Translation Invariance of the Lebesgue Outer Measure

Let μ_L^* be the Lebesgue outer measure, and let $E + x := \{ e + x : e \in E \}$ for $E \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$.

Then

$$\mu_L^*(E + x) = \mu_L^*(E).$$

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Proof

Let

$$I := \{ I_n : n \in \mathbb{N} \}$$

be an arbitrary set of left-closed right-open invervals such that

$$E \subset \bigcup_{n \in \mathbb{N}} I_n.$$

Then

$$\begin{aligned} E + x &\subset \left(\bigcup_{n \in \mathbb{N}} I_n \right) + x \\ &= \bigcup_{n \in \mathbb{N}} \left(I_n + x \right). \end{aligned}$$

Since μ_L^* is defined as infimum,

$$\begin{aligned} \mu_L^* \left(E + x \right) &\leq \sum_{n \in \mathbb{N}} l \left(I_n + x \right) \\ &= \sum_{n \in \mathbb{N}} l \left(I_n \right). \end{aligned}$$

Since I is arbitrary,

$$\mu_L^* \left(E + x \right) \leq \mu_L^* \left(E \right).$$

The following can be proved in the same way

$$\mu_L^* \left(E \right) \leq \mu_L^* \left(E + x \right).$$

Thus we proved.