Direct Model Reference Adaptive Controller

Maklen A. Estrada

October 1, 2024

1 Abstract

This report presents a Direct Model Reference Adaptive Controller for scalar systems previously developed in the book Robust and Adaptive Control by Eugene Lavretsky and Kevin A. Wise. The MRAC incorporates adaptive laws to estimate unknown parameters and ensure stability in the closed-loop system. The theoretical framework involves open-loop dynamics, a stable reference model, and error equations, leading to the derivation of adaptive laws based on Lyapunov stability analysis. The Lyapunov-based stability analysis demonstrates the asymptotic convergence of the tracking error, ensuring that the closed-loop tracking error dynamics are Globally Uniformly Asymptotically Stable. Subsequently, a simulation using Simulink is done, applying the MRAC system to the pitch dynamics of a hovering helicopter. The simulation results illustrate the effectiveness of the MRAC approach in practical scenarios.

2 Open Loop Dynamics

The general dynamics for a model reference adaptive controller for scalar systems are

$$\dot{x} = ax + b\left(u + f(x)\right) \tag{1}$$

where x is the system's state, and u is the control input. It is assumed that we know the sign of the constant b. The dynamics are dependent on f(x) which is an unknown function. This function is made up of unknown constants θ_i and the sum of N basis functions $\phi_i(x)$ that are assumed to be Lipschitz-continuous in x.

$$f(x) = \sum_{i=1}^{N} \theta_i \phi_i(x) = \theta^T \Phi(x) \tag{2}$$

By substituting (2) into (1) we get

$$\dot{x} = ax + b\left(u + \theta^T \Phi(x)\right) \tag{3}$$

3 Reference Model

We then use stable reference model whose dynamics are given by

$$\dot{x}_{ref} = a_{ref} x_{ref} + b_{ref} r(t) \tag{4}$$

where $a_{ref} < 0$ and b_{ref} are the desired constants and r(t) is the reference input command.

4 Open Loop Error

The open loop error equation used is with subsequent derivative

$$e = x - x_{ref} \tag{5}$$

$$\dot{e} = \dot{x} - \dot{x}_{ref} \tag{6}$$

5 Closed Loop Error

In order to get the closed loop error we first define a controller that is composed using the feedback and feedforward architecture. This is known as the ideal controller which assumes that the unknown parameters were known.

$$u_{ideal} = k_x x + k_r r - \theta^T \Phi(x) \tag{7}$$

where k_x is the ideal feedback gain and k_r is the ideal feedforward gain. We then substitute the ideal controller into the open loop dynamics (3)

$$\dot{x} = (a + bk_x)x + bk_r r(t) \tag{8}$$

We then compare (8) with the reference model (4) to get relations for the ideal gains.

$$a + bk_x = a_{ref} \qquad bk_r = b_{ref} \tag{9}$$

Then the tracking controller is based on the ideal controller (7) and we get

$$u = \hat{k}_x x + \hat{k}_r r - \hat{\theta}^T \Phi(x) \tag{10}$$

We then use (10) and substitute it into the open loop dynamics (3) and get

$$\dot{x} = (a + b\hat{k}_x) + b\left(\hat{k}_r r - \left(\hat{\theta} - \theta\right)^T \Phi(x)\right) \tag{11}$$

We rewrite (10) using the relations for the ideal gains (9) and get

$$\dot{x} = a_{ref} x + bk_r r + b\triangle k_x + b\triangle k_r - b\triangle\theta\Phi(x) \tag{12}$$

where

$$\triangle k_q = \hat{k_q} - k_q \tag{13}$$

$$\triangle k_{q_{cmd}} = \hat{k}_{q_{cmd}} - k_{q_{cmd}} \tag{14}$$

$$\triangle\theta = \hat{\theta} - \theta \tag{15}$$

Now we use the open loop error derivative equation (6) to arrive to the closed loop error by subtracting (4) from (12) and get

$$\dot{e} = a_{ref} e + b(\triangle k_x x + \triangle k_r r - \triangle\theta^T \Phi(x) \tag{16}$$

6 Lyapunov Based Stability

The candidate Lyapunov function chosen for the MRAC design is

$$V(e, \triangle k_x, \triangle k_r, \triangle\theta) = e^2 + |b|\left(\frac{1}{\gamma_x}\triangle k_x^2 + \frac{1}{\gamma_r}\triangle k_r^2 + \frac{1}{\Gamma_\theta}\triangle\theta^T\triangle\theta\right) \tag{17}$$

where $\gamma_x > 0$, $\gamma_r > 0$, and a constant symmetric positive definite matrix $\Gamma_\theta \in \mathbb{R}^{nxn}$ are the rates of adaptation. Taking the derivative of the Lyapunov function we get

$$\begin{aligned}
\dot{V} &= 2e\dot{e} + 2|b|\left(\frac{1}{\gamma_x}\triangle k_x \dot{\hat{k}}_x + \frac{1}{\gamma_r}\triangle k_r \dot{\hat{k}}_r \frac{1}{\Gamma_\theta}\triangle\theta^T \dot{\hat{\theta}}\right) \\
&= 2e\left(a_{ref} e + b\left(\triangle k_x x + \triangle k_r r - \triangle\theta^T \Phi(x)\right)\right) \\
&\quad +2|b|\left(\frac{1}{\gamma_x}\triangle k_x \dot{\hat{k}}_x + \frac{1}{\gamma_r}\triangle k_r \dot{\hat{k}}_r + \frac{1}{\Gamma_\theta}\triangle\theta^T \dot{\hat{\theta}}\right) \\
&= 2a_{ref} e^2 + 2|b|\left(\triangle k_x\left(xesgn(b) + \frac{1}{\gamma_x}\dot{\hat{k}}_x\right)\right) \\
&\quad +2|b|\left(\triangle k_r\left(resgn(b) + \frac{1}{\gamma_r}\dot{\hat{k}}_r\right)\right) \\
&\quad +2|b|\triangle\theta^T\left(-\Phi(x)esgn(b) + \frac{1}{\Gamma_\theta}\dot{\hat{\theta}}\right)
\end{aligned} \tag{18}$$

In order to enforce closed-loop stability the adaptive laws were chosen so that $\dot{V} \leq 0$

$$\begin{aligned}
\dot{\hat{k}}_x &= -\gamma_x x e \; sgn(b) \\
\dot{\hat{k}}_r &= -\gamma_r r e \; sgn(b) \\
\dot{\hat{\theta}} &= \Gamma_\theta \Phi(x) e \; sgn(b)
\end{aligned} \tag{19}$$

Using these adaptive laws along with the trajectory tracking chosen $\dot{V}$ becomes negative semi-definite.

$$\dot{V} = 2a_{ref} e(t)^2 \leq 0 \;\; where \;\; a_{ref} < 0 \tag{20}$$

This result implies that e, $\triangle k_x$,$\triangle k_r$, $\triangle\theta \in \mathcal{L}_\infty$.

6.1 Bounding

$$\begin{aligned}
e, \triangle k_x, \triangle k_r, \triangle\theta \in \mathcal{L}_\infty \\
x_{ref}, r \in \mathcal{L}_\infty \; and \; \theta \; constant \;\; \mapsto x, \hat{\theta} \in \mathcal{L}_\infty \\
\phi_i(x) \in x \in \mathcal{L}_\infty \mapsto \phi_i(x) \in \Phi(x) \mapsto \Phi(x) \in \mathcal{L}_\infty \\
\hat{k}_x, \hat{k}_r, \hat{\theta}, \Phi(x) \in \mathcal{L}_\infty \mapsto \hat{k}_x, \hat{k}_r, \hat{\theta}, \Phi(x) \in u \mapsto u \in \mathcal{L}_\infty \\
x, u, \Phi(x) \in \mathcal{L}_\infty \mapsto x, u, \Phi(x) \in \dot{x} \mapsto \dot{x} \in \mathcal{L}_\infty \\
x_{ref}, r \in \mathcal{L}_\infty \mapsto x_{ref}, r \in \dot{x}_{ref} \mapsto \dot{x}_{ref} \in \mathcal{L}_\infty
\end{aligned} \tag{21}$$

Taking the second derivative of the Lyapunov function we get

$$\ddot{V} = 4 \, a_{ref} \, e \, \dot{(e)} \tag{22}$$

Given this, $\ddot{V}$ is bounded and $\dot{V}$ is a uniformly continuous function. Now Barbalats Lemma is used to get that $\lim_{t\to\infty} \dot{V}(t) = 0$. Since $\dot{V} \leq 0$ it is concluded that the tracking error e tends to zero asymptotically as $t \to \infty$. The overall stability result of this analysis is that the closed-loop tracking error dynamics are Globally Uniformly Asymptotically Stable.

7 Controller and Adaptive Laws

The controller and ideal controller are given by

$$u_{ideal} = k_x x + k_r r - \theta^T \Phi(x) \tag{23}$$

$$u = \hat{k}_x x + \hat{k}_r r - \theta^T \Phi(x) \tag{24}$$

The adaptive laws were selected using an inverse Lyapunov design approach where first a candidate Lyapunov function is chosen, and then the adaptive laws are chosen such that the derivative of the Lyapunov function is non-positive. The adaptive laws are

$$\begin{aligned}
\dot{\hat{k}}_x &= -\gamma_x x e \; sgn(b) \\
\dot{\hat{k}}_r &= -\gamma_r r e \; sgn(b) \\
\dot{\hat{\theta}} &= \Gamma_\theta \Phi(x) e \; sgn(b)
\end{aligned} \tag{25}$$

This system does not have any filters or estimators.

8 Simulation

For the simulation, a helicopter's pitch dynamics and control during hover is used. Assuming constant thrust, and neglecting small forward and vertical speed components the pitch dynamics of a helicopter during a hover can be approximated by the following differential equation

$$\dot{q} = M_q q + M_d \delta + f(q) \tag{26}$$

where M_q represents the vehicle pitch damping, M_d is the elevator effectiveness, and $f(q)$ models uncertainties.

$$f(q) = \theta \Phi(q) = -0.01 \tanh\left(\frac{360}{\pi} \times q\right) \tag{27}$$

where $\theta = -0.01$ and $\Phi(q)$ is the known regressor. The adaptive pitch controller is given by:

$$\delta = \hat{k}_q q + \hat{k}_\delta \delta_{\text{cmd}} - \hat{\theta}^T \Phi(q) \tag{28}$$

with adaptive laws:

$$\dot{\hat{k}}_q = \gamma_q q (q - q_{ref}) \tag{29}$$

$$\dot{\hat{k}}_\delta = \gamma_{q_{cmd}} (q - q_{ref}) \tag{30}$$

$$\dot{\hat{\theta}} = -\Gamma_\theta \Phi(q)(q - q_{ref}) \tag{31}$$

where the rates of adaptation are $\gamma_q = \gamma_{qcmd} = 6000$, $\Gamma_\theta = 8$. The reference pitch rate is generated by:

$$\dot{q}_{\text{ref}} = 4 \times (q_{\text{cmd}} - q_{\text{ref}}) \tag{32}$$

The time-varying pitch rate command chosen for the simulation is

$$q_{cmd} = \frac{1}{\pi} \sin\left(\frac{1}{2} t\right) \tag{33}$$

The ideal controller is given by

$$\delta_{ideal} = \frac{1}{M_\delta} \left((a_{ref} - M_q) q + b_{ref} q_{cmd}\right) - f(q) \tag{34}$$

where we selected $a_{ref} = -4$ and $b_{ref} = 4$. The ideal gains are calculated by comparing the ideal controller (34) with the adaptive pitch controller (28), we get the following relations

$$\begin{aligned} \frac{a_{ref} - M_q}{M_\delta} &= k_q \\ \frac{b_{ref}}{M_\delta} &= k_{qcmd} \\ \theta &= -.01 \end{aligned} \tag{35}$$

The ideal gains are

k_q	0.50977
$k_{q_{cmd}}$	-0.60150
θ	-0.01000

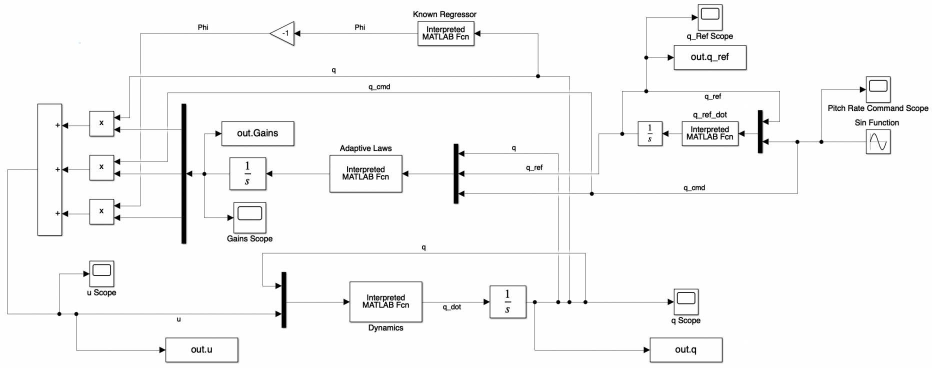

Figure 1: Simulink Model

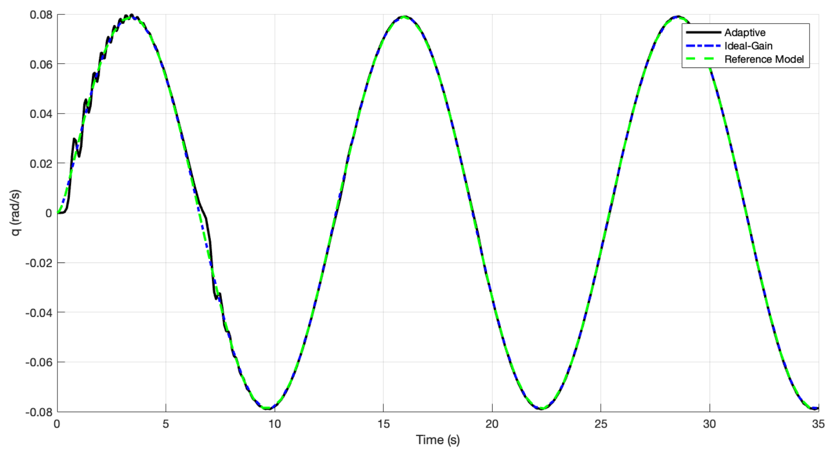

Figure 2: Pitch rate command tracking performance

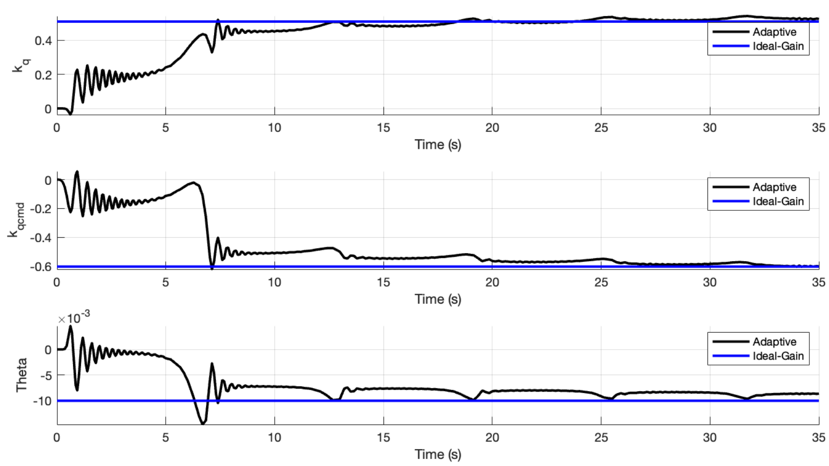

Figure 3: Adaptive parameters performance

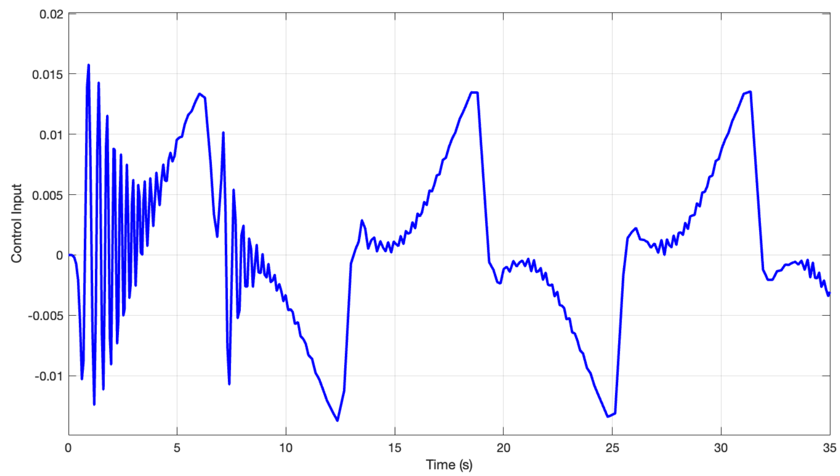

Figure 4: Control Input

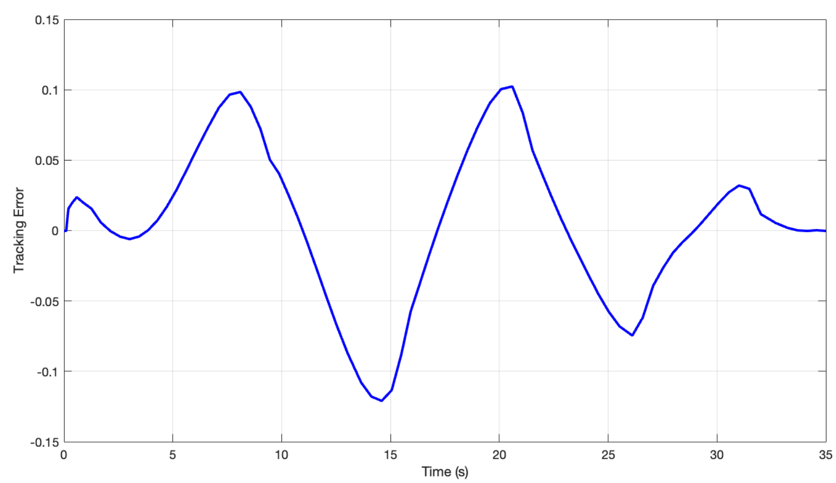

Figure 5: Tracking Error

9 Discussion

As shown in Figure 3 the feedback gain k_d, feedforward gain k_{cmd}, and estimate vector of parameters θ all converge to the ideal values relatively quickly showing the effectiveness of the model reference adaptive controller. The ideal values are given by the ideal controller (34), this controller assumes that the unknown parameters are known The tracking error performance is also relatively good, in Figure 6 we see that the tracking error will converge to 0 as $t \mapsto \infty$. For this simulation, we did not need to tune the gains because we used the ideal controller to find the ideal adaptive gains.

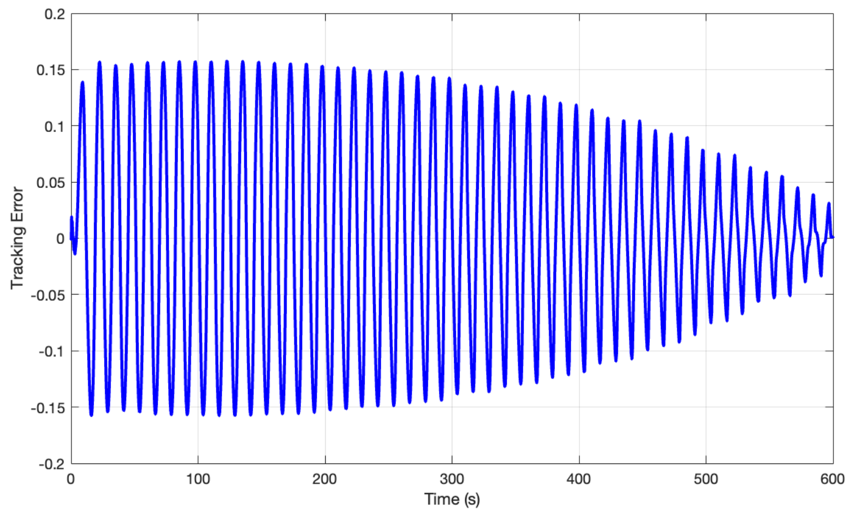

Figure 6: Tracking Error with 10 minute simulation time

10 Conclusion

In conclusion, the direct model reference adaptive controller for scalar systems shown in this report demonstrated the effectiveness in handling uncertainties in dynamics and ensuring stability for the closed-loop system. The Lyapunov stability analysis establishes asymptotic convergence of the tracking error, giving us the result of Global Uniformly Asymptotic Stability. The simulation with the pitch dynamics of a helicopter during hover further validates the MRAC approach. The controller successfully tracks the reference pitch rate command, and the adaptive laws converge to their ideal values, highlighting the controller's ability to adapt to unknown parameters. The tracking error will converge to zero as $t \mapsto \infty$, showcasing the practical effectiveness of the MRAC in real-world scenarios.

11 References

[1] Lavretsky, E. and Wise, K.A. (2023) '9.3 Direct MRAC Design for Scalar Systems', in Robust and adaptive control: With aerospace applications. S.l.: SPRINGER INTERNATIONAL PU.