Lesson 11. The Linear DE of order one

A DE is called linear first order ~~of order one~~ if it can be written as

$$y' + P(x)\cdot y = Q(x).$$

where $P(x)$ and $Q(x)$ are continuous functions.

By thm 2. Problem 4; we know that

$$\mu(x) = e^{\int p(x)\,dx}$$ is an integrating factor.

Hence $$\frac{dy}{dx} + P(x)\cdot y - Q(x) = 0$$

OR $$dy + [P(x)\cdot y - Q(x)]\,dx = 0$$

If we multiply this DE with $\mu(x)$ we get an exact DE

$$\mu(x)\cdot dy + \mu(x)\,[P(x)\cdot y - Q(x)]\,dx = 0$$

then we know how to solve this DE.

(31, 5)

Lesson 11D: Bernoulli Equation.

A special type of first order DE is called Bernoulli (A swiss mathematician) equation if it has the following form

$$y' + p(x)\, y = q(x)\, y^n \qquad \text{for } n \neq 0 \text{ or } 1.$$

Let's make the substitution ~~y~~ $u = y^{1-n}$.

Then ~~y~~ $u' = (1-n) \cdot y^{-n} \cdot y'$ and subs ~~y~~

Suppose $y \neq 0$ then divide the DE by y^n to get

$$\frac{y'}{y^n} + p(x) \cdot \frac{y}{y^n} = q(x)$$

It follows that (or equivalently $y' \cdot y^{-n} + p(x) \cdot y^{1-n} = q(x)$)

$$\frac{u'}{1-n} + p(x)\, u = q(x)$$

This is a linear first order DE. (32)

From the homework problem we know that

$$\mu(x) = e^{-\int \frac{3}{x} dx} = e^{-3\ln|x|} = e^{\ln|x|^{-3}} = x^{-3}$$

is the integrating factor.

Then

Now rewrite the DE.

$$-\frac{du}{dx} + \frac{3}{x}u - x^2 = 0$$

OR $$-du + \left(\frac{3}{x}u - x^2\right)dx = 0$$

If we multiply ~~to~~ with the integrating factor μ we get

$$\underbrace{-x^{-3}}_{M} du + \underbrace{\left(\frac{3 \cdot u}{x^4} - \frac{1}{x}\right)}_{N} dx = 0$$

$$f(x,u) = \int M\, du = -\int x^{-3} du = -ux^{-3} + h(x)$$

$$f_x = 3ux^{-4} + h'(x) = 3ux^{-4} - \frac{1}{x}$$

From this we get $h'(x) = \frac{1}{x}$ and $h(x) = \ln x$

Then as $n \neq 1$ we get

$$u' + (1-n)\,p(x)\,u = q(x).$$

Now this is a linear first order DE. and we know how to solve this DE.

Example: $y' + \frac{3}{x} y = x^2 y^2 \quad , \quad x > 0$

Consider the DE. Is it's a Bernoulli equation? ~~Fist~~ Solve the DE.

Solution: $y' + \frac{3}{x} y = x^2 y^2 \Leftarrow y' + P(x) y = q(x) y^n$

So $n = 2$ and it's a Bernoulli's equation.

Now we first make the substitution $u = y^{1-2} = y^{-1}$

then $u' = -y^{-2} \cdot y'$

If we divide everything by y^2 we get (assume $y \neq 0$)

$$\frac{y'}{y^2} + \frac{3}{x} \frac{y}{y^2} = \frac{x^2 y^2}{y^2} \quad \text{or} \quad y^{-2} \cdot y' + \frac{3}{x} y^{-1} = x^2$$

then DE becomes

$$-u' + \frac{3}{x} u = x^2. \quad \text{or} \quad u' - \frac{3}{x} u = -x^2$$

Therefore

$$0 = f(x,y) = -u\,x^{-3} + \ln x + c \qquad x>0.$$

OR solve for u

$$u = -x^{3}\ln x + cx^{3}$$

As $u = y^{-1}$ or $y = u^{-1}$ we get

$$y = \left(x^{3}(c - \ln x)\right)^{-1}.$$ is the general solution

Riccati Equation: An equation of the form

$$y' = p(x)\,y^{2} + q(x)\,y + r(x)$$ is called

Riccati equation.

Notice that when $r(x) = 0$ then we get the Bernoulli equation;

$$y' - q(x)\,y = p(x)\,y^{2}$$

Hence we get a Bernoulli's equation with $n=-1$.

In general if y_1 is a particular solution to $y' = p(x)y^2 + q(x)y + r(x)$

then consider $y = y_1(x) + \frac{1}{u}$.

Take x derivative; $y' = y_1'(x) - \frac{1}{u^2} \cdot u'$

Substitute this into the DE

$$y_1'(x) - \frac{1}{u^2}u' = p(x) \cdot \left(y_1(x) + \frac{1}{u}\right)^2 + q(x)\left[y_1(x) + \frac{1}{u}\right] + r(x)$$

$$y_1'(x) - \frac{1}{u^2}u' = p(x) \cdot y_1^2(x) + 2p(x)\frac{y_1}{u} + \frac{1}{u^2}p(x) + q(x)y_1(x) + q(x)\frac{1}{u} + r(x)$$

Since $y_1(x)$ is particular solution to the DE

then $y_1'(x) = p(x)y_1^2(x) + q(x)y_1(x) + r(x)$

we get

$$-\frac{1}{u^2}u' = 2p(x)\cdot\frac{y_1}{u} + \frac{1}{u^2}\cdot p(x) + q(x)\cdot\frac{1}{u}$$

Now if $u \neq 0$ then we multiply everything with u^2 to get

$$-u' = 2p(x)\cdot u\cdot y_1 + p(x) + q(x)u$$

or $$-u' = \left[2p(x)y_1 + q(x)\right]u + p(x)$$

which is a linear equation of order one.

Lesson 14B: Orthogonal Trajectories.

Defn: A curve which cuts every member of a given 1-parameter family of curves in a 90° angle is called orthogonal trajectory of the family.

Example: Find the orthogonal trajectories of the 1-parameter family of curves.

$$y = cx^5.$$

Solution: We first need to find slope of this 1-parameter family

$$y' = 5cx^4. \quad \text{and} \quad y' = 5\frac{y}{x}. \quad x \neq 0$$

or $c = \frac{y}{x^5}$ when $x \neq 0$

At any pt (x, y) the slope is

$$y' = \frac{5y}{x} \quad x \neq 0.$$

We are looking curves orthogonal to $y = cx^5$

Therefore slope of those curves should be $\frac{-1}{\text{the slope of the ren curves}}$

Hence $y' = -\frac{x}{5y}$ $y \neq 0$ (and $x \neq 0$)

or ~~Sstep~~ $\underbrace{-\frac{x}{5y}}_{\text{slope}} \cdot \underbrace{\frac{5y}{x}}_{\text{slope}} = -1$

Now we have a DE

$$y' = -\frac{x}{5y}$$ which is seperable

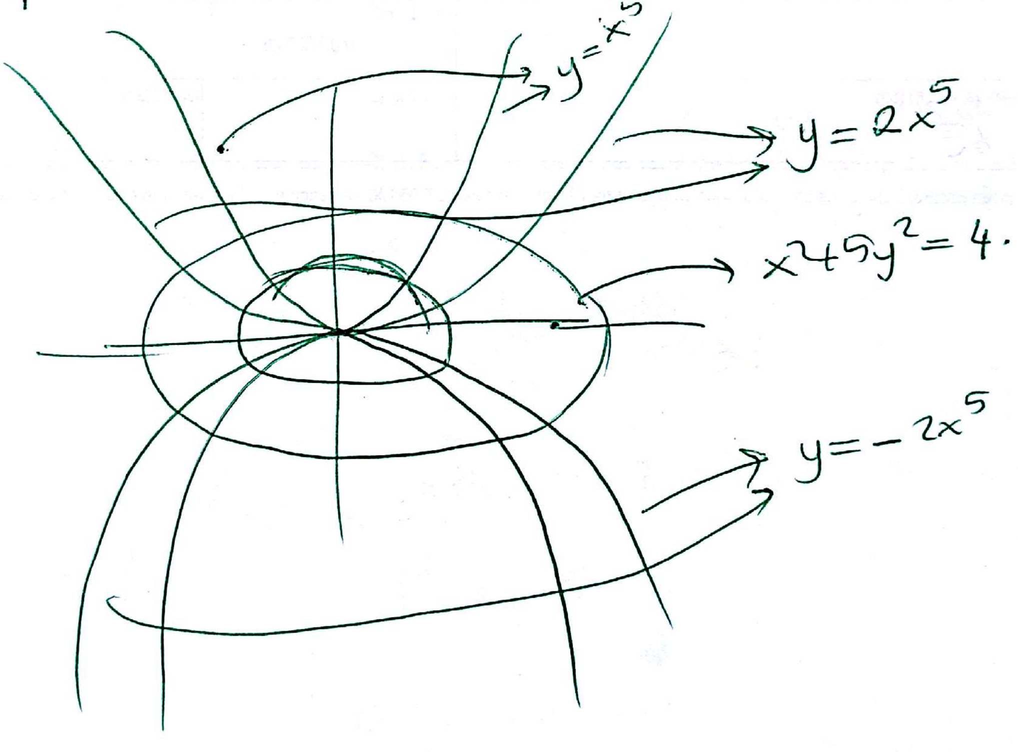
The solution is $yy' = -\frac{x}{5}$ or

$$y\,dy = -\frac{x}{5}\,dx$$

$$y^2 = -\frac{x^2}{5} + C$$

Hence 1-parameter family of solution is

Ellipses. $\longleftarrow x^2 + 5y^2 = C$ $x \neq 0$, $y \neq 0$.



Lesson 15C: Newton's law of cooling

Experimentally proven that

$$\textcircled{x} \quad \frac{dT}{dt} = k(T_s - T)$$

T = is the temperature of the body at any time t.

T_s = is the constant temperature of the environment

In order to solve the DE we need two values of T at two different times say t_0 and t_1

Then the solution is

$$T = T_s + (T_0 - T_s)e^{-kt}$$ solves the DE ⊗

where $T(0) = T_0$

with $T(t_1) = T_1$

Problem: In a murder investigation a corpse was found by a detective at exactly 4 AM. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F at what time did the murder occur? where did it happen? How is the #1 suspect

Solution: Take 4 AM to be $t=0$

then the DE is

$$\frac{dT}{dt} = k(50-T)$$

$$T(0) = 70$$

$$T(2) = 60$$

Then the solution is $T(t) = 50 + (\cancel{70}-50)e^{-kt}$

$$= 50 + 20e^{-kt}$$

As $T(2) = 60$ then $T(2) = 60 = 50 + 20 + e^{-2k}$

$$10 = 20 + e^{-2k}$$

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If we solve for k we get

$\frac{1}{2} = e^{2k}$ $\quad \log \frac{1}{2} = -2k$

or $k = -\frac{1}{2} \log \frac{1}{2} = \log\left(\frac{1}{2}\right)^{-\frac{1}{2}}$

Then $T(t) = 50 + 20\, e^{-t \log\left(\frac{1}{2}\right)^{-\frac{1}{2}}}$

$= 50 + 20\, e^{\log\left(\frac{1}{2}\right)^{\frac{t}{2}}}$

$= 50 + 20 \left(\frac{1}{2}\right)^{\frac{t}{2}}$

We are looking for t for which $T(t) = 98.6$
(at the time he was not dead temperature was 98.6)

$T(t) = 98.6 = 50 + 20\left(\frac{1}{2}\right)^{\frac{t}{2}}$

$48.6 = 20\left(\frac{1}{2}\right)^{\frac{t}{2}}$ $\rightsquigarrow t \approx -2.56$
(2 and a half hours ago)