lesson 40: Ordinary Points and ~~Singular~~ Singular Points

In this part we will consider second order linear differential equations of the form

$$A(x)\, y'' + B(x)\, y' + C(x)\, y = 0$$

or equivalently, $y'' + P(x)\, y' + Q(x)\, y = 0$

$$P(x) = \frac{B(x)}{A(x)} \quad \& \quad Q(x) = \frac{C(x)}{A(x)}$$

We know that when $P(x)$ and $Q(x)$ are analytic at a point $x=x_0$ then we have two linearly independents solutions which are also analytic at $x=x_0$.

Here the question is the following

__Question:__ What if either $P(x)$ or $Q(x)$ is not real analytic at $x=x_0$, does above differential equation have a solution at $x=x_0$?

This brings us to the singular points

Example: $x^2(x-2)^2 y'' + (x-2)y' + 3x^2 y = 0$

If we rewrite the DE as

$$y'' + \frac{(x-2)}{(x-2)^2}\frac{1}{x^2}y' + \frac{3x^2}{x^2(x-2)^2}y = 0$$

Don't do simplification!

Now $P(x) = \frac{x-2}{(x-2)^2}\cdot\frac{1}{x^2}$ & $Q(x) = \frac{3x^2}{(x-2)^2 x^2}$

then we know that at $x=2$, P is not analytic as P(2) is undefined. (and therefore P' or any derivative of P does not exist). ~~there~~ Similarly, at $x=0$, P is undefined and derivative of P does not exist at $x=0$. Q is not real analytic at $x=2$, but may be real analytic at $x=0$ (yes simplification is allowed but we need to be careful.)

Another question here is that once we understood $x_0=0$ & $x_0=2$ are irregular points, are they same in the sense that, is there any hope that there is still solution to above DE?

Regular Singular Points:

A point $x = x_0$ is called "regular singular point"
if

$$\lim_{x \to x_0} (x - x_0) P(x) \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 . Q(x)$$

both exist and are finite,

Example: $x^2 y'' + x(1+x) y' + (\pi + x^2) y = 0$

If we rewrite the DE. as

$$y'' + \underbrace{\frac{x(1+x)}{x^2}}_{P(x)} y' + \underbrace{\frac{(\pi + x^2)}{x^2}}_{Q(x)} y = 0$$

We see that P and Q are both have singularities at $x_0 = 0$. Therefore $x_0 = 0$ is a singular point.

We next check if it's regular singular point at $x_0 = 0$

$$\lim_{x \to 0} x P(x) = \lim_{x \to 0} \frac{x^2 (1+x)}{x^2} = 1$$

$$\lim_{x \to 0} x^2 Q(x) = \lim_{x \to 0} x^2 \frac{(\pi + x^2)}{x^2} = \pi .$$

As both limit exist then $x_0 = 0$ is a
a Regular singular point

Irregular Singular Points

If either of the limit;

$$\lim_{x \to x_0} (x-x_0) P(x) \quad \text{or} \quad \lim_{x \to x_0} (x-x_0)^2 Q(x)$$

is not finite or does not exist then $x = x_0$ is called irregular singular points.

Example: $x^2(x-2)^2 y'' + (x-2)y' + 3x^2 y = 0$

Find and classify all singular points of the DE.

Solution: Rewrite the DE as

$$y'' + \frac{(x-2)}{x^2(x-2)^2} y' + \frac{3x^2}{x^2(x-2)^2} y = 0$$

$$P(x) = \frac{x-2}{x^2(x-2)^2} \quad \text{and} \quad Q(x) = \frac{3x^2}{x^2(x-2)^2}$$

Now, P is not analytic at $x=0$ and $x=2$.

Similarly Q is not analytic at $x=0$ and $x=2$.

0, 2 are singular points.

To classify them we need to check the limits.

$x_0 = 0$;

$$\lim_{x \to 0} x \cdot P(x) = \lim_{x \to 0} \frac{x \cdot (x-2)}{x^2(x-2)^2} = \text{ is not finite}$$

Therefore $x_0 = 0$ is irregular singular point.

$x_0 = 2$.

$$\lim_{x \to 2} (x-2) P(x) = \lim_{x \to 2} \frac{(x-2)(x-2)}{x^2 (x-2)} = \frac{1}{4}$$

$$\lim_{x \to 2} (x-2)^2 Q(x) = \lim_{x \to 2} \frac{(x-2)^2 \cdot 3x^2}{x^2 (x-2)^2} = 3$$

Both limit exists and finite, therefore by definitions $x_0 = 2$ is a regular singular points.

Lesson 40B: Solution about a Regular Singular Points

— Method of Frobenius —

let x_0 be a regular singular point for the differential equation

$$A(x) y'' + B(x) y' + C(x) y = 0.$$

In this case, we can rewrite the DE as

$$(x-x_0)^2 y'' + (x-x_0) f_1(x) y' + (x-x_0)^2 f_2(x) y = 0$$

Where both $f_1(x)$ and $f_2(x)$ are analytic at $x=x_0$.

Theorem (Method of Frobenius):

If x_0 is a regular singular point to above differential equation then there exists at least one solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

A solution of this form is called a Frobenius-type solution.

Example: Consider the following differential eq.

$$4x^2 y'' - 4x^2 y' + (1-2x)y = 0.$$

Find and classify all singular points. If the singular point is regular singular point then try to find a Frobenius-type solution.

If we rewrite the DE as

$$y'' - \frac{4x^2}{4x^2} y' + \frac{(1-2x)}{4x^2} y = 0$$

then $x=0$ is a singular point as Q has no Taylor series expansion around $x_0=0$.

Now since,

$$\lim_{x\to 0} x\,P(x) = \lim_{x\to 0} x \cdot \frac{4x^2}{\cancel{4x^2}} = 0$$

and

$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} x^2 \frac{(1-2x)}{4x^2} = 1$$

both limit exists and are finite then $x_0=0$ is a regular singular point. (and it's the only singular point.)

By Frobenius-type solution we mean $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

Let's find y' & y'' and then substitute this into the DE to find a_n.

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2}$$

Now we write the DE

$$4x^2y'' - 4x^2y' + (1-2x)y = 0$$

in terms of the power series representation

$$4x^2 \sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n-2} - 4x^2 \sum_{n=0}^{\infty} (k+n) a_n x^{k+n-1}$$

$$(1-2x) \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

That becomes

$$\sum_{n=0}^{\infty} 4(k+n)(k+n-1) a_n x^{k+n} - 4 \sum_{n=0}^{\infty} (k+n) x^{k+n+1} + \sum_{n=0}^{\infty} a_n x^{k+n} - 2 \sum_{n=0}^{\infty} a_n x^{k+n}$$

OR

$$0 = \sum_{n=0}^{\infty} 4(k+n)(k+n-1) a_n x^{k+n} - 4 \sum_{n=1}^{\infty} a_{n-1}(k+n-1) x^{k+n} + \sum_{n=0} a_n x^{k+n} - 2 \sum_{n=1}^{\infty} a_{n-1} x^{k+n}$$

For $n=0$ (as n starts from 1 in the second and last summation)

we have

$$4k.(k-1)x^k a_0 + a_0 x^k = 0 \quad \text{OR } a_0 x^k (4k(k-1)+1) = 0$$

Assume that $a_0 \neq 0$, then we get $k(k-1)+1=0$

This equation is called "indicial equation".

$4k(k-1)+1=0$ or $4k^2-4k+1=0 \rightsquigarrow (2k-1)^2=0$

therefore indicial equation has a double root at $k=\frac{1}{2}$.

we have k and now we need to find a_n.
The remaining part in the ~~ODE~~ is

$$\sum_{n=1}^{\infty} 4(k+n)(k+n-1)a_n x^{k+n} - 4\sum_{n=1}^{\infty} a_{n-1}(k+n-1)x^{k+n} + \sum_{n=1}^{\infty} a_n x^{k+n} - 2\sum_{n=1}^{\infty} a_{n-1}x^{n+k}$$

$$\sum_{n=1}^{\infty}\left[4(k+n)(k+n-1)a_n - 4a_{n-1}(k+n-1) + a_n - 2a_{n-1}\right]x^{k+n} = 0$$

If this is true for any n we get

$$4(k+n)(k+n-1)a_n - 4a_{n-1}(k+n-1) + a_n - 2a_{n-1} = 0$$

Combining a_n & a_{n-1} we get

$$\left[4(k+n)(k+n-1)+1\right]a_n - \left(4(k+n-1)+2\right)a_{n-1} = 0$$

we know $k = \frac{1}{2}$

$$a_n = \frac{\left[4(k+n-1)+2\right]a_{n-1}}{\left[4(k+n)(k+n-1)\right]+1} \rightarrow \frac{1}{n}a_{n-1}$$

Suppose $y(0) = 1$ (i.e. $a_0 = 1$)

then $a_1 = \dfrac{\left(4\left(\frac{1}{2}+1-1\right)+2\right)a_0}{\left[4\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+1-1\right)\right]+1} = \dfrac{4}{4} = 1$

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$a_2 = \frac{1}{2}a_1 = \frac{1}{2}$, $a_3 = \frac{1}{3}a_2 = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{3!}$

$a_4 = \frac{1}{4}a_3 = \frac{1}{4}\cdot\frac{1}{3!} = \frac{1}{4!}$ then $a_n = \frac{1}{n}\cdot a_{n-1} = \frac{1}{n}\frac{1}{(n-1)!} = \frac{1}{n!}$

. We get $y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$

$$= x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} x^n = x^{\frac{1}{2}} . e^x .$$

which is ~~the~~ solution to the given initial value problem.

Notice that our differential equation is a second order and therefore we expect that it has ~~two~~ solutions. We only found one, how about the second solution?.

— The Frobenius Method —

Consider the DE $A(x) y'' + B(x) y' + C(x) y = 0$.

Assume that $x = x_0$ is a regular singular point.

For simplicity assume $x_0 = 0$

Step 1. We seek a solution "Frobenius-type" solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k} \quad \text{for some } k.$$

Aim is to find the "indicial equation" using the DE.

Use the DE (find y & y', y'' in terms of power series) to find the indicial equation. which will be a quadratic equation in "k".

Case 1: If the indicial equation has two roots say k_1 and k_2 ~~that~~ with the property that $k_1 - k_2$ is not an integer.

then we have two linearly independent solution

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

$$y_2(x) = x^{r_2} \sum_{k=0}^{\infty} b_k x^k$$

there a_k & b_k are unknowns and need to be found.

Case 2: If the indicial equation has two roots say k_1 and k_2 but $k_1 - k_2$ is an integer then two linearly independent solns are

$$y_1(x) = x^{k_1} \sum_{n=0}^{\infty} a_n x^{n} \quad \text{and}$$

$$y_2(x) = x^{k_2} \sum_{n=0}^{\infty} b_n x^n + c(\ln x) y_1$$

* here a_n, b_n & c are unknowns and needed to be found.

Case 3: Finaly if indicial equation have double root k then two independent solutions are;

$$y_1(x) = x^k \sum_{n=0}^{\infty} a_n x^n$$

and

$$y_2(x) = x^k \sum_{n=0}^{\infty} b_n x^n + (\ln x)\, y_1(x).$$

~~where~~ a_n and b_n are unknow and using the DE we needs to find them

Case 4: If indicial equations have complex roots then

$$y(x) = x^k \sum_{n=0}^{\infty} a_k x^k$$

will be the two linearly independent solutions.

Note here that a_k will be complex valued.

Example: (Page 576) Find a Frobenius-type solution of the DE; (at the singular point).

$$x^2 y'' + x(x+\tfrac{1}{2})y' - (x^2+\tfrac{1}{2})y = 0$$

Solution: We first rewrite the DE as

$$y'' + \frac{x}{x^2}(x+\tfrac{1}{2})y' - \frac{1}{x^2}(x^2+\tfrac{1}{2})y = 0$$

$P(x) = \frac{1}{x}(x+\frac{1}{2})$ and $Q(x) = -\frac{1}{x^2}(x^2+\frac{1}{2})$

P and Q are not analytic. Where they are singular at $x=0$

As a next ~~step~~ we need to classify this singular point.

Since $\lim_{x\to 0} x P(x) = \lim_{x\to 0} x \cdot \frac{1}{x}(x+\frac{1}{2}) = \frac{1}{2}$ and

$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} x^2 \cdot \frac{1}{x^2}(x^2+\tfrac{1}{2}) = \tfrac{1}{2}$$

Both limits exist and finite. Therefore $x=0$ is a regular singular point. In this case we need to find the indicial equation; to find it consider a solution of the form;

$$y(x) = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+k) x^{n+k-1}, \quad y''(x) = \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-2}$$

then the DE is

$$x^2 \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-2} + \left(x^2+\frac{x}{2}\right)\sum_{n=0}^{\infty} (n+k) a_n x^{n+k-1} - \left(x^2+\frac{1}{2}\right)\sum_{n=0}^{\infty} a_n x^{k+n} = 0$$

Some ~~little~~ algebra gives

$$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{k+n} + \sum_{n=0}^{\infty} (n+k) a_n x^{n+k+1} + \frac{1}{2}\sum (n+k) a_n x^{n+k} - \sum_{n=0}^{\infty} a_n x^{k+n+2} - \frac{1}{2}\sum_{n=0}^{\infty} a_n x^{k+n}$$

the second and fourt sumation were needed to be rewritten by changing the indices

$$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{k+n} + \sum_{n=1}^{\infty} (\cancel{n+1}k) a_{n-1} x^{n+k} + \frac{1}{2}\sum_{n=0}^{\infty} a_n (n+k) x^{n+k} - \sum_{n=2}^{\infty} a_{n-2} x^{k+n} - \frac{1}{2}\sum_{n=0}^{\infty} a_n x^{k+n} = 0$$

For $n=0$ we get terms from first third and last sumation

$$a_0 k(k-1) x^k + \frac{1}{2} a_0 k x^k - \frac{1}{2} a_0 x^k = 0$$

That is $a_0 x^k \left(k(k-1) + \frac{k}{2} - \frac{1}{2} \right) = 0$

if $a_0 \neq 0$ then $k(k-1) + \frac{k}{2} - \frac{1}{2} = 0$

equivalently $k^2 - k + \frac{k}{2} - \frac{1}{2} = 0 \sim k^2 - \frac{k}{2} - \frac{1}{2} = 0$

$$k_1 = 1 \quad \& \quad k_2 = -\frac{1}{2}$$

since $k_1 - k_2 = 1 + \frac{1}{2} = \frac{3}{2}$ is not an integer by the Thm we know that

$$y_1(x) = x^{k_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and}$$

$$= x \sum_{n=0}^{\infty} a_n x^n \quad \text{and}$$

$$y_2(x) = x^{k_2} \sum_{n=0}^{\infty} b_n x^n$$

$$= x^{-\frac{1}{2}} \sum_{n=0}^{\infty} b_n x^n$$

are two linearly independent solutions.

Now we return to our DE.

$$\sum_{n=2}^{\infty} \left[a_n (n+k)(n+k-1) + (n-1+k)a_{n-1} + \frac{1}{2} a_n (n+k) - \frac{1}{2} a_{n-2} - \frac{1}{2} a_n \right] x^{k+n}$$

Thus one must have

$$a_n (n+k)(n+k-1) + (n-1+k)a_{n-1} + \frac{1}{2} a_n (n+k) - \frac{1}{2} a_{n-2} - \frac{1}{2} a_n = 0$$

for all $n \geq 2$.

~~First~~ solve for a_n;

we will get two for each $k=1$ and $k=-\frac{1}{2}$

for $k=1$

$$a_n(n+1)n + n a_{n-1} + \frac{1}{2} a_n(n+1) - \frac{1}{2} a_{n-2} - \frac{1}{2} a_n = 0$$

Find a_n ✓.

For $k=-\frac{1}{2}$ we have another recursive formula to find b_n.

(I denote that one here to a_n as well)

$$a_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + \left(n-\frac{3}{2}\right)a_{n-1} + \frac{1}{2} a_n\left(n-\frac{1}{2}\right) - \frac{1}{2} a_{n-2} - \frac{1}{2} a_n = 0$$

This gives you the second series b_n..