Notice that $y = \frac{1}{x^2-x-6} = \frac{1}{(x-3)(x+2)}$

the interval containg 0 and y is defined the DE is defined.

Now y is defined when $x \in (-2,3)$

the DE is also well defined on $(-2,3)$.

Hence the interval is $(-2,3)$.

<u>Defn</u>: A function $f(x,y)=0$ is called an <u>implicit solution</u> of the DE

$$F(x, y, y', \dots y^{(n)})=0$$

on an interval I if

say

1. y is an implicit function of x

   i.e. $\exists\, g(x)$ s.t $f(x, g(x))=0$ $\forall x \in I$.

2. $g(x)$ satisfies $F(x, g(x), g'(x) \dots g^{(n)}(x))=0$.

   for all $x \in I$.

<u>Examples</u> Test whether

$$xy^2 - e^{-y} - 1 = 0$$

is an implicit solution of the differential equation

$$(xy^2+2xy-1)y' + y^2 = 0$$

Soln: Use the method of implicit differentiation

$$\frac{d}{dx}\left(xy^2 - e^{-y} - 1\right) = 0$$

$$= y^2 + 2xy \cdot y' + e^{-y} \cdot y' = 0 \quad (*)$$

Use the ~~DE~~ ~~explicit~~ equation above to ~~ge~~ find e^{-y};

solve for e^{-y}

$$xy^2 - e^{-y} - 1 = 0 \Rightarrow e^{-y} = 1 - xy^2$$

and use this in (*) to get

$$y^2 + 2xy \cdot y' + y'(1 - xy^2) = 0$$

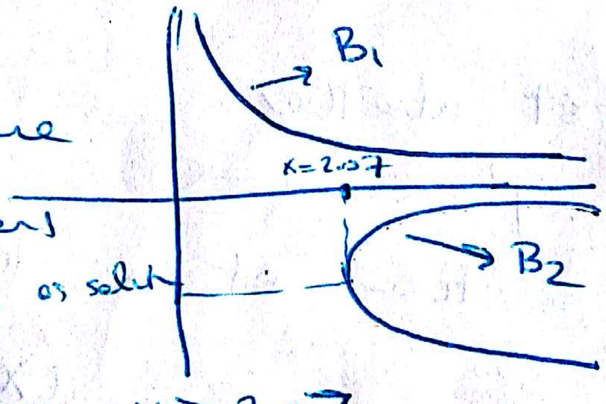
$$y^2 + (2xy + 1 - xy^2)\,y' = 0$$

which is our DE. Hence $xy^2 - e^{-y} - 1$ is a solution of the DE. Notice that

$$xy^2 = e^{-y} + 1 > 0 \quad \text{for all } x \text{ and } y$$

therefore RHS must be positive which gives us the restriction $x > 0$

Hence the implicit solution is valid for all $x > 0$.



Here B_1 & B_2 are both implicit solutions and if you choose B_1 as solution then $x>0$

if you choose B_2 then $x > 2.07$.

Exercise ~~Example~~: Test whether

$$f(x,y) = x^3+y^3-3xy=0 \quad -\infty<x<\infty.$$

Is an implicit solution of

$$F(x,y,y') = (y^2-x)y' - y + x^2 = 0 \quad -\infty<x<\infty.$$

Exercise: ~~Table~~ Page 27, Problems 1d, 2j, 4b

Lesson 4: The General Solution of a DE.

Lesson 4A: Multiplicity of solutions of a DE.

Suppose you have a differential equation $y' = f(x)$ and you want to solve! what do you do? To find y integrate it

$$\int y' \, d(x) = \int f(x) \, dx$$

$$y(x) = \int f(x) \, dx.$$

Example: Let $y' = x^2$.

By a simple integration we get

$$y = \int x^2 + c = \frac{x^3}{3} + c.$$

Example: $y' = e^x$

Then $y = e^x + c$

by integrating the DE

$y'' = e^x$

$y' = e^x + c_1$

by integrating the DE

$y = e^x + c_1 x + c_2$

by integrating twice.

~~The~~ DEs with

i) $(y')^2 + y^2 = 0$ has only one solution $y = 0$

ii) $|y'| + |y| + 1 = 0$ has no solutions

iii) $y' - \left(\frac{1}{3}\right)^{\frac{1}{3}} = 0$ has ∞-many solutions

$y = \left(\frac{2}{3}(x - \alpha)\right)^{\frac{3}{2}}$ is solution for any $\alpha \geq 0$.

Example: The first order DE

$(y' - y)(y' - 2y) = 0$ has the

solution $(y - c_1 e^x)(y - c_2 e^{2x}) = 0$

which has two parameters, c_1 and c_2. (9)

Defn! The functions defined by

$$y = f(x, c_1, c_2 \ldots c_n) \quad \circledast$$

of the $n+1$ variables, $x, c_1 -- c_n$ will be called an n-parameter family of solutions of the DE

$$\circledast \quad F(x, y, \ldots y^{(n)}) = 0$$

if for each choice of a set of values $c_1 .. c_n$ the resulting function $f(x)$ defined by $\circledast$ satisfies $\oplus$. if

$$F(x, f, f' \ldots f^{(n)}) = 0$$

Example1 Show that the functions defined by $y = f(x, c_1, c_2) = c_1 e^{-2x} + c_2 e^{-x} + 2e^{x}$ ~~[illegible]~~ of the three variables x, c_1, c_2 are 2-parameter family of solutions to the DE $F(x, y, y', y'') = \cancel{\quad} 0 = y'' + 3y' + 2y - 12e^{x}$

Soln! Let a, b any two values of c_1, c_2 resp.

Then $y = f(x) = ae^{-2x} + be^{-x} + 2e^{x}$

that is a function of x now!

$y' = f'(x) = -2ae^{-2x} - be^{-x} + 2e^{x}$

$y'' = f''(x) = 4ae^{-2x} + be^{-x} + 2e^{x}$

Substitute this into the DE

$4ae^{-2x} + be^{-x} + 2e^{x} - 6ae^{-2x} - 3be^{-x} + 6e^{x}$

$+2ae^{-2x} + 2be^{-x} + 2e^{x} - 12e^{x}$

$= 0e^{-2x} + 0e^{-x} + 0e^{x} = 0$

Hence $y = c_1e^{-2x} + c_2e^{-x} + 2e^{x}$ is a 2-parameter of family of solutions. ∎

Lesson 4B: Method of Finding a DE if its n-parameter family of solution is known.

Example: Find a differential equation whose 1-parameter family of solution is

$$y = c.\cos x + x.$$

Solution: Notice that we only have one parameter "c" therefore, it's expected that it should be a solution of first order