The Solutions to Bessel's Equation

the differential equation

$$x^2y''+xy'+(x^2-p^2)y=0$$ is called

the Bessel's equation of order p.

the p can be any number, (not just an integer),

~~but~~. Though integers and multiples of $\frac{1}{2}$

are most important in applications.

Check that $x=0$ is a regular singular point.

let's find a solution of the form

$$y(x)=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x)=\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}$$ and $y''=\sum(n+r)(n+r-1)a_n x^{n+r-2}$

Plug into the Bessel's equation to get

$$0=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+2+r}-p^2\sum_{n=0}^{\infty}a_n x^{n+r}$$

We have

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)a_n+(n+r)a_n-p^2a_n\right]x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

For $n=0$ we have the indicial equation

$\phi(r) = r(r-1) + r - p^2 = (r-p)(r+p).$

This gives us two roots $r_1 = p$, $r_2 = -p$.

If p is not integer & then cond $n \geq 2$) we get
and $r = p$ we get

$$a_n = \frac{-a_{n-2}}{n(n+2p)} \quad n \geq 2$$ (a₀ can be anything as the coefficient of $a_0 = 0$)

$$a_2 = \frac{-a_0}{2.(2p+2)} = \frac{-a_0}{2^2(p+1)}$$

$$a_4 = -\frac{a_2}{4(4+2p)} = \frac{-a_2}{2^3(2+p)} = \frac{a_0}{2^4 .2(p+1)(p+2)}$$

$$a_6 = \frac{-a_4}{6(6+2p)} = \frac{-a_0}{2^6 .2.3(p+1)(p+2)(p+3)}$$

$\vdots$

$$a_{2n} = \frac{(-1)^n . a_0}{2^{2n} . n!\,(p+1)(p+2)\dots(p+n)}$$

Therefore, for $r=p$ we have the Frobenius type solution

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n+p}}{2^{2n} n!\,(p+1)\dots(p+n)}$$

Similarly for $r = -p$ one gets the second linearly independent solution

$$y_2(x) = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n-p}}{2^{2n} n!\,(-p+1)(-p+2)\dots(-p+n)}$$

Let's define the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} . e^{-t} dt. \quad x \in \mathbb{R}.$$

Let's consider $\Gamma(x+1) = \int_0^\infty t^{(x+1)-1} e^{-t} dt$

$$= \int_0^\infty t^x . e^{-t} dt$$

$$\overset{\text{by integration by parts}}{=} \underbrace{\left(-t^x e^{-t}\right)\Big|_{t=0}^{\infty}}_{\substack{=0 \\ \text{at } t=0 \\ \text{and } t\to\infty}} + \int_0^\infty (x) t^{x-1} e^{-t} dt$$

$$= 0 + x \int_0^\infty t^{x-1} e^{-t} dt$$

$$= x \, \Gamma(x)$$

Therefore $\Gamma(x)$ has property that for any x

$$\Gamma(x+1) = x\,\Gamma(x)$$

Note that when $x = n$ integer then we get $\Gamma(n+1) = n!$

We can rewrite, for integer n,

$$\Gamma(n+p+1) = (n+p)\,\Gamma(n+p)$$
$$= (n+p)(n+p-1)\,\Gamma(n+p-1)$$
$$= (n+p)(n+p-1) \cdots \Gamma(1+p)$$

$$\Gamma(n-p+1) = (n-p)(n-p-1) \cdots \Gamma(1-p).$$

Remember that the Frobenius type solution for $r_1 = p$ is

$$y_1(x) = \sum \frac{(-1)^n \, x^{2n+p}}{2^{2n}\, n!\, (p+1)\cdots(p+n)}$$

let $$J_p(x) = \frac{1}{2^p\,\Gamma(1+p)}\, y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{k!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

is also solution (as y_1 is solution so is constant multiple of y_1)

Similarly
$$J_{-p}(x) = \frac{1}{2^{-p}\,\Gamma(1-p)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{k!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}$$

is also solution of the Bessel equation corresponding to $r_2 = -p$

When p is not integer

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x)$$ is the general

solution of the Bessel equation. However,

when p is an integer, then we have

$$J_n(x) = (-1)^n J_{-n}$$. Hence we do not

get second linearly independent solution.

The second solution is the so-called

Bessel function of second kind.

The second linearly independ solution is defined

as

$$Y_n(x) = \lim_{p \to n} \frac{\cos(p\pi) J_p(x) - J_{-p}(x)}{\sin(p\pi)}$$

Notice that J_p and J_{-p} are both solutions, and

therefore Y_n is also solution. It turns out that

Y_n is a linearly independent solution.

Therefore when p is integer the general solution is

$$y(x) = A J_n(x) + B Y_n(x).$$

Like trig functions, Bessel functions satisfy some identities;

i) $J_p'(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$

ii) $J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$

iii) $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$

iv) $J_{p-1}(x) = \frac{2p}{x} J_p(x) - J_{p+1}(x)$.

Here $J_p'(x) = \frac{d}{dx}\left(J_p(x)\right)$ = derivative of $J_p(x)$ w.r.t to x