# Probability Review

Gonzalo Mateos
Dept. of ECE and Goergen Institute for Data Science
University of Rochester
gmateosb@ece.rochester.edu
http://www.ece.rochester.edu/~gmateosb/

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# Axioms of probability

- Define a function $P(E)$ from a sigma-algebra $\mathcal{F}$ to the real numbers
- $P(E)$ qualifies as a probability if
  A1) Non-negativity: $P(E) \geq 0$
  A2) Probability of universe: $P(S) = 1$
  A3) Additivity: Given sequence of disjoint events $E_1, E_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

  $\Rightarrow$ Disjoint (mutually exclusive) events means $E_i \cap E_j = \emptyset$, $i \neq j$
  $\Rightarrow$ Union of countably infinite many disjoint events

- Triplet $(S, \mathcal{F}, P(\cdot))$ is called a probability space

# Consequences of the axioms

- Implications of the axioms A1)-A3)
  - $\Rightarrow$ Impossible event: $P(\emptyset) = 0$
  - $\Rightarrow$ Monotonicity: $E_1 \subset E_2 \;\; \Rightarrow P(E_1) \leq P(E_2)$
  - $\Rightarrow$ Range: $0 \leq P(E) \leq 1$
  - $\Rightarrow$ Complement: $P(E^c) = 1 - P(E)$
  - $\Rightarrow$ Finite disjoint union: For disjoint events $E_1, \ldots, E_N$

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P\left(E_i\right)$$

  - $\Rightarrow$ Inclusion-exclusion: For any events $E_1$ and $E_2$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

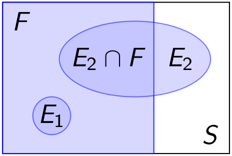
# Conditional probability

- Consider events $E$ and $F$, and suppose we know $F$ occurred
- Q: What does this information imply about the probability of $E$?
- **Def:** Conditional probability of $E$ given $F$ is (need $P(F) > 0$)

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow$ In general $P(E|F) \neq P(F|E)$

- Renormalize probabilities to the set $F$
  - Discard a piece of $S$
  - May discard a piece of $E$ as well




- For given $F$ with $P(F) > 0$, $P(\cdot|F)$ satisfies the axioms of probability

# Conditional probability example

- The name I wrote is male. What is the probability of name $x_n$?
- Assume male names are $F = \{x_1, \ldots, x_M\}$ $\Rightarrow P(F) = \frac{M}{N}$
- If name $x_n$ is male, $x_n \in F$ and we have for event $E = \{x_n\}$

$$P(E \cap F) = P(\{x_n\}) = \frac{1}{N}$$

$\Rightarrow$ Conditional probability is as you would expect

$$P(E \,\big|\, F) = \frac{P(E \cap F)}{P(F)} = \frac{1/N}{M/N} = \frac{1}{M}$$

- If name is female $x_n \notin F$, then $P(E \cap F) = P(\emptyset) = 0$

$\Rightarrow$ As you would expect, then $P(E \,\big|\, F) = 0$

# Law of total probability

- Consider event $E$ and events $F$ and $F^c$
  - $F$ and $F^c$ form a partition of the space $S$ ($F \cup F^c = S$, $F \cap F^c = \emptyset$)
- Because $F \cup F^c = S$ cover space $S$, can write the set $E$ as

$$E = E \cap S = E \cap [F \cup F^c] = [E \cap F] \cup [E \cap F^c]$$

- Because $F \cap F^c = \emptyset$ are disjoint, so is $[E \cap F] \cap [E \cap F^c] = \emptyset$

$$\Rightarrow P(E) = P([E \cap F] \cup [E \cap F^c]) = P(E \cap F) + P(E \cap F^c)$$

- Use definition of conditional probability

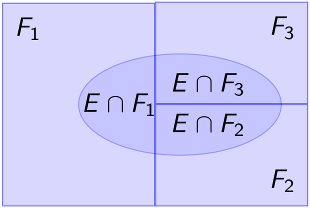
$$P(E) = P(E \,\big|\, F)P(F) + P(E \,\big|\, F^c)P(F^c)$$

- Translate conditional information $P(E \,\big|\, F)$ and $P(E \,\big|\, F^c)$

$\Rightarrow$ Into unconditional information $P(E)$

# Law of total probability (continued)

- In general, consider (possibly infinite) partition $F_i$, $i = 1, 2, \ldots$ of $S$
- Sets are disjoint $\Rightarrow F_i \cap F_j = \emptyset$ for $i \neq j$
- Sets cover the space $\Rightarrow \cup_{i=1}^{\infty} F_i = S$




- As before, because $\cup_{i=1}^{\infty} F_i = S$ cover the space, can write set $E$ as

$$E = E \cap S = E \cap \left[\bigcup_{i=1}^{\infty} F_i\right] = \bigcup_{i=1}^{\infty} [E \cap F_i]$$

- Because $F_i \cap F_j = \emptyset$ are disjoint, so is $[E \cap F_i] \cap [E \cap F_j] = \emptyset$. Thus

$$P(E) = P\left(\bigcup_{i=1}^{\infty} [E \cap F_i]\right) = \sum_{i=1}^{\infty} P(E \cap F_i) = \sum_{i=1}^{\infty} P(E \mid F_i) P(F_i)$$

# Total probability example

- Consider a probability class in some university
  - $\Rightarrow$ Seniors get an A with probability (w.p.) 0.9, juniors w.p. 0.8
  - $\Rightarrow$ An exchange student is a senior w.p. 0.7, and a junior w.p. 0.3
- Q: What is the probability of the exchange student scoring an A?
- Let $A =$ "exchange student gets an A," $S$ denote senior, and $J$ junior
  - $\Rightarrow$ Use the law of total probability

$$P(A) = P(A \mid S)P(S) + P(A \mid J)P(J)$$

$$= 0.9 \times 0.7 + 0.8 \times 0.3 = 0.87$$

# Bayes' rule

- From the definition of conditional probability

$$P(E \mid F)P(F) = P(E \cap F)$$

- Likewise, for $F$ conditioned on $E$ we have

$$P(F \mid E)P(E) = P(F \cap E)$$

- Quantities above are equal, giving Bayes' rule

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}$$

- Bayes' rule allows time reversion. If $F$ (future) comes after $E$ (past),
  - $\Rightarrow P(E \mid F)$, probability of past ($E$) having seen the future ($F$)
  - $\Rightarrow P(F \mid E)$, probability of future ($F$) having seen past ($E$)
- Models often describe future $\mid$ past. Interest is often in past $\mid$ future

# Bayes' rule example

- Consider the following partition of my email
  - $\Rightarrow$ $E_1 =$ "spam" w.p. $P(E_1) = 0.7$
  - $\Rightarrow$ $E_2 =$ "low priority" w.p. $P(E_2) = 0.2$
  - $\Rightarrow$ $E_3 =$ "high priority" w.p. $P(E_3) = 0.1$
- Let $F =$ "an email contains the word *free*"
  - $\Rightarrow$ From experience know $P(F \mid E_1) = 0.9$, $P(F \mid E_2) = P(F \mid E_3) = 0.01$
- I got an email containing "free". What is the probability that it is spam?
- Apply Bayes' rule

$$P(E_1 \mid F) = \frac{P(F \mid E_1)P(E_1)}{P(F)} = \frac{P(F \mid E_1)P(E_1)}{\sum_{i=1}^{3} P(F \mid E_i)P(E_i)} = 0.995$$

$\Rightarrow$ Law of total probability very useful when applying Bayes' rule

# Independence

- **Def:** Events $E$ and $F$ are independent if $P(E \cap F) = P(E)P(F)$

  $\Rightarrow$ Events that are not independent are dependent

- According to definition of conditional probability

$$P(E \,|\, F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

  $\Rightarrow$ Intuitive, knowing $F$ does not alter our perception of $E$

  $\Rightarrow$ $F$ bears no information about $E$

  $\Rightarrow$ The symmetric is also true $P(F \,|\, E) = P(F)$

- Whether $E$ and $F$ are independent relies strongly on $P(\cdot)$

- Avoid confusing with disjoint events, meaning $E \cap F = \emptyset$
- Q: Can disjoint events with $P(E) > 0$, $P(F) > 0$ be independent? No

# Independence example

- Wrote one name, asked a friend to write another (possibly the same)
- Probability space $(S, \mathcal{F}, P(\cdot))$ for this experiment
  - $\Rightarrow$ $S$ is the set of all pairs of names $[x_n(1), x_n(2)]$, $|S| = N^2$
  - $\Rightarrow$ Sigma-algebra is (cartesian product) power set $\mathcal{F} = 2^S$
  - $\Rightarrow$ Define $P(E) = \frac{|E|}{|S|}$ as the uniform probability distribution
- Consider the events $E_1 =$'I wrote $x_1$' and $E_2 =$'My friend wrote $x_2$'
  Q: Are they independent? Yes, since

$$P(E_1 \cap E_2) = P\big(\{(x_1, x_2)\}\big) = \frac{|\{(x_1, x_2)\}|}{|S|} = \frac{1}{N^2} = P(E_1)P(E_2)$$

- Dependent events: $E_1 =$'I wrote $x_1$' and $E_3 =$'Both names are male'

# Independence for more than two events

- **Def:** Events $E_i$, $i = 1, 2, \ldots$ are called mutually independent if

$$P\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} P(E_i)$$

  for every finite subset $I$ of at least two integers

- Ex: Events $E_1$, $E_2$, and $E_3$ are mutually independent if all the following hold

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$$
$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$
$$P(E_1 \cap E_3) = P(E_1)P(E_3)$$
$$P(E_2 \cap E_3) = P(E_2)P(E_3)$$

- If $P(E_i \cap E_j) = P(E_i)P(E_j)$ for all $(i, j)$, the $E_i$ are pairwise independent

  $\Rightarrow$ Mutual independence $\rightarrow$ pairwise independence. Not the other way

# Random variable (RV) definition

- **Def:** RV $X(s)$ is a function that assigns a value to an outcome $s \in S$

  $\Rightarrow$ Think of RVs as measurements associated with an experiment

Example

- Throw a ball inside a $1m \times 1m$ square. Interested in ball position
- Uncertain outcome is the place $s \in [0, 1]^2$ where the ball falls
- Random variables are $X(s)$ and $Y(s)$ position coordinates

- RV probabilities inferred from probabilities of underlying outcomes

$$P(X(s) = x) = P(\{s \in S : X(s) = x\})$$

$$P(X(s) \in (-\infty, x]) = P(\{s \in S : X(s) \in (-\infty, x]\})$$

- $X(s)$ is the random variable and $x$ a particular value of $X(s)$

# Example 1

- Throw coin for head ($H$) or tails ($T$). Coin is fair $P(H) = 1/2$, $P(T) = 1/2$. Pay \$1 for $H$, charge \$1 for $T$. Earnings?
- Possible outcomes are $H$ and $T$
- To measure earnings define RV $X$ with values

$$X(H) = 1, \qquad X(T) = -1$$

- Probabilities of the RV are

$$\begin{aligned} P(X = 1) \quad &= P(H) = 1/2, \\ P(X = -1) &= P(T) = 1/2 \end{aligned}$$

$\Rightarrow$ Also have $P(X = x) = 0$ for all other $x \neq \pm 1$

# Example 2

- Throw 2 coins. Pay \$1 for each $H$, charge \$1 for each $T$. Earnings?
- Now the possible outcomes are $HH$, $HT$, $TH$, and $TT$
- To measure earnings define RV $Y$ with values

$$Y(HH) = 2, \quad Y(HT) = 0, \quad Y(TH) = 0, \quad Y(TT) = -2$$

- Probabilities of the RV are

$$\begin{aligned} P(Y = 2) &= P(HH) &&= 1/4, \\ P(Y = 0) &= P(HT) + P(TH) &&= 1/2, \\ P(Y = -2) &= P(TT) &&= 1/4 \end{aligned}$$

# About Examples 1 and 2

- RVs are easier to manipulate than events

- Let $s_1 \in \{H, T\}$ be outcome of coin 1 and $s_2 \in \{H, T\}$ of coin 2

  $\Rightarrow$ Can relate $Y$ and $X$s as

$$Y(s_1, s_2) = X_1(s_1) + X_2(s_2)$$

- Throw $N$ coins. Earnings? Enumeration becomes cumbersome

- Alternatively, let $s_n \in \{H, T\}$ be outcome of $n$-th toss and define

$$Y(s_1, s_2, \ldots, s_N) = \sum_{n=1}^{N} X_n(s_n)$$

  $\Rightarrow$ Will usually abuse notation and write $Y = \sum_{n=1}^{N} X_n$

# Example 3

- Throw a coin until landing heads for the first time. $P(H) = p$
- Number of throws until the first head?
- Outcomes are $H$, $TH$, $TTH$, $TTTH$, $\ldots$ Note that $|S| = \infty$

  $\Rightarrow$ Stop tossing after first $H$ (thus $THT$ not a possible outcome)
- Let $N$ be a RV counting the number of throws

  $\Rightarrow$ $N = n$ if we land $T$ in the first $n-1$ throws and $H$ in the $n$-th

$$
\begin{aligned}
P(N = 1) &= P(H) && = p \\
P(N = 2) &= P(TH) && = (1-p)p \\
&\vdots \\
P(N = n) &= P(\underbrace{TT\ldots T}_{n-1 \text{ tails}}H) && = (1-p)^{n-1}p
\end{aligned}
$$

# Example 3 (continued)

- From A2) we should have $P(S) = \sum_{n=1}^{\infty} P(N\!=\!n) = 1$
- Holds because $\sum_{n=1}^{\infty}(1-p)^{n-1}$ is a geometric series

$$\sum_{n=1}^{\infty}(1-p)^{n-1} = 1 + (1-p) + (1-p)^2 + \ldots = \frac{1}{1-(1-p)} = \frac{1}{p}$$

- Plug the sum of the geometric series in the expression for $P(S)$

$$\sum_{n=1}^{\infty} P(N=n) = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p \times \frac{1}{p} = 1 \checkmark$$

# Indicator function

- The indicator function of an event is a random variable
- Let $s \in S$ be an outcome, and $E \subset S$ be an event

$$\mathbb{I}\{E\}(s) = \begin{cases} 1, & \text{if } s \in E \\ 0, & \text{if } s \notin E \end{cases}$$

$\Rightarrow$ Indicates that outcome $s$ belongs to set $E$, by taking value 1

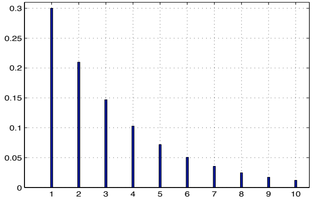
## Example

- Number of throws $N$ until first H. Interested on $N$ exceeding $N_0$
  - $\Rightarrow$ Event is $\{N : N > N_0\}$. Possible outcomes are $N = 1, 2, \ldots$
  - $\Rightarrow$ Denote indicator function as $\mathbb{I}_{N_0} = \mathbb{I}\{N : N > N_0\}$
- Probability $P(\mathbb{I}_{N_0} = 1) = P(N > N_0) = (1-p)^{N_0}$
  - $\Rightarrow$ For $N$ to exceed $N_0$ need $N_0$ consecutive tails
  - $\Rightarrow$ Doesn't matter what happens afterwards

# Probability mass and cumulative distribution functions

- Discrete RV takes on, at most, a countable number of values
- Probability mass function (pmf) $p_X(x) = P(X = x)$
  - If RV is clear from context, just write $p_X(x) = p(x)$

- If $X$ supported in $\{x_1, x_2, \ldots\}$, pmf satisfies
  (i) $p(x_i) > 0$ for $i = 1, 2, \ldots$
  (ii) $p(x) = 0$ for all other $x \neq x_i$
  (iii) $\sum_{i=1}^{\infty} p(x_i) = 1$
  - Pmf for "throw to first heads" $(p = 0.3)$

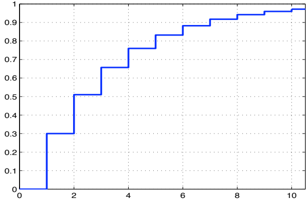



- Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x) = \sum_{i: x_i \leq x} p(x_i)$$

$\Rightarrow$ Staircase function with jumps at $x_i$

  - Cdf for "throw to first heads" $(p = 0.3)$



# Bernoulli

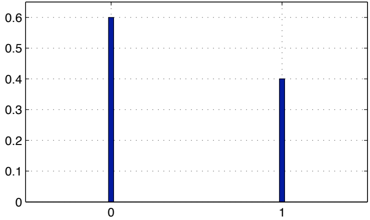
- A trial/experiment/bet can succeed w.p. $p$ or fail w.p. $q := 1 - p$

  $\Rightarrow$ Ex: coin throws, any indication of an event

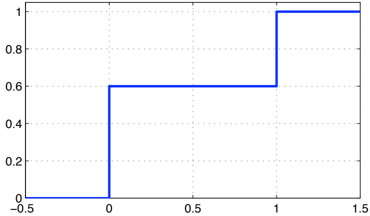
- Bernoulli $X$ can be 0 or 1. Pmf is $p(x) = p^x q^{1-x}$
- Cdf is

$$F(x) = \begin{cases} 0, & x < 0 \\ q, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

pmf ($p = 0.4$)



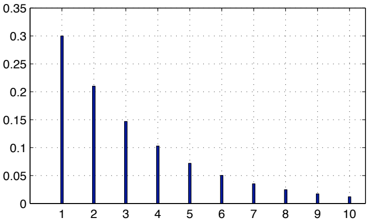

cdf ($p = 0.4$)



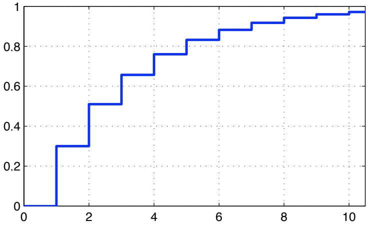
# Geometric

- Count number of Bernoulli trials needed to register first success
  - $\Rightarrow$ Trials succeed w.p. $p$ and are independent
- Number of trials $X$ until success is geometric with parameter $p$
- Pmf is $p(x) = p(1-p)^{x-1}$
  - One success after $x-1$ failures, trials are independent
- Cdf is $F(x) = 1 - (1-p)^x$
  - Recall $\mathrm{P}(X > x) = (1-p)^x$; or just sum the geometric series

pmf ($p = 0.3$)




cdf ($p = 0.3$)



# Binomial

- Count number of successes $X$ in $n$ Bernoulli trials

  $\Rightarrow$ Trials succeed w.p. $p$ and are independent

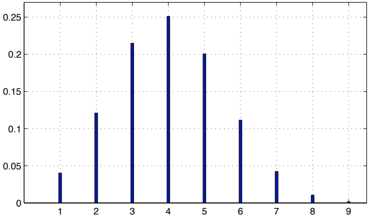
- Number of successes $X$ is binomial with parameters $(n, p)$. Pmf is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

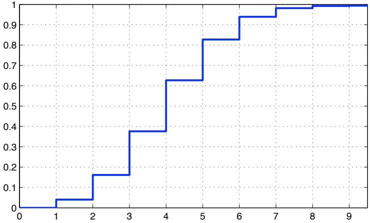
  $\Rightarrow$ $X = x$ for $x$ successes ($p^x$) and $n - x$ failures ($(1-p)^{n-x}$).

  $\Rightarrow$ $\binom{n}{x}$ ways of drawing $x$ successes and $n - x$ failures

pmf ($n = 9,\ p = 0.4$)




cdf ($n = 9,\ p = 0.4$)



# Binomial (continued)

- Let $Y_i$, $i = 1, \ldots n$ be Bernoulli RVs with parameter $p$

  $\Rightarrow$ $Y_i$ associated with independent events

- Can write binomial $X$ with parameters $(n, p)$ as $\Rightarrow X = \sum_{i=1}^{n} Y_i$

Example

- Consider binomials $Y$ and $Z$ with parameters $(n_Y, p)$ and $(n_Z, p)$

  $\Rightarrow$ Q: Probability distribution of $X = Y + Z$?

- Write $Y = \sum_{i=1}^{n_Y} Y_i$ and $Z = \sum_{i=1}^{n_Z} Z_i$, thus

$$X = \sum_{i=1}^{n_Y} Y_i + \sum_{i=1}^{n_Z} Z_i$$

  $\Rightarrow$ $X$ is binomial with parameter $(n_Y + n_Z, p)$

# Poisson

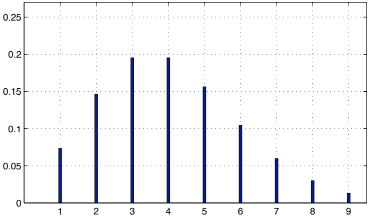
- Counts of rare events (radioactive decay, packet arrivals, accidents)
- Usually modeled as Poisson with parameter $\lambda$ and pmf

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

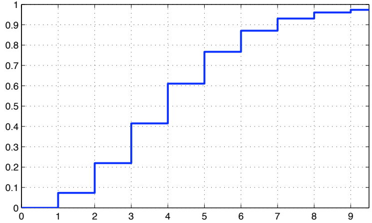
- Q: Is this a properly defined pmf? Yes
- Taylor's expansion of $e^x = 1 + x + x^2/2 + \ldots + x^i/i! + \ldots$. Then

$$P(S) = \sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1 \checkmark$$

pmf ($\lambda = 4$)

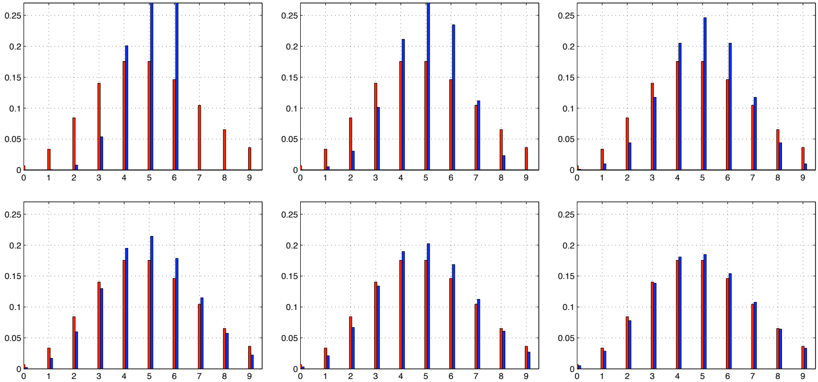



cdf ($\lambda = 4$)



# Poisson approximation of binomial

- $X$ is binomial with parameters $(n, p)$
- Let $n \to \infty$ while maintaining a constant product $np = \lambda$
  - If we just let $n \to \infty$ number of successes diverges. Boring
- Compare with Poisson distribution with parameter $\lambda$
  - $\lambda = 5,\ n = 6, 8, 10, 15, 20, 50$



# Poisson and binomial (continued)

- This is, in fact, the motivation for the definition of a Poisson RV
- Substituting $p = \lambda/n$ in the pmf of a binomial RV

$$p_n(x) = \frac{n!}{(n-x)!x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$
$$= \frac{n(n-1)\dots(n-x+1)}{n^x}\,\frac{\lambda^x}{x!}\,\frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

$\Rightarrow$ Used factorials' defs., $(1-\lambda/n)^{n-x} = \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$, and reordered terms

- In the limit, red term is $\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}$
- Black and blue terms converge to 1. From both observations

$$\lim_{n\to\infty} p_n(x) = 1\,\frac{\lambda^x}{x!}\,\frac{e^{-\lambda}}{1} = e^{-\lambda}\frac{\lambda^x}{x!}$$

$\Rightarrow$ Limit is the pmf of a Poisson RV

# Closing remarks

- Binomial distribution is motivated by counting successes
- The Poisson is an approximation for large number of trials $n$

  $\Rightarrow$ Poisson distribution is more tractable (compare pmfs)
- Sometimes called "law of rare events"
  - Individual events (successes) happen with small probability $p = \lambda/n$
  - Aggregate event (number of successes), though, need not be rare
- Notice that all four RVs seen so far are related to "coin tosses"

# Continuous RVs, probability density function

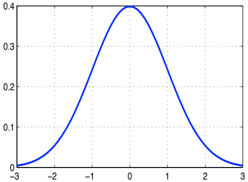
- Possible values for continuous RV $X$ form a dense subset $\mathcal{X} \subseteq \mathbb{R}$

  $\Rightarrow$ Uncountably infinite number of possible values

- Probability density function (pdf) $f_X(x) \geq 0$ is such that for any subset $\mathcal{X} \subseteq \mathbb{R}$ (Normal pdf to the right)

$$P(X \in \mathcal{X}) = \int_{\mathcal{X}} f_X(x) dx$$

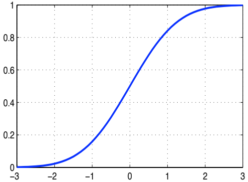
  $\Rightarrow$ Will have $P(X = x) = 0$ for all $x \in \mathcal{X}$



- Cdf defined as before and related to the pdf (Normal cdf to the right)

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$\Rightarrow P(X \leq \infty) = F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1$$



# More on cdfs and pdfs

- When the set $\mathcal{X} = [a, b]$ is an interval of $\mathbb{R}$

$$\mathrm{P}\left(X \in [a, b]\right) = \mathrm{P}\left(X \leq b\right) - \mathrm{P}\left(X \leq a\right) = F_X(b) - F_X(a)$$

- In terms of the pdf it can be written as

$$\mathrm{P}\left(X \in [a, b]\right) = \int_a^b f_X(x)\, dx$$

- For small interval $[x_0, x_0 + \delta x]$, in particular

$$\mathrm{P}\left(X \in [x_0, x_0 + \delta x]\right) = \int_{x_0}^{x_0+\delta x} f_X(x)\, dx \approx f_X(x_0)\delta x$$

$\Rightarrow$ Probability is the "area under the pdf" (thus "density")

- Another relationship between pdf and cdf is $\Rightarrow \dfrac{\partial F_X(x)}{\partial x} = f_X(x)$

$\Rightarrow$ Fundamental theorem of calculus ("derivative inverse of integral")

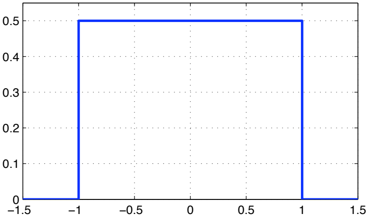
# Uniform

- Model problems with equal probability of landing on an interval $[a, b]$
- Pdf of uniform RV is $f(x) = 0$ outside the interval $[a, b]$ and

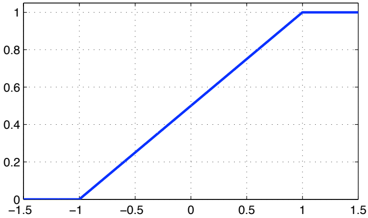
$$f(x) = \frac{1}{b-a}, \quad \text{for } a \leq x \leq b$$

- Cdf is $F(x) = (x-a)/(b-a)$ in the interval $[a, b]$ (0 before, 1 after)
- Prob. of interval $[\alpha, \beta] \subseteq [a, b]$ is $\int_\alpha^\beta f(x)dx = (\beta - \alpha)/(b-a)$
  $\Rightarrow$ Depends on interval's width $\beta - \alpha$ only, not on its position

pdf $(a = -1,\ b = 1)$




cdf $(a = -1,\ b = 1)$



# Exponential

- Model duration of phone calls, lifetime of electronic components
- Pdf of exponential RV is

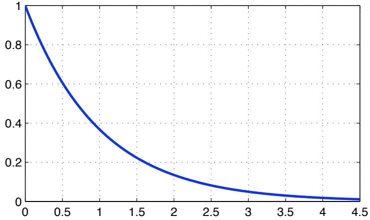
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\Rightarrow$ As parameter $\lambda$ increases, "height" increases and "width" decreases

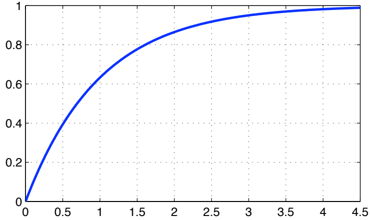
- Cdf obtained by integrating pdf

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \int_{0}^{x} \lambda e^{-\lambda u}\, du = -e^{-\lambda u}\Big|_{0}^{x} = 1 - e^{-\lambda x}$$

pdf ($\lambda = 1$)




cdf ($\lambda = 1$)



# Normal / Gaussian

- Model randomness arising from large number of random effects
- Pdf of normal RV is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

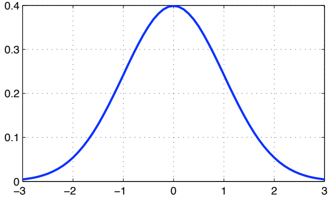
$\Rightarrow$ $\mu$ is the mean (center), $\sigma^2$ is the variance (width)

$\Rightarrow$ 0.68 prob. between $\mu \pm \sigma$, 0.997 prob. in $\mu \pm 3\sigma$

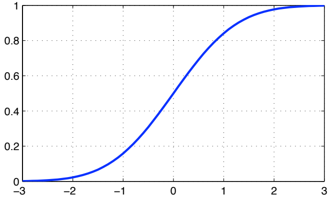
$\Rightarrow$ Standard normal RV has $\mu = 0$ and $\sigma^2 = 1$

- Cdf $F(x)$ cannot be expressed in terms of elementary functions

pdf ($\mu = 0$, $\sigma^2 = 1$)




cdf ($\mu = 0$, $\sigma^2 = 1$)



# Expected values

- We are asked to summarize information about a RV in a single value

  $\Rightarrow$ What should this value be?

- If we are allowed a description with a few values

  $\Rightarrow$ What should they be?

- Expected (mean) values are convenient answers to these questions

- **Beware:** Expectations are condensed descriptions

  $\Rightarrow$ They overlook some aspects of the random phenomenon

  $\Rightarrow$ Whole story told by the probability distribution (cdf)

# Definition for discrete RVs

- Discrete RV $X$ taking on values $x_i$, $i = 1, 2, \ldots$ with pmf $p(x)$
- **Def:** The expected value of the discrete RV $X$ is

$$\mathbb{E}[X] := \sum_{i=1}^{\infty} x_i p(x_i) = \sum_{x:p(x)>0} x p(x)$$

- Weighted average of possible values $x_i$. Probabilities are weights
- Common average if RV takes values $x_i$, $i = 1, \ldots, N$ equiprobably

$$\mathbb{E}[X] = \sum_{i=1}^{N} x_i p(x_i) = \sum_{i=1}^{N} x_i \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

# Expected value of Bernoulli and geometric RVs

Ex: For a Bernoulli RV $p(x) = p^x q^{1-x}$, for $x \in \{0, 1\}$

$$\mathbb{E}[X] = 1 \times p + 0 \times q = p$$

Ex: For a geometric RV $p(x) = p(1-p)^{x-1} = pq^{x-1}$, for $x \geq 1$

- Note that $\partial q^x / \partial q = x q^{x-1}$ and that derivatives are linear operators

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} x p q^{x-1} = p \sum_{x=1}^{\infty} \frac{\partial q^x}{\partial q} = p \frac{\partial}{\partial q} \left( \sum_{x=1}^{\infty} q^x \right)$$

- Sum inside derivative is geometric. Sums to $q/(1-q)$, thus

$$\mathbb{E}[X] = p \frac{\partial}{\partial q} \left( \frac{q}{1-q} \right) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

- Time to first success is inverse of success probability. Reasonable

# Expected value of Poisson RV

Ex: For a Poisson RV $p(x) = e^{-\lambda}(\lambda^x/x!)$, for $x \geq 0$

- First summand in definition is 0, pull $\lambda$ out, and use $\frac{x}{x!} = \frac{1}{(x-1)!}$

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

- Sum is Taylor's expansion of $e^{\lambda} = 1 + \lambda + \lambda^2/2! + \ldots + \lambda^x/x!$

$$\mathbb{E}[X] = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

- Poisson is limit of binomial for large number of trials $n$, with $\lambda = np$

  $\Rightarrow$ Counts number of successes in $n$ trials that succeed w.p. $p$

- Expected number of successes is $\lambda = np$

  $\Rightarrow$ Number of trials $\times$ probability of individual success. Reasonable

# Definition for continuous RVs

- Continuous RV $X$ taking values on $\mathbb{R}$ with pdf $f(x)$
- **Def:** The expected value of the continuous RV $X$ is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f(x)\, dx$$

- Compare with $\mathbb{E}[X] := \sum_{x:p(x)>0} x p(x)$ in the discrete RV case
- Note that the integral or sum are assumed to be well defined

  $\Rightarrow$ Otherwise we say the expectation does not exist

# Expected value of normal RV

Ex: For a normal RV add and subtract $\mu$, separate integrals

$$\begin{aligned}
\mathbb{E}[X] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x+\mu-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \mu\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx
\end{aligned}$$

- First integral is 1 because it integrates a pdf in all $\mathbb{R}$
- Second integral is 0 by symmetry. Both observations yield

$$\mathbb{E}[X] = \mu$$

- The mean of a RV with a symmetric pdf is the point of symmetry

# Expected value of uniform and exponential RVs

Ex: For a uniform RV $f(x) = 1/(b-a)$, for $a \leq x \leq b$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \frac{b^2-a^2}{2(b-a)} = \frac{(a+b)}{2}$$

- Makes sense, since pdf is symmetric around midpoint $(a+b)/2$

Ex: For an exponential RV (non symmetric) integrate by parts

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^{\infty} x\lambda e^{-\lambda x}\,dx \\
&= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx \\
&= -xe^{-\lambda x}\Big|_0^{\infty} - \frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda}
\end{aligned}$$

# Expected value of a function of a RV

- Consider a function $g(X)$ of a RV $X$. Expected value of $g(X)$?
- g(X) is also a RV, then it also has a pmf $p_{g(X)}(g(x))$

$$\mathbb{E}[g(X)] = \sum_{g(x):p_{g(X)}(g(x))>0} g(x)p_{g(X)}(g(x))$$

$\Rightarrow$ Requires calculating the pmf of $g(X)$. There is a simpler way

**Theorem**

*Consider a function $g(X)$ of a discrete RV $X$ with pmf $p_X(x)$. Then*

$$\mathbb{E}[g(X)] = \sum_{i=1}^{\infty} g(x_i)p_X(x_i)$$

- Weighted average of functional values. No need to find pmf of $g(X)$
- Same can be proved for a continuous RV

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx$$

# Expected value of a linear transformation

- Consider a linear function (actually affine) $g(X) = aX + b$

$$\begin{aligned}
\mathbb{E}\left[aX + b\right] &= \sum_{i=1}^{\infty} (ax_i + b) p_X(x_i) \\
&= \sum_{i=1}^{\infty} ax_i p_X(x_i) + \sum_{i=1}^{\infty} b p_X(x_i) \\
&= a \sum_{i=1}^{\infty} x_i p_X(x_i) + b \sum_{i=1}^{\infty} p_X(x_i) \\
&= a\mathbb{E}\left[X\right] + b1
\end{aligned}$$

- Can interchange expectation with additive/multiplicative constants

$$\mathbb{E}\left[aX + b\right] = a\mathbb{E}\left[X\right] + b$$

$\Rightarrow$ Again, the same holds for a continuous RV

# Expected value of an indicator function

- Let $X$ be a RV and $\mathcal{X}$ be a set

$$\mathbb{I}\left\{X \in \mathcal{X}\right\} = \begin{cases} 1, & \text{if } x \in \mathcal{X} \\ 0, & \text{if } x \notin \mathcal{X} \end{cases}$$

- Expected value of $\mathbb{I}\left\{X \in \mathcal{X}\right\}$ in the discrete case

$$\mathbb{E}\left[\mathbb{I}\left\{X \in \mathcal{X}\right\}\right] = \sum_{x:p_X(x)>0} \mathbb{I}\left\{x \in \mathcal{X}\right\} p_X(x) = \sum_{x \in \mathcal{X}} p_X(x) = \mathrm{P}\left(X \in \mathcal{X}\right)$$

- Likewise in the continuous case

$$\mathbb{E}\left[\mathbb{I}\left\{X \in \mathcal{X}\right\}\right] = \int_{-\infty}^{\infty} \mathbb{I}\left\{x \in \mathcal{X}\right\} f_X(x) dx = \int_{x \in \mathcal{X}} f_X(x) dx = \mathrm{P}\left(X \in \mathcal{X}\right)$$

- Expected value of indicator RV = Probability of indicated event
  - $\Rightarrow$ Recall $\mathbb{E}\left[X\right] = p$ for Bernoulli RV (it "indicates success")

# Moments, central moments and variance

- **Def:** The $n$-th moment ($n \geq 0$) of a RV is

$$\mathbb{E}\left[X^n\right] = \sum_{i=1}^{\infty} x_i^n p(x_i)$$

- **Def:** The $n$-th central moment corrects for the mean, that is

$$\mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^n\right] = \sum_{i=1}^{\infty} \left(x_i - \mathbb{E}\left[X\right]\right)^n p(x_i)$$

- 0-th order moment is $\mathbb{E}\left[X^0\right] = 1$; 1-st moment is the mean $\mathbb{E}\left[X\right]$
- 2-nd central moment is the variance. Measures width of the pmf

$$\text{var}\left[X\right] = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X]$$

Ex: For affine functions

$$\text{var}\left[aX + b\right] = a^2 \text{var}\left[X\right]$$

# Variance of Bernoulli and Poisson RVs

Ex: For a Bernoulli RV $X$ with parameter $p$, $\mathbb{E}[X] = \mathbb{E}\left[X^2\right] = p$

$$\Rightarrow \operatorname{var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] = p - p^2 = p(1-p)$$

Ex: For Poisson RV $Y$ with parameter $\lambda$, second moment is

$$\begin{aligned}
\mathbb{E}\left[Y^2\right] &= \sum_{y=0}^{\infty} y^2 e^{-\lambda} \frac{\lambda^y}{y!} = \sum_{y=1}^{\infty} y \frac{e^{-\lambda}\lambda^y}{(y-1)!} \\
&= \sum_{y=1}^{\infty} (y-1) \frac{e^{-\lambda}\lambda^y}{(y-1)!} + \sum_{y=1}^{\infty} \frac{e^{-\lambda}\lambda^y}{(y-1)!} \\
&= e^{-\lambda}\lambda^2 \sum_{y=2}^{\infty} \frac{\lambda^{y-2}}{(y-2)!} + e^{-\lambda}\lambda \sum_{y=1}^{\infty} \frac{\lambda^{y-1}}{(y-1)!} \\
&= e^{-\lambda}\lambda^2 e^{\lambda} + e^{-\lambda}\lambda e^{\lambda} = \lambda^2 + \lambda
\end{aligned}$$

$$\Rightarrow \operatorname{var}[Y] = \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

# Joint cdf

- Want to study problems with more than one RV. Say, e.g., $X$ and $Y$
- Probability distributions of $X$ and $Y$ are not sufficient

  $\Rightarrow$ Joint probability distribution (cdf) of $(X, Y)$ defined as

$$F_{XY}(x, y) = \mathrm{P}(X \leq x, Y \leq y)$$

- If $X, Y$ clear from context omit subindex to write $F_{XY}(x, y) = F(x, y)$
- Can recover $F_X(x)$ by considering all possible values of $Y$

$$F_X(x) = \mathrm{P}(X \leq x) = \mathrm{P}(X \leq x, Y \leq \infty) = F_{XY}(x, \infty)$$

  $\Rightarrow$ $F_X(x)$ and $F_Y(y) = F_{XY}(\infty, y)$ are called marginal cdfs

# Joint pmf

- Consider discrete RVs $X$ and $Y$
  $X$ takes values in $\mathcal{X} := \{x_1, x_2, \ldots\}$ and $Y$ in $\mathcal{Y} := \{y_1, y_2, \ldots\}$
- Joint pmf of $(X, Y)$ defined as

$$p_{XY}(x, y) = \mathrm{P}\left(X = x, Y = y\right)$$

- Possible values $(x, y)$ are elements of the Cartesian product $\mathcal{X} \times \mathcal{Y}$
  - $(x_1, y_1)$, $(x_1, y_2)$, …, $(x_2, y_1)$, $(x_2, y_2)$, …, $(x_3, y_1)$, $(x_3, y_2)$, …
- Marginal pmf $p_X(x)$ obtained by summing over all values of $Y$

$$p_X(x) = \mathrm{P}\left(X = x\right) = \sum_{y \in \mathcal{Y}} \mathrm{P}\left(X = x, Y = y\right) = \sum_{y \in \mathcal{Y}} p_{XY}(x, y)$$

$\Rightarrow$ Likewise $p_Y(y) = \sum_{x \in \mathcal{X}} p_{XY}(x, y)$. Marginalize by summing

# Joint pdf

- Consider continuous RVs $X$, $Y$. Arbitrary set $\mathcal{A} \in \mathbb{R}^2$
- Joint pdf is a function $f_{XY}(x, y) : \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$\mathrm{P}\left((X, Y) \in \mathcal{A}\right) = \iint_{\mathcal{A}} f_{XY}(x, y)\, dxdy$$

- Marginalization. There are two ways of writing $\mathrm{P}\left(X \in \mathcal{X}\right)$

$$\mathrm{P}\left(X \in \mathcal{X}\right) = \mathrm{P}\left(X \in \mathcal{X}, Y \in \mathbb{R}\right) = \int_{X \in \mathcal{X}} \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy\, dx$$

$\Rightarrow$ Definition of $f_X(x)$ $\Rightarrow \mathrm{P}\left(X \in \mathcal{X}\right) = \int_{X \in \mathcal{X}} f_X(x)\, dx$

# Joint pdf

- Consider continuous RVs $X$, $Y$. Arbitrary set $\mathcal{A} \in \mathbb{R}^2$
- Joint pdf is a function $f_{XY}(x, y) : \mathbb{R}^2 \to \mathbb{R}^+$ such that

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- Marginalization. There are two ways of writing $\mathrm{P}\left(X \in \mathcal{X}\right)$

$$\mathrm{P}\left(X \in \mathcal{X}\right) = \mathrm{P}\left(X \in \mathcal{X}, Y \in \mathbb{R}\right) = \int_{X \in \mathcal{X}} \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy\, dx$$

$\Rightarrow$ Definition of $f_X(x)$ $\Rightarrow \mathrm{P}\left(X \in \mathcal{X}\right) = \int_{X \in \mathcal{X}} f_X(x)\, dx$

- Lipstick on a pig (same thing written differently is still same thing)

$$\Rightarrow f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dx$$

# Example

- Consider two Bernoulli RVs $B_1, B_2$, with the same parameter $p$
  $\Rightarrow$ Define $X = B_1$ and $Y = B_1 + B_2$
- The pmf of $X$ is

$$p_X(0) = 1 - p, \quad p_X(1) = p$$

- Likewise, the pmf of $Y$ is

$$p_Y(0) = (1-p)^2, \quad p_Y(1) = 2p(1-p), \quad p_Y(2) = p^2$$

- The joint pmf of $X$ and $Y$ is

$$\begin{aligned} &p_{XY}(0,0) = (1-p)^2, \quad &&p_{XY}(0,1) = p(1-p), \quad &&p_{XY}(0,2) = 0 \\ &p_{XY}(1,0) = 0, &&p_{XY}(1,1) = p(1-p), &&p_{XY}(1,2) = p^2 \end{aligned}$$

# Random vectors

- For convenience often arrange RVs in a vector
  $\Rightarrow$ Prob. distribution of vector is joint distribution of its entries
- Consider, e.g., two RVs $X$ and $Y$. Random vector is $\mathbf{X} = [X, Y]^T$
- If $X$ and $Y$ are discrete, vector variable $\mathbf{X}$ is discrete with pmf

$$p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{X}}\left([x, y]^T\right) = p_{XY}\left(x, y\right)$$

- If $X$, $Y$ continuous, $\mathbf{X}$ continuous with pdf

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}\left([x, y]^T\right) = f_{XY}\left(x, y\right)$$

- Vector cdf is $\Rightarrow F_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}}\left([x, y]^T\right) = F_{XY}\left(x, y\right)$
- In general, can define $n$-dimensional RVs $\mathbf{X} := [X_1, X_2, \ldots, X_n]^T$
  $\Rightarrow$ Just notation, definitions carry over from the $n = 2$ case

# Joint expectations

- RVs $X$ and $Y$ and function $g(X, Y)$. Function $g(X, Y)$ also a RV
- Expected value of $g(X, Y)$ when $X$ and $Y$ discrete can be written as

$$\mathbb{E}\left[g(X,Y)\right] = \sum_{x,y:p_{XY}(x,y)>0} g(x,y)p_{XY}(x,y)$$

- When $X$ and $Y$ are continuous

$$\mathbb{E}\left[g(X,Y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)\,dxdy$$

$\Rightarrow$ Can have more than two RVs and use vector notation

Ex: Linear transformation of a vector RV $\mathbf{X} \in \mathbb{R}^n$: $g(\mathbf{X}) = \mathbf{a}^T\mathbf{X}$

$$\Rightarrow \mathbb{E}\left[\mathbf{a}^T\mathbf{X}\right] = \int_{\mathbb{R}^n} \mathbf{a}^T\mathbf{x}f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}$$

# Expected value of a sum of random variables

- Expected value of the sum of two continuous RVs

$$\mathbb{E}[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f_{XY}(x,y)\,dxdy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\,dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\,dxdy$$

- Remove $x$ $(y)$ from innermost integral in first (second) summand

$$\mathbb{E}[X+Y] = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx + \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy$$
$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

$\Rightarrow$ Used marginal expressions

- Expectation $\leftrightarrow$ summation $\Rightarrow \mathbb{E}\left[\sum_i X_i\right] = \sum_i \mathbb{E}[X_i]$

# Expected value is a linear operator

- Combining with earlier result $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ proves that

$$\mathbb{E}[a_x X + a_y Y + b] = a_x \mathbb{E}[X] + a_y \mathbb{E}[Y] + b$$

- Better yet, using vector notation (with $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{R}^n$, $b$ a scalar)

$$\mathbb{E}\left[\mathbf{a}^T \mathbf{X} + b\right] = \mathbf{a}^T \mathbb{E}[\mathbf{X}] + b$$

- Also, if $\mathbf{A}$ is an $m \times n$ matrix with rows $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$ and $\mathbf{b} \in \mathbb{R}^m$ a vector with elements $b_1, \ldots, b_m$, we can write

$$\mathbb{E}[\mathbf{A}\mathbf{X} + \mathbf{b}] = \begin{pmatrix} \mathbb{E}\left[\mathbf{a}_1^T \mathbf{X} + b_1\right] \\ \mathbb{E}\left[\mathbf{a}_2^T \mathbf{X} + b_2\right] \\ \vdots \\ \mathbb{E}\left[\mathbf{a}_m^T \mathbf{X} + b_m\right] \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^T \mathbb{E}[\mathbf{X}] + b_1 \\ \mathbf{a}_2^T \mathbb{E}[\mathbf{X}] + b_2 \\ \vdots \\ \mathbf{a}_m^T \mathbb{E}[\mathbf{X}] + b_m \end{pmatrix} = \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b}$$

- Expected value operator can be interchanged with linear operations

# Independence of RVs

- Events $E$ and $F$ are independent if $\mathrm{P}(E \cap F) = \mathrm{P}(E)\,\mathrm{P}(F)$
- **Def:** RVs $X$ and $Y$ are independent if events $X \le x$ and $Y \le y$ are independent for all $x$ and $y$, i.e.

$$\mathrm{P}(X \le x, Y \le y) = \mathrm{P}(X \le x)\,\mathrm{P}(Y \le y)$$

$\Rightarrow$ By definition, equivalent to $F_{XY}(x, y) = F_X(x)F_Y(y)$

- For discrete RVs equivalent to analogous relation between pmfs

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

- For continuous RVs the analogous is true for pdfs

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

- Independence $\Leftrightarrow$ Joint distribution factorizes into product of marginals

# Sum of independent Poisson RVs

- Independent Poisson RVs $X$ and $Y$ with parameters $\lambda_x$ and $\lambda_y$
- Q: Probability distribution of the sum RV $Z := X + Y$?

- $Z = n$ only if $X = k$, $Y = n - k$ for some $0 \leq k \leq n$
(use independence, Poisson pmf, rearrange terms, binomial theorem)

$$
\begin{aligned}
p_Z(n) &= \sum_{k=0}^{n} \mathrm{P}\left(X = k, Y = n-k\right) &&= \sum_{k=0}^{n} \mathrm{P}\left(X = k\right) \mathrm{P}\left(Y = n-k\right) \\
&= \sum_{k=0}^{n} e^{-\lambda_x} \frac{\lambda_x^k}{k!} e^{-\lambda_y} \frac{\lambda_y^{n-k}}{(n-k)!} &&= \frac{e^{-(\lambda_x+\lambda_y)}}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \lambda_x^k \lambda_y^{n-k} \\
&= \frac{e^{-(\lambda_x+\lambda_y)}}{n!} (\lambda_x + \lambda_y)^n
\end{aligned}
$$

- $Z$ is Poisson with parameter $\lambda_z := \lambda_x + \lambda_y$
  $\Rightarrow$ Sum of independent Poissons is Poisson (parameters added)

# Expected value of a product of independent RVs

**Theorem**

*For independent RVs $X$ and $Y$, and arbitrary functions $g(X)$ and $h(Y)$:*

$$\mathbb{E}\left[g(X)h(Y)\right] = \mathbb{E}\left[g(X)\right]\mathbb{E}\left[h(Y)\right]$$

*The expected value of the product is the product of the expected values*

- Can show that $g(X)$ and $h(Y)$ are also independent. Intuitive

Ex: Special case when $g(X) = X$ and $h(Y) = Y$ yields

$$\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]$$

- Expectation and product can be interchanged if RVs are independent
- Different from interchange with linear operations (always possible)

# Variance of a sum of independent RVs

- Let $X_n$, $n = 1, \ldots N$ be independent with $\mathbb{E}[X_n] = \mu_n$, $\text{var}[X_n] = \sigma_n^2$
- Q: Variance of sum $X := \sum_{n=1}^{N} X_n$?

- Notice that mean of $X$ is $\mathbb{E}[X] = \sum_{n=1}^{N} \mu_n$. Then

$$\text{var}[X] = \mathbb{E}\left[\left(\sum_{n=1}^{N} X_n - \sum_{n=1}^{N} \mu_n\right)^2\right] = \mathbb{E}\left[\left(\sum_{n=1}^{N} (X_n - \mu_n)\right)^2\right]$$

- Expand square and interchange summation and expectation

$$\text{var}[X] = \sum_{n=1}^{N} \sum_{m=1}^{N} \mathbb{E}\Big[(X_n - \mu_n)(X_m - \mu_m)\Big]$$

# Variance of a sum of independent RVs (continued)

- Separate terms in sum. Then use independence and $\mathbb{E}(X_n - \mu_n) = 0$

$$\text{var}[X] = \sum_{n=1, n\neq m}^{N} \sum_{m=1}^{N} \mathbb{E}\Big[(X_n - \mu_n)(X_m - \mu_m)\Big] + \sum_{n=1}^{N} \mathbb{E}\Big[(X_n - \mu_n)^2\Big]$$

$$= \sum_{n=1, n\neq m}^{N} \sum_{m=1}^{N} \mathbb{E}(X_n - \mu_n)\mathbb{E}(X_m - \mu_m) + \sum_{n=1}^{N} \sigma_n^2 = \sum_{n=1}^{N} \sigma_n^2$$

- If RVs are independent $\Rightarrow$ Variance of sum is sum of variances
- Slightly more general result holds for independent $X_i$, $i = 1, \ldots, n$

$$\text{var}\left[\sum_i (a_i X_i + b_i)\right] = \sum_i a_i^2 \text{var}[X_i]$$

# Covariance

- **Def:** The covariance of $X$ and $Y$ is (generalizes variance to pairs of RVs)

$$\text{cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

- If $\text{cov}(X, Y) = 0$ variables $X$ and $Y$ are said to be uncorrelated

- If $X, Y$ independent then $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ and $\text{cov}(X, Y) = 0$

  $\Rightarrow$ Independence implies uncorrelated RVs

- Opposite is not true, may have $\text{cov}(X, Y) = 0$ for dependent $X, Y$
  - Ex: $X$ uniform in $[-a, a]$ and $Y = X^2$

  $\Rightarrow$ But uncorrelatedness implies independence if $X, Y$ are normal

- If $\text{cov}(X, Y) > 0$ then $X$ and $Y$ tend to move in the same direction

  $\Rightarrow$ Positive correlation

- If $\text{cov}(X, Y) < 0$ then $X$ and $Y$ tend to move in opposite directions

  $\Rightarrow$ Negative correlation

# Covariance example

- Let $X$ be a zero-mean random signal and $Z$ zero-mean noise
  $\Rightarrow$ Signal $X$ and noise $Z$ are independent
- Consider received signals $Y_1 = X + Z$ and $Y_2 = -X + Z$

(I) $Y_1$ and $X$ are positively correlated ($X$, $Y_1$ move in same direction)

$$\begin{aligned} \operatorname{cov}(X, Y_1) &= \mathbb{E}[XY_1] - \mathbb{E}[X]\,\mathbb{E}[Y_1] \\ &= \mathbb{E}[X(X+Z)] - \mathbb{E}[X]\,\mathbb{E}[X+Z] \end{aligned}$$

- Second term is 0 ($\mathbb{E}[X] = 0$). For first term independence of $X$, $Z$

$$\mathbb{E}[X(X+Z)] = \mathbb{E}\left[X^2\right] + \mathbb{E}[X]\,\mathbb{E}[Z] = \mathbb{E}\left[X^2\right]$$

- Combining observations $\Rightarrow \operatorname{cov}(X, Y_1) = \mathbb{E}\left[X^2\right] > 0$

# Covariance example (continued)

(II) $Y_2$ and $X$ are negatively correlated ($X$, $Y_2$ move opposite direction)

- Same computations $\Rightarrow \text{cov}(X, Y_2) = -\mathbb{E}\left[X^2\right] < 0$

(III) Can also compute correlation between $Y_1$ and $Y_2$

$$\begin{aligned}\text{cov}(Y_1, Y_2) &= \mathbb{E}\left[(X+Z)(-X+Z)\right] - \mathbb{E}\left[(X+Z)\right]\mathbb{E}\left[(-X+Z)\right] \\ &= -\mathbb{E}\left[X^2\right] + \mathbb{E}\left[Z^2\right]\end{aligned}$$

$\Rightarrow$ Negative correlation if $\mathbb{E}\left[X^2\right] > \mathbb{E}\left[Z^2\right]$ (small noise)

$\Rightarrow$ Positive correlation if $\mathbb{E}\left[X^2\right] < \mathbb{E}\left[Z^2\right]$ (large noise)

- Correlation between $X$ and $Y_1$ or $X$ and $Y_2$ comes from causality
- Correlation between $Y_1$ and $Y_2$ does not. Latent variables $X$ and $Z$

$\Rightarrow$ Correlation does not imply causation

Plausible, indeed commonly used, model of a communication channel