lesson 37.

Power Series Solutions of Linear DEs.

Introduction to Power Series

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

This is a power series in a.

When $a=0$ then this is a power series in x.

In general we consider power series in x.

But the properties which will be discussed also valid for power series in $(x-a)$.

We say that a power series converges on the interval I provided that the limit

$$\lim_{N \to \infty} \sum_{n=0}^{N} c_n x^n = \sum_{n=0}^{\infty} c_n x^n$$

is defined for all $x \in I$. In this case

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$ is defined for all $x \in I$.

Consider

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$f(x_0) = a_0$$

$$f'(x) = a_1 + 2a_2(x-x_0) + \cdots \qquad f''(x) = 2a_2 + 3.2a_3(x-x_0)$$

$$f'(x_0) = a_1 \qquad f''(x_0) = 2a_2$$

$$f'''(x_0) = 3.2\,a_3 + 4.3.2\,a_4(x-x_0) + \cdots$$

$$f'''(x_0) = 3.2\,a_3$$

Then Taylor series expansion f around x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Ideas to check convergence of a power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

We check the absolute values (or absolute convergence)

$$\lim_{N\to\infty} \sum_{n=0}^{N} |a_n (x-x_0)^n| \to \text{Converges}$$

One way to check is ratio test!

Ratio test: $\lim_{n\to\infty} \left| \dfrac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-x_0| \lim \left|\dfrac{a_{n+1}}{a_n}\right| < 1$

Once we determine L, convergence or divergence follows as:

$L<1$ series converges for that x

$L=1$ convergence/divergence can not be determined

$L>1$ series diverges for that x.

Example1 Find the interval of convergence of

$$\sum \frac{(x+1)^n}{n\,2^n} = \sum \frac{(x-(-1))^n}{n\,2^n}$$

then $x_0 = -1$ & $a_n = \frac{1}{n\,2^n}$

To check the interval of convergence we calculate

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)2^{(n+1)}} \cdot \frac{n\,2^n}{(x+1)^n} \right| = |x+1| \lim_{n\to\infty} \frac{n\,2^n}{(n+1)\,2^{n+1}}$$

$$= |x+1| \lim_{n\to\infty} \frac{n}{(n+1)\,2}$$

$$= \frac{|x+1|}{2} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{|x+1|}{2}$$

which gives $\Bigg\{ \; L<1$

which gives

$L < 1$ ~~when~~ $\frac{|x+1|}{2} < 1$

$|x+1| < 2 \rightsquigarrow -3 < x < 1 \Rightarrow$ converges for all x in $(-3, 1)$.

$L = 1 = \frac{|x+1|}{2} = 1$ or $|x+1| = 2$ $x = -3$ or $x = 1$
inconclusive

$L > 1 \Rightarrow \frac{|x+1|}{2} > 1$ $|x+1| > 2$ or $x > 1$ or $x < -3$

We need to check borderline case
$x = -3$ & $x = 1$

When $x = 1$ we have $\sum \frac{(x+1)^n}{n 2^n} = \sum \frac{2^n}{n 2^n} = \sum \frac{1}{n}$

which is known to be divergent.

When $x = -3$ then $\sum \frac{(x+1)^n}{n 2^n} = \sum (-1)^n$ which is known to be convergent

Therefore the interval of convergence is
$[-3, 1)$. Radius of convergence is 2.
around $x_0 = -1$.

Algebraic operations let $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

1) $f(x) \mp g(x) = \sum (a_n \mp b_n)(x-x_0)^n$

$$f(x)\,g(x) = \left(\sum a_n (x-x_0)^n\right)\left(\sum b_n (x-x_0)\right)^n$$

$$f(x)/g(x) = \left(\sum a_n (x-x_0)^n\right) \Big/ \sum \left(b_n (x-x_0)^n\right).$$

<u>Example!</u> Find Taylor series of e^x around $x=0$.

Since $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$

we need to find $f^{(n)}(x_0) \to$ n^{th} derivative of f at x_0

$f(x) = e^x$ & $f'(x) = e^x \leadsto f^{(n)}(x) = e^x$

e^x has taylor series around 0 as (at $x_0=0$)

$$\sum_{n=0}^{\infty} \frac{e^{x_0}}{1} \frac{(x-x_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example: let $f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

If $f(x) = g(x)$ then what are a_n?

Since ~~$n a_n x^{n-1} \neq 0$ which $n \neq$~~

Since $f(x) = g(x)$ we then have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n.$$

$$\Downarrow \quad \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

or $$\sum_{n=0}^{\infty} \big((n+1) a_{n+1} - a_n\big) x^n = 0$$

therefore $(n+1) a_{n+1} - a_n = 0$. Hence

$a_{n+1} = \dfrac{a_n}{n+1}$. Therefore $a_1 = \dfrac{a_0}{1}$, $a_2 = \dfrac{a_1}{2}$, ~~$= a_3$~~ $= \dfrac{a_0}{2}$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{3.2} = \frac{a_0}{6} = \frac{a_0}{3!}$$

$$a_4 = \frac{a_0}{4!}$$

Power Series Solution

Now $a_n = \frac{a_0}{n!}$

Which gives

$$f(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} \cdot x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

Power Series Solution.

Using power series, we will try to guess a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

(unknown here a_n's, $a_0, a_1 \ldots$) In order to find the solution $y(x)$, we need to find all a_i's.

Consider the DE.

$$A(x)\, y'' + B(x)\, y' + C(x)\, y = 0$$

Using power series method we look for a solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Now we need to find

y' & y'' in terms of the power series.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3.2\, a_3 x + 4.3\, a_4 x^2 + \cdots$$

Now then we will collect anything

$$A(x) y'' + B(x) y' + C(x) y = 0$$

$$A(x) \sum_{n=0}^{\infty} a_n x^n + B(x) \sum_{n=1}^{\infty} n a_n x^{n-1} + C(x) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 0$$

~~Example~~

Theorem: let $y^{(n)} + f_{n-1}(x) y^{n-1} + \cdots + f_1(x) y' + f_0(x) y = Q(x)$

If each function $f_0(x) \ldots f_{n-1}(x)$, $Q(x)$ is analytic at $x = x_0$, i.e. each function has a ~~power series~~ Taylor series expansion in power of $(x - x_0)$ valid for $|x - x_0| < r$ then there is a unique solution to above DE which is also analytic at $x = x_0$ satisfying the initial conditions.

$$y(x_0) = a_0, \quad y'(x_0) = a_1 \quad \ldots \quad y^{(n-1)}(x_0) = a_{n-1}.$$

Comment: A polynomial is a finite series. Hence the series is valid for all x. If $f_0(x) \ldots f_{n-1}(x), Q(x)$ are all polynomials then Theorem says that any solution of above DE has a Taylor series expansion valid for all x.

Example:
$$y'' - (x+1)y' + x^2 y = x$$
with $y(0) = 1 \quad y'(0) = 1.$

Find a solution to the DE using power series method.

Solution: Here
$f_0(x) = 1$
$f_1(x) = -(x+1)$
$f_2(x) = x^2$
$Q(x) = x$
they are all polynomials therefore series solutions must exist, and that solution will be valid for all x.

Note that here $x_0 = 0$
$$\begin{pmatrix} y(0) = 1 \\ y'(0) = 1 \end{pmatrix}$$
x_0

Therefore we will look for a power series solution around $x=0$.

$$y'' - (x+1)y' + x^2 y = x$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - (x+1) \sum n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = x$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2} - x = 0$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n - x =$$

$$= \underbrace{2.1.\, a_2}{} + (3)(2) a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n$$

$$- a_1 - 2a_2 x - \sum_{n=2}^{\infty} (n+1) a_{n+1} x^n + \sum_{2=n}^{\infty} a_{n-2} x^n - x = 0$$

From this the $\sum b_n x^n = 0$ implies $b_n = 0 \ \forall n$

Therefore the sum of constant terms on the left has to zero

$$2a_2 - a_1 = 0.$$

Now the coefficient of x

$$6a_3 x - a_1 x - 2a_2 x - x = 0 \cdot x \quad \text{for all } x.$$

$$6a_3 - a_1 - 2a_2 - 1 = 0$$

Now the remain terms in the identity above is

$$\cancel{\sum_{n=2}^{\infty} (n+1) a_{n+1} x^n}$$

$$\sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2} n a_n x^n - \sum_{n=2}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$= \sum_{n=2}^{\infty} \Big((n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} + a_{n-2} \Big) x^n = 0 \quad \forall n$$

therefore we get a recursive identity

$$(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} + a_{n-2} = 0$$

Now we have

$$2a_2 - a_1 = 0$$

$$6a_3 - a_1 - 2a_2 - 1 = 0$$

$$(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} + a_{n-2} = 0$$

Now we know that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$ and we want to find $a_n, n \geq 0 \ldots$

Using the initial value $y(0)=1$ we get

$$y(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$y(0) = a_0 = 1$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$y'(0) = a_1 = 1.$$

Using the first identity in ~~po~~

$$2a_2 - \underbrace{a_1}_{1} = 0$$ we get $a_2 = \frac{1}{2}$

Using the second identity

$$6a_3 - a_1 - 2a_2 - 1 = 0$$ we get

$$6a_3 - 1 - \frac{1}{2}.2 - 1 = 0 \Rightarrow a_3 = \frac{1}{2}$$

Using the last identity we get (to find a_4 $n=2$)

$$4.3\, a_4 - 2a_2 - 3a_3 + a_0 = 0$$

$$12a_4 - 2.\frac{1}{2} - 3.\frac{1}{2} + 1 = 0 \qquad a_4 = \frac{1}{8}$$

Then using for $n=3$ we can find a_5. Recursively we can get all $a_n \ldots$

Now the solution we are looking for is

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \cdots \quad \text{for all } x.$$

Example: $y'' - 2xy' + y = 0$. Find a power series solution to the DE. with $y(0) = 2$ & $y'(0) = 1$

Solution: Find a solution in the form (as $x_0 = 0$)

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Now $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $\quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Then substitute in the DE to get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Equivalently

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \quad \text{by reindexing the sum}$$

$$2x \sum_{n=1}^{\infty} n a_n x^{n-1} = 2 n a_n x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

As summations start from 0 and this summation starts at $n=1$ we can rewrite these functions as

$$2 \cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} 2n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

That is

$$2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2n a_n + a_n \right] x^n = 0$$

From this we get

$$2a_2 + a_0 = 0 \quad \text{and}$$

$$(n+2)(n+1)a_{n+2} - 2n a_n + a_n = 0 \quad \text{for all } n = 1, 2, \ldots$$

Solve for a_{n+2} to get

$$a_{n+2} = \frac{(2n-1)\,a_n}{(n+2)(n+1)}.$$

Now using the initial conditions $y(0) = 2$ & $y'(0) = 1$ we get

$y(x) = a_0 + a_1 x + \ldots \qquad y(0) = a_0 = 2$

$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \qquad y'(0) = a_1 = 1$

Using these two information $a_0 = 2$ and $a_1 = 1$

we can find all a_n.

for example

$$a_2 = \frac{(2.2-1)}{4.3} a_0 = \frac{3}{4.3} \cdot 2 = \frac{1}{2}$$

$$a_3 = \frac{(2.3-1)}{5.4} a_1 = \frac{5}{5.4} \cdot 1 = \frac{1}{4}$$

Therefore

$$y(x) = 2 + x + \frac{x^2}{2} + \frac{x^3}{4} + \cdots$$ is valid for all x.

<u>Example!</u> $$y'' + \frac{x}{1-x^2} \cdot y' - \frac{1}{1-x^2} y = 0 \quad , \quad |x| \neq 1$$

$y(0) = 1$ & $y'(0) = 1$

Find a power series solution to the DE. Write the largest possible interval of convergence.

<u>Solution:</u> the initial condition is given at the point $x_0 = 0$. Then we will look for a power series solution around $x = 0$

for a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Note that the function $\frac{x}{1-x^2}$ is defined for all $x \in (-1,1)$ similarly $\frac{-1}{1-x^2}$ too. Therefore solution exists for ~~interval~~ values of $x \in (-1,1)$.

Since $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Now the DE becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \underset{\sim}{\frac{x}{1-x^2}} \sum n a_n x^{n-1} - \underset{\sim}{\frac{1}{1-x^2}} \sum_{n=0}^{\infty} a_n x^n = 0$$

We need to write these terms as a power series and then find the product.

Since
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

Then
$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} \quad \text{for } |x|<1$$

and
$$\frac{x}{1-x^2} = x \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} x^{2n+1} \quad \text{for } |x|<1.$$

OR equivalently
$$\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + \cdots$$

$$\sum_{n=0}^{\infty} x^{2n+1} = x + x^3 + x^5 + \cdots$$

Now find the product of the power series in the middle

$$\frac{x}{1-x^2}\sum n a_n x^{n-1} = (x+x^3+x^5+x^7+\dots)\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} n a_n x^{n+2} + \sum_{n=1}^{\infty} n a_n x^{n+4} + \dots$$

The last term in the DE becomes

$$\frac{1}{1-x^2}\sum_{n=0}^{\infty} a_n x^n = (1+x+x^2+x^3+\dots)\sum a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} + \dots$$

Rewrite the DE to get;

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} n a_n x^{n+2} + \dots$$

$$\cancel{+} \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} + \dots = 0$$

Now combine terms (first rewrite the first power series as

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Now once you try to combine terms it will be too complicated.

A different approach.

Rember that $\quad$ by Taylor series expansion of y

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)\, x^n}{n!}$$

So if I can find $y^{(n)}(0)$ then I can find y

for $n = 0, 1, 2 \ldots$

$y(0) = 1$ ✓ $\quad y'(0) = 1$ are given

How can I find $y''(0) = ?$

go back to the DE.

$$y''(x) + \frac{x}{1-x^2} y'(x) - \frac{1}{1-x^2} y(x) = 0 \quad \text{for all } -1<x<1.$$

substitute $x=0$ to get $\quad$ solve for $y''(0)$.

$$y''(0) + \frac{0}{1-0} y'(0) - \frac{1}{1-0} y(0) = 0$$

$$y''(0) = y(0) = 1.$$

Now we need to find $y'''(0)$. To find this first a little trick; as $-1<x<1$, then $1-x^2 \neq 0$. Then I can multiply the DE with ~~and positive~~ $1-x^2$ to get

$$(1-x^2)\, y''(x) + x\, y'(x) - y(x) = 0$$

To find $y'''(0)$, first take implicit derivative ~~of~~ wrt x to get a term with 3rd order derivative!

$$\frac{d}{dx}\left[(1-x^2)\,y''(x)+xy'(x)-y(x)\right]=0$$

becomes

(*) $$\underbrace{-2x\,y''(x)}+(1-x^2)\,y'''(x)+\cancel{y'(x)}+\underbrace{xy''(x)}_{x\,y''(x)}-\cancel{y'(x)}=0$$

Now substitute $x=0$

$$0+y'''(0)+\cancel{y''(0)}^{\,0}=0$$ ~~since $y''(0)=1$~~

we get $y'''(0)=0$.

To find $y^{(4)}(0)$ take another derivative in (*) and keep going.

Now we have

$$y(x)=y(0)+y'(0)\,x+\frac{y''(0)}{2!}\,x^2+\frac{y'''(0)\,x^3}{3!}+\cdots$$

$$=1+x+\frac{x^2}{2!}+0+\cdots$$