6 Von Mises plasticity with isotropic and kinematic hardening for small strains

References

A. Menzel. Theory of Materials. Lecture Notes, TU Dortmund, Institute of Mechanics.

J.C. Simo and T.J.R. Hughes. Computational Inelasticity. Volume 7 of Interdisciplinary Applied Mathematics, Springer, 1998.

6.1 Theory

6.1.1 Structure of algorithmic material models with internal variables

In the framework of continuum mechanics the concept of internal variables is employed to describe inelastic deformation processes. The assumption of local material behaviour leads to the continuous stress formulation

$$\boldsymbol{\sigma} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \boldsymbol{k}) \,, \tag{6.1}$$

where $\widehat{\boldsymbol{\sigma}}$ is a continuous stress function, the arguments of which are the strain tensor $\boldsymbol{\varepsilon}$ as well as a set of internal variables $\boldsymbol{k}$ which characterise the dissipative properties of a particular material model. The evolution in time of the internal variables $\boldsymbol{k}$ is given by the evolution equation

$$\dot{\boldsymbol{k}} = \boldsymbol{\nu}(\boldsymbol{\varepsilon}, \boldsymbol{k}) \quad \text{with} \quad \boldsymbol{k}(t = t_0) = \boldsymbol{k}_0 \,, \tag{6.2}$$

where $\boldsymbol{\nu}$ is a continuous evolution operator and $\boldsymbol{k}_0$ is an initial value for time step t_0. This class of inelastic material models is represented by a system of coupled first-order differential equations. Such problems can be solved using a framework of numerical integration in which a particular time interval $[t_n, t_{n+1}]$ is considered, knowing all variables at time step t_n. Classical stress update algorithms compute the stress $\boldsymbol{\sigma}_{n+1}$ at time step t_{n+1} with a preset deformation $\boldsymbol{\varepsilon}_{n+1}$ at time step t_{n+1} and are referred to as *deformation driven algorithmic material models*. A decisive factor for the algorithmic evaluation of the model is the choice of the numerical integration scheme to be used for the set of evolution equations presented in (6.2). The integration algorithm

$$\boldsymbol{k}_{n+1} = \boldsymbol{k}_n + \Delta t \left[[1-\alpha] \, \boldsymbol{\nu}(\boldsymbol{\varepsilon}_n, \boldsymbol{k}_n) + \alpha \, \boldsymbol{\nu}(\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{k}_{n+1}) \right] \tag{6.3}$$

represents a suitable form of the stress update algorithm. Here, $\Delta t = t_{n+1} - t_n$ is the time increment and $\alpha \in [0, 1]$ is an integration operator which provides the explicit Euler method for $\alpha = 0$ and relates to the implicit Euler method for $\alpha = 1$. For $\alpha = 0.5$ the trapezoidal rule is obtained which is a second-order method unlinke the Euler methods which are first-order methods. Reformulation of equation (6.2) as a deformation driven update algorithm $\widehat{\boldsymbol{k}}_{n+1}$ leads to the symbolic expression

$$\boldsymbol{k}_{n+1} = \widehat{\boldsymbol{k}}_{n+1}(\boldsymbol{\varepsilon}_{n+1}; \boldsymbol{\varepsilon}_n, \boldsymbol{k}_n) \tag{6.4}$$

which calculates the algorithmic internal variables $\boldsymbol{k}_{n+1}$ at time step t_{n+1} based on the preset deformation $\boldsymbol{\varepsilon}_{n+1}$ at time step t_{n+1} and the known variables $\boldsymbol{\varepsilon}_n$ and $\boldsymbol{k}_n$ at the previous time step t_n. For linear evolution equations the update $\boldsymbol{k}_{n+1}$ follows directly from equation (6.4) whereas, in the case of non-linear evolution operators $\widehat{\boldsymbol{k}}_{n+1}$, equation (6.4) is an algorithmic box whereby $\boldsymbol{k}_{n+1}$ is determined iteratively. Now we carry out the algorithmic formulation of equation (6.1)

$$\boldsymbol{\sigma}_{n+1} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{k}_{n+1}) \tag{6.5}$$

and insert the expression (6.4)

$$\boldsymbol{\sigma}_{n+1} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}_{n+1}, \widehat{\boldsymbol{k}}_{n+1}(\boldsymbol{\varepsilon}_{n+1}; \boldsymbol{\varepsilon}_n, \boldsymbol{k}_n))\,. \tag{6.6}$$

The algorithmic stress now can be transformed to the function

$$\boldsymbol{\sigma}_{n+1} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\varepsilon}_n, \boldsymbol{k}_n)\,. \tag{6.7}$$

The algorithmic stresses $\boldsymbol{\sigma}_{n+1}$ at the end of time step t_{n+1} are calculated by the preset deformation $\boldsymbol{\varepsilon}_{n+1}$ at time step t_{n+1} and by the history variables $\boldsymbol{\varepsilon}_n$ and $\boldsymbol{k}_n$ at the beginning of the time interval. It is important that, in the framework of a deformation driven update algorithm, the algorithmic constitutive equations of inelasticity are formally considered to be non-linear depending on the total deformation $\boldsymbol{\varepsilon}_{n+1}$ while the history variables $\boldsymbol{\varepsilon}_n$ and $\boldsymbol{k}_n$ are considered to be „frozen“.

For the iterative solution of boundary value problems by means of the Finite Element Method and, respectively, the constitutive driver algorithm developed within tutorials 3 and 4, the sensitivities of the algorithmic stress $\mathrm{d}\boldsymbol{\sigma}_{n+1}$ with respect to changes of the total deformation $\mathrm{d}\boldsymbol{\varepsilon}_{n+1}$ are needed. By means of algorithmic consistent tangent operators $\mathsf{E}_{\mathrm{a}\,n+1}$, the sensitivities result in

$$\mathrm{d}\boldsymbol{\sigma}_{n+1} = \mathsf{E}_{\mathrm{a}\,n+1} : \mathrm{d}\boldsymbol{\varepsilon}_{n+1}\,. \tag{6.8}$$

Formally they are calculated as the derivatives of the algorithmic stress $\boldsymbol{\sigma}_{n+1}$ with respect to the total deformation $\boldsymbol{\varepsilon}_{n+1}$ at time step t_{n+1}. The tangent moduli

$$\mathsf{E}_{\mathrm{a}\,n+1} \doteq \frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} \tag{6.9}$$

at time step t_{n+1} are fourth-order tensors which are generally provided in an analytical manner by the constitutive routine. However, by using calculations of perturbations, these can be approximated by so-called „numerical tangent moduli“.

The driver algorithm dealt with in earlier exercises is now to be extended by a constitutive law with internal variables. The underlying structure of the algorithm preserves as it is shown in box 6.1. At each iteration step the constitutive box 6.1 is evaluated for a retained axial deformation ε_{n+1}. The iteration is proceeded as long as the convergence criterion

$$\|\bar{\boldsymbol{\sigma}}\| < tol \tag{6.10}$$

is met, with $tol \ll 1$. After the convergence criterion is met, the internal variables are updated $\boldsymbol{k}_n \leftarrow \boldsymbol{k}_{n+1}$ and the time step is increased $t_{n+1} \leftarrow t_{n+1} + \Delta t$ with a new preset axial deformation $\varepsilon_{n+1}(t_{n+1})$.

6.1.2 Basic equations of classical rate-independent plasticity

1. **Kinematics: Additive decomposition of the strain tensor**
   In the framework of plasticity for small strains, one assumes that the strain tensor $\boldsymbol{\varepsilon}$ can be decomposed into an elastic part $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and a plastic part $\boldsymbol{\varepsilon}^{\mathrm{p}}$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}} \,. \tag{6.11}$$

   Herein, the plastic strains $\boldsymbol{\varepsilon}^{\mathrm{p}}$ are *internal variables*, the time evolution of which is given by the flow rule which is an evolution equation given below, see 4.

2. **Elastic stress-strain-relation**
   The stress tensor $\boldsymbol{\sigma}$ is related to the elastic strains $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}$ by means of the strain-energy function Ψ via the hyperelastic relationship

$$\boldsymbol{\sigma} = \frac{\partial \Psi(\boldsymbol{\varepsilon}^{\mathrm{e}})}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} \,. \tag{6.12}$$

   In the framework of linear elasticity, the strain-energy function is expressed by a quadratic function in dependence of $\boldsymbol{\varepsilon}$, i.e. $\Psi = 1/2\, \boldsymbol{\varepsilon}^{\mathrm{e}} : \mathsf{E}^{\mathrm{e}} : \boldsymbol{\varepsilon}^{\mathrm{e}}$, where E^{e} is the constant, fourth-order elastic tangent moduli tensor. Then, equations (6.11) and (6.12) result in

$$\boldsymbol{\sigma} = \mathsf{E}^{\mathrm{e}} : [\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}] \,. \tag{6.13}$$

3. **Elastic domain in stress space and yield condition**
   The so-called *yield function* $\Phi(\boldsymbol{\sigma},\, \boldsymbol{\kappa})$ limits the admissible states of $\{\boldsymbol{\sigma},\, \boldsymbol{\kappa}\}$ in stress space to a domain $\mathbb{E}$. Mathematically, this is expressed by

$$\mathbb{E} = \{(\boldsymbol{\sigma},\, \boldsymbol{\kappa}) | \Phi(\boldsymbol{\sigma},\, \boldsymbol{\kappa}) \leq 0\} \,. \tag{6.14}$$

   The interior of $\mathbb{E}$ is described by $\Phi(\boldsymbol{\sigma},\, \boldsymbol{\kappa}) < 0$ and is also referred to as *elastic region*, whereas $\Phi(\boldsymbol{\sigma},\, \boldsymbol{\kappa}) = 0$ characterizes the boundary of this domain. That boundary is referred to as *yield surface* which represents plastic states. As a remark, all states $\{\boldsymbol{\sigma},\, \boldsymbol{\kappa}\}$ outside $\mathbb{E}$ are non-admissible and therefore excluded within the framework of classical rate-independent plasticity.

4. **Flow rule and hardening law. Loading/unloading conditions**
   The irreversibility of plastic flow is represented by the following, not necessarily associative, evolution equations for $(\boldsymbol{\varepsilon}^{\mathrm{p}},\, \boldsymbol{\kappa})$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda\, \boldsymbol{\nu}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{6.15}$$
$$\dot{\boldsymbol{k}} = \lambda\, \boldsymbol{\zeta}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \,, \tag{6.16}$$

   where equation (6.15) is denoted as *flow rule* and equation (6.16) is referred to as *hardening law*. Here, $\boldsymbol{\nu}$ and $\boldsymbol{\zeta}$ are prescribed functions which define the direction of plastic flow as well as the type of hardening.

Within the framework of plasticity, it is common to choose a so-called „associated“ structure, i.e. the functions $\boldsymbol{\nu}$ and $\boldsymbol{\zeta}$ are derivatives of the *yield function* $\Phi(\boldsymbol{\sigma},\,\boldsymbol{\kappa})$ with respect to $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$, as a result of the postulate of maximum plastic dissipation, cf. also the dissipation inequality $\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{\kappa} \bullet \dot{\boldsymbol{k}} \geq 0$.

The special associated structure of the flow rule and the hardening law result in

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda\,\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \lambda\,\boldsymbol{\nu} \tag{6.17}$$

$$\dot{\boldsymbol{k}} = -\lambda\,\frac{\partial \Phi}{\partial \boldsymbol{\kappa}} = -\lambda\,\boldsymbol{\zeta}\,. \tag{6.18}$$

The consistency parameter, or, Lagrange multiplier $\lambda \geq 0$ has to satisfy the so-called *Kuhn-Tucker complementary conditions*

$$\lambda \geq 0\,, \quad \Phi(\boldsymbol{\sigma},\boldsymbol{\kappa}) \leq 0\,, \quad \lambda\,\Phi(\boldsymbol{\sigma},\boldsymbol{\kappa}) = 0\,, \tag{6.19}$$

which are also referred to as *loading/unloading conditions*. In addition, λ has to satisfy the *consistency condition*

$$\lambda\,\dot{\Phi}(\boldsymbol{\sigma},\boldsymbol{\kappa}) = 0\,. \tag{6.20}$$

The Kuhn-Tucker complementary conditions can be explained and interpreted according to the following settings:

a) A state $\{\boldsymbol{\sigma},\,\boldsymbol{\kappa}\}$ *inside* of $\mathbb{E}$ is considered such that $\Phi(\boldsymbol{\sigma},\,\boldsymbol{\kappa}) < 0$ holds. From condition $(6.19)_3$ we conclude that

$$\lambda\,\Phi = 0 \quad \text{and} \quad \Phi < 0 \quad \longrightarrow \quad \lambda = 0\,. \tag{6.21}$$

From equation (6.15) we observe that the internal variables are not evolving within the considered time interval, such that the material behaviour is *purely elastic* and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \boldsymbol{0}$ as well as $\dot{\boldsymbol{k}} = \boldsymbol{0}$.

b) A state $\{\boldsymbol{\sigma},\,\boldsymbol{\kappa}\}$ *on the boundary* of $\mathbb{E}$ is considered such that $\Phi(\boldsymbol{\sigma},\,\boldsymbol{\kappa}) = 0$ holds. In this case, condition $(6.19)_3$ is fulfilled automatically, even if $\lambda > 0$. Whether λ is positive, negative or zero, can be decided by means of condition (6.20). Hereby, two different cases are to be differed:

  i. If $\dot{\Phi}(\boldsymbol{\sigma},\,\boldsymbol{\kappa}) < 0$, from condition (6.20) we obtain

$$\lambda\,\dot{\Phi} = 0 \quad \text{and} \quad \dot{\Phi} < 0 \quad \longrightarrow \quad \lambda = 0\,. \tag{6.22}$$

  From equation (6.15), in turn, it follows that the internal variables do not evolve over time, i.e. $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \boldsymbol{0}$ and $\dot{\boldsymbol{k}} = \boldsymbol{0}$. Because $\{\boldsymbol{\sigma},\,\boldsymbol{\kappa}\}$ lies *on the boundary* of $\mathbb{E}$, this state is called *elastic unloading*.

  ii. If $\dot{\Phi}(\boldsymbol{\sigma},\,\boldsymbol{\kappa}) = 0$, condition (6.20) is fulfilled automatically. If $\lambda > 0$, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \neq \boldsymbol{0}$ and $\dot{\boldsymbol{k}} \neq \boldsymbol{0}$ hold. This state is called *plastic loading*. In the case of $\lambda = 0$ one speaks of *neutral loading*.

6.1.3 Von Mises plasticity with isotropic and kinematic hardening

Within the framework of metal plasticity, the internal plastic variables are typically given as $\boldsymbol{k} = \{k\,,\,\boldsymbol{a}\}$. Here, k indicates the equivalent plastic strain that defines the isotropic hardening of the von Mises yield surface, and $\boldsymbol{a}$ defines the center of the von Mises yield surface in stress deviator space at kinematic hardening. By application of the quadratic form of the free energy

$$\Psi = \frac{K}{2}\,[\boldsymbol{\varepsilon}^{\mathrm{e}} : \boldsymbol{I}]^2 + G\,[\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}}]^2 : \boldsymbol{I} + \frac{H}{2}\,r\,k^2 + \frac{H}{2}\,[1-r]\,a_{\mathrm{e}}^2 \tag{6.23}$$

with

$$a_{\mathrm{e}} = \sqrt{\frac{2}{3}}\,\|\boldsymbol{a}\|\,, \tag{6.24}$$

the elastic stresses can be expressed by

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = \boldsymbol{\sigma}_{\mathrm{vol}} + \boldsymbol{\sigma}_{\mathrm{dev}} = \sigma_{\mathrm{m}}\,\boldsymbol{I} + \boldsymbol{\sigma}_{\mathrm{dev}} \tag{6.25}$$

with the volumetric and deviatoric parts

$$\sigma_{\mathrm{m}} = K\,\varepsilon^{\mathrm{e}}_{\mathrm{vol}} = K\,[\boldsymbol{\varepsilon}^{\mathrm{e}} : \boldsymbol{I}]\,, \quad \text{respectively} \quad \boldsymbol{\sigma}_{\mathrm{dev}} = 2\,G\,\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} = 2\,G\,[\boldsymbol{\varepsilon}_{\mathrm{dev}} - \boldsymbol{\varepsilon}^{\mathrm{p}}] \tag{6.26}$$

as well as the hardening stresses

$$\kappa = -\,\frac{\partial \Psi}{\partial k} = -\,r\,H\,k \tag{6.27}$$

$$\boldsymbol{\alpha} = -\,\frac{\partial \Psi}{\partial \boldsymbol{a}} = -\,\frac{2}{3}\,[1-r]\,H\,\boldsymbol{a}\,. \tag{6.28}$$

Furthermore, combining the von Mises yield function

$$\Phi = \|\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}\| - \sqrt{\frac{2}{3}}\,[\sigma_{\mathrm{y}} - \kappa] \le 0 \tag{6.29}$$

which includes the reduced deviatoric stress

$$\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}} = \boldsymbol{\sigma}_{\mathrm{dev}} + \boldsymbol{\alpha}\,, \tag{6.30}$$

with the hardening laws (6.18), leads to the associated evolution equations

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda\,\boldsymbol{\nu}\,, \qquad \boldsymbol{\nu} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}}{\|\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}\|} \tag{6.31}$$

$$\dot{k} = \lambda\,\zeta_\kappa\,, \quad \zeta_\kappa = \frac{\partial \Phi}{\partial \kappa} = \sqrt{\frac{2}{3}} \tag{6.32}$$

$$\dot{\boldsymbol{a}} = \lambda\,\boldsymbol{\zeta}_\alpha\,, \quad \boldsymbol{\zeta}_\alpha = \frac{\partial \Phi}{\partial \boldsymbol{\alpha}} = \frac{\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}}{\|\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}\|} = \boldsymbol{\nu}\,. \tag{6.33}$$

Here, H represents the so-called hardening modulus, σ_{y} represents the yield stress, $r \in [0,1]$ is a parameter which specifies the hardening mechanism, i.e. $r = 1$: isotropic hardening $r = 0$: kinematic hardening, and $\boldsymbol{\nu}$ represents the normalised tensor orthogonal to the von Mises yield surface. The relationships specified above are briefly summarised in Box 6.2.

6.1.4 Algorithmic structure

The algorithm developed within this section is based on the equations given in Box 6.2. It applies to three-dimensional stress states as well as to plane-strain cases. The simplicity of the von Mises yield criterion (6.29) enables a closed-form solution without the need of additional local Newton iterations. This algorithm is called „radial return algorithm“.

Radial return mapping

Within the previous section, the internal plastic variables were introduced as $\boldsymbol{k} = \{k\,,\,\boldsymbol{a}\}$. Furthermore, the reduced deviatoric stress was defined as

$$\boldsymbol{\sigma}^{\text{red}}_{\text{dev}} = \boldsymbol{\sigma}_{\text{dev}} + \boldsymbol{\alpha}\,. \tag{6.34}$$

At the the end of a time step $[t_n,\, t_{n+1}]$, the normalised tensor, orthogonal to the von Mises yield surface, reads

$$\boldsymbol{\nu}_{n+1} = \frac{\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1}}{\|\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1}\|}\,. \tag{6.35}$$

By means of the Backward Euler Method, the evolution equations (6.31)-(6.33) transform to the discretised structure

$$\boldsymbol{\varepsilon}^{\text{p}}_{n+1} = \boldsymbol{\varepsilon}^{\text{p}}_{n} + \Delta\lambda_{n+1}\,\boldsymbol{\nu}_{n+1} \tag{6.36}$$

$$k_{n+1} = k_n + \Delta\lambda_{n+1}\sqrt{2/3} \tag{6.37}$$

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta\lambda_{n+1}\,\boldsymbol{\nu}_{n+1} \tag{6.38}$$

where $\Delta\lambda = \Delta t\,\lambda$.

Deviatoric reduced stresses. The devatoric reduced stresses $\boldsymbol{\sigma}^{\text{red}}_{\text{dev}\,n+1}$ specified in equation (6.34) are calculated as

$$\boldsymbol{\sigma}^{\text{red}}_{\text{dev}\,n+1} = \boldsymbol{\sigma}_{\text{dev}\,n+1} + \boldsymbol{\alpha}_{n+1} \tag{6.39}$$

$$= 2\,G\,[\boldsymbol{\varepsilon}_{\text{dev}\,n+1} - \boldsymbol{\varepsilon}^{\text{p}}_{n+1}] - \frac{2}{3}\,[1-r]\,H\,\boldsymbol{a}_{n+1}\,, \tag{6.40}$$

by means of equation (6.33). Inserting the evolution equations (6.36) and (6.38) leads to

$$\boldsymbol{\sigma}^{\text{red}}_{\text{dev}\,n+1} = \boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1} - \Delta\lambda_{n+1}\left[2\,G + \frac{2}{3}\,[1-r]\,H\right]\boldsymbol{\nu}_{n+1}\,, \tag{6.41}$$

where

$$\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1} = \boldsymbol{\sigma}^{\text{tr}}_{\text{dev}\,n+1} + \boldsymbol{\alpha}^{\text{tr}}_{n+1} \tag{6.42}$$

with

$$\boldsymbol{\sigma}^{\text{tr}}_{\text{dev}\,n+1} = 2\,G\,[\boldsymbol{\varepsilon}_{\text{dev}\,n+1} - \boldsymbol{\varepsilon}^{\text{p}}_{n}] \tag{6.43}$$

and

$$\boldsymbol{\alpha}^{\text{tr}}_{n+1} = -\frac{2}{3}\,[1-r]H\,\boldsymbol{a}_n\,. \tag{6.44}$$

Consistency parameter. To determine the consistency parameter $\Delta\lambda_{n+1}$, we recall that by definition $\boldsymbol{\sigma}^{\text{red}}_{\text{dev}\,n+1} = \|\boldsymbol{\sigma}^{\text{red}}_{\text{dev}\,n+1}\|\,\boldsymbol{\nu}_{n+1}$ holds. Hence, the normalised tensor $\boldsymbol{\nu}_{n+1}$ is solely calculated by means of the elastic trial-stresses $\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1}$

$$\boldsymbol{\nu}_{n+1} = \frac{\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1}}{\|\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1}\|}\,. \tag{6.45}$$

The von Mises yield criterion according to equation (6.29) reads

$$\Phi_{n+1} = \|\boldsymbol{\sigma}^{\text{red}}_{\text{dev}\,n+1}\| - \sqrt{\frac{2}{3}}\,[\sigma_{\text{y}} - \kappa_{n+1}]\,. \tag{6.46}$$

By means of the hardening law (6.27) and by the evolution equation (6.37) as well as with the definition

$$\Phi^{\text{tr}}_{n+1} = \|\boldsymbol{\sigma}^{\text{red tr}}_{\text{dev}\,n+1}\| - \sqrt{\frac{2}{3}}\,[\sigma_{\text{y}} - \kappa^{\text{tr}}_{n}] \quad \text{with} \quad \kappa^{\text{tr}}_{n} = -r\,H\,k_n\,, \tag{6.47}$$

the consistency parameter is calculated by

$$\Delta\lambda_{n+1} = \frac{\Phi^{\text{tr}}_{n+1}}{2\,G + \frac{2}{3}\,H}\,. \tag{6.48}$$

The relations stated above are briefly summarised in Box 6.3. The next step is the determination of the consistent elasto-plastic tangent moduli by exact linearisation of the return mapping algorithm. These moduli link incremental strains and incremental stresses and are decisive for the solution of general boundary value problems.

Exact linearisation of the algorithm

Differentiation of the algorithmic expression of the stress tensor

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{\text{vol}\,n+1} + \boldsymbol{\sigma}_{\text{dev}\,n+1} \tag{6.49}$$

leads to

$$\text{d}\boldsymbol{\sigma}_{n+1} = \mathsf{\mathbf{E}}_{\text{a}\,n+1} : \text{d}\boldsymbol{\varepsilon}_{n+1} \tag{6.50}$$

with the elasto-plastic tangent moduli

$$\mathsf{\mathbf{E}}_{\text{a}\,n+1} := \frac{\text{d}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_{n+1})}{\text{d}\boldsymbol{\varepsilon}_{n+1}}\,, \tag{6.51}$$

which is specified below.

By partial differentiation of the volumetric stresses

$$\boldsymbol{\sigma}_{\text{vol}\,n+1} = K\,[\boldsymbol{\varepsilon}_{n+1} : \boldsymbol{I}]\,\boldsymbol{I} \tag{6.52}$$

as well as the deviatoric stresses

$$\boldsymbol{\sigma}_{\mathrm{dev}\,n+1} = \boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1} - \Delta\lambda_{n+1}\, 2\, G\, \boldsymbol{\nu}_{n+1} \tag{6.53}$$

we obtain

$$\mathsf{E}_{\mathrm{a}\,n+1} = \frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} = \frac{\mathrm{d}\boldsymbol{\sigma}^{\mathrm{vol}}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} + \frac{\mathrm{d}\boldsymbol{\sigma}^{\mathrm{dev}}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} \tag{6.54}$$

with the volumetric part

$$\frac{\mathrm{d}\boldsymbol{\sigma}^{\mathrm{vol}}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} \left[K \left[\boldsymbol{\varepsilon}_{n+1} : \boldsymbol{I} \right] \boldsymbol{I} \right] = K\, \boldsymbol{I} \otimes \boldsymbol{I} \tag{6.55}$$

and the deviatoric part

$$\frac{\mathrm{d}\boldsymbol{\sigma}^{\mathrm{dev}}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} \left[\boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1} - \Delta\lambda_{n+1}\, 2\, G\, \boldsymbol{\nu}_{n+1} \right] \tag{6.56}$$

$$= \frac{\partial \boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} - 2\, G \left[\frac{\partial \Delta\lambda_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\, \boldsymbol{\nu}_{n+1} + \Delta\lambda_{n+1} \frac{\partial \boldsymbol{\nu}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \right] . \tag{6.57}$$

Here, the partial derivatives of the deviatoric trial-stresses

$$\frac{\partial \boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial}{\partial \boldsymbol{\varepsilon}_{n+1}} \left[2G [\boldsymbol{\varepsilon}^{\mathrm{dev}}_{n+1} - \boldsymbol{\varepsilon}^{\mathrm{p}}_{n}] \right] = \frac{\partial}{\partial \boldsymbol{\varepsilon}_{n+1}} \left[2\, G\, [\mathsf{I}^{\mathrm{sym}}_{\mathrm{dev}} : \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^{\mathrm{p}}_{n}] \right] = 2\, G\, \mathsf{I}^{\mathrm{sym}}_{\mathrm{dev}}\,, \tag{6.58}$$

the partial derivatives of the Lagrange multipliers

$$\frac{\partial \Delta\lambda_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial}{\partial \boldsymbol{\varepsilon}_{n+1}} \left[\frac{\Phi^{\mathrm{tr}}_{n+1}}{2G + \frac{2}{3}H} \right] = \frac{1}{2G + \frac{2}{3}H} \underbrace{\left[\frac{\partial \Phi^{\mathrm{tr}}_{n+1}}{\partial \boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}} : \frac{\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \right]}_{=\,\boldsymbol{\nu}_{n+1} \,:\, 2\,G\,\mathsf{I}^{\mathrm{sym}}_{\mathrm{dev}} \,=\, 2\,G\,\boldsymbol{\nu}_{n+1}} = \frac{2G}{2G + \frac{2}{3}H}\, \boldsymbol{\nu}_{n+1} \tag{6.59}$$

as well as the yield direction

$$\begin{aligned} \frac{\partial \boldsymbol{\nu}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} &= \frac{\partial}{\partial \boldsymbol{\varepsilon}_{n+1}} \left[\frac{\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}}{\|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|} \right] = \frac{\partial \|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|^{-1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \otimes \boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1} + \|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|^{-1} \frac{\partial \boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \\ &= \frac{\partial \|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|^{-1}}{\partial \boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}} : \frac{\partial \boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \otimes \boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1} + \frac{2\,G}{\|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|}\, \mathsf{I}^{\mathrm{sym}}_{\mathrm{dev}} \\ &= \frac{2\,G}{\|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|} \left[\mathsf{I}^{\mathrm{sym}}_{\mathrm{dev}} - \boldsymbol{\nu}_{n+1} \otimes \boldsymbol{\nu}_{n+1} \right] \end{aligned} \tag{6.60}$$

are calculated as specified above. Combining the single expressions leads to

$$\mathsf{E}_{\mathrm{a}\,n+1} = K\, \boldsymbol{I} \otimes \boldsymbol{I} + c_1\, \mathsf{I}^{\mathrm{sym}}_{\mathrm{dev}} + c_2\, \boldsymbol{\nu}_{n+1} \otimes \boldsymbol{\nu}_{n+1} \tag{6.61}$$

with

$$c_1 = 2\,G \left[1 - \frac{2\,G}{\|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|}\, \Delta\lambda_{n+1} \right] \tag{6.62}$$

$$c_2 = 4\,G^2 \left[\frac{\Delta\lambda_{n+1}}{\|\boldsymbol{\sigma}^{\mathrm{red\;tr}}_{\mathrm{dev}\,n+1}\|} - \frac{1}{2\,G + \frac{2}{3}\,H} \right] \tag{6.63}$$

6.1.5 Programming task

As exercise, the formulation of the von Mises plasticity consitutive law with associated isotropic and kinematic hardening specified in box 6.3 is to be implemented into Matlab. As a framework, please use the constitutive driver algorithm for one-dimensional tension-compression simulations, specified in box 6.1 and already developed during tutorials 3, 4 and 5. The task is structured in the following steps:

1. Implement an array `sdv` (dimension: number of internal variables $\times$ 1) for the internal variables or **s**tate **d**ependent **v**ariables into the existing constitutive driver `constit_driver`. Please bear in mind that the internal variables are only updated after equilibrium has been reached.
2. Implement the radial return algorithm for the von Mises plasticity constitutive law specified in box 6.3.
3. Study the material behaviour for the tempering steel 25CrMo4 for linear as well as cycling $(2-4$ cycles) axial loading with $\varepsilon_{11,\mathrm{max}} = 5\%$. Use the following material parametres

   Young's modulus $E = 205$ GPa

   Poisson ratio $\nu = 0.29$

   Initial yield stress $\sigma_y^0 = 695$ MPa

   Hardening modulus $H = 2091$ MPa.

   for solely isotropic hardening $(r = 1)$, solely kinematic hardening $(r = 0)$ as well as for arbitrary intermediate hardening $(r \in]0,1[)$. Discuss the results by comparing these to the content of the lecture.

Box 6.1: Axial stress driver for constitutive laws with internal variables for tension-compression simulations

a) For a given axial deformation ε_{n+1} at time t_{n+1} and $\{\boldsymbol{\varepsilon}_n, \boldsymbol{k}_n\}$ at time t_n, the partition of the strain tensor $\bar{\boldsymbol{\varepsilon}}_{n+1} = \bar{\boldsymbol{\varepsilon}}_n$ is to be initialised.

b) Compute the total deformation: $\boldsymbol{\varepsilon}_{n+1} = \varepsilon_{n+1}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \bar{\boldsymbol{\varepsilon}}_{n+1}$.

c) Determine the updated internal variables, algorithmic stresses and moduli by evaluation of the three dimensional constitutive box

$$\{\boldsymbol{\sigma}_{n+1}, \boldsymbol{k}_{n+1}\} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}_{n+1}; \{\boldsymbol{\varepsilon}_n, \boldsymbol{k}_n\}) \quad \text{and} \quad \mathbf{E}_{\mathrm{a}\,n+1} = \frac{\mathrm{d}\widehat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}_{n+1}; \{\boldsymbol{\varepsilon}_n, \boldsymbol{k}_n\})}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} \tag{6.64}$$

d) Partitioning

$$\boldsymbol{\sigma}_{n+1} = \sigma_{n+1} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \bar{\boldsymbol{\sigma}}_{n+1} \quad \text{and} \quad \bar{\mathbf{E}}_{\mathrm{a}\,n+1} = \frac{\mathrm{d}\bar{\boldsymbol{\sigma}}}{\mathrm{d}\bar{\boldsymbol{\varepsilon}}_{n+1}} \tag{6.65}$$

e) Update of the transverse deformation

$$\bar{\boldsymbol{\varepsilon}}_{n+1} \quad \leftarrow \quad \bar{\boldsymbol{\varepsilon}}_{n+1} - \bar{\mathbf{E}}_{\mathrm{a}\,n+1}^{-1} : \bar{\boldsymbol{\sigma}} \tag{6.66}$$

f) Check for convergence: if $\|\bar{\boldsymbol{\sigma}}\| > tol$ go to b), else quit computation and update internal variables $\boldsymbol{k}_n \leftarrow \boldsymbol{k}_{n+1}$ at global level.

Box 6.2: Prototype model: Von Mises plasticity with associated isotropic and kinematic hardening

a) Kinematics

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}} \tag{6.67}$$

b) Internal variables

$$\{\boldsymbol{\varepsilon}^{\mathrm{p}}\,,\, k\,,\, \boldsymbol{a}\} \tag{6.68}$$

c) Free energy

$$\varPsi = \frac{K}{2}\,[\boldsymbol{\varepsilon}^{\mathrm{e}} : \boldsymbol{I}]^2 + G\,[\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}}]^2 : \boldsymbol{I} + \frac{H}{2}\,r\,k^2 + \frac{H}{2}\,[1-r]\,a_{\mathrm{e}}^2 \tag{6.69}$$

with

$$a_{\mathrm{e}} = \sqrt{\frac{2}{3}}\,\|\boldsymbol{a}\| \tag{6.70}$$

d) Elastic stresses

$$\boldsymbol{\sigma} = \frac{\partial\varPsi}{\partial\boldsymbol{\varepsilon}^{\mathrm{e}}} = \boldsymbol{\sigma}_{\mathrm{vol}} + \boldsymbol{\sigma}_{\mathrm{dev}} = \sigma_{\mathrm{m}}\,\boldsymbol{I} + \boldsymbol{\sigma}_{\mathrm{dev}} \tag{6.71}$$

with

$$\sigma_{\mathrm{m}} = K\,\varepsilon^{\mathrm{e}}_{\mathrm{vol}} = K\,[\boldsymbol{\varepsilon}^{\mathrm{e}} : \boldsymbol{I}]\,, \quad \text{and} \quad \boldsymbol{\sigma}_{\mathrm{dev}} = 2\,G\,\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} = 2\,G\,[\boldsymbol{\varepsilon}_{\mathrm{dev}} - \boldsymbol{\varepsilon}^{\mathrm{p}}] \tag{6.72}$$

e) Hardening stresses

$$\kappa = -\frac{\partial\varPsi}{\partial k} = -r\,H\,k \tag{6.73}$$

$$\boldsymbol{\alpha} = -\frac{\partial\varPsi}{\partial\boldsymbol{a}} = -\frac{2}{3}\,[1-r]\,H\boldsymbol{a} \tag{6.74}$$

f) Dissipation

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \kappa\,\dot{k} + \boldsymbol{\alpha} : \dot{\boldsymbol{a}} \geq 0 \tag{6.75}$$

g) Yield function

$$\varPhi = \|\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}\| - \sqrt{\frac{2}{3}}\,[\sigma_{\mathrm{y}} - \kappa] \leq 0 \tag{6.76}$$

with

$$\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}} = \boldsymbol{\sigma}_{\mathrm{dev}} + \boldsymbol{\alpha} \tag{6.77}$$

h) Associated evolution equations

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda\,\boldsymbol{\nu}\,, \qquad \boldsymbol{\nu} = \frac{\partial\varPhi}{\partial\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}}{\|\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}\|} \tag{6.78}$$

$$\dot{k} = \lambda\,\zeta_\kappa\,, \quad \zeta_\kappa = \frac{\partial\varPhi}{\partial\kappa} = \sqrt{\frac{2}{3}} \tag{6.79}$$

$$\dot{\boldsymbol{a}} = \lambda\,\boldsymbol{\zeta}_\alpha\,, \quad \boldsymbol{\zeta}_\alpha = \frac{\partial\varPhi}{\partial\boldsymbol{\alpha}} = \frac{\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}}{\|\boldsymbol{\sigma}^{\mathrm{red}}_{\mathrm{dev}}\|} = \boldsymbol{\nu} \tag{6.80}$$

i) Kuhn-Tucker-conditions

$$\lambda \geq 0\,, \quad \varPhi \leq 0\,, \quad \lambda\,\varPhi = 0 \tag{6.81}$$

j) Material parametres

$$K\,,\, G\,,\, H\,,\, \sigma_{\mathrm{y}}\,,\, r \tag{6.82}$$

Box 6.3: Radial return algorithm: Von Mises plasticity with associated isotropic and kinematic hardening

a) Given: Strains and internal variables of the previous state of equilibrium

$$\boldsymbol{\varepsilon}_{n+1}\,,\ \{\boldsymbol{\varepsilon}^{\mathrm{p}}_n\,,\ k_n\,,\ \boldsymbol{a}_n\}$$

b) Computation of the trial values

$$\begin{aligned}
\boldsymbol{\sigma}_{\mathrm{vol}\,n+1} &= K\,[\boldsymbol{\varepsilon}_{n+1}:\boldsymbol{I}]\,\boldsymbol{I}\\
\boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1} &= 2\,G\,[\boldsymbol{\varepsilon}^{\mathrm{dev}}_{n+1}-\boldsymbol{\varepsilon}^{\mathrm{p}}_n]\\
\kappa^{\mathrm{tr}}_{n+1} &= -r\,H\,k_n\\
\boldsymbol{\alpha}^{\mathrm{tr}}_{n+1} &= -\frac{2}{3}\,[1-r]\,H\,\boldsymbol{a}_n\\
\boldsymbol{\sigma}^{\mathrm{red\ tr}}_{\mathrm{dev}\,n+1} &= \boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1}+\boldsymbol{\alpha}^{\mathrm{tr}}_{n+1}\\
\Phi^{\mathrm{tr}}_{n+1} &= \|\boldsymbol{\sigma}^{\mathrm{red\ tr}}_{\mathrm{dev}\,n+1}\|-\sqrt{\frac{2}{3}}\,[\sigma_{\mathrm{y}}-\kappa^{\mathrm{tr}}_{n+1}]
\end{aligned}$$

c) Check yield condition

if $\Phi^{\mathrm{tr}}_{n+1}\le 0$ set $[\bullet]=[\bullet]^{\mathrm{tr}}_{n+1}$ and go to e) with $c_1=2\,G$ and $c_2=0$, else go to d).

d) Radial return

$$\begin{aligned}
\Delta\lambda_{n+1} &= \frac{\Phi^{\mathrm{tr}}_{n+1}}{2\,G+\frac{2}{3}\,H}\\
k_{n+1} &= k_n+\Delta\lambda_{n+1}\,\sqrt{2/3}\\
\boldsymbol{\nu}_{n+1} &= \frac{\boldsymbol{\sigma}^{\mathrm{red\ tr}}_{\mathrm{dev}\,n+1}}{\|\boldsymbol{\sigma}^{\mathrm{red\ tr}}_{\mathrm{dev}\,n+1}\|}\\
\boldsymbol{a}_{n+1} &= \boldsymbol{a}_n+\Delta\lambda_{n+1}\,\boldsymbol{\nu}_{n+1}\\
\boldsymbol{\varepsilon}^{\mathrm{p}}_{n+1} &= \boldsymbol{\varepsilon}^{\mathrm{p}}_n+\Delta\lambda_{n+1}\,\boldsymbol{\nu}_{n+1}\\
\boldsymbol{\sigma}_{\mathrm{dev}\,n+1} &= \boldsymbol{\sigma}^{\mathrm{tr}}_{\mathrm{dev}\,n+1}-\Delta\lambda_{n+1}\,2\,G\,\boldsymbol{\nu}_{n+1}\\
\boldsymbol{\sigma}_{n+1} &= \boldsymbol{\sigma}_{\mathrm{vol}\,n+1}+\boldsymbol{\sigma}_{\mathrm{dev}\,n+1}\\
\kappa_{n+1} &= -r\,H\,k_{n+1}\\
\boldsymbol{\alpha}_{n+1} &= -\frac{2}{3}\,[1-r]\,H\,\boldsymbol{a}_{n+1}
\end{aligned}$$

e) Consistent elasto-plastic tangent moduli

$$\mathbf{E}_{\mathrm{a}\,n+1}=K\,\boldsymbol{I}\otimes\boldsymbol{I}+c_1\,\mathbf{I}^{\mathrm{sym}}_{\mathrm{dev}}+c_2\,\boldsymbol{\nu}_{n+1}\otimes\boldsymbol{\nu}_{n+1}$$

with

$$\begin{aligned}
c_1 &= 2\,G\left[1-\frac{2\,G}{\|\boldsymbol{\sigma}^{\mathrm{red\ tr}}_{\mathrm{dev}\,n+1}\|}\,\Delta\lambda_{n+1}\right]\\
c_2 &= 4\,G^2\left[\frac{\Delta\lambda_{n+1}}{\|\boldsymbol{\sigma}^{\mathrm{red\ tr}}_{\mathrm{dev}\,n+1}\|}-\frac{1}{2\,G+\frac{2}{3}\,H}\right]
\end{aligned}$$