

Note on Neoclassical Growth Model: Introduction

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January 2007

1 Introduction

We set up the so-called neoclassical growth model, which we are going to solve in a variety of ways. We also overview different solution methods.

2 Neoclassical Growth Model

Time is discrete, starts from 0, and infinite. There is only one agent in the economy. There is no uncertainty. There is no productivity growth.

2.1 Preference

The agent has the following utility over the consumption path.

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

$\beta \in (0, 1)$ is the time discount factor. We make the following assumptions for $u(C)$. These are standard in macro.

1. C^2
2. Strictly increasing in C
3. Strictly concave in C
4. Inada conditions

As an example, we assume the constant relative risk aversion (CRRA) utility function, which takes the following form:

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

σ is called the coefficient of relative risk aversion. It can be shown that the elasticity of intertemporal substitution (EIS) is $\frac{1}{\sigma}$ with this utility form.

2.2 Endowment

The agent is endowed with capital K_0 in period 0 and amount of time $\bar{N} = 1$ each period. Since the leisure is not valued, the agent spends all the available time in working.

2.3 Technology

Following production technology is available for the agent:

$$Y = zF(K, N)$$

where z is total factor productivity, Y is output, K is capital input, and N is labor supply. z is assumed to be constant. $F()$ satisfies usual neoclassical conditions, namely:

1. C^2
2. $F(0, 0) = 0$
3. Constant Returns to Scale (CRS)
4. Strictly increasing in both arguments (positive marginal products)
5. Strictly concave in both arguments (diminishing marginal products)
6. Inada conditions

Due to our assumption on the preference, N is always 1. As an example, we assume the most popular functional form, which is Cobb-Douglas form:

$$F(K, N) = K^\alpha N^{1-\alpha}$$

Finally, capital depreciates at a constant rate $\delta \in [0, 1]$.

2.4 Solution Concept

We are going to solve for the Pareto Optimal allocation. In other words, we will solve the problem of the social planner. However, remember that, for this economy, you can show the equivalence between the allocation of a competitive equilibrium and Pareto Optimal allocation (Basic Welfare Theorems).

So, the problem can be represented as follows:

Problem 1 (Neoclassical Growth Model: Sequential Formulation)

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to

$$C_t + K_{t+1} = zF(K_t, 1) + (1 - \delta)K_t$$

$$C_t \geq 0 \quad \forall t$$

$$K_{t+1} \geq 0 \quad \forall t$$

$$K_0 \geq 0 \text{ given}$$

3 Overview of Different Solution Methods

First of all, can we solve the problem with pencil and paper? Yes, under some very strict conditions. If we assume a perfect depreciation ($\delta = 1$) and unit EIS ($\sigma = 1$), it is known that an analytical solution exists. To learn some analytical feature of the model, this might be enough. However, obviously, it is not enough for quantitative questions. For example, we know that $\delta = 1$ is unrealistic. Therefore, we need to get out of the region of assumptions where there is an analytical solution, to be able to answer quantitative questions in a persuasive or reasonable way.

The original sequential formulation of the problem requires finding a sequence of consumption which optimizes the life-time utility function, subject to infinite number of constraints. Obviously, it is really hard to solve the problem directly. It is especially so when there is a shock to the economy. Standard ways to go are the followings:

1. **Value function iteration** Use recursive formulation (Bellman equation). The recursive formulation of the problem enables us to use dynamic programming method (value function iteration) to find a solution. We are going to see three popular methods to solve this class of problems. The difference is basically how to approximate the value function.
 - (a) Discretization (Discretized state space method)
 - (b) Finite element method
 - (c) Weighted-residual method
2. **Policy Function Iteration** Alternatively called the Euler Equation method. Use Lagrange Theorem (or Kuhn-Tucker Theorem) to obtain First Order Conditions (FOCs, intertemporal FOCs are called Euler Equations) for the problem. FOCs enable us to characterize the optimal solution in a way we can easily handle. We are going to see three solution methods. The difference is basically how to approximate the optimal decision rule.
 - (a) Solving system of equations for a optimal path
 - (b) Policy function approximation using finite element method
 - (c) Policy function approximation using weighted residual method

Notice that all the methods are categorized into the **global method**, in the sense that the solution is a valid approximation not only locally around some point in the state space (**local method**) but rather the solution is valid all over the state space. We are going to see local methods when we talk about the real business cycle models, where the economic fluctuation is considered to be a perturbation around some steady state (or balanced growth path) of the economy.