Partial Differential Equations (PDEs)

* There is more than one independent variable $x, y, \ldots$
* There is a dependent variable that is an unknown function of these variables $u(x, y, \ldots)$.

Notation $\quad \dfrac{\partial u}{\partial x} = u_x \quad$ partial derivative of u with respect to x.

$\dfrac{\partial u}{\partial y} = u_y \quad$ partial derivative of u wrt variable y.

In general, the partial differential equation can be written as

$$0 = F(x, y, u(x, y), u_x(u, y), u_y(x, y)) = F(x, y, u, u_x, u_y)$$

This is the most general PDE in two independent variables of first order. (The highest order of derivative)

$$0 = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

is the most general second order-PDE in two variables.

A solution of a PDE is a function $u(x, y \ldots)$ that satisfies the equation identically, at least in some region of the $x, y \ldots$ variables.

Some examples of PDEs

1. $u_x + u_y = 0$ Transport equation, 1^{st} order.
2. $u_x + y u_y = 0$ Transport equation, 1^{st} order.
3. $u_x + u u_y = 0$ Shock wave equation 1^{st} order.
4. $u_{xx} + u_{yy} = 0$ Laplace's equation 2^{nd} order.
5. $u_{tt} - u_{xx} + u^3 = 0$ Wave equation with interaction 2^{nd} order.
6. $u_t + u u_x + u_{xxx} = 0$ Dispersive wave equation 3^{rd} order.
7. $u_{tt} + u_{xxxx} = 0$ Vibrating Bar equation 4^{th} order.
8. $u_{xx} - u_t = 0$ Heat conduction equation 2^{nd} order.

Linearity: Linearity means the following. Write the equation in the form $\mathcal{L}u = 0$, where $\mathcal{L}$ is an operator. That is $\mathcal{L}v$ is a new function if v is a function. For example.

For 1. $\mathcal{L} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ $\quad \mathcal{L}u = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u = u_x + u_y$

2. $\mathcal{L} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $\quad \mathcal{L}u = \frac{\partial}{\partial x}u + y\frac{\partial}{\partial y}u = u_x + y u_y$

Then the linearity for $\mathcal{L}$ is

(*) $\quad \mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v \quad$ & $\quad \mathcal{L}(cv) = c\mathcal{L}v$

for any functions u, v and constant c.

Now $\mathcal{L}$ is called linear operator if (*) holds

$\mathcal{L}u = 0$ is called Homogeneous equation.

$\mathcal{L}u = g$ is called non-Homogeneous equation.

<u>Example:</u> Verify that $u_x + y u_y = 0$ is a linear PDE.

<u>Solution:</u> let $\mathcal{L} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

Then $\mathcal{L}u = u_x + y\, u_y$.

Hence we need to check

- $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}v$.

* $\mathcal{L}(u+v) = \frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v)$

$$= u_x + v_x + y[u_y + v_y]$$

$$= u_x + y u_y + v_x + y v_y$$

$$= \mathcal{L}u + \mathcal{L}v \checkmark$$

* $\mathcal{L}(cu) = \frac{\partial}{\partial x}(cu) + y\frac{\partial}{\partial y}(cu)$

$$= c u_x + c y u_y = c\mathcal{L}u$$

Hence $\mathcal{L}$ is a linear operator.

Example: Verify that $u_x + u u_y = 0$ is a non linear PDE.

Solution: check if $\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v$

$\mathcal{L}u = \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y}$

$\mathcal{L}(u+v) = \frac{\partial}{\partial x}(u+v) + (u+v)\frac{\partial}{\partial y}(u+v)$

$$= u_x + v_x + (u+v)u_y + (u+v)v_y$$

$\neq \mathcal{L}u + \mathcal{L}v$. Hence it's not a linear PDE.

The advantage of linearity is that if

$\mathcal{L}u = 0$ is a linear PDE and if

u and v are both solutions to $\mathcal{L}u = 0$
then $u+v$ is also solution to this PDE;

$$\mathcal{L}(u+v) = 0.$$

Example 1: Find all $u(x,y)$ satisfying the equation

$$u_{xx} = 0.$$

As u has two independent variable if we integrate this PDE we get

$$u_x = f(y) \quad \text{for some arbitrary } y.$$

Integrate again to get

$$u(x,y) = x\,f(y) + g(y).$$

for some arbitrary $g(y)$. Hence the general solution is

$$u(x,y) = x\,f(y) + g(y) \quad \text{for some arbitrary functions } f(y) \text{ and } g(y).$$

<u>Example 2:</u> Solve the PDE $u_{xx} + u = 0$

If u had a one variable we know that the solution was $u(x) = a\cos x + b\sin x$

Since $u = u(x,y)$, now the constants can depend on y: $u(x,y) = f(y)\cos x + g(y).\sin x$

Example 1 Solve the PDE $u_{xy}=0$

This is not hard as we can first integrate with respect to x to get

$$u_y = f(y). \text{ for some arbitrary } f(y).$$

Now integrate one more time with respect to y this ~~time~~ time to get

$$u(x,y) = F(y) + G(x) \quad \text{where } F' = f.$$

First-Order Linear PDEs

Let us solve $a u_x + b u_y = 0$
where a, b are not both zero.

then the quantity $a u_x + b u_y$ is the directional derivative of u in the direction ~~of the~~ vector $\vec{v} = (a, b)$. and it must be zero.

Hence u must be zero in the direction of $\vec{v}$.
Now the lines parallel to $\vec{v}$ have the equations $bx - ay = \text{constant}$

the $bx - ay = \text{constant}$ is called characteristic lines.

Our solution u is constant on these lines. Hence $u(x,y)$ depends only on $bx-ay$ only.

Thus the solution is

$$u(x,y) = f(bx-ay) \quad (*)$$

where f is any function of one variable.

Review: We know that u is constant on the line $bx-ay=c$;

$$u(x,y) = f(bx-ay) = f(c).$$

Since c is arbitrary, we have formula $(*)$ for all values of x and y.

Check the solution:

$$u_x = f'(bx-ay)\cdot b$$
$$u_y = f'(bx-ay)\cdot(-a)$$

Now

$$a u_x + b u_y = a f'\cdot b + b(f'\cdot(-a))$$
$$= abf' - abf' = 0.$$

Example: Solve the PDE $4u_x - 3u_y = 0$ together with the auxiliary condition $u(0,y) = y^3$.

Solution: Here $a=4$ and $b=-3$ from $(*)$ we get

$$u(x,y) = f(-3x-4y) \text{ is a solution.}$$

This is the general solution. Set $x=0$ to get

$$u(0,y) = f(-4y) = y^3$$

there we need rewrite $f(-4y) \rightsquigarrow f(y)$

$$f(y) = \left(-\frac{y}{4}\right)^3 = -\frac{y^3}{64}$$

<u>OR:</u> $f(-4y) = y^3$, let $w = -4y$ then $y = -\frac{w}{4}$

Now $f(-4y) = f(w) = y^3 = \left(-\frac{w}{4}\right)^3 = -\frac{w^3}{64}$

Hence $f(w) = -\frac{w^3}{64}$

We knew that our general solution is

$$u(x,y) = f(-3x-4y) = -\frac{(-3x-4y)^3}{64} = \frac{(3x+4y)^3}{64}$$

is the solution with the given auxiliary condition.

Verify your soltion: $u(x,y) = \frac{(3x+4y)^3}{64}$ solves the PDE $4u_x - 3u_y = 0$.

(Find u_x & u_y and plug in to the PDE.)

Check the auxiliary conditions; $u(0,y) = y^3$

$$u(0,y) = \frac{(3.0+4y)^3}{64} = \frac{4^3 . y^3}{64} = y^3 \checkmark$$

Hence $u(x,y) = \frac{(3x+4y)^3}{64}$ is the solution.