E.I.S.T.I. - Mathematics Department
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Introduction to Monte- Carlo Simulation
Part I

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Contents

List of Figures

Chapter 1

Generalities

1 Principle - Simulation and Optimization

In general, the methods of simulation allow one to artificially reproduce the realizations of various random variables which in terms of precise models describe the evolution of some physical systems following corresponding probability laws.

These realizations allow to estimate the corresponding statistical parameters (expectation values, variances and statistical errors etc..) consequently help to analyze and study more efficiently the characteristics of a system in order to improve the peformances and optimize the results of the process' work.

Very often the modelling and optimization can be done by using deterministic variables together with stochastic processes as the example given below:

$$(P.0) \qquad \begin{cases} \quad \text{Opt. } f(X_1 \ldots X_n) \\ \text{with} \quad \rho_1(X_i \; i=1,\ldots,n) = b_1 \\ \quad \vdots \ldots\ldots\ldots\ldots \vdots \\ \quad \vdots \ldots\ldots\ldots\ldots \vdots \\ \quad \rho_m(X_i \; i=1,\ldots,n) = b_m \end{cases}$$

Among the most interesting methods we shall use the **Monte- Carlo method of Simulation** as the most appropriate to a computer and mathematical scientist student.

Chapter 2

The Method of Monte-Carlo Simulation

1 The role of the (continuous) Uniform random variable $\mathcal{U}(]0,1])$

1 Reminder: The law

Probability distribution function - Density distribution function

Definition 2.1

Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space. A continuous random variable X defined on $(\Omega,\ \mathcal{A})$ follows the Uniform law with values in the interval $]a,\ b] \subset \mathbb{R}$

$$\Leftrightarrow X : \mathcal{U}(]a,\ b]),$$

*if it admits as **support**:*

$$C_X =]a,\ b]$$

*and **probability density distribution function**:*

$$f_X(x) = \left\{ \begin{array}{cc} \dfrac{1}{b-a} & \textit{si } x \in\ C_X \\ 0 & \textit{otherwise} \end{array} \right\}$$

*** In other words:** X describes the random experience of choosing an arbitrary number x in the interval $]a,\ b]$ in such a way that the probability $P[\{x \in]a,\ b]\}]$ is **independent of** the chosen sub-interval in which x varies.

* Let $X : \mathcal{U}(]a,\ b])$, then:

Probability distribution function is given by:

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq a \\ \dfrac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b \end{cases}$$

Law of Uniform random variable, $X \ : \ U(]0,1])$

In an analogous way:

Definition 2.2
Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space. A continuous random variable X defined on $(\Omega,\ \mathcal{A})$ follows the Uniform law with values in the interval $]0,\ 1] \subset \mathbb{R}$

$$\Leftrightarrow X : \mathcal{U}(]0,\ 1]),$$

if it admits as support:

$$C_{\mathcal{U}} =]0,\ 1]$$

and density distribution function:

$$f_{\mathcal{U}}(x) = \left\{ \begin{array}{cc} 1 \ \textit{if} \ x \in & C_{\mathcal{U}} \\ 0 & \textit{otherwise} \end{array} \right\}$$

- **Probability distribution function** $$F_{\mathcal{U}}(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x \in]0,1[\\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

- Expectation value; $$\mathbb{E}[\mathcal{U}] = \frac{1}{2}$$

- Variance: $$\sigma^2_{\mathcal{U}} = \frac{1}{12}$$

2 Uniform Law and Simulation

1 Theorem (Reminder)- the role of $\mathcal{U}(]0,\ 1])$

Theorem 2.1 *Let X be a continuous random variable with probability distribution function F_X*

$$\Rightarrow$$

The "transform" random variable" $Y = F_X(x)$ follows a uniform probability law $\mathcal{U}(]0,\ 1])$

Important! Thanks to this theorem we "simulate" a random variable X, by taking the **inverse transform of the probability distribution function**:

$$F_X^{-1}(u) = x,$$

This is the Monte- Carlo Method

Reminder:
C , $JAVA$ and $SCILAB$ provide the corresponding computer programs for the simulation of the uniform random variable $\mathcal{U}(]0,1])$, here is an example:

Definition 2.3 *Function* <u>*random*</u> *("RANDOM")*

(Simulation of U : $\mathcal{U}(]\prime,\infty]))$
(Algorithm in C)

We choose an integer (num) and:

$$\left\{ \begin{array}{c} \textit{float } x \text{ ;} \\ x = \textit{(float) RANDOM (num)/(float)num ;} \end{array} \right.$$

$Remark$

As the num becomes greater and greater, the approximation becomes better.

We now present two applications:
:

a) Simulation of a continuous random variable;

b) Simulation of a discrete random variable.

3 Simulations

1 Simulation of a continuous random variable

Proposition 2.1 *Let X be a continuous random variable and $F_X(x)$ the corresponding Probability distribution function. In order to realize m values $(\alpha_1, \ldots, \alpha_m)$ of X, we call m times the function "RANDOM" and we obtain m values $\{u_1, \ldots, u_m\}$; $\forall\, u_j$, we look for an α_j such that:*

$$\begin{array}{rcl} F_X(\alpha_j) & = & u_j \\ & \Leftrightarrow & \\ \alpha_j & = & F_X^{-1}(u_j) \end{array}$$

Example 2.1 ***Exponential random variable:***
We consider the probability distribution function

$$F_X(x) = \left\{ \begin{array}{ccc} 0 & \textit{if} & x \leq 0 \\ 1 - e^{-\theta x} & \textit{if} & x \geq 0 \end{array} \right.$$

and $\forall$ valeur u_j generated by "RANDOM", we look for

$$\begin{array}{lcl} \textit{one } \alpha_j & \textit{such that} & u_j = F_X(\alpha_j) \\ & \Leftrightarrow & u_j = 1 - e^{-\theta \alpha_j} \\ & \Leftrightarrow & e^{-\theta \alpha_j} = 1 - u_j \\ & & -\theta \alpha_j = \ln|1 - u_j| \\ & \Leftrightarrow & \boxed{\alpha_j = \ln|1 - u_j| / -\theta} \end{array}$$

Exercise 2.1 Simulate an exponential random variable of parameter $\theta = 0,5$ and estimate by simulation the values of the statistic parameters: expectation value μ and variance σ^2.

Compare your results with the corresponding theoretical values of these parameters.

2 Simulation of a discrete random variable.

The principle is the same but: *"the inversion " of the probability distribution function* is realized by comparing two subsequent values of this function, $F_X(x_{i-1}) = \sum_{k=1}^{i-1} p_X(x_k)$ and $F_X(x_i) = \sum_{k=1}^{i} p_X(x_k)$ with the randomly obtained number u_j "RANDOM" ; then, the simulated value a_i of X is the one which corresponds to the largest value of F_X. More explicitly:

Proposition 2.2

Let $(\Omega,\ \mathcal{A},\ P)$ *be a probability space. Let* X *be a discrete random variable defined on* $(\Omega,\ \mathcal{A})$ *with:*

$$\begin{array}{l} \textit{Support} \quad D_X = \{x_1, \dots, x_n\} \\ \textit{Mass distribution function:} \quad p_X(x_k) \\ \textit{and Probability distribution function} \quad F_X(x_i) = \displaystyle\sum_{k=1}^{i} p_X(x_k) \\ \textit{(Equivalent notations:)} \quad p_k(x_k) \equiv p_X(x_k) \equiv p_k. \end{array}$$

In order to realize m *independent valiues:* $(\alpha_1, \dots, \alpha_m)$ *of* X*, we call* m *times the function "RANDOM" and obtain* m *values* $\{u_1, \dots, u_m\}$ *;* $\forall\ u_j$*, we look for one* $i \in \{1, \dots, n\}$ *such that*

$$\sum_{k=1}^{i-1} p_k \leq u_j < \sum_{k=1}^{i} p_k$$

and we put

$$\boxed{\alpha_j = x_i}$$

Example 2.2 (without computer) *Let* X *be the discrete random variable with* $(k \in \{1, 2, 3\})$*:*

x_k	*1*	*2*	*3*
P_k	*0,3*	*0,5*	*0,2*

"RANDOM" : $j \in \{1, 2, \dots, 5\}$ *:*

$$\begin{cases} u_1 = 0,4 \ ; \\ u_2 = 0,65 \ ; \\ u_3 = 0,15 \ ; \\ u_4 = 0,85 \ ; \\ u_5 = 0,79 \end{cases}$$

$\Rightarrow$ *we find:*

$$\begin{array}{l} u_1 \Rightarrow \alpha_1 = x_2 \equiv 2 \\ u_2 \Rightarrow \alpha_2 = x_2 \equiv 2 \\ u_3 \Rightarrow \alpha_3 = x_1 \equiv 1 \\ u_4 \Rightarrow \alpha_4 = x_3 \equiv 3 \\ u_5 \Rightarrow \alpha_5 = x_2 \equiv 2 \end{array}$$

Exercise 2.2

i). Simulate a discrete random variable which follows the Poisson law with parameter $\lambda = 3.$ and estimate by simulation the values of the statistic parameters: expectation value μ and variance σ^2 . Compare your results with the corresponding theoretical values of these parameters.

ii). Repeat the same exercise for a discrete random variable which follows the Binomial law $\mathcal{B}(n;\ p)$, with parameters $n = 50; p = 0,02$.

Chapter 3

Simulation of the Normal distribution (Gaussian law)

1 1^{rst} method of simulation of the Normal distribution $\mathcal{N}(0;1)$

1 Reminders

Sample

Definition 3.1 (Sample) *Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space. A sample is defined by the following data:*

1. *The set of n independent random variables $(X_1,\ \ldots,\ X_n)$ which follow the same law of probability with the same parameters $(\mu,\ \sigma^2)$.*
2. *An abstract random variable X called the "associated", random variable, which follows the same law with the same parameters $(\mu,\ \sigma^2)$.*
3. $\bar{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$ *the "arithmetic mean" of the sample which verifies the following properties:*

$$\mathbb{E}[\bar{X}] = \mu \qquad Var[\bar{X}] = \frac{\sigma^2}{n}.$$

Example 3.1 *In Statistics a* sample *is a portion drawn from a population, the study of which is intended to lead to* ***statistical estimates*** *of the attributes of the whole population.*

2 Reminder: Central Limit Theorem

Theorem 3.1 (T.C.L.)

Let $(\Omega,\ \mathcal{A},\ P)$, be a probability space and let us consider a sequence of independent random variables $X_1,\ X_2,\ \ldots\ X_n$ defined on $(\Omega,\ \mathcal{A})$, which follow the same probability distribution law and such that:

$$\forall\ i \in \{1,\ 2, \ldots\ \ n\}\ \mathbb{E}[X_i] = \mu,\ \ Var[X_i] = \sigma^2$$

exist,

$$\Rightarrow$$

the sequence $Y_1,\ Y_2,\ \ldots, Y_n,\ \ldots$ *where*

$$Y_n = \sqrt{n}\ \frac{\bar{X} - \mu}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)$$

converges in law to a standard normal random variable $Y : \mathcal{N}(0,\ 1)$.

3 Simulation of the standard Normal (standard Gaussian) N(0, 1) distribution (1^{st} method)

For this Simulation we apply the "Central Limit Theorem" by using a sample of Continuous Uniform random variables on $]0,\ 1[$

$$\Leftrightarrow \qquad (X_1,\ \ldots,\ X_n) = (\mathcal{U}_1,\ \ldots,\ \mathcal{U}_n)$$

Then, taking into account the facts that:

$$\mathbb{E}[\mathcal{U}] = \frac{1}{2}; \qquad \sigma^2_{\mathcal{U}} = \frac{1}{12}$$

the **"Central Limit Theorem"** implies that for sufficiently large n the variable

$$Y_n = \sqrt{n}\ \frac{\bar{\mathcal{U}} - 0,5}{1/\sqrt{12}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left((\mathcal{U}_i - 0,5)\sqrt{12} \right)$$

follows the standard Gaussian law: $Y : \mathcal{N}(0,\ 1)$.

Exercise 3.1 *i). Simulate a standard Gaussian distribution law:* $Y : \mathcal{N}(0,\ 1)$.. *and estimate the expectation value and the variance. Compare your results with the corresponding theoretical values.*

ii). Repeat the same exercise for a Gaussian law: $X : \mathcal{N}(\mu,\ \sigma^2)$*, of parameters* $\mu = 10;\ \sigma = 2$

2 2^{nd} method of Simulation of standard Gaussian distribution $\mathcal{N}(0; 1)$ – (The method of Polar Coordinates)

1 Reminders - A

Proposition 3.1
Let U *be a continuous uniform random variable on* $]0, 1]$

$$\Rightarrow\ \ \tilde{U} = 1 - U$$

is a uniform random variable on $]0, 1]$.

Proof

$$\begin{aligned}
&\Phi(U) = \tilde{U} : \tilde{U} = 1 - U \\
&\psi(\tilde{U}) = U = 1 - \tilde{U} \\
&\Rightarrow \qquad \psi'(\tilde{U}) = -1 \\
&f_{\tilde{U}} = f_U(\psi(\tilde{U}))|\psi'(\tilde{U})| = \begin{cases} 1 & \text{if } \tilde{U} \in]0, 1] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Corollary 3.1
Following Lemma 3.1

$$X = -\frac{\ln(1-U)}{\lambda} \quad \textit{is an exponential distribution with parameter } \lambda$$

$$Y = -\frac{\ln(\tilde{U})}{\lambda} \quad \textit{exponential distribution law with parameter } \lambda \textit{ and expectation value } \frac{1}{\lambda}$$

2 Reminders - B–(Transformation of multidimensional random variables)

Proposition 3.2
Let $(\Omega,\ \mathcal{A},\ P)$*, be a probability space and let* X_1 *and* X_2 *be two joined continuous random variables defined on* Ω *with joint density distribution function* f_{X_1,X_2} *. Let* $Y_1 = \rho_1(X_1X_2)$; $Y_2 = \rho_2(X_1X_2)$ *be two transforms random variables functions of* $(X_1 and X_2)$.

If the following conditions are verified

C.1 the system of equations $\begin{cases} y_1 = \rho_1(x_1, x_2) \\ y_2 = \rho_2(x_1, x_2) \end{cases}$

admits a solution : $\begin{matrix} x_1 = h_1(y_1 y_2) \\ x_2 = h_2(y_1 y_2) \end{matrix}$

C.2 the functions $\rho_1,\ \rho_2$ *are continuously differentiable and the Jacobian* $\neq 0$ *everywhere* $\Leftrightarrow$

$$J(x_1, x_2) = \begin{vmatrix} \dfrac{\partial \rho_1}{\partial x_1} & \dfrac{\partial \rho_1}{\partial x_2} \\ \\ \dfrac{\partial \rho_2}{\partial x_1} & \dfrac{\partial \rho_2}{\partial x_2} \end{vmatrix} \neq 0 \ (\forall x_1\ x_2)$$

$$\Rightarrow$$

Y_1 *and* Y_2 *are continuously joined "good random variables" with density distribution function:*

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}$$

$$\left\{ \begin{matrix} \textit{where} & x_1 = h_1(y_1, y_2) \\ & x_2 = h_2(y_1, y_2) \end{matrix} \right\}$$

Remark 3.1 Notice the analogy with the corresponding transformation of a random variable in one dimension.

Example 3.2 *Let* $(\Omega,\ \mathcal{A},\ P)$, *be a probability space and let* X_1 *and* X_2 *be two joined continuous random variables defined on* Ω *with given joint density distribution function* f_{X_1,X_2}.

If $Y_1 = X_1 + X_2$ *and* $Y_2 = X_1 - X_2$ $\quad \left(J = -2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right)$

$$\Rightarrow$$

$$f_{Y_1 Y_2}(y_1 y_2) = \frac{1}{2} f_{X_1 X_2}\left(\frac{y_1 + y_2}{2},\ \frac{y_1 - y_2}{2} \right)$$

because $\left(x_1 = \dfrac{y_1 + y_2}{2},\ x_2 = \dfrac{y_1 - y_2}{2} \right)$

Some particular interesting cases

a) If $X_1,\ X_2$ are independent uniform continuous variables.

$$\Rightarrow \quad f_{Y_1,Y_2}(y_1 y_2) = \begin{cases} \frac{1}{2} \text{ if} & 0 \leqslant y_1 + y_2 \leqslant 2\ ;\ 0 \leqslant y_1 - y_2 \leqslant 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

b) If $X_1,\ X_2$ are independent exponential random variables with parameters ($\lambda_1,\ \lambda_2$ respectively)

$$\Rightarrow \quad f_{Y_1 Y_2}(y_1 y_2) = \begin{cases} \frac{\lambda_1 \lambda_2}{2} \exp\{-\lambda_1 \left(\frac{y_1 + y_2}{2}\right) - \lambda_2 \left(\frac{y_1 - y_2}{2}\right) & \text{if } \begin{cases} y_1 + y_2 \geqslant 0 \\ y_1 - y_2 \geqslant 0 \end{cases} \\ 0 \quad \text{ailleurs} \end{cases}$$

c) If $X_1,\ X_2$ are independent standard Gaussian random variables

$$\Rightarrow \quad f_{Y_1Y_2}(y_1y_2) = \frac{1}{4\pi}\exp\left[-\frac{(y_1+y_2)^2}{8} - \frac{(y_1-y_2)^2}{8}\right] = \frac{e^{-y_1^2/4}}{\sqrt{4\pi}}\,\frac{e^{-y_2^2/4}}{\sqrt{4\pi}}$$

We find again that $X_1 + X_2$ **is independent** of $X_1 - X_2$.

Theorem 3.2

Let $(\Omega,\ \mathcal{A},\ P)$, be a probability space and let X_1 and X_2 be two independent random variables with the same probability distribution function F

*$\Rightarrow$ The random variables $X_1 + X_2$ and $X_1 - X_2$ are independent **iff** F **is the probability distribution function of a gaussian random variable**.*

Example 3.3 *Let $(\Omega,\ \mathcal{A},\ P)$, be a probability space and let X_1 and X_2 be two standard gaussian random variables.*

Let $r = \rho_1(x,y),\ \theta = \rho_2(x,y)$ with $r = \sqrt{x^2+y^2}$, $\theta = \arctan y/x$

$$J = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{2x}{2(x^2+y^2)^{1/2}} & \dfrac{2y}{2(x^2+y^2)^{1/2}} \\ \\ -\dfrac{1}{x^2}\dfrac{y}{(1+\dfrac{y^2}{x^2})} & \dfrac{1}{x(1+\dfrac{y^2}{x^2})} \end{vmatrix}$$

so:

$$J = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

and from the joint density distribution function of X_1, X_2 :

$$f(x,y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2}$$

$\Rightarrow$ joint density distribution function of (R,θ) the new couple of random variables

$$f(r,\theta) = \frac{r}{2\pi}\, e^{-r^2/2} \quad 0 < \theta < 2\pi, \quad 0 < r <= \infty$$

$\Rightarrow$ f can be decomposed in a product of independent random variables R and θ $\Rightarrow$:
R **follows the law of Rayleigh with density distribution function**

$$f(r) = re^{-r^2/2} \quad 0 < r < +\infty,$$

and θ **follows the continuous uniform law on** $[0, 2\pi]$.

If we are interested on the random variable (R^2,θ)

$$d = x^2 + y^2 \quad \theta = \arctan(y/x)$$

$$\Rightarrow \quad J = \begin{vmatrix} 2x & 2y \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = 2$$

$\Rightarrow$ joint density distribution

$$f_{R^2,\theta}(d,\theta) = \frac{1}{2}\, e^{-d/2}\, \frac{1}{2\pi} \quad 0 < d < +\infty \quad 0 < \theta < 2\pi$$

$\Leftrightarrow$ R^2 et θ **are independent.**

R^2 follows an exponential law of parameter $\lambda = 1/2$ and θ follows a continuous uniform law on $[0, 2\pi]$.

Coherence with the result: $R^2 = X^2 + Y^2$ follows a law of
χ^2 **with 2 degrees of freedom**
$\Leftrightarrow$ **exponential law of parameter** $\frac{1}{2}$.

3 2^{nd} method of Simulation of standard Gaussian distribution $\mathcal{N}(0;1)$ – (The method of Polar Coordinates)

Let $U_1,\ U_2$ be two continuous uniform random variables on $]0,1]$.

We determine 2 transformations of U_1 and U_2 which yield 2 **standard gaussian random variables** $(X_1,\ X_2)$ by using polar coordinates:

$$\left.\begin{array}{r} R^2 = X_1^2 + X_2^2 \\ \text{and } \theta = \arctan \dfrac{X_2}{X_1} \end{array}\right\} \text{which are independent if } X_1,\ X_2 \text{ are independent.}$$

$-2\ln U_1$ follows an exponential distribution law of parameter $\frac{1}{2} \Rightarrow \underline{R^2 = -2\ln U_1}$ and for θ we take $\underline{2\pi U_2}$ (Uniform on $[0, 2\pi]$).

Now

$$\left.\begin{array}{l} X_1 = R\cos\theta \\ X_2 = R\sin\theta \end{array}\right\} \Leftrightarrow$$

$X_1,\ X_2$ **are independent standard Gaussian random variables** (cf. example 3.3).

Method of polar coordinates

Conclusion For two independent gaussian random variables $X_1,\ X_2$ we have:

$$\left.\begin{array}{l} X_1 = (-2\ln U_1)^{1/2}\cos(2\pi U_2) \\ X_2 = (-2\ln U_1)^{1/2}\sin(2\pi U_2) \\ (\textbf{independent standard gaussian random variables}) \end{array}\right\} \tag{3.1}$$

Remark :

We want to reduce the time of calculations (because of the sines and cosines)

If U Uniform on (0,1) $\Rightarrow$ $2U$ is Uniform on [0,2]

$$\Rightarrow\ 2U - 1 \text{ Uniform on } [-1, 1].$$

If we generate 2 numbers $U_1,\ U_2$ and put the vector:

$$\left.\begin{array}{l} V_1 = 2U_1 - 1 \\ V_2 = 2U_2 - 1 \end{array}\right\}$$

$\Rightarrow$

$(V_1,\ V_2)$ is uniformly distributed inside the square of area 4 and of center $(0,0)$.

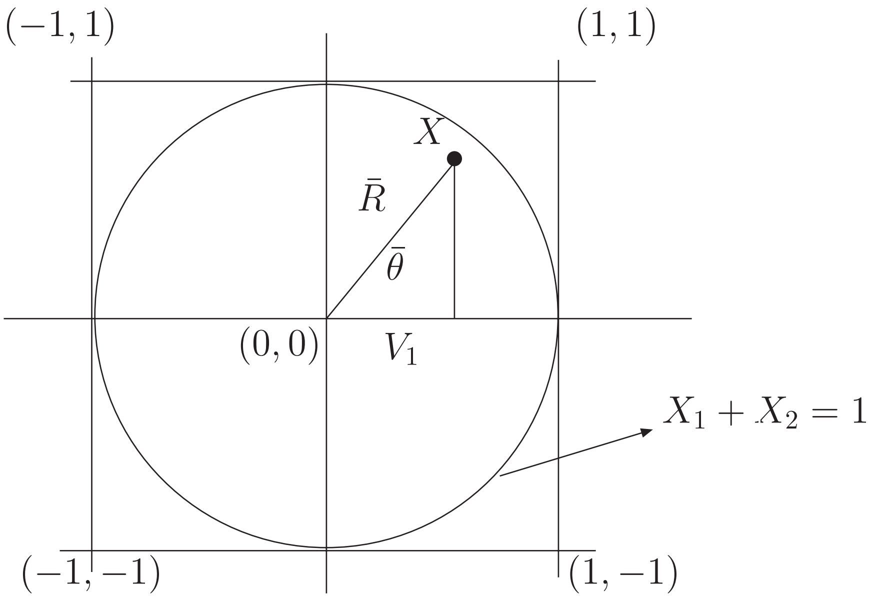

Figure 3.1: The circle $\mathcal{C}$ of radius $R = 1$ centered at the origin.

If we generate a sequence of couples $(V_1,\ V_2)$ until that $(V_1,\ V_2)$ should be:

$$V_1^2 + V_2^2 \leqslant 1$$

$\Rightarrow$ the vector $(V_1,\ V_2)$ is inside the cercle $\mathcal{C}$.

Let (R, θ) be the polar coordinates:

$$\sin\theta = V_2/\bar{R} = \frac{-V_2}{\sqrt{V_1^2 + V_2^2}}$$

$$\cos\theta = V_1/\bar{R} = \frac{-V_1}{\sqrt{V_1^2 + V_2^2}}$$

$\Rightarrow$

We verify that $\bar{R}^2$ is uniformly distributed on $]0,\ 1]$ and, θ is also uniformly distributed on $]0,\ 2\pi]$.

$\Rightarrow$

From the equation 3.1 and the previous remarks we conclude that :

We can generate **(simulate) two independent standard gaussian random variables** $(X,\ Y)$ by generating another number U **(random)**, by choosing $U \Leftrightarrow \bar{R}^2$ and by defining:

$$\begin{aligned} X &= (-2\ln\bar{R}^2)^{1/2}\,\frac{V_1}{\bar{R}} = \sqrt{\frac{-2\ln S}{S}}V_1 \\ Y &= (-2\ln\bar{R}^2)^{1/2}\,\frac{V_2}{\bar{R}} = \sqrt{\frac{-2\ln S}{S}}V_2 \end{aligned}$$

with: $\qquad S = \bar{R}^2 = V_1^2 + V_2^2$

Procedure of the Simulation

Step 1 : Generate the random numbers U_1 and U_2

Step 2 : Put $V_1 = 2U_1 - 1,\ V_2 = 2U_2 - 1,\ S = V_1^2 + V_2^2 = \bar{R}^2$

Step 3 : If $S > 1$ come back to step 1

Step 4 : Simulate the 2 independent standard gaussian random variables

$$X = \sqrt{\frac{-2\ln S}{S}}V_1, \quad Y = \sqrt{\frac{-2\ln S}{S}}V_2$$

A remark about the time of calculus

Now, probability to find a point inside the circle is equal to :

$$P[z \in\ \mathcal{C}] = \frac{\text{Area circle}}{\text{Area square}} = \frac{\pi}{4}$$

$\Rightarrow$ The method of polar coordinates needs $4/\pi \simeq 1,273$ for the step 1

$\Rightarrow$ $2,546$ rundom numbers,

$\Rightarrow$ 1 logarithm,

$\Rightarrow$ 1 square root ,

$\Rightarrow$ 1 division and

$\Rightarrow$ $2,546$ multiplications in order to generate 2 independent standard gaussian random variables.

Exercise 3.2

i). Simulate (following the method of "polar coordinates" 2 independent standard gaussian random variables $X : \mathcal{N}(0,\ 1)$. *et* $Y : \mathcal{N}(0,\ 1)$.. *and estimate the corresponding expected values and variances. Compare your results with respect to the theoretical values of these parameters.*

ii). Repeate the analogous exercise for two gaussian random variables: $X : \mathcal{N}(\mu,\ \sigma^2)$.*, and* $Y : \mathcal{N}(\mu,\ \sigma^2)$..*with* $\mu = 10;\ \sigma = 2$

iii). Using the same values of the parameters compare your results with those qbtained by the first method 1^{st} *method (application of TCL theorem)l by using a sample of uniform* U_i *(Random) variables).*

The comparison is realized with respect to:

a) The precision of estimations, and

b) La rate of convergence to the best estimation.

3 3^d method of Simulation of standard Gaussian distribution $\mathcal{N}(0;1)$ – The method of reject (or simply "reject")

The principle

a) We know *how to simulate* a continuous random variable Y with a density distribution function: $g(y)$ and,

b) By using the density $g(y)$, we want to simulate another random variable X with known density distribution function $f(x)$ but with not explicitly given the corresponding probability distribution function F_X.

c) We first simulate Y and accept this value with a probability proportional to the ratio $f(y)/g(y)$;

$$\Leftrightarrow \qquad \text{given a previously defined constant } C \text{ we require } \forall y, \ \frac{f(y)}{g(y)} \leq C$$

The steps:

1. We simulate $Y = y$ and calculate $\dfrac{f(y)}{g(y)}$

2. We simulate a random variable *Random* $U = u$

3.
$$\text{If} \quad u \leq \frac{f(y)}{Cg(y)} \quad \text{we put } \ x = y$$

   If not, we go back to step 1 .

   In figure 3.2 graphically represent the steps of the reject method.

Theorem 3.3 *The random variable generated by the method of "reject" admits as density distribution function f.*

Exercise 3.3 *Give the proof of this theorem.*

Exercise 3.4 *Apply the reject method to simulate a standard gaussian random variable.*

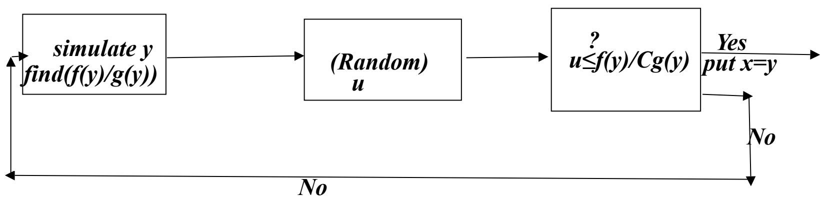

Figure 3.2: Graphical representation of the simulation of X by the “reject”.

Chapter 4

Simulation of the Brownian motion

1 Reminders

1 Laws of large numbers

Definition 4.1

Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space and X $(X_1,\ \ldots,\ X_n,\ \ldots)$ random variables defined on $(\Omega,\ \mathcal{A})$ (not necessarily independent) and such that $\forall\ i\ \mathbb{E}[X_i]$ exists.
If the transformed random variable defined by:

$$Y_n = \frac{1}{n}\sum_{i=1}^{n}[X_i - \mathbb{E}[X_i]] = \bar{X} - \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i]$$

converges to 0 following one of the previously defined type of convergence, then the sequence obeys one of the laws of large numbers:

* *If $\lim\limits_{n\to+\infty} Y_n = 0(P) \Rightarrow$ then we say that the sequence obeys the **weak law of large numbers**.*

* *If $\lim\limits_{n\to+\infty} Y_n = 0(a.s) \Leftrightarrow$ then we say that the sequence obeys the **strong law of large numbers**.*

Application to a sample

Theorem 4.1 *Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space and X $(X_1,\ \ldots,\ X_n,)$ independent random variables defined on $(\Omega,\ \mathcal{A})$ following the same law of probability and,*

$$\forall\ i \in \{1,\ 2, \ldots\quad n\}\ \ \mathbb{E}[X_i] = \mu,\ \ Var[X_i] = \sigma^2 \quad \textit{exist}$$

$$\Rightarrow$$

the sequence $\{X_1,\ \ldots,\ X_n\}$ obeys both the strong and weak law of large numbers.

For the proof of this theorem we use the generalization of Tchebycheff inequalities $\Leftrightarrow$ Kolmogorov inequalities.

2 Convergence in law

Definition 4.2

Let $(\Omega,\ \mathcal{A},\ P)$ *be a probability space and* X $(X_1,\ \dots,\ X_n,\ \dots)$ *random variables defined on* $(\Omega,\ \mathcal{A})$ *with corresponding probability distribution functions* F_X, $\{F_{X_1},\ \dots\ F_{X_n}\}$. *We say that the sequence* $\{X_1,\ X_2,\ \dots,\ X_n\}$ ***converges in law*** *to* X if

$$\lim_{n\to+\infty} F_{X_n}(x) = F_X(x)$$

at every point x of continuity of F_X.

$$X_n \underset{\mathcal{L}}{\to} X \ \Leftrightarrow \lim_{n\to\infty} X_n = X(\mathcal{L})$$

Central Limit Theorem

Theorem 4.2 (T.C.L.)

Let $(\Omega,\ \mathcal{A},\ P)$, *be a probability space and let us consider a sequence of independent random variables* $X_1,\ X_2,\ \dots\ \ X_n$ *defined on* $(\Omega,\ \mathcal{A})$, *which follow the same probability distribution law and such that:*

$$\forall\ \ i \in \{1,\ 2, \dots\ \ n\}\ \ \mathbb{E}[X_i] = \mu,\ \ Var[X_i] = \sigma^2$$

exist,

$$\Rightarrow$$

the sequence $Y_1,\ Y_2,\ \dots, Y_n,\ \dots$ *where*

$$Y_n = \sqrt{n}\ \ \frac{\bar{X} - \mu}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)$$

converges in law to a standard normal random variable $Y : \mathcal{N}(0,\ 1)$.

2 The standard Brownian motion $\{W(t)\}$

- $W(t=0) = 0$
- $t \to W_t(\omega)$ is a continuous function (continuity of trajectories)
- $\forall\ s \leqslant t \quad W_t - W_s$ is $\mathcal{F}_s(\omega)$-independent ($\Leftrightarrow$ independence of increments)
- $W_t - W_s$ follows the same law as $W_{t-s} - W_0$ ($\Leftrightarrow$stationarity of increments)
- $$E[W_t] = 0 \qquad Var[W_t] = E[W_t^2] = t$$

  and density distribution function:

  $$f_{W_t}(x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \quad \text{(centered Gaussian distribution).}$$

3 1^{st} method of simulation of the Brownian motion 'The random walk"

1 The steps

(a) We consider a sequence $\{X_i\}_{i\geqslant 0} \Leftrightarrow$ a random sample of (independent) discrete variables with corresponding support, probability mass distribution function and corresponding mean and variance:

$$\begin{array}{lll} D_{X_i} = \{-1, 1\}, & & P_{X_i}(1) = \frac{1}{2}\ ;\ P_{X_i}(-1) = \frac{1}{2} \\ E[X_i] = 0\ \forall i & \text{and} & Var[X_i] = E[X_i^2] = 1 \end{array}$$

We simulate n independent random variables of type X_i and, we define the random variable :

$$S_n = \sum_{i=1}^{n} X_i$$

(b) We **approximate the Brownian motion** $\{W(t)\}$ by the process $\{X_t^n\}_{t\geqslant 0}$ with $X_n^t = \frac{1}{\sqrt{n}} S_{[nt]}$.
where $[nt]$ means the **integer part of** nt.

2 Justification of the method

The standard Gaussian random variable $\mathcal{N}(0,1)$ which corresponds to the Brownian motion is the following:

$$\boxed{Y = \frac{W_t - 0}{\sqrt{t}}}$$

$\forall$ fixed t, we use the sequence $\{X_i\}_{i=1,\ldots,n}$ (of the previously defined n independent random variables with mean zero variance $\sigma_i^2 = 1$) and we define:

$$S_{[nt]} = \sum_{i=1}^{[nt]} X_i \quad \text{and} \quad \bar{S}_{[nt]} = \frac{\sum_{i=1}^{[nt]} X_i}{[nt]}$$

with:

$$E[\bar{S}_{[nt]}] = 0, \quad Var[\bar{S}_{[nt]}] = \frac{[nt]}{[nt]^2} = \frac{1}{[nt]}$$

Then, we apply the "**Central Limit Theorem**" to the sequence:

$$Y_n = \frac{\sum_{i=1}^{[nt]} X_i/[nt] - 0}{([nt])^{-1/2}} = \frac{\sum_{i=1}^{[nt]} X_i}{\sqrt{[nt]}}$$

$$\begin{array}{l} \Rightarrow \quad \lim_{n\to\infty} Y_n = \lim_{n\to\infty} \frac{\sum^{[nt]} X_i}{\sqrt{n}\sqrt{t}} = \lim_{n\to\infty} \frac{S_{[nt]}}{\sqrt{n}\sqrt{t}} = Y : \mathcal{N}(0,1) \\ \qquad\qquad \text{and} \qquad Y\sqrt{t} = W_t \end{array}$$

It follows that after a sufficient large number n of iterations the process

$$\left\{ \frac{S_{[nt]}}{\sqrt{n}} \right\}$$

is a good approximation of the Brownian motion.

4 2^{nd} method of simulation of the Brownian motion: the sequences of "Gaussians"

1 The steps

(a) We simulate a sequence of independent standard Gaussian random variables $\{g_i\}_{i \geqslant 0}$.

(b) With $\Delta t > 0$ we put:

$$\begin{array}{rcl} S_0 & = & 0 \\ S_{n+1} - S_n & = & g_n \end{array} \tag{4.1}$$

(c) $\Rightarrow$ the law of $(\sqrt{\Delta t} S_0,\ \sqrt{\Delta t} S_1, ..., \sqrt{\Delta t} S_n)$ is identical to the one of :

$$(W_0,\ W_{\Delta t},\ W_{2\Delta t}, ...,\ W_{n\Delta t})$$

In other words we approximate the standard Brownian motion by the process :

$$X_t^n = \sqrt{\Delta t} S_{[t/\Delta t]}$$

$$\text{(Remark:} \qquad n\Delta t = t \ \Rightarrow n = \frac{t}{\Delta t} \quad \Rightarrow \quad n \to \infty \Leftrightarrow \Delta t \to 0)$$

2 Justification of the method

Following 4.1 we have :

$$S_0 = 0,\ S_1 = g_0,\ S_2 = g_1 + g_0,\ ... \ ... S_n = \sum_{i=0}^{n-1} g_i$$

and we define:

$$\bar{S}_n = \sum_{i=0}^{n-1} \frac{g_i}{n}$$

with,

$$\mathbb{E}(\bar{S}_n) = 0 \qquad \text{et} \qquad \sigma^2_{\bar{S}_n} = \frac{n \times 1}{n^2} = \frac{1}{n}$$

and, (in the same way as in the first method) the "central limit theorem" yields:

$$\lim_{n\to\infty} Y_n \equiv \lim_{n\to\infty} \frac{\sum \frac{g_i}{n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{g_i}{\sqrt{n}} = Z :\ \mathcal{N}(0,1)$$

Now, the standard gaussian random variable associated to $W_{n\Delta t}$ is:

$$\frac{W_{n\Delta t}}{\sqrt{n\Delta t}} = Z$$

$$W_{n\Delta t} = Z\sqrt{n\Delta t} \quad \text{or} \quad W_t = S_n\sqrt{\Delta t}$$

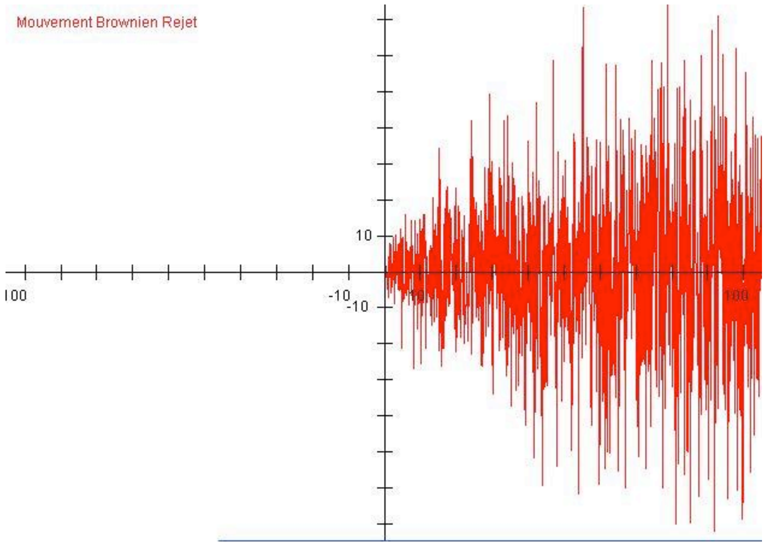

Figure 4.1: Graphical representation of the Brownian motion simulated by the method of Gaussians (reject method for the simulation of the standard normal gaussians).

Exercise 4.1

i). Simulate W_t by the first method and estimate the corresponding expectation value and variance (at fixed t). Represent graphically the results of your iterations.

By varying t represent graphically a trajectory of the Brownian motion.

ii). Repeat the same exercise by using the second method ("Gaussians") of simulation. Apply the three different methods you know for the simulation of the standard normal distribution $\mathcal{N}(0,1)$.

$\Leftrightarrow$ *TCL theorem with "random sample of $U[]0,1[]$"'s, the "Polars" and the "Reject"*

Compare the corresponding precisions and rapidity of your estimates following the three methods and then compare each one of them with the method of random walk first method) (Display graphically the three different trajectories.

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