Second Order Linear PDEs

Examples:

$\alpha^2 u_{xx} = u_t$ one dimensional Heat eqn.

$a^2 u_{xx} = u_{tt}$ wave equation

$u_{xx} + u_{yy} = 0$ Laplace equation.

Classification of Second Order Linear PDEs.

The generic form of linear PDEs with constant coefficients has the following form

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g(x,y).$$

For the equation to be second order, a, b, c can not be zero at at the same time.

Define its discriminant to be $b^2 - 4ac$.

- If $b^2 - 4ac > 0$ then equation is called Hyperbolic.

  Example: $\alpha^2 u_{xx} = u_{tt}$.
  $a = \alpha^2$, $b = 0$, $c = -1$

- If $b^2 - 4ac = 0$ then equation is called Parabolic

  Example: $u_{xx} = u_t$
  $a = 1$, $b = 0$, $c = 0$

• If $b^2-4ac<0$ then the equation is called **elliptic**

Example: $u_{xx}+u_{yy}=0$

$a=1, \; b=0, \; c=1$

The One Dimensional Heat Conduction Equation

Model: Consider a thin bar of length L of uniform cross-section and constructed of homogeneous material. Suppose the side of the bar is perfectly insulated, so no heat transfer could occur through two ends of the bar.

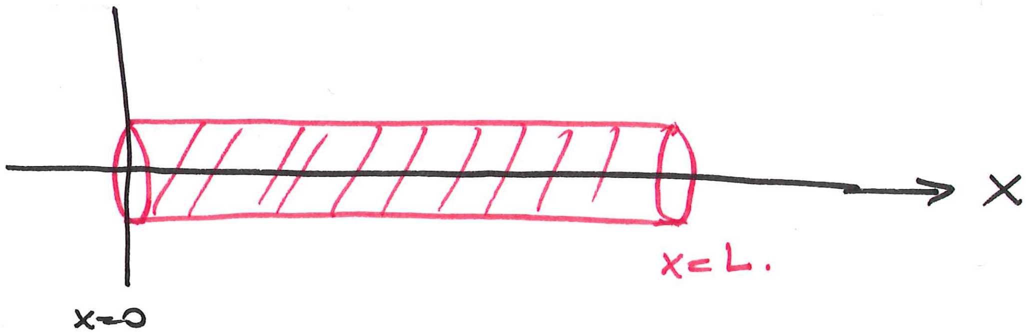
Thus the movement of the heat inside the bar could occur only in the x-direction.

Then the amount of the heat content at any place inside the bar, $0<x<L$, at any time t is given by the temperature distribution $u(x,t)$.

By experiments, it's shown that $u(x,t)$ satisfies the homogeneous heat equation

$$\alpha^2 u_{xx} = u_t .$$

α: constant, called thermo diffusivity of the bar



Further, assume that the both ends of the bar are kept constantly at 0 degree temperature.

That is, $u(0,t)=0$ and $u(L,t)=0$ $t>0.$

These two conditions are called boundary conditions

In addition, the initial temperature distribution within the bar, $u(x,0)$, (at time $t=0$). It will be given by a function $f(x)$.

That is $u(x,0)=f(x)$.

Review

Heat equation: $\alpha^2 u_{xx} = u_t$, $0<x<L$, $t>0$.

Boundary Conditions $u(0,t)=0$ and $u(L,t)=0$

Initial Conditions $u(x,0)=f(x)$

This is what is called an initial value problem.

If the boundary conditions are given by u;

$$u(0,t) = f(t) \quad \& \quad u(L,t) = g(t)$$

then these conditions is called Dirichlet conditions

If the boundary conditions are given by x derivative of u;

$$u_x(0,t) = f(t) \quad \& \quad u_x(L,t) = g(t)$$

then these conditions is called Neumann conditions

Lastly, if the boundary conditions are linear combinations of u and u_x;

$$\alpha u(0,t) + \beta u_x(0,t) = f(t)$$

$$a u(L,t) + b u_x(L,t) = g(t)$$

then these conditions are called Robin conditions.

We are going to solve the heat equation with the given initial and boundary conditions.

There are different ways; Laplace's method, seperation of variables. We will ~~see~~ the seperation of variables method to solve this initial value problem

Consider the Heat equation

$$\alpha^2 u_{xx} = u_t$$

We are looking for a solution $u(x,t)$ which has the following form;

$u(x,t) = X(x)\,T(t)$ where

X is a function of x alone.
T is a function of t alone.

Then

$$u = X.T \qquad u_x = X'T \qquad u_t = XT'$$

$$u_{xx} = X''T \qquad u_{tt} = XT''$$

$$u_{xt} = u_{tx} = X'T'.$$

Hence we can rewrite the Heat equation
$\alpha^2 u_{xx} = u_t$ as

$$\alpha^2 X'' T = X T'$$

Dividing both sides $\alpha^2 X T$ we get $\left(\begin{array}{c}\text{assume} \\ \alpha \neq 0 \\ X \neq 0 \\ T \neq 0\end{array}\right)$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\,\frac{T'}{T}$$

Note when $X=0$ or $T=0$ then

$u(x,t)=0$ is the trivial solution.

To find the non-trivial solutions assume $X \neq 0$, $T \neq 0$.

Hence we have

$$\otimes \qquad \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$$

Remember that X is a function of $\underline{\underline{x}}$ alone.
T is a function of $\underline{\underline{t}}$ alone.

Therefore X'' is a function of x alone.
T' is a function of t alone.

Now the left hand side of $\otimes$ is a function of x and the right hand side of $\otimes$ is function of t alone.

This is possible only if

$$\frac{X''}{X} = -\lambda = \frac{1}{\alpha^2}\frac{T'}{T}$$

where $-\lambda$ is a constant. λ can be positive, negative or zero.

Now we have two identities;

$$\frac{X''}{X} = -\lambda \longrightarrow X'' = -\lambda X \rightarrow X'' + \lambda X = 0$$

and

$$\frac{T'}{\alpha^2 T} = -\lambda \longrightarrow T' = -\alpha^2 \lambda T \rightarrow T' + \alpha^2 \lambda T = 0$$

These are ordinary differential equations
(second order and first order).

Next step, we are going to solve these differential equations.

Remember that we had boundary data;

$$u(0,t) = 0 \longrightarrow u(0,t) = X(0).T(t) = 0$$
$$X(0) = 0 \quad \text{or} \quad T(t) = 0$$

This gives us the trivial solution

$$u(L,t) = 0 \longrightarrow u(L,t) = X(L).T(t) = 0$$
$$X(L) = 0 \quad \text{or} \quad T(t) = 0$$

Not interesting.

Hence the boundary conditions are

$$X(0) = 0 \quad \& \quad X(L) = 0$$

What do we have now

$$\boxed{\begin{array}{ll} X'' + \lambda X = 0 \qquad X(0)=0 \ \ \& \ \ X(L)=0 \\ T' + \alpha^2 \lambda T = 0 \end{array}}$$

The general solution that satisfies the boundary conditions will be first solved from this system of differential equations.

Then the initial condition $u(x,0) = f(x)$ will be used to get the particular solution.

Now the first differential equation

$$X'' + \lambda X = 0 \qquad X(0)=0 \ \ \& \ \ X(L)=0$$

let $\lambda = k^2$, for some $k>0$.

Then $X'' + k^2 X = 0$ has solutions
$X(0) = 0 = X(L)$

$$\underset{\text{eigen function}}{X(x) = \sin\frac{n\pi x}{L}} \qquad \text{with} \qquad \underset{\text{eigenvalue.}}{\lambda = \frac{n^2\pi^2}{L^2}}$$

Hence for $\lambda_n = \frac{n^2\pi^2}{L^2}$ we have, for $n=1,2\ldots$

$$X_n(x) = \sin \frac{n\pi x}{L}$$ is a solution

to $X'' + \lambda X = 0 \quad X(0) = 0 = X(L)$

For this $\lambda_n = \frac{n^2\pi^2}{L^2}$, try to solve the second equation:

$$T' + \alpha^2 \lambda T = 0, \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

The general solution is

$$T(t) = c\, e^{-\lambda \alpha^2 t}$$

Now for each n;

$$T_n(t) = C_n e^{-\lambda_n \alpha^2 t} = C_n e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t}$$

solves $T' + \alpha^2 \lambda_n T = 0 \qquad \lambda_n = \frac{n^2\pi^2}{L^2}.$

$n=1,2\ldots$

For each $n=1,2\ldots$ we have

$$u_n(x,t) = X_n(x)\cdot T_n(t) = C_n e^{-\frac{n^2\pi^2}{L^2}\alpha^2 t} \sin \frac{n\pi x}{L}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\alpha^2 \frac{n^2\pi^2}{L^2} t} \sin \frac{n\pi x}{L}$$

solves the boundary value problem

$$\alpha^2 u_{xx} - u_t = 0$$
$$u(0,t) = 0 \,, \quad u(L,t) = 0 .$$

One last thing to check; the initial value.

$u(x,0) = f(x)$ for a given f.

plug in $t=0$ in the solution

$$u(x,0) = \sum_{n=1}^{\infty} C_n e^{0} \sin \frac{n\pi x}{L} = f(x)$$

To find C_n we will use the Fourier series method!

<u>Summary</u>

1. Separate the PDE into two ordinary differential equations; one with x variable, one with t variable. Then Rewrite the boundary conditions with respect to X & T.

2. Solve the first ode with the given boundary conditions;

$$X'' + \lambda X = 0$$

$\longrightarrow X(0) = X(L) = 0$

$\longrightarrow X'(0) = X(L) = 0$

$\longrightarrow X(0) = X'(L) = 0$

$\longrightarrow X'(0) = X'(L) = 0$

there could be four different boundary conditions

This will give you eigenvalues λ_n

and corresponding eigen functions X_n

3. For each values of λ_n, solve the second equation, equation with $T(t)$ and find corresponding solution $T_n(t)$ corresponding to λ_n.

4. Since $u_n = X_n T_n$ is solution, the general solt.

$$u(x,t) = \sum_{n=1}^{\infty} X_n T_n \,.$$

Now using the initial condition

$$u(x,0) = \underbrace{\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}}_{\text{Fourier series of } f(x)} = f(x)$$

Hence $f(x)$ has to periodic function with period $2L$. Since Fourier's series has only sine terms, it has to be an odd function

That is $$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

Now we know that the coefficient c_n can be found by

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x) \sin\frac{n\pi x}{L}\, dx.$$

Now we have the particular solution (c_n is as above.)

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \frac{n^2\pi^2}{L^2} t} \cdot \sin\frac{n\pi x}{L}$$

<u>Example:</u> Solve the heat equation

$$8\, u_{xx} = u_t \qquad 0<x<5 \ , t>0.$$

$$u(0,t)=0 \quad \text{and} \quad u(5,t)=0$$

$$u(x,0) = 2\sin(\pi x) - 4\sin(2\pi x) + \sin(5\pi x)$$

<u>Solution:</u> Our model equation was

$\alpha^2 u_{xx} = u_t$ and in this case the general solution is when the boundary condition is Dirichlet

$$u(x,t) = \sum_{n=1}^{\infty} c_n \, e^{-\alpha^2 \frac{n^2\pi^2}{L^2} t} \sin \frac{n\pi x}{L}$$

$\alpha^2=8$, $L=5$

$$= \sum_{n=1}^{\infty} c_n \, e^{-8 \frac{n^2\pi^2}{25} t} \sin \frac{n\pi x}{5}$$

Now use the initial conditions

$$u(x,0) = 2\sin(\pi x) - 4\sin(2\pi x) + \sin(5\pi x)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{5} = 2\sin \pi x - 4\sin(2\pi x) + \sin 5\pi x$$

$$\frac{n\pi x}{5} = \pi x \rightarrow n=5$$

Notice that all $c_n = 0$ except

$n=5$, $c_5 = 2$ $\qquad$ $n=25$ $\quad c_{25}=1$

$n=10$ $\quad c_{10} = -4$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-8\frac{n^2\pi^2}{25}t} \sin\frac{n\pi x}{5}$$

$C_n = 0$ except C_5, C_{10}, C_{25}

$$= C_5 e^{-8\frac{25\pi^2}{25}t} \sin\frac{5\pi x}{5}$$

$$+ C_{10} e^{-8\frac{100\pi^2}{25}t} \sin\frac{10\pi x}{5}$$

$$+ C_{25} e^{-8\frac{25^2\pi^2}{25}t} \cdot \sin\frac{25\pi x}{5}$$

Use $C_5 = 2$, $C_{10} = -4$, $C_{25} = 1$

$$= 2e^{-8\pi^2 t} \sin \pi x - 4e^{-32\pi^2 t} \sin 2\pi x$$

$$+ e^{-200\pi^2 t} \sin 5\pi x.$$

is the particular solution.

Example! ~~Co~~nsider the same problem

$$8\, u_{xx} = u_t \qquad 0 < x < 5\ , \quad t > 0$$

$$u(0,t) = 0 \quad \text{and} \quad u(5,t) = 0$$

change the initial conditions

$$u(x,0) = x$$

Since this is the Dirichlet problem we know the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-\frac{8 n^2 \pi^2}{L^2} t} \sin \frac{n\pi x}{5}$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n\, e^{0} \cdot \sin \frac{n\pi x}{5} = x$$

To find c_n (notice that $f(x) = x$ and we can find it's Fourier series as

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} = \frac{2}{5} \int_0^5 x \sin \frac{n\pi x}{5}\, dx$$

Find c_n. Then write the particular solution