Solving RBC models with Linearized Euler Equations: Blanchard-Kahn Method

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1 Introduction

Very often, real business cycle (RBC) models or dynamic stochastic general equilibrium (DSGE) models are characterized by a set of non-linear equations, including Euler equations. In order to derive approximated optimal decision rules out of the system of non-linear equations, various methods are proposed.

The methods based on Euler equations cover substantially larger class of models since the models with complete markets but distortions (and thus non Pareto Optimal) can be characterized by Euler equations, while the method based on the value function iteration requires the economy to be Pareto Optimal, so that we can solve the problem of the Social Planner instead of directly solving the equilibrium of the model.

In this note, I am going to cover the methods which are based on linearized Euler equations and matrix decomposition. Once the non-linear equations are linearized, we have a system of linear equations, some of which are difference equations. Once the equilibrium os characterized by a system of linear equations, there are variety of methods to solve the system. The classic (and still a standard) reference is Blanchard and Kahn (1980). Burnside (1999) employs a slightly different solution method but it gives an overall picture of how the research in the area is done. King and Watson (1998) expanded the class of models that can be solved using matrix decomposition. Klein (2000) proposed a more general matrix decomposition method, based on Schur decomposition. Section 2-3 of Heer and Maussner (2005) also covers the method.

A closely-related method is to use the system of linearized equations but apply the method of undetermined coefficients instead of using matrix decomposition. By not using matrix decomposition, the method is applicable even when certain matrix operation is unavailable. A useful paper is Uhlig (1997). The method is covered in a separate note.

Below I will show how the standard real business cycle model can be solved using the linearized Euler equations and matrix decomposition.

2 The Standard RBC Model

Let's start by describing the competitive equilibrium of the standard RBC model, using the recursive formulation.

Definition 1 (Competitive equilibrium of the standard RBC model: recursive formulation) *Competitive equilibrium is the list of functions* $R(z, K)$, $W(z, K)$, $V(z, K, k)$, $g_k(z, K, k)$, $g_\ell(z, K, k)$, $G_K(z, K)$, $G_L(z, K)$, *such that:*

1. *The representative consumer solves the following problem with* $V(z, K, k)$. *The associated optimal decision rules are* $g_k(z, K, k)$, $g_\ell(z, K, k)$. *In particular, the recursive formulation of the*

consumer's problem is as follows:

$$V(z,K,k) = \max_{k'\geq 0, \ell \in [0,1]} \left\{ u(c, 1-\ell) + \beta E_{z'|z} V(z', k') \right\}$$

subject to

$$c + k' = R(z,K,L)k + W(z,K,L)\ell$$
$$L = G_L(z,K)$$
$$z' = \rho z + \epsilon' \quad \epsilon' \sim iidN(0, \sigma_\epsilon^2)$$

2. *The representative firm maximizes its profit. The consequence is that the rental price of capital and the wage are determined as follows:*

$$R(z,K,L) = e^z F_K(K,L) + 1 - \delta$$
$$W(z,K,L) = e^z F_L(K,L)$$

3. *Optimal decision rules of the consumer are consistent with the aggregate laws of motion:*

$$K' = G_K(z,K) = g_k(z,K,K)$$
$$L = G_L(z,K) = g_\ell(z,K,K)$$

For concreteness, let's give functional forms to the utility function and the production function:

$$u(1, 1-\ell) = \log(c) + \mu \log(1-\ell)$$
$$F(K,L) = K^\theta L^{1-\theta}$$

3 The Procedure

You need to take the following steps to solve a model using linearized Euler equations and matrix decomposition.

1. Find the system of (potentially non-linear) equations that characterize the model.
2. Find the steady state of the model.
3. Approximate the non-linear equations in the system around the steady state, using 1st order Taylor approximation.
4. Express the system of equations that characterize the solution using a linear system of control and (exogenous and endogenous) state variables.
5. Fit the system of equations into some matrix representation. Since the representation is closely related to the solution method, the representation differs for each solution method.
6. Derive the optimal decision rules (linear functions from the state variables to the control variables) and the laws of motion for endogenous state variables (linear function from the state variables to the state variables in the next period), using some matrix decomposition method.

We will see the steps above one by one.

4 Characterize the Solution

The Euler equation and the first order condition of the consumer's problem are:

$$\frac{1}{c} = \beta E_{z'|z} R(Z', K', L') \frac{1}{c'} \tag{1}$$

$$\frac{1}{c} W(z, K, L) = \frac{\mu}{1-\ell} \tag{2}$$

The dynamics of the capital stock and the TFP shock are characterized by the following laws of motion.

$$c + k' = R(z, K, L)k + W(z, K, L)\ell \tag{3}$$

$$z' = \rho z + \epsilon' \tag{4}$$

The interest rate and the wage are characterized by the following equations:

$$R(z, K, L) = 1 - \delta + \theta e^z K^{\theta-1} L^{1-\theta} \tag{5}$$

$$W(z, K, L) = (1-\theta) e^z K^{\theta} L^{-\theta} \tag{6}$$

Let's impose the consistency conditions so that the individual variables are all replaced by their aggregate counterparts. Besides, for simplicity of notation, let's eliminate the arguments of the functions. Thus the list of equations that characterize is as follows:

$$\frac{1}{C} = \beta E_{z'|z} \frac{R'}{C'} \tag{7}$$

$$(1-L)W = \mu C \tag{8}$$

$$C + K' = RK + WL \tag{9}$$

$$z' = \rho z + \epsilon' \tag{10}$$

$$R = 1 - \delta + \theta e^z K^{\theta-1} L^{1-\theta} \tag{11}$$

$$W = (1-\theta) e^z K^{\theta} L^{-\theta} \tag{12}$$

Notice that there are other possibilities for the choice of the variables. For example, you could use Lagrangian and use Lagrange multiplier as a variable. You could also use Lagrange multiplier and eliminate consumption out of the system after linearizing the system.

5 Finding the Steady State

Assume that in the steady state of the model, the shock z stays at its unconditional mean, which is zero. By the definition of the steady state, all the variables are assumed to be constant over time. That is:

$$\overline{z} = z' = z = 0$$

$$\overline{K} = K' = K$$
$$\overline{C} = C' = C$$
$$\overline{L} = L' = L$$
$$\overline{R} = R' = R$$
$$\overline{W} = W' = W$$

With the steady state conditions, the Euler equation becomes:

$$\overline{R} = \frac{1}{\beta}$$

This equation and the equation for the interest rate together yields:

$$\frac{\overline{K}}{\overline{L}} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\theta}\right)^{\frac{1}{\theta-1}}$$

Let's simplify the notation by denoting $\frac{\overline{K}}{\overline{L}} = a$

With $\frac{\overline{K}}{\overline{L}} = a$, the equation for wage can be easily solved for the steady state wage as follows:

$$\overline{W} = (1-\theta)a^{\theta}$$

Combining the budget constraint and the first order condition with respect to the labor supply yields:

$$(1-\overline{L})\overline{W} = \mu((\overline{R}-1)\overline{K} + \overline{WL})$$

Or:

$$(1-\overline{L})\overline{W} = \mu((\overline{R}-1)\overline{K} + \overline{WL})$$

The only unknown in this equation is $\overline{L}$, so $\overline{L}$ can be obtained by solving this equation. Once $\overline{L}$ is obtained, $\overline{K}$ is automatically obtained. Finally, $\overline{C}$ can be obtained from the budget constraint:

$$\overline{C} = \overline{R}(\overline{K} - 1) + \overline{WL}$$

6 Log-linearization

We will construct the system of linear equations which characterize the model. Basically, for every non-linear equation, you need to take the total differentiation of the equation around the steady state. However, there is an easy (and almost automatic) way to do this. What can be done instead is to replace each variable by the product of its steady state level and the deviation from it. For example, a variable X can be rewritten as follows:

$$X = \overline{X}e^{x} \simeq \overline{X}(1+x)$$

where x represents the deviation from $\overline{X}$. In addition, use the following approximation whenever there is a product of two deviation variables, let's say x and y.

$$xy \simeq 0$$

Using the method, it's easy to create a system of equations all of which are linear in deviation (lower letter) variables, rather than using the original (upper letter) variables representing the level.

For the Euler equation (7), first replace the level variables by the deviation variables. Then we get:

$$\frac{1}{\overline{C}e^{c}} = \beta E_{z'|z} \frac{\overline{R}e^{r'}}{\overline{C}e^{c'}}$$

This equation can be simplified into:

$$e^{-c} = E_{z'|z}[e^{r'} e^{-c'}]$$

Taking natural log of both sides, and we get:

$$-c = E_{z'|z}[r' - c']$$

Similarly, the first order condition (8) is:

$$(1 - \overline{L}(1+\ell))\overline{W}(1+w) = \mu \overline{C}(1+c)$$

Using the steady state conditions, this can be simplified into:

$$w + \frac{\overline{L}}{1-\overline{L}}\ell = c$$

After replacing the level variables, the budget constraint (9) is:

$$\overline{C}(1+c) + \overline{K}(1+k') = \overline{R}(1+r)\overline{K}(1+k) + \overline{W}(1+w)\overline{L}(1+\ell)$$

Using the steady state condition, we get:

$$\overline{C}c + \overline{K}k' = \overline{RK}(r+k) + \overline{WL}(w+\ell)$$

The equation for the interest rate (11) is:

$$\overline{R}(1+r) = 1 - \delta + \theta e^{z}(\overline{K}e^{k})^{\theta-1}(\overline{L}e^{\ell})^{1-\theta}$$

Using the steady state equation, it is equivalent to:

$$\overline{R}r = 1 - \delta + \theta e^{z}(\overline{K}e^{k})^{\theta-1}(\overline{L}e^{\ell})^{1-\theta} - [1 - \delta + \theta \overline{K}^{\theta-1}\overline{L}^{1-\theta}]$$

Or:

$$\overline{R}r = \theta e^{z}(\overline{K}e^{k})^{\theta-1}(\overline{L}e^{\ell})^{1-\theta} - \theta \overline{K}^{\theta-1}\overline{L}^{1-\theta}$$

Or:

$$\overline{R}r = \theta\overline{K}^{\theta-1}\overline{L}^{1-\theta}[e^z e^{k(\theta-1)}e^{\ell(1-\theta)} - 1]$$

Or:

$$\overline{R}r = \theta\overline{K}^{\theta-1}\overline{L}^{1-\theta}[z + (\theta-1)k + (1-\theta)\ell]$$

The equation for the wage (12) is:

$$\overline{W}e^w = (1-\theta)e^z(\overline{K}e^k)^\theta(\overline{L}e^\ell)^{-\theta}$$

Using the steady state condition:

$$e^w = e^z e^{\theta k}e^{-\theta\ell}$$

Taking natural log of both sides, we get:

$$w = z + \theta k - \theta\ell$$

For equation (10), it is trivial:

$$z' = \rho z + \epsilon'$$

In sum, we have the following five linear equations that characterize the solution:

$$-c = E_{z'|z}[r' - c'] \tag{13}$$
$$w + \frac{\overline{L}}{1-\overline{L}}\ell = c \tag{14}$$
$$\overline{C}c + \overline{K}k' = \overline{RK}(r+k) + \overline{WL}(w+\ell) \tag{15}$$
$$z' = \rho z + \epsilon' \tag{16}$$
$$\overline{R}r = \theta\overline{K}^{\theta-1}\overline{L}^{1-\theta}[z + (\theta-1)k + (1-\theta)\ell] \tag{17}$$
$$w = z + \theta k - \theta\ell \tag{18}$$

As a preparation to fit the system of equations into the matrix representation that Blanchard and Kahn (1980) use, let us identify the state and control variables in the equations. First of all, let's denote n_s and n_v as the number of endogenous and exogenous state variables, and m as the number of control variables. The number of exogenous state variables n_v coincides with the number of shocks, naturally. Since we have only one shock, $n_v = 1$. The number of endogenous state variables n_s coincides with the number of equations which include an expectation operator. We only have one, which is the Euler equation. It means $n_s = 1$ in our current example. There is some degree of freedom in how to determine the number of control variables m. In our current example, we have four more variables (c, ℓ, w, and r), implying that $m = 4$. However, we can substitute out two equations (for example, r and w), and leaving only c, and ℓ. In this case, we have $m = 2$. As an example, let's leave four control variables as they are. Also define $n = n_s + n_v$, which is the number of state variables (including both endogenous and exogenous ones).

7 Convert to the Matrix Representation

The method by Blanchard and Kahn (1980) requires that the system of linear equations which characterizes the solution must take the following matrix representation. Once this is done, the rest of the process is almost automatic.

$$A \begin{bmatrix} x' \\ Ey' \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix} + Cv' \tag{19}$$

where x is an $(n \times 1)$ vector of state variables, y is an $(m \times 1)$ vector of control variables, and v is an $(n_v \times 1)$ vector of shocks. Again, since the vector x contains both endogenous and exogenous state variables, $n = n_v + n_s$. The total number of equations is $n + m$. A and B are $(n+m) \times (n+m)$ matrices. C is a $(n+m) \times n_v$ matrix. E is an expectation operator with information at the current period.

Notice that there is an expectation operator only for y'. It means that an next-period state variable cannot be included in the expectation operator. If you have one endogenous state variable in the expectation operator, you need to replace by control variables, either by substituting out the endogenous state variable or choosing the endogenous state variable as another control variable.

For our current example, $n_v = 1$ (z), $n_s = 1$ (k), $n = n_v + n_s = 2$, $m = 4$ (c, ℓ, r, and w). If we fit the system of equations into the matrix representation, we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{K} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z' \\ k' \\ Ec' \\ E\ell' \\ Er' \\ Ew' \end{bmatrix} =$$

$$\begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \frac{\overline{L}}{1-\overline{L}} & 0 & 1 \\ 0 & \overline{RK} & -\overline{C} & \overline{WL} & \overline{RK} & \overline{WL} \\ \frac{\theta\overline{K}^{\theta-1}}{\overline{L}^{\theta-1}} & \frac{\theta(\theta-1)\overline{K}^{\theta-1}}{\overline{L}^{\theta-1}} & 0 & \frac{\theta(1-\theta)\overline{K}^{\theta-1}}{\overline{L}^{\theta-1}} & -\overline{R} & 0 \\ 1 & \theta & 0 & -\theta & 0 & -1 \end{bmatrix} \begin{bmatrix} z \\ k \\ c \\ \ell \\ r \\ w \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \epsilon' \end{bmatrix} \tag{20}$$

In order to avoid messy notation again, and present the solution method in a general manner, I will use the compact notation (19) from now on.

8 Solving the System of Equations

First of all, we multiply from the left A^{-1}. This requires that the matrix A is not singular. In case A is singular, methods proposed by King and Watson (1998) or Uhlig (1997) could be used. For now, assume that A is non-singular. After multiplying A^{-1} from the left, we get:

$$\begin{bmatrix} x' \\ Ey' \end{bmatrix} = A^{-1}B \begin{bmatrix} x \\ y \end{bmatrix} + A^{-1}Cv' \tag{21}$$

Let's simplify the notation by redefining the matrices:

$$\begin{bmatrix} x' \\ Ey' \end{bmatrix} = F \begin{bmatrix} x \\ y \end{bmatrix} + Gv' \tag{22}$$

F is a $(n+m) \times (n+m)$ matrix, and G is a $(n+m) \times n_v$ matrix.

Now, we apply the spectrum decomposition (Jordan decomposition) to the matrix F and get the following. The form is called the *Jordan canonical form*:

$$F = HJH^{-1} = \begin{bmatrix} d_1 & d_2 & \ldots & d_{n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_{n+m} \end{bmatrix} \begin{bmatrix} d_1 & d_2 & \ldots & d_{n+m} \end{bmatrix}^{-1} \tag{23}$$

where $\{\lambda_i\}_{i=1}^{n+m}$ are eigenvalues and the column vectors $\{d_i\}_{i=1}^{n+m}$ are associated eigenvectors. $\{\lambda_i\}_{i=1}^{n+m}$ are ordered such that

$$|\lambda_1| < |\lambda_2| < |\lambda_3| < ...|\lambda_{n+m}|$$

Let the number of eigenvalues outside the unit circle as h. Then the following can be said about the solution.

Proposition 1 (Blanchard-Kahn condition)
(i) *If $h = m$, the solution of the system is unique.* (ii) *If $h > m$, there is no solution to the system.* (iii) *If $h < m$, there is an infinity of solutions (indeterminacy).*

Assume that we have the a unique solution to the system ($h = m$). Let's partition the matrix J such that the upper-left partition contains only the eigenvalues inside the unit circle. G is also partitioned accordingly. Then we have:

$$\begin{bmatrix} x' \\ Ey' \end{bmatrix} = H \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} H^{-1} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} v' \tag{24}$$

Multiply H^{-1} from the left and we get:

$$H^{-1} \begin{bmatrix} x' \\ Ey' \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} H^{-1} \begin{bmatrix} x \\ y \end{bmatrix} + H^{-1} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} v' \tag{25}$$

Partition H and H^{-1} matrices as follows:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix}$$

In order to simplify the notation again, let:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = H^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} = H^{-1} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

Using the new notation with tildes, (25) becomes:

$$\begin{bmatrix} \tilde{x}' \\ E\tilde{y}' \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} v' \tag{26}$$

Notice two things. First, because the matrix J is diagonal, we can separate the top and bottom part of the equations nicely. Second, due to the way we reordered the eigenvectors, equations associated with the first n rows are stable (corresponding to eigenvalues inside the unit circle), while the equations associated with the remaining m rows are unstable (corresponding to eigenvalues outside the unit circle).

Look at the unstable part of the system, which is:

$$E\tilde{y}' = J_2\tilde{y} + \tilde{G}_2 v' \tag{27}$$

Solving for $\tilde{y}$ yields:

$$\tilde{y} = J_2^{-1}E\tilde{y}' - J_2^{-1}\tilde{G}_2 v' \tag{28}$$

Forwarding the equation by one period yields:

$$\tilde{y}' = J_2^{-1}E\tilde{y}'' - J_2^{-1}\tilde{G}_2 v'' \tag{29}$$

Take expectation with respect to the current period:

$$E\tilde{y}' = J_2^{-1}E\tilde{y}'' - J_2^{-1}E\tilde{G}_2 v'' \tag{30}$$

Substituting back the equation into (28) yields:

$$\tilde{y} = J_2^{-2}E\tilde{y}'' - J_2^{-2}E\tilde{G}_2 v'' - J_2^{-1}\tilde{G}_2 v' \tag{31}$$

Keep iterating forward and we obtain:

$$\tilde{y} = -J_2^{-1}\tilde{G}_2 v' - J_2^{-2}E\tilde{G}_2 v'' - J_2^{-3}E\tilde{G}_2 v''' - \ldots \tag{32}$$

In our current example (and in many macro models), $E[v'] = E[v''] = E[v'''] = \ldots = 0$. Therefore, the last equation can be simplified into:

$$\tilde{y} = -J_2^{-1}\tilde{G}_2 v' \tag{33}$$

Plugging back the formula for $\tilde{y}$ and we obtain:

$$\hat{H}_{21}x + \hat{H}_{22}y = -J_2^{-1}[\hat{H}_{21}G_1 + \hat{H}_{22}G_2]v' \tag{34}$$

Which gives:

$$y = -\hat{H}_{22}^{-1}\hat{H}_{21}x - \hat{H}_{22}^{-1}J_2^{-1}[\hat{H}_{21}G_1 + \hat{H}_{22}G_2]v' \tag{35}$$

The equation (35) gives the optimal decision rules (mappings from x and v' to y). Once we get the function from x and v' to y, we can use the law of motion for x', which is:

$$x' = F_{11}x + F_{12}y + Gv' \tag{36}$$

to obtain x' from x, v', and y. If we want to obtain the law of motion for x', as a mapping from x and v', plugging in (35) into (36) and get:

$$x' = [F_{11} - F_{12}\hat{H}_{22}^{-1}\hat{H}_{21}]x + [G - \hat{H}_{22}^{-1}J_2^{-1}[\hat{H}_{21}G_1 + \hat{H}_{22}G_2]]v' \tag{37}$$

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