A Geometrically Motivated Introductory Notes on Finite Element Method

Maksymilian Frankiewicz

2025

Contents

Abstract

The Finite Element Method (FEM) is a powerful numerical technique for solving differential equations. This document provides an introduction to FEM, emphasizing its geometric interpretation rooted in the Galerkin method. We explore how FEM can be understood as a projection of an infinite-dimensional problem onto a finite-dimensional subspace, making complex problems computationally tractable.

1 Introduction: The Problem Setting

Many physical phenomena can be described by differential equations of the form: $Au = f$ where u is the unknown function we seek, A is a linear differential operator, and f is a known function (often called the source term or forcing function). The function u typically resides in an infinite-dimensional function space, and the equation is to hold over some domain $\Omega \subset \mathbb{R}^d$.

2 The Galerkin Method: A Foundational Idea

The Finite Element Method, in its standard version, is based on the Galerkin method. The Galerkin method leverages a seemingly simple observation:

Remark 1 (Equivalence in a Weak Sense)**.** *A function u satisfying the differential equation $Au = f$ is equivalent to the inner product equality $\langle Au, v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)}$ holding for all functions v in a suitable test function space (typically, the space of square-integrable functions, $L^2(\Omega)$).*

Definition 1 (Inner Product)**.** *An **inner product** on a real vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties for all vectors $x, y, z \in V$ and all scalars $c \in \mathbb{R}$:*

1. ***Symmetry:*** $\langle x, y\rangle = \langle y, x\rangle$
2. ***Linearity in the first argument:*** $\langle cx + y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$
3. ***Positive-definiteness:*** $\langle x, x\rangle \geq 0$*, and* $\langle x, x\rangle = 0$ *if and only if* $x = \mathbf{0}$ *(the zero vector).*

The standard inner product for square-integrable functions on a domain Ω is the $L^2(\Omega)$ inner product, defined as: $\langle f, g\rangle_{L^2(\Omega)} \coloneqq \int_\Omega f(x)g(x), \mathrm{d}x$

The Central Role of Inner Products in Structuring Vector Spaces: The inner product plays a pivotal role in endowing a vector space with geometric structure, transforming it into an inner product space (and, upon completion, a Hilbert space). From the inner product, we derive a **norm** that quantifies the "length" of vectors: $|u| = \sqrt{\langle u, u\rangle}$ This norm, in turn, induces a **metric** (distance function) between any two vectors u and v: $d(u, v) = |u - v| = \sqrt{\langle u - v, u - v\rangle}$ Crucially, the inner product defines the notion of **orthogonality**: two vectors u and v are orthogonal if $\langle u, v\rangle = 0$. This concept is fundamental for decompositions and projections. Orthogonality enables the definition of **orthogonal projections**, which are key to understanding approximations in subspaces.

Geometric Motivation for Inner Products and Projections: In finite-dimensional Euclidean spaces (like $\mathbb{R}^n$), the dot product ($x \cdot y = \sum x_i y_i$) is a familiar inner product. It allows us to define lengths (norms: $|x| = \sqrt{x \cdot x}$) and angles (via $x \cdot y = |x||y| \cos\theta$). Crucially, the properties of an inner product are precisely what's needed to decompose a vector w into components along

a chosen basis $\mathbf{e}_i$. If we want to find the component of w along $\mathbf{e}_j$, we compute $\langle w, \mathbf{e}_j\rangle / \langle \mathbf{e}_j, \mathbf{e}_j\rangle$ (assuming an orthogonal basis, it's simpler). More generally, finding the orthogonal projection of a vector w onto a subspace spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_k$ involves solving a system of equations derived from inner products: the projection $Pw = \sum c_i \mathbf{e}_i$ is found by requiring $w - Pw$ to be orthogonal to each $\mathbf{e}_j$, i.e., $\langle w - Pw, \mathbf{e}_j\rangle = 0$. This leads to $\langle w, \mathbf{e}_j\rangle = \sum c_i \langle \mathbf{e}_i, \mathbf{e}_j\rangle$. The inner product formalizes this geometric intuition for more general vector spaces, including function spaces. To make this more precise, let us define projections in general and then focus on orthogonal projections.

Definition 2 (Projection Operator)**.** *A linear operator $P : V \to V$ on a vector space V is a* ***projection*** *if it is idempotent, i.e., $P^2 = P$. This means that applying P twice yields the same result as applying it once. In terms of eigenvalues, a projection operator has eigenvalues that are only 0 or 1. If P projects onto a subspace $W \subset V$, then $Pv \in W$ for all $v \in V$, and $Pw = w$ for all $w \in W$.*

Definition 3 (Orthogonal Projection with Respect to an Inner Product)**.** *Given an inner product space V with inner product $\langle \cdot, \cdot \rangle$ and a closed subspace $W \subset V$, the* ***orthogonal projection*** *$P_W : V \to W$ is the unique projection onto W such that the error $v - P_W v$ is orthogonal to W, i.e., $\langle v - P_W v, w\rangle = 0 \quad \forall w \in W$. Equivalently, $P_W v$ is the unique element in W that minimizes the distance to v in the norm induced by the inner product: $P_W v = \arg\min_{w \in W} |v - w|$. This best-approximation property follows directly from the orthogonality condition and the Pythagorean theorem in inner product spaces: for any $w \in W$, $|v - w|^2 = |v - P_W v + P_W v - w|^2 = |v - P_W v|^2 + |P_W v - w|^2 \geq |v - P_W v|^2$, with equality if and only if $w = P_W v$.*

The inner product thus provides the geometric tools—norms, distances, orthogonality, and projections—essential for approximating solutions in subspaces, which is at the heart of the Galerkin method and FEM.

The "Almost Everywhere" Equivalence: The statement "if vectors u_1 and u_2 have equal inner products with all vectors in the space, they have to be the same vector" is strictly true in finite-dimensional spaces. In infinite-dimensional function spaces like $L^2(\Omega)$, if $\langle u_1, v\rangle_{L^2(\Omega)} = \langle u_2, v\rangle_{L^2(\Omega)}$ for all $v \in L^2(\Omega)$, then $u_1 - u_2$ is orthogonal to every function in $L^2(\Omega)$, including itself. Thus $\langle u_1 - u_2, u_1 - u_2\rangle_{L^2(\Omega)} = |u_1 - u_2|^2_{L^2(\Omega)} = 0$. This implies that $u_1 - u_2 = 0$ "almost everywhere" (a.e.), meaning $u_1(x)$ and $u_2(x)$ can differ only on a set of points $S \subset \Omega$ whose Lebesgue measure is zero. For practical purposes in physics and engineering, functions equal almost everywhere are often considered equivalent.

2.1 Relaxing the Condition: Finite-Dimensional Subspaces

The Galerkin method then relaxes the condition $\langle Au, v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)}$ to hold not for *all* $v \in L^2(\Omega)$, but only for v in a carefully constructed finite-dimensional subspace, say $V_h \subset L^2(\Omega)$, with dimension N. Let $\phi_1, \phi_2, \ldots, \phi_N$ be a basis for V_h. $\langle Au, v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)}, \quad \forall v \in V_h$ This is known as the **weak form** or **variational formulation** of the problem. Obviously, there are now infinitely many functions u in the original infinite-dimensional space that could satisfy this relaxed set of N conditions (one for each basis function $\phi_j \in V_h$). To obtain a unique solution from these N equations, it is natural to seek an approximate solution u_h also within a finite-dimensional space. If we seek u_h in the *same* N-dimensional space V_h (the "trial space" for the solution is the same as the "test space" for v), this is precisely the **Galerkin method**. Find $u_h \in V_h$ such that $\langle Au_h, v_h\rangle_{L^2(\Omega)} = \langle f, v_h\rangle_{L^2(\Omega)}, \quad \forall v_h \in V_h$ If u_h is sought in a different finite-dimensional space $W_h \neq V_h$ (but typically $\dim(W_h) = \dim(V_h) = N$), this constitutes the more general **Petrov-Galerkin method**.

2.2 Function Spaces and Sobolev Spaces

When we write $\langle Au, v\rangle_{L^2(\Omega)}$, the spaces to which u, f, and v belong are crucial. If A is a differential operator of order $2m$ (e.g., $m = 1$ for the Laplacian Δ, which involves second derivatives), then for Au to be well-defined and in $L^2(\Omega)$ (so the inner product $\langle Au, v\rangle_{L^2(\Omega)}$ makes sense with $v \in L^2(\Omega)$), the function u generally needs to possess $2m$ weak derivatives that are square-integrable. Such functions belong to a **Sobolev space**, denoted $H^k(\Omega)$.

Definition 4 (Sobolev Space (Informal))**.** *The Sobolev space $H^k(\Omega)$ consists of functions $u \in L^2(\Omega)$ such that all their weak (distributional) derivatives up to order k also belong to $L^2(\Omega)$. The norm typically involves the $L^2(\Omega)$ norms of the function and its derivatives. For example, for $H^1(\Omega)$: $|u|^2_{H^1(\Omega)} \coloneqq |u|^2_{L^2(\Omega)} + |\nabla u|^2_{L^2(\Omega)} = \int_\Omega u^2, \mathrm{d}x + \int_\Omega |\nabla u|^2, \mathrm{d}x$ For $H^2(\Omega)$, it would also include integrals of squares of second derivatives. Spaces like $H^1_0(\Omega)$ or $H^2_0(\Omega)$ are subspaces of $H^1(\Omega)$ or $H^2(\Omega)$ respectively, where functions (and possibly their derivatives up to a certain order) vanish on the boundary $\partial\Omega$ in a trace sense. These are crucial for incorporating homogeneous Dirichlet boundary conditions.*

So, for $Au = f$ where A is second order (like the Laplacian), we might initially consider $u \in H^2(\Omega)$, $f \in L^2(\Omega)$, and $v \in L^2(\Omega)$.

3 The Role of Boundary Conditions

So far, we have avoided the matter of boundary conditions, which are crucial for ensuring the solution u is unique. For simplicity, let's initially consider **essential boundary conditions** (often Dirichlet conditions), where the value of u is prescribed on (part of) the boundary $\partial\Omega$: $u = g_D \quad \text{on } \Gamma_D \subseteq \partial\Omega$ One might initially think of seeking a solution in the space of functions that satisfy this boundary condition. However, if $g_D \neq 0$, this set of functions is an affine manifold, not a vector space (e.g., if u_1 and u_2 satisfy $u = g_D$, then $u_1 + u_2$ satisfies $u = 2g_D$, not g_D). Our assumption of the solution u_h being a linear combination of basis functions relies on V_h being a vector space. There is a critical exception: the space of functions satisfying a *homogeneous* essential boundary condition ($u = 0$ on Γ_D) *is* a vector space. We exploit this by decomposing the desired solution u as: $u = u_0 + u_D$ where:

- u_0 is a function satisfying the homogeneous boundary condition ($u_0 = 0$ on Γ_D). It will be sought in a space like $H^1_0(\Omega)$ (if A is second order and after integration by parts, see later).
- u_D is any specific, known function that satisfies the non-homogeneous boundary condition ($u_D = g_D$ on Γ_D) and has sufficient regularity (e.g., $u_D \in H^1(\Omega)$ for second-order problems).

Substituting this into $Au = f$, we get $A(u_0 + u_D) = f$, or $Au_0 = f - Au_D$. The weak form becomes: $\langle Au_0, v\rangle_{L^2(\Omega)} = \langle f - Au_D, v\rangle_{L^2(\Omega)}$ Since u_D is a known function, the term $\langle Au_D, v\rangle_{L^2(\Omega)}$ can be moved to the right-hand side. The term $\langle f, v\rangle_{L^2(\Omega)} - \langle Au_D, v\rangle_{L^2(\Omega)}$ is a linear functional of v. We now seek u_0 in a vector space of functions satisfying homogeneous boundary conditions. This outlines the core idea of the Galerkin method. In principle, it is a mesh-free method. However, it would be very difficult to apply in practice on arbitrary domains for two main reasons:

1. It is challenging to construct basis functions for V_h that can approximate the solution of a differential equation reasonably well over the entire complex domain and correctly incorporate boundary conditions.
2. Even if we devise such global basis functions, computing their inner products (integrals like $\int_\Omega (A\phi_i)\phi_j, \mathrm{d}x$) over complicated domains Ω would be non-trivial.

4 FEM as a Practical Implementation of Galerkin's Method

The Finite Element Method (FEM) can be viewed as a practical and systematic way to implement the Galerkin method. The core idea is to **discretize** the domain Ω by dividing it into a finite number of smaller, simpler subdomains called **elements** (e.g., triangles or quadrilaterals in 2D; tetrahedra or hexahedra in 3D). This collection of elements is called a **mesh**. The mesh is then used to construct basis functions $\phi_i(x)$ for V_h that are **locally supported**. This means each ϕ_i is non-zero only over a small patch of elements connected to a specific node i in the mesh, and zero everywhere else. Typically, these are low-order polynomials on each element (e.g., linear or quadratic). This local support property is key:

- It simplifies the construction of basis functions that can approximate a wide variety of solutions.
- It makes the resulting system matrix (from $\langle A\phi_j, \phi_i \rangle_{L^2(\Omega)}$) sparse, which is highly advantageous for computational efficiency.
- Integrals are now computed element-wise and summed up, which is much more manageable.

FEM effectively reduces the problem of finding suitable global basis functions to a computational geometry problem (generating a mesh for Ω), which, while still challenging for arbitrary geometries, can be automated with relatively little input from a software user (e.g., using libraries like Gmsh or CGAL).

5 Geometric Interpretation of Galerkin's Method

The Galerkin condition is: find $u_h \in V_h$ such that $\langle Au_h - f, v_h \rangle_{L^2(\Omega)} = 0, \quad \forall v_h \in V_h$ This means that the **residual**, $R_h \coloneqq Au_h - f$, is required to be $L^2(\Omega)$-orthogonal to the entire subspace V_h. Geometrically, this is equivalent to stating that the orthogonal projection of Au_h onto the subspace V_h must be equal to the orthogonal projection of f onto V_h: $P_{V_h}(Au_h) = P_{V_h}(f)$ where P_{V_h} denotes the $L^2(\Omega)$-orthogonal projection operator onto V_h. We don't typically compute these projections explicitly by, for instance, constructing and inverting the Gram matrix $G_{ij} = \langle \phi_i, \phi_j \rangle_{L^2(\Omega)}$ for V_h. Instead, we know that the components of any vector w that lie in V_h are determined by its inner products with the basis vectors ϕ_j of V_h. If $\langle Au_h, \phi_j \rangle_{L^2(\Omega)} = \langle f, \phi_j \rangle_{L^2(\Omega)}$ for all basis functions ϕ_j, then their projections onto V_h must be identical. Effectively, we are projecting our original problem, which is naturally set in an infinite-dimensional function space, into a chosen finite-dimensional subspace V_h.

6 The Role of Integration by Parts (Green's Formulas)

Often, but not always, the left-hand side of the weak form $\langle Au_h, v_h \rangle_{L^2(\Omega)}$ is transformed using integration by parts (or its higher-dimensional analogues, Green's identities). This step has several important consequences:

1. **Reduced Continuity Requirements:** It typically reduces the order of derivatives acting on the trial function u_h and transfers some derivatives to the test function v_h. For example, a term involving second derivatives of u_h (like $\int (\Delta u_h) v_h, \mathrm{d}x$) might become a term involving first derivatives of both u_h and v_h (like $-\int \nabla u_h \cdot \nabla v_h, \mathrm{d}x$, plus boundary terms). This means u_h might only need to be in $H^1(\Omega)$ instead of $H^2(\Omega)$, which allows for simpler basis functions (e.g., continuous piecewise linear functions are in $H^1(\Omega)$ but not $H^2(\Omega)$).

2. **Symmetric Bilinear Forms:** For many common differential operators (like the Laplacian), integration by parts can lead to a symmetric bilinear form $a(u_h, v_h)$ on the left-hand side: $a(u_h, v_h) = a(v_h, u_h)$. This results in a symmetric system matrix, which has computational advantages.

3. **Natural Incorporation of Boundary Conditions:** Integration by parts naturally introduces boundary integral terms. These terms are where **natural boundary conditions** (e.g., Neumann conditions, specifying $\nabla u \cdot \mathbf{n}$) are incorporated into the weak formulation. Essential conditions (Dirichlet) are imposed directly on the function space for u_h (and u_0).

After integration by parts, the problem often takes the form: Find $u_h \in V_{h0}$ (a subspace of, e.g., $H_0^1(\Omega)$ incorporating homogeneous Dirichlet BCs) such that $\mathrm{a}(\mathrm{u}_h, v_h) = L(v_h), \quad \forall v_h \in V_{h0} where \mathrm{a}(\cdot,\cdot)$ is a bilinear form (resulting from Au_h part) and $L(\cdot)$ is a linear functional (resulting from f and any non-homogeneous boundary data from u_D or Neumann conditions).

6.1 Galerkin Orthogonality

In the literature on FEM (e.g., Brenner Scott, "The Mathematical Theory of Finite Element Methods"), a central concept is the **Galerkin orthogonality** of the error. If u is the true solution to $Au = f$ (or $a(u, v) = L(v)$ after integration by parts) and $u_h \in V_h$ is the FEM solution, then $a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h$ This states that the error $e_h = u - u_h$ is orthogonal to the subspace V_h with respect to the bilinear form $a(\cdot,\cdot)$. This is a direct consequence of $a(u, v_h) = L(v_h)$ and $a(u_h, v_h) = L(v_h)$. This orthogonality is fundamental for proving convergence results, such as Céa's Lemma, which states that the FEM solution u_h is the best approximation to u in V_h with respect to the norm induced by $a(\cdot,\cdot)$ (if $a(\cdot,\cdot)$ is coercive and continuous, even if not symmetric). While the residual orthogonality $\langle Au_h - f, v_h\rangle_{L^2(\Omega)} = 0$ gives the geometric picture of equal $L^2(\Omega)$-projections of Au_h and f onto V_h, the Galerkin orthogonality $a(u - u_h, v_h) = 0$ (after integration by parts) often provides a more direct link to best approximation properties in an "energy" norm associated with the operator A.

7 Example: The Poisson Equation

Let us consider the Poisson equation with mixed boundary conditions:

$$-\Delta u = f \quad \text{in } \Omega \; u \qquad\qquad = g_D \quad \text{on } \Gamma_D \; \nabla u \cdot \mathbf{n} = g_N \quad \text{on } \Gamma_N$$

where $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$ and $\Gamma_D \cap \Gamma_N = \emptyset$. $\mathbf{n}$ is the outward unit normal vector. We decompose the solution $u = u_0 + u_D$, where u_D is a function satisfying $u_D = g_D$ on Γ_D (and can be arbitrary on Γ_N, often chosen for convenience, e.g. $u_D = 0$ there if g_D allows smooth extension), and u_0 must satisfy $u_0 = 0$ on Γ_D. The equation for u_0 becomes: $-\Delta u_0 = f + \Delta u_D \quad$ in Ω with $u_0 = 0$ on Γ_D and $\nabla u_0 \cdot \mathbf{n} = g_N - \nabla u_D \cdot \mathbf{n}$ on Γ_N. Let $\tilde{f} = f + \Delta u_D$ and $\tilde{g}N = g_N - \nabla u_D \cdot \mathbf{n}$. The problem for u_0 is:

$$-\Delta u_0 = \tilde{f} \quad \text{in } \Omega \; u_0 \qquad\qquad = 0 \quad \text{on } \Gamma_D \; \nabla u_0 \cdot \mathbf{n} = \tilde{g}N \quad \text{on } \Gamma_N$$

We now seek u_0 in a space like $V_0 = w \in H^1(\Omega) \mid w|\Gamma_D = 0$. For the strong form $-\Delta u_0 = \tilde{f}$ to hold, and for $\tilde{f} \in L^2(\Omega)$, we would typically need $u_0 \in H^2(\Omega) \cap V_0$. The weak form is obtained by multiplying by a test function $v \in V_0$ and integrating over Ω: $\int_\Omega (-\Delta u_0)v, \mathrm{d}x = \int_\Omega \tilde{f}v, \mathrm{d}x, \quad \forall v \in V_0$ Now, we apply integration by parts (Green's first identity) to the left-hand side: $\int_\Omega (-\Delta u_0)v, \mathrm{d}x =$

$\int_\Omega \nabla u_0 \cdot \nabla v, \mathrm{d}x - \int_{\partial\Omega} (\nabla u_0 \cdot \mathbf{n}) v, \mathrm{d}S$ Since $v \in V_0$, $v = 0$ on Γ_D. So the boundary integral $\int \partial\Omega (\nabla u_0 \cdot \mathbf{n}) v, \mathrm{d}S$ simplifies to $\int_{\Gamma_N} (\nabla u_0 \cdot \mathbf{n}) v, \mathrm{d}S$. Using the Neumann condition $\nabla u_0 \cdot \mathbf{n} = \tilde{g} N$ on Γ_N, the weak formulation becomes: Find $u_0 \in V_0$ such that $\int_\Omega \nabla u_0 \cdot \nabla v, \mathrm{d}x = \int_\Omega \tilde{f} v, \mathrm{d}x + \int_{\Gamma_N} \tilde{g}_N v, \mathrm{d}S, \quad \forall v \in V_0$ This is of the form $a(u_0, v) = L(v)$, where:

- The bilinear form is $a(w, z) \coloneqq \int \Omega \nabla w \cdot \nabla z, \mathrm{d}x$.
- The linear functional is $L(z) \coloneqq \int_\Omega \tilde{f} z, \mathrm{d}x + \int_{\Gamma_N} \tilde{g} N z, \mathrm{d}S$.

After integration by parts, the continuity requirements are relaxed. Both u_0 and v need only have first derivatives that are square-integrable, so they belong to $V_0 \subset H^1(\Omega)$. (Specifically, $V_0 = H^1 \Gamma_D(\Omega) \coloneqq w \in H^1(\Omega) \mid w|\Gamma_D = 0$. If $\Gamma_D = \partial\Omega$, then $V_0 = H_0^1(\Omega)$).

Is $a(\cdot, \cdot)$ an inner product on V_0? The bilinear form $a(w, z) = \int \Omega \nabla w \cdot \nabla z, \mathrm{d}x$ is clearly bilinear and symmetric. For positive-definiteness on V_0: $a(w, w) = \int_\Omega |\nabla w|^2, \mathrm{d}x \geq 0$. If $a(w, w) = 0$, then $\nabla w = \mathbf{0}$ almost everywhere in Ω, which implies w is constant a.e. in Ω. If Γ_D has positive measure (i.e., it's not empty or just a set of points), then any function $w \in V_0$ that is constant and also zero on Γ_D must be the zero function $w \equiv 0$. Thus, if Γ_D is non-empty in a meaningful way, $a(\cdot, \cdot)$ is indeed an inner product on V_0. This induces the "energy norm" $|w|_a = \sqrt{a(w, w)}$.

Discretization: We now choose a finite-dimensional subspace $V_{h,0} \subset V_0$, spanned by basis functions $\phi_i i = 1^N$ (e.g., piecewise linear "hat" functions that are zero on Γ_D). We seek an approximate solution $uh, 0 = \sum_{j=1}^N c_j \phi_j \in V_{h,0}$. Substituting this into the weak form and testing against each basis function $\phi_i \in V_{h,0}$: $a\left(\sum_{j=1}^N c_j \phi_j, \phi_i\right) = L(\phi_i), \quad \text{for } i = 1, \ldots, N$ By linearity of $a(\cdot, \cdot)$ in its first argument: $\sum_{j=1}^N c_j a(\phi_j, \phi_i) = L(\phi_i), \quad \text{for } i = 1, \ldots, N$ This is an $N \times N$ system of linear equations $\mathbf{Kc} = \mathbf{F}$, where:

- $\mathbf{K}$ is the stiffness matrix with entries $K_{ij} = a(\phi_j, \phi_i)$.
- $\mathbf{c}$ is the vector of unknown coefficients c_j.
- $\mathbf{F}$ is the load vector with entries $F_i = L(\phi_i)$.

Since $a(\cdot, \cdot)$ is symmetric, the stiffness matrix $\mathbf{K}$ will be symmetric. If $a(\cdot, \cdot)$ is an inner product on $V_{h,0}$ (which it is if it's an inner product on V_0), then $\mathbf{K}$ will also be positive definite, guaranteeing a unique solution for $\mathbf{c}$.

Geometric Interpretation after Integration by Parts: The true solution $u_0 \in V_0$ satisfies $a(u_0, v) = L(v)$ for all $v \in V_0$. The approximate solution $u_{h,0} \in V_{h,0}$ satisfies $a(u_{h,0}, v_h) = L(v_h)$ for all $v_h \in V_{h,0}$. Since $V_{h,0} \subset V_0$, it must be that $a(u_0, v_h) = L(v_h)$ for all $v_h \in V_{h,0}$. Therefore, $a(u_0, v_h) = a(u_{h,0}, v_h)$ for all $v_h \in V_{h,0}$, which can be rewritten as: $a(u_0 - u_{h,0}, v_h) = 0, \quad \forall v_h \in V_{h,0}$ This is the Galerkin orthogonality condition with respect to the bilinear form $a(\cdot, \cdot)$. It means that the error $u_0 - u_{h,0}$ is a-orthogonal to the subspace $V_{h,0}$. Since $a(\cdot, \cdot)$ is an inner product on V_0 (under the assumptions on Γ_D), $u_{h,0}$ is precisely the orthogonal projection of the true solution u_0 onto the subspace $V_{h,0}$ with respect to the inner product $a(\cdot, \cdot)$. By the very definition of orthogonal projection, $u_{h,0}$ is the best approximation to u_0 in $V_{h,0}$ with respect to the energy norm $|\cdot|_a$: $|u_0 - u_{h,0}|_a = \min_{w_h \in V_{h,0}} |u_0 - w_h|_a$. This best-approximation property is a special case of the more general **Céa's Lemma**, which provides similar quasi-best approximation guarantees for bilinear forms that are continuous and coercive (elliptic) but not necessarily symmetric.

8 Conclusion

The Finite Element Method, grounded in the Galerkin method, offers a robust framework for numerically approximating solutions to differential equations. Its geometric interpretation as a projection method—either projecting Au_h to match f in an $L^2(\Omega)$ sense, or (often after integration by parts) projecting the true solution u onto a finite-dimensional subspace V_h with respect to an energy inner product $a(\cdot, \cdot)$—provides powerful intuition. By discretizing the domain and using locally supported basis functions, FEM transforms complex infinite-dimensional problems into manageable systems of algebraic equations, making it a versatile tool across science and engineering.

It is important to note that the applicability of FEM extends far beyond the linear, stationary problem discussed here. The framework can be adapted to solve **nonlinear problems**, which typically result in systems of nonlinear algebraic equations requiring iterative solution schemes (e.g., Newton's method), as well as **nonstationary (time-dependent) problems**, which necessitate the discretization of the time domain (e.g., using finite differences) and solving a system of equations at each time step. These advanced topics are beyond the focus of this brief introduction, which is intended to provide the mathematical context for the author's accompanying C++ solver for the 2D Poisson equation, available at https://github.com/maksf113/FEMSolver.