The Laplace Equation

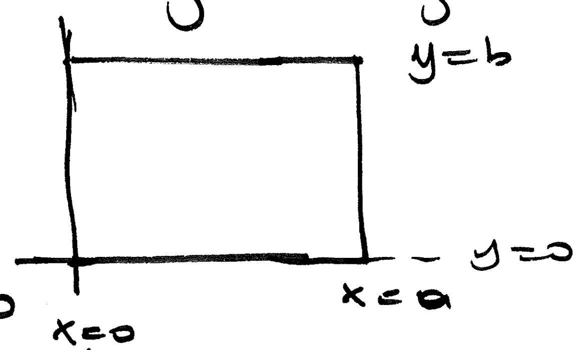
Let $u(x,y)$ be the potential function. Then it's governed by the two dimensional Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Any function having continuous first and second order partial derivatives that satisfies the two-dimensional Laplace's equation is called a harmonic function.

Laplace's equation for a rectangular region.

$$u_{xx} + u_{yy} = 0 \quad ; \quad \begin{matrix} 0 < x < a \\ 0 < y < b \end{matrix}$$



Boundary Conditions $\left\{ \begin{matrix} u(x,0) = 0 \\ u(x,b) = 0 \end{matrix} \right.$ $\Bigg|$ $\begin{matrix} u(0,y) = 0 \\ u(a,y) = f(y) \end{matrix}$

The separation process is similar

Let $u(x,y)$ be the solution with the property

$$u(x,y) = X(x) \cdot Y(y)$$

$$u_x = X'Y, \quad u_{xx} = X''Y$$

$$u_y = XY', \quad u_{yy} = XY''$$

Then

$$u_{xx} + u_{yy} = X''Y + XY'' = 0.$$

Divide both sides by XY to get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \text{or} \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

Notice that the left-hand side is function of x and the right hand side is function of y. Only posibility is that both are constant λ. that is,

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

$$\frac{X''}{X} = \lambda \longrightarrow X'' = \lambda X \longrightarrow X'' - \lambda X = 0$$

$$-\frac{Y''}{Y} = \lambda \longrightarrow -Y'' = \lambda Y \longrightarrow Y'' + \lambda Y = 0.$$

The boundary conditions:

$u(x,0) = X(x)\cdot Y(0) = 0 \rightarrow X(x)=0$ or $Y(0)=0$

$u(x,b) = 0 = X(x)\cdot Y(b) \rightarrow X(x)=0$ or $Y(b)=0$

$u(0,y) = 0 = X(0)\cdot Y(y) = 0 \longrightarrow X(0)=0$ or $Y(y)=0$

$u(a,y) = f(y) \rightarrow X(a)\cdot Y(y) = f(y)$.

The boundary conditions $X(x)=0$ will give us only the trivial solution!

So we will ~~consider~~ consider;
$Y(0)=0$ ($u(x,0)=0$ gives)
$Y(b)=0$ ($u(x,b)=0$ ")
$X(0)=0$ ($u(0,y)=0$)

Hence we have

$X''-\lambda X=0$, $X(0)=0$

$Y''+\lambda Y=0$ $Y(0)=0$ and $Y(b)=0$

Plus the fourth boundary condition $u(a,y)=f(y)$.

The next step is to solve the eigen value problem.

$$Y'' + \lambda Y = 0, \qquad Y(0) = 0 \quad \& \quad Y(b) = 0$$

$$\lambda = \sigma^2 = \frac{n^2\pi^2}{b^2} \qquad n = 1, 2 \ldots$$

are the eigenvalues and the corresponding eigenfunctions are

$$Y_n = \sin\frac{n\pi y}{b} \qquad n = 1, 2, \ldots$$

Once we found the eigenvalues, substitute λ into the equation of X.

$$X'' - \lambda X = X'' - \frac{n^2\pi^2}{b^2} X = 0.$$

Its characteristics are (from 2.4.10)

$$r = \mp \frac{n\pi}{b}$$ and the general solution

$$X = c_1 e^{\frac{n\pi}{b}x} + c_2 e^{-\frac{n\pi}{b}x}.$$

The only boundary condition

$$X(0) = 0 = c_1 e^0 + c_2 e^0 \rightarrow c_2 = -c_1$$

Therefore for $n = 1, 2 \ldots$

$$X_n(x) = c_n\left(e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x}\right)$$

As $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$.

hyperbolic sine function

thence $X_n = K_n \sinh\frac{n\pi x}{h}$ $n=1,2\ldots$

We replace $2c_n = K_n$.

Now if we combine the solutions we get

$$U_n(x,y) = X_n(x) \cdot Y_n(y)$$

$$= K_n \sinh\frac{n\pi x}{b} \cdot \sin\frac{n\pi y}{b} \qquad n=1,2\ldots$$

As the general solution is the linear combination of all the solutions $n=1\ldots$

$$u(x,y) = \sum_{k=1}^{\infty} K_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b}$$

we have used all boundary values
$u(x,0)=0$, $u(0,y)=0$, $u(x,b)=0$
except $u(a,y)=f(y)$.

Now $u(a,y) = f(y)$

$$u(a,y) = \sum_{n=1}^{\infty} K_n \sinh \frac{a n \pi}{b} \cdot \sin \frac{n \pi y}{b} = f(y)$$

Now notice that, above power series whose Fourier sine coefficients are

$$b_n = K_n \sinh\left(\frac{a n \pi}{b}\right).$$

Hence, above conditions tells us that

f(y) must be either an odd periodic function with period $= 2b$, or it needs to be expended into one.

Notice that $b_n = K_n \sinh \frac{a n \pi}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}$

If we solve for K_n;

Therefore, $K_n = \dfrac{2}{b \sinh \frac{a n \pi}{b}} \displaystyle\int_0^b f(y) \sin \frac{n \pi y}{b}\, dy.$

Example: Solve the following Laplace's equation

$$u_{xx}+u_{yy}=0 \qquad \begin{array}{l} 0<x<1 \\ 0<b<2\pi . \end{array}$$

Boundary Conditions

$$\begin{cases} u(x,0)=0 & u(0,y)=0 \\ u(x,\pi)=0 & u(\pi,y)= \end{cases}$$

Solution: Here $a=1 \quad b=2\pi$

The general solution is

$$u(x,y)=\sum_{k=1}^{\infty} K_n \sinh\frac{n\pi x}{b}\,\sin\frac{n\pi y}{b}$$

$$=\sum_{k=1}^{\infty} K_n \sinh\frac{n\pi x}{2\pi}\,\sin\frac{n\pi y}{2\pi}$$

$$=\sum_{k=1}^{\infty} K_n \sinh\frac{nx}{2}\,\sin\frac{ny}{2}$$

$$u(\pi,y)=\sum_{k=1}^{\infty} K_n \sinh\frac{n\pi}{2}\,\sin\frac{ny}{2}=$$

$L = 2\pi$ so $2L = 4\pi$

$$u(x,y) = \sum_{k=1}^{\infty} b_n \sin \frac{n\pi y}{2\pi} = f(x)$$

$$b_n = K_n \sinh \frac{\pi n.}{\pi}$$

Since $f(x) = 5 \sin \frac{3y}{2} + \sin 2y + 7 \sin 5y$
is already in its ~~fourier form~~
we get

$$f(x) = 5 \sin \frac{3\pi y}{2\pi} + \sin \frac{4\pi y}{2\pi} + 7 \sin \frac{10\pi y}{2\pi}$$

$b_3 = 5$, $b_4 = 1$, $b_{10} = 7$ all other $b_n = 0$ for all $n = 1..$

$$u(x,y) = K_3 \sinh \frac{3x}{2} \cdot \sin \frac{3y}{2} +$$

$$K_4 \sinh 2x \quad \sin 2y$$

$$K_{10} \sinh 5x \quad \sin 5y$$

$$K_3 = \frac{b_3}{\sinh \frac{1}{2}} \qquad K_4 = \frac{b_4}{\sinh 2} \qquad K_{10} = \frac{b_{10}}{\sinh 5}.$$

Hence the particular solution is

$$u(x,y) = \frac{5}{\sinh\frac{1}{2}} \sinh\frac{3x}{2} \sin\frac{3y}{2}$$

$$+ \frac{1}{\sinh 2} \sin 2x \sin 2y$$

$$+ \frac{7}{\sinh 5} \cdot \sin 5x \sin 5y.$$