Kernel Methods in Machine Learning
Homework Assignment 2.

student: Donayorov Maksad
id: 571335

Question 1:

$S = \{x_1, x_2, x_3 \dots x_l\}$

$k: X \cdot X \to \mathbb{R}$ kern. func.

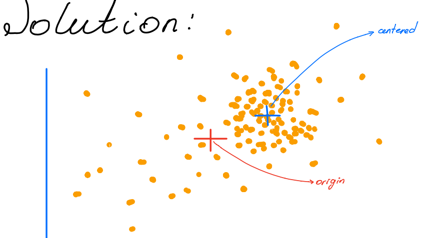
$\phi: X \to F$ feature map

Show that:

$$k(x_i, x_j) = k(x_i, x_j) - \frac{1}{l}\sum_{p=1}^{l}(x_p, x_j) - \frac{1}{l}\sum_{q=1}^{l}(x_j, x_q) + \frac{1}{l^2}\sum_{p,q=1}^{l} k(x_p, x_q)$$

where: $k_c(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$ is kernel value after centering.

Solution:



$\frac{1}{l}\sum_{i=1}^{l}\phi(x_i)$

centering to the mass

feature map: $\phi_c(x) = \phi(x) - \frac{1}{l}\sum_{i=1}^{l}\phi(x_i)$

$k_c(x_i, x_j) = \langle \phi_c(x_i), \phi_c(x_j) \rangle$

we know that $\phi_c(x) = \phi(x) - \frac{1}{l} \cdot \sum_{i=1}^{l}\phi(x_i)$

plugging that into the above equation, we get:

$$k_c(x_i, x_j) = \langle \phi(x_i) - \frac{1}{l}\sum_{p=1}^{l}\phi(x_p), \phi(x_j) - \frac{1}{l} \cdot \sum_{q=1}^{l}\phi(x_q) \rangle$$

we know that $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

expanding: $\langle \underbrace{\phi(x_i)}_{x} \underbrace{-\frac{1}{l}\sum_{p=1}^{l}\phi(x_p)}_{y}, \underbrace{\phi(x_j) - \frac{1}{l}\cdot\sum_{q=1}^{l}\phi(x_q)}_{z}\rangle$

we get:

$$k_c(x_i,x_j) = \langle \underbrace{\phi(x_i)}_{x}, \underbrace{\phi(x_j)}_{y} \underbrace{-\frac{1}{l}\cdot\sum_{q=1}^{l}\phi(x_q)}_{z}\rangle + \langle \underbrace{-\frac{1}{l}\sum_{p=1}^{l}\phi(x_p)}_{x}, \underbrace{\phi(x_j)}_{y} \underbrace{-\frac{1}{l}\sum_{q=1}^{l}\phi(x_q)}_{z}\rangle$$

$\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

$$= \underbrace{\langle \phi(x_i), \phi(x_j)\rangle}_{\text{orig. kernel.}} + \underbrace{\langle \phi(x_j), -\frac{1}{l}\sum_{q=1}^{l}\phi(x_q)\rangle}_{\text{second term}} + \underbrace{\langle -\frac{1}{l}\sum_{p=1}^{l}\phi(x_p), \phi(x_j)\rangle}_{\text{first term}} + \underbrace{\langle -\frac{1}{l}\sum_{p=1}^{l}\phi(x_p), -\frac{1}{l}\sum_{q=1}^{l}(x_q)\rangle}_{\mu}$$

Hence

$$k_c(x_i,x_j) = k(x_i,x_j) - \frac{1}{l}\sum_{p=1}^{l}k(x_p,x_i) - \frac{1}{l}\sum_{q=1}^{l}k(x_i,x_q) + \frac{1}{l^2}\sum_{p,q=1}^{l}k(x_p,x_q)$$

Question 2:

$$P(Y_i = k \mid X = x_i) = \frac{1}{Z}\cdot \exp(\langle w_i, x_i\rangle)$$

we know that $\frac{1}{Z}\cdot\sum_{i=1}^{K}\exp(\langle w_i, x_i\rangle) = 1.$

so: $Z = \sum_{i=1}^{K}\exp(\langle w_i, x_i\rangle)$

consequently the function would be:

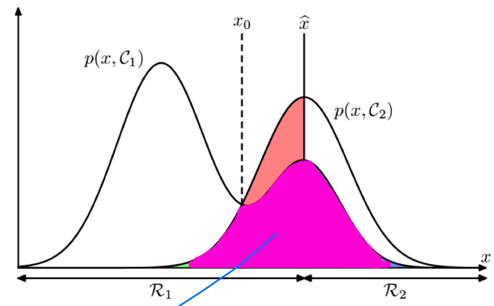
$$P(Y_i = k \mid X = x_i) = \frac{\exp(\langle w_i, x_i\rangle)}{\sum_{i=1}^{K}\exp(\langle w_i, x_i\rangle)}$$

Question 3:

$$\int_{x \in X} \frac{\min(P(x, c_1), P(x, c_2)\, dx}{p(x, c_1) + p(x, c_2)} \leqslant$$

$$\leqslant \int \min(p(x, c_1), p(x, c_2))$$

$$\leqslant \int \min(p(x, c_1), p(x, c_2))$$



error to be minimized.
this error is always less than:
$p(x, c_1) + p(x, c_2)$