1 Introduction

Simpson's rule is a method for performing numerical integration. While there are several different approximations under the name "Simpson's rule", the most common version, Simpson's 1/3 rule, is equivalent fitting a 2'nd degree polynomial to three points so that the integration can be carried out analytically. It is usually written as:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{1}$$

This can be extended to larger domains with more function evaluations by breaking the domain into subdomains and applying 1 to each. However, equation 1 assumes that the function $f(x)$ is evaluated at equally spaced points. If we have a discretized function that is *not* evaluated at equally spaced points, equation 1 cannot be used.

2 Generalizing to Non-Uniform Spacing

To generalize equation 1 to non-equally spaced points, we just need to perform the polynomial fit using non-equally spaced points an perform the integration on the new fit. Using Lagrange polynomial interpolation, we have

$$f(x) \approx f(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \tag{2}$$

where x_1, x_2, and x_3 are the points at which the function is evaluated.

The Lagrange basis polynomials can be written as

$$l(x, A, B, C) = \frac{(x-A)(x-B)}{(C-A)(C-B)} \tag{3}$$

and we can rewrite the interpolation as

$$f(x) \approx f(x_1)l(x, x_2, x_3, x_1) + f(x_2)l(x, x_1, x_3, x_2) + f(x_3)l(x, x_1, x_2, x_3) \tag{4}$$

2.1 Method 1: Integrating the Lagrange polynomials directly

Integrating the Lagrrange basis polynomials from a to b gives

$$L(A, B, C, a, b) = \int_a^b l(x, A, B, C)\mathrm{d}x = \frac{1}{(C-A)(C-B)}\left[\frac{1}{3}\left(b^3 - a^3\right) + \frac{A+B}{2}\left(b^2 - a^2\right) + AB(b-a)\right] \tag{5}$$

and the integral of $f(x)$ is then approximated as

$$\int_a^b f(x)\mathrm{d}x \approx f(x_1)L(x_2, x_3, x_1, a, b) + f(x_2)L(x_1, x_3, x_2, a, b) + f(x_3)L(x_1, x_2, x_3, a, b) \tag{6}$$

To reproduce equation 1, we let $x_1 = a$, $x_2 = \frac{a+b}{2}$, and $x_3 = b$. For example,

$$L(x_2, x_3, x_1, a, b) = \frac{1}{(a - \frac{a+b}{2})(a-b)}\left[\frac{1}{3}\left(b^3 - a^3\right) + \frac{\frac{a+b}{2} + b}{2}\left(b^2 - a^2\right) + \frac{a+b}{2}b(b-a)\right] \tag{7}$$

apparently reduces to $\frac{b-a}{6}$...

2.2 Method 2: Integrating the Lagrange polynomials by parts

Integrating the Lagrange polynomials by parts gives

$$L(A,B,C,a,b) = \int_a^b l(x,A,B,C)\mathrm{d}x = \frac{1}{(C-A)(C-B)}\int_a^b (x-A)(x-B)\mathrm{d}x \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(x-A)(x-B)^2\right] = \frac{1}{2}(x-B)^2 + (x-A)(x-B) \tag{9}$$

$$\int (x-A)(x-B)\mathrm{d}x = \frac{1}{2}(x-A)(x-B)^2 - \frac{1}{2}\int (x-B)^2\mathrm{d}x \tag{10}$$

$$L(A,B,C,a,b) = \frac{1}{(C-A)(C-B)}\left[\frac{1}{2}(x-A)(x-B)^2 - \frac{1}{6}(x-B)^3\right]\Bigg|_a^b \tag{11}$$

2.3 Interpolating to uniform spacing

Since Simpon's rule is derived by interpolating a function with a polynomial and integrating, we can use the polynomial to interpolate to the medpoint of $[a, b]$ and just use the usual formula. Given the function at points x_1, x_2, and x_3, the interpolated function value at $\frac{x_1+x_2}{2}$ is

$$f\left(\frac{x_1+x_3}{2}\right) \approx f(x_1)l(\frac{x_1+x_2}{2}, x_2, x_3, x_1) + f(x_2)l(\frac{x_1+x_2}{2}, x_1, x_3, x_2) + f(x_3)l(\frac{x_1+x_2}{2}, x_1, x_2, x_3) \tag{12}$$