~~[illegible]~~ Seperable Differentiable Equations

Suppose we have a differential equation

$$\frac{dy}{dx} = y' = f(x,y)$$

and this can be written in the form

$$p(y)\frac{dy}{dx} = q(x)$$

where $p(y)$ and $q(x)$ are some function of y and x respectively. Suppose that $y = g(x)$ is a solution to the differential equation

then $$p(g(x)) \cdot g'(x) = q(x)$$

Integrate both sides w.r.t x we get

$$\int \underbrace{p(g(x))\, g'(x)}\,dx = \int q(x)$$

$$\int \frac{d}{dx} p(g(x))\, dx = \int q(x)\, dx + c.$$

$$\cancel{p(g(x)) = \int q(x)\,dx + c}$$

Example: Solve the differential equations

$$y' = -2xy^3$$

Then rewrite the DE

$$\frac{y'}{y^3} = -2x. \quad * \text{ assume } y \neq 0$$

Here $p(y) = \frac{1}{y^3}$ & $g(x) = -2x$

Integrate the DE ~~exact~~ ~~that~~

$$\int \frac{1}{y^3}\, dy = -\int 2x\, dx$$

$$-\frac{1}{2}\frac{1}{y^2} = -x^2 + c \quad \text{OR} \quad y^2 = \frac{1}{2(x^2 - c)}$$

is a general solution to the DE.

Note we divide y ~~to~~ therefore we assume $y \neq 0$

we have to check if $y=0$ is a solution

$$y' = -2xy^3$$

if $y=0$ then the lefthand side is zero so is the RHS. Therefore $y=0$ is a particular solution.

Therefore $y^2 = \frac{1}{2}\left(\frac{1}{x^2 - c}\right)$ is a general solution

$y = 0$ is a particular solution

Exercise: Solve the initial value problem

$$y' = \frac{\sec x \tan x}{5y^4 + 2e^{2y}} \quad \text{with } y(0) = 3$$

lesson 9: Exact Differential Equations.

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (*)$$

checking a differential equation is exact is the exact same process as checking whether a 2D-vector field is conservative, that is, a vector field V is conservative if $\exists f = f(x,y)$ s.t

$$V = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{bmatrix}$$

think that $V = \begin{bmatrix} M(x,y) \\ N(x,y) \end{bmatrix}$ then find f

(*) is due to clairaut's theorem which states for C^2 function f

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

A function $f(x,y)$ is a solution to the DE

$$M(x,y)dx + N(x,y)dy = 0$$

if the differential of f, $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$

<u>Example:</u> Show that

$$(3x^2 - 2y^2)\, dx + (1 - 4xy)\, dy = 0$$

is exact.

<u>Solution:</u> Here $M(x,y) = 3x^2 - 2y^2$

$N(x,y) = 1 - 4xy$

Then $\frac{\partial M}{\partial y} = -4y$ & $\frac{\partial N}{\partial x} = -4y$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence the DE is exact.

How to solve exact DEs?

One needs (and should be able to) find f

for which $\frac{\partial f}{\partial x} = M(x,y)$ and

$\frac{\partial f}{\partial y} = N(x,y)$

Then $f(x,y)=0$ is an implicit solution to the exact DE.

Now since $\frac{\partial f}{\partial x}(x,y) = M(x,y)$

Then integrate wrt x you get

$$\int \frac{\partial f}{\partial x}(x,y)\,dx = \int M(x,y) + g(y)$$

Then $f(x,y) = \int M(x,y)\,dx + g(y)$

Then using $\frac{\partial f}{\partial y}(x,y) = N(x,y)$

find $g(y)$.

Example: Remember the previous example, we already knew it's an exact DE. Find the solution.

$$M(x,y) = 3x^2 - 2y^2 = \frac{\partial f}{\partial x}$$

To find f integrate wrt x;

$$f(x,y) = \int (3x^2 - 2y^2)\,dx + g(y) = 0$$

$$= x^3 - 2xy^2 + g(y) = 0$$

Now we need to find $g(y)$ by using

$$\frac{\partial f}{\partial y} = N(x,y) = 1-4xy$$

Since $f(x,y) = x^3 - 2xy^2 + g(y)$

$$\frac{\partial f}{\partial y}(x,y) = -4xy + g'(y) = 1-4xy$$

Therefore $g'(y) = 1$, Hence $g(y) = y + c$.

From this we get $f(x,y) = x^3 - 2xy^2 + y + c = 0$

<u>Exercises:</u> Determine whether

$$(xy^2 + 4x^2y)dx + (3x^2y + 4x^3)dy = 0$$

is exact, if it's, find the solution.

<u>Lesson 10B:</u> Integrating Factor

The previous example is not exact.

But if we multiply M and N by

$$\mu(x,y) = x^{-1}y$$

Then $M(x,y) \cdot x^{-1}y = y^3 + 4xy^2 = \tilde{M}(x,y)$

$N(x,y) \cdot x^{-1}y = 3xy^2 + 4x^2y = \tilde{N}(x,y)$

and $\frac{\partial \tilde{M}}{\partial y} = 3y^2 + 8xy$

$\frac{\partial \tilde{N}}{\partial x} = 3y^2 + 8xy$

$\Rightarrow \frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x}$

therefore $(y^3 + 4xy^2)dx + (3xy^2 + 4x^2y)dy = 0$

is an exact ~~DE~~ DE.

<u>Defn:</u> A function $\mu(x,y)$ is called an integrating factor for the DE

$$M(x,y)dx + N(x,y)dy = 0$$

If the DE

$$\mu(x,y)\, M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact.

* Finding the integrating factor is not EASY.

Consider the new DE which is exact.

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

Since this DE is exact then

$$\frac{\partial}{\partial y}\left(\mu(x,y)\,M(x,y)\right) = \frac{\partial}{\partial x}\left(\mu(x,y)\,N(x,y)\right)$$

Equivalently;

$$\mu_y M + \mu \cdot M_y = \mu_x N + \mu \cdot N_x$$

Capture μ and derivatives of μ

$$\circledast \quad \mu(M_y - N_x) = \mu_x N - \mu_y M$$

Further assume that the integrating factor

$$\mu(x,y) = P(x)\,Q(y).$$

Then $\mu_x = P'(x)\,Q(y)$ and $\mu_y = P(x) \cdot Q'(y)$.

Plugging these into $\circledast$ we get

$$P(x)\,Q(y) \cdot (M_y - N_x) = P'(x)\,Q(y)\,N - P(x)\,Q'(y)\,M$$

Divide everything by $P(x)$ and $Q(y)$

To have

$$M_y - N_x = \frac{P'(x)}{P(x)} N - \frac{Q'(y)}{Q(y)} \cdot M \quad (*)$$

let $p(x) = \frac{P'(x)}{P(x)}$ and $q(y) = \frac{Q'(y)}{Q(y)}$

$(*)$ becomes

$$M_y - N_x = p(x) N - q(y) N$$

Focus on $p(x)$ and $q(y)$

$$\int p(x)dx = \ln P(x) \quad \text{or} \quad P(x) = e^{\int p(x)dx}$$

$$\int q(x)dx = \ln Q(x) \quad \text{or} \quad Q(x) = e^{\int q(x)dx}$$

$(*)$ No guarantee that we may find $p(x)$ and $q(y)$. there are some conditions with which we can find $p(x)$ and $q(y)$!

Thm: Consider the DE $M\,dx + N\,dy = 0$

a) if $\frac{M_y - N_x}{N}$ is independent of y

define $p(x) = \dfrac{M_y - N_x}{N}$

Then $\mu(x) = e^{\int p(x)dx}$ is an integrating factor for $M\,dx + N\,dy = 0$

b) If $\dfrac{N_x - M_y}{M}$ is independent of x

then $q(y) = \dfrac{N_x - M_y}{M}$ ~~is an integrating factor~~

$\mu(y) = e^{\int q(y)\,dy}$ is an integrating factor for the DE $M\,dx + N\,dy = 0$.

<u>Example!</u> Find an integrating factor for DE $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0$ and find the solution.

<u>Solution!</u>