1 A Quantitative Theory of Unsecured Consumer Credit With Risk of Default, Supplementary Material: Proofs of Lemmas 7, 19-21, and 23

This document provides the proof of Lemma 7 and the proofs of Lemmas 19-21 and 23 which were omitted from the Appendix to the main article. The notation is the same as that used in the Appendix and equation numbers refer to equations in this document or to equations in the main article or in the Appendix to the main article.

Lemma A7. The goods market clearing condition (ixA) is implied by the other conditions for an equilibrium in Definition 2.

Proof: First note that the household budget sets (2)-(5) imply

$$\begin{aligned}
&c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s} \cdot \ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*) \cdot \left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right] \\
&= \left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + (\ell - \zeta(s))\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right].
\end{aligned}$$

Then aggregating over all households yields

$$\begin{aligned}
&\int \left\{c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]d\mu^*\right\} \\
&+ \int \left\{\zeta(s)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* \\
&= \int \left\{\left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^*.
\end{aligned} \tag{47}$$

Condition (v) along with (47) imply

$$\begin{aligned}
&\int \left\{c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]d\mu^*\right\} \\
&+ \int \left\{\zeta(s)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* \\
&= \int \left\{\left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* \\
&+ \int \left\{\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\zeta(s) + \ d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\max\{\ell,0\} - \zeta(s)/m^*\right\}d\mu^*
\end{aligned}$$

or

$$\begin{aligned}
&\int \left\{c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* + \int \frac{\zeta(s)}{m^*}d\mu^*. \\
&= \int \left\{\left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* \\
&+ \int \left\{d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\max\{\ell,0\}\right\}d\mu^*
\end{aligned} \tag{48}$$

Since $d^*_{\ell,h,s}(e;\alpha,q^*,w^*) = 1$ implies $\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*) = 0$, it follows that the product of $\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)$ and $d^*_{\ell,h,s}(e;\alpha,q^*,w^*)$ is 0 for all ℓ, h, s, e. Hence, the left hand side of (48) can be written

$$\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int \frac{\zeta(s)}{m^*}d\mu^*.$$

Next the first term on the right hand side can be written

$$w^*\left[\int e d\mu^* - \gamma\int e\mu^*(d\ell,1,ds,de) - \alpha\int(e-e_{\min})\cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de)\right]$$

Finally, the remaining term on the right hand side of (48) can be written

$$\begin{aligned}
&\sum_{\ell,s}\ell\int(1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*))\mu^*(\ell,dh,s,de) + \sum_{\ell\geq 0,s}\int d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\ell\mu^*(\ell,dh,s,de)\\
&= \sum_{\ell,s}\ell\int\mu^*(\ell,dh,s,de) - \sum_{\ell<0,s}\ell\int d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\mu^*(\ell,dh,s,de)\\
&= \sum_{\ell>0,s}\ell\int\mu^*(\ell,dh,s,de) + \sum_{\ell<0,s}\ell\int(1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*))\mu^*(\ell,dh,s,de) \qquad (49)
\end{aligned}$$

Next, observe that for $x \neq 0$, we have from (x), (6), and (vii)

$$\begin{aligned}
&\int\mu^*(x,dh',\tilde{s},de';q^*,w^*)\\
&= \rho\int\left[\mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=x\}}\sum_{h'}H^*(\ell,h,s,e;h')\int_E\Phi(e'|\sigma)de'\Gamma(s;\sigma)\right]d\mu^*\\
&= \rho\int\left[\mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=x\}}\Gamma(s;\sigma)\right]\mu^*(d\ell,dh,ds,de)\\
&= \rho\sum_s a^*_{x,s}\Gamma(s;\tilde{s}),
\end{aligned}$$

where for ease of notation we have replaced s_{-1} with $\tilde{s}$. Hence, the first term in (49):

$$\begin{aligned}
\sum_{x>0,\tilde{s}}x\int\mu^*(x,dh,\tilde{s},de) &= \sum_{x>0,\tilde{s}}x\rho\sum_s a^*_{x,s}\Gamma(s;\tilde{s})\\
&= \rho\sum_{x>0,s}xa^*_{x,s}\sum_{\tilde{s}}\Gamma(s;\tilde{s})\\
&= \rho\sum_{x>0,s}xa^*_{x,s}
\end{aligned}$$

Now consider the second term in (49):

$$\begin{aligned}
&\int(1-d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*))\mu^*(x,dh,\tilde{s},de)\\
&= \int\mu^*(x,dh,\tilde{s},de) - \int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de)
\end{aligned}$$

We can re-write the latter part of this expression as

$$\int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de;\alpha,q^*,w^*) =$$

$$\rho \int \left[\mathbf{1}_{\{(\ell,\eta,s,\varepsilon):(\ell'^*_{\ell,\eta,s}(\varepsilon;q^*,w^*)=x\}} \sum_h H(\ell,\eta,s,\varepsilon;h) \int_E d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de\Gamma(s;\tilde{s}) \right] \mu^*(d\ell,d\eta,ds,d\varepsilon).$$

Since $x<0$, it follows that $\eta=0$ and $h=0$ so that $H(\ell,0,s,\varepsilon;0)=1$ and $H(\ell,0,s,\varepsilon;1)=0, \forall \ell,s,\varepsilon$. Therefore

$$\begin{aligned}
&\int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de;\alpha,q^*,w^*) \\
&= \rho \int \left[\mathbf{1}_{\{(\ell,0,s,\varepsilon):(\ell'^*_{\ell,0,s}(\varepsilon;q^*,w^*)=x\}} \int_E \sum_h H(\ell,0,s,\varepsilon;h) d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de\Gamma(s;\tilde{s}) \right] \mu^*(d\ell,0,ds,d\varepsilon) \\
&= \rho \int \left[\mathbf{1}_{\{(\ell,0,s,\varepsilon):(\ell'^*_{\ell,0,s}(\varepsilon;q^*,w^*)=x\}} \int_E d^*_{x,0,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de\Gamma(s;\tilde{s}) \right] \mu^*(d\ell,0,ds,d\varepsilon).
\end{aligned}$$

Let $p^{*\tilde{s}}_x = \int_E d^*_{x,0,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de$ be the probability of default on a loan of size x by households with characteristic $\tilde{s}$. Then

$$\begin{aligned}
&\int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de;\alpha,q^*,w^*) \\
&= \sum_s \rho \int \left[\mathbf{1}_{\{(\ell,0,s,e):(\ell'^*_{\ell,0,s}(e;\alpha,q^*,w^*)=x\}} p^{*\tilde{s}}_x \Gamma(s;\tilde{s}) \right] \mu^*(d\ell,0,s,de;\alpha,q^*,w^*) \\
&= \rho \sum_s p^{*\tilde{s}}_x \Gamma(s;\tilde{s}) a^*_{x,s}.
\end{aligned}$$

The second equality follows from (vii) recognizing that $\mu^*(Z)=0$ for all $Z \in L_{--} \times \{1\} \times S \times \mathcal{B}(E)$. Thus the second part of (49) can be written

$$\begin{aligned}
&\sum_{x>0,\tilde{s}} x \int \mu^*(x,dh,\tilde{s},de) + \sum_{x<0,\tilde{s}} x \int (1-d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*))\mu^*(x,dh,\tilde{s},de) \\
&= \rho \sum_{x>0,s} x a^*_{x,s} + \rho \sum_{x<0,s} x a^*_{x,s} - \sum_{x<0,s} x\rho \sum_{\tilde{s}} p^{*\tilde{s}}_x \Gamma(s;\tilde{s}) a^*_{x,s} \\
&= \rho \left[\sum_{x>0,s} x a^*_{x,s} + \sum_{x<0,s} x a^*_{x,s}(1-p^*_{x,s}) \right].
\end{aligned}$$

Thus, re-writing (48) we have

$$\begin{aligned}
&\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s} \ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int \frac{\zeta(s)}{m^*} d\mu^* \\
&= w^* \int e d\mu^* - \gamma w^* \int e\mu^*(d\ell,1,ds,de) - \alpha w^* \int (e-e_{\min}) \cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de) \\
&+ \rho \sum_{\ell,s} \ell a^*_{\ell,s}(1-p^*_{\ell,s}).
\end{aligned}$$

But

$$\int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* = \sum_{\ell'}\int \mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=\ell'\}}q_{\ell',s}\ell'\mu^*(d\ell,dh,ds,de)$$
$$= \sum_{\ell',s} q^*_{\ell',s}a^*_{\ell',s}\ell'$$
$$= K^*$$

where the last inequality follows from (20). Another implication of (20) is

$$(1+r^*-\delta)\,K^* = \rho\sum_{(\ell',s)\in L\times S}(1-p^*_{\ell',s})a^*_{\ell',s}\ell'.$$

Thus, we have

$$\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + K^* + \int\frac{\zeta(s)}{m^*}d\mu^*$$
$$= w^*N^* - \gamma w^*\int e\mu^*(d\ell,1,ds,de) + (1+r^*-\delta)\,K^*$$
$$= F(N^*,K^*) + (1-\delta)\,K^* - \gamma w^*\!\int\! e\mu^*(d\ell,1,ds,de) - \alpha w^*\!\int\!(e-e_{\min})\cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de).$$

So that the goods market clears. ■

Lemma A19. (i) $\Phi_s(\overline{ES}^{(\ell',d)}) \le \bar{x}^{(\ell',d)}_{\ell,h,s}$ (ii) $\Phi_s(\overline{ED}^{(\ell',d)}) \le (1-\bar{x}^{(\ell',d)}_{\ell,h,s})$ and (iii) $\sum_{\ell'\in L}\Phi_s(\overline{ES}^{(\ell',0)}) + \Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) = 1 = \sum_{\ell'\in L}\bar{x}^{(\ell',0)}_{\ell,h,s} + \bar{x}^{(0,1)}_{\ell,h,s}$.

Proof. To prove (i) we first establish that $\overline{ES}^{(\ell',d)} \subseteq \cup_{m=1}^{\infty}\left(\cap_{k\ge m}ES_k^{(\ell',d)}\right)$. Consider $\hat{e}\in\overline{ES}^{(\ell',d)}$. Then $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};0,\bar{q},\bar{w}) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)}\phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};0,\bar{q},\bar{w}) > 0$. By Lemma A2 it follows that there exists $N(\hat{e})$ such that for all $m\ge N(\hat{e})$, $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};\alpha_m,q_m,w_m) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)}\phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};\alpha_m,q_m,w_m) > 0$. Therefore, $\hat{e}\in\cap_{k\ge N(\hat{e})}ES_k^{(\ell',d)}$. Hence we must have $\hat{e}\in\cup_{m=1}^{\infty}\left(\cap_{k\ge m}ES_k^{(\ell',d)}\right)$. Next, observe that for each m, $\cap_{k\ge m}ES_k^{(\ell',d)}$ is Borel measurable since it is a countable intersection of Borel measurable sets. Therefore, $\Phi_s(\overline{ES}^{(\ell',d)}) \le \Phi_s\left(\cup_m\left(\cap_{k\ge m}ES_k^{(\ell',d)}\right)\right) = \lim_{m\to\infty}\Phi_s(\cap_{k\ge m}ES_k^{(\ell',d)})$. The last equality follows because the sets $\cap_{k\ge m}ES_k^{(\ell',d)}$ are increasing in m. Next, observe that $\Phi_s(\cap_{k\ge m}ES_k^{(\ell',d)}) \le \Phi_s(ES_m^{(\ell',d)}) = x^{(\ell'd)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m)$, where the last equality follows from Lemma A8 which implies the set $E_m^{(\ell',d)}\cap\left(ES_m^{(\ell',d)}\right)^c$ is finite and therefore of Φ_s-measure 0. Thus, $\lim_{m\to\infty}\Phi_s(\cap_{k\ge m}ES_k^{(\ell',d)}) \le \lim_{m\to\infty}x^{(\ell',d)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m) = \bar{x}^{(\ell',d)}_{\ell,h,s}$. Therefore $\Phi_s(\overline{ES}^{(\ell',d)}) \le \bar{x}^{(\ell',d)}_{\ell,h,s}$. This establishes (i).

To prove (ii) we first establish that $\overline{ED}^{(\ell',d)} \subseteq \cup_{m=1}^{\infty}\left(\cap_{k\ge m}ED_k^{(\ell',d)}\right)$. Consider $\hat{e}\in\overline{ED}^{(\ell',d)}$. Then $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};0,\bar{q},\bar{w}) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)}\phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};0,\bar{q},\bar{w}) < 0$. By Lemma A2, there exists $N(\hat{e})$ such that for all $m \geq N(\hat{e})$, $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};\alpha_m,q_m,w_m) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)} \phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};\alpha_m,q_m,w_m) < 0$. Therefore, $\hat{e} \in \cap_{k\geq N(\hat{e})} ED^{(\ell',d)}_k$. Hence we must have $\hat{e} \in \cup_{m=1}^{\infty}\left(\cap_{k\geq m} ED^{(\ell',d)}_k\right)$. Next, observe that for each m, $\cap_{k\geq m} ED^{(\ell',d)}_k$ is Borel measurable since it is a countable intersection of Borel measurable sets. Therefore, $\Phi_s(\overline{ED}^{(\ell',d)}) \leq \Phi_s\left(\cup_m\left(\cap_{k\geq m} ED^{(\ell',d)}_k\right)\right) = \lim_{m\to\infty}\Phi_s(\cap_{k\geq m} ED^{(\ell',d)}_k)$. The last equality follows because the sets $\cap_{k\geq m} ED^{(\ell',d)}_k$ are increasing in m. Next, observe that $\Phi_s(\cap_{k\geq m} ED^{(\ell',d)}_k) \leq \Phi_s(ED^{(\ell',d)}_m) = 1 - x^{(\ell'd)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m)$, where the last equality follows from Lemma A8 which implies $\left(E^{(\ell',d)}_m\right)^c \cap \left(ED^{(\ell',d)}_m\right)^c$ is a finite set and therefore of Φ_s-measure 0. Thus $\lim_{m\to\infty}\Phi_s(\cap_{k\geq m} ES^{(\ell',d)}_k) \leq \lim_{m\to\infty}[1 - x^{(\ell',d)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m)] = 1 - \bar{x}^{(\ell',d)}_{\ell,h,s}$. Therefore $\Phi_s(\overline{ED}^{(\ell',d)}) \leq 1 - \bar{x}^{(\ell',d)}_{\ell,h,s}$. This establishes (ii).

To prove (iii), consider the set $\left(\cup_{\ell'\in L}\overline{ES}^{(\ell',0)} \cup \overline{ES}^{(0,1)} \cup \overline{I}^{(0,1)}\right)^c$. A member of this set is any e for which there is more than one optimal action none of which involve default. By Lemma A8 this is a finite set and therefore of Φ_s-measure 0. Hence $\Phi_s\left(\cup_{\ell'\in L}\overline{ES}^{(\ell',0)} \cup \overline{ES}^{(0,1)} \cup \overline{I}^{(0,1)}\right) = 1$. Since any pair of sets in the union is disjoint, it follows that $\sum_{\ell'\in L}\Phi_s(\overline{ES}^{(\ell',0)}) + \Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) = 1$. Next, consider the set $\left(\cup_{\ell'\in L} ES^{(\ell',0)}_m \cup ES^{(0,1)}_m\right)^c$. A member of this set is any e for which there is more than one optimal action. By Lemma A8 again this is a finite set. Therefore $\Phi_s\left(\cup_{\ell'\in L} ES^{(\ell',0)}_m \cup ES^{(0,1)}_m\right) = 1$. Since any pair of sets in this union is disjoint, it follows that $\sum_{\ell'\in L}\Phi_s(ES^{(\ell',0)}_m) + \Phi_s(ES^{(0,1)}_m) = 1$. Since $ES^{(\ell',d)}_m$ and $E^{(\ell',d)}_m$ can differ by at most a finite set of points (by Lemma A8), it follows that $\sum_{\ell'\in L} x^{(\ell',0)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m) + x^{(0,1)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m) = 1$. Taking limits on both sides yields $\sum_{\ell'\in L}\bar{x}^{(\ell',0)}_{\ell,h,s} + \bar{x}^{(0,1)}_{\ell,h,s} = 1$. This establishes (iii). ■

Lemma A20. For all $(\ell,h,s)\in\mathcal{L}$ there exist measurable functions $c_{\ell,h,s}(e)$, $\ell'_{\ell,h,s}(e)$, and $d_{\ell,h,s}(e)$ for which the implied choice probabilities $\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}}\Phi(de|s) = \overline{x}^{(\ell',d)}_{(\ell,h,s)}$ and the triplet $\left(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\right) \in \chi_{\ell,h,s}(e;0;\overline{q},\overline{w})$.

Proof. The decision rules are constructed for two mutually exclusive cases. First, consider the case where $\Phi_s(\bar{I}^{(0,1)}) = 0$. For this case construct the decision rules as follows. Assign to action (ℓ',d) all e such that $e \in \overline{ES}^{(\ell',d)}$. This step leaves unassigned the set $\overline{I}^{0,1} \cup \left(\cup_{\ell'\in L}\overline{I}^{(\ell',0)}\right)$. To complete the assignment, assign all elements of $\overline{I}^{0,1}$ to $(0,1)$ and assign any remaining elements to actions in any manner provided that each element is assigned to an action only once and an element is assigned to an action (ℓ',d) only if it belongs to $\overline{I}^{(\ell',d)}$. Since $\overline{ES}^{(\ell',d)}$ are disjoint, the assignment maps each e to exactly one action (ℓ',d). Let $\ell'_{\ell,h,s}(e)$ and $d_{\ell,h,s}(e)$ be the resulting decision rules for ℓ' and d and let $c_{\ell,h,s}(e)$ be the decision rule for c implied by the household budget constraint given $\ell'_{\ell,h,s}(e)$ and $d_{\ell,h,s}(e)$.

We will now establish that these decision rules are measurable, optimal and imply the limiting choice probability vector $\bar{x}$. To establish measurability it is sufficient to establish that for each action (ℓ',d) the set $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}$ is Borel measurable. For $(0,1)$, the corresponding set is the union of $\overline{ES}^{(0,1)}$ and $\overline{I}^{(0,1)}$ both of which are Borel measurable and therefore the union is Borel measurable. Furthermore, $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = 0 \text{ and } d_{\ell,h,s}(e) = 1\}\right) = \Phi_s\left(\overline{ES}^{(0,1)}\right)$ since $\Phi_s(\overline{I}^{(0,1)}) = 0$. For $(\ell, 0)$, the corresponding set is the union of $\overline{ES}^{(\ell',0)}$, which is Borel measurable, and some subset of $\overline{I}^{(\ell',0)}$. By Lemma A8, $\overline{I}^{(\ell',0)}$ is a finite set and therefore any subset of it is Borel measurable. Hence $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = 0\}$ is also a union of Borel measurable sets and therefore Borel measurable. Furthermore, $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = 0\}\right) = \Phi_s\left(\overline{ES}^{(\ell',0)}\right)$ since $\Phi_s(\overline{I}^{(\ell,0)}) = 0$ (being a finite set). The decision rules are optimal by construction. Finally, note that by Lemma A19(iii) we have $\sum_{\ell' \in L}[\Phi_s(\overline{ES}^{(\ell,0)}) - \bar{x}^{(\ell',0)}_{\ell,h,s}] + [\Phi_s(\overline{ES}^{(0,1)}) - \bar{x}^{(0,1)}_{\ell,h,s}] = 0$. By Lemma A19(i) each term in this sum is nonnegative. It follows immediately that $\Phi_s(\overline{ES}^{(\ell',d)}) = \bar{x}^{(\ell',d)}_{\ell,h,s}$. Hence, $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}\right) = \Phi_s\left(\overline{ES}^{(\ell',d)}\right) = \bar{x}^{(\ell',d)}_{\ell,h,s}$.

Next, consider the case where $\Phi_s(\overline{I}^{(0,1)}) = \delta > 0$. The assignment has to distribute members $\overline{I}^{(0,1)}$ in such a way that choice probabilities induced by the assignment are the limiting choice probabilities $\bar{x}$. To begin, we first claim that there must exist exactly one action $(\hat{\ell}', 0)$ for which $\overline{I}^{(0,1)} = \overline{I}^{(\hat{\ell}',0)}$. Suppose there were two such actions $(\hat{\ell}', 0)$ and $(\tilde{\ell}', 0)$. Then, $I^{(\hat{\ell}',0),(\tilde{\ell}',0)}_{\ell,h,s}(0, \bar{q}, \bar{w}) \supseteq \overline{I}^{(0,1)}$ implying that $I^{(\hat{\ell}',0),(\tilde{\ell}',0)}_{\ell,h,s}(0, \bar{q}, \bar{w})$ has strictly positive measure which, by Lemma A8, is impossible.

Next, we claim that $\Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) + \Phi_s(\overline{ES}^{(\hat{\ell}',0)}) = \bar{x}^{(0,1)}_{\ell,h,s} + \bar{x}^{(\hat{\ell},0)}_{\ell,h,s}$. To see this, suppose that $\Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) + \Phi_s(\overline{ES}^{(\hat{\ell}',0)}) < \bar{x}^{(0,1)} + \bar{x}^{(\hat{\ell},0)}$. But by Lemma A19(iii) this implies that $\sum_{\ell' \neq \hat{\ell}'} \Phi_s(\overline{ES}^{(\ell',0)}) > \sum_{\ell' \neq \hat{\ell}'} \bar{x}^{(\ell',0)}_{\ell,h,s}$, which contradicts the bound in Lemma A19(i). Suppose then that $\Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) + \Phi_s(\overline{ES}^{(\hat{\ell}',0)}) > \bar{x}^{(0,1)} + \bar{x}^{(\hat{\ell},0)}$. By Lemma A19(iii) $\sum_{\ell' \neq \hat{\ell}'} \Phi_s(\overline{ES}^{(\ell',0)}) < \sum_{\ell' \neq \hat{\ell}'} \bar{x}^{(\ell',0)}_{\ell,h,s}$. But this implies $\sum_{\ell' \neq \hat{\ell}'}[1 - \Phi_s(\overline{ES}^{(\ell',0)})] > \sum_{\ell' \neq \hat{\ell}'}[1 - \bar{x}^{(\ell',0)}_{\ell,h,s}]$, which contradicts the bound in Lemma A19(ii). This establishes the claim.

We can now proceed with the assignment. To (ℓ', d) distinct from $(0, 1)$ or $(\hat{\ell}', 0)$, assign all e such that $e \in \overline{ES}^{(\ell',0)}$. Next, partition the set $\overline{I}^{(0,1)}$ into two disjoint (measurable) sets I_1 and I_2 such that $\Phi_s\left(\overline{ES}^{(\hat{\ell},0)} \cup I_1\right) = \bar{x}^{(\hat{\ell},0)}_{\ell,h,s}$ and $\Phi_s\left(\overline{ES}^{(0,1)} \cup I_2\right) = \bar{x}^{(0,1)}_{\ell,h,s}$ (since Φ_s is atomless such a partition exists). Finally, assign in any manner all remaining elements provided that each element is assigned to an action only once and an element is assigned to an action (ℓ', d) only if it belongs to $\overline{I}^{(\ell',d)}$.

These assignments assign each e to exactly one action (ℓ, d) and therefore imply decision rules $\ell'_{\ell,h,s}(e)$, $d_{\ell,h,s}(e)$ and, via the household budget constraint, $c_{\ell,h,s}(e)$. The measurability of these decision rules can be established by expressing the sets $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}$ as unions of measurable sets as was done for the first case. By construction, the decision rules are optimal. Finally, note that by our earlier claim $\sum_{\ell' \neq \hat{\ell}'}[\Phi_s(\overline{ES}^{(\ell',0)}) - \bar{x}^{(\ell',0)}_{\ell,h,s}] = 0$. By Lemma A19(i) each term in this sum is nonnegative and, therefore, $\Phi_s(\overline{ES}^{(\ell',0)}) = \bar{x}^{(\ell',0)}_{\ell,h,s}$ for $\ell' \neq \hat{\ell}'$. Hence,

$\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}\right) = \Phi_s\left(\overline{ES}^{(\ell',d)}\right) = \bar{x}^{(\ell',d)}_{\ell,h,s}$ where the first equality uses the fact that the set $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}$ differs from the set $\overline{ES}^{\ell',d)}$ by at most a finite set of points. Finally, by construction $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = \hat{\ell}' \text{ and } d_{\ell,h,s}(e) = 0\}\right) = \bar{x}^{(\hat{\ell}',0)}_{\ell,h,s}$ and $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = 0 \text{ and } d_{\ell,h,s}(e) = 1\}\right) = \bar{x}^{(0,1)}_{\ell,h,s}$. ■

We now establish the analogs of Lemma A12 and A15 for the sequence $\{\alpha_m, q^*_m, w^*_m\}$ converging to $(0, \overline{q}, \overline{w})$.

Lemma A21. Let $\overline{\pi}_{(0,\overline{q},\overline{w})}$ be the invariant distribution of the Markov chain $\overline{P}$ defined by the decision rules $(\ell'_{\ell,h,s}(e), d_{\ell,h,s}(e))$. Then the sequence $\pi_{(\alpha_m,q^*_m,w^*_m)}$ converges weakly to $\overline{\pi}_{(0,\overline{q},\overline{w})}$.

Proof. We apply Theorem 12.13 in Stokey, Lucas, and Prescott (1989). Part a of the requirements follows since $\mathcal{L}$ is compact. Part b requires that $P^*_{(\alpha_m,q^*_m,w^*_m)}[(\ell_n, h_n, s_n), \cdot]$ converge weakly to $\overline{P}_{(0;\overline{q},\overline{w})}[(\ell, h, s), \cdot]$ as $(\ell_n, h_n, s_n, \alpha_m, q^*_m, w^*_m) \to (\ell, h, s, 0, \overline{q}, \overline{w})$. By Theorem 12.3d of Stokey, Lucas, and Prescott (1989) it is sufficient to show that for any (ℓ', h', s'),

$$\lim_{k\to\infty} P^*_{(\alpha_m,q^*_m,w^*_m)}\left[(\ell_n, h_n, s_n), (\ell', h', s')\right] = \overline{P}_{(0;\overline{q},\overline{w})}\left[(\ell, h, s), (\ell', h', s')\right].$$

By definition

$$\begin{aligned} &P^*_{(\alpha_m,q^*_m,w^*_m)}\left[(\ell, h, s), (\ell', h', s')\right] \\ &= \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}} H^*_{(\alpha_m,q^*_m,w^*_m)}(\ell, h, s, e, h')\Phi(de|s)\Gamma(s, s') \\ +(1-\rho)\int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s', de') \end{array}\right] \end{aligned}$$

and

$$H^*_{(q,w)}(\ell, h, s, e, h' = 1) = \begin{cases} 1 & \text{if } d^*_{\ell,h,s}(e; q, w) = 1 \\ \lambda & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 1 \\ 0 & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 0 \end{cases},$$

$$H^*_{(q,w)}(\ell, h, s, e, h' = 0) = \begin{cases} 0 & \text{if } d^*_{\ell,h,s}(e; q, w) = 1 \\ 1-\lambda & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 1 \\ 1 & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 0 \end{cases}.$$

By construction, the Markov chain $\overline{P}$ is

$$\begin{aligned} &\overline{P}\left[(\ell, h, s), (\ell', h', s')\right] \\ &= \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell'\}} H^*_{(0,\overline{q},\overline{w})}(\ell, h, s, e, h')\Phi(de|s)\Gamma(s, s') \\ +(1-\rho)\int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s', de') \end{array}\right] \end{aligned}$$

where $H^*_{(0,\overline{q},\overline{w})}(\ell, h, s, e, h')$ is determined by $d_{\ell,h,s}(e)$.

Since $\mathcal{L}$ is finite, without loss of generality consider the sequence $(\alpha_m, q^*_m, w^*_m) \to (0, \overline{q}, \overline{w})$. Since the second term on the r.h.s. is independent of (α, q, w), it is sufficient to consider the limiting behavior of the integral

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}} H^*_{(\alpha_m,q^*_m,w^*_m)}(\ell, h, s, e, h')\Phi(de|s).$$

For $h = 0$ and $h' = 0$, this integral in P^* is

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=0\}} \Phi(de|s) = x^{(\ell',0)}_{(\ell,0,s)}(\alpha_m, q^*_m, w^*_m)$$

and in $\overline{P}$ it is

$$\int_E \mathbf{1}_{\{\ell'_{\ell,0,s}(e)=\ell'\}} H^*_{(0,\overline{q},\overline{w})}(\ell, 0, s, e, 0)\Phi(de|s) = \int_E \mathbf{1}_{\{\ell'_{\ell,0,s}(e)=\ell', d_{\ell,0,s}(e)=0\}} \Phi(de|s).$$

By Lemma A20 we have

$$\lim_{k\to\infty} x^{(\ell',0)}_{(\ell,0,s)}(\alpha_m, q^*_m, w^*_m) = \overline{x}^{(\ell',0)}_{(\ell,0,s)} = \int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=0\}} \Phi(de|s).$$

Hence

$$\lim_{k\to\infty} P^*_{(\alpha_m,q^*_m,w^*_m)} \left[(\ell, 0, s), (\ell', 0, s')\right] = \overline{P}_{(0;\overline{q},\overline{w})} \left[(\ell, 0, s), (\ell', 0, s')\right].$$

The remaining cases can be dealt with in exactly the same way. We simply note here which choice probabilities are involved in each case and omit the details.

For $h = 0$ and $h' = 1$, the integral in P^* is

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,0,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,0,s}(e;\alpha_m,q^*_m,w^*_m)=1\}} \Phi(de|s) = x^{(\ell',1)}_{(\ell,0,s)}(\alpha_m, q^*_m, w^*_m).$$

For $h = 1$ and $h' = 0$, the integral in P^* is

$$(1 - \lambda) \int_E \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=0\}} \Phi(de|s) = x^{(\ell',0)}_{(\ell,0,s)}(\alpha_m, q^*_m, w^*_m).$$

For $h = 1$ and $h' = 1$, the integral in P^* is

$$\int_E \left[\begin{array}{c} \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=1\}} \\ +\lambda\mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=0\}} \end{array} \right] \Phi(de|s) = \left[\begin{array}{c} x^{(\ell',1)}_{(\ell,1,s)}(\alpha_m, q^*_m, w^*_m) \\ +\lambda x^{(\ell',0)}_{(\ell,1,s)}(\alpha_m, q^*_m, w^*_m) \end{array} \right].$$

■

Lemma A23. Let $K_{(0,\overline{q},\overline{w})} \equiv \sum_{(\ell',s)\in L\times S} \ell'\overline{q}_{\ell',s} \int \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}}\overline{\mu}_{(0,\overline{q},\overline{w})}(d\ell, dh, s, de)$, $N_{(0,\overline{q},\overline{w})} \equiv \int ed\overline{\mu}_{(0,\overline{q},\overline{w})}$, and $p_{(0,\overline{q},\overline{w})}(\ell', s) \equiv \int d_{\ell',0,s'}(e')\Phi(e'|s')\Gamma(s; ds')de'$. Then (i) $\lim_m K(\alpha_m, q^*_m, w^*_m) = K_{(0,\overline{q},\overline{w})}$, (ii) $\lim_m N(\alpha_m, q^*_m, w^*_m) = N_{(0,\overline{q},\overline{w})}$, and (iii) $\lim_m p_{(\alpha_m,q^*_m,w^*_m)}(\ell', s) = p_{(0,\overline{q},\overline{w})}(\ell', s)$.

Proof. To prove (i), note that we know by Lemma A13,

$$\begin{aligned}
&\int_{L\times H\times E} \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}} \mu_{(\alpha_m,q^*_m,w^*_m)}(d\ell, dh, s, de) \\
&= \sum_{\ell,h} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}} \Phi(de|s)\pi_{(\alpha_m,q^*_m,w^*_m)}(\ell, h, s)
\end{aligned}$$

By Lemma A20

$$
\begin{aligned}
&\lim_{n_k\to\infty}\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=d\}}\Phi(de|s)\\
&=\int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell',d_{\ell,h,s}(e)=d\}}\Phi(de|s).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}}\Phi(de|s)\\
&=\sum_{d\in\{0,1\}}\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=d\}}\Phi(de|s),
\end{aligned}
$$

then

$$
\begin{aligned}
&\lim_{n_k\to\infty}\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}}\Phi(de|s)\\
&=\sum_{d\in\{0,1\}}\int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell',d_{\ell,h,s}(e)=d\}}\Phi(de|s)=\int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}}\Phi(de|s).
\end{aligned}
$$

Next, by Lemma A21,

$$
\lim_{n\to\infty}\pi_{(\alpha_m,q^*_m,w^*_m)}(\ell,h,s)=\pi_{(0,\overline{q},\overline{w})}(\ell,h,s).
$$

Therefore $\lim_{n_k\to\infty}K_{(\alpha_m,q^*_m,w^*_m)}=K_{(0,\overline{q},\overline{w})}$. To prove (ii) simply apply Lemma A21. To prove (iii), note that by Lemma A20

$$
\lim_{n_k\to\infty}\int_E d^*_{\ell',0,s'}(e';\alpha_m,q^*_m,w^*_m)\Phi(de'|s')=\int_E d_{\ell',0,s'}(e')\Phi(de'|s').
$$

Thus,

$$
\begin{aligned}
&\lim_{m\to\infty}\int_{E\times S} d^*_{\ell',0,s'}(e';\alpha_m,q^*_m,w^*_m)\Phi(de'|s')\Gamma(s;ds')\\
&=\int_{E\times S} d_{\ell',0,s'}(e')\Phi(de'|s')\Gamma(s;ds').
\end{aligned}
$$

■

References

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.