Homework 3
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BIL622-Numerical Analysis II

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1 Problem

Given the second order differential equation $(x^2 + 1) \times y'' = 2 \times x \times y'$ with initial conditions $y(0) = 1, y'(0) = 3$ at the interval $[0; 1]$, find the solution using Runge-Kutta 4^{th} order method with $h = 0.2$. Compare results with the solution of the equation given by $y = -x^3 + 3 \times x + 1$.

2 Solution

It works by splitting the problem into 2 first-order differential equations:
$y' = u$
$u' = f(x, y, u)$
In our case it will be as follows:
Equations is given as: $(x^2 + 1)y\text{"} - 2xy' = 0; y(0) = 1; y'(0) = 3$
Splitting into 2:
$y\text{"} = 2xy'/(x^2 + 1)$ or the $f(x, y, y')$
$u' = 2xu/(x^2 + 1)$ or the $f(x, y, u)$
Solution for 6 points as a table in the interval $[0, 0.5]$ is as follows:

x	y
0.0	1.0000000000000000
0.2	1.6079992157631604
0.4	2.2639946460128000
0.6	3.0159859627546286
0.8	3.9119736243068348
1.0	4.9999579899703832

We can use the following expression to evaluate the absolute error, which is the sum of the absolute values of the residuals:

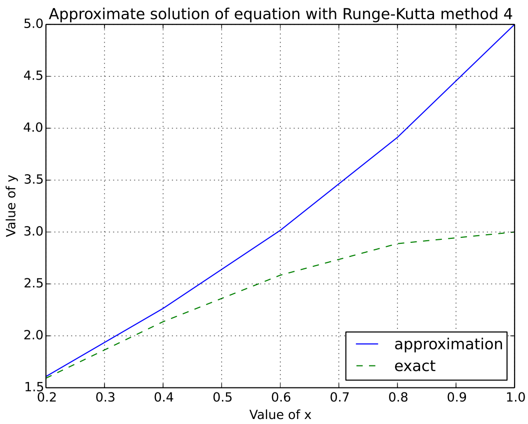
$$\varepsilon_{abs} = \sum_{i=1}^{N} |y(x_i) - w_i|$$

I have used the following Python code [1] to evaluate values and plot the graph.

```
from matplotlib import pyplot as plt
import numpy as np

x = 0.0
y = 1.0
u = 3.0

h = 0.2
vx=[]
vy=[]

print "x   ", "y"
print("%4.1f %10.16f" % (x, y))

while(x<1.0):
        m1 = u
        k1 = 2*x*u/(x**2+1)
        m2 = u + (h/2.)*k1
        x_2 = x + (h/2.)
        y_2 = y + (h/2.)+m1
        u_2 = m2
        k2 = 2*x_2*u_2/(x_2**2+1)
        m3 = u + (h/2.)*k2
        x_3 = x + (h/2.)
        y_3 = y + (h/2.) * m2
        u_3 = m3
        k3 = 2*x_3*u_3/(x_3**2+1)
        m4 = u + h * k3
        x_4 = x+h
        y_4 = x+h*m3
        u_4 = m4
        k4 = 2*x_4*u_4/(x_4**2+1)
        x = x + h
        y = y + (h/6.)*(m1+(2.*m2)+(2.*m3) + m4)
        u = u + (h/6.)*(k1+(2.*k2)+(2.*k3) + k4)
        vx.append(x)
        vy.append(y)

def y(x):
        return -x**3+3*x+1

yx = []
for v in vx:
        yx.append(y(v))

def error_abs(v,w):
        total_err = 0
        for i in range(0,len(v)):
                total_err = total_err + np.abs(v[i]-w[i])
        return total_err
```

```
for x, y in list(zip(vx, vy))[::1]:
    print("%4.1f %10.16f" % (x, y))
plt.plot(vx, vy, label='approximation' )
plt.plot( vx, yx, linestyle='--',label='exact' )
plt.title( "Approximate solution of equation with Runge-Kutta method 4")
plt.xlabel('Value of x')
plt.ylabel('Value of y')
plt.legend(loc=4)
plt.grid()
plt.savefig( '1_1.eps', fmt='EPS', dpi=100 )
plt.show()

err = error_abs(yx, vy)
print err
```

Error value equals to 3.59991143881.
If we try to plot these values and the exact function as a graph result is as follows:



References

[1] Second order ODE solved with RK4 in Python,
`https://www.youtube.com/watch?v=-k64Xa0onfQ`