# Note on Type Distribution of Heterogeneous Agents

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## 1 Introduction

In dealing with variety of heterogeneous agent models where agents might have zero measure, and where agents' type includes continuous variable (like asset holding), we need a consistent way to keep track of the type distribution of agents. That's why we need the help of the measure theory. We will study the minimum set of knowledge on the measure theory to deal with the issue. The part of this note is based on the Appendix of Ríos-Rull (1999), or the note written by Josep Pijoan-Mas.

Once we acquired how to store the type distribution theoretically, we will learn how to store the type distribution in computers.

## 2 Quick Course in Measure Theory

Let $A$ a set. For example, $A \subset \mathbb{R}$.

**Definition 1 (power wet)**
*Power set $\mathcal{P}(A)$ is a set of all subsets of $A$.*

**Definition 2 (family of sets over $A$)**
*$\mathcal{A}$ is a family of sets over $A$ if $\mathcal{A}$ is a non-empty subset of $\mathcal{P}(A)$.*

Notice that, if $a \in \mathcal{A}$, $a \subset A$.

**Definition 3 (algebra)**
*$\mathcal{A}$ is an algebra if* (i) *$\mathcal{A}$ is closed under complements ($a \in \mathcal{A} \Rightarrow a^c \in \mathcal{A}$), and* (ii) *$\mathcal{A}$ contains the empty set ($\varnothing \in \mathcal{A}$).*

Notice that the two conditions together imply $A \in \mathcal{A}$.

**Definition 4 ($\sigma$-algebra)**
*$\mathcal{A}$ is a $\sigma$-algebra if $\mathcal{A}$ is an algebra and $\mathcal{A}$ is closed under countable unions ($\{a_i\} \in \mathcal{A} \Rightarrow \cup_i a_i \in \mathcal{A}$).*

**Definition 5 (Borel $\sigma$-algebra)**
*Borel $\sigma$-algebra is a $\sigma$-algebra generated by a family of open sets.*

An important Borel $\sigma$-algebra is the one defined over $A = \mathbb{R}$.

**Definition 6 (measure)**
*A function $x : \mathcal{A} \to \mathbb{R}_+$ is a measure if* (i) *the empty set has measure zero ($x(\varnothing) = 0$), and* (ii) *$x$ satisfies countable additivity ($\sigma$-additivity) ($x(\cup_i a_i) = \cup_i x(a_i)$ for pairwise disjoint sets $a_i \in \mathcal{A}$)*

**Definition 7 (probability measure)**
*Probability measure is a measure with $x(A) = 1$.*

**Definition 8 (measure space)**
*The triple* $(A, \mathcal{A}, x)$ *is called a measure space.*

**Definition 9 (probability space)**
*A probability space is a measure space with probability measure.*

In storing the type distribution of heterogeneous agents, we use a probability space $(A, \mathcal{A}, x)$, where $A$ is the space of agents' type (an implicit assumption here is that the size of population is normalized to one).

**Definition 10 (measurable function)**
*A function* $g : A \to \mathbb{R}$ *is measurable with respect to* $(A, \mathcal{A})$ *if* $D \equiv \{a \in A | g(a) \leq c\} \in \mathcal{A} \quad \forall c \in \mathbb{R}$

**Definition 11 (transition function)**
*A function* $Q : A \times \mathcal{A} \to \mathbb{R}$ *is a transition function if* (i) *given* $a \in A$, $Q(a, .)$ *is a probability measure, and* (ii) *given* $B \in \mathcal{A}$, $Q(., B)$ *is measurable function.*

As the names indicates, a transition function is used to update the type distribution of agents. If $x$ is updated to $x'$ by $Q$, we can write down $x'$ as follows:

$$x'(B) = \int_A Q(a, B)\ dx$$

Naturally, the invariant distribution associated with $Q$ can be easily defined as follows:

**Definition 12 (invariant distribution)**
*A measure* $x^*$ *is an invariant distribution with respect to the transition function* $Q$ *if, for all* $B \in \mathcal{A}$*:*

$$x^*(B) = \int_A Q(a, B)\ dx^*$$

Roughly speaking, an invariant distribution $x^*$ is a distribution which ends up the same distribution when updated with $Q$.

# 3 The Convenience of Measure Space

When we use a probability space $(A, \mathcal{A}, x)$ to store the type distribution of agents, it's easy to define variety of aggregate statistics. To see the point, let's assume that the type space is defined as $A = \mathcal{S} \times \mathcal{K}$, where $\mathcal{S} = \{s_1, s_2, ..., s_n\}$ and $\mathcal{K} \subset \mathbb{R}_+$. $\mathcal{S}$ is the space of individual productivity, and $\mathcal{K}$ is the space of capital stock. An agent is characterized by $(s, k) \in \mathcal{S} \times \mathcal{K} = A$. Assume that there is no labor-leisure choice, so that $s$ represents the productivity-adjusted individual labor supply. Also assume that the transition probabilities from $s_i$ to $s_{i'}$ is $\pi_{ii'}$

Suppose the type distribution is $x$. Then the aggregate capital stock $K$, and aggregate labor supply $L$ can be compactly expressed as follows:

$$K = \int_A k \ dx$$

$$L = \int_A s \ dx$$

Because the total population size is normalized to one, we have:

$$1 = \int_A \ dx$$

Notice that, since the total population size is one, the aggregate and the average can be used interchangeably.

The measure of agents whose capital stock holding is higher than the average is:

$$\int_A I_{k>K} \ dx$$

where $I$ is an indicator function which takes the value 1 if the condition attached to the indicator function is true, and 0 otherwise. The measure of agents with $s = s_i$ is:

$$\int_A I_{s=s_i} \ dx$$

The average capital stock holding of type $s_i$ agents is:

$$\frac{\int_A k I_{s=s_i} \ dx}{\int_A I_{s=s_i} \ dx}$$

The proportion of capital stock owned by type $s_i$ agents is:

$$\frac{\int_A k I_{s=s_i} \ dx}{\int_A k \ dx}$$

# 4 Approximation of Type Distribution

Let's keep using the example above. We have the type space $A = \mathcal{K} \times \mathcal{S}$ and use the probability space $(A, \mathcal{A}, x)$ to represent the type distribution of agents. Obviously, computers cannot handle the type distribution as it is, if the type includes continuous variable, like $k$ in the current example. Naturally, we need to somehow approximate the type distribution. How? Three ways are presented in the following three sections below. In each of the sections, the approximation method is explained. And the algorithms to update the distribution using decision rules and transition matrices are also explained.

# 5 Discretization or Step Function Approximation of CDF

## 5.1 Approximation

The most intuitive way is to discretize $\mathcal{K}$ using a large (but finite, of course) number of grid points. Let's define $\hat{\mathcal{K}} = \{k_j\}_{j=1}^m$. Then the type distribution is approximated by $\{p_{ij}\}$, which is the measure assigned to $(s_i, k_j)$. By construction:

$$p_{ij} \geq 0 \quad \forall i, j$$

$$\sum_i \sum_j p_{ij} = 1$$

The method can be interpreted as using step functions to approximate the cumulative density function with respect to $k$. To see the point more clearly, let's assume $\mathcal{K}$ is a compact set and denote $\Psi_i(k)$ as the cumulative density function conditional on $s = s_i$. Further assume that $\hat{\mathcal{K}} = \{k_j\}_{j=1}^m$ is a set of grid points over the space $\mathcal{K}$. If we use the discretization discussed above to approximate the type distribution, it is equivalent to approximate the type distribution using the following step function to approximate $\Psi_i(k)$.

$$\Psi_i(k) = \sum_{j=1}^{\ell} p_{ij} \quad \text{for } k_\ell \leq k < k_{\ell+1}$$

$$\Psi_i(k) = \sum_{j=1}^{m} p_{ij} \quad \text{for } k = k_m$$

By construction:

$$\sum_i \Psi_i(k_m) = 1$$

And $\Psi_i(k_m)$ is the total measure of type $s = s_i$ agents.

Using this approximation method, various statistics can be easily computed as follows:

$$\text{Aggregate (=average) asset holding} = \sum_i^n \sum_j^m k_j p_{ij}$$

$$\text{Aggregate (=average) labor supply} = \sum_i^n \sum_j^m s_i p_{ij}$$

## 5.2 Updating

Suppose the distribution is approximated using this method. If the space of $\mathcal{K}$ is discretized in the same way when the optimal decision rule is found, it is very easy to update the distribution. Suppose the optimal decision rule takes the following form:

$$j' = g_k(i, j)$$

where $j'$ represents the grid point of the asset holding in the next period. If we want to update the distribution, we only need to use the following simple algorithm:

**Algorithm 1 (Updating distribution with discretized decision rules )**

1. *Set* $p'_{ij} = 0$

2. *For* $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, $i' = 1, 3, ..., n$, *do the following:*

$$p'_{i'g_k(i,j)} = p'_{i'g_k(i,j)} + \pi_{ii'} p_{ij}$$

This is basically constructing a transition matrix of size $(n \times m) \times (n \times n)$ and multiply the initial distribution by the transition matrix to update a distribution.

Next, consider the case when the optimal decision rule is not discretized. Assume the following form of the optimal decision rule:

$$k' = g_k(s, k)$$

You can you the following algorithm to update a distribution:

**Algorithm 2 (Updating distribution with continuous decision rules )**

1. *For* $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, *do the followings:*

2. *Find a grid point* $k_{j'}$ *which satisfies the following:*

$$k_{j'} \leq g_k(s_i, k_j) \leq k_{j'+1}$$

3. *For* $i' = 1, 2, ..., n$, *implement the following addition:*

$$p'(s_{i'}, k_{j'}) = p'(s_{i'}, k_{j'}) + p(s_i, k_j)\pi_{ii'} \frac{k_{j'+1} - g_k(s_i, k_j)}{k_{j'+1} - k_{j'}}$$

$$p'(s_{i'}, k_{j'+1}) = p'(s_{i'}, k_{j'+1}) + p(s_i, k_j)\pi_{ii'} \frac{g_k(s_i, k_j) - k_{j'}}{k_{j'+1} - k_{j'}}$$

Notice that the last step is a kind of lottery.

Notice that what is done at step 3 is a kind of lottery. Almost surely, an optimal choice $g_k(s, k)$ does not happen to be one of the grid points. Suppose $g_k(s, k)$ falls between grid $k_{j'}$ and $k_{j'+1}$. If this is the case, what we do here is to let the agents to draw a lottery and the proportion $\frac{k_{j'+1} - g_k(s_i, k_j)}{k_{j'+1} - k_{j'}}$ are forced to choose $k_{j'}$ and the rest are forced to choose $k_{j'+1}$. This is a nice trick to deal with the finite grid point approximation of the distribution.

# 6 Piecewise-Linear Approximation of CDF

## 6.1 Approximation

The method is proposed by Ríos-Rull (1999). Instead of using a step function to approximate the CDF, he proposes using piecewise-linear approximation. Formally, put grids on the space of capital stock holding $\{k_j\}_{j=1}^{m}$. $m$ need not be as large as in the case of discretization to achieve the same level of precision. Let the cumulative density with respect to $k_j$, conditional on $s_i$, as $p_{ij}$. Then the distribution is stored by $\{p_{ij}\}$. $p_{ij}$ represents the measure of agents with $s = s_i$ and $k \leq k_j$. To evaluate the cumulative density not on one of the grid points, use the piecewise-linear approximation. More formally:

$$\Psi_i(k) = p_{ij} + \frac{p_{ij+1} - p_{ij}}{k_{j+1} - k_j}(k - k_j) \quad \text{for } k_j \leq k \leq k_{j+1}$$

By construction:

$$\sum_i \Psi_i(k_m) = 1$$

And $\Psi_i(k_m)$ is the total measure of type $s = s_i$ agents.

Using piecewise-linear approximation is important because the piecewise-linear approximation is shape-preserving. For example, if Chebyshev polynomial or cubic spline function is used to approximate the true CDF, it could be the case that the approximated function decreases for some $k$, which should not happen. There is no such problem with piecewise-linear approximation because the piecewise-linear approximation preserves monotonicity of the original function.

A natural extension of the previous method is to use other kinds of shape-preserving spline approximation to approximate CDF with respect to $k$, conditional on $s_i$. Please see the note for finite element method for alternatives.

## 6.2 Updating

If the optimal decision rule $k' = g_k(s, k)$ is an increasing function, we can use the following algorithm, proposed by Ríos-Rull (1999) to efficiently update the distribution function $\Psi_i(k)$:

**Algorithm 3 (Updating distribution: CDF $\Psi(s_i, a)$ )**

1. *For $i = 1, 2, ..., n$, $j' = 1, 2, ..., m$, do the followings:*

2. *Find $\tilde{k}$ which satisfies the following:*

   $$k_{j'} = g_k(s_i, \tilde{k})$$

   *Or, using the inverse function representation:*

   $$\tilde{k} = g_k^{-1}(s_i, k_{j'})$$

   *Notice that you need a numerical root finder to implement this step in general, but the use of piecewise-linear function makes the search for $\tilde{k}$ an easy task (think yourself why).*

3. *For $i' = 1, 2, ..., n$, execute the following:*

   $$\Psi'(s_i', k_{j'}) = \Psi'(s_{i'}, k_{j'}) + \pi_{ii'}\Psi(s_i, \tilde{k})$$

   *$\Psi(s_i, \tilde{k})$ can be computed easily, using the piecewise-linear approximation. Notice that you can do this because the optimal decision rule is increasing; those who have $s_i$ and $k \leq \tilde{k}$ chooses $k' \leq k_{j'}$ optimally.*

# 7 Monte-Carlo Simulation

## 7.1 Approximation

The method is not directly related to other methods, or the way we handle type distribution. But one popular way to approximate the distribution is to create finite (but large) number of agents and approximate the type distribution of agents by implementing Monte-Carlo simulation of the model economy with the agents. Suppose we use $N$ agents. Then we basically approximate the type distribution using $\{(s^i, k^i)\}_{i=1}^N$. The method is useful for computing some statistics, but the method is stochastic in the sense that the type distribution depends on the realization of shocks for each agent. Therefore, it's relatively hard to obtain convergence when we are looking for the stationary distribution.

## 7.2 Updating

Since we are storing the type distribution by the type of large number of agents, it's straightforward to update the distribution. Basically, all you need is to simulate the economy for one period. Below is how:

**Algorithm 4 (Updating Distribution: Monte-Carlo Simulation )**

1. *For $i = 1, 2, ..., N$, update the individual state $(s^i, k^i)$ using the following procedure:*
2. *Updating the asset holding $k'^i$ is easy, as we have already obtained the optimal decision rule:*

$$k'^i = g_k(s^i, k^i)$$

3. *Updating $s^i$ is a bit tricky. We want to update $s^i$ using the transition probability of the Markov chain, and a random number generator. Let's use the uniform $[0, 1]$ random generator, which is available for any computer language. Let $d \in [0, 1]$ is a draw from a random number generator. Suppose an agent has $s^i = s_j$. Let $\tilde{j}'$ be the smallest integer that satisfies the following:*

$$d \leq \sum_{j'=1}^{\tilde{j}'} p_{jj'}$$

*What does the $j'$ mean? $s_{j'}$ is the next period shock to the agent $i$, according to the transition probability and a random number $d$.*

4. *At the end, the new individual state for agent $i$ is $(s'^i = s_{j'}, k'^i)$.*

# References

**Ríos-Rull, José-Víctor**, "Computation of Equilibria in Heterogeneous-Agent Models," in Ramon Marimon and Andrew Scott, eds., *Computational Methods for the Study of Economic Dynamics*, Oxford: Oxford University Press, 1999, chapter 11.