Wave Equation: Vibration of an Elastic String

Consider a piece of thin flexible string of length L, of negligible weight.

Suppose two ends of the string are firmly secured, so that they will not move.

Assume the set-up has no damping.

The vertical displacement of the string $0<x<L$, and at any time $t>0$, is given by the displacement function $u(x,t)$.

It satisfies the homogeneous one-dimensional undamped wave equation

$$a^2 u_{xx} = u_{tt}$$

where constant coefficient a^2 is given by

$a^2 = \frac{T}{\rho}$: such that a = phase velocity

T = force of tension exerted on the string

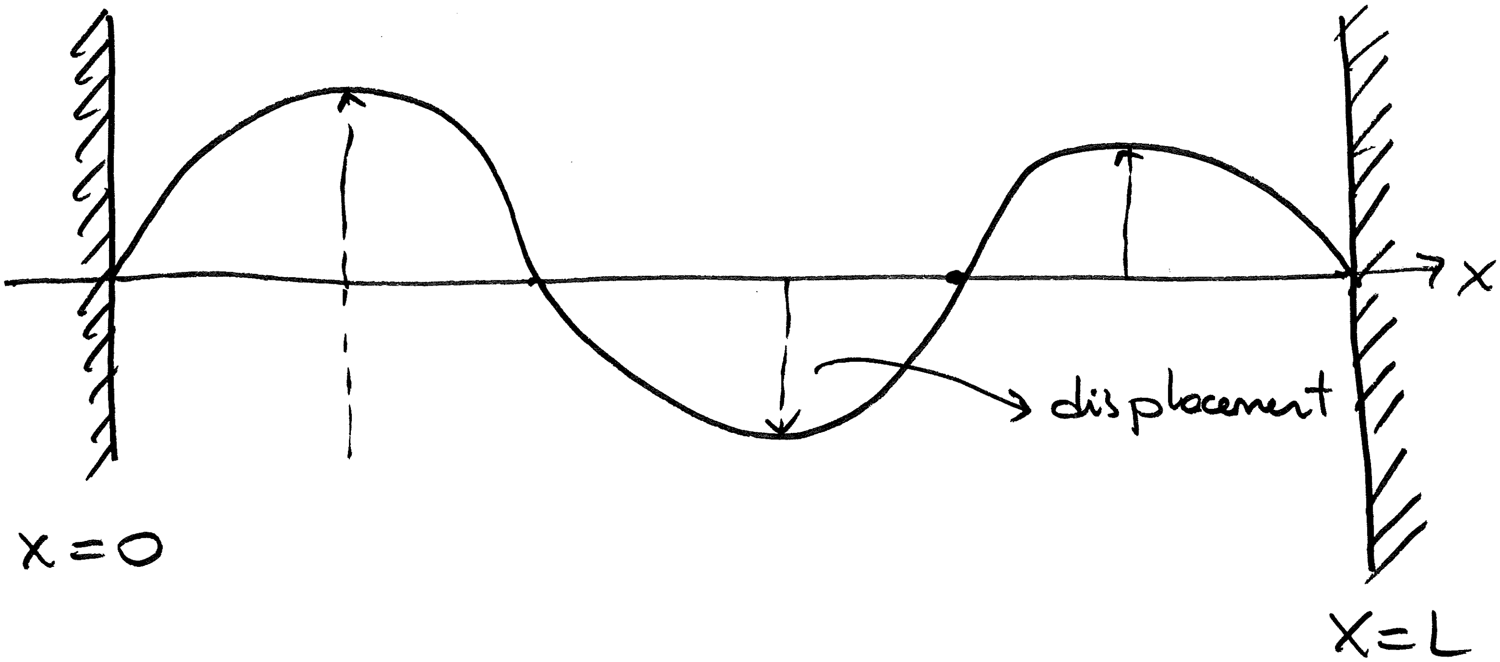
ρ = mass density.

It's subject to homogeneous boundary conditions

$$u(0,t)=0 \quad , \quad u(L,t)=0 \qquad t>0.$$

These two are the usual boundary conditions.

There will be two initial conditions (as we have two t derivatives). These two initial conditions are the initial displacement $u(x,0)$ and the initial velocity $u_t(x,0)$ both are functions of x alone.



Wave equation: $a^2 u_{xx} = u_{tt}$ $\quad 0<L<x,\ t>0$

Boundary conditions $u(0,t)=0$ and $u(L,t)=0$

Initial conditions $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$.

<u>Solution:</u> Let $u(x,t) = X(x)T(t)$ and separate the wave equation into two ordinary differential equations.

$u_x = X'T \quad \& \quad u_{xx} = X''T$

$u_t = XT' \quad \& \quad u_t = XT''$

$a^2 X''T = XT''.$

Dividing both sides a^2XT gives

$$\frac{X''}{X} = \frac{T''}{a^2T}.$$

Again with the same idea, left-hand side is a function of x and right-hand is function of t, this is possible only if they are constant $-\lambda$;

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda.$$

$$\rightarrow \frac{X''}{X} = -\lambda \longrightarrow X'' = -\lambda X \longrightarrow X'' + \lambda X = 0$$

$$\frac{T''}{a^2T} = -\lambda \longrightarrow T'' = -\lambda a^2 T \rightarrow T'' + \lambda a^2 T = 0$$

Now we rewrite the boundary conditions;

$u(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$ or $T(t) = 0$

$u(L,t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$ or $T(t) = 0$

As usual, in order to obtain nontrivial solutions we need to choose

$X(0)=0$ & $X(L)=0$ as the new boundary conditions.

Therefore,

$$X''+\lambda X=0 \quad , \quad X(0)=0 \quad \text{and} \quad X(L)=0$$

$$T''+a^2\lambda T=0.$$

Now, we first solve

$$X''+\lambda X=0 \quad , \quad X(0)=0 \quad \& \quad X(L)=0$$

we already solved this;

Eigenvalues $\lambda=\dfrac{n^2\pi^2}{L^2}$ $n=1,2\ldots$

Eigenfunctions $X_n=\sin\dfrac{n\pi x}{L}$ $n=1,2\ldots$

Now substitute this eigenvalue into the second differential equation

$$T''+a^2\lambda T=T^2+a^2\frac{n^2\pi^2}{L^2}T=0$$

It has characteristic equation

$$r^2+\frac{a^2n^2\pi^2}{L^2}=0$$

It's characteristic have a pair of purely imaginary complex conjugate roots

$$r = \mp \frac{a n \pi}{L} i .$$

Thus, the solutions

$$T_n(t) = A_n \cos\frac{a n \pi t}{L} + B_n \sin\frac{a n \pi t}{L} \qquad n = 1, 2 \ldots$$

From this T_n & $\cancel{X}_n$ we get

$$U_n(x,t) = X_n(x) \cdot T_n(t)$$
$$= \sin\frac{n\pi x}{L} \left[A_n \cos\frac{a n \pi t}{L} + B_n \sin\frac{a n \pi t}{L} \right]$$

As this is a solution for any n, we get

$$U(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left[A_n \cos\frac{a n \pi t}{L} + B_n \sin\frac{a n \pi t}{L} \right]$$

We have not used the initial conditions yet to find A_n & B_n.

The first initial condition $u(x,0) = X(x)\,T(0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} \left(A_n \cos(0) + B_n \sin(0) \right) \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x)$$

Hence, we again observe that the initial displacement $f(x)$ needs to be a Fourier series.

We know that the Fourier coefficient of $f(x)$ can be found

$$A_n = b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Each A_n can be found through the Fourier series of $f(x)$.

Now using the second initial conditions will give B_n:

$$u_t(x,0) = g(x).$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-A_n \frac{a n\pi}{L} \sin\frac{a n\pi t}{L} + B_n \frac{a n\pi}{L} \cos\frac{a n\pi t}{L}\right) \sin\frac{n\pi x}{L}.$$

set $t=0$;

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(-A_n \frac{a n\pi}{L} \overset{0}{\overset{\shortparallel}{\sin(0)}} + B_n \frac{a n\pi}{L} \cos(0)\right) \sin\frac{n\pi x}{L}$$

$$= g(x)$$

Therefore

$$g(x) = \sum_{n=1}^{\infty} \underbrace{B_n \frac{a n\pi}{L}}_{b_n} \sin\frac{n\pi x}{L}$$

Now we observe that $g(x)$ also needs to be a Fourier sine series. In order to find Fourier coefficient we use

$$B_n \frac{a n\pi}{L} = b_n = \frac{2}{L}\int_0^L g(x) . \sin\frac{n\pi x}{L}\, dx$$

From this we get

$$B_n = \frac{2}{a n\pi}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx.$$

<u>Example</u>! Solve the one-dimensional wave problem

$$9u_{xx} = u_{tt} \qquad 0<x<5 \;, t>0$$

$$u(0,t)=0 \quad \text{and} \quad u(5,t)=0$$

$$u(x,0)= 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x)$$

$$u_t(x,0)=0.$$

<u>Solution</u>! $a^2=9$, so $a=3$ and $L=5$.

Then the general solution is

$$u(x,t)= \sum_{n=1}^{\infty} \left(A_n \cos \frac{an\pi t}{L} + B_n \sin \frac{an\pi t}{L} \right) \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \left(A_n \cos \frac{3n\pi t}{5} + B_n \sin \frac{3n\pi t}{5} \right) \sin \frac{n\pi x}{5}.$$

Note that $u(x,0) = f(x) = 4\sin\pi x - \sin 2\pi x - 3\sin 5\pi x$ is already in the form of a Fourier sine series.

The Fourier sine series normally

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = b_1 \sin \frac{\pi x}{5} + b_2 \sin \frac{2\pi x}{5} + \cdots$$

$$+ b_5 \sin \frac{5\pi x}{5} + \cdots + b_{10} \sin \frac{10\pi x}{5}$$

$$+ \cdots + b_{25} \sin \frac{25\pi x}{5} + \cdots$$

Therefore we should get

$A_5 = b_5 = 4$, $A_{10} = b_{10} = -1$, $A_{25} = b_{25} = -3$

all other $A_n = b_n = 0$.

Hence

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{5} = 4\sin \pi x - \sin 2\pi x - 3\sin 5\pi x$$

gives us $A_5 = b_5 = 4$ | for all other

$A_{10} = b_{10} = -1$ | $A_n = b_n = 0$

$A_{25} = b_{25} = -3$

$\cancel{u(x,t) = \sum_{n=1}^{\infty} A_n \cos}$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5} \right) \sin\frac{n\pi x}{5}$$

$$= \left(A_5 \cos\frac{3.5\pi t}{5} + B_5 \sin\frac{3.5\pi t}{5} \right) \sin\frac{5\pi x}{5}$$

$$+ \left(A_{10} \cos\frac{3.10\pi t}{5} + B_{10} \sin\frac{3.10\pi t}{5} \right) \sin\frac{\pi 10x}{5}$$

$$+ \left(A_{25} \cos\frac{3.25\pi t}{5} + B_{25} \sin\frac{3.25\pi t}{5} \right) . \sin\frac{25\pi x}{5}$$

Now we use the second ~~boundary~~ initial conditions

$u_t(x,0) = g(x) = 0$

take derivative in the general solution

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(-A_n \frac{3n\pi}{5} \underbrace{\sin(0)}_{=0} + B_n \frac{3n\pi}{5} \underbrace{\cos \frac{3n\pi 0}{5}}_{=1}\right) \sin \frac{n\pi x}{5}$$

$= 0$

This is zero for any

$$u_t(x,0) = \sum_{n=1}^{\infty} - B_n \frac{3n\pi}{5} . 1 . \sin \frac{n\pi x}{5} = 0$$

iff $B_n = 0$ for all n.

Therefore implies $B_5 = B_{10} = B_{25} = 0$

$$u(x,t) = A_5 \cdot \cos 3\pi t \sin \pi x + A_{10} \cos 6\pi t \sin 2\pi x$$
$$+ A_{25} \cos 15\pi t \sin 5\pi x .$$

Use now $A_5 = 4$, $A_{10} = -1$, $A_{25} = -3$

$$u(x,t) = 4 \cos 3\pi t \sin \pi x - \cos 6\pi t \sin 2\pi x$$
$$- 3 \cos 15\pi t \sin 5\pi x .$$