Elliptic operators on rough domains

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Special Session on Special Session on Regularity Theory of PDEs and Calculus of Variations on Domains with Rough Boundaries

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Hartford

Introduction

Introduction

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Theorem (Dahlberg 1977)

$$\Omega \subset \mathbb{R}^{n+1}$$
 Lipschitz domain $\implies \left\{ \begin{array}{l} \omega \in RH_2(\sigma) \\ \left(\oint_{\Delta} k^2 \, d\sigma \right)^{\frac{1}{2}} \lesssim \oint_{\Delta} k \, d\sigma = \frac{w(\Delta)}{\sigma(\Delta)} \end{array} \right.$

$$\omega \in RH_2(\sigma) \iff \text{Solvability of } (D_2) \begin{cases}
\mathcal{L}u = 0 \text{ in } \Omega \\
u|_{\partial\Omega} = f \in L^2(\sigma)
\end{cases}$$

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$$u(X) = \int_{\partial \Omega} f(y) d\omega^X(y)$$
 solves (D_2)

• Can we go beyond Lipschitz domains

$$\omega \in A_{\infty}(\sigma) = \bigcup_{p>1} RH_p(\sigma) \rightsquigarrow (D_q)$$
 solvable for some q (large)

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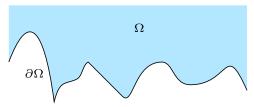
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- $\Omega \subset \mathbb{R}^{n+1}$, n > 2, open

$$u(X) = \int_{\partial\Omega} f(x) d\omega^{X}(x) \quad \text{solves} \quad (D) \begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} \text{``} = \text{''} f \in C_{c}(\partial\Omega) \end{cases}$$

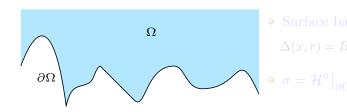


$$\Delta(x,r) = B(x,r) \cap \partial\Omega, \ x \in \partial\Omega$$

$$\bullet \ \sigma = \mathcal{H}^n \big|_{\partial \Omega}$$

- $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, open
- Harmonic measure $\{\omega^X\}_{X\in\Omega}$ family of "probabilities" on $\partial\Omega$

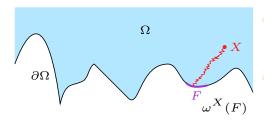
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 $\partial \Omega \text{ ADR } \leadsto \sigma(\Delta(x,r)) \approx r^n, \quad x \in \partial \Omega$

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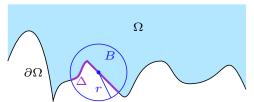
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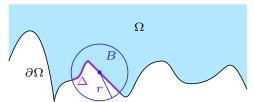


Surface ball

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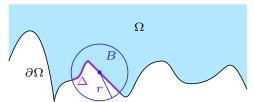
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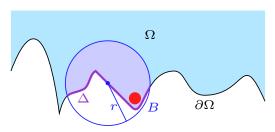
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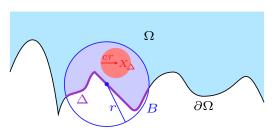
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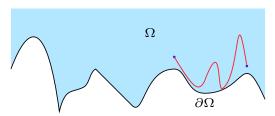
- Openness → Corkscrew condition
- Path-connectedness \rightsquigarrow Harnack chain condition



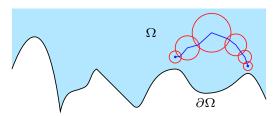
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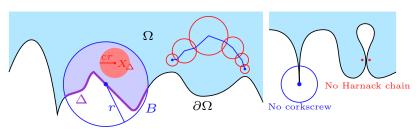
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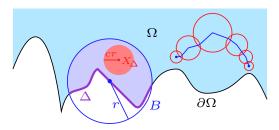
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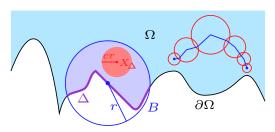


- Ω is CAD \equiv • Interior Corkscrew and Harnack Chain Exterior Corkscrew $\partial\Omega$ ADR

Introduction

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- Openness → Corkscrew condition
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- Ω is CAD \equiv • Interior Corkscrew and Harnack Chain Exterior Corkscrew $\partial\Omega$ ADR
- Ω is 1-sided CAD \equiv • Interior Corkscrew and Harnack Chain $\partial\Omega$ ADR

Theorem (David-Jerison 1990; Semmes 1989)

$$\bullet \ \Omega \subset \mathbb{R}^{n+1} \ \mathbf{CAD} \quad \Longrightarrow \quad \left\{ \begin{array}{l} \omega \in A_{\infty}(\sigma) = \bigcup_{p>1} RH_p(\sigma) \\ \left(\oint_{\Delta} k^p \, d\sigma \right)^{\frac{1}{p}} \lesssim \oint_{\Delta} k \, d\sigma = \frac{w(\Delta)}{\sigma(\Delta)} \end{array} \right.$$

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• Can we consider operators with variable coefficients?

•
$$Lu(X) = -\operatorname{div}(A\nabla u)(X), X \in \Omega$$

•
$$A(X) = (a_{i,j}(X))_{1 \le i,j \le n+1}$$
 real (symmetric)
 $A(X)\xi \cdot \xi > \Lambda^{-1}|\xi|^2$ and $|A(X)\xi \cdot \eta| < \Lambda|$

• ω_L elliptic measure

•
$$\Omega \subset \mathbb{R}^{n+1}$$
 CAD • $\delta(X) := \operatorname{dist}(X, \partial \Omega)$
• $|\nabla A| \, \delta \in L^{\infty}(\Omega)$

$$\bullet$$
 $|\nabla A|^2 \delta$ is a Carleson measure $\Big]$

• A is a Carleson perturbation of
$$A_0$$
 \Longrightarrow $\omega_L \in A_{\infty}(\sigma)$

Can we go beyond CAD?

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$$\omega_{L_0} \in A_{\infty}(\sigma)$$

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THEOLETTI (Manual)

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- 0 A :- O---I
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Theorem (Kenig-Pipher 01; Fefferman-Kenig-Pipher 91, Milakis-Pipher-Toro 13)

and

• $\Omega \subset \mathbb{R}^{n+1}$ CAD

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- |∇A| δ ∈ L[∞](Ω)
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$$\longrightarrow w_{r} \in \Lambda (\sigma$$

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- $\bullet \ \omega_{L_0} \in A_{\infty}(\sigma)$
 - $\bullet \ A \ is \ a \ Carleson \ perturbation \ of \ A_0 \ \ \Longrightarrow \ \omega_L \in A_{\infty}(\sigma)$

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- Can we go beyond CAD?

Section 2

Characterizations of CAD

•
$$L = -\operatorname{div}(A\nabla) \in \mathbb{L}_0$$
:

•
$$|\nabla A| \delta \in L^{\infty}(\Omega) + |\nabla A|^2 \delta$$
 is a Carleson measure:

$$\sup_{\substack{x \in \partial \Omega \\ 0 < r < \operatorname{diam}(\partial \Omega)}} \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r) \cap \Omega} |\nabla A(Y)|^2 \delta(Y) \, dY < \infty$$

•
$$\Omega \subset \mathbb{R}^{n+1}$$
 1-sided CAT

•
$$L \in \mathbb{L}_0$$

$$\bigcirc$$
 Ω CAD

$$0 \implies 2$$
 [Kenig-Pipher] $\leadsto \mathbb{L}_0$; [David-Jerison; Semmes] \leadsto Laplacian

$$2 \implies 1$$
 [Hofmann-M.-Mayboroda-Toro-Zhao] $\rightsquigarrow L_0$

• [Hofmann-M.-Toro]
$$\rightsquigarrow |\nabla A| \delta \in L^{\infty}(\Omega) + |\nabla A|$$
 is a Carleson measure

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Theorem

•
$$\Omega \subset \mathbb{R}^{n+1}$$
 1-sided CAD

•
$$L \in \mathbb{L}_0$$

$$\mathbf{0} \Omega \text{ CAD}$$



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$$2 \implies 1$$
 [Hofmann-M.-Mayboroda-Toro-Zhao] $\rightsquigarrow \mathbb{L}_0$

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Theorem

• $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

- $2\omega_L \in A_{\infty}(\sigma)$
- $1 \implies 2$ [Kenig-Pipher] $\rightsquigarrow \mathbb{L}_0$; [David-Jerison; Semmes] \rightsquigarrow Laplacian

- $L = -\operatorname{div}(A\nabla) \in \mathbb{L}_0$:
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Theorem

• $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

- $L \in \mathbb{L}_0$
- $\bigcirc \Omega \text{ CAD} \iff$
- $2\omega_L \in A_{\infty}(\sigma)$
- $1 \implies 2$ [Kenig-Pipher] $\rightsquigarrow L_0$; [David-Jerison; Semmes] \rightsquigarrow Laplacian
- $2 \implies 1$ [Hofmann-M.-Mayboroda-Toro-Zhao] $\rightsquigarrow L_0$

- $L = -\operatorname{div}(A\nabla) \in \mathbb{L}_0$:
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Theorem

• $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

- $L \in \mathbb{L}_0$
- $\bigcirc \Omega \text{ CAD} \iff$
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- $1 \implies 2$ [Kenig-Pipher] $\rightsquigarrow L_0$; [David-Jerison; Semmes] \rightsquigarrow Laplacian
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$$u(X) = \int_{\partial\Omega} f(y) d\omega_L^X(y), f \in C_c(\partial\Omega) \quad \leadsto \quad Lu = 0$$

•
$$(D_p)$$

$$\begin{cases} Lu = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \in L^p(\sigma) \\ \|Nu\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} \end{cases}$$

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$$\Omega$$
 CAD

$$\Longrightarrow$$

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•00000

Perturbation

Elliptic operators on rough domains

Theorem

• $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

- Ω CAD \iff $\omega_L \in A_{\infty}$

- (D_n) solvability $p \gg 1$

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- Goal: Develop perturbation theory on 1-sided CAD

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• Good operator: $L_0 u = -\operatorname{div}(A_0 \nabla u)$

- Perturbed operator: $Lu = -\operatorname{div}(A\nabla u)$
- Question 1: When $\omega_L \in A_{\infty}$?
 - Fefferman-Kenig-Pipher → Lipschitz
 Milakis-Pipher-Toro → CAL
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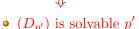
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Disagreement between L_0 and L

$$\rho(A_0,A)(Y) := \sup_{Z \in B(Y,\frac{\delta(Y)}{2})} \frac{|A_0(Z) - A(Z)|}{\delta(Z)}$$

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•
$$\Omega$$
 1-sided CAD • L_0 , L elliptic operators • $\|\rho(A_0,A)\| < \varepsilon$

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Corollar

• $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAT

 \bullet $L \in I$

 \bigcirc Ω CAD

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$$1 \implies 2$$

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Large constant perturbation and Characterization of CAD

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Corollary

• $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD

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 $\bigcirc \Omega$ CAD

 \iff



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 $\mathbf{1}$ Ω CAD

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Small constant perturbation

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Theorem (Cavero-Hofmann-M.)

$$\Omega$$
 1-sided CAD Ω Ω 1-sided CAD Ω 1-sided CAD Ω Ω 1-sided CAD Ω 1-sided CAD

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Theorem (Cavero-Hofmann-M.)

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 1-sided CAD \circ L_0 , L elliptic operators

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• Ω 1-sided CAD • L_0 , L elliptic operators • $\|\rho(A_0,A)\| \ll 1$

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Theorem (Cavero-Hofmann-M.)

• Ω 1-sided CAD • L_0 , L elliptic operators • $\|\rho(A_0,A)\| \ll 1$

$$\omega_{L_0} \in RH_q(\sigma) \implies \omega_L \in RH_q(\sigma)$$

 (D_p) solvable for $p \ge q'$ \Longrightarrow (D_p) solvable for $p \ge q'$

Section 4

Other A_{∞} properties

- 1-sided CAD $\leadsto \sigma = H^n \big|_{\partial\Omega}$ good measure $\leadsto \omega_L \in A_{\infty}(\sigma)$
 - What happens if $H^n|_{\partial\Omega}$ is a bad object?
 - Ω a 1-sided CAD and $\sup_{\Delta \subset \partial \Omega} \frac{1}{\sigma(\Delta)} \iint_{B \cap \Omega} \rho(A_0, A)^2 \, \delta \, dY < \infty$

$$\left(\omega_{L_0} \in A_{\infty}(\sigma) \implies \omega_L \in A_{\infty}(\sigma)\right) \implies \omega_L \in A_{\infty}(\omega_{L_0})$$

- Can we directly prove $\omega_L \in A_{\infty}(\omega_{L_0})$?
- [Fefferman-Kenig-Pipher: Milakis-Pipher-Toro] $\leadsto \Omega$ CAD

$$\sup_{\Delta \subset \partial \Omega} \frac{1}{\omega_{L_0}(\Delta)} \iint_{B \cap \Omega} \rho(A_0, A)^2 G_{L_0} dY \ll 1 \implies \omega_L \in RH_2(\omega_{L_0}) \subset A_{\infty}(\omega_{L_0})$$

This condition does not involve $\sigma!!!$

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- Ω satisfies CDC (capacity density condition, aka uniform 2-fat)

$$\frac{\operatorname{Cap}_2(\overline{B} \setminus \Omega, 2B)}{\operatorname{Cap}_2(\overline{B}, 2B)} \gtrsim 1$$

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