Spring 2019 - Math 3150
Exam 1 - March 1
Time Limit: 50 Minutes

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right.

1. (10 points) Prove that $1+\sqrt{1+\sqrt{2}}$ is irrational.

Solution: Suppose $x = 1+\sqrt{1+\sqrt{2}}$. Then $x-1=\sqrt{1+\sqrt{2}}$ or $(x-1)^2 = 1+\sqrt{2}$ or $(x-1)^2-1=\sqrt{2}$ or equivalently $((x-1)^2-1)^2 = 2$. From this we have that x is a solution to

$$(x^2-2x+1-1)^2-2 = x^4-4x^3+4x^2-2 = 0.$$

From the corollary of the Rational zeros theorem we know that any rational solution of this equation must be an integer which divides 2. The only possible rational solutions are ± 1 and ± 2. It is clear that none of these numbers are solutions. Therefore x is not a rational number.

2. Let S be the set $S := \{x \text{ is irrational} : \ x \geq 0 \text{ and } x^2 \leq 2\}$ and let (s_n) be the sequence defined by $s_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Decide which of the following statements are true or false. **Justify your answer**.

   (a) (2 points) S is a bounded set.

   > **Solution:** True. S is a bounded set bounded set and bounded above by $\sqrt{2}$ and bounded below by 0.

   (b) (2 points) S has a supremum which is in S.

   > **Solution:** True. Since $\sup(S) = \sqrt{2}$ and it is an irrational number hence $\sup(S) \in S$. i.e, S has a maximum.

   (c) (2 points) S has a minimum.

   > **Solution:** False. The infimum of S is 0 but 0 is not in S. Hence the minimum does not exist.

   (d) (2 points) (s_n) is a bounded sequence.

   > **Solution:** True. Since $|s_n| \leq 1$. Hence it is bounded.

   (e) (2 points) (s_n) is a Cauchy sequence.

   > **Solution:** This is also true as (s_n) converges to 0 and convergent sequences are bounded.

3. (a) (5 points) Write **the definition** of being a convergent sequence.

Solution: We say that a sequence (s_n) is convergent with limit s provided that given $\epsilon > 0$ there exists $N > 0$ such that if $n > N$ then $|s_n - s| < \epsilon$.

(b) (5 points) Using the definition, show that $\lim \frac{n+5}{n^2-7} = 0$.

Solution: Informal proof: Using the definition above for a given $\epsilon >$ we need to find N so that

$$\left|\frac{n+5}{n^2-7} - 0\right| < \epsilon.$$

Now we are going to estimate that sequence from above. We always have $n + 5 \leq n + 5n = 6n$. On the other hand, we need a lower estimate for $n^2 - 7$. Let us see for which n we have $n^2 - 7 \geq n^2/2$. Now $2n^2 - 14 \geq n^2$ or equivalently $n^2 \geq 14$ or $n \geq 4$ (as n is integer). Hence we have to pick $N \geq 4$. Using these two estimates we get

$$\frac{n+5}{n^2-7} \leq \frac{6n}{\frac{n^2}{2}} = \frac{12}{n} < \epsilon.$$

Now we can solve for n in terms of ϵ. That is,

$$\frac{12}{\epsilon} < n \quad \text{implies} \quad N = \frac{12}{\epsilon}.$$

Formal proof: Given $\epsilon > 0$, choose $N = \max\{4, \frac{12}{\epsilon}\}$. If $n > N$ then $n \geq 4$ (therefore we can use $n^2 - 7 \geq n^2/2$) and $n > N \geq \frac{12}{\epsilon}$, i.e., $\epsilon > \frac{12}{n}$. Using these we have

$$\epsilon > \frac{12}{n} = \frac{6n}{\frac{n^2}{2}} \geq \frac{n+5}{n^2-7}$$

whenever $n > N = \max\{4, \frac{12}{\epsilon}\}$. This proves that the limit is zero.

4. (a) (3 points) Write **the definition** of being a Cauchy sequence.

Solution: A sequence (s_n) is called Cauchy provided that given $\epsilon > 0$ there exists $N > 0$ such that if $n, m > N$ then $|s_n - s_m| < \epsilon$.

(b) (5 points) Using the definition, show that $s_n = \frac{1}{2^n}$ is a Cauchy sequence.

Solution: Given $\epsilon > 0$ choose N such that $\frac{1}{2^N} < \epsilon/2$. Then if $m, n > N$ we have

$$|s_n - s_m| = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| \leq \frac{1}{2^n} + \frac{1}{2^m} \leq \frac{1}{2^N} + \frac{1}{2^N} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

where we have the triangle inequality on the third estimate. This shows that (s_n) is a Cauchy sequence.

(c) (2 points) Is (s_n) a convergent sequence?

Solution: Yes, since we know that every Cauchy sequence is a convergent sequence.

5. (a) (5 points) Prove that convergent sequences are bounded.

Solution: See the lecture notes or the book.

(b) (5 points) Provide a counterexample for the converse direction. (i.e., a bounded sequence which is not convergent)

Solution: Let $(s_n) = (-1)^n$. This is a bounded sequence. $|s_n| \leq 1$ for all $n \in \mathbb{N}$. But we showed that (s_n) is not convergent sequence. Hence we have a bounded sequence which is not convergent.