Solving RBC models with Linearized Euler Equations: Method of Undetermined Coefficients

Makoto Nakajima, UIUC

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1 Introduction

In this note I will present another method to solve real business cycle models or dynamic stochastic general equilibrium models, called the method of undetermined coefficients. The method is proposed by Uhlig (1997). He has many papers on this method and nice codes to implement the method easily. Please visit his web page for more information on the method. This note is basically the summary of Uhlig (1997).

The method is similar to the method by Blanchard and Kahn (1980) in the sense that the method crucially depends on linearizing the equations that characterize the solution. In this sense, both methods are categorized as the linearizing euler equation method. Moreover, both methods are *local method*. The (potentially) non-linear equations that characterize the solution of the model are linearized around some state, most likely the steady state of the model. Remember that the approximation is valid only around the steady state. Besides, the methods necessarily imply *certainty equivalence*.

We are going to use the standard RBC model with indivisible labor by Hansen (1985). The representative consumer makes consumption-savings choice as well as labor-leisure choice each period. First, we solve the model using the baby version of the solution method. The model is used to show the basic concept of the solution method. Then we show the general formulation of the method.

2 The Standard RBC Model

Let's start by describing the social planner's problem in the economy of Hansen (1985). Remember that the solution to the social planner's problem is equivalent to the allocation in the competitive equilibrium of the economy where labor supply is indivisible (choice set of the individual labor supply decision is $\{0, \overline{n}\}$) but consumers can use a lottery to smooth out the indivisibility.

Problem 1 (Social planner's problem of the standard RBC model: recursive formulation)

$$V(Z, K) = \max_{C,K',N} \left\{ u(C, N) + \beta E_{Z'|Z} V(Z', K') \right\}$$

subject to

$$\begin{aligned}
&C + K' = ZF(K, N) + (1 - \delta)K \\
&\ln Z' = (1 - \rho)\overline{Z} + \rho \ln Z + \epsilon' \quad \epsilon' \sim iidN(0, \sigma_\epsilon^2) \\
&C \geq 0 \\
&K' \geq 0 \\
&N \in [0, 1]
\end{aligned}$$

As usual, prime denotes a variable in the next period. Following Hansen (1985), let's assume:

$$u(C, N) = \log(C) - \mu N$$

$$Y = ZK^{\theta}N^{1-\theta}$$

3 The Procedure

You need to take the following steps to solve a model using linearized Euler equations and matrix decomposition. The remaining part of the note explains the procedure to the details step by step.

1. Find the system of (potentially non-linear) equations that characterize the solution of the model.
2. Find the steady state of the model.
3. Approximate the non-linear equations in the system around the steady state, using 1st order Taylor approximation.
4. Fit the system of equations into some matrix representation. Since the representation is closely related to the solution method, the representation differs for each solution method.
5. Derive the optimal decision rules (linear functions from the state variables to the control variables) and the laws of motion for endogenous state variables (linear function from the state variables to the state variables in the next period). Once the system of equations is fit into the matrix representation associated with the method of undetermined coefficients, the solution is automatically obtained (solution method doesn't depend on the characteristics of the model).

4 Characterizing the Solution

The solution of the social planner's problem can include, most importantly, the first order conditions (including the Euler equation), and laws of motion for state variables. Other equations might be included depending on the state and control variables chosen. In general, if we have k exogenous state variables, m endogenous state variables, and n control (jump) variables, we have $k + n + m$ equations or more.

In the current example, we choose $k = 1$ (Z), $m = 1$ (K), and $n = 5$ (C, N, Y, R, and I). Obviously, there is a degree of freedom in how to choose the control variables.

For our current examples, the following system of equations characterize the solution to the social planner's problem:

$$
\begin{aligned}
&\mu = C^{-\sigma}(1-\theta)\frac{Y}{N} \\
&C^{-\sigma} = \beta\mathbb{E}[C'^{-\sigma}R'] \\
&R = \theta\frac{Y}{K} + 1 - \delta \\
&Y = ZK^{\theta}N^{1-\theta} \\
&K' = I + (1-\delta)K \\
&Y = C + I \\
&\ln Z' = (1-\rho)\ln\overline{Z} + \rho\ln Z + \epsilon'
\end{aligned}
$$

5 Finding Steady State

I will skip the details. For more details, please see the lecture note for Blanchard-Kahn method. The key of this step is to assume that Z stays at its unconditional mean ($Z = Z' = \overline{Z}$). With this assumption, we can solve for the steady state values of all the other variables ($\overline{K}$, $\overline{N}$, $\overline{Y}$, $\overline{C}$, $\overline{R}$, and $\overline{I}$)

6 Log-linearization

Again I skip the details. For details, please see the lecture note for Blanchard-Kahn method. Basically, what you do in this step is to apply 1st order Taylor expansion around the steady state to all the non-linear equations so that all the equations become linear in state and control variables.

It sounds tedious, but practically, all you need is to apply the following rules to log-linearize the equations.

$$X = \overline{X}e^{x} \simeq \overline{X}(1+x)$$

$$xy \simeq 0$$

where X is the original variable, $\overline{X}$ is its steady state value, and x represents the percentage deviation from the steady state value. y is a deviation variable for Y.

If we apply the rules to log-linearize to the system of equations for our current model, we obtain the following system of log-linearized equations:

$$
\begin{align}
0 &= -\sigma c + y - n \tag{1}\\
0 &= \mathbb{E}[\sigma c - \sigma c' + r'] \tag{2}\\
\overline{R}r &= \theta \frac{\overline{Y}}{\overline{K}}(y-k) \tag{3}\\
y &= z + \theta k + (1-\theta)n \tag{4}\\
\overline{K}k' &= \overline{I}i + (1-\delta)\overline{K}k \tag{5}\\
\overline{Y}y &= \overline{I}i + \overline{C}c \tag{6}\\
z' &= \rho z + \epsilon' \tag{7}
\end{align}
$$

7 Simple Method of Undetermined Coefficients

Before presenting the formal matrix representation of the method and showing the general method to solve the model, let us take a simpler approach, to understand the essence of the method. The formal solution method is just a generalization of what is presented in this section.

Currently, we have the six linear equations: (1), (2), (3), (4), (5), (6), and (7). First of all, try to minimize the number of equations by substituting out some variables. In particular, we can use the equations (1), (3), (4) and (6) to substitute out y, i, c and r from the system. Now we have:

$$
\begin{align}
0 &= \mathbb{E}[(z+\theta k - \theta n) - (z' + \theta k' - \theta n') + \theta \frac{\overline{Y}}{\overline{KR}}(z' + (\theta-1)k' + (1-\theta)n')] \tag{8}\\
\overline{K}k' &= \overline{Y}(z + \theta k + (1-\theta)n) - \frac{\overline{C}}{\sigma}(z + \theta k - \theta n) + (1-\delta)\overline{K}k \tag{9}\\
z' &= \rho z + \epsilon' \tag{10}
\end{align}
$$

Notice that this is the minimum number of equations which characterize the solution, since we have one exogenous state variable z, one endogenous state variable k, and two control variables (k' and n), one of which is a endogenous state variable.

To ease the notation, express the equations above by the following:

$$
\begin{align}
0 &= \mathbb{E}[k' + \phi_1 k + \phi_2 n' + \phi_3 n + \phi_4 z' + \phi_5 z] \tag{11}\\
0 &= k' + \eta_1 k + \eta_2 n + \eta_3 z \tag{12}\\
z' &= \rho z + \epsilon' \tag{13}
\end{align}
$$

Now, we postulate (correctly) that the optimal decision rule for k and n are linear in state variables (k and z). Specifically, let's assume that following functional forms:

$$
\begin{align}
k' &= \psi_1 k + \psi_2 z \tag{14}\\
n &= \psi_3 k + \psi_4 z \tag{15}
\end{align}
$$

Now our goal is to find $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ which are consistent with (11), (13), and (13). Let's plug in (13), (14), and (15) into (11) and (12) so that the resulting two equations contain only z and k. We obtain the followings:

$$0 = \mathbb{E}[(\psi_1 + \phi_1 + \phi_2\psi_3\psi_1 + \phi_3\psi_3)k + (\psi_2 + \phi_2\psi_3\psi_2 + \phi_2\psi_4\rho + \phi_3\psi_4 + \phi_4\rho + \phi_5)z + (\phi_2\psi_4 + \phi_4)\epsilon'] \tag{16}$$

$$0 = (\psi_1 + \eta_1 + \eta_2\psi_3)k + (\psi_2 + \eta_2\psi_4 + \eta_3)z \tag{17}$$

Notice that $\mathbb{E}[\epsilon'] = 0$, wherefore, the first equation becomes:

$$0 = (\psi_1 + \phi_1 + \phi_2\psi_3\psi_1 + \phi_3\psi_3)k + (\psi_2 + \phi_2\psi_3\psi_2 + \phi_2\psi_4\rho + \phi_3\psi_4 + \phi_4\rho + \phi_5)z \tag{18}$$

Since both (18) and (17) have to be satisfied for all k and z, we get the following equations, which can be used to solve for the four coefficients:

$$0 = \psi_1 + \phi_1 + \phi_2\psi_3\psi_1 + \phi_3\psi_3 \tag{19}$$

$$0 = \psi_2 + \phi_2\psi_3\psi_2 + \phi_2\psi_4\rho + \phi_3\psi_4 + \phi_4\rho + \phi_5 \tag{20}$$

$$0 = \psi_1 + \eta_1 + \eta_2\psi_3 \tag{21}$$

$$0 = \psi_2 + \eta_2\psi_4 + \eta_3 \tag{22}$$

Notice that (19) and (21) contain only ψ_1 and ψ_3. Solve (21) for ψ_3 and plugging into (19), and we get:

$$0 = \psi_1 + \phi_1 + \phi_2(-\frac{1}{\eta_2}\psi_1 - \frac{\eta_1}{\eta_2})\psi_1 + \phi_3(-\frac{1}{\eta_2}\psi_1 - \frac{\eta_1}{\eta_2})$$

Equivalently:

$$\psi_1^2 + \left(\eta_1 - \frac{\eta_2}{\phi_2} + \frac{\phi_3}{\phi_2}\right)\psi_1 + \left(\frac{\phi_3\eta_1}{\phi_2} - \frac{\eta_2\phi_1}{\phi_2}\right) = 0$$

This is a normal quadratic equation. Solve it. Notice that the product of two roots is:

$$\frac{\phi_3\eta_1}{\phi_2} - \frac{\eta_2\phi_1}{\phi_2} = \frac{\theta\eta_1}{\phi_2} - \frac{\eta_2\phi_1}{\phi_2}$$

Notice (it's a bit tedious though) that the product of two roots is:

$$\frac{\phi_3\eta_1}{\phi_2} - \frac{\eta_2\phi_1}{\phi_2} = \overline{R} = \frac{1}{\beta}$$

It implies that there is at last one root inside the unit circle. Since we are interested in the stable solution, take the root inside the unit circle. Once ψ_1 is obtained, ψ_3 can be computed from (21). Combining the remaining two equations, we can solve for ψ_2 and ψ_4. Specifically, we can obtain:

$$\psi_4 = \frac{\rho\phi_1 + \phi_5 + \eta_3 + \eta_3\psi_3\phi_2}{\eta_2 + \eta_2\phi_2\psi_3 - \rho\phi_2 - \phi_3}$$

and

$$\psi_2 = -\eta_2\psi_4 - \eta_3$$

8 General Solution Method: Fitting into the Matrix Representation

The general solution method using the undetermined coefficients is presented. Basically it is a generalization of the method in the previous section. First of all, fit the system of linearized equations into the following matrix representation:

$$0 = Ax' + Bx + Cy + Dz \tag{23}$$
$$0 = \mathbb{E}[Fx'' + Gx' + Hx + Jy' + Ky + Lz' + Mz] \tag{24}$$
$$z' = Nz + \epsilon' \quad \mathbb{E}[\epsilon'] = 0 \tag{25}$$

where x is a vector of endogenous state variables (size $m \times 1$), y is a vector of control variables (size $n \times 1$), z is a vector of exogenous state variables (size $k \times 1$). ϵ' is a vector of shocks (size $k \times 1$). C is of size $n \times n$. F is of size $m \times n$, and N has only stable eigenvalues.[1]

The optimal decision rules can be expressed as follows:

$$x' = Px + Qz \tag{26}$$
$$y = Rx + Sz \tag{27}$$

9 General Solution Method: Solving the Matrix Equations

As we did for the simple case, let's substitute out x', x'', y, y', and z' using (26), (27), and (25). Then the equations (23) and (24) become the followings:

$$0 = A(Px + Qz) + Bx + C(Rx + Sz) + Dz \tag{28}$$

$$\begin{aligned} 0 = \mathbb{E}[&F(P(Px + Qz) + Q(Nz + \epsilon')) + G(Px + Qz) + Hx + \\ &J(R(Px + Qz) + S(Nz + \epsilon')) + K(Rx + Sz) + L(Nz + \epsilon') + Mz] \end{aligned} \tag{29}$$

Collecting terms and using$\mathbb{E}[\epsilon'] = 0$:

$$0 = (AP + B + CR)x + (AQ + CS + D)z \tag{30}$$
$$0 = (FP^2 + GP + H + JRP + KR)x + (FPQ + FQN + GQ + JRQ + JSN + KS + LN + M)z \tag{31}$$

Since the two equations have to be satisfied for any x and z:

$$0 = AP + B + CR \tag{32}$$
$$0 = AQ + CS + D \tag{33}$$
$$0 = FP^2 + GP + H + JRP + KR \tag{34}$$
$$0 = FPQ + FQN + GQ + JRQ + JSN + KS + LN + M \tag{35}$$

[1] Solution method for more general case (where the number of equations in (23) is larger than n) can be found in Uhlig (1997).

Notice (32) and (34) contain only P and R. Solve (32) for R and substitute into (34) and we obtain:

$$FP^2 + GP + H + J(-C^{-1}AP - C^{-1}B)P + K(-C^{-1}AP - C^{-1}B) = 0$$

Collecting terms:

$$(F - JC^{-1}A)P^2 - (JC^{-1}B + KC^{-1}A - G)P - (KC^{-1}B - H) = 0 \tag{36}$$

To simplify the notation, let's express (36) as follows:

$$\Psi P^2 - \Gamma P - \Theta = 0 \tag{37}$$

This is a matrix quadratic equation of P. There are many ways to solve the equation in general, but an often-used method is to use the generalized eigenvalue problem (also called the QZ decomposition). One of the attractive features for the method is that the method does not require invertibility of Ψ matrix.

In general, suppose we have two matrices of the same size X and Y. The generalized eigenvalue problem is to find the generalize eigenvalues λ_i and generalized eigenvectors d_i satisfying:

$$Xd_i = \lambda_i Y d_i$$

If we use $Y = I$, we go back to the standard eigenvalue problem.

How do we apply the generalized eigenvalue problem to our matrix quadratic equation? Let's define the following two matrices:

$$\Xi = \begin{pmatrix} \Gamma & \Theta \\ I_m & 0_m \end{pmatrix}$$

$$\Delta = \begin{pmatrix} \Psi & 0_m \\ 0_m & I_m \end{pmatrix}$$

where I_m represents the identity matrix of size $m \times m$ and 0_m represents the $m \times m$ matrix with only zero entries. Since both Γ and Psi are $m \times n$ matrices, both Ξ and Δ are $2m \times 2m$ matrices. Now, apply the generalized eigenvalue problem to the pair of matrices.

$$\begin{pmatrix} \Gamma & \Theta \\ I_m & 0_m \end{pmatrix} \begin{pmatrix} d_{i,1} \\ d_{i,2} \end{pmatrix} = \lambda_i \begin{pmatrix} \Psi & 0_m \\ 0_m & I_m \end{pmatrix} \begin{pmatrix} d_{i,1} \\ d_{i,2} \end{pmatrix}$$

If we separate the top and bottom half of the equation, we get:

$$\Gamma d_{i,1} + \Theta d_{i,2} = \lambda_i \Psi d_{i,1}$$
$$d_{i,1} = \lambda_i d_{i,2}$$

The second equation can be used to substitute out $d_{i,1}$. Now we have only one equation (To clean up the notation, let's redefine $d_i = d_{i,2}$):

$$\Gamma \lambda_i d_i + \Theta d_i = \lambda_i \Psi \lambda_i d_i \tag{38}$$

Suppose we can find m eigenvalues corresponding m linearly independent eigenvectors. Then we have the counterpart of Blanchard-Kahn condition.

Proposition 1
If all of the m eigenvalues are inside the unit circle (i.e., $\max_i |\lambda_i| < 1$), the solution is stable.

Suppose the condition is satisfied. We can combine (38) for all i as follows:

$$\Psi\Omega\Lambda^2 - \Gamma\Omega\Lambda - \Theta\Omega = 0 \tag{39}$$

where Ω is $m \times m$ matrix which contains all the eigenvectors in each column, and Λ is $m \times m$ diagonal matrix with m eigenvalues. Specifically:

$$\Omega = \begin{bmatrix} d_1 & d_2 & \dots & d_m \end{bmatrix} \tag{40}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_m \end{bmatrix} \tag{41}$$

Multiply $(39) by \Omega^{-1}$ from the right, and we get:

$$\Psi\Omega\Lambda^2\Omega^{-1} - \Gamma\Omega\Lambda\Omega^{-1} - \Theta = 0 \tag{42}$$

Compare (42) with (37). The equation (42) implies that $P = \Omega\Lambda\Omega^{-1}$. In sum, once we implement the generalized eigenvalue decomposition with respect to Ξ and Δ, we basically got P.

Once P is obtained, (32) is used to obtain R as

$$R = -C^{-1}(AP + B) \tag{43}$$

The remaining two equations, (33) and (35), contains Q and S. Solve (33) for S and we get:

$$S = -C^{-1}(AQ + D) \tag{44}$$

Plugging into (35), and we get:

$$(FP + G + JR - KC^{-1}A)Q + (F - JC^{-1}A)QN - JC^{-1}DN - KC^{-1}D + LN + M = 0 \tag{45}$$

The equation contains only Q as unknown, but it's not trivial as Q is sandwiched in some terms. In this case, we can use the vectorization. The following is useful:

$$vec(AXB) = (B^T \otimes A)vec(X)$$

where $\otimes$ denotes the Kronecker product of the two matrices. Apply vectorization to (45) and we get:

$$vec(Q) = (I \otimes (FP + G + JR - KC^{-1}A) + (N^T \otimes (F - JC^{-1}A)))^{-1} vec(JC^{-1}DN + KC^{-1}D - LN - M) \tag{46}$$

Once Q is obtained, we can use (44) to compute S.

References

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