expression

$$
\begin{aligned}
&L(u_{n+1}, v_{n+1}, \varepsilon^e_{n+1}, \alpha_{n+1}) - L(u_n, v_n, \varepsilon^e_n, \alpha_{n+1}) \\
&\quad = \underbrace{-\int_{\mathcal{B}} \Delta\gamma \sigma_Y \, dx}_{\text{Dissipation}\geq 0} -(2\vartheta - 1) \underbrace{\left[T(v_{n+1} - v_n) + V_{\text{int}}(\varepsilon^e_{n+1} - \varepsilon^e_n, \alpha_{n+1} - \alpha_n)\right]}_{\text{Quadratic form}\geq 0} \\
&\qquad - \int_{\mathcal{B}} \Delta\gamma f(\sigma_{n+\vartheta}, \alpha_{n+\vartheta}) \, dx.
\end{aligned}
\tag{1.6.17}
$$

Then the stability of the numerical approximation depends on the schemes adopted in enforcing the consistency condition:

a. *Consistency enforced for* $\delta = \vartheta$. Then the term $\Delta\gamma f(\sigma_{n+\vartheta}, \alpha_{n+\vartheta}) = 0$ as a result of the Kuhn–Tucker conditions. Inspection of (1.6.17) reveals that (1.6.1) holds provided that $\vartheta \geq \frac{1}{2}$. Therefore, for this class of algorithms, unconditional stability holds if $\vartheta \in [\frac{1}{2}, 1]$.
b. *Consistency enforced for* $\delta = 1$. Then result (1.6.17) is inconclusive except for $\vartheta = \delta = 1$ which corresponds to the standard return-mapping algorithms. To see this, we observe that convexity of the yield function implies that

$$
\Delta\gamma f(\sigma_{n+\vartheta}, \alpha_{n+\vartheta}) \leq \vartheta \Delta\gamma f(\sigma_{n+1}, \alpha_{n+1}) + (1 - \vartheta)\Delta\gamma f(\sigma_n, \alpha_n) \leq 0,
\tag{1.6.18}
$$

since $\Delta\gamma f(\sigma_{n+1}, \alpha_{n+1}) = 0$, $\Delta\gamma \geq 0$ and $f(\sigma_n, \alpha_n) \leq 0$ as a result of the design condition $\delta = 1$. Therefore, the last term on the right-hand side of equality (1.6.17) is positive and one cannot conclude that the left-hand side is nonpositive.

Clearly, both schemes include the classical return-mapping algorithms which, according to the preceding analysis, are unconditionally stable. Additional topics, such as uniqueness of the solution to the algorithmic problem and contractivity of solutions obtained for different initial data, are addressed in detail in subsequent chapters.

## 1.7 One-Dimensional Viscoplasticity

In this section, in the spirit of our elementary discussion in Section **1.2**, we illustrate the mathematical structure of the constitutive equations for classical viscoplasticity by a simple rheological model. Our objective here is merely to motivate in the simplest possible context the formulation of the general viscoplastic models undertaken in Chapter **2**.

In addition we examine in some detail the structure of a general class of recently proposed integrative algorithms, again within the context of a simple one-dimensional model problem. We show that these algorithms are obtained from the return-mapping algorithms for rate-independent plasticity examined in Section **1.4**, by an *explicit closed-form expression*.

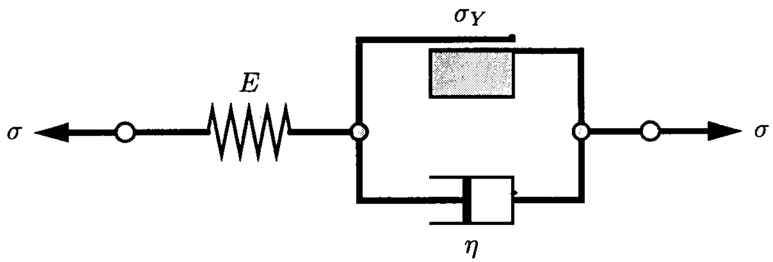


FIGURE 1-14. One-dimensional rheological model illustrating the response of a one-dimensional viscoplastic solid.

### *1.7.1 One-Dimensional Rheological Model*

The mathematical structure underlying classical (rate-dependent) viscoplasticity is motivated by examining the response of the mechanical device arranged as illustrated in Figure **1.14**.

The device possesses unit length (and unit area) and consists of a spring with elastic constant $E$, which is connected to a dashpot with constant $\eta$, in parallel with a coulombic frictional device with constant $\sigma_Y$.

Let $\sigma$ be the applied stress on the device, and let $\varepsilon$ be the total strain. As in Section **1.2** we consider the additive decomposition

$$\boxed{\varepsilon = \varepsilon^e + \varepsilon^{\mathrm{vp}},} \tag{1.7.1}$$

where $\varepsilon^e$ is the strain in the spring, so that

$$\boxed{\sigma = E\varepsilon^e = E\left(\varepsilon - \varepsilon^{\mathrm{vp}}\right).} \tag{1.7.2}$$

Next, we examine the rate of change of $\varepsilon^{\mathrm{vp}} := \varepsilon - \varepsilon^e$. To this end, consider the set of all possible stresses whose absolute value is less than or equal to the frictional constant $\sigma_Y$. This set is the closed interval $[-\sigma_Y, \sigma_Y]$. As before we use the notation

$$\mathbb{E}_\sigma = \left\{\tau \in \mathbb{R} \mid f(\tau) = |\tau| - \sigma_Y \leq 0\right\}, \tag{1.7.3}$$

and call the function $f(\sigma) := |\sigma| - \sigma_Y$ the *loading function*. Further, we recall that $\operatorname{int}(\mathbb{E}_\sigma)$ and $\partial\mathbb{E}_\sigma$ denote the interior and boundary of $\mathbb{E}_\sigma$, respectively, i.e.,

$$\operatorname{int}(\mathbb{E}_\sigma) = (-\sigma_Y, \sigma_Y), \quad \partial\mathbb{E}_\sigma = \{-\sigma_Y, \sigma_Y\}. \tag{1.7.4}$$

Finally we recall that $\mathbb{E}_\sigma$ and $\partial\mathbb{E}_\sigma$ are called the closure and boundary of the *elastic range* $\operatorname{int}(\mathbb{E}_\sigma)$ respectively. With these notations at hand, we consider the following two possibilities.

a. First, let $\sigma \in \operatorname{int}(\mathbb{E}_\sigma)$. Then $f(\sigma) \equiv |\sigma| - \sigma_Y < 0$ and *no instantaneous change should take place in* $\varepsilon^{\mathrm{vp}} = \varepsilon - \varepsilon^e$, that is,

$$\dot{\varepsilon}^{\mathrm{vp}} = 0 \ \text{ if } \ f(\sigma) \equiv |\sigma| - \sigma_Y < 0. \tag{1.7.5}$$

b. Second, assume that $\sigma \notin \mathbb{E}_\sigma$, that is, $f(\sigma) \equiv |\sigma| - \sigma_Y > 0$. Then, the stress in the frictional device is $\sigma_Y$ and the stress on the dashpot, called the *extra stress* and denoted by $\sigma_{\text{ex}}$, is given as

$$\sigma_{\text{ex}} = \left\{ \begin{array}{lll} \sigma - \sigma_Y & \text{if} & \sigma \geq \sigma_Y \\ \sigma + \sigma_Y & \text{if} & \sigma \leq -\sigma_Y \end{array} \right\} = (|\sigma| - \sigma_Y)\,\text{sign}(\sigma). \tag{1.7.6}$$

Using the fact that the stress $\sigma_{\text{ex}}$ on the dashpot is connected to the strain through the viscous relationship $\sigma_{\text{ex}} = \eta \dot{\varepsilon}^{\text{vp}}$ from (1.7.6), we obtain

$$\dot{\varepsilon}^{\text{vp}} = \frac{1}{\eta} f(\sigma)\,\text{sign}(\sigma) \quad \text{if} \quad f(\sigma) = |\sigma| - \sigma_Y \geq 0. \tag{1.7.7}$$

If we denote the ramp function by $\langle x \rangle = \frac{(x+|x|)}{2}$, (1.7.5) and (1.7.7) combine to yield the expression

$$\boxed{\begin{aligned} \dot{\varepsilon}^{\text{vp}} &= \frac{\langle f(\sigma) \rangle}{\eta} \frac{\partial f(\sigma)}{\partial \sigma}, \\ f(\sigma) &:= |\sigma| - \sigma_Y. \end{aligned}} \tag{1.7.8}$$

We refer to (1.7.8) as a *viscoplastic constitutive equation of the Perzyna type*. An alternative formulation of the rate equation (1.7.8), which is particularly useful in a numerical analysis context, is considered next.

1.7.1.1 Viscoplastic flow rule and closest point projection.

An important interpretation of (1.7.7) is derived by rewriting this evolutionary equation as follows. First introduce a time constant, denoted by $\tau$ defined as

$$\boxed{\tau := \frac{\eta}{E}.} \tag{1.7.9}$$

The ratio $\tau$ of the viscosity coefficient in the dashpot to the spring constant in the device in Figure **1.14** is called the *relaxation time* of the device. Its physical significance is illustrated in the example below. Now rewrite (1.7.7) as

$$\begin{aligned} \dot{\varepsilon}^{\text{vp}} &= \frac{E^{-1}}{\tau} \left[ |\sigma|\,\text{sign}(\sigma) - \sigma_Y \text{sign}(\sigma) \right] \\ &= \frac{E^{-1}}{\tau} \left[ \sigma - \sigma_Y \text{sign}(\sigma) \right]. \end{aligned} \tag{1.7.10}$$

In view of this expression, we set

$$\boxed{\dot{\varepsilon}^{\text{vp}} = \frac{E^{-1}}{\tau} \left[ \sigma - \mathbf{P}\sigma \right],} \tag{1.7.10}$$

where $\mathbf{P} : \mathbb{R} \to \partial\mathbb{E}_\sigma$ is the mapping defined by

$$\boxed{\mathbf{P}\sigma = \sigma_Y\,\text{sign}(\sigma),} \tag{1.7.11}$$

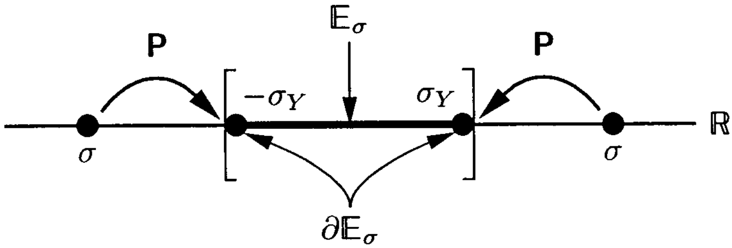


FIGURE 1-15. The map $\mathbf{P} : \mathbb{R} \to \partial\mathbb{E}_\sigma$ "returns" $\sigma \in \mathbb{R}$ to the boundary of $\mathbb{E}_\sigma$.

with a geometric interpretation illustrated in Figure **1.15**. One can easily show that $\mathbf{P} : \mathbb{R} \to \partial\mathbb{E}_\sigma$ is a projection in the sense that

$$\mathbf{P}(\mathbf{P}\sigma) = \mathbf{P}^2\sigma = \mathbf{P}\sigma \iff \mathbf{P}^2 = \mathbf{P}. \tag{1.7.12}$$

The physical significance of (1.7.11) should be clear. $\mathbf{P}$ maps a stress point $\sigma$ onto the *closest point* of the boundary $\partial\mathbb{E}_\sigma$ of the elastic range. This interpretation of the viscoplastic flow rule, which is the result of the alternative expression (1.7.10) is attributed to Duvaut and Lions [1972].

1.7.1.2 Example: Relaxation test.

To further illustrate the physical significance of the constitutive model just outlined, we consider the following experiment.

At time $t = 0$ to the device in Figure **1.14** we apply an instantaneous strain which is held constant throughout time, that is, we consider the strain history (see Figure **1.16**)

$$\varepsilon(t) = \varepsilon_0 H(t),$$

where

$$H(t) := \begin{cases} 1 \text{ if } t > 0 \\ 0 \text{ otherwise,} \end{cases} \tag{1.7.13}$$

and $\varepsilon_0 > 0$. The discontinuous function $H(t)$ is the Heaviside step function. Let $\sigma_0 := E\varepsilon_0$. Consequently, $\sigma_0 > 0$. Clearly, by (1.7.10), if

$$\sigma_0 := E\varepsilon_0 \begin{cases} < \sigma_Y \Rightarrow \dot{\varepsilon}^{\mathrm{vp}} = 0 & \text{(elastic response),} \\ > \sigma_Y \Rightarrow \dot{\varepsilon}^{\mathrm{vp}} \neq 0 & \text{(viscoplastic response).} \end{cases} \tag{1.7.14}$$

Since the elastic case corresponding to the condition $\sigma_0 - \sigma_Y < 0$ is elementary, we consider the situation illustrated in Figure **1.16** for which $\sigma_0 - \sigma_Y > 0$.

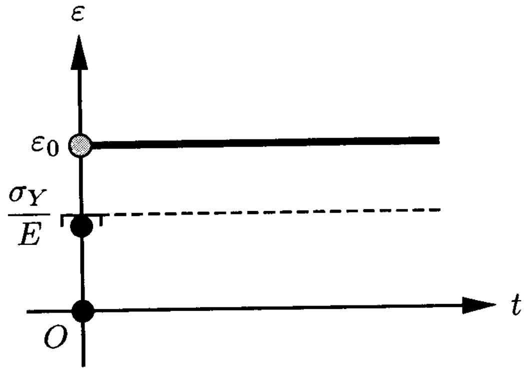


FIGURE 1-16. Strain history for a relaxation test.

To compute the stress history, we need to integrate the constitutive model as follows. From (1.7.2), (1.7.10) and (1.7.9),

$$\left.\begin{aligned} \dot{\sigma} &= E\dot{\varepsilon} - E\dot{\varepsilon}^{\text{vp}}, \\ \dot{\varepsilon}^{\text{vp}} &= \frac{1}{\tau} E^{-1}\left[\sigma - \sigma_Y\right]. \end{aligned}\right\} \tag{1.7.15}$$

Combining these equations, we obtain

$$\left.\begin{aligned} \dot{\sigma} + \frac{1}{\tau}\sigma &= E\dot{\varepsilon} + \frac{1}{\tau}\sigma_Y \\ \varepsilon &= \varepsilon(0) > \frac{\sigma_Y}{E} \end{aligned}\right\} \text{ in } (0, \infty). \tag{1.7.16}$$

Equation (1.7.16), integrated in closed form (note that $e^{\frac{t}{\tau}}$ is the integrating factor), yields

$$e^{t/\tau}\sigma - \sigma(0) = \int_0^t e^{s/\tau}\, E\dot{\varepsilon}(s)ds + \sigma_Y\left(e^{t/\tau} - 1\right). \tag{1.7.17}$$

Now, since $\dot{\varepsilon}(t) = 0$ in $(0, \infty)$, it is easily shown that the integral in (1.7.17) vanishes identically. (One needs to be a bit careful with the singularity of $\dot{\varepsilon}(t)$ at $t = 0$.) Consequently, since $\sigma(0) = E\varepsilon(0) = E\varepsilon_0$,

$$\boxed{\sigma(t) = \left[E\varepsilon_0 - \sigma_Y\right]e^{-\frac{t}{\tau}} + \sigma_Y.} \tag{1.7.18}$$

The stress response given by (1.7.18) is shown in Figure **1.17**. Note that the stress decays exponentially with time. In fact, as $t/\tau \to \infty$, $\sigma(t) \to \sigma_Y$.

From a physical standpoint, it is important to realize that the controlling factor in the *relaxation* process illustrated in Figure **1.17** is the *relative time* $t/\tau$. The absolute time $t \in [0, \infty)$ is regarded as short or long only when compared with $\tau = \eta/E$. Equivalently, what counts is the *ratio* of the viscosity $\eta$ in the dashpot

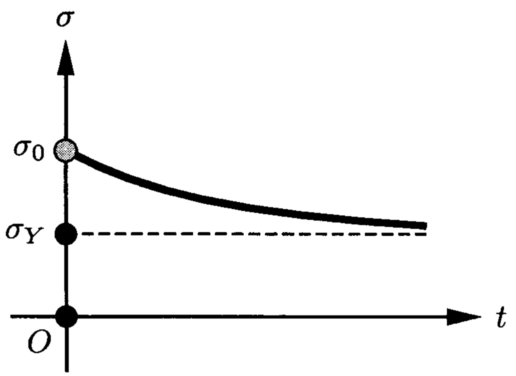


FIGURE 1-17. Stress response in a relaxation test

to the stiffness $E$ in the spring in the device in Figure **1.14**. Because of this, $\tau$ is called the *natural relaxation time*.

1.7.1.3 Extensions to account for strain hardening.

Strain-hardening effects are incorporated in the model outlined above by a procedure similar to that discussed in detail in Section **1.2.2.1**. The simplest *linear isotropic hardening* model is obtained by appending an internal variable, denoted by $\alpha$, to an evolutionary equation given by

$$\boxed{\dot{\alpha} = \left|\dot{\varepsilon}^{\mathrm{vp}}\right| \geq 0,} \tag{1.7.19}$$

and modifying the loading function as

$$f(\sigma, \alpha) := |\sigma| - \left[\sigma_Y + K\alpha\right]. \tag{1.7.20}$$

The closure of the elastic range is the time-dependent (*closed convex*) set defined by

$$\boxed{\mathbb{E}_\sigma = \left\{(\sigma, \alpha) \in \mathbb{R} \times \mathbb{R}_+ \mid f(\sigma, \alpha) \leq 0\right\}.} \tag{1.7.21}$$

1.7.1.4 The viscoplastic regularization.

As illustrated in the example 1.7.1.2, within the framework of (rate-dependent) viscoplasticity, the variables $(\sigma, \alpha)$ are no longer constrained to lie within the closure of the elastic range $\mathbb{E}_\sigma$, in sharp contrast with the situation found in the rate-independent plasticity model.

On the other hand, on physical grounds, by inspecting Figure **1.14**, one concludes that, as $\eta \to 0$, the effect of the dashpot disappears and one recovers the rate–independent model illustrated in Figure **1.1**. In the next chapter we rigorously show that this intuition is correct. This important fact is exploited analytically and numerically and leads to the notion of *viscoplastic regularization* (which is closely related to the Yoshida regularization; see, e.g., Pazy [1983, p.9]) of rate-independent plasticity.

To elaborate further, observe that by setting

$$\gamma := \frac{\langle f(\sigma, \alpha)\rangle}{\eta}, \tag{1.7.22}$$

the equations of evolution (1.7.7) and (1.7.19) are written as

$$\dot{\varepsilon}^{\mathrm{vp}} = \gamma \operatorname{sign}(\sigma),$$

and (1.7.23)

$$\dot{\alpha} = \gamma,$$

which is the exact counterpart of the evolutionary equations of classical rate-independent plasticity, but with the Kuhn–Tucker conditions (1.2.26) and the consistency condition (1.2.27) now replaced by (1.7.22).

Because the consistency parameter is no longer determined by the consistency condition but directly through constitutive equation (1.7.22), one speaks of a viscoplastic regularization. For convenience and subsequent reference, the one-dimensional viscoplastic model developed above is summarized in BOX **1.6**.

**BOX 1.6.** One-Dimensional Classical Viscoplasticity.

**1.** Elastic stress strain relationship

$$\sigma = E\left(\varepsilon - \varepsilon^{\mathrm{vp}}\right).$$

**2.** Closure of the elastic range and loading function

$$\mathbb{E}_\sigma := \left\{(\sigma, \alpha) \in \mathbb{R} \times \mathbb{R}_+ \quad | \quad f(\sigma, \alpha) \le 0\right\}$$
$$f(\sigma, \alpha) := |\sigma| - \left[\sigma_Y + K\alpha\right].$$

**3.a.** Flow rule and hardening law (Perzyna formulation)

$$\dot{\varepsilon}^{\mathrm{vp}} = \frac{\left\langle f(\sigma, \alpha)\right\rangle}{\eta} \operatorname{sign}(\sigma)$$

$$\dot{\alpha} = \frac{\left\langle f(\sigma, \alpha)\right\rangle}{\eta}.$$

**3.b.** Flow rule and hardening law (Duvaut–Lions formulation)

$$\dot{\varepsilon}^{\mathrm{vp}} = \begin{cases} \dfrac{E^{-1}}{\tau}\left[\sigma - \mathbf{P}\sigma\right]; & f(\sigma, \alpha) > 0 \\[2ex] 0 & \text{otherwise,} \end{cases}$$

$$\dot{\alpha} = \left|\dot{\varepsilon}^{\mathrm{vp}}\right|,$$

where $\mathbf{P} : \mathbb{R} \to \partial\mathbb{E}_\sigma$ is the closest point projection onto $\partial\mathbb{E}_\sigma$, the boundary of the elastic range.