Homework 1
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BIL622-Numerical Analysis II

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1 Solve $y' = (x \times y^2 + x)/(y - x^2 \times y)$ using Euler method and Runge–Kutta method 4

1.1 Solution using Euler method

Euler's method for numerically approximating the solution of a first-order initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ as a table of values. To start, we must decide the interval $[x_0, x_f]$ that we want to find a solution on, as well as the number of points n that we wish to approximate in that interval. Then taking $h = (x_f - x_0)/(n - 1)$ we have n evenly spaced points $x_0, x_1, ..., x_n$, with $x_j = x_0 + j \times h$. Then our objective is then to fill in the values of $y(x_i)$ in the table below.

x	y
x_0	y_0
x_1	$y(x_1)$
x_2	$y(x_2)$
.	.
.	.
.	.
x_n	$y(x_n)$

Let's use Euler's method to approximate the value of the function in the interval [0.0, 0.9] with 10 points. Then $x_0 = 0, y_0 = 1, x_f = 0.9, n = 10$, and $h = (x_f - x_0)/(n - 1) = 0.1$. I have the following Python script [2] to evaluate these values:

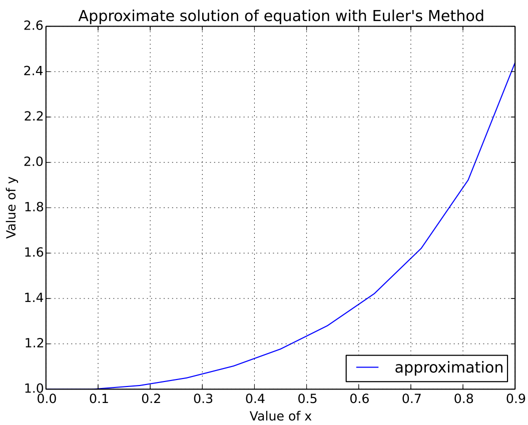
```
import numpy as np
from matplotlib import pyplot as plt

def euler( f, x0, y0, x1, n):
        h = (x1-x0)/float(n)
        t = np.arange(x0, x1+h, h )
        w = np.zeros((n+1,))
        t[0] = x0
        w[0] = y0
```

```
        for i in range(1,n+1):
                w[i] = w[i-1] + h * f(t[i-1], w[i-1])
                t[i] = x0 + i * h
        return t,w
def f(x,y):
        return (x*y**2+x)/(y-x**2*y)
vx,vy = euler(f, 0, 1, 0.9, 10)
for x, y in list(zip(vx, vy))[::1]:
    print("%4.1f %10.16f" % (x, y))
plt.plot(vx, vy, label='approximation' )
plt.title( "Euler's Method Example")
plt.xlabel('Value of x')
plt.ylabel('Value of y')
plt.legend(loc=4)
plt.grid()
plt.savefig( '1_1.eps', fmt='EPS', dpi=100 )
plt.show()
```

I have chosen the interval [0.0, 0.9] because at point 1.0 value of function will be infinite. Following is the table after filling out the values of $y(x_i)$:

x	y
0.0	1.0
0.1	1.0202020202020201
0.2	1.0618770210354367
0.3	1.1279299548267183
0.4	1.223858994411555
0.5	1.3599221002436146
0.6	1.5563525693735585
0.7	1.8581596972015748
0.8	2.3906767135460782
0.9	3.7212406540001011

If we try to plot the graph of this function result will be as follows:



1.2 Solution using Runge-Kutta method 4

Runge-Kutta 4 method is stated as follows: Starting with a given y_n and x_n calculate:
$k_1 = h \times y'(x_n, y_n)$
$k_2 = h \times y'(x_n, +1/2 \times h, y_n + 1/2 \times k_1)$
$k_3 = h \times y'(x_n, +1/2 \times h, y_n + 1/2 \times k_2)$
$k_4 = h \times y'(x_n, +h, y_n + k_3)$
then:
$y_{n+1} = y_n + 1/6 \times (k_1 + 2 \times k_2 + 2 \times k_3 + k_4)$
$x_{n+1} = x_n + h$

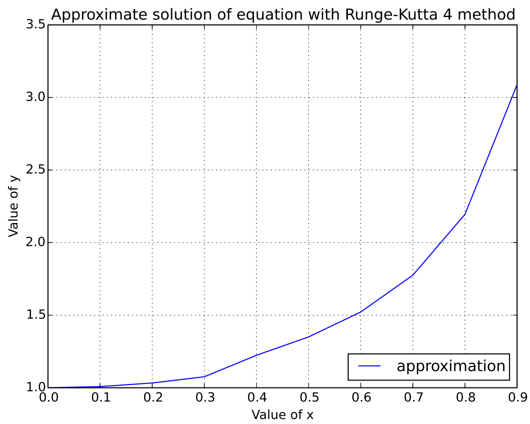
Let's approximate the value of the function in the interval [0.0, 0.9] with 10 points as it was with Euler's method. Then $x_0 = 0, y_0 = 1, x_f = 0.9, n = 10$, and $h = (x_f - x_0)/(n-1) = 0.1$. I have the following Python script [1] to evaluate these values:

```
from math import sqrt
from matplotlib import pyplot as plt

def rk4(f, x0, y0, x1, n):
    vx = [0] * (n + 1)
    vy = [0] * (n + 1)
    h = (x1 - x0) / float(n)
    vx[0] = x = x0
    vy[0] = y = y0
    for i in range(1, n + 1):
        k1 = h * f(x, y)
        k2 = h * f(x + 0.5 * h, y + 0.5 * k1)
        k3 = h * f(x + 0.5 * h, y + 0.5 * k2)
        k4 = h * f(x + h, y + k3)
        vx[i] = x = x0 + i * h
        vy[i] = y = y + (k1 + k2 + k2 + k3 + k3 + k4) / 6
    return vx, vy

def f(x, y):
    return (x*y*y+x)/(y-x*x*y)

vx, vy = rk4(f, 0, 1, 0.9, 10)
for x, y in list(zip(vx, vy))[::1]:
    print("%4.1f %10.16f" % (x, y))

vx2, vy2 = rk4(f, 0, 1, 0.9, 100)
plt.plot(vx2,vy2,'o')
plt.xlabel("Value of x")
plt.ylabel("Value of y")
plt.title("Approximate solution of with Runge-Kutta 4 method")
plt.show()
```

Following is the table after filling out the values of $y(x_i)$:

x	y
0.0	1.0
0.1	1.0081331169094654
0.2	1.0329425001919301
0.3	1.0757627926056470
0.4	1.223858994411555
0.5	1.3502868471569234
0.6	1.5219160643979874
0.7	1.7756474787541259
0.8	2.1945860424968737
0.9	3.0879623126444544

If we try to plot the graph of this function result will be as follows:



2 Solve $y' = (1 - 2 * x)/y^2$ using Modified Euler method 1 and Runge-Kutta method 3

2.1 Solution using Modified Euler method 1

Modified Euler method 1 is stated as follows:
$y_{i+1} = y_i + h/2(y'_i + y'_{i+1})$
$= y_i + h/2(f(x_i, y_i) + f(x_{i+1}, y_{i+1}))$

Let's approximate the value of the function in the interval [0.0, 1.0] with 10 points. Then $x_0 = 0, y_0 = 1, x_f = 1.0, n = 10$, and $h = (x_f - x_0)/(n - 1) = 0.1$. I have the following Python script [3] to evaluate these values:

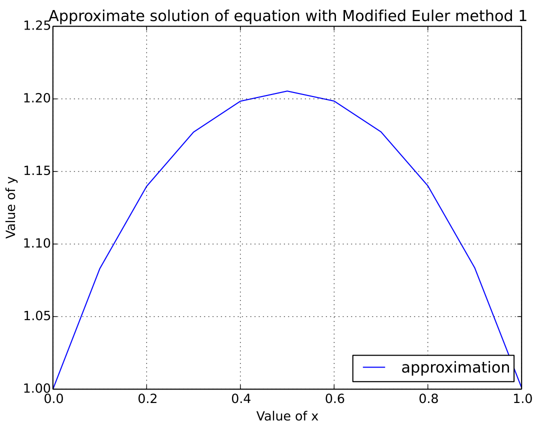
```
import numpy as np
from matplotlib import pyplot as plt

def mod_euler( f, x0, y0, x1, n):
        h = (x1-x0)/float(n)
        t = np.arange(x0, x1+h, h )
        w = np.zeros((n+1,))
        t[0] = x0
```

```
        w[0] = y0
        for i in range(1,n+1):
                f1 = f( t[i-1], w[i-1] )
                f2 = f( t[i], w[i-1] + h * f1 )
                w[i] = w[i-1] + h * ( f1 + f2 ) / 2.0
                t[i] = x0 + i * h
        return t,w
def f(x,y):
        return (1-2*x)/y**2
vx,vy = mod_euler(f, 0, 1, 1.0, 10)
plt.plot(vx, vy, label='approximation' )
plt.title( "Approximate solution of equation with Modified Euler method 1")
plt.xlabel('Value of x')
plt.ylabel('Value of y')
plt.legend(loc=4)
plt.grid()
plt.show()
```

Following is the table after filling out the values of $y(x_i)$:

x	y
0.0	1.0
0.1	1.0830578512396694
0.2	1.1397927856631667
0.3	1.1771044659164211
0.4	1.1984147285629976
0.5	1.2053775574789269
0.6	1.1984949373933313
0.7	1.1772799940498253
0.8	1.1401030742252725
0.9	1.0835983072329833
1.0	1.0010435760606788

If we try to plot the graph of this function result will be as follows:

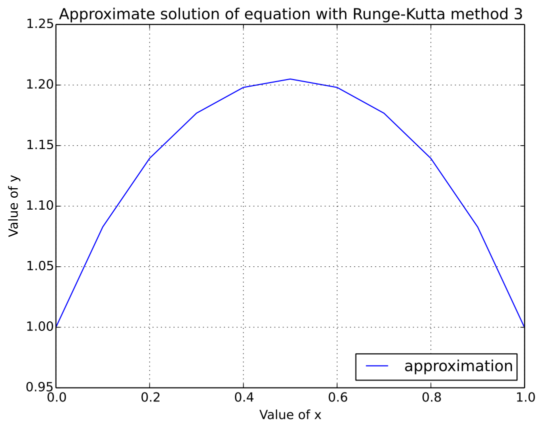


2.2 Solution using Runge-Kutta method 3

Runge-Kutta 3 method is stated as follows: Starting with a given y_n and x_n calculate:
$k_1 = h \times f(x_n, y_n)$
$k_2 = h \times f(x_n + h/2, y_n + k_1/2)$
$k_3 = h \times f(x_n + h, y_n - k_1 + 2 \times k_2)$
then:
$y_{n+1} = y_n + 1/6 \times (k_1 + 4 \times k_2 + k_3)$
$x_{n+1} = x_n + h$
Let's approximate the value of the function in the interval [0.0, 1.0] with 10 points. Then $x_0 = 0, y_0 = 1, x_f = 1.0, n = 10$, and $h = (x_f - x_0)/(n-1) = 0.1$. I have modified Python script [1] to evaluate the values to get the following table of values:

x	y
0.0	1.0
0.1	1.0828822813050640
0.2	1.1395306642445624
0.3	1.1767842783825406
0.4	1.1980457654770715
0.5	1.2049593292333054
0.6	1.1980185265294063
0.7	1.1767247315617080
0.8	1.1394256727804197
0.9	1.0827006273059878
1.0	0.9996565522637323

If we try to plot the graph of this function result will be as follows:



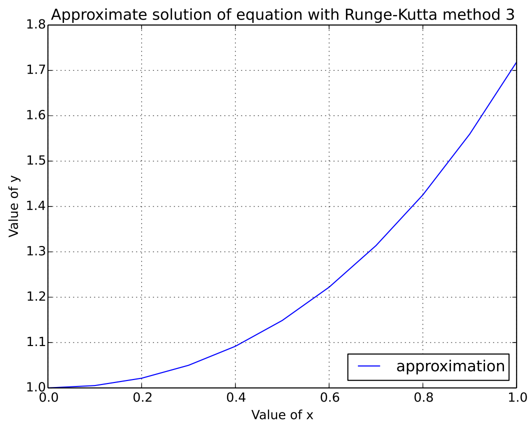
3 Solve $y' = e^x - 1$ using Modified Euler method 2 and Runge-Kutta method 4

3.1 Solution using Runge-Kutta method 4

Let's approximate the value of the function in the interval [0.0, 1.0] with 10 points. Then $x_0 = 0, y_0 = 1, x_f = 1.0, n = 10$, and $h = (x_f - x_0)/(n-1) = 0.1$. Following is the table after filling out the values of $y(x_i)$:

x	y
0.0	1.0
0.1	1.0051709217263292
0.2	1.0214027658454783
0.3	1.0498588197202638
0.4	1.0918247147134355
0.5	1.1487212932184701
0.6	1.2221188289278071
0.7	1.3137527426597500
0.8	1.4255409710333107
0.9	1.5596031618225337
1.0	1.7182818881038566

If we try to plot the graph of this function result will be as follows:

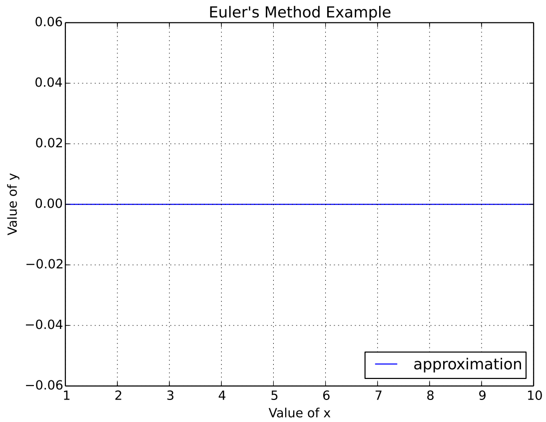


4 Solve $y' = (y^2 - y)/x^2$ using Euler method and Runge-Kutta 4 method

Let's try to approximate the value of the function in the interval [0.1, 1.0] with 10 points (if we start with 0 it will give division by zero). Then $x_0 = 0.1, y_0 = 1.0, x_f = 0, n = 10$, and $h = (x_f - x_0)/(n-1) = 0.1$. Following is the table after filling out the values of $y(x_i)$:

x	y
0.0	0.0
0.1	0.0
0.2	0.0
0.3	0.0
0.4	0.0
0.5	0.0
0.6	0.0
0.7	0.0
0.8	0.0
0.9	0.0
1.0	0.0

If we try to plot the graph of this function result will be as follows:



Both methods give the same solution.

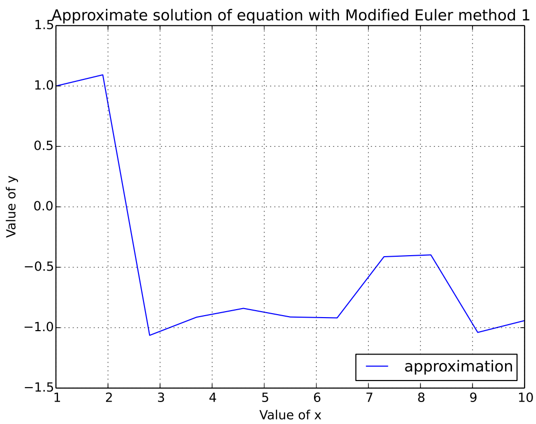
5 Solve $y' = (1 + y)/tanx$ using Modified Euler method 1 and Runge-Kutta method 4

5.1 Solution using Modified Euler method 1

Let's try to approximate the value of the function in the interval [1.0, 10.0] with 10 points. Then $x_0 = 1.0, y_0 = 1.0, x_f = 10.0, n = 10$, and $h = (x_f - x_0)/(n - 1) = 0.9$. Following is the table after filling out the values of $y(x_i)$:

x	y
1.0	1.0000000000000000
1.9	1.0927286877177949
2.8	−1.0633684619181731
3.7	−0.9132600104639936
4.6	−0.8400285429342458
5.5	−0.9115549541314385
6.4	−0.9189695157150598
7.3	−0.4126897959522442
8.2	−0.3975307758698257
9.1	−1.0392086151305469
10.0	−0.9412789357986875

If we try to plot the graph of this function result will be as follows:

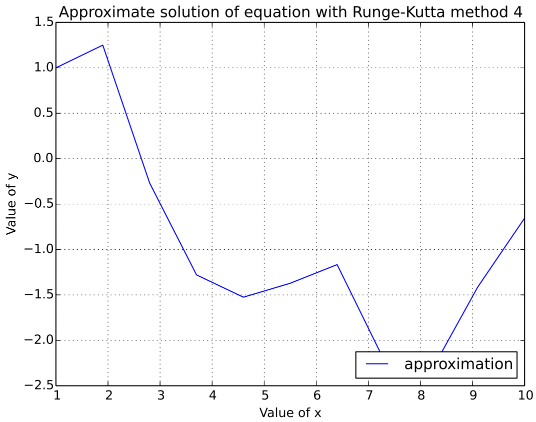


5.2 Solution using Runge-Kutta method 4

I will be using the same interval as was above and we get the following values:

x	y
1.0	1.0000000000000000
1.9	1.2499889202642265
2.8	−0.2682690087484034
3.7	−1.2796185692468598
4.6	−1.5251095569943645
5.5	−1.3714977166209588
6.4	−1.1663710130927201
7.3	−2.2165022273451749
8.2	−2.3461039207834338
9.1	−1.4133945810421138
10.0	−0.6546702603992544

If we try to plot the graph of this function result will be as follows:



References

[1] Rosetta Code Link,
`https://rosettacode.org/wiki/Runge-Kutta-method`

[2] Euler Method PDF,
`https://sites.math.washington.edu/ wcasper/math307-win16/review/euler-method/euler-meth`

[3] Numerical Solutions to ODEs,
`http://connor-johnson.com/2014/02/21/numerical-solutions-to-odes/`