Note on Heterogeneous Agent Model: Labor-Leisure Choice and Fiscal Policy

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1 Introduction

This note explains how to compute an extended versions of Aiyagari (1994) model. In the first part, we add nontrivial labor-leisure choice in the model. Next, we will add the government which needs to collect a fixed amount of taxes.

2 Model with Labor-Leisure Choice

We start by describing the environment.

Model 1 (Aiyagari (1994) with Endogenous Labor Supply)

1. *Time is discrete ($t = 0, 1, ...$). There are continuum of consumers. Total number of consumers is normalized to one. Each consumers has the following preference:*

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - \ell_t)$$

where c_t is consumption at period t, and ℓ_t is labor supply at period t.

2. *Consumers have one unit of time each period, and are given asset a_0. Consumers can hold asset $a_t \in A = [\underline{a}, \infty)$ which yields return r_t.*

3. *A consumer's labor income in period t is $w_t\ \ell_t\ s_t$ where w_t is the wage for efficiency unit of labor. s_t follows a Markov chain (S, P) ($S = \{s_1 < s_2 < ... < s_{n_s}\}$. An element of P, p_{ij}, represents $p_{ij} = prob(s_{t+1} = s_j | s_t = s_i)$). s_t of each agent is independent of others' s_t.*

4. *There are continuum of firms which have access to the following CRS technology:*

$$Y_t = F(K_t, L_t)$$

where K_t is capital input and L_t is labor input measured in efficiency unit of labor. CRS technology means that the size of the firms does not matter, or we can assume a single representative firm, without loss of generality. Firms rent inputs in competitive markets. capital depreciates at a constant rate δ.

Lets assume the following functional forms:

1. Utility function is separable between consumption and leisure.

$$u(c) = \frac{c^{1-\sigma_c}}{1-\sigma_c} + \gamma \frac{(1-\ell)^{1-\sigma_\ell}}{1-\sigma_\ell}$$

2. Production function is the standard Cobb-Douglas.

$$Y = K^{\theta} L^{1-\theta}$$

The individual state is the same as in the basic model $x = (s_i, a)$, because the labor-leisure choice is an instantaneous choice. Let's use the cumulative density function $\Psi(s_i, a)$ to store type distribution. As before, we restrict our attention to the stationary equilibrium, so the time scripts do not matter anymore. Individual consumer's problem can be formulated recursively as follows:

Problem 1 (Recursive Formulation of Consumer's Problem)

$$V(s_i, a) = \max_{c,\ell,a'} \left\{ u(c, 1-\ell) + \beta \sum_{j=1}^{n_s} p_{ij} V(s_j, a') \right\}$$

subject to

$$a(1+r) + w\ \ell\ s_i = a' + c$$
$$\ell \in [0, 1]$$
$$c \geq 0$$
$$a' \geq \underline{a}$$
$$r \text{ and } w \text{ are taken as given}$$

The associated optimal decision rules are $g_a(s_i, a)$, $g_c(s_i, a)$, and $g_\ell(s_i, a)$. Let's define the stationary equilibrium.

Definition 1 (Stationary Equilibrium)
A stationary equilibrium consists of prices r, w, value function $V(s_i, a)$, optimal decision rules $g_a(s_i, a)$, $g_c(s_i, a)$, $g_\ell(s_i, a)$, type distribution of agents, $\Psi(s_i, a)$, and aggregate capital stock K and labor supply L, such that:

1. ***Consumers' optimization:*** *Given prices r and w, the value function $V(s_i, a)$ is a solution to the consumer's optimization problem, and $g_a(s_i, a)$, $g_c(s_i, a)$, and $g_\ell(s_i, a)$ are the associated optimal decision rules.*

2. ***Firm's optimization:*** *Prices r and w satisfy the following marginal conditions:*

$$r = F_K(K, L) - \delta$$

$$w = F_L(K, L)$$

3. ***Consistency:*** *$\Psi(s_i, a)$ is a stationary distribution consistent with the optimal decision rule $g_a(s_i, a)$ and Markov chain (S, P).*

4. ***Aggregation:*** *Aggregate capital stock and labor supply are consistent with the stationary distribution* $\Psi(s_i, a)$*, or:*

$$K = \sum_{i=1}^{n_s} \int_A a \; d\Psi(s_i, a)$$

$$L = \sum_{i=1}^{n_s} \int_A g_\ell(s_i, a) \; s_i \; d\Psi(s_i, a)$$

Notice that the most notable difference from the basic model is the definition of the aggregate labor supply L, which includes not only s_i but also ℓ, which appears as the optimal labor supply decision $g_\ell(s_i, a)$.

In computing the stationary equilibrium of this model, we first have to understand how to solve the consumer's optimization problem. It's not as trivial as in the basic model, because there are two independent choice variables, namely a' (or c) and ℓ.

In general, we need to use one of the two methods:

1. Value function iteration with 2-variable optimization. We can search for the optimum using one of the multidimensional optimization routines, or nest one of the one-dimensional optimization routine to the other one-dimensional optimization routine.
2. Euler Equations, combined with finding a root of system of two nonlinear equations. We first obtain two first order conditions associated with the two choice variables, and either use multidimensional root-finding subroutine and nest one one-dimensional root-finding subroutine to the other one-dimensional root-finding subroutine.

However, thanks to the nature of one of the decisions (remember that labor-leisure choice is instantaneous decision, and does not affect the future value directly), the optimal labor supply actually takes really simple formula. To see that, let's take the first order condition with respect to ℓ. We obtain:

$$s_i \; w \; c^{-\sigma_c} = \gamma(1-\ell)^{-\sigma_\ell}$$

This equation is associated with the particular functional form that we assumed, but we can obtain similarly simple formula for other types of functional form for preferences (like Cobb-Douglas between consumption and leisure). Anyhow, the equation above, together with the budget constraint, implicitly defines the optimal ℓ as a function of states (s_i, a) and a'. Notice that there is a nonnegativity constraint for ℓ. If the equation implies a negative ℓ, it should be interpreted as $\ell = 0$. In the end, we can construct a function:

$$\ell = \tilde{g}_\ell(s_i, a, a')$$

If we plug in this function back into the original optimization problem, we basically have the very similar problem of choosing only the optimal a'. Therefore, we can find the optimal a' in the same way as in the basic model and can obtain:

$$a' = g_a(s_i, a)$$

Let's see using the example of the Euler Equation. The original Euler Equation of the problem looks like the following:

$$u_c(s_i w\ell + (1+r)a - a', 1-\ell) = \beta \sum_{j=1}^{n_s} p_{ij} u_c(s_j w\ell' + (1+r)a' - g_a^0(s_j, a'), 1-\ell')(1+r)$$

If we plug $\ell = \tilde{g}_\ell(s_i, a, a')$ into the equation, we obtain the following:

$$\begin{aligned} & u_c(s_i w\tilde{g}_\ell(s_i, a, a') + (1+r)a - a', 1 - \tilde{g}_\ell(s_i, a, a')) \\ &= \beta \sum_{j=1}^{n_s} p_{ij} u_c(s_j w\tilde{g}_\ell(s_j, a', g_a^0(s_j, a')) + (1+r)a' - g_a^0(s_j, a'), 1 - \tilde{g}_\ell(s_j, a', g_a^0(s_j, a')))(1+r) \end{aligned}$$

This is not an easy formula, but it actually contains only one unknown, which is a'. Therefore, we can obtain $a' = g_a(s_i, a)$ by solving the equation above. Once we get the optimal decision rule $a' = g_a(s_i, a)$, we can obtain the optimal decision rule for ℓ by plugging it into $\tilde{g}_\ell(s_i, a, a')$ as follows:

$$\ell = \tilde{g}_\ell(s_i, a, g_a(s_i, a)) = g_\ell(s_i, a)$$

Finally, the optimal decision rule for consumption $g_c(s_i, a)$ is obtained as the residual.

Now we are ready to write an excess demand function $\Phi(r)$:

Definition 2 (Excess demand function $\Phi(r)$)

1. *Given r, the function does the followings.*
2. *Compute the capital-labor ratio k, which satisfies the following:*

   $$r = \theta k^{\theta-1} - \delta$$

3. *Compute w which is consistent with the obtained capital-labor ratio k as follows:*

   $$w = (1-\theta)k^\theta$$

4. *Given r and w, solve the consumers' optimization problem. Denote the optimal decision rules as $g_a(s_i, a)$, $g_c(s_i, a)$, and $g_\ell(s_i, a)$.*
5. *Using the obtained optimal decision rule $g_a(s_i, a)$ and the exogenous law of motion for the idiosyncratic shock s_i, find the stationary distribution. Suppose the stationary distribution is characterized by a cumulative density function $\Psi(s_i, a)$.*
6. *Compute the aggregate supply of capital K and aggregate labor supply L as follows:*

   $$K = \sum_{i=1}^{n_s} \int_A a \; d\Psi(s_i, a)$$

   $$L = \sum_{i=1}^{n_s} \int_A g_\ell(s_i, a) \; s_i \; d\Psi(s_i, a)$$

7. *Excess demand can be defined as*

$$\Phi(r) = k - \frac{K}{L}$$

The excess demand function is slightly different from the basic model, because L is an equilibrium object as well. We define the excess demand function as the per-labor-supply basis. Finding a root of the function $\Phi(r)$ is equivalent to finding a fixed point of the capital-labor ratio $\frac{K}{L}$.

3 Fiscal Policy

We can add the government in the model. As usual, assume that the government has to expend an amount G in each period and thus has to collect taxes of the amount G. We can add variety of taxes in the model. We can add lump-sum tax, capital income tax, labor income tax, general income tax, or consumption tax.

As an example, let's add a flat-rate general income tax in the model. The recursive formulation of the consumer's problem looks like the following:

Problem 2 (Recursive Formulation of Consumer's Problem with Taxes)

$$V(s_i, a) = \max_{c,\ell,a'} \left\{ u(c, 1-\ell) + \beta \sum_{j=1}^{n_s} p_{ij} V(s_j, a') \right\}$$

subject to

$$a(1 + (1-\tau)r) + w\ \ell\ s_i(1-\tau) = a' + c$$
$$\ell \in [0, 1]$$
$$c \geq 0$$
$$a' \geq \underline{a}$$
$$r \text{ and } w \text{ are taken as given}$$

where τ is the flat general tax rate. The government budget balance condition is:

$$G = Kr\tau + wL\tau$$

In this economy, we need to iterate on two objects, namely r (or k) and τ. Therefore, the algorithm to compute the stationary equilibrium for this economy looks like the following:

Algorithm 1 (Computing Stationary Equilibrium of Economy with Taxation)

1. *Set the tolerance parameter* ϵ.
2. *Make a guess for the interest rate* r^0 *and the government budget balance tax rate* τ^0.

3. *Compute the capital-labor ratio* k*, which satisfies the following:*

$$r^0 = \theta k^{\theta-1} - \delta$$

4. *Compute* w *which is consistent with the obtained capital-labor ratio* k *as follows:*

$$w^0 = (1-\theta)k^{\theta}$$

5. *Given* r^0 *and* w^0*, and tax rate* τ*, solve the consumers' optimization problem. Denote the optimal decision rules as* $g_a(s_i, a)$*,* $g_c(s_i, a)$*, and* $g_\ell(s_i, a)$*.*

6. *Using the obtained optimal decision rule* $g_a(s_i, a)$ *and the exogenous law of motion for the idiosyncratic shock* s_i*, find the stationary distribution. Suppose the stationary distribution is characterized by a cumulative density function* $\Psi(s_i, a)$*.*

7. *Compute the aggregate supply of capital* K *and aggregate labor supply* L *as follows:*

$$K = \sum_{i=1}^{n_s} \int_A a \; d\Psi(s_i, a)$$

$$L = \sum_{i=1}^{n_s} \int_A g_\ell(s_i, a) \; s_i \; d\Psi(s_i, a)$$

8. *Compute government budget balance tax rate* τ^1 *by solving the following:*

$$G = Kr^0\tau^1 + w^0 L\tau^1$$

9. *Compute the interest rate associated with the* K *and* L *just obtained as follows:*

$$r^1 = \theta \left(\frac{K}{L}\right)^{\theta-1} - \delta$$

10. *Compute the error between* (r^0, τ^0) *and* (r^1, τ^1) *using the following:*

$$\text{max error} = \max(|r^0 - r^1|, |\tau^0 - \tau^1|)$$

11. *If max error* $< \epsilon$ *then we are done. Otherwise, update* r *and* τ *and go back to step 3. Updating should be conservative, in the sense that you better not jump to the new* (r^1, τ^1)*. Instead, it's better to use some intermediate values between the old guesses and the new guesses.*

Models with other kinds of taxes can be solved in a similar fashion. Notice that, if there are two types of taxes simultaneously, there will be an indeterminacy. We can compute the stationary economy with different types of taxes and see how the choice of taxes affect macroeconomic aggregates.

References

Aiyagari, S. Rao, "Uninsured Idiosyncratic Risk, and Aggregate Saving," *Quarterly Journal of Economics*, 1994, *109*, 659–684.