Lecture 1. Numerical stability, sources of errors

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If we approximate x with $\tilde{x}$ there are two types of errors:

- Absolute error

$$e = x - \tilde{x}, \quad \tilde{x} = x + e$$

- Relative error

$$\epsilon = \frac{x - \tilde{x}}{x}, \quad \tilde{x} = x \cdot (1 + \epsilon)$$

Most of the times only absolute value of the error is relevant.

The same classification applies to performance metrics comparison:

	224×224		320×320 / 299×299	
	top-1 err	top-5 err	top-1 err	top-5 err
ResNet-101 [14]	22.0	6.0	-	-
ResNet-200 [15]	21.7	5.8	20.1	4.8
Inception-v3 [39]	-	-	21.2	5.6
Inception-v4 [37]	-	-	20.0	5.0
Inception-ResNet-v2 [37]	-	-	19.9	4.9
ResNeXt-101 (**64** × **4d**)	20.4	5.3	**19.1**	**4.4**

Table 5. State-of-the-art models on the ImageNet-1K validation set (single-crop testing). The test size of ResNet/ResNeXt is 224×224 and 320×320 as in [15] and of the Inception models is 299×299.

Model		
Cimpoi '15 [4]		66.7
Zhang '14 [30]		74.9
Branson '14 [2]		75.7
Lin '15 [20]		80.9
Simon '15 [24]		81.0
CNN (ours)	224px	82.3
2×ST-CNN	224px	83.1
2×ST-CNN	448px	83.9
4×ST-CNN	448px	**84.1**

Some common sources of errors in computational processes:

1. *Rounding errors* from inexact computer arithmetics.
2. *Truncation/discretization error* from approximate formulas.
3. *Termination of iterations*
4. *Randomness induced errors* (e.g. in Monte-Carlo methods)

Consider approximating function's derivative (finite difference method).

$$f_{diff}(x; h) = \frac{f(x+h) - f(x)}{h}$$

From Taylor expansion

$$f(x+h) = f(x) + hf'(x) + f''(\theta)\frac{h^2}{2}, \quad \text{where } \theta \in [x, x+h]$$

we can see that

$$f_{diff}(x; h) = \frac{f(x+h) - f(x)}{h} = f'(x) + f''(\theta)\frac{h}{2}$$

If second derivative is bounded on $[x, x+h]$ then

$$|f'(x) - f_{diff}(x; h)| \leq Mh \ \text{ for some } M$$

Rounding error vs. truncation error

On the other hand there is a computer arithmetics we are dealing with. Let $\tilde{f}_{diff}$ denote an approximation of f_{diff} due to a finite precision.
Numerator of $\tilde{f}_{diff}$ introduces rounding error $\leq \epsilon |f(x)|$

$$|f_{diff}(x;h) - \tilde{f}_{diff}(x;h)| \leq \left| \frac{f(x+h) - f(x)}{h} - \frac{f(x+h) - f(x) + \epsilon f(x)}{h} \right|$$

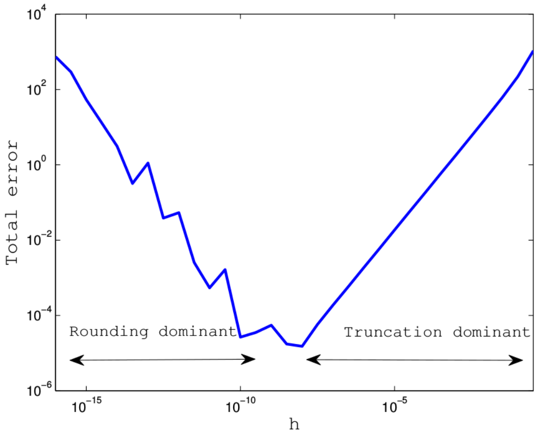
$$\leq \epsilon \frac{|f(x)|}{h}$$

Therefore total error is

$$|f'(x) - \tilde{f}_{diff}(x;h)| \leq Mh + \frac{\epsilon |f(x)|}{h}$$

Rounding error vs. truncation error (example)

Let's assume $f(x) = e^{5x}$, and we want to estimate derivative at $x = 1$.



$$\underbrace{\pm}_{\text{1 sign bit}} \quad \underbrace{d_1, d_2, \ldots, d_p}_{\text{p mantissa bits}} \quad \underbrace{E}_{\text{exponent bits}}$$

Exponent resides in the interval $L \leq E \leq U$.

	total bits	p	L	U
IEEE single precision	32	23	-126	127
IEEE double precision	64	52	-1022	1023

Rounding error ϵ (or ϵ_{mach}) is introduced when mantissa requires more than p bits.
In IEEE double precision, $\epsilon = 2^{-52} \approx 2.22 \times 10^{-16}$

Well-posed problem (according to Hadamard definition):

1. a solution exists
2. the solution is unique
3. solution's behavior changes continuously with respect to changes in initial conditions

Otherwise a problem is **ill-posed**. It's solution might be sensitive to noise in input data.

Well-posed problems	Ill-posed problems
Multiplication by a small number $$y = ax, \quad a \ll 1$$	Division by a small number $$y = x/a, a \ll 1$$
Multiplication by a matrix $$y = Ax$$	Inversion of the matrix when matrix is nearly singular $$x = A^{-1}y$$
Integration $y(t) = y(0) + \int_0^t x(t)dt$	Differentiation $x(t) = y'(t)$

Many numerical operations can be represented as

$$y = f(x)$$

Small change in x leads to some change in y:

$$y + \Delta y = f(x + \Delta x)$$

Condition number quantifies the relative change

$$k = \frac{|\Delta y / y|}{|\Delta x / x|}$$

Condition number: function evaluation

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function.

$$\frac{\Delta y}{y} = \frac{f(x + \Delta x) - f(x)}{f(x)} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \frac{\Delta x}{f(x)}$$

When Δx is small, we have

$$\frac{\Delta y}{y} \approx \frac{f'(x) \Delta x}{f(x)}$$

Therefore,

$$k \approx \left| \frac{x f'(x)}{f(x)} \right|$$

Let's consider system of equations

$$Ax = b$$

From linearity of matrix-vector multiplication we have

$$A(x + \Delta x) = b + \Delta b \quad \Longrightarrow \quad A\Delta x = \Delta b$$

Recalling the matrix norm definition, we have

$$k = \frac{||\Delta b||/||b||}{||\Delta x||/||x||} = \frac{||A\Delta x||}{||\Delta x||}\frac{||x||}{||Ax||} \leq ||A|| \cdot ||A^{-1}||$$

Problem: Given an array X of numbers find variance s^2
Solution: Relies on formula

$$s^2 = \frac{\sum_{i=0}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2}{N-1}$$

$n \leftarrow$ size(X)
$Sum \leftarrow 0$, $SumSq \leftarrow 0$
for all $x \in X$ **do**
    $Sum \leftarrow Sum + x$
    $SumSq \leftarrow SumSq + x^2$
end for
$Var \leftarrow (SumSq - Sum \times Sum/n)/(n-1)$

What is the problem with this approach?

Algorithm example: sample variance

In [1]:
```
import numpy as np

def variance_vanilla(data):
    Ex = 0
    Ex2 = 0
    n = data.shape[0]
    for i in range(n):
        Ex += data[i]
        Ex2 += data[i]*data[i]
    var = (Ex2 - Ex*Ex/n)/(n-1)
    return var
```

In [2]:
```
sigma = 10.0
N = 100000
```

In [4]:
```
mu1 = 1e15
data1 = np.random.normal(loc = mu1, scale=sigma, size=N)
print('Data1\nmax = %0.3f\nmin = %0.3f\ntrue variance = %0.3f'%(data1.max(), data1.min(), np.var(data1)))
var = variance_vanilla(data1)
print('Vanilla Variance = %f\n'%(var))

mu2 = 100.0
data2 = np.random.normal(loc = mu2, scale=sigma, size=N)
print('Data2\nmax = %0.3f\nmin = %0.3f\ntrue variance = %0.3f'%(data2.max(), data2.min(), np.var(data2)))
var = variance_vanilla(data2)
print('Vanilla Variance = %f\n'%(var))
```

```
Data1
max = 1000000000000041.500
min = 999999999999953.750
true variance = 99.610
Vanilla Variance = 698031776066930048.000000

Data2
max = 143.500
min = 56.895
true variance = 100.596
Vanilla Variance = 100.596730
```

Problem with that solution: catastrophic cancellation on large numbers with small variance

Solution: Variance is invariant to shift:

$$Var(X - K) = Var(X)$$

$n \leftarrow \text{size}(X)$
$K \leftarrow X[0]$
$Sum \leftarrow 0$, $SumSq \leftarrow 0$
for all $x \in X$ **do**
  $Sum \leftarrow Sum + (x - K)$
  $SumSq \leftarrow SumSq + (x - K)^2$
end for
$Var \leftarrow (SumSq - Sum \times Sum/n)/(n - 1)$

Does that solve all the problems?

Algorithm example: sample variance

```
In [5]: def variance_shift(data):
            K = data[0]
            Ex = 0
            Ex2 = 0
            n = data.shape[0]
            for i in range(n):
                Ex += data[i]-K
                Ex2 += (data[i]-K)*(data[i]-K)
            var = (Ex2 - Ex*Ex/n)/(n-1)
            return var
```

```
In [6]: sigma = 10.0
        N = 100000
```

```
In [7]: mu1 = 1e15
        data1 = np.random.normal(loc = mu1, scale=sigma, size=N)
        print('Data1\nmax = %0.3f\nmin = %0.3f\ntrue variance = %0.3f'%(data1.max(), data1.min(), np.var(data1)))
        var = variance_shift(data1)
        print('Vanilla Variance = %f\n'%(var))

        mu2 = 100.0
        data2 = np.random.normal(loc = mu2, scale=sigma, size=N)
        print('Data2\nmax = %0.3f\nmin = %0.3f\ntrue variance = %0.3f'%(data2.max(), data2.min(), np.var(data2)))
        var = variance_shift(data2)
        print('Vanilla Variance = %f\n'%(var))
```

```
Data1
max = 1000000000000043.250
min = 999999999999956.375
true variance = 100.852
Vanilla Variance = 100.852711

Data2
max = 146.156
min = 59.814
true variance = 99.726
Vanilla Variance = 99.726858
```

The Patriot Missile Failure

February 25, 1991: an American Patriot Missile battery in Dharan, Saudi Arabia, failed to track and intercept an incoming Iraqi Scud missile.

- Time was stored in tenths of seconds since boot time
- Multiplied by 0.1 to get time in seconds
- 0.1 was stored in 24-bit representation leading to 0.95×10^{-7} abs. error
- In 100 hours total error was 0.34 seconds
- Incoming missile's speed was $1,676$ mps
- 28 dead soldiers, about 100 injured

Explosion of the Ariane 5

June 4, 1996: Ariane 5 rocket launched by the European Space Agency exploded just forty seconds after lift-off

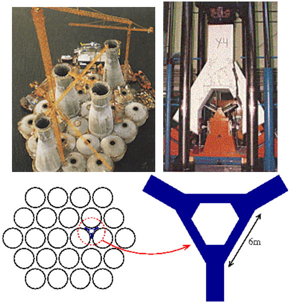
- 64 bit floating point number (vertical speed) was converted to a 16 bit signed integer
- Floating point number was greater than $32,768$
- The destroyed rocket and its cargo were valued at $500 million

Real world examples

The sinking of the Sleipner A offshore platform

August 23, 1991: concrete base structure for Sleipner A sprang a leak and sank under a controlled ballasting operation during preparation for deck mating in Gandsfjorden outside Stavanger, Norway

- Error to inaccurate finite element approximation of the linear elastic model of the tricell
- The shear stresses were underestimated by 47
- Total economic loss of about $700 million



Real world examples

The Vancouver Stock Exchange, 1982

- New index introduced at initial value of 1000.00
- Index was updated after each transaction
- Because of truncation index had fallen to 520
- If rounded instead, it would be 1098.892

German parliament election, 1992

- The Green party was calculated to have 5% of votes (min necessary)
- Means no seats for people from major party (Social Democrats) list
- Actual result was 4.97% of votes, rounded up due to one digit after decimal point