Boundary Value Problems and Fourier Series

Consider the following differential equation with boundary values

$$y'' + \lambda y = 0 \qquad y(0) = 0 \quad \text{and} \quad y(L) = 0$$

where λ is a real number and $L > 0$.

Here obviously $y = 0$ is a solution to the above boundary value problem for any value of λ. We will observe that most other values of λ, there are no other solutions.

The question we want to answer is that

<u>Question!</u> For what values of λ does the above problem have nontrivial solutions? and what are they?

A value of λ for which above problem has a nontrivial solution is an $\boxed{\text{eigenvalue}}$ of the problem, and in this case nontrivial solutions are $\boxed{\lambda\text{-eigenfunctions}}$ or $\boxed{\text{eigenfunctions associated to } \lambda}$.

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Example: Find all eigenvalues and eigen function of the Boundary Value problem

$$y'' + \lambda y = 0 \qquad y(0) = 0 \text{ and } y(L) = 0.$$

Solution. Let $\lambda = k^2$. for some k.

From HW 5, problem 5, we know that the the differential equation has solution

$$y(x) = A\cos(kx) + B\sin(kx).$$

Let's find A and B.

Since $y(0) = A = 0$ we choose $A = 0$

Also $y(L) = B\sin(kL) = 0$

Since $\sin x$ is zero when $x = 0, \mp\pi, \mp 2\pi, \ldots$ $= n\pi$, for $n \in \mathbb{N}$. enough as $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

then $kL = n\pi$

$k = \frac{n\pi}{L}$ or $\lambda_n = k^2 = \frac{n^2\pi^2}{L^2}$ are the eigenvalues (for each n, we have different eigenvalue)

and the corresponding solution is (or eigen function)

$y(x) = \sin(kx) = \sin\left(\frac{n\pi}{L}x\right)$ are the eigenfunction.

Therefore $\boxed{\lambda_n = \frac{n^2\pi^2}{L^2} \text{ eigenvalues for } n = 1, 2, 3. \quad y(x) = \sin\left(\frac{n\pi}{L}x\right) \text{ are the corresponding eigen functions.}}$

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Example! Find all eigenvalues and corresponding eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0 \quad , \quad y'(0) = 0 \quad \text{and} \quad y'(L) = 0.$$

Solution! Note that assuming again $\lambda = k^2$ the general solution of the differential equation is

$$y(x) = A \sin(kx) + B\cos(kx).$$

Since the boundary values are given as terms of derivative of y; we need to find y'

$$y'(x) = Ak \cos kx - Bk \sin(kx)$$

$$y'(0) = Ak = 0$$

$$y'(L) = Bk \sin(kL) = 0 \qquad \text{since } \sin x = 0 \text{ when } x = n\pi \text{ for } n \in \mathbb{N}.$$

then we get $kL = n\pi$, $k = \frac{n\pi}{L}$, $n = 1, 2, \ldots$

Hence $y(x) = B\cos\left(\frac{n\pi}{L}x\right)$.

As $\lambda = k^2 = \frac{n^2\pi^2}{L^2}$ are eigenvalues for $n \in \mathbb{N}$. $n = 1, 2, \ldots$

and the corresponding eigenfunctions are

$$y_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, \ldots$$

There we have another eigenvalue; $\lambda_0 = 0$ and the corresponding eigenfunction $y_0 = 1$. (check!)

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The common thing about these eigenfunctions are that they are all periodic. Therefore, the solutions of the differential equation with boundary value are periodic.

Orthogonality: We say that two (integrable) functions f and g are orthogonal on an interval $[a,b]$ if

$$\int_a^b f(x)\,g(x)\,dx = 0.$$

More generally, we say that the functions

$\{\phi_1, \phi_2, \ldots \phi_n, \ldots\} \longrightarrow$ there may be finitely many of them or infinitely many of them.

are orthogonal on $[a,b]$ if

$$\int_a^b \phi_i(x)\cdot\phi_j(x)\,dx = 0 \quad \text{whenever } i \neq j.$$

Example: Show that $1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots$

which are the eigenfunction for the differential eq.

$y'' + \lambda y = 0 \qquad y(-L) = y(L) \;\;\&\;\; y'(-L) = y'(L)$

are orthogonal on $[-L, L]$.

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Solution! To show this $1, \left\{\cos\frac{n\pi x}{L}\right\}_{n=1}^{\infty}, \left\{\sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$

are the functions. For some n, consider

$$\int_{-L}^{L} 1.\cos\frac{n\pi x}{L}\,dx = \frac{1}{\frac{n\pi}{L}}\sin\frac{n\pi x}{L}\Big|_{x=-L}^{x=L} = 0$$

Therefore 1, and $\cos\frac{n\pi x}{L}$ are orthogonal for any $n\geq 1$.

Similarly $$\int_{-L}^{L} 1.\sin\frac{n\pi x}{L}\,dx = -\frac{1}{\frac{n\pi}{L}}\cos\frac{n\pi x}{L}\Big|_{x=-L}^{x=L} = 0$$

To show for arbitrary m and n, $m\geq 1, n\geq 1$

$\cos\frac{n\pi x}{L}$ & $\sin\frac{m\pi x}{L}$ are orthogonal we check

$$\int_{-L}^{L} \underbrace{\underbrace{\cos\frac{n\pi x}{L}}_{\text{even function}} . \underbrace{\sin\frac{m\pi x}{L}}_{\text{odd function}}}_{\text{odd function}}\,dx = \quad = 0$$

$$\int_{-L}^{L} \text{odd function}\,dx = 0$$

One can also check $$\int_{-L}^{L} \cos\frac{n\pi x}{L} . \cos\frac{m\pi x}{L}\,dx = 0$$

for $m\neq n$ $$\int_{-L}^{L} \sin\frac{n\pi x}{L}\,\sin\frac{n\pi x}{L}\,dx = 0$$

Therefore above functions are orthogonal.

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From now on we use the following notation

$$< f(x), g(x) > = \int_{-L}^{L} f(x) g(x) \, dx .$$

This is called inner product! It has good properties

It's positive definite : $< f, f > = \int_{-L}^{L} f^2(x) dx \geq 0$

and $<f, f> = 0$ iff $f = 0$.

It's symetric $< f, g > = \int_{-L}^{L} f(x) g(x) \, dx = < g, f >$.

It's bilinear since it's symetric and linear

$$< f, ag + bh > = \int_{-L}^{L} f(x) [ag(x) + bh(x)] \, dx$$

$$= a \int_{-L}^{L} f(x) g(x) dx + b \int_{-L}^{L} f(x) h(x) dx$$

$$= a < f, g > + b < f, h >$$

An inner product provides the notion of an angle in a "vector space".

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Fourier Series of a Function:

When f is periodic function with period $2L$

$$f(x) = f(x+2L)$$

then f admits a series expansion ("Fourier series" of f)

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.}$$

where
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x).\cos\frac{n\pi x}{L}\,dx \qquad n \geq 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x).\sin\frac{n\pi x}{L}\,dx \qquad n \geq 1$$

and
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

Here a_n's are called Fourier cosine coefficients

b_n's are called Fourier sine coefficients.

Above series is called Fourier series of $f(x)$.

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Remark: If f is piecewise continuous on $[-L, L]$

then f admits a fourier series on $[-L, L]$.

One needs to be careful, at the discontinuity point

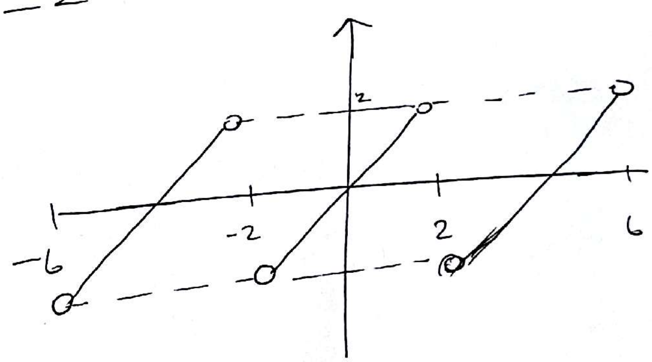
(as we only assume f is piecewise continuous)

the function f and its ~~fourier~~ series ~~may~~ do

not agree!.

Example: Find a fourier series for

$f(x) = x$, $-2 < x < 2$, $f(x+4) = f(x)$.



here the function f is $2L = 4$ periodic.

Therefore $L = 2$

Now $a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \frac{1}{2} \int_{-2}^{2} x\,dx = 0$ as x is odd.

$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx = \frac{1}{2} \int_{-2}^{2} \underbrace{\underbrace{x}_{\text{odd}} . \underbrace{\cos \frac{n\pi x}{L}}_{\text{even}}}_{\text{odd}}\,dx$

$= 0$

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$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} \cancel{f(x)}\cdot\sin\frac{n\pi x}{2}\,dx$$

int. by parts

$$= \frac{1}{2}x\left(-\cos\frac{n\pi x}{2}\right)\frac{1}{\frac{\pi n}{2}}\Bigg|_{x=-2}^{x=2} + \frac{1}{2}\frac{1}{\frac{n\pi}{2}}\int_{-2}^{2}\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{\cancel{2}}\cancel{2}\left(-\cos\frac{n\pi \cancel{2}}{\cancel{2}}\right)\frac{2}{n\pi} - \frac{1}{\cancel{2}}\cdot(-\cancel{2})\left(-\cos\frac{n\pi(-2)}{2}\right)\frac{2}{\pi n} + \frac{1}{\cancel{2}}\frac{\cancel{2}}{\pi n}\frac{1}{\frac{n\pi}{2}}\sin\frac{n\pi x}{2}\Bigg|_{x=-2}^{x=2}$$

$$= (-1)^n\frac{2}{n\pi} + \frac{2(-1)^n}{n\pi} + \frac{2}{(n\pi)^2}\cdot\Big(\underbrace{\sin n\pi}_{=0} - \underbrace{\sin(-n\pi)}_{=0}\Big)$$

$$= \frac{4(-1)^n}{n\pi}$$

$\cos(n\pi) = (-1)^n$.

$\cos(0) = 1$

$\cos(\pi) = -1$

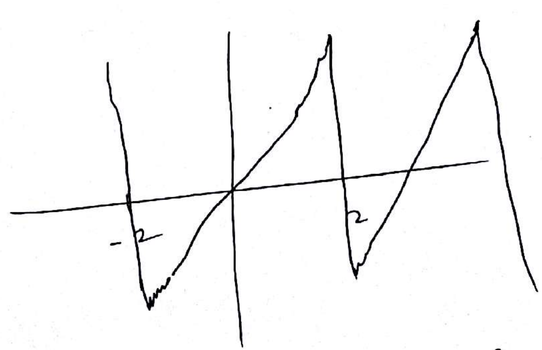
$\cos(2\pi) = 1$

$\vdots$

Therefore we get

$$f(x) = \underset{a_0}{0} + \underset{\sum a_n \cos\frac{n\pi x}{L}}{0} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\sin\frac{n\pi x}{2} \quad \text{for } -2<x<2.$$



This is the partial sum of the first 30 terms of Fourier series of f.

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<u>Example!</u> Find a fourier series representing

$$f(x) = 5 + \cos(4x) - \sin 5x.$$

with a period of 2π.

Note that this function π itself 2π periodic and

It's fourier series of $f(x)$.

<u>Example!</u> Find a Fourier series expansion of the function

$$f(x) = \begin{cases} 1+x & x \in [-1,0) \\ 1-x & x \in [0,1]. \end{cases}$$

$$f(x) = f(x+2).$$

the f is $2 = 2L$ periodic therefore $L = 1$.

Now we want to find Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1}$$

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx.$$

$$= \left(x + \frac{x^2}{2}\right)_{x=-1}^{x=0} + \left(x - \frac{x^2}{2}\right)_{x=0}^{x=1}$$

$$= (0) - \left(-1 + \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) - 0$$

$$= 1$$

Therefore $a_0 = 1$

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$$a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx = \int_{-1}^{0} (1+x)\cos n\pi x \, dx + \int_{0}^{1} (1-x)\cos n\pi x \, dx$$

$$= (1+x)\frac{(\cancel{+}\sin n\pi x)}{n\pi}\Big|_{x=-1}^{x=0} + \int_{-1}^{0} \frac{\sin n\pi x}{n\pi} dx + (1-x).\frac{\sin n\pi x}{n\pi}\Big|_{x=0}^{x=1} \cancel{+} \int_{0}^{1} \frac{\sin n\pi x}{n\pi} dx$$

$$= 0 - \underbrace{\frac{\sin 0}{n\pi}}_{=0} + \int_{-1}^{0} \frac{\sin n\pi x}{n\pi} dx + 0 - \underbrace{\frac{\sin n\pi}{n\pi}}_{=0} + \int_{0}^{1} \frac{\sin n\pi}{n\pi} . dx$$

$$= -\frac{\cos n\pi x}{(n\pi)^2}\Big|_{x=-1}^{x=0} - \frac{\cos n\pi x}{(n\pi)^2}\Big|_{x=0}^{x=1}$$

$$= \frac{1}{(n\pi)^2} + \frac{\cos(-n\pi)}{(n\pi)^2} - \frac{\cos n\pi}{(n\pi)^2} + \frac{1}{(n\pi)^2} = \frac{2}{(n\pi)^2} + \frac{2(-1)^n}{(n\pi)^2}$$

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx = \int_{-1}^{0} (1+x)\sin n\pi x + \int_{0}^{1} (1-x)\sin n\pi x \, dx$$

$$= (1+x)\frac{(-\cos n\pi x)}{n\pi}\Big|_{x=-1}^{x=0} + \int_{-1}^{0} \frac{\cos n\pi x}{n\pi} dx + (1-x)\frac{(-\cos n\pi x)}{n\pi} + \int_{0}^{1} \frac{\cos n\pi x}{n\pi} dx$$

$$= -\cancel{\frac{\cos 0}{n\pi}} + 0 + \int_{-1}^{0} \frac{\cos n\pi x}{n\pi} dx + \cancel{\frac{\cos 0}{n\pi}} + \int_{0}^{1} \frac{\cos n\pi x}{n\pi} dx$$

$$= + \frac{\sin n\pi x}{(n\pi)^2}\Big|_{-1}^{0} + \frac{\sin n\pi x}{(n\pi)^2}\Big|_{0}^{1} = 0$$

Therefore $b_n = 0$

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The Fourier Convergence Theorem

Theorem: Suppose f and f' are piecewise continuous on the interval $-L \le x \le L$. Further assume that f is defined elsewhere so that it is periodic with period $2L$. Then f has Fourier series in the form

$$\cancel{f(x) =} \quad a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

This Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous and it converges to

$$\frac{1}{2}\left[\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right]$$

at every point x_0 where f is discontinuous.

(*) f being piecewise continuous guarantees that the Fourier coefficients can be found.

(*) f' being piecewise continuous is a sufficient condition the ~~Fourier~~ series defined above will converges for every x.

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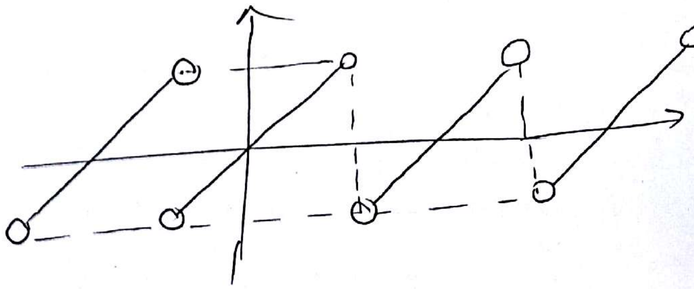
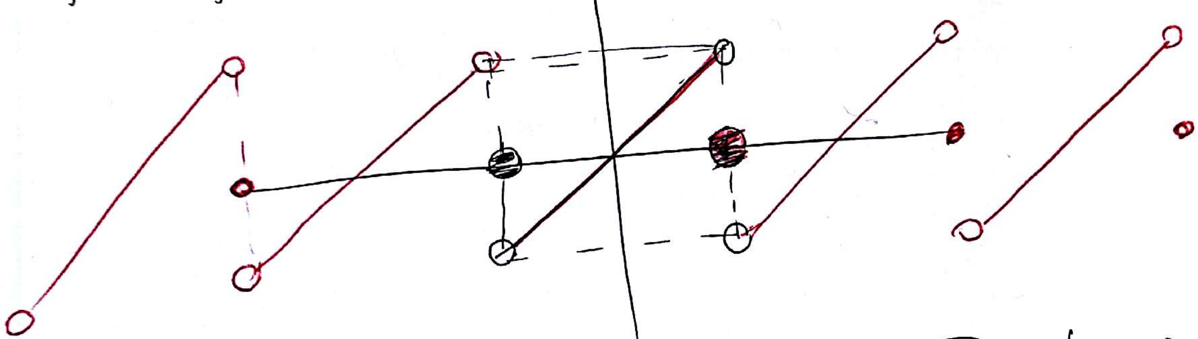
Example: Let $f(x)=x$, $-2<x<2$, $f(x+4)=f(x)$.

This function has the graph



If we draw the Fourier series we have found for $f(x)$ we get



Even and Odd Functions

Recall that an even is a function with the property $f(x)=f(-x)$ for all x in its domain

Examples: $\cos x$, x^2, x^4, constant function, x^{-2}, x^{-10}

An odd function is a function with the property $f(x)=-f(-x)$ for all x in its domain

Examples: $\sin x$, x, x^3, $\ldots$ x^{-1}, $x^{-3}\ldots$

Some functions are neither even nor odd

Examples: $1+x$, $x+x^2$, ..., e^x, ...

$f(x) = e^x \neq e^{-x} = f(-x)$ ~~...~~

$f(x) = e^{x} \neq -e^{-x} = -f(-x)$.

Therefore linear combination of an odd and an even function is not necessarily even or odd.

In general, linear combination of odd functions are odd

$\underset{\text{odd}}{x} + \underset{\text{odd}}{x^3}$ is odd, $\underset{\text{odd}}{100x} + \underset{\text{odd}}{\sin x}$ is odd.

Similarly, linear combination of even functions is even

$\underset{\text{even}}{5x^2} + \underset{\text{even}}{\cos x}$ is even, $\underset{\text{even}}{1} + \underset{\text{even}}{x^2}$ is even

Remark: Let $f(x)$ be even and $g(x)$ be odd

Then product $f(x) \cdot g(x)$ is odd function

If $f(x)$ and $g(x)$ are both even function then $f(x)\, g(x)$ is even

If $f(x)$ and $g(x)$ are both odd then $f(x)\, g(x)$ is even.

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Example: x is odd x^2 is even $\rightarrow$ $x \cdot x^2 = x^3$ is odd

x is odd $\sin x$ is odd $\rightarrow$ $x \sin x$ is even

x^2 is even $\cos x$ is even $\rightarrow$ $x^2 \cos x$ is even.

Some nice properties of even and odd functions:

let $f(x)$ be an even function, continuous on $-L<x<L$

then $$\boxed{\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.}$$

let $g(x)$ be an odd function, continuous on $-L<x<L$

then $$\boxed{\int_{-L}^{L} f(x)\,dx = 0.}$$

The Fourier Cosine Series

Suppose f is an even periodic function of period $2L$, then its Fourier series contains only cosine (and possibly the constant) terms. In this case

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

(No sine series!)

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Conversely, any periodic function whose Fourier series has the form of a cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

must be even function. So f has to be an even function.

Normally $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

This holds since $f(x)$ is even and $\cos\frac{n\pi x}{L}$ is even

then product is even and the prop. of even function gives us the last identity.

The Fourier Sine Series

If f is an odd function of period 2L then its Fourier series contains only sine terms. That is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Conversely, any periodic function whose Fourier series has the form of a sine series as above

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must be an odd periodic function.

<u>Example:</u> $f(x) = x \ , \ -2 < x < 2 \ , \ f(x) = f(x+4)$

This is an odd function with period 2. therefore its Fourier series should only contain sine terms

$$f(x) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{2}.$$

(which it was indeed; $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin \frac{n\pi x}{2}.$

<u>Example 1</u> $f(x) = x^4 + x^2 \ . \ -1 < x < 1$ is an even function
$f(x) = f(x+2)$
with period 2 and its Fourier series will be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}.$$

Half - Range Expansions

If f and f' are precewise continuous in an interval $0 \leq t \leq L$ then f can be extended into an even periodic function, F, of period $2L$ s.t $f(x) = F(x)$ on $[0, L]$. whose Fourier series therefore a cosine series.

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Similarly $f(x)$ can be extend into an odd function F, of period $2L$ for which $f(x) = F(x)$ on $[0,L]$. ~~Then $f(x)$~~ Then this function's Fourier series contains only sine terms.

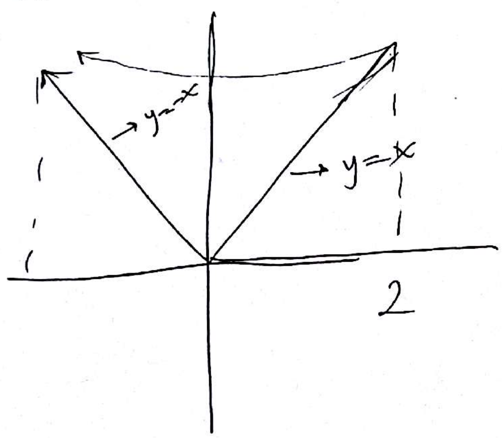
<u>Example:</u> let $f(x) = x$ for $0 \le x < 2$. (Ln 2)

We want to extend f into an $2L$ period function F which is <u>even</u>. $\left\{ \begin{array}{l} \underline{F(x) = f(x)} \text{ on } 0 \le x < 2 \\ F(-x) = F(x) = f(x) \end{array} \right.$

Since F is even then $F(-x) = F(x) = f(x)$ when $-x \in [0,2)$ i.e. $x \in$

Hence we should set

$$F(x) = \begin{cases} f(x), & 0 \le x \le 2 \\ f(-x) & -2 < x \le 0 \end{cases}$$



In the even case Reflect f around y-axis

$$F(x) = \begin{cases} x & 0 \le x < 2 \\ -x & -2 < x \le 2. \end{cases}$$

In general Given $f(x)$ defined on $[0,L]$. Its <u>even</u> extension of period $2L$ is

$$F(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < \end{cases} \qquad F(x+2L) = F(x)$$

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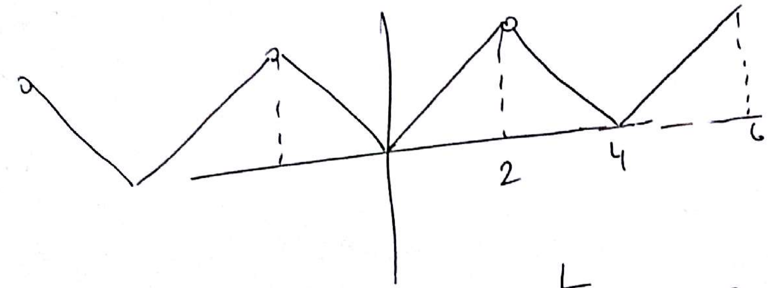
The function $f(x)=x$ has the even extension on $(-2,2)$



In this case $a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx$

$b_n = 0.$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

If we want to extend $f(x)=x$ on $0 \le x \le 2$ with period of 2 to an odd function we then reflect through origin i.e.

$$F(x) = \begin{cases} f(x) & 0<x<L \\ 0 & x=0,L \\ -f(-x) & -L<x<0 \end{cases} \qquad F(x+2L)=F(x).$$

Hence $f(x)=x$ has extension

$$F(x) = \begin{cases} f(x)=x & 0<x<L \\ x=0 & \text{when } x=0 \ \& \ x=L \\ -f(-x)=x & \text{when } -L<x<0. \end{cases}$$

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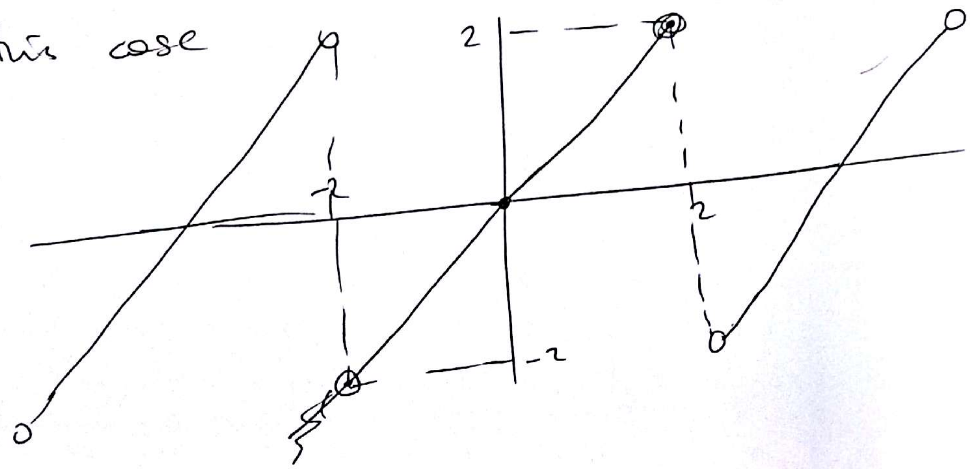
In this case



In general F has Fourier series (when we extend f to be periodic ~~function of~~ function of 2L to odd function then

$a_0 = 0$ & $a_n = 0$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

Example: Extend $f(x) = x^3$ $0 \le x < 2$ into a ~~function~~

1. odd function with period 4
2. even function with period 4.

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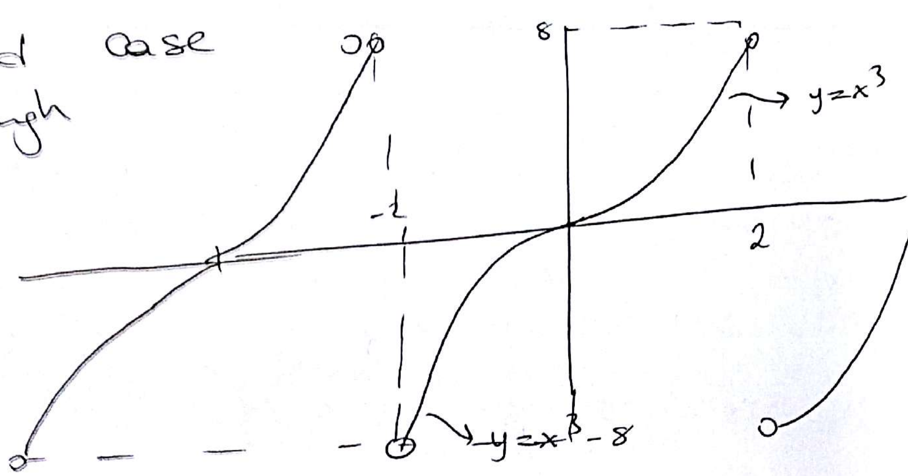
1. Odd case

Reflect through origin



$$F(x)=\begin{cases} x^3 & 0\le x<2 \\ -(-x)^3=x^3 & -2<x<0. \end{cases}$$

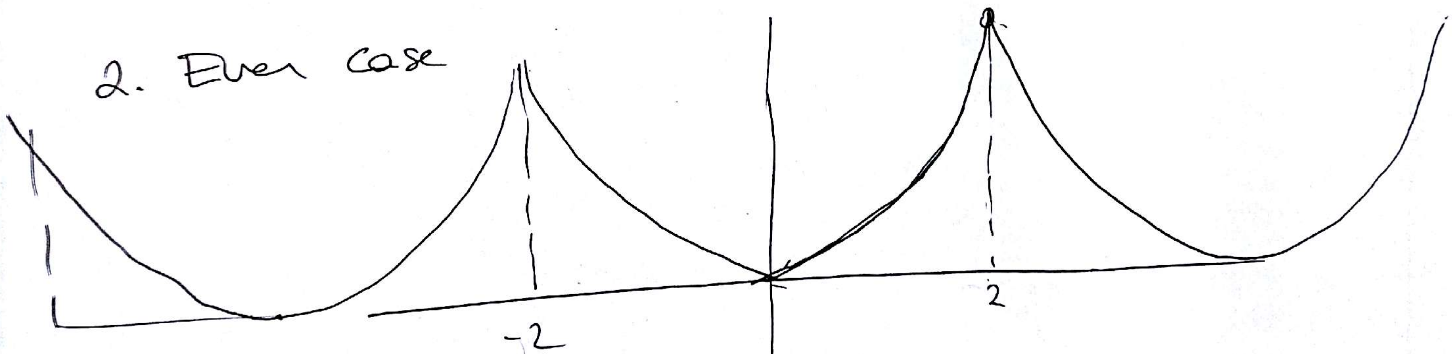
This is an odd function $F(x)=f(x)$ when $x\in[0,2)$ and F has period 4

$F(x)=F(x+4)$.

then $F(x)$ has only sine terms (as it's odd)

$$F(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

2. Even case



Reflect thru y-axis.

$$F(x)=\begin{cases} x^3 & 0\le x<2 \\ (-x)^3=-x^3 & -2<x<0. \end{cases}$$

$F(x)=F(x+4)$.

Notice that this function is an even function with period 4.

Hence F has power series with only cosine terms

$$F(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}.$$

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