Solving RBC models with L-Q approximation

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1 Introduction

This is a classic method of solving dynamic stochastic general equilibrium models. A good reference is Díaz-Gimenez (1999). I will follow the exposition of the chapter, though I will use the standard RBC model as the example, instead of using the stochastic growth model without labor-leisure choice.

Basically, the method locally approximates the period utility function using a quadratic function, when the utility function is not quadratic. If the utility function is (approximated as) quadratic, we know that the value function is going to be quadratic, too. If the period utility function is (approximated as) quadratic, value function is quadratic, and constraints are linear, we know that the optimal decision rules are going to be linear in state variables. Therefore, it's very easy to find the optimal decision rule, with a guess of the value function. Once we solve the optimization problem, we can update the value function using Bellman equation. We keep iterating on the value function until we get the convergence.

The method is easy to implement, but it is usually only applicable to the economy where the welfare theorems hold; the competitive equilibrium is Pareto Optimal. This is because the method is based on the value function iteration and thus can be used only to solve the Social Planner's problem. For solving equilibrium of wider class of economies, in particular models with distortions, various methods approximating the system of equations, including euler equations and other first order conditions, which characterize the equilibrium are used.

For the class of economies where the linear-quadratic approximation is applicable, linear-quadratic approximation and the method of linearizing euler equation around the steady state are equivalent. In both methods, the optimal decision rules are going to be linear in state variables. Since both methods rely on the local approximation, both methods are valid only locally around some state (usually the steady state, after de-trending the model).

2 The Standard RBC Model

Consider the following problem of the Social Planner. Since there is no distortion in the economy, the economy is easily decentralized into a competitive equilibrium with complete markets.

Problem 1 (Standard RBC model: sequential formulation)

$$\max_{\{C_t,K_{t+1},L_t,H_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, L_t)$$

subject to

$$
\begin{aligned}
& K_0, Z_0 \;\; \textit{given} \\
& C_t + K_{t+1} = e^{Z_t} F(K_t, H_t) + (1-\delta) K_t \quad \forall t \\
& Z_{t+1} = \rho Z_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim iidN(0, \sigma_\epsilon^2) \quad \forall t \\
& C_t \geq 0 \quad \forall t \\
& K_{t+1} \geq 0 \quad \forall t \\
& L_t \in [0,1] \quad \forall t \\
& H_t \in [0,1] \quad \forall t \\
& L_t + H_t = 1 \quad \forall t
\end{aligned}
$$

The problem can be transformed into the recursive formulation, as follows:

Problem 2 (Standard RBC model: recursive formulation)

$$
V(Z,K) = \max_{C,K',L,H} \left\{ u(C,L) + \beta E_{Z'|Z} V(Z',K') \right\}
$$

subject to

$$
\begin{aligned}
& C + K' = e^{Z} F(K, H) + (1-\delta) K \\
& Z' = \rho Z + \epsilon' \quad \epsilon' \sim iidN(0, \sigma_\epsilon^2) \\
& C \geq 0 \\
& K' \geq 0 \\
& L \in [0,1] \\
& H \in [0,1] \\
& L + H = 1
\end{aligned}
$$

where, as usual, prime denotes a variable in the next period. For concreteness, let's give functional forms to the utility function and the production function:

$$
u(C,L) = \frac{(C^{\mu} L^{1-\mu})^{1-\sigma}}{1-\sigma}
$$

$$
F(K,H) = K^{\theta} H^{1-\theta}
$$

As is easily seen, the utility function is not quadratic, so we need to locally approximate the utility function to make the linear-quadratic method to work.

3 The Procedure

We follow the following steps to solve a model using linear-quadratic approximation.

1. Set-up the recursive formulation (we are done already).
2. Solve for the steady state of the model.
3. Identify the (endogenous and exogenous) state and choice variables.
4. Re-define the utility as a function of (endogenous and exogenous) state variables and choice variables. Then, approximate the utility function around the steady state, using quadratic function. Basically 2nd order Taylor approximation is used.
5. Use value function iteration to find the optimal value function.

We will see the steps above one by one.

4 Finding Steady State

In the steady state, the shock Z is assumed to be its unconditional mean (which is zero), and all the variables are assumed to be constant over time. That is:

$$\overline{Z} = Z' = Z = 0$$

$$\overline{K} = K' = K$$
$$\overline{C} = C' = C$$
$$\overline{Y} = Y' = Y$$
$$\overline{L} = L' = L$$
$$\overline{H} = H' = H$$

In order to obtain the value of these variables in the steady state, we need to obtain the first order conditions, and the budget constraint. The euler equation for the problem is:

$$u_C(C, L) = \beta E_{Z'|Z}(1 - \delta + e^{Z'} F_K(K', H'))u_C(C', L')$$

If we plug-in the steady state conditions, we get:

$$\beta(1 - \delta + F_K(\overline{K}, \overline{H})) = 1$$

The first order condition with respect to labor supply is:

$$u_C(C, L)e^Z F_H(K, H) = u_L(C, L)$$

Imposing the steady state conditions:

$$u_C(\overline{C}, \overline{L})F_H(\overline{K}, \overline{H}) = u_L(\overline{C}, \overline{L})$$

Using the functional forms:

$$\mu L(1 - \theta)\overline{K}^{\theta}\overline{H}^{-\theta} = (1 - \mu)\overline{C}$$

Notice, from the budget constraint:

$$\overline{C} = F(\overline{K}, \overline{H}) - \delta \overline{K}$$

Combining the last two equations and $L = 1 - H$:

$$\mu(1-\overline{H})(1-\theta)\overline{K}^{\theta}\overline{H}^{-\theta} = (1-\mu)(F(\overline{K}, \overline{H}) - \delta \overline{K})$$

This equation plus the euler equation gives the two equations for two unknowns ($\overline{K}$ and $\overline{H}$). Once you obtain $\overline{K}$ and $\overline{H}$, derivation of other steady state variables is trivial.

5 Quadratic Approximation of the Utility Function

The utility function that we have is obviously not quadratic. Therefore, we need to approximate the utility function around the steady state, using quadratic function.

First of all, in order to use a general notation. Let's define the vectors of exogenous state variables z, endogenous state variables s, and control variables d. In our current example, $z = [Z]$, $s = [K]$, and $d = [K', H]$. Using (z, s, d), let's redefine the utility function as $r(z, s, d)$. Further assume:

$$\overline{R} = r(\overline{z}, \overline{s}, \overline{d})$$

$$W = [z, s, d]^T$$
$$\overline{W} = [\overline{z}, \overline{s}, \overline{d}]^T$$

Using second order Taylor approximation, we can approximate the utility function (z, s, d) as follows:

$$r(z, s, d) \simeq \overline{R} + (W - \overline{W})^T \overline{J} + \frac{1}{2}(W - \overline{W})^T \overline{H}(W - \overline{W})$$

where $\overline{J}$ and $\overline{H}$ are the Jacobian and the Hessian evaluated at $(\overline{z}, \overline{s}, \overline{d})$. Now, we can manipulate the approximated utility function to get a quadratic form:

$$\begin{aligned}
r(z, s, d) &\simeq \overline{R} + (W - \overline{W})^T \overline{J} + \frac{1}{2}(W - \overline{W})^T \overline{H}(W - \overline{W}) \\
&\simeq (\overline{R} - \overline{W}^T \overline{J} + \frac{1}{2}\overline{W}^T \overline{HW}) + W^T(\overline{J} - \overline{HW}) + \frac{1}{2} W^T \overline{H} W \\
&\simeq \begin{bmatrix} 1 & W^T \end{bmatrix} \begin{bmatrix} \overline{R} - \overline{W}^T \overline{J} + \frac{1}{2}\overline{W}^T \overline{HW} & \frac{1}{2}(\overline{J} - \overline{HW})^T \\ \frac{1}{2}(\overline{J} - \overline{HW}) & \frac{1}{2}\overline{H} \end{bmatrix} \begin{bmatrix} 1 \\ W \end{bmatrix} \\
&\simeq \begin{bmatrix} 1 & W^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12}^T \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} 1 \\ W \end{bmatrix} \\
&\simeq \begin{bmatrix} 1 & W^T \end{bmatrix} Q \begin{bmatrix} 1 \\ W \end{bmatrix}
\end{aligned}$$

At the end we got:

$$r(z, s, d) \simeq \begin{bmatrix} 1 & W^T \end{bmatrix} Q \begin{bmatrix} 1 \\ W \end{bmatrix} \tag{1}$$

6 Deriving the Optimal Decision Rule

As we discussed, when the utility function is quadratic, the value function which solves the Bellman equation is going to be quadratic, too. Consider the following functional form for the value function.

$$V(z,s) = \begin{bmatrix} 1 & z & s \end{bmatrix} P \begin{bmatrix} 1 \\ z \\ s \end{bmatrix} = F^T P F$$

Suppose we make a guess for the value function, which is characterized by a matrix P. In the current linear-quadratic framework, we know that P which solves the Bellman equation is a negative semi-definite symmetric matrix. Let's call the guess P_n. The Bellman equation looks like the following:

$$V_{n+1}(z,s) = \max_d \left\{ \begin{bmatrix} 1 & z & s & d \end{bmatrix} Q \begin{bmatrix} 1 \\ z \\ s \\ d \end{bmatrix} + \beta E_{Z'|Z} \begin{bmatrix} 1 & z' & s' \end{bmatrix} P_n \begin{bmatrix} 1 \\ z' \\ s' \end{bmatrix} \right\}$$

Notice that the state variables in the next period are linear in the current state and the control variables. Therefore the following expression holds:

$$\begin{bmatrix} 1 \\ z' \\ s' \end{bmatrix} = B \begin{bmatrix} 1 \\ z \\ s \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon' \\ 0 \end{bmatrix}$$

Plugging this back into the Bellman equation yields:

$$V_{n+1}(z,s) = \max_d \left\{ \begin{bmatrix} 1 & z & s & d \end{bmatrix} Q \begin{bmatrix} 1 \\ z \\ s \\ d \end{bmatrix} + \beta \begin{bmatrix} 1 & z & s & d \end{bmatrix} B^T P_n B \begin{bmatrix} 1 \\ z \\ s \\ d \end{bmatrix} + \beta \, tr(P\Sigma) \right\}$$

where Σ is the covariance matrix of the shocks. tr represents the trace operator. In the current example, since we have only one shock ϵ', the last term is equivalent to $\beta P_n^{zz} \sigma_\epsilon^2$ where P_n^{zz} is the diagonal element of P_n associated with z. In case there are more than one shocks, we just sum up $\beta P_n^{zz} \sigma_\epsilon^2$ across all shocks. That is equivalent to $\beta \, tr(P\Sigma)$.

Notice that the expectation operator disappears and a constant term $\beta \, tr(P\Sigma)$ appears. The only term associated with the shocks turns out to be $\beta \, tr(P\Sigma)$ because all the other terms including the shocks are expected to be zero. We can further simplify the formula as follows:

$$V_{n+1}(z,s) = \max_d \left\{ \begin{bmatrix} 1 & z & s & d \end{bmatrix} [Q + \beta B^T P_n B] \begin{bmatrix} 1 \\ z \\ s \\ d \end{bmatrix} + \beta \, tr(P\Sigma) \right\}$$

or

$$V_{n+1}(z,s) = \max_d \left\{ \begin{bmatrix} 1 & z & s & d \end{bmatrix} [Q + \beta M] \begin{bmatrix} 1 \\ z \\ s \\ d \end{bmatrix} \right\}$$

In the last formula, the term $\beta\ tr(P\Sigma)$ disappears because we define M as the following:

$$M = B^T P_n B + \begin{bmatrix} \beta tr(P\Sigma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Let's simplify the notation as follows:

$$\begin{aligned} V_{n+1}(z,s) &= \max_d \left\{ \begin{bmatrix} F^T & d^T \end{bmatrix} \left[\begin{bmatrix} Q_{FF} & Q_{Fd}^T \\ Q_{Fd} & Q_{dd} \end{bmatrix} + \beta \begin{bmatrix} M_{FF} & M_{Fd}^T \\ M_{Fd} & M_{dd} \end{bmatrix} \right] \begin{bmatrix} F \\ d \end{bmatrix} \right\} \\ &= \max_d \left\{ F^T(Q_{FF} + \beta M_{FF})F + 2d^T(Q_{Fd} + \beta M_{Fd})F + d^T(Q_{dd} + \beta M_{dd})d \right\} \end{aligned}$$

Take the first order condition with respect to d and we obtain:

$$2(Q_{Fd} + \beta M_{Fd})F + 2(Q_{dd} + \beta M_{dd})d = 0$$

Or

$$d = -(Q_{dd} + \beta M_{dd})^{-1}(Q_{Fd} + \beta M_{Fd})F \tag{3}$$

Let's denote the optimal decision rule as:

$$d = J_n^T F$$

Notice that the expression for d does not include M_{FF}, implying that σ_ϵ^2 does not matter for the optimal decision rule. The property that the optimal decision rule does not depend on the size of the shock is called *certainty equivalence*. Certainty equivalence implies that there is no precautionary motive for savings, and thus it's not a desirable property when we want to talk about the precautionary savings. But the property makes it easy to solve the problem. The bottom line is, it's useful for an easy solution, but use with caution. In general, methods which include linearized optimal decision rule, like linearizing euler equation, has the same property. One of the method which does not have certainty equivalence is higher-order approximation, like second order perturbation method. We will look at the method later.

If we substitute the optimal decision rule back to the Bellman equation, we get:

$$\begin{aligned} V_{n+1}(z,s) &= F^T(Q_{FF} + \beta M_{FF})F + 2F^T J_n(Q_{Fd} + \beta M_{Fd})F + F^T J_n(Q_{dd} + \beta M_{dd})J_n^T F \\ &= F^T(Q_{FF} + \beta M_{FF} + 2J_n(Q_{Fd} + \beta M_{Fd}) + J_n(Q_{dd} + \beta M_{dd})J_n^T)F \\ &= F^T(Q_{FF} + \beta M_{FF} - (Q_{Fd} + \beta M_{Fd})^T(Q_{dd} + \beta M_{dd})^{-1}(Q_{Fd} + \beta M_{Fd}))F \end{aligned}$$

Remember that the updated value function takes the following quadratic form:

$$V_{n+1}(z, s) = F^T P_{n+1} F$$

This implies that, given P_n, P_{n+1} can be updated as follows:

$$P_{n+1} = Q_{FF} + \beta M_{FF} - (Q_{Fd} + \beta M_{Fd})^T (Q_{dd} + \beta M_{dd})^{-1} (Q_{Fd} + \beta M_{Fd}) \tag{4}$$

Now we can update the quadratic value function, given a guess for the value function.

7 Value Function Iteration

Once we know how to update the value function, it is easy to implement value function iteration. Let's summarize the steps to implement the value function iteration.

1. Guess P_0. Since the value function is assumed to be quadratic, We assume a symmetric negative semi-definite matrix. An easy choice is $P_0 = -I$.

2. Suppose we have P_n. Given P_n, update the value function using equation (4) and obtain P_{n+1}. Q matrix is defined by (1). M matrix is defined by (2).

3. Compare P_n and P_{n+1}. If the distance (measured in sup-norm) is smaller than the predetermined tolerance level, stop. Otherwise let P_{n+1} and go back to the updating step (step 2).

4. With the optimal P^*, we can compute the optimal decision rule, using equation (3).

5. We can use the optimal decision rule to simulate the economy and get whatever the statistics that we are interested in.

References

Díaz-Gimenez, Javier, "Linear Quadratic Approximations: An Introduction," in Ramon Marimon and Andrew Scott, eds., *Computational Methods for the Study of Economic Dynamics*, Oxford: Oxford University Press, 1999, chapter 2.