total consumption is thus made up by the two components $C(t) = c_n Y^n(t) + cY(t) = c_k K(t) + cY(t)$, with $c_k = c_n u^n$. The corresponding market clearing condition reads $Y(t) = A(t) + vY(t) + c_k K(t) + cY(t)$. Again putting $s := 1 - c - v$, eq. (10) changes to

$$Y(t) \;=\; \frac{1}{s}\,\big[A(t) + c_k K(t)\big] \;=\; \frac{1}{s}\,\bigg[\frac{K(t+\theta) - K(t)}{\theta} + c_k K(t)\bigg] \tag{21}$$

In comparison to (10), s is here a marginal propensity to save out of current income, rather than an average rate. Equation (21) leads to a new version of (11) and determines the IS output-capital ratio as

$$u(t) \;=\; \frac{1}{s\theta\,k(t)}\,\big[\exp(\theta g^n)\,k(t+\theta) \;-\; (1 - \theta c_k)\,k(t)\big] \tag{22}$$

(the reasoning is the same as above). The analogue of the consistency condition (13) for normal utilization in a steady state is obvious. Treating the new coefficient c_k as given and solving the condition for s, the saving propensity results as

$$s \;=\; \big[\exp(\theta g^o) - (1 - \theta c_k)\big] / \theta\, u^n \tag{23}$$

Clearly, the stronger the influence of permanent income in consumption (the higher c_k), the higher the marginal propensity s. Hence, *via* the parameter c_k, there is some scope to reduce the unpleasantly strong multiplier effects implied by (19), where we had $c_k = 0$ in the present notation.

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[22]A systematic study of this effect is Franke (2016b). However, it also points out that this approach adds a new dynamic dimension to a taxless economy, because the bonds that have to finance the government budget deficit will generally change over time (or more precisely in a growth model, the bond-to-capital ratio).

[23]Incidentally, this specification is comparable to Kalecki's (1935, p. 327) constant term in his capitalist consumption.

In sum, the growth version (KG) of Kalecki's model is augmented by a non-linear investment function and the permanent income hypothesis in the consumption function. The dynamics of this economy is described by eq. (8) for the changes $\dot{k}(t)$ in the capital variable, where one substitutes eq. (20) rather than (12) for $g(t-\theta)$, and eq. (22) dated $(t-\theta)$ rather than (11) for $u(t-\theta)$ in the investment function. For convenience, the three constituent components are assembled in an extra block. We may refer to it as (KGA), for **K**alecki **G**rowth model **A**ugmented:

$$
\begin{aligned}
\dot{k}(t) &= k(t)\left[g(t-\theta)-g^n\right] \\
g(t-\theta) &= g^n + \mu \tanh\left\{\frac{1}{\mu}\left[\alpha\left[u(t-\theta)-u^n\right]-\beta\left[k(t)-1\right]\right]\right\} \quad \text{(KGA)} \\
u(t-\theta) &= \frac{1}{s\theta\,k(t-\theta)}\left[\exp(\theta g^n)\,k(t)-(1-\theta c_k)\,k(t-\theta)\right]
\end{aligned}
$$

By construction, the linearization of (KGA) yields exactly the same coefficients A and B on $[k(t)-1]$ and $[k(t-\theta)-1]$ as for (KG). This is due to the fact that $\tanh'(0)=1$ for the derivative of the hyperbolic tangent, and $(1-\theta c_k)$ is not multiplied by $k(t)$ or $k(t-\theta)$ in the expression for $u(t-\theta)$. The new coefficient c_k plays nevertheless an indirect role in the sense that it may increase the saving propensity in (23). However, if with the translation rules of Table 1 and eq. (23) one starts out from the parameters m and n, then, as far as local stability is concerned, this issue remains in the background.

We are thus ready to study the global behaviour of (KGA), for which we have recourse to numerical simulations. The result is quickly told: with the newly intro-duced nonlinear investment function (20), (KGA) works out as sketched above. There is a wide range of parameters that generate locally divergent oscillations while, on the other hand, the flexible ceiling and floor in (20) take care that the motions remain bounded. Moreover, independently of the initial shocks to the steady state path with which we set the system in motion, the trajectories converge to a limit cycle which, by all appearances, is uniquely determined. (An illustration is given in a moment.) Hence (KGA) is indeed able to produce persistent cyclical behaviour that is robust across initial conditions and across parameter values.

4.2 Numerical issues

The claim that the unique a limit cycle represents a business cycle gains in credibil-ity if the model can be calibrated. There are three targets that we want the model to achieve. The first one is the period T. Looking at the post-war cycles of economic activity in the U.S., we may require $T=8.50$ years. The second one is the desired maximal deviation of the capital growth rate from its equilibrium value (designated

gDev), where we refer to the smooth and regular oscillations as we find them in the simulations. Adopting the data and reasoning from Franke (2016b), let us consider gDev = 1.17%. The third target concerns the amplitude of utilization u. Regarding uDev, the distance between the peak value of u and normal utilization u^n, we borrow a desired ratio uDev/gDev = 2.20 from the same paper. The three desired summary statistics are repeated in the first row of Table 2.[24]

	m	n	μ	c_k	$\longrightarrow$	T	Div	gDev	uDev/gDev
Target:						8.50	—	0.0117	2.20
Scenario 1									
KG:	0.940	0.168	—	—		8.47	10.09		divergent
KGA:	0.940	0.168	0.071	0.000		8.50	—	0.0118	30.13
KGA:	0.940	0.168	0.071	0.384		8.50	—	0.0118	2.20
Scenario 2									
KG:	0.946	0.169	—	—		8.46	19.56		divergent
KGA:	0.946	0.169	0.051	0.000		8.50	—	0.0116	30.13
KGA:	0.946	0.169	0.051	0.384		8.50	—	0.0116	2.20

Table 2: Calibrating the period and amplitudes of the limit cycle of (KGA).

Note: Div is the rate at which utilization increases from one peak value to the next (in per cent).

The steady state values of the model are unproblematic. We set the natural rate of growth at $g^n = 3\%$ and normal utilization at $u^n = 0.90$ (see Franke, 2016a, for a discussion of this order of magnitude). Essential for the dynamics of (KGA) are the four behavioural parameters m, n, μ and c_k. We calibrate them in a way that allows us to achieve the targets in three successive steps. To begin with, m and n determine the period of the oscillations as well as the speed at which the quasi-linear system (KG) diverges. The latter dynamics is practically tantamount to (KGA) with $c_k = 0$ and μ excessively large, or to the local behaviour of (KGA) with any values of c_k and μ. While the speed of local divergence will be of no significance for our present purpose where we are concerned with the global dynamics of (KGA) near the limit cycle, it may be informative for a wider characterization of the dynamic

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[24]These statistics are given for concreteness. If preferred, it is easily possible to apply the following procedure to alternative values and obtain similar matches.

properties of (KGA). In a second step, m and n are fixed along with $c_k = 0$ and μ is chosen such as to bring about the desired value of gDev. The shortcoming that this will strongly disregard the desired amplitude of utilization can be overcome in a third step, which brings c_k into play.

In the stage of our first exploratory simulations we noticed that a lowering of the coefficient μ tends to reduce the period T, although only marginally so. For this reason we strive for a period slightly less than 8.50 years when setting m and n in (KG). The speed of divergence (Div) is measured by the rate at which utilization increases from one peak value to the next. To decide on a specific number, let us set our sights on a desired value of Div = 10%. Being content with three significant digits in the parameters, $m = 0.940$ and $n = 0.168$ are found to give rise to a period of 8.47 years and a divergence of 10.09%. This is the first row of what is called Scenario 1 in Table 2.

The second row of Scenario 1 in the table shows the effect of making the non-linearity (20) effective. Setting $\mu = 0.071$ raises the period to exactly the desired length and produces the limit cycle behaviour that has already been pointed out, with an amplitude of the capital growth rate of $g = 3\% \pm 1.18\%$ (with $\mu = 0.072$, gDev = 1.16% would result). The table also shows that the oscillations of the output-capital ratio are entirely unreasonable. They are brought in line with our target by increasing the consumption coefficient from zero to $c_k = 0.384$, a change that leaves the other two statistics T and gDev unaffected.
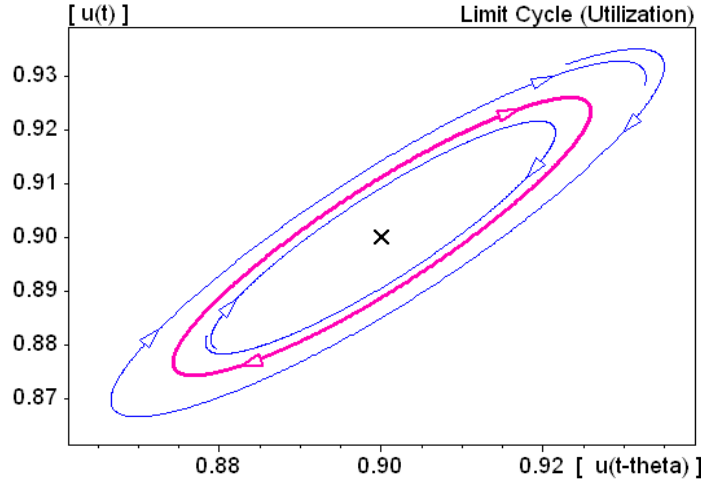
Figure 2: The attractive limit cycle of Scenario 1 projected onto the plane $(u(t-\theta), u(t))$.

These parameters, reported in the third line of Scenario 1, constitute our

showcase for a successful calibration of (KGA). The resulting limit cycle is presented in Figure 2, which plots utilization $u(t)$ against its lagged values $u(t-\theta)$. The diagram also illustrates the uniqueness of the cycle and that the other trajectories converge toward it.

It should, however, be added that a limit cycle with practically the same statistics T, gDev and uDev/gDev is obtained by Scenario 2 given in Table 2. The only difference from the first scenario is its stronger local divergence, which requires a stronger nonlinearity (a lower coefficient μ) to constrain the oscillations to their desired range. Incidentally, the coefficient c_k responsible for uDev/gDev is the same as before. In detail, the shapes of the two limit cycles arising from Scenario 1 and 2 are not perfectly identical, even though they are still so similar that for the naked eye it is all the same.[25]

Scenario 1 and 2 are just two examples for what we want to achieve. Certainly, there is a continuum of parameter vectors (m, n, μ, c_k) producing the same three desired statistics. In this sense the calibration is overdetermined or, from a slightly different point of view, there is still a degree of freedom for m and n that might serve to achieve an additional target. While there is no further meaningful candidate in the present framework, this option might become useful when extending the model.

# 5   Long-range effects in the dynamics

In explaining the mechanism of the business cycle, Kalecki (1933/71, Figure I.4, pp. 9ff; 1935, Figure 2, pp. 341f) also portrays the oscillations of the capital stock. He points out that because gross investment is still above replacement even when it (together with output) is already declining after its peak, there will be a phase shift of about a quarter of a cycle of K relative to I and Y. This regularity carries over to the growth model (KGA). In precise terms, on the benchmark limit cycle of Scenario 1 with its period of 8.50 years, the capital variable $k(t)$ lags utilization $u(t)$ by 2.42 years. The phenomenon, which will also play a role in a moment, is illustrated in Figure 3 by the vertical dotted lines indicating the peak in utilization at $t = 10.07$; $k(t)$ itself peaks at $t = 12.49$.[26]

It may furthermore be mentioned that the capital growth rate lags utilization by 0.30 years (in both Scenario 1 and 2). Considering that Chiarella et al. (2005, p. 206) report maximal cross-correlations between the two for US data of 0.84 and

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[25]This is also the reason that the second scenario is not included in Figure 2 or another diagram.

[26]For the limit cycle of Scenario 2, the lag is 2.44 years. The small difference between the two scenarios will certainly not be overrated.

0.86 at a lag of one and two quarters, respectively, this is not at odds with empirical regularities.
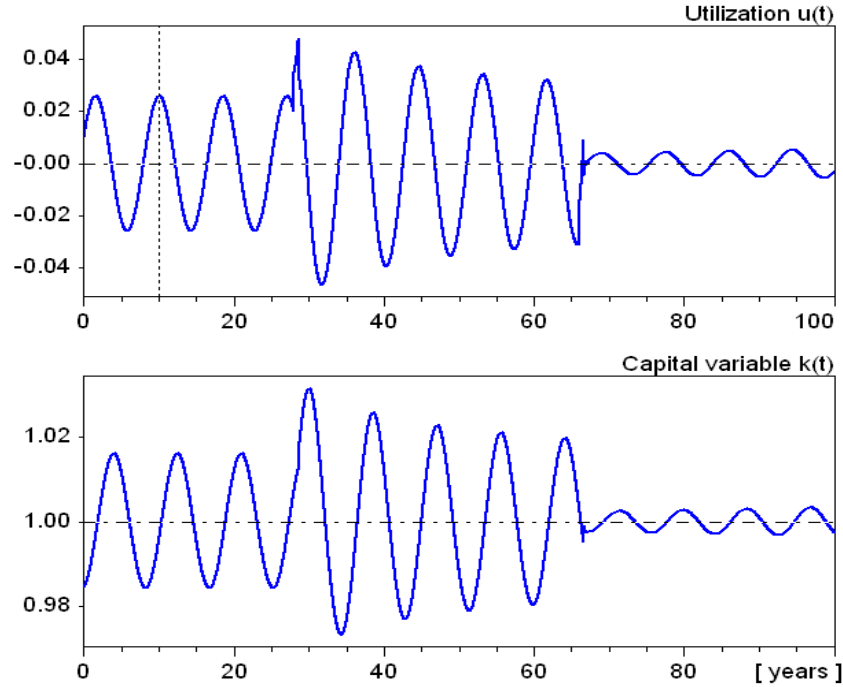
Figure 3: The effect of two positive shocks of equal size in Scenario 1.

From the observation in Figure 2 that off the limit cycle successive turning points are relatively close in the phase plane, it can be inferred that convergence to the limit cycle will typically take quite a while. This property has an important implication if we widen our perspective and consider the phenomenon that the economy continuously experiences exogenous shocks not explained by a model. To reveal in greater clarity what we like to emphasize, suppose the economy is already on its limit cycle and then perturbed by a one-time shock. The fluctuations then setting in (which are again deterministic) will be different from the limit cycle and, in particular, exhibit different amplitudes. Figure 3 demonstrates what may possibly occur.

Concretely, we subject investment (i.e. the planned capital growth rate) to a positive shock at $t_1 = 28$, which is shortly after the peak in utilization. It is so strong that the beginning downturn in $u(t)$ is immediately reversed; utilization rises again and even exceeds the previous peak. This, however, means that also the ensuing and inevitable contraction is stronger than before. Not only the next cycle is in this way more pronounced, but also the following ones. While, of course, their amplitude

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decreases over time, this readjustment to the limit cycle is so slow that it would actually take some fifty years until it is completed.

Let furthermore in this process the economy be hit by a second positive shock of the same size but in a different stage of the business cycle. We choose $t_2 = 66$, which is shortly after the trough in utilization. As expected, this impulse accelerates the recovery. Soon after passing its normal level, however, utilization suddenly falls again. The perhaps surprising switch can be explained by the fact that, because of the phase shift mentioned above, the capital variable is still declining in this stage. Therefore, when $k(t)$ falls short of unity, we have a negative effect on investment and thus economic activity as a whole, which for a moment is even dominant. These changes remain nevertheless a short intermezzo, afterwards utilization resumes its uprise.

Important is another and more permanent effect on the dynamics. Figuratively speaking, the delay in the expansion shortens the run-up for the expansion. As a consequence, the next upper turning point is much lower than before. Moreover, also the oscillations to follow exhibit a much smaller amplitude. An observer from the outside may even be tempted to conclude that the economy has entered a new regime, although we know that nothing in the structure of the economy has changed and the dampening of the amplitude is only the effect from a single shock event.[27]

A very similar pattern to Figure 3, even quantitatively, is obtained for Scenario 2. We take this (and other explorations not shown) as evidence that the features in the diagram are not dependent on a set of very special numerical parameters. Hence, to sum up the little experiment, we have here a most elementary explanation of the variations in the width of the business cycle as they are observed in reality. Already the impulses of infrequent shocks are sufficient to bring them about. However, no complicated shock structure is needed. For lack of other mechanisms in the model (besides the straightforward investment function and the multiplier), we can attribute the shock propagation with its long-range effects to an implementation lag of nonnegligible length. This is a remarkable aspect because it is not captured in other heterodox approaches to business cycle modelling.[28]

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[27]It takes longer than the thirty years shown in Figure 3 until the naked eye safely realizes that the fluctuations begin to widen.

[28]Or at least, comparably strong propagation effects have as yet not been pointed out in the literature.

6 Conclusion

Kalecki was not only the first who set up a rigorous model of macroeconomic dynamics, he also revealed a most fundamental mechanism to generate oscillations of a business cycle frequency. This is an elementary multiplier-accelerator (which in itself does not give rise to cycles) together with the empirical phenomenon of a non-negligible implementation lag in building up capital equipment. Kalecki moreover found numerical parameters for his model that produce a desired cycle period; fifty years before the idea of calibration began to play a role in macroeconomic analysis. Nevertheless, although in appraisals of his life's work the model is highly appreciated by Keynesian or Kaleckian economists, it was practically never resumed and further developed in the business cycle literature. The reason for this neglect is just the implementation lag, which severely limits the scope for a mathematical treatment. It was, however, indicated that with today's prospects of computer simulations this need no longer be a critical hindrance.

An attempt to revive Kalecki's model with its merits has to take account of the fact that it is formulated in the framework of a stationary economy, whereas most heterodox macrodynamic models find no difficulty incorporating long-run growth.[29] This feature should thus represent a standard for a reconsideration of Kalecki's model, and this was exactly the present paper's concern. Actually, it is no insurmountable problem. The paper was able to show that a translation of the model into a growth context is fairly straightforward.

The procedure leads to a nonlinearity, but it so inessential that the properties of the original linear model are preserved. In a next step, therefore, a flexible ceiling and floor were introduced into the investment function, which is a more direct device than what Kalecki (1937) considered in his informal discussion. Their specification turns out to be a full success in the sense that, with parameter sets implying local instability, the nonlinearity prevents the dynamics from exploding. In this way, persistent cyclical behaviour comes about. The nonlinearity is furthermore so effective that it generates limit cycles, unique by all appearances, to which all motions off the steady state growth path converge. Of course, this is a convenient property because such a cycle can be unambiguously referred to as *the* business cycle of the model.

This type of behaviour is robust. Assigning selected numerical values to the parameters can serve to give rise to three desired summary statistics characterizing the business cycle. This is its period and the amplitudes of utilization and the capital growth rate. A last feature that we checked is the effect of infrequent shocks to the economy. They have different consequences depending on the stage of the cycle in

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[29]In contrast to the New-Keynesian mainstream modelling, we cannot resist to add as an aside.

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which they occur. In particular, they can decrease or increase the amplitude of the oscillations, where these effects are rather long-lasting (longer than what is typically obtained in dynamic models without a time delay). This characteristic provides us with an elementary explanation for the time-varying amplitudes observed in the real world.

To sum up, Kalecki's model from 1935 presents the implementation lag of investment as the, so to speak, most natural source of persistence, and it introduces a negative influence of the current level of capacity in the investment function as a most elementary mechanism to put a curb on the cumulative forces in an expansion or contraction. These are two strong reasons why his model, adapted to growth and possibly furnished with a suitable investment nonlinearity, deserves a revival. No doubt, the model lacks all of the features discussed in the contemporaneous post-Keynesian or neo-Kaleckian models. The present contribution may, however, open up a new perspective for future ambitious business cycle research: Take the nonlinear growth version of Kalecki's model and its calibration as a reliable point of departure which is now qualitatively and quantitatively well understood, and step by step incorporate into it some of the variables and feedback effects that have been of interest in this literature.

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