3.7.1 Motivation. J_2-Viscoplasticity

To develop the extension of the algorithm presented in Section **3.3.1**, we recall that classical viscoplasticity is obtained from rate-independent plasticity by replacing the consistency parameter $\gamma > 0$ with the constitutive equation

$$\boxed{\gamma = \frac{\langle f(\boldsymbol{\sigma}, \boldsymbol{q}) \rangle}{\eta} \ , \qquad \eta \in (0, \infty) \, .} \tag{3.7.1}$$

For J_2 viscoplasticity the flow potential in BOX **3.7** has the expression $f = \|\boldsymbol{\xi}\| - \sqrt{\frac{2}{3}}\big[\sigma_Y + \beta \bar{H}'\alpha\big]$ where, for simplicity, we have assumed *linear* isotropic/kinematic hardening. As in the rate-independent theory, $\beta \in [0, 1]$ is a material parameter. Assuming that $f^{\text{trial}}_{n+1} > 0$ so that viscoplastic loading takes place, then an implicit backward-Euler difference scheme yields the counterpart of the algorithmic equations (3.3.1):

$$\left.\begin{aligned}
\boldsymbol{\epsilon}^{\text{vp}}_{n+1} &= \boldsymbol{\varepsilon}^{\text{vp}}_n + \frac{f_{n+1}}{\eta} \Delta t \boldsymbol{n}_{n+1} \\
\alpha_{n+1} &= \alpha_n + \sqrt{\tfrac{2}{3}} \frac{f_{n+1}}{\eta} \Delta t \\
\boldsymbol{\beta}_{n+1} &= \boldsymbol{\beta}_n + \tfrac{2}{3} (1 - \beta) \bar{H}' \frac{f_{n+1}}{\eta} \Delta t \boldsymbol{n}_{n+1} \, ,
\end{aligned}\right\} \tag{3.7.2}$$

where $\boldsymbol{n}_{n+1} := \boldsymbol{\xi}_{n+1} / \|\boldsymbol{\xi}_{n+1}\|$. An argument identical to that presented in Section **3.3.1** leads to the result

$$\boxed{\boldsymbol{n}_{n+1} = \boldsymbol{\xi}^{\text{trial}}_{n+1} \big/ \big\| \boldsymbol{\xi}^{\text{trial}}_{n+1} \big\|,} \tag{3.7.3}$$

along with the condition

$$\boxed{\|\boldsymbol{\xi}_{n+1}\| = \big\|\boldsymbol{\xi}^{\text{trial}}_{n+1}\big\| - \frac{2\mu\Delta t}{\eta} \langle f_{n+1} \rangle \left[1 + (1 - \beta)\frac{\bar{H}'}{3\mu}\right] \, ,} \tag{3.7.4}$$

where $\boldsymbol{\xi}_{n+1} := \boldsymbol{\sigma}_{n+1} - \boldsymbol{\beta}_{n+1}$, and $\boldsymbol{\xi}^{\text{trial}}_{n+1}$ is defined by (3.3.2). From an algorithmic standpoint the only difference with the rate-independent case concerns the enforcement of the counterpart of the consistency condition. Now it follows from (3.7.4) that

$$\boxed{\Delta\gamma_{n+1} := \langle f_{n+1} \rangle \frac{\Delta t}{\eta} = \frac{\langle f^{\text{trial}}_{n+1} \rangle \, /2\mu}{\frac{\tau}{\Delta t} + \left[1 + \frac{\bar{H}'}{3\mu}\right]} \ , \qquad \tau := \frac{\eta}{2\mu} \ ,} \tag{3.7.5}$$

where f^{trial}_{n+1} is defined by (3.2.3) and τ is the *relaxation time* (see (2.7.12)). The preceding analysis easily extends to the case of nonlinear kinematic/isotropic hardening by considering a local iterative procedure analogous to that summarized in BOX **3.1.**

3.7.1.1 Linearization.

By differentiating the algorithm along the lines discussed in Section **3.3.2**, one obtains the *algorithmic consistent viscoplastic tangent moduli*. In particular, for $\Delta\gamma_{n+1}$,

$$\frac{\partial f_{n+1}^{\text{trial}}}{\partial \varepsilon_{n+1}} = 2\mu \boldsymbol{n}_{n+1} \;\Rightarrow\; \frac{\partial \Delta\gamma_{n+1}}{\partial \varepsilon_{n+1}} = \frac{\boldsymbol{n}_{n+1}}{\frac{\eta}{2\mu\Delta t} + \left[1 + \frac{\bar{H}'}{3\mu}\right]} \,. \tag{3.7.6}$$

Moreover, since $\boldsymbol{\sigma}_{n+1} = \kappa \operatorname{tr}[\varepsilon_{n+1}]\mathbf{1} + 2\mu(\boldsymbol{e}_{n+1}^p - \boldsymbol{e}_n^p - \Delta\gamma_{n+1}\boldsymbol{n}_{n+1})$, Lemma **3.2** and (3.7.6) yield the expression recorded in BOX **3.7**, which also includes a step-by-step summary of the algorithm.

Remark 3.7.1. From expression (3.7.5),

$$\Delta\gamma_{n+1} := f_{n+1}\frac{\Delta t}{\eta} \;\to\; \frac{f_{n+1}^{\text{trial}}/2\mu}{1 + \frac{\bar{H}'}{3\mu}} \,, \quad \text{as } \tau/\Delta t \;\to\; 0 \,, \tag{3.7.7}$$

which coincides with expression (3.3.7) for γ_{n+1} in the rate-independent case. This illustrates the fact that, as the ratio of the relaxation time over the time step goes to zero, i.e., $\tau/\Delta t \to 0$, one recovers the rate-independent limit; in agreement with the conclusions obtained in Section **1.9.1** in analyzing the continuum problem.

3.7.2 Closest Point Projection

The iterative procedure is analogous to that for the rate-independent case. One simply needs to observe the following:

1. For perfect viscoplasticity, $\Delta\gamma_{n+1} = \Delta t f_{n+1}/\eta$, where $f_{n+1} = f(\boldsymbol{\sigma}_{n+1})$.
2. Since $\boldsymbol{R}_{n+1} = -\varepsilon_{n+1}^{\text{vp}} + \varepsilon_n^{\text{vp}} + \Delta\gamma_{n+1}\nabla f_{n+1}$, the linearization yields

$$\frac{\partial \boldsymbol{R}_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}} = \left[\mathbf{C}^{-1} + \Delta\gamma_{n+1}\nabla^2 f_{n+1}\right] + \frac{\Delta t}{\eta}\left[\nabla f_{n+1} \otimes \nabla f_{n+1}\right] \tag{3.7.8}$$

where, for simplicity, we have restricted our attention to perfect viscoplasticity. The general case is handled along similar lines. With these observations in mind, for convenience the iterative scheme is summarized in BOX **3.8**.

3.7.3 A Note on Notational Conventions

In order to minimize confusion, we wish to review our notational conventions. The symbol Δ is used to denote an incremental quantity, such as the increment over a time step, or an increment between successive iterations. We typically do not use separate notations to distinguish between these two cases, relying on context to make the intent clear. On the other hand, we often encounter situations in which the rate-of-slip, γ, appears in both incremental forms within an alogrithm, and even a single equation; see, e.g., Box **3.5**. In these cases, we adhere to the following conventions: $\Delta\gamma = \gamma\,\Delta t$ denotes the increment of γ over a time step, and $\Delta^2\gamma$ denotes

BOX 3.7. J_2-Viscoplasticity.
Linear Isotropic/Kinematic Hardening.

1. Compute *trial elastic* stress:

$$\begin{aligned} \boldsymbol{e}_{n+1} &= \boldsymbol{\varepsilon}_{n+1} - \tfrac{1}{3}(\mathrm{tr}[\boldsymbol{\varepsilon}_{n+1}])\mathbf{1} \\ \boldsymbol{s}^{\mathrm{trial}}_{n+1} &= 2\mu\big(\boldsymbol{e}_{n+1} - \boldsymbol{e}^{p}_{n}\big) \\ \boldsymbol{\xi}^{\mathrm{trial}}_{n+1} &= \boldsymbol{s}^{\mathrm{trial}}_{n+1} - \boldsymbol{\beta}_{n} \end{aligned}$$

2. Check viscoplastic flow potential

$$f^{\mathrm{trial}}_{n+1} := \left\|\boldsymbol{\xi}^{\mathrm{trial}}_{n+1}\right\| - \sqrt{\tfrac{2}{3}}\,(\sigma_Y + \beta\bar{H}'\alpha_n)$$

IF: $f^{\mathrm{trial}}_{n+1} \le 0$

Set $(\bullet)_{n+1} = (\bullet)^{\mathrm{trial}}_{n+1}$ & *EXIT*

ENDIF

3. Compute $\boldsymbol{n}_{n+1}$ and $\Delta\gamma_{n+1} := f_{n+1}\Delta t/\eta$

$$\boldsymbol{n}_{n+1} := \boldsymbol{\xi}^{\mathrm{trial}}_{n+1}\Big/\left\|\boldsymbol{\xi}^{\mathrm{trial}}_{n+1}\right\|$$

$$\Delta\gamma_{n+1} := \frac{f^{\mathrm{trial}}_{n+1}/2\mu}{\frac{\eta}{2\mu\Delta t} + \left[1 + \frac{\bar{H}'}{3\mu}\right]}$$

4. Update back stress, viscoplastic strain, and stress

$$\begin{aligned} \boldsymbol{\beta}_{n+1} &= \boldsymbol{\beta}_{n} + \tfrac{2}{3}(1-\beta)\bar{H}'\Delta\gamma_{n+1}\boldsymbol{n}_{n+1} \\ \alpha_{n+1} &= \alpha_n + \sqrt{\tfrac{2}{3}}\,\Delta\gamma_{n+1} \\ \boldsymbol{\varepsilon}^{\mathrm{vp}}_{n+1} &= \boldsymbol{\varepsilon}^{\mathrm{vp}}_{n} + \Delta\gamma_{n+1}\boldsymbol{n}_{n+1} \\ \boldsymbol{\sigma}_{n+1} &= \kappa\,\mathrm{tr}[\boldsymbol{\varepsilon}_{n+1}]\mathbf{1} + \boldsymbol{s}^{\mathrm{trial}}_{n+1} - 2\mu\Delta\gamma_{n+1}\boldsymbol{n}_{n+1} \end{aligned}$$

5. Compute *consistent viscoplastic tangent moduli*

$$\mathbf{C}_{n+1} = \kappa\mathbf{1}\otimes\mathbf{1} + 2\mu\theta_{n+1}\big[\mathbf{I} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1}\big] - 2\mu\bar{\theta}_{n+1}\boldsymbol{n}_{n+1}\otimes\boldsymbol{n}_{n+1}$$

$$\theta_{n+1} := \left[1 - \frac{2\mu\Delta\gamma_{n+1}}{\left\|\boldsymbol{\xi}^{\mathrm{trial}}_{n+1}\right\|}\right]$$

$$\bar{\theta}_{n+1} := \left[\frac{1}{\frac{\eta}{2\mu\Delta t} + \left(1 + \frac{\bar{H}'}{3\mu}\right)} - \frac{2\mu\Delta\gamma_{n+1}}{\left\|\boldsymbol{\xi}^{\mathrm{trial}}_{n+1}\right\|}\right]$$

the increment of $\Delta\gamma$ between iterations. For example, we write

$$\Delta\gamma^{(k+1)} = \Delta\gamma^{(k)} + \Delta^2\gamma^{(k)}, \tag{3.7.9}$$

where k is the iteration number.

BOX 3.8. Perfect Viscoplasticity.
Closest Point Projection Iteration.

1. Initialize: $k = 0$, $\varepsilon_{n+1}^{\mathrm{vp}(0)} = \varepsilon_n^{\mathrm{vp}}$, $\boldsymbol{\sigma}_{n+1}^{(0)} := \nabla W\big(\nabla^s \boldsymbol{u}_{n+1} - \varepsilon_n^{\mathrm{vp}}\big)$.

IF: $f(\boldsymbol{\sigma}_{n+1}^{(0)}) \leq 0$

Set $\varepsilon_{n+1}^{\mathrm{vp}} = \varepsilon_n^{\mathrm{vp}}$ & *EXIT*

ELSE:

2. Return mapping iterative algorithm

2.a. Compute residuals

$$\begin{aligned}
\boldsymbol{\sigma}_{n+1}^{(k)} &:= \nabla W\big(\nabla^s \boldsymbol{u}_{n+1} - \varepsilon_{n+1}^{\mathrm{vp}^{(k)}}\big)\\
f_{n+1}^{(k)} &:= f\big(\boldsymbol{\sigma}_{n+1}^{(k)}\big)\\
\Delta\gamma_{n+1}^{(k)} &:= \Delta t f_{n+1}^{(k)}/\eta\\
\boldsymbol{R}_{n+1}^{(k)} &:= -\varepsilon_{n+1}^{\mathrm{vp}(k)} + \varepsilon_n^{\mathrm{vp}} + \Delta\gamma_{n+1}^{(k)} \nabla f\big(\boldsymbol{\sigma}_{n+1}^{(k)}\big)
\end{aligned}$$

2.b. Check convergence

IF: $\big\|\boldsymbol{R}_{n+1}^{(k)}\big\| < \mathrm{TOL}_1$ THEN:

Set $\varepsilon_{n+1}^{\mathrm{vp}} = \varepsilon_{n+1}^{\mathrm{vp}^{(k)}}$ & *EXIT*

ELSE:

2.b. Compute consistent (algorithmic) tangent moduli

$$\begin{aligned}
\mathbf{C}_{n+1}^{(k)} &:= \nabla^2 W\big(\nabla^s \boldsymbol{u}_{n+1} - \varepsilon_{n+1}^{\mathrm{vp}^{(k)}}\big)\\
\boldsymbol{\Xi}_{n+1}^{(k)^{-1}} &:= \Big[\mathbf{C}_{n+1}^{-1} + \Delta\gamma_{n+1}^{(k)} \nabla^2 f\big(\boldsymbol{\sigma}_{n+1}^{(k)}\big)\Big]\\
\boldsymbol{\Xi}_{n+1}^{\mathrm{vp}^{(k)}} &:= \left[\boldsymbol{\Xi}_{n+1}^{(k)^{-1}} + \frac{\Delta t}{\eta} \nabla f_{n+1}^{(k)} \otimes \nabla f_{n+1}^{(k)}\right]^{-1}
\end{aligned}$$

2.c. Compute kth increments

$$\Delta\varepsilon_{n+1}^{\mathrm{vp}^{(k)}} = \mathbf{C}_{n+1}^{-1} : \boldsymbol{\Xi}_{n+1}^{\mathrm{vp}^{(k)}} : \boldsymbol{R}_{n+1}^{(k)}$$

2.d. Update viscoplastic strain

$$\varepsilon_{n+1}^{\mathrm{vp}^{(k+1)}} = \varepsilon_{n+1}^{\mathrm{vp}^{(k)}} + \Delta\varepsilon_{n+1}^{\mathrm{vp}^{(k)}}$$

Set $k \leftarrow k + 1$ & GO TO **2.a**.