Math Notes

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1 Envelope Theorem

Consider the following maximization problem with a single constraint.

$$\max_x f(x;a) \;\; s.t. \; g(x;a) = b. \tag{1}$$

Defined a value function as $v(a) = f(x(q);q)$. Now consider taking a derivative with respect to a parameter.

$$\frac{\partial v}{\partial a} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial a}. \tag{2}$$

The maximization problem (1) is characterized by Lagrangian

$$\mathscr{L} = f(x;a) + \lambda(b - g(x;a)).$$

The first order condition of this problem is then

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}.$$

Substitute this into (2)

$$\frac{\partial v}{\partial a} = \frac{\partial f}{\partial a} + \lambda\frac{\partial g}{\partial x}\frac{\partial x}{\partial a}. \tag{3}$$

Moreover at the optimum, the constraint is also satisfied

$$g(x(a);a) = b,$$

which implies

$$\frac{\partial g}{\partial x}\frac{\partial x}{\partial a} = -\frac{\partial g}{\partial a}.$$

By substituting this relation into (3), I establish the envelope theorem with constraint.

$$\frac{\partial v}{\partial a} = \frac{\partial f}{\partial a} - \lambda\frac{\partial g}{\partial a}.$$

1.1 Application.

Consider the following most common utility maximization problem.

$$\max_x u(x) \;\; s.t. \; p \cdot x = m.$$

Here the indirect utility is defined as

$$v(p, m) = u(x(p, m)),$$

where $x(p, m)$ is a Marshallian demand. Applying the envelope theorem I have the following result

$$\begin{aligned} \frac{\partial v}{\partial p} &= -\lambda x(p, m) \\ \frac{\partial v}{\partial m} &= \lambda. \end{aligned}$$

Combining the two equations, I establish Roy's identity.

$$x(p, m) = -\frac{\partial v / \partial p}{\partial v / \partial m}.$$

This result is coming from the fact that the utility function does not depend on paramters.