# Autonomous Equations and Stability of Equilibrium Points

**Autonomous Equations:** A differential equation where the independent variable does not explicitly appear in its expression. It has the general form

$$y' = f(y)$$

**Examples:**

$$y' = e^{2y} - y^3$$

$$y' = y^3 - 4y$$

$$y' = y^4 - 81 + \sin y$$

Note that any autonomous DE is seperable. Becase

$$\frac{dy}{dx} = f(y) \quad \text{equivalently} \quad \frac{dy}{f(y)} = dx$$

Therefore

$$\int \frac{dy}{f(y)} = \int dt.$$

As long as we can integrate the LHS. there we know how to solve these Autonomous DE. What we are interested here is to predict the behaviour of autonomous equation's solution without solving the DE. By using it's Direction Fields.

In order the DE to make sense we assume that $f(y) \neq 0$. Now we want to understand what happend when $f(y) = 0$?

## Equilibrium Solution

Equilibrium solutions (or Critical points) occur where $y' = f(y) = 0$. That is they are the roots of $f(y) = 0$

**Example!** $y' = y^3 - 2y^2$

What are the equilibrium solutions? ~~Are they stable?~~

**Solution!**

Since $f(y) = y^3 - 2y^2 = 0$

Therefore $y^2(y-2) = 0$ therefore $y = 0$ and $y = 2$

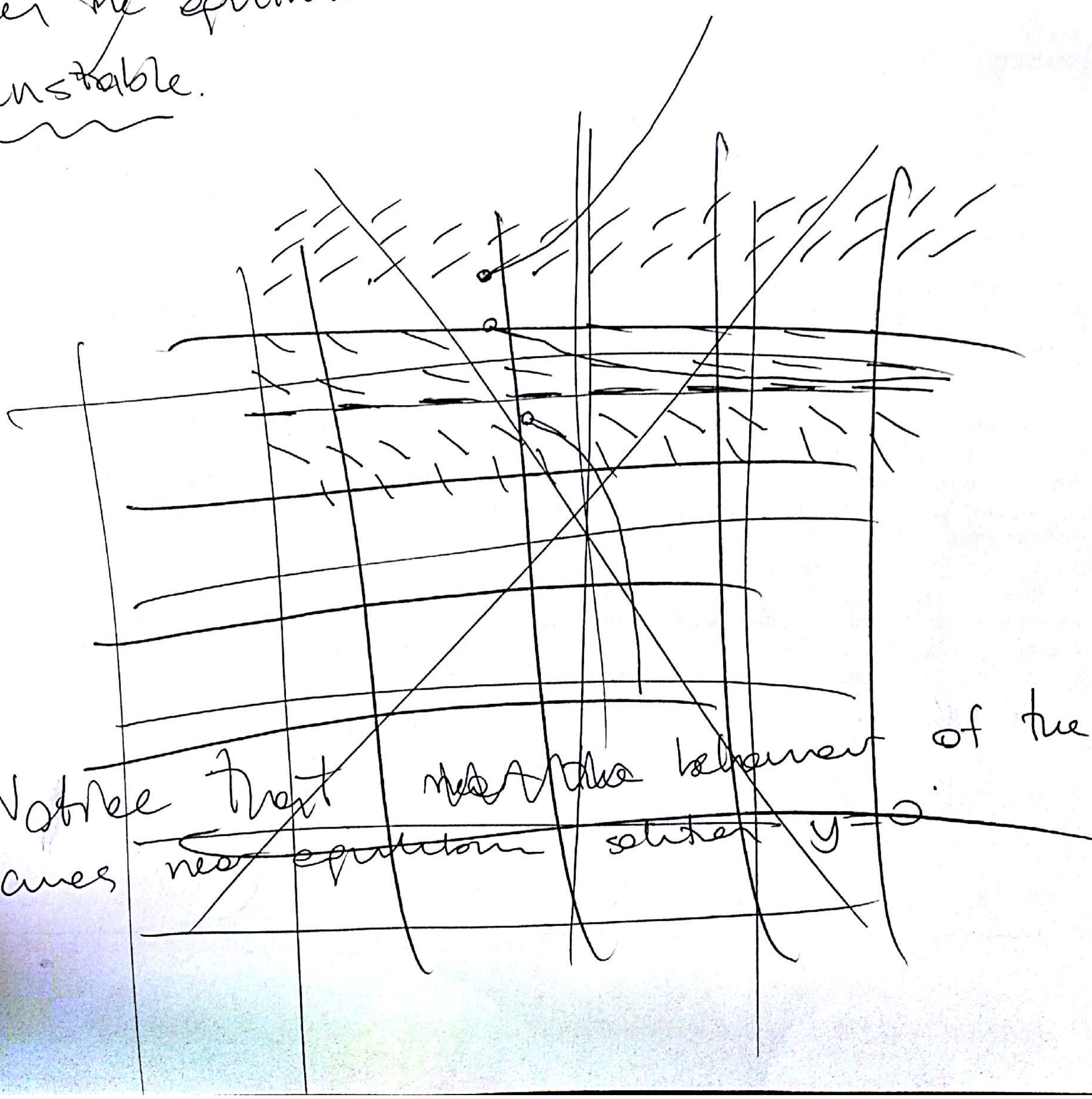
Hence $y = 0$ & $y = 2$ are equilibrium solutions.

## Stability of Equilibrium Solution

Stability of a equilibrium solution is classified according to the behavior of the integral curves near it.

# If the nearby integral curves all converge towards an equilibrium solution as t increases, then the equilibrium solution is said to be <u>stable</u>, or <u>asymptotically stable</u>

# If the nearby integral curves all diverge away from an equilibrium solution as $t$ increases then the equilibrium solution is said to be <u>unstable</u>.

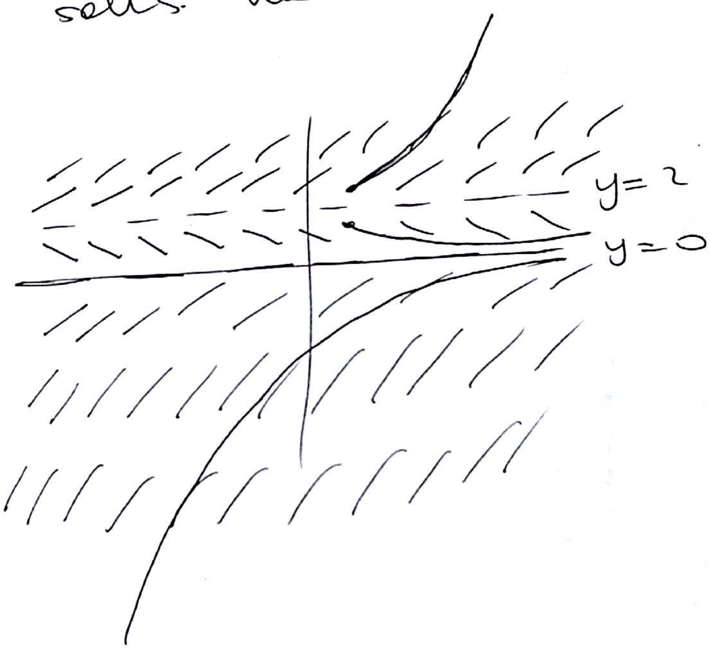


Notice that the behaviour of the integral curves near equilibrium solution $y=0$.

Example: $y' = y^2 - 2y$. Find the equilibrium solutions
$(y-2)y$. and classify them

$y' = f(y) = y^2 - 2y = 0$

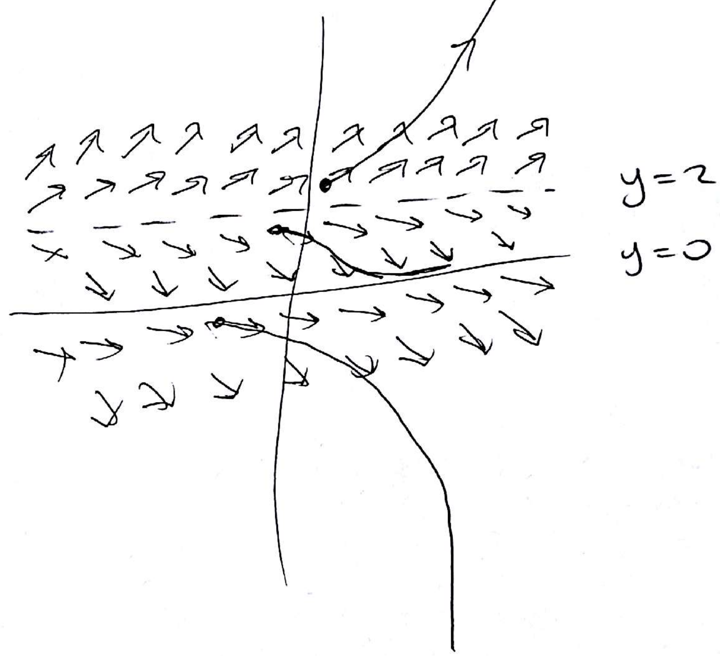
then $y = 2$ and $y = 0$ are
equilibrium sols. We draw the direction field




Any solution starting near $y = 0$
then as $t \to \infty$ solutions get closer and closer
to $y = 0$. therefore $y = 0$ is a stable
or asymptotically stable equilibrium solution.
For solutions starting near $y = 2$ runaway from
from $y = 2$ lines and therefor it's unstable.

* There is also semistable equilibrium solution.
What that means is that solutions starting ~~above~~ from one side approaches to the equilibrium solution and the other side goes away from the equilibrium solution.

Example! $y' = y^3 - 2y^2$ Then $y=0$ and $y=2$ are equilibrium solutions. The direction field shows that




So solutions starting above $y=2$ goes to $\infty$ as $x \to \infty$
Solutions starting between $0<y<2$ goes to the equilibrium solution $y=0$
Solutions starting below $y=0$ goes to $-\infty$ as $x \to \infty$.

The Existence and Uniqueness Theorem for the linear DE

Does an initial value problem have always a solution? How many solutions are there? The following theorem states precise conditions under which exactly one solution would always exist for a given initial value problem.

<u>Thm!</u> Let $y' + P(x)y = q(x)$.

Let $x_0 \in (a,b) = I$. Let $P(x)$ and $q(x)$ be continuous on $I$. Then the DE has a unique solution for each $x_0 \in I$. with the initial value

$$y(x_0) = y_0$$ for arbitrary prescribed $y_0$.

Example! Consider the DE with initial value

$$\cos x\, y' - \sin x\, y = 3x \cos x \qquad y(2\pi) = 0$$

Find the largest interval for which unique solution exists.

<u>Solution!</u> Rewrite the DE as

$$y' - \frac{\sin x}{\cos x} = 3x$$

Now $P(x) = -\dfrac{\sin x}{\cos x}$ and $Q(x) = 3x$

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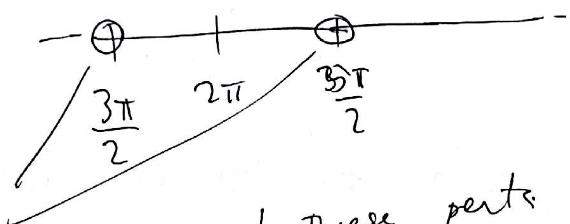
Here $3x$ is a nice and continuous function for all $x$

But $P(x)$ is discont at points

$$x = \mp\frac{\pi}{2},\ \mp\frac{3\pi}{2},\ \mp\frac{5\pi}{2},\ \mp\cdots$$

According to the existence and uniqueness thm

~~At least~~ A unique solution exists to the initial value problem




discontin of these points

Therefore the largest interval containing $2\pi$ is

$$\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right).$$

Existence and Uniqueness Thm for Nonlinear Equations.

A theorem analogous to the prev one is

Thm! ~~Let the~~ Consider the DE

$$y' = f(x,y).$$

Let the function $f$ and $\frac{\partial f}{\partial y}$ ~~be~~ continuous in some rectangle

$a<x<b$ and $s<y<t$, containing point $(x_0, y_0)$. Then in some interval $x_0-h<x<x_0+h$ contained in $a<x<b$

there is a unique solution $y(x)$ of the above initial value problem.

Example: Consider the nonlinear DE with initial value

$$y' = x^2 \cdot y^{\frac{1}{2}} \qquad y(0)=0$$

Since $f(x,y) = x^2 y^{\frac{1}{2}}$ and

$$\frac{\partial f}{\partial y} = \frac{1}{2} x^2 y^{-\frac{1}{2}}.$$ But this function is not continuous when $y=0$

there unique solution is not guaranteed

In fact $y(t) = \frac{t^6}{36}$ and $y(0)=0$

are both solutions.