Spring 2019 - Math 3150
Final Exam - May 8
Time Limit: 120 Minutes

Name (Print): ____________________

This exam contains 12 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	20	
3	8	
4	15	
5	10	
6	12	
7	10	
8	10	
9	0	
10	0	
Total:	100	

1. (15 points) Decide which of the following statements are true or false. (No justification is needed).

   F Bounded sequences are convergent.

   T If (s_n) is a unbounded increasing sequence then $\lim s_n = +\infty$.

   T A sequence (s_n) is called Cauchy sequence if

   for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

   F There are Cauchy sequences which are not convergent.

   F A convergent sequence can have a subsequence with a different limit.

   T Every sequence has a monotonic subsequence.

   F There are bounded sequences which has no convergent subsequences.

   T A function f is continuous at x_0 if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

   F The function $f(x)$ defined below is continuous at $x = 1$.

$$f(x) = \begin{cases} x & \text{when } 0 \leq x < 1, \\ 2 & \text{when } x = 1. \end{cases}$$

   T $\max(f, g)$ is a continuous function whenever f and g are continuous functions.

   T If f is a continuous function on $[a, b]$ and $f(a) = 1$ and $f(b) = 3150$ with $a < b$ then there exists $c \in (a, b)$ such that $f(c) = 2019$.

   T $f(x) = x^2$ is a uniformly continuous function on $[0, 3]$.

   F If $\lim_{x \to a} f(x) = 3$ and $\lim_{x \to a} g(x) = 5$ then $\lim_{x \to a} \frac{3f(x) + g^2(x)}{g(x)} = 5$.

   F The function $f(x)$ defined below is not integrable on $[0, 1]$.

$$f(x) = \begin{cases} x & \text{when } 0 \leq x < 1, \\ 2 & \text{when } x = 1. \end{cases}$$

   T Every continuous function is integrable.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^4 \sin(\frac{1}{x^2}) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) (6 points) Show that $f(x)$ is continuous at every point $x \in \mathbb{R}$.

Solution: We first observe that when $x \neq 0$ then $sin(x)$ and $1/x^2$ are continuous and therefore, by composition $\sin(1/x^2)$ is continuous. Moreover, x^4 is continuous. Therefore, $x^4 \sin(\frac{1}{x^2})$ is continuous when $x \neq 0$. When $x = 0$ and if (x_n) is any sequence with $x_n \to 0$ (assuming (x_n) is not the zero sequence in which case $f(x_n) = 0$) then observe that

$$|f(x_n)| = |x_n^4 \sin(\frac{1}{x_n^2})| \leq |x_n^4| \to 0 = f(0).$$

Hence we see that $f(x_n) \to f(0) = 0$ whenever $x_n t ø 0$. By definition f is continuous at $x = 0$ as well. We conclude that f is continuous.

(b) (6 points) Show that $f(x)$ is differentiable at $x = 0$ and compute $f'(x)$ for all $x \in \mathbb{R}$.

Solution: At $x = 0$,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^4 \sin(\frac{1}{x^2}) - 0}{x - 0} = \lim_{x\to 0} x^3 \sin(\frac{1}{x^2}).$$

Since

$$|x^3 \sin(\frac{1}{x^2})| \leq |x^3| \to 0 \quad \text{as } x \to 0$$

we see that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

When $x \neq 0$, then $sin(x)$ and $1/x^2$ are differentiable and therefore, by composition $\sin(1/x^2)$ is differentiable. Moreover, x^4 is differentiable. Therefore, $x^4 \sin(\frac{1}{x^2})$ is differentiable when $x \neq 0$ and

$$f'(x) = 4x^3 \sin(\frac{1}{x^2}) - 2x \cos(\frac{1}{x^2}) \quad \text{when } x \neq 0.$$

(c) (4 points) Show that $f'(x)$ is continuous at $x = 0$.

Solution: From part (b) we got that

$$f'(x) = \begin{cases} 4x^3 \sin(\frac{1}{x^2}) - 2x\cos(\frac{1}{x^2}) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

Once again if (x_n) is a sequence with $x_n \to 0$, assuming (x_n) is not the zero sequence (as in this case $f'(x_n) = 0$ clearly), we have

$$|4x_n^3 \sin(\frac{1}{x_n^2}) - 2x_n \cos(\frac{1}{x_n^2})| \leq 4|x_n^3| + 2|x_n| \to 0.$$

Therefore, we have that $f'(x_n) \to 0$ whenever $x_n \to 0$. This shows that f' is continuous at $x = 0$.

(d) (4 points) Show that $f'(x)$ is not differentiable at $x = 0$.

Solution: If it was differentiable,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{4x^3 \sin(\frac{1}{x^2}) - 2x\cos(\frac{1}{x^2})}{x} = \lim_{x\to 0} 4x^2 \sin(\frac{1}{x^2}) - 2\cos(\frac{1}{x^2}).$$

Observe that $\cos(\frac{1}{x^2})$ does not have a limit as $x \to 0$. Therefore, the limit

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0}$$

does not exist and f' is not differentiable at 0.

3. (a) (4 points) Carefully state the Fundamental Theorem of Calculus I.

Solution: Fundamental Theorem of Calculus I: let f be a continuous function on $[a, b]$ and differentiable on (a, b). If f' is integrable on $[a, b]$ then

$$\int_a^b f' = f(b) - f(a).$$

(b) (4 points) Carefully state the Fundamental Theorem of Calculus II.

Solution: Fundamental Theorem of Calculus II: Let f be an integrable function on $[a, b]$. For $x \in [a, b]$, let

$$F(x) = \int_a^x f(t)dt.$$

1. Then F is continuous on $[a, b]$.
2. If f is continuous at $x_0 \in (a, b)$ then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

4. Let

$$f(t) = \begin{cases} t & \text{when } t < 0, \\ t^2 + 1 & \text{when } 0 \le t \le 2, \\ 0 & \text{when } t > 2. \end{cases}$$

(a) (6 points) Determine the function $F(x) = \int_0^x f(t)dt$.

Solution: For $x < 0$ we have

$$F(x) = \int_0^x f(t)dt = \int_0^x t dt = \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2}.$$

For $0 \le x \le 2$ we have

$$F(x) = \int_0^x f(t)dt = \int_0^x (t^2 + 1)dt = (\frac{t^3}{3} + t)|_0^x = \frac{x^3}{3} + x.$$

For $x > 2$, we get

$$F(x) = \int_0^x f(t)dt = \int_0^2 f(t)dt + \int_2^x f(t)dt = \int_0^2 (t^2+1)dt + \int_2^x 0dt = (\frac{t^3}{3}+t)|_0^2 = \frac{8}{3}+2.$$

Hence

$$F(x) = \begin{cases} \frac{x^2}{2} & \text{when } x < 0, \\ \frac{x^3}{3} + x & \text{when } 0 \le x \le 2, \\ \frac{8}{3} + 2 & \text{when } x > 2. \end{cases}$$

(b) (3 points) Find points at which F is continuous. *Justify your answer.*

Solution: From the Fundamental theorem of Calculus II, we know that if f is integrable then F is continuous. Therefore, F is continuous everywhere.

(c) (6 points) Find points at which F is differentiable and find F' at those points. *Justify your answer*

Solution: From the Fundamental theorem of Calculus II, we know that if f is continuous at a point x_0 then F is differentiable at x_0. Since f is continuous everywhere except $x = 0$ and $x = 2$. Hence F is differentiable at every points except $x = 0$ and $x = 2$.

At $x = 0$,

$$1 = \lim_{x \to 0^+} \frac{F(x) - F(0)}{x - 0} \neq \lim_{x \to 0^-} \frac{F(x) - F(0)}{x - 0} = 0.$$

At $x = 2$,

$$0 = \lim_{x \to 2^+} \frac{F(x) - F(0)}{x - 0} \neq \lim_{x \to 2^-} \frac{F(x) - F(0)}{x - 0} = 5.$$

Hence F is differentiable everywhere except at $x = 0$ and $x = 2$. Moreover,

$$F'(x) = \begin{cases} x & \text{when } x < 0, \\ x^2 + 1 & \text{when } 0 < x < 2, \\ 0 & \text{when } x > 2. \end{cases}$$

5. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying

$$|f(x) - f(y)| \leq |x - y|^{1+\epsilon} \quad \text{for all } x, y \in \mathbb{R}$$

for some $\epsilon > 0$.

(a) (8 points) Show that f is a differentiable function at every point $x \in \mathbb{R}$.

Solution: Let $y \in \mathbb{R}$ be fixed. Observe that for every y (with $y \neq x$) we have

$$\frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|^{\epsilon}.$$

Now given $\eta > 0$, let $\delta = \eta^{1/\epsilon}$. If $|x - y| < \delta$ then

$$|\frac{f(x) - f(y)}{|x - y| - 0}| \leq |x - y|^{\epsilon} < \delta^{\epsilon} = \eta.$$

As η is arbitrary, this shows that

$$f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$$

at every $y \in \mathbb{R}$.

(b) (2 points) Show that f is a constant function.

Solution: We observe that $f'(y) = 0$.

6. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 & \text{when } 0 < x < 3, \\ 3 & \text{when } x = 0, \\ 1 & \text{when } x = 3. \end{cases}$$

(a) (5 points) Compute the lower Riemann sum $L(f)$.

Solution: Let P be any partitioning $P = \{x_0 = a < x_1 < \ldots < x_n = b\}$. Then

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} f(x_k - x_{k-1}) = 2(x_1 - x_0) + \sum_{k=2}^{n-1} 2(x_k - x_{k-1}) + 1(x_n - x_{n-1}) \\ &= 2 \sum_{k=1}^{n} (x_k - x_{k-1}) - (x_n - x_{n-1}) = 2 * 3 - (x_n - x_{n-1}). \end{aligned}$$

Then we obtain that

$$L(f) = \sup_P L(f, P) = \sup_P (6 - (x_n - x_{n-1})) = 6.$$

(b) (5 points) Compute the upper Riemann sum $U(f)$.

Solution: Similarly,

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} \sup_{[x_{k-1}, x_k]} f(x_k - x_{k-1}) = 3(x_1 - x_0) + \sum_{k=2}^{n-1} 2(x_k - x_{k-1}) + 3(x_n - x_{n-1}) \\ &= (x_1 - x_0) + 2 \sum_{k=1}^{n} (x_k - x_{k-1}) = (x_1 - x_0) + 6 \end{aligned}$$

Then we obtain

$$U(f) = \inf_P U(f, P) = \inf_P (x_1 - x_0 + 6) = 6.$$

(c) (2 points) Use Parts (a) and (b) to conclude that f is integrable and find $\int_0^3 f$.

Solution: Since $L(f) = U(f) = 6$ we see that f is integrable and $\int_0^3 f = 6$.

7. Find the following limits if they exist.

(a) (5 points) (Hint: you may want to use L'Hospital's rule).

$$\lim_{x\to\infty}(e^x+x)^{1/x}.$$

Solution:

$$\lim_{x\to\infty}(e^x+x)^{1/x}=\lim_{x\to\infty}e^{\ln(e^x+x)^{1/x})}=e^{\lim_{x\to\infty}(\frac{\ln(e^x+x)}{x})}.$$

Now we have $(\frac{\infty}{\infty})$ indeterminate form and by L'Hospital's rule we get

$$\lim_{x\to\infty}(e^x+x)^{1/x}=e^{\lim_{x\to\infty}(\frac{e^x+1}{e^x+x})}.$$

Now we again have $(\frac{\infty}{\infty})$ indeterminate form and by L'Hospital's rule once again we get

$$\lim_{x\to\infty}(e^x+x)^{1/x}=e^{\lim_{x\to\infty}(\frac{e^x}{e^x+1})}.$$

Applying one more time we have

$$\lim_{x\to\infty}(e^x+x)^{1/x}=e.$$

(b) (5 points) (Hint: you may want to use the L'Hospital's rule and the Fundamental Theorem of Calculus II).

$$\lim_{x\to 0}\frac{1}{x^2}\int_0^{x^2}e^{\sqrt{t}}dt.$$

Solution: Observe that when $x=0$ we have $(\frac{\infty}{\infty})$ indeterminate form. Therefore we can use the L'Hospital's rule. To take the derivative of numerator we let

$$F(x^2)=\int_0^{x^2}e^{\sqrt{t}}dt$$

Since $e^{\sqrt{t}}$ is a continuous function we see that F is differentiable and

$$\frac{d}{dx}F(x^2)=e^{\sqrt{x^2}}2x=2xe^x$$

Hence

$$\lim_{x\to 0}\frac{1}{x^2}\int_0^{x^2}e^{\sqrt{t}}dt=\lim_{x\to 0}\frac{2xe^x}{2x}=1.$$

8. Let f be a continuous function on $[a, b]$ such that f is a twice differentiable function on (a, b) (that is, f is differentiable and f' is also differentiable). Let x_1, x_2, x_3 be three distinct points in (a, b) such that $f(x_1) = f(x_2) = f(x_3) = 0$. (For simplicity you can assume $x_1 < x_2 < x_3$).

   (a) (5 points) Show that there exists $y_0 \in (a, b)$ such that $f'(y_0) = 0$.

   Solution: Since f is continuous on $[a, b]$ and differentiable on (a, b) we can apply the Mean Value theorem to get

   $$f'(y_0) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0$$

   for some $y_0 \in (x_1, x_2)$. Hence $f'(y_0) = 0$ for some $y_0 \in (x_1, x_2)$.
   Similarly, by the Mean Value theorem we get $y_1 \in (x_2, x_3)$ such that

   $$f'(y_1) = \frac{f(x_2) - f(x_3)}{x_2 - x_3} = 0$$

   Hence $f'(y_1) = 0$ for some $y_1 \in (x_2, x_3)$.

   (b) (5 points) Show that there exists $z_0 \in (a, b)$ such that $f''(z_0) = 0$.

   Solution: Since f' is also continuous on $[a, b]$ and differentiable on (a, b) and using part (a) and applying the Mean Value theorem for f' for points y_0 and y_1 we get

   $$f''(z_0) = \frac{f'(y_0) - f'(y_1)}{y_0 - y_1} = 0.$$

   for some $z_0 \in (a, b)$. Hence we have $f''(z_0) = 0$ for some $z_0 \in (a, b)$.

9. (5 points (bonus)) Let f be a continuous function on $[a, b]$. If $\int_a^b f^2(x)dx = 0$ then $f(x) = 0$ for every $x \in [a, b]$.

Solution:

10. (5 points (bonus)) Let f be a continuous function on $[a, b]$. If $\int_a^b f(x)g(x)dx = 0$ for every continuous function $g(x)$ on $[a, b]$ then $f(x) = 0$ for every $x \in [a, b]$.

Solution: