Homework 3
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BIL622-Numerical Analysis II

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# 1 Problem

Given the second order differential equation $(x^2 + 1) \times y'' = 2 \times x \times y'$ with initial conditions $y(0) = 1, y'(0) = 3$ at the interval $[0; 1]$, find the solution using Runge-Kutta $4^{th}$ order method with $h = 0.2$. Compare results with the solution of the equation given by $y = -x^3 + 3 \times x + 1$.

# 2 Solution

It works by splitting the problem into 2 first-order differential equations:
$y' = u$
$u' = f(x, y, u)$
In our case it will be as follows:
Equations is given as: $(x^2 + 1)y" - 2xy' = 0; y(0) = 1; y'(0) = 3$
Splitting into 2:
$y" = 2xy'/(x^2 + 1)$ or the $f(x, y, y')$
$u' = 2xu/(x^2 + 1)$ or the $f(x, y, u)$
Solution for 6 points as a table in the interval $[0, 0.5]$ is as follows:

| $x$ | $y$ |
|---|---|
| 0.0 | 1.0000000000000000 |
| 0.2 | 1.6079992157631604 |
| 0.4 | 2.2639946460128000 |
| 0.6 | 3.0159859627546286 |
| 0.8 | 3.9119736243068348 |
| 1.0 | 4.9999579899703832 |

We can use the following expression to evaluate the absolute error, which is the sum of the absolute values of the residuals:

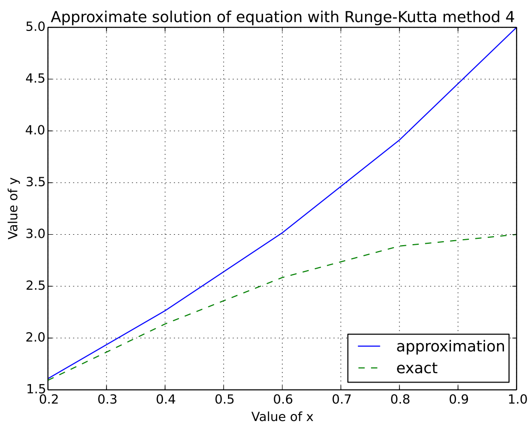
$$\varepsilon_{abs} = \sum_{i=1}^{N} |y(x_i) - w_i|$$

I have used the following Python code [1] to evaluate values and plot the graph.

```
from matplotlib import pyplot as plt
import numpy as np

x = 0.0
y = 1.0
u = 3.0

h = 0.2
vx=[]
vy=[]

print "x   ", "y"
print("%4.1f %10.16f" % (x, y))

while(x<1.0):
        m1 = u
        k1 = 2*x*u/(x**2+1)
        m2 = u + (h/2.)*k1
        x_2 = x + (h/2.)
        y_2 = y + (h/2.)+m1
        u_2 = m2
        k2 = 2*x_2*u_2/(x_2**2+1)
        m3 = u + (h/2.)*k2
        x_3 = x + (h/2.)
        y_3 = y + (h/2.) * m2
        u_3 = m3
        k3 = 2*x_3*u_3/(x_3**2+1)
        m4 = u + h * k3
        x_4 = x+h
        y_4 = x+h*m3
        u_4 = m4
        k4 = 2*x_4*u_4/(x_4**2+1)
        x = x + h
        y = y + (h/6.)*(m1+(2.*m2)+(2.*m3) + m4)
        u = u + (h/6.)*(k1+(2.*k2)+(2.*k3) + k4)
        vx.append(x)
        vy.append(y)

def y(x):
        return -x**3+3*x+1

yx = []
for v in vx:
        yx.append(y(v))

def error_abs(v,w):
        total_err = 0
        for i in range(0,len(v)):
                total_err = total_err + np.abs(v[i]-w[i])
        return total_err
```

```
for x, y in list(zip(vx, vy))[::1]:
    print("%4.1f %10.16f" % (x, y))
plt.plot(vx, vy, label='approximation' )
plt.plot( vx, yx, linestyle='--',label='exact' )
plt.title( "Approximate solution of equation with Runge-Kutta method 4")
plt.xlabel('Value of x')
plt.ylabel('Value of y')
plt.legend(loc=4)
plt.grid()
plt.savefig( '1_1.eps', fmt='EPS', dpi=100 )
plt.show()

err = error_abs(yx, vy)
print err
```

Error value equals to 3.59991143881.
If we try to plot these values and the exact function as a graph result is as follows:




# References


[1] Second order ODE solved with RK4 in Python, `https://www.youtube.com/watch?v=-k64Xa0onfQ`