Spring 2019 - Math 3150
Exam 2 - April 5
Time Limit: 50 Minutes

Name (Print): ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	0	
Total:	50	

Do not write in the table to the right.

1. Let $s_n = \sin(\frac{n\pi}{2})$ for $n \in \mathbb{N}$.

   (a) (3 points) Find the set of subsequential limits S of (s_n).

   Solution: Since
   $$(s_n) = (1, 0, -1, 0, 1, 0, -1, \ldots).$$
   We see that $S = (0, 1, -1)$.

   (b) (4 points) Find $\limsup s_n$ and $\liminf s_n$. Justify your answer.

   Solution: We know that
   $$\limsup s_n = \sup S = 1 \quad \text{and} \quad \liminf s_n = \inf S = -1.$$

   (c) (3 points) Is (s_n) a Cauchy sequence? Justify your answer.

   Solution: Since (s_n) is not a convergent sequence (as S has three elements) it is not a Cauchy sequence.

2. (a) (5 points) Carefully state the *Intermediate Value Theorem.*

Solution: Let f be a continuous real valued function. If $f(a) < f(b)$ for some a, b in domain of f with $a < b$ and if there is c such that $f(a) < c < f(b)$ then there exists $x \in (a, b)$ such that $f(x) = c$.

(b) (5 points) Let f be a real valued *continuous* function with $f : [0, 1] \mapsto [0, 1]$. Show that f has a fixed point, i.e., $f(x) = x$ for some $x \in [0, 1]$.

Solution: We consider two cases. If $f(1) = 1$ then we are done. So assume $f(1) < 1$. Also if $f(0) = 0$ then we are also done. Hence we also assume that $f(0) > 0$. We next define $g(x) = f(x) - x$. Then Since f is continuous and x is also continuous g is a continuous function. Then $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$. Hence $g(1) < 0 < g(0)$ and then by the Intermediate value theorem there is $c \in (0, 1)$ such that $g(c) = 0$. Now $g(c) = f(c) - c = 0$ gives that f has a fixed point.

3. Let f be a real valued function defined on a set $S \subset \mathbb{R}$.

   (a) (3 points) State the $\epsilon - \delta$ definition of *continuity* of f at a point $x_0 \in S$.

   Solution: Given $\epsilon > 0$ there exists δ depending on x_0 and ϵ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

   (b) (3 points) State the $\epsilon - \delta$ definition of *uniform continuity* of f on a set S. Justify the difference between a continuous function and uniformly continuous function.

   Solution: Given $\epsilon > 0$ there exists δ depending only on ϵ such that if $|x - y| < \delta$ for $x, y \in S$ then $|f(x) - f(y)| < \epsilon$.

   (c) (4 points) Given an example of a continuous function which is not uniformly continuous.

   Solution: Let $f(x) = \frac{1}{x}$ on $(0, 1)$. Now f is continuous on $(0, 1)$ but not uniformly continuous.

4. Let

$$f(x) = \begin{cases} x^2, & \text{when } x \in \mathbb{Q}, \\ 0, & \text{when } x \notin \mathbb{Q}. \end{cases}$$

(a) (5 points) Show that f is continuous at 0.

Solution: Given $\epsilon > 0$ let $\delta = \sqrt{\epsilon}$. If $x \in \mathbb{Q}$ with $|x - 0| < \delta$ then $|x|^2 < \delta^2 = \epsilon$. Hence

$$\epsilon = \delta^2 > |x^2| = |x^2 - 0| = |f(x) - f(0)|.$$

If $x \notin \mathbb{Q}$ with $|x - 0| < \delta$ then $f(x) = 0$ and $|f(x) - f(0)| = |0 - 0| = 0 < \delta^2 = \epsilon$ clearly holds. Hence f is continuous at 0.

(b) (5 points) Show that f is discontinuous at every other points.

Solution: Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. If x_0 is a rational point then we can find a sequence (x_n) of irrationals converging to x_0. But $f(x_n) = 0$ for all $n \in \mathbb{N}$ whereas $f(x_0) = x_0^2 > 0$. Hence f can not be continuous at x_0. If x_0 is an irrational point then we can find a sequence (x_n) of rationals converging to x_0 with $x_n > x_0$ if $x_0 > 0$ and $x_n < x_0$ if $x_0 < 0$ for all $n \in \mathbb{N}$. Now $x_n^2 > x_0^2$ for all $n \in \mathbb{N}$. But $f(x_n) = x_n^2 > x_0^2 > 0 = f(x_0)$. Hence $f(x_n) \not\to f(x_0)$. f is discontinuous in this case as well.

5. (10 points) Show that $f(x) = x^2 \sin(\frac{1}{x})$ is uniformly continuous on $(0,1)$.

Solution: One can directly try to show using $\epsilon - \delta$ definition. Another way to show this is to extend f to a continuous function $\tilde{f}$ on $[0,1]$ and use Theorem we proved in class. Consider

$$\tilde{f}(x) = \begin{cases} f(x) & \text{when } x \in (0,1), \\ 0 & \text{when } x = 0, \\ \sin(1) & \text{when } x = 1. \end{cases}$$

Now $\tilde{f}$ is a function defined on $[0,1]$ and $\tilde{f}(x) = f(x)$ when $x \in (0,1)$. Now we can show that $\tilde{f}$ is continuous by checking limits at $x = 0$ and $x = 1$. For $x = 0$ we can see that

$$0 \leq |x^2 \sin(\frac{1}{x})| \leq x^2$$

and as $x^2 \to 0$ as $x \to 0$ we see that f is continuous at 0. We can show that f is continuous at 1 either by checking that if (x_n) is a sequence with $x_n \to 1$ in $(0,1)$ then $\tilde{f}(x_n) = f(x_n) = x_n \sin(1/x_n) \to \sin 1$. Hence $\tilde{f}$ is continuous at 1 as well. Now $\tilde{f}$ is a continuous function on $[0,1]$ we see that it is uniformly continuous and therefore f is uniformly continuous.

6. (5 points (bonus)) Find a bounded sequence (s_n) satisfying

$$\inf\{s_n : n \in \mathbb{N}\} < \liminf s_n < \limsup s_n < \sup\{s_n : n \in \mathbb{N}\}.$$

Solution: Let $s_n = (-1)^n(1 + \frac{1}{n})$ $n \in \mathbb{N}$. It can be easily shown that $\inf\{s_n\} = -2$, $\sup\{s_n\} = 3/2$, $\limsup s_n = 1$ and $\liminf s_n = -1$.