Super Resolution for Automated Target Recognition

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Abstract

Super resolution is the process of producing high-resolution images from low-resolution images while preserving ground truth about the subject matter of the images and potentially inferring more such truth. Algorithms that successfully carry out such a process are broadly useful in all circumstances where high-resolution imagery is either difficult or impossible to obtain. In particular we look towards super resolving images collected using longwave infrared cameras since high resolution sensors for such cameras do not currently exist. We present an exposition of motivations and concepts of super resolution in general, and current techniques, with a qualitative comparison of such techniques. Finally we suggest directions for future research.

1 Appendix

1.1 Rayleigh Criterion

1.1.1 Wave Equation in a Vacuum

Starting with Maxwell's equations in a vacuum (in differential form) for the electric field $\boldsymbol{E}(x,y,z,t)$ and the magnetic field $\boldsymbol{B}(x,y,z,t)$:

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

Note that

$$\nabla \times (\nabla \times \boldsymbol{E}) = \frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{3}$$

and with the identity

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \tag{4}$$

we have the vector E-field *vector wave equation*:

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{5}$$

This decouples each components of the $\boldsymbol{E}$ field. We therefore arbitrarily choose the z component and solve the *scalar wave equation* for $E \coloneqq E_z$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = 0 \tag{6}$$

We proceed by *separation of variables*:

$$E(\boldsymbol{x},t) = U(\boldsymbol{x})T(t) \tag{7}$$

Then U obeys the *Helmholtz equation*

$$(\nabla^2 + \beta^2)U = 0 \tag{8}$$

where β is separation constant. The solution for T is straightforward

$$T(t) = e^{-\mathrm{i}\omega t} \tag{9}$$

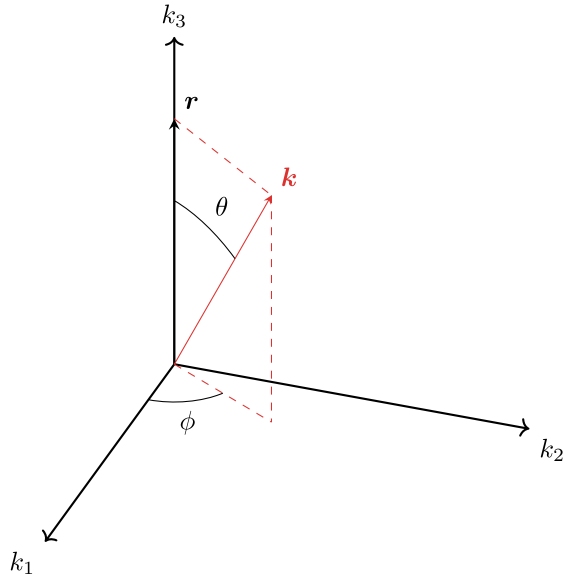


Figure 1: Rotated/aligned spherical coordinate system for computing the contour integral.

where $\omega = \beta c$. For U we seek a *Green's function* $G(\boldsymbol{x}, \boldsymbol{x}')$ for eqn. (8)

$$\nabla^2 G + \beta^2 G = -\delta(\boldsymbol{r}) \tag{10}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. Substituting the Fourier transform $\tilde{G}(\boldsymbol{k})$ of G

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_{\mathbb{R}^3} \tilde{G} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}\boldsymbol{k} \tag{11}$$

where $\boldsymbol{k} = (k_1, k_2, k_3)$ and the Fourier representation of $\delta(\boldsymbol{r})$

$$\delta(\boldsymbol{r}) = \int\limits_{\mathbb{R}^3} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}\boldsymbol{k} \tag{12}$$

into eqn. (10)

$$\int\limits_{\mathbb{R}^3} \left(-k^2 + \beta^2\right) \tilde{G} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}\boldsymbol{k} = -\int\limits_{\mathbb{R}^3} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}\boldsymbol{k} \tag{13}$$

where $k = \boldsymbol{k} \cdot \boldsymbol{k}$. Comparing both sides of eqn. (13) we conclude that

$$\tilde{G}(\boldsymbol{k}) = \frac{1}{k^2 - \beta^2} \tag{14}$$

and hence

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_{\mathbb{R}^3} \frac{e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2 - \beta^2} \,\mathrm{d}\boldsymbol{k} \tag{15}$$

Rotate the $\boldsymbol{k}$ coordinate system such that the k_3-axis and $\boldsymbol{r}$ are aligned and then transform to spherical coordinates (see figure 1). That implies

$$\boldsymbol{k} \cdot \boldsymbol{r} = k|\boldsymbol{r}|\cos(\theta)$$

and the differential volume element is

$$\mathrm{d}V = k^2 \sin(\theta) \,\mathrm{d}\phi \,\mathrm{d}\theta \,\mathrm{d}k \tag{16}$$

Then

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_0^\infty \int\limits_0^\pi \int\limits_0^{2\pi} \frac{e^{\mathrm{i}k|\boldsymbol{r}|\cos(\theta)}}{k^2 - \beta^2} k \sin(\theta) \,\mathrm{d}\phi \,\mathrm{d}\theta \,\mathrm{d}k = \frac{2\pi}{\mathrm{i}|\boldsymbol{r}|} \int\limits_0^\infty \frac{e^{\mathrm{i}k|\boldsymbol{r}|} - e^{-\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} k \,\mathrm{d}k \tag{17}$$

Since

$$\int\limits_0^\infty \frac{e^{\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} k \,\mathrm{d}k = \int\limits_{-\infty}^0 \frac{-e^{-\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} k \,\mathrm{d}k \tag{18}$$

equation (17) is

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{2\pi}{\mathrm{i}|\boldsymbol{r}|} \int\limits_{-\infty}^\infty \frac{k e^{\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} \,\mathrm{d}k \tag{19}$$

Note that there are there are two singularities or *poles* in the integrand in eqn. (17): the integrand goes to infinity as $k \to +\beta$ or $k \to -\beta$. To perform the integral, despite the poles, we need to use complex integration and the residue theorem[1]: we perform a line integral (called a *contour integral*) in the k-complex plane such that its value along the real axis equals the integral in eqn. (19) and its value elsewhere is zero.

A critical requirement of the residue theorem is that the poles are wholly contained in the contour. To that end we shift the poles $-\beta, +\beta$ by a pure imaginary component $+\mathrm{i}\epsilon$ and then take the limit as $\epsilon \to 0$. Consider the contour C in figure 2. It's composed of the portion C_1 along $\mathrm{Re}\{k\}$ and the semi-circle C_2 in the positive $\mathrm{Im}\{k\}$ half-plane. If we take the limit as $R \to \infty$ then the integral along C_1 agrees with the integrand in

[1] Consider a contour in the complex plane that encloses poles a_j. *Cauchy's residue theorem* dictates that for a *holomorphic* function f

$$\oint\limits_C f(z)\,dz = 2\pi\mathrm{i} \sum \mathrm{Res}(f, a_j)$$

where $\mathrm{Res}(f, a_k)$ are the *residues* at the poles a_k:

$$\mathrm{Res}(f, a_j) := \frac{1}{(n-1)!} \lim_{a \to a_j} \left[\frac{d^{(n-1)}}{da^{(n-1)}} \left((a - a_j)^n f(a) \right) \right]$$

where n is the order of the pole.

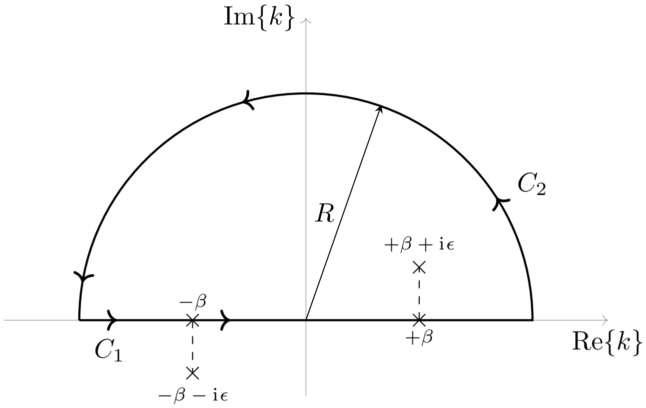


Figure 2: Contour $C = C_1 \cup C_2$ for the Helmholtz Green's function contour integral with poles $-\beta, +\beta$ and shifted poles $-\beta - \mathrm{i}\epsilon, +\beta + \mathrm{i}\epsilon$.

eqn. (19) and by *Jordan's lemma*[2], with

$$g(k) = \frac{1}{k^2 + \beta^2}$$

the integral along C_2 vanishes. Therefore

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \lim_{\epsilon \to 0} \frac{(2\pi)^2}{|\boldsymbol{r}|} \operatorname{Res}(f, \beta + \mathrm{i}\epsilon) \tag{20}$$

$$= \frac{(2\pi)^2}{|\boldsymbol{r}|} \lim_{\epsilon \to 0} \frac{\beta e^{\mathrm{i}\beta|\boldsymbol{r}|}}{2(\beta + \mathrm{i}\epsilon)} \tag{21}$$

$$= A \frac{e^{\mathrm{i}\beta|\boldsymbol{r}|}}{|\boldsymbol{r}|} \tag{22}$$

$$= A \frac{e^{\mathrm{i}kr}}{r} \tag{23}$$

where A absorbs constant factors, $r = |\boldsymbol{r}|$, and $k \equiv \beta$ to better align with conventional notation. Finally, recalling eqn. (7), we have the *mono-chromatic spherical wave* (due to the spherical symmetry, i.e., dependence on only r and a single frequency ω) solution

$$E(r, t) = A \frac{e^{\mathrm{i}(kr - \omega t)}}{r} \tag{24}$$

[2]Consider a complex-valued, continuous function f, defined on a semicircular contour

$$C_R = \{Re^{i\theta} \mid \theta \in [0, \pi]\}$$

of positive radius R lying in the upper half-plane, centered at the origin. If the function f is of the form

$$f(z) = e^{iaz} g(z), \quad z \in C_R$$

with a positive parameter a, then Jordan's lemma states the following upper bound for the contour integral:

$$\left| \int_{C_R} f(z)\, dz \right| \le \frac{\pi}{a} M_R \quad \text{where} \quad M_R \coloneqq \max_{\theta \in [0, \pi]} \left| g\left(Re^{i\theta}\right) \right|$$

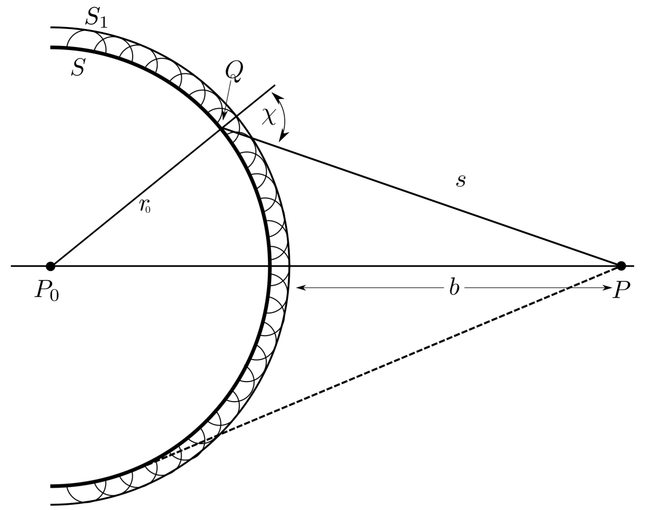


Figure 3: Illustration of the Huygens-Fresnel principle. P_0 and P are the light source and the receiver, respectively, r_0 is the radius of the wave front with surface S, S_1 is the wave front at a later instant and χ is the inclination angle at some point Q.

1.1.2 *Kirchhoff-Helmholtz integral theorem*

Let P_0 be a point source emitting spherical waves and an arbitrary wave-front S with radius r_0. According to the *Huygens-Fresnel principle* each point on S is a secondary emitter. Consider figure 3. Let S_1 be the secondary wave-front (envelope of secondary emitters). Then the field at a point P is the superposition of secondary waves that originate from the surface S. Kirchoff's derived Huygens-Fresnel by computing the field at point P in terms of the field and its derivatives at all points on an arbitrary surface S enclosing P (see figure 4).

First we derive a general fact about fields; if we assume U, U' are continuously differentiable to second order on S (and on its boundary) and we have by *Green's theorem*[3]

$$\int_v \left(U \nabla^2 U' - U' \nabla^2 U\right) dV = -\int_S \left(U \frac{\partial U'}{\partial \boldsymbol{n}} - U' \frac{\partial U}{\partial \boldsymbol{n}}\right) \mathrm{d}S$$

where $\frac{\partial}{\partial \boldsymbol{n}}$ is the directional derivative of in the direction of the **inward**[4] pointing normal $\boldsymbol{n}$. Since U' also

[3] Itself a special case of the *divergence theorem*, Green's theorem states that for ψ, φ both twice continuously differentiable on $U \subset \mathbb{R}^3$

$$\int_U \left(\psi\, \nabla^2 \varphi - \varphi\, \nabla^2 \psi\right) dV = \oint_{\partial U} \left(\psi \frac{\partial \varphi}{\partial \mathbf{n}} - \varphi \frac{\partial \psi}{\partial \mathbf{n}}\right) \mathrm{d}S$$

where $\frac{\partial}{\partial \boldsymbol{n}}$ is the directional derivative of in the direction of the outward pointing normal $\boldsymbol{n}$ to the surface element $\mathrm{d}S$, for example

$$\frac{\partial \varphi}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot \nabla \varphi$$

[4]We use inward rather than outward in order to handle a forthcoming nuance.

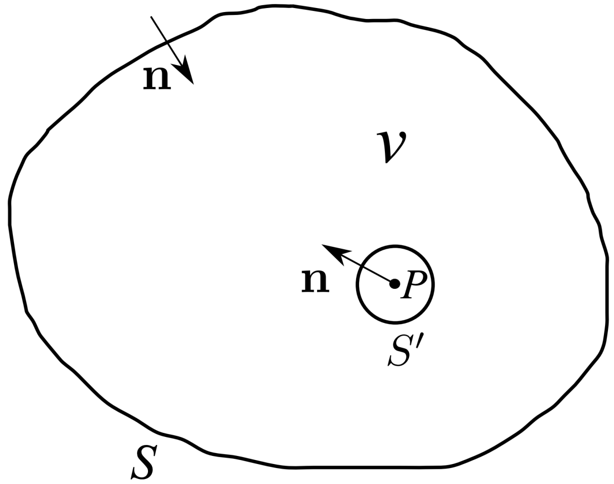


Figure 4: Schematic for derivation of Kirchhoff-Helmholtz integral.

satisfies eqn. (8) we have that

$$\int_S \left(U \frac{\partial U'}{\partial \boldsymbol{n}} - U' \frac{\partial U}{\partial \boldsymbol{n}} \right) \mathrm{d}S = 0 \qquad (25)$$

Suppose now take our Green's function from eqn. (23)

$$U' = \frac{e^{\mathrm{i}ks}}{s} \qquad (26)$$

to be the field at a secondary emitter (s being the distance from P to the emitter as in figure 3). Note that U' has a singularity at $s = 0$ (i.e. at P); since an assumption of eqn. (25) is U' continuous and differentiable in the entire volume v we must exclude P from the domain of integration. We shall therefore surround P by a small sphere S' of radius ϵ, treat the integration on the left-hand side eqn. (25) **over the volume between the exterior surface S and the interior surface S'**, and then take the limit as $\epsilon \to 0$.

Hence the integration on the right-hand side eqn. (25)

$$\left\{ \int_S + \int_{S'} \right\} \left[U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right] \mathrm{d}S = 0 \qquad (27)$$

Note that surface normal bmn points inwardly (into the volume) at both surfaces. Therefore

$$\begin{aligned} \int_S U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \mathrm{d}S = \\ - \int_{S'} \left(U \frac{e^{\mathrm{i}ks}}{s} \left(\mathrm{i}k - \frac{1}{s} \right) - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right) \mathrm{d}S = \\ - \int_{\Omega} \left(U \frac{e^{\mathrm{i}k\epsilon}}{\epsilon} \left(\mathrm{i}k - \frac{1}{\epsilon} \right) - \frac{e^{\mathrm{i}k\epsilon}}{\epsilon} \frac{\partial U}{\partial \boldsymbol{n}} \right) \epsilon^2 \,\mathrm{d}\Omega \end{aligned} \qquad (28)$$

where $\mathrm{d}\Omega$ is *solid angle*[5] and we have used the fact

$$\frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} = \boldsymbol{n} \cdot \nabla \left(\frac{e^{\mathrm{i}ks}}{s} \right) = \boldsymbol{n} \cdot \hat{\boldsymbol{s}} \left(\frac{\partial}{\partial s} \frac{e^{\mathrm{i}ks}}{s} \right) = \frac{\partial}{\partial s} \frac{e^{\mathrm{i}ks}}{s} \qquad (29)$$

since $\boldsymbol{n} \cdot \hat{\boldsymbol{s}} = 1$ (since S' is a sphere). Taking the limit of the right-hand side of eqn. (28) we see that terms proportional to ϵ don't contribute (first and third) and therefore

$$\begin{aligned} \int_S \left(U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right) \mathrm{d}S &= \lim_{\epsilon \to 0} \int_{\Omega} U e^{\mathrm{i}k\epsilon} \,\mathrm{d}\Omega \qquad (30) \\ &= 4\pi U(P) \qquad (31) \end{aligned}$$

since $\lim_{\epsilon \to 0} e^{\mathrm{i}k\epsilon} = 1$, the integration over $\mathrm{d}\Omega$ doesn't depend on ϵ. Therefore the field U at point P is determined by the *Kirchhoff-Helmholtz integral theorem*

$$U(P) = \frac{1}{4\pi} \int_S \left(U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right) \mathrm{d}S \qquad (32)$$

Note that the Kirchhoff-Helmholtz integral theorem determines $U(P)$ as a function of the values of U and $\partial U / \partial \boldsymbol{n}$ on the surface S alone.

1.1.3 Fresnel-Kirchhoff 's diffraction

We now apply eqn. (32) to the typical case of waves propagating through an aperture in a screen (see figure ??). We take as our surface $S = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ where $\mathcal{A}$ is the aperture itself, $\mathcal{B}$ is the unilluminated side of the screen, and $\mathcal{C}$ is a portion of a large sphere centered at P. The Kirchhoff-Helmholtz integral dictates that

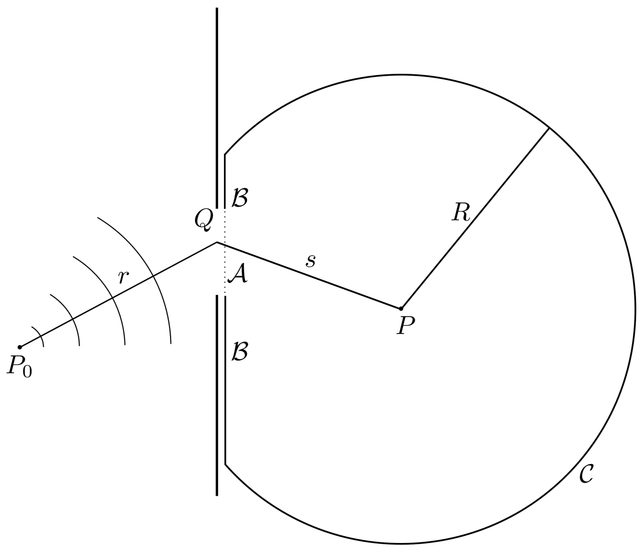
$$U(P) = \frac{1}{4\pi} \left\{ \int_{\mathcal{A}} + \int_{\mathcal{B}} + \int_{\mathcal{C}} \right\} \left[U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right] \mathrm{d}S \qquad (33)$$

Note that a fortiori the field is ~ 0 on $\mathcal{B}$ because it is unilluminated. Similarly we'd hope the field on $\mathcal{C}$ is approximately zero as well, since we are free to choose R as large as we want. Unfortunately this isn't exactly the case; while it is true that $U, \partial U / \partial \boldsymbol{n} \to 0$ as $R \to \infty$ it's also the case that the area of $S \to \infty$ and so we don't necessarily have that the integral on $\mathcal{C}$ vanishes. To assure this we need to further assume *Sommerfeld's radiation condition*:

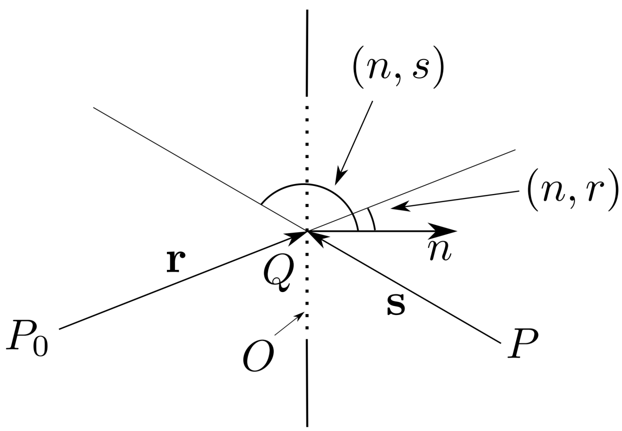
$$\lim_{r \to \infty} r \left(U(r) - \frac{\partial U}{\partial \boldsymbol{n}} \right) = 0 \qquad (34)$$

[5] The solid angle element is the angular component of the differential surface element

$$d\Omega = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$



(a) The surface S from consists of portions $\mathcal{A}, \mathcal{B}, \mathcal{C}$.



(b) Definition of angles and directions. $\mathbf{r}$ and $\mathbf{s}$ point in the direction of increasing r and s, respectively, and $\mathbf{n}$ is normal to the screen. (n, s) and (n, r) indicate the angles between the normal $\boldsymbol{n}$ and $\boldsymbol{s}, \boldsymbol{r}$ respectively.

Figure 5: Illustrating the derivation of the Fresnel-Kirchhoff diffraction formula of propagation through an aperture.

which assumes we are only dealing with *outgoing* waves. With this assumption included the integral on $\mathcal{C}$ indeed vanishes.

This leaves the field on $\mathcal{A}$ alone. Kirchoff assumed that the field on $\mathcal{A}$ is the same as if the screen were absent, i.e. the field is determined by a spherical wave emanating from P_0 (see eqn. (23)). Therefore we have

$$U(P) = \frac{1}{4\pi} \int_S \left(U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right) \mathrm{d}S \tag{35}$$

$$= \frac{A}{4\pi} \int_{\mathcal{A}} \left(\frac{e^{\mathrm{i}kr}}{r} \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \left(\frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}kr}}{r} \right) \right) \mathrm{d}S \tag{36}$$

Assuming the wavelength of the field is much smaller than the distances involved, i.e. $\lambda \ll r, s$ (and therefore $k \gg 1/r, 1/s$), we can approximate $\mathrm{i}k - 1/r \approx \mathrm{i}k - 1/s \approx \mathrm{i}k$. Then the *Fresnel-Kirchhoff diffraction formula* reads

$$U(P) = \frac{1}{4\pi} \int_S \left(U \frac{\partial}{\partial \boldsymbol{n}} \frac{e^{\mathrm{i}ks}}{s} - \frac{e^{\mathrm{i}ks}}{s} \frac{\partial U}{\partial \boldsymbol{n}} \right) \mathrm{d}S \tag{37}$$

$$= \frac{\mathrm{i}A}{2\lambda} \int_{\mathcal{A}} \left(\frac{e^{\mathrm{i}k(r+s)}}{rs} \big(\cos(\theta) - \cos(\phi)\big) \right) \mathrm{d}S \tag{38}$$

where $\theta = (n, s)$ the angle between $\boldsymbol{n}$ and $\boldsymbol{s}$ and similarly $\phi = (n, r)$ (see figure 5b).

For example, for a surface integral over a sphere of radius r (in spherical coordinates)

$$\int_{dA} = \int_{\Omega} r^2 d\Omega = r^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

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