A Minkowski problem and the Brunn-Minkowski inequality for capacity

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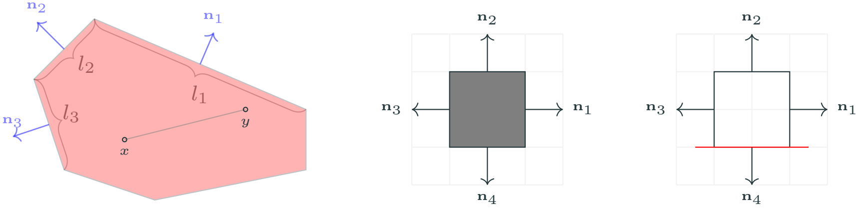
Outline

1. Minkowski Problem
   - Discrete Minkowski Problem
   - Classical Minkowski Problem
   - Minkowski Problem for p-capacity
2. Existence: Dark side of the p-capacity
3. Uniqueness: Detour; The Brunn-Minkowski inequality
4. Regularity: Detour; Monge-Ampère equation [if time permits]

Motivation

Question: Let m distinct unit vectors $\mathbf{n}_1, \ldots, \mathbf{n}_m$in $\mathbb{R}^2$ which spans $\mathbb{R}^2$ and m positive numbers $l_1, \ldots, l_m$ be given.

Can you find a convex polygon whose edges have the outer unit normals $\mathbf{n}_1, \ldots, \mathbf{n}_m$ with corresponding lengths $l_1, \ldots, l_m$?



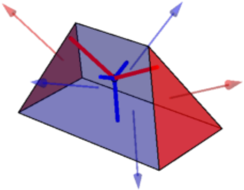
Example:

- For $\mathbf{n}_1 = (1, 0)$, $\mathbf{n}_2 = (0, 1)$, $\mathbf{n}_3 = (-1, 0)$, and $\mathbf{n}_4 = (0, -1)$, $l_1 = l_2 = l_3 = l_4 = 2$. $\rightarrow$ YES: square with sidelength 2.
- For $\mathbf{n}_1 = (1, 0)$, $\mathbf{n}_2 = (0, 1)$, $\mathbf{n}_3 = (-1, 0)$, $\mathbf{n}_4 = (0, -1)$, and $l_1 = l_2 = l_3 = 2$ and $l_4 = 3$. $\rightarrow$ NO??

Discrete Minkowski Problem

Let unit normal vectors $\mathbf{n}_1, \ldots, \mathbf{n}_m$ which span $\mathbb{R}^n$ and positive numbers $A_1, \ldots, A_m$ be given.

Question: Does there exist a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n$ whose faces have the given unit normals $\mathbf{n}_1, \ldots, \mathbf{n}_m$ and surface areas $A_1, \ldots, A_m$?



- Minkowski (1903):

Yes, iff $A_1 \mathbf{n}_1 + \ldots + A_m \mathbf{n}_m = \vec{0}$.

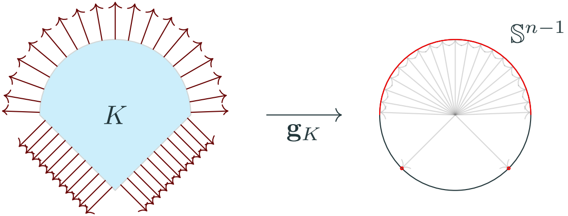
- Reformulation: given discrete measure μ on $\mathbb{S}^{n-1}$

$$\mu(\cdot) = \sum_{i=1}^{m} A_i \, \delta_{\mathbf{n}_i}(\cdot) \quad \text{where } \delta_{\mathbf{n}_i} \text{ is Dirac-delta point mass measure at } \mathbf{n}_i$$

is there a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n$ with desired properties?

Surface area measure on $\mathbb{S}^{n-1}$

Let K be a convex body $\equiv$ convex, compact, and non-empty interior.



$\mathbf{g}_K : \partial K \to \mathbb{S}^{n-1}$ is the Gauss map; $x \mapsto \mathbf{n}_x$.

- Define a measure μ_K on $\mathbb{S}^{n-1}$ associated to K by

$$\mu_K(E) := \int_{\mathbf{g}_K^{-1}(E)} d\sigma \quad \text{whenever} \quad E \subset \mathbb{S}^{n-1} \text{ is Borel set}$$

where $\sigma := \mathcal{H}^{n-1}|_{\partial K} = (n-1)$-Hausdorff measure on ∂K.

- If $\partial K \in C^2$ and has positive Gauss curvature everywhere then

$$d\mu_K(X) = \frac{1}{\kappa(X)} d\sigma(X) \quad \text{where } \kappa(X) \text{ is the Gauss curvature.}$$

Classical Minkowski Problem

Minkowski Problem
Let μ be a given positive finite Borel measure on $\mathbb{S}^{n-1}$.

Find necessary and sufficient conditions on μ such that there exists a convex body $K \subset \mathbb{R}^n$ satisfying

$$\mu(E) = \mu_K(E) = \int_{\mathbf{g}_K^{-1}(E)} d\sigma \qquad \text{whenever } E \subset \mathbb{S}^{n-1} \text{ is Borel set.}$$

Necessary conditions:

- **C1**: The centroid of the measure μ is at the origin;

$$\int_{\mathbb{S}^{n-1}} w \, d\mu(w) = \vec{0}. \qquad \left(\sum_{i=1}^{m} A_i \mathbf{n}_i = \vec{0} \text{ in the discrete setting.} \right)$$

- **C2**: The support of μ can not be contained in an equator of $\mathbb{S}^{n-1}$; (for every $\theta \in \mathbb{S}^{n-1}$)

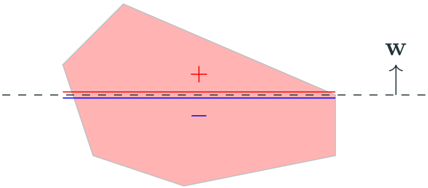
$$\int_{\mathbb{S}^{n-1}} |\theta \cdot w| d\mu(w) > 0. \qquad \left(\sum_{i=1}^{m} A_i |\theta \cdot \mathbf{n}_i| > 0 \text{ in the discrete setting.} \right)$$

Necessity of the condition C1

- The centroid of the measure μ is at the origin;

$$\int_{\mathbb{S}^{n-1}} w\, d\mu(w) = \vec{0}. \qquad \left(\sum_{i=1}^{m} A_i \mathbf{n}_i = \vec{0} \text{ in the discrete setting.}\right)$$

If $\mathbf{w} \in \mathbb{S}^{n-1}$ then the surface area of the projection of ith face to $\mathbf{w}^{\perp}$ is $A_i(\mathbf{n}_i \cdot \mathbf{w})$.



Therefore, for every $\mathbf{w} \in \mathbb{S}^{n-1}$,

$$\sum_{i=1}^{m} A_i(\mathbf{n}_i \cdot \mathbf{w}) = \sum_{\mathbf{n}_i \cdot \mathbf{w} > 0} A_i(\mathbf{n}_i \cdot \mathbf{w}) + \sum_{\mathbf{n}_i \cdot \mathbf{w} < 0} A_i(\mathbf{n}_i \cdot \mathbf{w}) = 0.$$

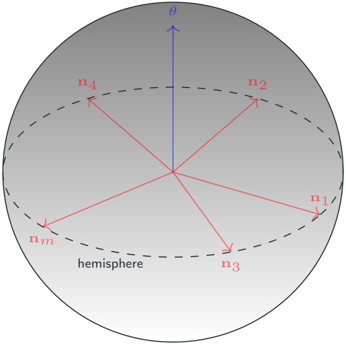
Necessity of the condition C2

- The support of μ can not be contained in an equator of $\mathbb{S}^{n-1}$;

(for every $\theta \in \mathbb{S}^{n-1}$)

$$\int_{\mathbb{S}^{n-1}} |\theta \cdot w| d\mu(w) > 0. \qquad \left(\sum_{i=1}^{m} A_i |\theta \cdot \mathbf{n}_i| > 0 \text{ in the discrete setting.} \right)$$

Otherwise,



there exists $\theta \in \mathbb{S}^{n-1}$ such that $\theta \cdot \mathbf{n}_i = 0$ for every $i = 1, \ldots, m$.

Then polyhedron will not be closed in the $-\theta$ direction.

Solution to Classical Minkowski Problem

Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ satisfying **C1-C2**.

- **Existence**: There is a convex body $K \subset \mathbb{R}^n$ such that

$$\mu(E) = \mu_K(E) = \int_{\mathbf{g}_K^{-1}(E)} d\sigma \quad \text{whenever} \quad E \subset \mathbb{S}^{n-1} \text{ is Borel set.}$$

- **Uniqueness**: K is unique up to translation.
- **Regularity**: If $d\mu = (1/\kappa)d\sigma$ for some strictly positive $\kappa \in C^{m,\alpha}(\mathbb{S}^{n-1})$, $m \in \mathbb{Z}^{\geq 0}$, and $\alpha \in (0,1)$, then $\partial K \in C^{m+2,\alpha}$.

▶ Existence and Uniqueness: Minkowski 1903 for the case of polyhedron, in general Alexandrov '37, Fenchel-Jessen '38, ...

▶ C^∞ regularity: Lewy '38, Pogorelov '53, Nirenberg '53, Cheng-Yau '76, ...

▶ $C^{2,\alpha}$ regularity: The precise gain of two derivatives and the treatment of small values of m is due to Caffarelli in '90.

▶ Sobolev regularity: Philippis-Figalli in '13.

Minkowski-type problems for other measures

- L_p **Minkowski problem**; (L_0 is the classical Minkowski problem) due to Andrews in '99, Chou-Wang in '06, Hug-Lutwak-Yang-Zhang in '05, Ludwig in '11, Lutwak-Oliker in '95 and many more recent results on this.

- **First eigenvalue of the Laplace operator with Dirichlet boundary conditions**; existence due to Jerison in '96, uniqueness due to Brascamp-Lieb in '76 and Colesanti in '05.

- Electrostatic capacity associated to Laplace's equation; due to Jerison in '96 and p-capacity associated to p-Laplace equation.

PDE Background

- For fixed p, $1 < p < \infty$, the p-Laplace equation

$$0 = \Delta_p u = \nabla\cdot(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-4}[(p-2)\sum_{i,k=1}^{n} u_{x_i}u_{x_k}u_{x_ix_k} + |\nabla u|^2\Delta u]$$

is the Euler-Lagrange equation of $\int_\Omega |\nabla u|^p dx$.

- Applications: image processing, glacelogy, plastic moulding etc
- $\Delta_n u =$ is invariant under Möbius transformations.
- For $p = 2$ one gets Laplace equation $\Delta u = u_{x_1x_1} + \ldots + u_{x_nx_n} = 0$.
- In the complex plane, relation with quasiconformal mappings.
- $\mathcal{A}$-harmonic PDEs: $-\text{div}\cdot\mathcal{A}(\nabla u) = 0$.
- In general, p-harmonic functions are in the class $C^{1,\alpha}$ due to Ural'ceva, Uhlenbeck, Tolksdorf, Lewis, Evans, ... in '80s.
- $u \in W^{1,p}_{\text{loc}}(\Omega)$ is called a weak solution in a domain Ω if

$$\int \langle |\nabla u|^{p-2}\nabla u, \nabla\eta\rangle dx = 0 \quad \text{for all } \eta \in C_0^\infty(\Omega).$$

p-capacity

If $K \subset \mathbb{R}^n$ is closed set, then p-capacity of K is

$$\mathrm{Cap}_p(K) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla v|^p dx,\ v \in C_0^\infty(\mathbb{R}^n) :\ v \geq 1 \text{ on } K \right\}.$$

- It is $(n-p)$-homogeneous: $\mathrm{Cap}_p(\rho K + z) = \rho^{n-p} \mathrm{Cap}_p(K)$.
- A ball of radius r has p-capacity $\approx r^{n-p}$.
- If u is the minimizer then

$$\begin{cases} \Delta_p u = 0 & \text{in } \quad \mathbb{R}^n \setminus K, \\ u = 1 & \text{on } \quad \partial K, \\ \lim_{|x| \to \infty} u(x) = 0. & \end{cases}$$

- Sets with empty interiors (a line segment in $\mathbb{R}^2$ or $\mathbb{R}^3$) can have positive p-capacity and p-harmonic functions can see those sets (Any set K with $0 < \mathcal{H}^l(K)$ has $\mathrm{Cap}_p(K) > 0$ for $n - p < l < n$).
- When $p \geq n$ then any set has zero p-capacity.

A Minkowski problem for p-capacitary measure

Given a convex body $K \subset \mathbb{R}^n$, let u_K be the p-harmonic function $(1 < p < n)$;

$$\begin{cases} \Delta_p u_K = 0 & \text{in} \quad \mathbb{R}^n \setminus K, \\ u_K = 1 & \text{on} \quad \partial K, \\ \lim_{|x|\to\infty} u_K(x) = 0. \end{cases}$$

- Define p-capacitary measure $\mu_{p,K}$ on $\mathbb{S}^{n-1}$ associated to K by

$$\mu_{p,K}(E) := \int_{\mathbf{g}_K^{-1}(E)} |\nabla u_K|^p d\sigma \quad \text{whenever} \quad E \subset \mathbb{S}^{n-1} \text{ is Borel.}$$

- It is well-defined: $|\nabla u_K|^p \sigma \sim \sigma$ on ∂K (by Dahlberg in '77 for $p = 2$ and for other p by Lewis-Nyström in '12).

The Minkowski problem for p-capacitary measure

Let μ be a given positive finite Borel measure on $\mathbb{S}^{n-1}$.

Find *necessary and sufficient* conditions on μ for which there exists a convex body K in $\mathbb{R}^n$ such that $\mu_{p,K} = \mu$.

Solution to the Minkowski Problem

Let μ be a positive finite Borel measure on $\mathbb{S}^{n-1}$ satisfying **C1-C2**.

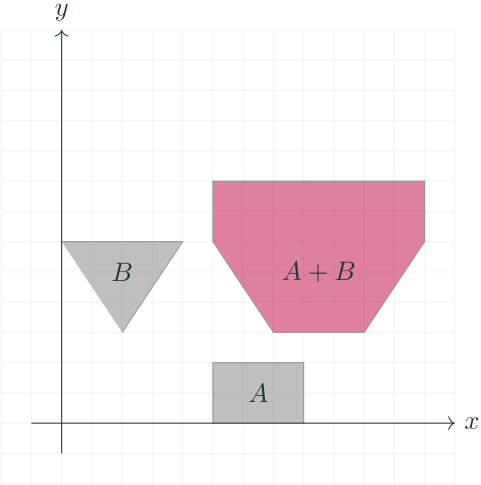
- **Existence**: There is a convex body K such that
$$\mu(E) = \mu_{p,K}(E) = \int_{\mathbf{g}_K^{-1}(E)} |\nabla u_K|^p d\sigma \quad \text{whenever} \quad E \subset \mathbb{S}^{n-1} \text{ is Borel.}$$
- **Uniqueness**: K is unique up to translation (and dilation when $p = n-1$).
- **Regularity**: If $d\mu = (1/\kappa)d\sigma$ for some strictly positive $\kappa \in C^{m,\alpha}(\mathbb{S}^{n-1})$, $m \in \mathbb{Z}^{\geq 0}$, and $\alpha \in (0,1)$, then $\partial K \in C^{m+2,\alpha}$.

- Jerison '96 completely solved for $p = 2$.
- Colesanti-Nyström-Salani-Xiao-Yang-Zhang '15 solved existence and regularity for $1 < p < 2$ and uniqueness for $1 < p < n$.
- Akman-Gong-Hineman-Lewis-Vogel '17 solved the existence and uniqueness when $1 < p < n$.
- Akman-Lewis-Saari-Vogel '19 solved existence and uniqueness for a related problem when $n \leq p < \infty$.
- Akman-Lewis-Vogel '20 solved the regularity for all p in $\mathbb{R}^3$.

Minkowski addition of sets

Minkowski addition of two sets A and B is defined as

$$A + B := \{a + b \,|\, a \in A,\, b \in B\} = \bigcup_{b \in B} A + \{b\}.$$



$sA := \{sa \,|\, a \in A\}$.

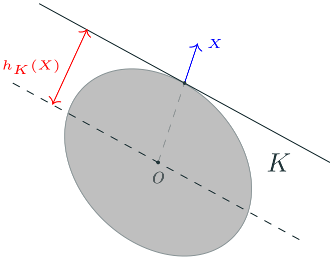
$A + A \neq 2A$ in general.

The support function of a convex body

- The support function h_K of a convex domain K in $\mathbb{R}^n$ is

$$h_K : \mathbb{S}^{n-1} \to \mathbb{R}, \quad h_K(X) = \sup\{\langle X, x\rangle;\, x \in K\}$$

i.e.: $h_K(X)$ is the distance of supporting hyperplane at a point on ∂K from the origin whose normal is X.



-h_K is homogeneous of degree 1.
-h_K is convex.
-Any convex body K is uniquely determined by h_K.

For every convex bodies K, L and constants $\alpha, \beta \geq 0$;

$$h_{\alpha K + \beta L} = \alpha h_K + \beta h_L.$$

The Hadamard Variational Formula

Let K and K_1 be convex bodies.

- The Hadamard variational formula for p-capacity

$$\begin{aligned}\frac{d}{dt}\mathsf{Cap}_p(K+tK_1)|_{t=0} &= (p-1)\int_{\partial K} h_{K_1}(\mathbf{g}_K(x))|\nabla u_K(x)|^p\, d\sigma \\ &= (p-1)\int_{\mathbb{S}^{n-1}} h_{K_1}(\xi)d\mu_{p,K}(\xi).\end{aligned}$$

- Hence $d\mu_{p,K} = |\nabla u_K|^p\, d\sigma$ is the first variation of p-capacity at K.
- Since

$$\frac{d}{dt}\mathsf{Cap}_p(K+tK)|_{t=0} = \frac{d}{dt}(1+t)^{n-p}|_{t=0}\mathsf{Cap}_p(K) = (n-p)\mathsf{Cap}_p(K).$$

- Hence

$$\mathsf{Cap}_p(K) = \frac{p-1}{n-p}\int_{\partial K} h_K|\nabla u_K|^p\, d\sigma = \frac{p-1}{n-p}\int_{\mathbb{S}^{n-1}} h_K d\mu_{p,K}$$

Existence for Minkowski problem

Strategy for proof of Existence

Given finite positive Borel measure μ on $\mathbb{S}^{n-1}$ satisfying C1-C2;

- Define a functional from set of convex bodies $\mathcal{K}$;

$$\mathcal{F} : \mathcal{K} \to \mathbb{R} \quad \text{by} \quad \mathcal{F}(L) = \int_{\mathbb{S}^{n-1}} h_L(X)\, d\mu(X) \quad \text{for} \quad L \in \mathcal{K}.$$

- Consider the following minimization problem

$$\inf\{\mathcal{F}(L) \quad \text{subject to the constraint} \quad \mathrm{Cap}_p(L) \geq 1\}.$$

- If $\tilde{K}$ is a minimizer then Lagrange multiplier implies $\exists \lambda \in \mathbb{R}$

$$d\mathcal{F}(\tilde{K}) = \lambda d\mathrm{Cap}_p(L).$$

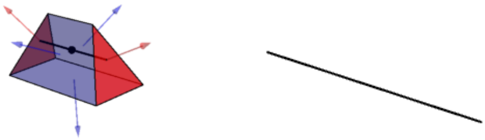
- Use the fact that the first variation of the p-capacity is $d\mu_{\tilde{K}}$;

$$d\mathcal{F}(\tilde{K}) = d\mu \quad \text{and} \quad d\mathrm{Cap}_p \tilde{K} = d\mu_{p,\tilde{K}} \quad \implies d\mu = \lambda d\mu_{p,\tilde{K}}.$$

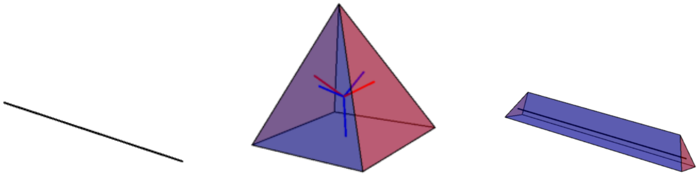
- A re-scaled copy K of $\tilde{K}$ will be the solution.

What can possible go wrong?

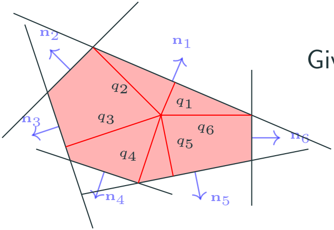
Imagine the 3 blue faces moving to the origin and giving the minimizer the black 1-d segment.



- For appropriate p, it can have p-capacity 1.
- For every minimizer K with empty interior find a competitor $\hat{K}$ with non-empty interior s.t. $\text{Cap}_p(\hat{K}) = 1$ and $\mathcal{F}(\hat{K}) < \mathcal{F}(K)$ $\quad \Rightarrow\!\Leftarrow$.



Proof of Existence in the discrete case



Given $q = (q_1, \ldots, q_m) \in \mathbb{R}^m$ with $q_i \geq 0$.

$$\mathcal{P}(q) := \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n;\ \langle x, \mathbf{n}_i \rangle \leq q_i\}.$$

- Let $\mathbf{n}_1, \ldots, \mathbf{n}_m$ be unit normals spans $\mathbb{R}^n$ and let $c_1, \ldots, c_m$ be positive numbers $\to$ $\mu = \sum_{i=1}^{m} c_i \delta_{\mathbf{n}_i}$.
- Prove that the minimization problem

$$\inf \left\{ \sum_{i=1}^{m} c_i q_i \ \mid\ q = (q_1, \ldots, q_m) \text{ satisfying } \mathrm{Cap}_p(\mathcal{P}(q)) \geq 1 \right\}.$$

has a solution for some $\tilde{q} \in \mathbb{R}^m$.

- $\mathrm{Cap}_p(\mathcal{P}(\tilde{q})) = 1$ and $\mathcal{P}(\tilde{q})$ has faces F_k with normal $\mathbf{n}_k$;

$$\mu(\mathbf{n}_k) = c_k = \lambda \mu_{p,\mathcal{P}(\tilde{q})}(\mathbf{n}_k) \quad \text{and} \quad c_k = \frac{\lambda(n-1)}{n-p} \int_{F_k} |\nabla u|^p d\mathcal{H}^{n-1}.$$

How to rule out low dimensional candidates

- $\mathcal{P}(\tilde{q})$ has non-empty interior then we are done.

- $\mathcal{P}(\tilde{q})$ may have empty interior; i.e., some of $\tilde{q}_i = 0$;

i.e., $0 < \mathcal{H}^k(\mathcal{P}(\tilde{q})) < \infty$ for some $n - p < k \leq n - 1$.

Construct $\mathcal{P}(\bar{q})$ with non-empty interior and $\mathrm{Cap}_p(\mathcal{P}(\bar{q})) = 1$ s.t.

$$\sum_{i=1}^{m} \bar{q}_i c_i < \sum_{i=1}^{m} \tilde{q}_i c_i.$$

Let $a = \frac{1}{4} \min\{\tilde{q}_i : i \in \{1, \dots m\} \,\&\, \tilde{q}_i \neq 0\}$ and for small $t > 0$,

$$\tilde{E}(t) := \bigcap_{i=1}^{m} \{x : \langle x, \mathbf{n}_i \rangle \leq \tilde{q}_i + at\}, \quad E := \bigcap_{i=1}^{m} \{x : \langle x, \mathbf{n}_i \rangle \leq a\}.$$

$$\text{Put} \qquad E_t = \frac{\tilde{E}(t)}{\mathrm{Cap}_p(\tilde{E}(t))^{1/(n-p)}}.$$

- $E_t = E(q(t))$ with $\mathrm{Cap}_p(E_t) = 1$ where $q(t) = (q_1(t), \dots, q_m(t))$ and

$$q_j(t) = \frac{\tilde{q}_j + at}{\mathrm{Cap}_p(\tilde{E}(t))^{1/(n-p)}} \quad \text{for} \quad 1 \leq j \leq m.$$

- Enough to show for small $t > 0$ near 0 that

$$k(t) = \mathsf{Cap}_{\mathcal{A}}(\tilde{E}(t))^{\frac{-1}{(n-p)}} \sum_{i=1}^{m} c_i(\tilde{q}_i + at) < \sum_{i=1}^{m} \tilde{q}_i c_i = k(0).$$

- Show that

$$\lim_{\tau \to 0} \frac{d}{dt} \mathsf{Cap}_p(\tilde{E}(t) + tE)\Big|_{t=\tau} = \lim_{\tau \to 0} \int_{\partial(\tilde{E}(\tau)+tE)} h_E |\nabla u_{\tilde{E}(\tau)+\tau E}|^p d\sigma = \infty.$$

- Use this in

$$[\mathsf{Cap}_p(\tilde{E}(t) + tE)]^{1+\frac{1}{(n-p)}} \frac{d}{dt} k(t) \,|_{t=\tau} = \mathsf{Cap}_p(\tilde{E}(t) + \tau E) \sum_{i=1}^{m} c_i a$$

$$- \frac{(p-1)}{(n-p)} [\sum_{i=1}^{m} c_i(\tilde{q}_i + a\tau)] \frac{d}{dt} \mathsf{Cap}_p(\tilde{E}(t) + tE) \,|_{t=\tau} .$$

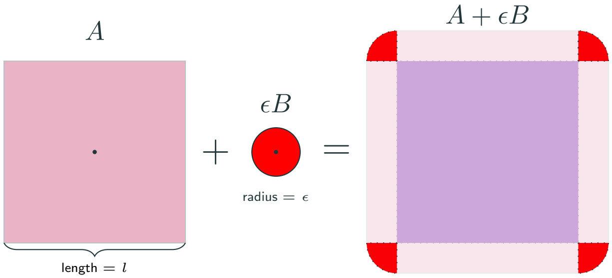
- Clearly, for some $t_0 > 0$ small

$$\frac{d}{dt} k(t)\Big|_{t=\tau} < 0 \quad \text{for} \quad \tau \in (0, t_0].$$

- This finishes the proof of existence in the discrete case. One has to run a similar argument for the general case.

Uniqueness for Minkowski Problem, a detour: Brunn-Minkowski inequality

Minkowski sum of a square and a disk



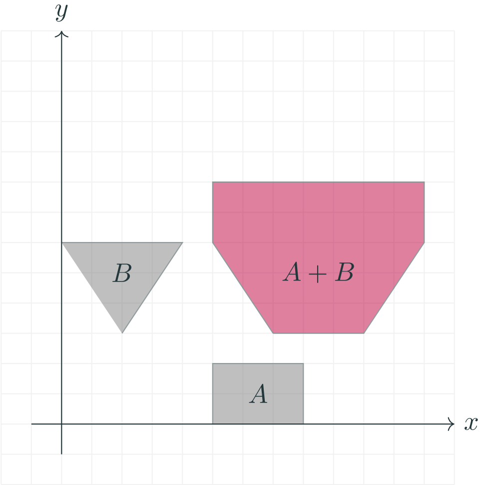
$$\begin{aligned} \text{area}(A+\epsilon B) &= \text{area}(A) + 4l\epsilon + \text{area}(\epsilon B) \\ &\geq \text{area}(A) + 2\sqrt{\pi} l\epsilon + \text{area}(\epsilon B) \\ &= \text{area}(A) + 2\sqrt{\text{area}(A)\text{area}(\epsilon B)} + \text{area}(\epsilon B) \\ &= \left((\text{area}(A))^{1/2} + (\text{area}(\epsilon B))^{1/2}\right)^2 . \end{aligned}$$

Hence

$$\text{area}(A+\epsilon B)^{1/2} \geq \text{area}(A)^{1/2} + \text{area}(\epsilon B)^{1/2}.$$

Is this a coincidence? How about in $\mathbb{R}^n$?

Another Example



area$(A) = 6.$
area$(B) = 6.$
area$(A+B) = 29.$

Hence

$$\sqrt{29} = \text{area}(A+B)^{1/2} \geq \text{area}(A)^{1/2} + \text{area}(B)^{1/2} = \sqrt{6} + \sqrt{6} = \sqrt{24}.$$

Classical Brunn-Minkowski inequality

- For $\lambda \in [0,1]$

$$|(1-\lambda)A + \lambda B)|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}} + \lambda |B|^{\frac{1}{n}}.$$

whenever A, B bounded open sets in $\mathbb{R}^n$. Equality holds precisely when A is a translation and dilation of B (i.e. A is homothetic to B).

- Due to Brunn in 1887 and Minkowski in 1896 for convex sets and (for bounded and open sets by Lyusternik in '35).

Equivalent forms of the Brunn-Minkowski inequality

- Elegant

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

- Multiplicative

$$|(1-\lambda)A + \lambda B| \geq |A|^{1-\lambda}|B|^{\lambda}.$$

- Minimal

$$|(1-\lambda)A + \lambda B| \geq \min\{|A|, |B|\}.$$

The Brunn-Minkowski inequality for p-capacity

- For $1 < p < n, \lambda \in [0, 1]$,

$$[\mathrm{Cap}_p((1-\lambda)A + \lambda B)]^{\frac{1}{n-p}} \geq (1-\lambda)[\mathrm{Cap}_p(A)]^{\frac{1}{n-p}} + \lambda[\mathrm{Cap}_p(B)]^{\frac{1}{n-p}}$$

whenever $A, B \subset \mathbb{R}^n$ are convex compact sets with non-empty interiors. Equality holds precisely when A is homothetic to B.

- Borell in '83, Caffarelli-Jerison-Lieb in '96, Colesanti-Salani in '03.
- Akman-Gong-Hineman-Lewis-Vogel '17: Inequality holds for any convex compact sets. Equality holds when A is homothetic to B.

Equivalent forms of the Brunn-Minkowski inequality for p-capacity

- Elegant: $\mathrm{Cap}_p(A + B)^{\frac{1}{n-p}} \geq \mathrm{Cap}_p(A)^{\frac{1}{n-p}} + \mathrm{Cap}_p(B)^{\frac{1}{n-p}}$.
- Multiplicative: $\mathrm{Cap}_p((1-\lambda)A + \lambda B) \geq \mathrm{Cap}_p(A)^{1-\lambda}\mathrm{Cap}_p(B)^{\lambda}$.
- Minimal: $\mathrm{Cap}_p((1-\lambda)A + \lambda B) \geq \min\{\mathrm{Cap}_p(A), \mathrm{Cap}_p(B)\}$.
- Akman-Lewis-Saari-Vogel '18: For $n \leq p < \infty$, certain quantity associated to p-harmonic function satisfies a Brunn-Minkowski type inequality.

Sketch of Uniqueness of Minkowski Problem

- Let E_0 and E_1 are two convex body with $\mu_{p,E_0} = \mu_{p,E_1} = \mu$.
- For $t \in [0,1]$, let $E_t = (1-t)E_0 + tE_1$
- Define $\mathbf{m}(t) = \mathrm{Cap}_p(E_t)^{\frac{1}{n-p}} = \mathrm{Cap}_p((1-t)E_0 + tE_1)^{\frac{1}{n-p}}$.
- The Brunn-Minkowski inequality says that $\mathbf{m}(t)$ is a concave function on $[0,1]$ therefore $\mathbf{m}'(0) \geq \mathbf{m}(1) - \mathbf{m}(0)$ with strict inequality unless $\mathbf{m}$ is linear in t.
- Aim: show that $\mathbf{m}(t)$ is linear in $t \to$ Equality in Brunn-Minkowski inequality.
- By the Hadamard variational formula

$$\begin{aligned}
\mathbf{m}'(0) &= \mathrm{Cap}_p(E_0)^{\frac{1}{n-p}-1} \left. \frac{d}{dt}\mathrm{Cap}_p(E_t)\right|_{t=0} \\
&= (p-1)\int_{\mathbb{S}^{n-1}} (h_1(\xi) - h_0(\xi))d\mu(\xi) \\
&= (n-p)[\mathrm{Cap}_p(E_1) - \mathrm{Cap}_p(E_0)].
\end{aligned}$$

Cont'

$$\mathbf{m}(1) - \mathbf{m}(0) \le \mathbf{m}'(0) = \mathsf{Cap}_p(E_0)^{\frac{1}{n-p}-1}[\mathsf{Cap}_p(E_1) - \mathsf{Cap}_p(E_0)]$$
$$= \mathbf{m}(0)^{1-n+p}[\mathbf{m}(1)^{n-p} - \mathbf{m}(0)^{n-p}].$$

Let $l = \left(\frac{\mathsf{Cap}_p(E_1)}{\mathsf{Cap}_p(E_0)}\right)^{\frac{1}{n-p}} = \left(\frac{\mathbf{m}(1)}{\mathbf{m}(0)}\right)$.

Now $l^{n-p} - 1 \ge l - 1$. Reversing the roles of E_0, E_1 we also get

$$l^{p-n} - 1 \ge l^{-1} - 1.$$

Clearly, both these inequalities can only hold if $l = 1$. Thus $\mathsf{Cap}_p(E_0) = \mathsf{Cap}_p(E_1)$ and hence

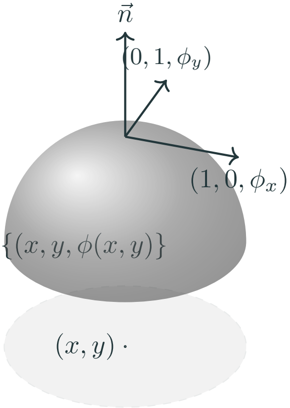
$$\mathbf{m}'(0) = 0 = \mathbf{m}(1) - \mathbf{m}(0).$$

Thus $\mathbf{m}(t)$ is linear in t which is implies equality in the Brunn-Minkowski inequality;

- E_0 is a translation and dilation of E_1.

Regularity for Minkowski problem A detour: Monge-Ampère equation

A brief introduction to Monge-Ampère equation



Surface = Graph of a convex Lipschitz ϕ.

- At $(x, y, \phi(x, y))$,

$$\vec{n} = \frac{(-\phi_x, -\phi_y, 1)}{\sqrt{1 + |\nabla\phi|^2}}.$$

- The Gauss curvature at $(x, y, \phi(x, y))$ is

$$\kappa(x, y) := \det d\vec{n} = \frac{\phi_{xx}\phi_{yy} - \phi_{xy}^2}{(1 + |\nabla\phi|^2)^2}.$$

- One gets Monge-Ampère equation

$$\det(\nabla^2\phi) = \kappa(x, y)(1 + |\nabla\phi|^2)^2 = f(x, \phi, \nabla\phi)$$

Smoothness of the surface = regularity of ϕ.

Monge-Ampère measure

Definition
Let Ω be a open convex set. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ be a convex function. $\phi : \Omega \to \mathbb{R}$ is called an (Alexandrov) solution to the Monge-Ampère equation $\det(\nabla^2 \phi) = f(x, \phi, \nabla\phi)$ in Ω if

$$\mu_\phi(A) = \int_A f(x, \phi, \nabla\phi)dx \quad A \subset \Omega \text{ Borel.}$$

For any C^2 function ϕ in a domain D, we can write

$$\int_E \det(\nabla^2\phi)dx = \int_{\nabla\phi(E)} dy = \text{Measure of } \nabla\phi(E).$$

For a C^1 function ϕ, define Monge-Ampère measure μ as

$$\mu_\phi(E) = \text{Lebesgue measure of } \nabla\phi(E).$$

For a convex function ϕ,

$$\mu_\phi(E) = \text{Lebesgue measure of } \partial\phi(E)$$

where $\partial\phi(x) = \{y \in \mathbb{R}^n;\ \phi(z) \geq \phi(x) + \langle y, z - x\rangle,\ \forall z\}$.

Regularity of Minkowski Problem - Classical case

Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ satisfying **C1-C2**. Let K be the convex body with non-empty interior s.t.

$$\mu(E) = \mu_K(E) = \int_{\mathbf{g}_K^{-1}(E)} d\sigma \quad \text{whenever} \quad E \subset \mathbb{S}^{n-1} \text{ is Borel set.}$$

- Let $d\mu = d\mu_K = (1/\kappa)d\sigma$ for some strictly positive $\kappa \in C^{m,\alpha}(\mathbb{S}^{n-1})$, $m \in \mathbb{Z}^{\geq 0}$, and $\alpha \in (0,1)$.

- Let ϕ be a convex Lipschitz function defined on an open subset O of $\mathbb{R}^{n-1}$ whose graph $\{(x, \phi(x)) : x \in O\}$ is a portion of ∂K.

Then ϕ satisfies the **Monge-Ampère equation**

$$\det(\nabla^2 \phi) = (1 + |\nabla \phi|^2)^{\frac{n+1}{2}} \kappa = d\mu$$

- Caffarelli '90: If $\mu(F) \leq C\mu(F/2)$ for every $F = \partial K \cap H$ for some half-space H then $\partial K \in C^{2,\alpha}$.

Regularity of Minkowski Problem - Our case

Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ satisfying **C1-C2**. Let K be the convex body with non-empty interior s.t.
$\mu(E) = \mu_{p,K}(E) = \int_{\mathbf{g}_K^{-1}(E)} |\nabla u_K(x)|^p d\sigma$ whenever $E \subset \mathbb{S}^{n-1}$ is Borel set.

Then ϕ satisfies the **Monge-Ampère equation**

$$\det(\nabla^2\phi) = (1 + |\nabla\phi|^2)^{\frac{n+1}{2}}\kappa|\nabla u_K|^p = d\mu.$$

- Jerison '91 + Gutiérrez and Hartenstine '03: When $p = 2$, μ satisfies "weak doubling". That is, for every $F = \partial K \cap H$ for some half-space H, for some $\epsilon \in (0, 1]$ such that

$$\int_F \delta(x, F)^{1-\epsilon}\, d\mu(x) \leq C\mu(F/2).$$

Then one can run Caffarelli's argument to show $\partial K \in C^{2,\alpha}$.

- Akman-Lewis-Vogel '20: Run Jerison's argument and study p-harmonic function in cone domains with arbitrarily small aperture to show for all p in $\mathbb{R}^3$.

THANKS!