Elliptic operators on rough domains

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INSTITUTO DE CIENCIAS MATEMÁTICAS

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Hartford

Section 1

Introduction

Lipschitz domains

Theorem (Dahlberg 1977)

$$\Omega \subset \mathbb{R}^{n+1} \textbf{ Lipschitz domain} \implies \begin{cases} \omega \in RH_2(\sigma) \\ \left(\fint_\Delta k^2\,d\sigma\right)^{\frac{1}{2}} \lesssim \fint_\Delta k\,d\sigma = \dfrac{w(\Delta)}{\sigma(\Delta)} \end{cases}$$

- $\omega \in RH_2(\sigma) \iff$ Solvability of (D_2) $\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} = f \in L^2(\sigma) \end{cases}$
- $u(X) = \int_{\partial\Omega} f(y)\,d\omega^X(y)$ solves (D_2)
- Can we go beyond Lipschitz domains?

$$\omega \in A_\infty(\sigma) = \bigcup_{p>1} RH_p(\sigma) \rightsquigarrow (D_q) \text{ solvable for some } q \text{ (large)}$$

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Harmonic measure

- $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, open
- Harmonic measure $\{\omega^X\}_{X\in\Omega}$ family of "probabilities" on $\partial\Omega$

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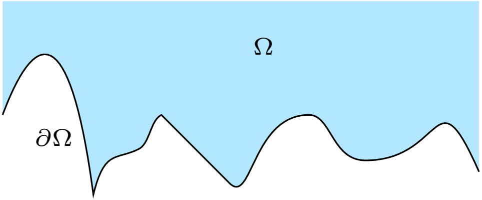

- Surface ball

$$\Delta(x,r) = B(x,r) \cap \partial\Omega,\ x \in \partial\Omega$$

- $\sigma = \mathcal{H}^n\big|_{\partial\Omega}$

- $\partial\Omega$ ADR $\rightsquigarrow \sigma(\Delta(x,r)) \approx r^n, \quad x \in \partial\Omega$

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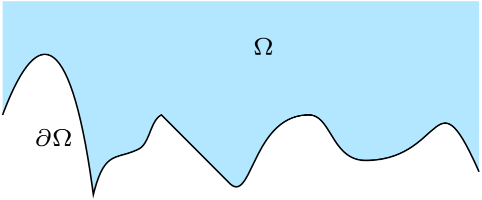

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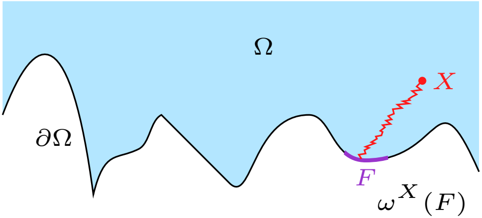

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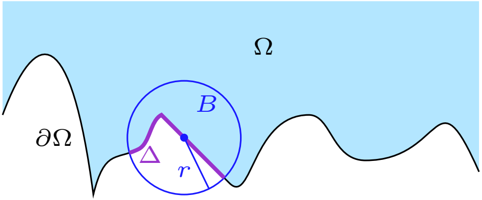

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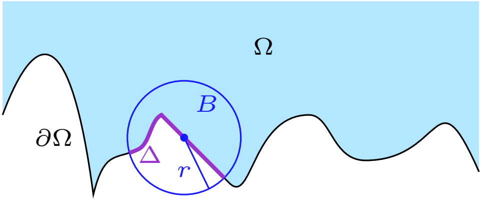

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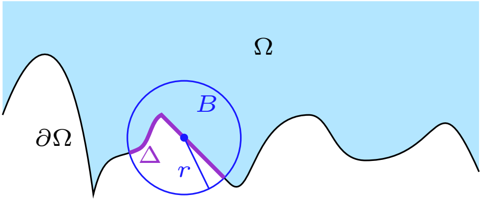

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[Jerison-Kenig 1982] NTA/CAD domains

- Openness ⇝ Corkscrew condition
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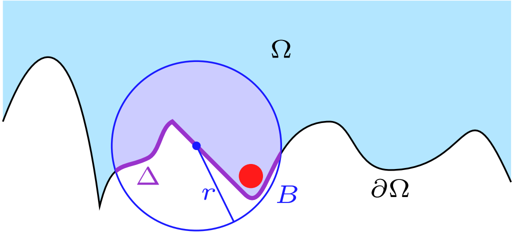

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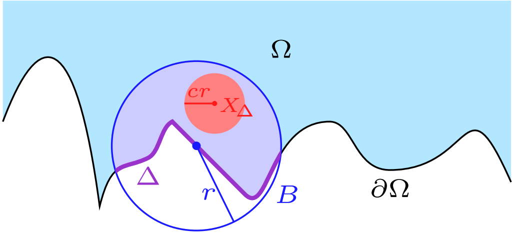

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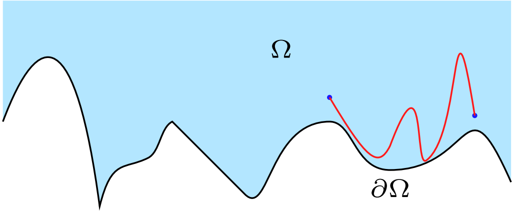

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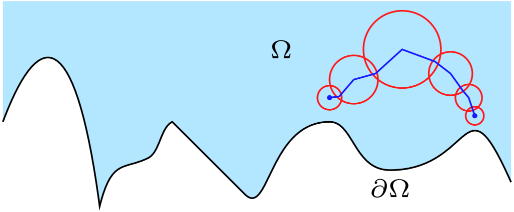

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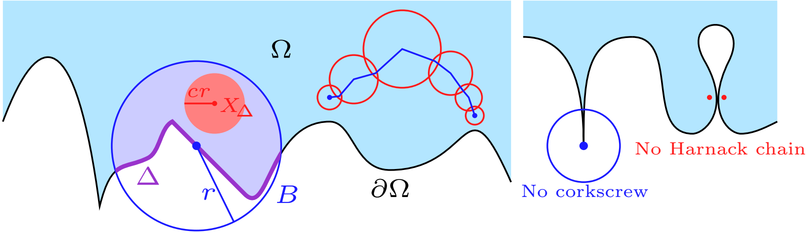

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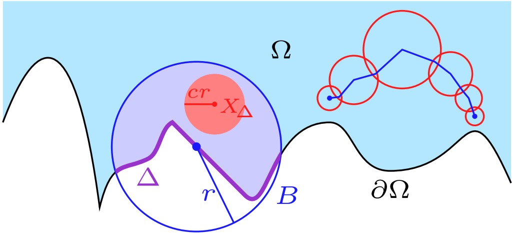

- Ω is CAD $\equiv$
  - **Interior** Corkscrew and Harnack Chain
  - **Exterior** Corkscrew
  - $\partial\Omega$ ADR

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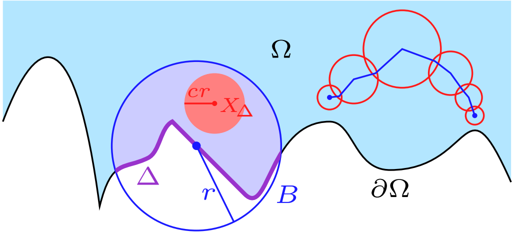

- Ω is CAD $\equiv$
  - **Interior** Corkscrew and Harnack Chain
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- Ω is 1-sided CAD $\equiv$
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NTA/CAD domains

Theorem (David-Jerison 1990; Semmes 1989)

- $\Omega \subset \mathbb{R}^{n+1}$ CAD $\Longrightarrow \begin{cases} \omega \in A_\infty(\sigma) = \bigcup_{p>1} RH_p(\sigma) \\ \left(⨍_\Delta k^p \, d\sigma\right)^{\frac{1}{p}} \lesssim ⨍_\Delta k \, d\sigma = \dfrac{w(\Delta)}{\sigma(\Delta)} \end{cases}$

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Elliptic operators

- $Lu(X) = -\operatorname{div}(A\nabla u)(X)$, $X \in \Omega$
- $A(X) = \big(a_{i,j}(X)\big)_{1\le i,j\le n+1}$ **real** (symmetric)

$$A(X)\xi\cdot\xi \ge \Lambda^{-1}|\xi|^2 \qquad \text{and} \qquad |A(X)\xi\cdot\eta| \le \Lambda|\xi|\,|\eta|$$

- ω_L elliptic measure

Theorem (Kenig-Pipher 01; Fefferman-Kenig-Pipher 91, Milakis-Pipher-Toro 13)

- $\Omega \subset \mathbb{R}^{n+1}$ CAD
- $\delta(X) := \operatorname{dist}(X, \partial\Omega)$

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- $|\nabla A|\,\delta \in L^\infty(\Omega)$ and $|\nabla A|^2\,\delta$ *is a Carleson measure* $\Longrightarrow \omega_L \in A_\infty(\sigma)$
- $\omega_{L_0} \in A_\infty(\sigma)$ and A *is a Carleson perturbation of* A_0 $\Longrightarrow \omega_L \in A_\infty(\sigma)$

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$$A(X)\xi\cdot\xi \ge \Lambda^{-1}|\xi|^2 \qquad \text{and} \qquad |A(X)\xi\cdot\eta| \le \Lambda|\xi|\,|\eta|$$

- ω_L elliptic measure

Theorem (Kenig-Pipher 01; Fefferman-Kenig-Pipher 91, Milakis-Pipher-Toro 13)

- $\Omega \subset \mathbb{R}^{n+1}$ CAD
- $\delta(X) := \operatorname{dist}(X, \partial\Omega)$

- $\left.\begin{array}{l} \bullet\ |\nabla A|\,\delta \in L^\infty(\Omega) \\ \bullet\ |\nabla A|^2\,\delta \textit{ is a Carleson measure} \end{array}\right\} \Longrightarrow \omega_L \in A_\infty(\sigma)$

- $\left.\begin{array}{l} \bullet\ \omega_{L_0} \in A_\infty(\sigma) \\ \bullet\ A \textit{ is a Carleson perturbation of } A_0 \end{array}\right\} \Longrightarrow \omega_L \in A_\infty(\sigma)$

- Can we go beyond CAD?

Section 2

Characterizations of CAD

Elliptic operators

- $L = -\operatorname{div}(A\nabla) \in \mathbb{L}_0$:
  - $|\nabla A|\,\delta \in L^\infty(\Omega)$ + $|\nabla A|^2\delta$ is a Carleson measure:

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r)\cap\Omega} |\nabla A(Y)|^2\delta(Y)\,dY < \infty$$

Theorem

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
- $L \in \mathbb{L}_0$

1 Ω CAD $\Longleftrightarrow$ 2 $\omega_L \in A_\infty(\sigma)$

1 $\Longrightarrow$ 2 [Kenig-Pipher] $\rightsquigarrow$ $\mathbb{L}_0$; [David-Jerison; Semmes] $\rightsquigarrow$ Laplacian

2 $\Longrightarrow$ 1 [Hofmann-M.-Mayboroda-Toro-Zhao] $\rightsquigarrow$ $\mathbb{L}_0$

- [Hofmann-M.-Uriarte; Azzam-Hofmann-M.-Nyström-Toro] $\rightsquigarrow$ Laplacian
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- $u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y)$, $f \in C_c(\partial\Omega)$ $\rightsquigarrow$ $Lu = 0$

- (D_p) $\begin{cases} Lu = 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} = f \in L^p(\sigma) \\ \|Nu\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} \end{cases}$

- (D_p) solvable $\iff$ $\omega_L \in RH_{p'}(\sigma)$

Theorem

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
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Section 3

Perturbation

Motivation

Theorem

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
- $L \in \mathbb{L}_0$
  - Ω CAD $\iff$ $\omega_L \in A_\infty$ $\iff$ (D_p) solvability $p \gg 1$

- Can we consider other operators? (e.g., non-smooth coefficients)
- [Fefferman-Kenig-Pipher 1991, Milakis-Pipher-Toro 2013]

  $A_\infty(\sigma)$ is stable under Carleson perturbation on CAD

- Goal: Develop perturbation theory on 1-sided CAD

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- Perturbed operator: $Lu = -\operatorname{div}(A \nabla u)$

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Large constant perturbation

- Disagreement between L_0 and L

$$\rho(A_0,A)(Y) := \sup_{Z\in B(Y,\frac{\delta(Y)}{2})} \frac{|A_0(Z)-A(Z)|}{\delta(Z)}$$

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Theorem (Cavero-Hofmann-M.; Cavero-Hofmann-M.-Toro)

- Ω 1-sided CAD
- *L_0, L elliptic operators*
- $\|\rho(A_0,A)\|<\infty$

$$\omega_{L_0}\in A_\infty(\sigma) \quad\Longrightarrow\quad \omega_L\in A_\infty(\sigma)$$

$$(D_p) \text{ solvable for } p>p_0 \quad\Longrightarrow\quad (D_p) \text{ solvable for } p>q_0$$

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Large constant perturbation and Characterization of CAD

- $L_0 = -\operatorname{div}(A_0\nabla) \in \mathbb{L}_0$: $|\nabla A_0|\,\delta \in L^\infty(\Omega) + |\nabla A_0|^2\delta$ Carleson meas.
- $L = -\operatorname{div}(A\nabla) \in \mathbb{L}$: $\|\rho(A_0, A)\| < \infty$ for some $L_0 \in \mathbb{L}_0$

Corollary

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD
- $L \in \mathbb{L}$

$$① \ \Omega \text{ CAD} \iff ② \ \omega_L \in A_\infty(\sigma)$$

$\boxed{② \Longrightarrow ①}$

- [Cavero-Hofmann-M.]: $\omega_L \in A_\infty(\sigma) \Longrightarrow \omega_{L_0} \in A_\infty(\sigma)$
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Corollary

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- Disagreement between L_0 and L

$$\rho(A_0, A)(Y) := \sup_{Z \in B(Y, \frac{\delta(Y)}{2})} \frac{|A_0(Z) - A(Z)|}{\delta(Z)}$$

$$\|\rho(A_0, A)\| = \sup_{\Delta(x,r) \subset \partial\Omega} \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r)\cap\Omega} \rho(A_0, A)(Y)^2 \, \delta(Y) \, dY$$

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Section 4

Other A_∞ properties

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- Ω a 1-sided CAD and $\displaystyle\sup_{\Delta\subset\partial\Omega} \frac{1}{\sigma(\Delta)} \iint_{B\cap\Omega} \rho(A_0, A)^2\, \delta\, dY < \infty$

$$\Big(\omega_{L_0} \in A_\infty(\sigma) \implies \omega_L \in A_\infty(\sigma)\Big) \implies \omega_L \in A_\infty(\omega_{L_0})$$

  - Can we directly prove $\omega_L \in A_\infty(\omega_{L_0})$?
- [Fefferman-Kenig-Pipher; Milakis-Pipher-Toro] $\rightsquigarrow$ Ω CAD

$$\sup_{\Delta\subset\partial\Omega} \frac{1}{\omega_{L_0}(\Delta)} \iint_{B\cap\Omega} \rho(A_0, A)^2\, G_{L_0}\, dY \ll 1 \implies \omega_L \in RH_2(\omega_{L_0}) \subset A_\infty(\omega_{L_0})$$

This condition does not involve σ!!!

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1-sided NTA and CDC

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided NTA (aka uniform domain)
  - Interior Corkscrew
  - Interior Harnack Chain
- Ω satisfies CDC (capacity density condition, aka uniform 2-fat)

$$\frac{\mathrm{Cap}_2(\overline{B}\setminus\Omega, 2\,B)}{\mathrm{Cap}_2(\overline{B}, 2\,B)} \gtrsim 1$$

where $\mathrm{Cap}_2(K,D) = \inf\left\{ \iint_D |\nabla v|^2\,dX :\ v\in C_c^\infty(\mathcal{D}),\ v\ge 1 \text{ in } K\right\}$

- CDC is a quantitative version of Wiener regularity
- Examples:
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- Ω satisfies CDC (capacity density condition, aka uniform 2-fat)

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where $\mathrm{Cap}_2(K, D) = \inf\left\{ \iint_D |\nabla v|^2\, dX : \; v \in C_c^\infty(\mathcal{D}), \; v \geq 1 \text{ in } K \right\}$

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  - Ω NTA $\Longrightarrow$ CDC $\rightsquigarrow \overline{B'} \subset \overline{B}\setminus\overline{\Omega} \subset \overline{B}\setminus\Omega$ (exterior)
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1-sided NTA and CDC

- $\Omega \subset \mathbb{R}^{n+1}$ 1-sided NTA (aka uniform domain)
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- Disagreement between L_0 and L

$$\rho(A_0, A)(Y) := \sup_{Z \in B(Y, \frac{\delta(Y)}{2})} \frac{|A_0(Z) - A(Z)|}{\delta(Z)}$$

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Theorem (Akman, Hofmann, M., Toro)

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$$\mathcal{A}(\rho(A_0, A))(x) := \left(\iint_{\Gamma(x)} \frac{\rho(A_0, A)(Y)^2}{\delta(Y)^{n-1}} \, dY \right)^{\frac{1}{2}}, \qquad x \in \partial\Omega$$

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