Notes

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Contents

1 Differential Geometry

Contents

1.1 Directional Derivative

Elements of the *tangent space* $T_p(\mathbb{R}^n)$ anchored at a point $p = (p^1, \ldots, p^n) \in \mathbb{R}^n$ can be visualized as arrows emanating from p. These arrows are called *tangent vectors* and represented by column vectors:

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \tag{1}$$

The line through a point p with direction $\boldsymbol{v}$ has parameterization

$$c(t) = \left(p^1 + tv^1, \ldots, p^n + tv^n\right) \tag{2}$$

If $f \in C^\infty$ in a neighborhood of p and $\boldsymbol{v}$ is a tangent vector at p, the *directional derivative* of f in the direction of $\boldsymbol{v}$ at p is defined

$$D_{\boldsymbol{v}} f = \lim_{t\to 0} \frac{f(c(t)) - f(p)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t))\bigg|_{t=0} \tag{3}$$

By the chain rule

$$D_{\boldsymbol{v}} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\bigg|_p \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \tag{4}$$

$$= \sum_{i=1}^{n} \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f}{\partial x^i}\bigg|_p \tag{5}$$

$$= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}\bigg|_p \tag{6}$$

$$\tag{7}$$

The directional derivative operator at p is defined

$$D_{\boldsymbol{v}} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{8}$$

The association $\boldsymbol{v} \mapsto D_{\boldsymbol{v}}$ offers a way to *isomorphically* identify tangent vectors with operators on functions. The following makes this rigorous.

1.2 Derivations

For each tangent vector $\boldsymbol{v}$ at a point $p \in \mathbb{R}^n$, the directional derivative at p gives a map of vector spaces

$$D_{\boldsymbol{v}} \colon C_p^\infty \to \mathbb{R}$$

$D_{\boldsymbol{v}}$ is a linear map that satisfies the *Leibniz rule*

$$D_{\boldsymbol{v}}(fg) = (D_{\boldsymbol{v}} f)g(p) + f(p)(D_{\boldsymbol{v}} g) \tag{9}$$

because the partial derivative satisfy the product rule. In general, any linear map $L \colon C_p^\infty \to \mathbb{R}$ that satisfies the Leibniz rule is called a *derivation* at p. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. **This set is also a real vector space**.

So far we know directional derivatives $D_{\boldsymbol{v}}$ at p are derivations at p. Thus, there is a map

$$\begin{aligned} \phi \colon T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ \boldsymbol{v} &\mapsto D_{\boldsymbol{v}} \end{aligned}$$

Theorem 1.1. *The linear map ϕ is an isomorphism of vector spaces.*

The implication is that we may identify tangent vectors at p with derivations at p (by way of directional derivatives against germs). Under this isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \ldots, e_n\}$ for $T_p(\mathbb{R}^n)$ maps to

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \tag{10}$$

Therefore from now on we write a tangent vector as

$$\boldsymbol{v} = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p \tag{11}$$

The point being that, while not as geometrically intuitive as arrows, $\mathcal{D}_p(\mathbb{R}^n)$ generalizes to manifolds.

1.3 Vector Fields

A *vector field* X on an open $U \subset \mathbb{R}^n$ is function that assigns to $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Notice, carefully, that the vector field assigns at each point a vector in the tangent space anchored at that point. Using the tangent basis (eqn. (10))

$$X \colon p \mapsto \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \tag{12}$$

Note that both the coefficients **and** the partial derivatives are evaluated at p. Having said that, we often omit p in the specification of a vector field when it clear from context.

Example 1.1. On $\mathbb{R}^n - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$\begin{aligned} X &= \frac{-y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}^T \end{aligned}$$

See figure ??

In general we can identify vector fields with parameterized column vectors

$$X = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \leftrightarrow \begin{bmatrix} a^1(p) \\ \vdots \\ a^n(p) \end{bmatrix} \tag{13}$$

1.4 Dual Space

The *dual space* $V^\wedge$ of V is the set of all real-valued linear functions on V i.e. all $f \colon V \to \mathbb{R}$. Elements of $V^\wedge$ are called *covectors*.

Assume V is finite dimensional and let $\{e_1, \ldots, e_n\}$ be a basis V. Recall that $e_i \coloneqq \partial_{x_i}$. Then $X = \sum a^i \partial_{x_i}$ for all $X \in T_p$. Let $\alpha^i \colon V \to \mathbb{R}$ be the linear function that picks out the ith coordinate of a **vector**, i.e. $\alpha^i(X) = a^i(p)$. Note that

$$\begin{aligned} \alpha^i(\partial_j) &= \alpha^i(1 \cdot \partial_j) && (14) \\ &= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} && (15) \\ &= \delta^i_j && (16) \end{aligned}$$

Note that position of indices is important – upper indices are for covectors.

Proposition 1.1. *$\{\alpha^i\}$ form a basis for $V^\wedge$.*

Proof. We first prove that $\{\alpha^i\}$ span $V^\wedge$. If $f \in V^\wedge$ and $X = \sum a^i \partial_{x_i} \in V$, then

$$\begin{aligned} f(X) &= \sum a^i f(\partial_{x_i}) && (17) \\ &= \sum \alpha^i(X) f(\partial_{x_i}) && (18) \\ &= \sum f(\partial_{x_i}) \alpha^i(X) && (19) \end{aligned}$$

which shows that any f can be expanded as a linear sum of α^i. To show linear independence, suppose $\sum c_i\alpha^i = 0$ with at least one c_i non-zero. Applying this to an arbitrary ∂_{x_i} gives

$$0 = \left(\sum_i c_i\alpha^i\right)(\partial_{x_i}) = \sum_i c_i\alpha^i(\partial_{x_i}) = \sum_i c_i\delta^i_j = c_j \tag{20}$$

which is a contradiction. Hence α^i are linearly independent. □

This basis $\{\alpha^i\}$ for $V^\wedge$ is said to be *dual* to the basis $\{\partial_{x_i}\}$ for V.

Example 1.2. (*Coordinate functions*) With respect to a basis $\{\partial_{x_i}\}$ for V, every $X \in V$ can be written uniquely as a linear combination $X = \sum a^i\partial_{x_i}$ with $a^i \in \mathbb{R}$. Let $\{\alpha^i\}$ be the dual basis (i.e. the basis for $V^\wedge$). Then

$$\begin{aligned}\alpha^i(X) &= \alpha^i\left(\sum_j a^j\partial_{x_j}\right)\\ &= \sum_j a^j\alpha^i(\partial_{x_j})\\ &= \sum_j a^j\delta^i_j\\ &= a^i\end{aligned}$$

Thus, the dual basis $\{\alpha^i\}$ to $\{\partial_{x_i}\}$ is the set of coordinate functions. The sense here is that since tangent vectors are directional derivatives (i.e. operators on functions) and the dual space is a mapping from those operators to $\mathbb{R}$, then a mapping from operators to scalars means hitting an operator with a function (or vice-versa). **And** the coordinate functions are constant with respect to each other coordinate (and hence partials wrt them are naturally zero).

2 Appendix

Contents

2.1 Definitions

2.1.1 Linear Operator

A map $L\colon V \to W$ between vector spaces over a field K is a *linear operator* if

1. **distributivity**: $L(u+v) = L(u) + L(v)$
2. **homogeneity**: $L(rv) = rL(v)$

To emphasize the field, L is said to be $K-$linear.

2.1.2 Germs

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f\colon U \to \mathbb{R}$ is a C^∞ function. We say that $(f, U) \sim (g, U')$ if there is an open W such that $p \in W \subset U \cap U'$ and $f = g$ when restricted to W. The equivalence class $[(f, U)]$ of (f, U) is the *germ* of f at p. We write

$$C^\infty_p(\mathbb{R}^n) := \{[(f, U)]\} \tag{21}$$

for the set all germs of C^∞ functions on $\mathbb{R}^n$ at p.

2.1.3 Algebra

An *algebra* over *field* K is a vector space A over K with a multiplication map

$$\mu\colon A \times A \to A \tag{22}$$

usually written $\mu(a, b) = a \cdot b$, such that μ is associative, distributive, and homogeneous, where homogeneity is defined:

1. **associativity**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. **distributivity**: $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + a \cdot c$
3. **homogeneity**: $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$

If A, A' are algebras then an *algebra homomorphism* is a linear operator L that respects algebra multiplication

$L(ab) = L(a)L(b)$. It's the case that addition and multiplication of functions **induces addition and multiplication on the set of germs** C_p^∞, making it into an algebra over $\mathbb{R}^n$.

2.1.4 Module

If R is a commutative ring with identity, then a (left) $R-$*module* is an abelian group A with a scalar multiplication map

$$\mu\colon R \times A \to A \tag{23}$$

such that μ is

1. **associative**: $(rs)a = r(sa)$ for $r, s \in R$
2. **identity**: $1 \in R \implies 1a = a$
3. **distributive**: $(r+s)a = ra+sa$ and $r(a+b) = ra+rb$

If R is a field, then an $R-$module is a vector space over R; in this sense modules generalize vector space to scalars from a ring rather than a field.

Let A, A' be $R-$modules. An $R-$*module homomorphism* $f\colon A \to A'$ is a map that preserves both addition and scalar multiplication.

2.1.5 Tensor Product

Let f be k-linear function and g be an ℓ-linear function on a vector space V. Then, their *tensor product* is the $(k+\ell)$-linear function $f \otimes g$

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) \coloneqq f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}) \tag{24}$$

Example 2.1. (*Bilinear maps*) Let $\{e_i\}$ be a basis for a vector space V and $\{\alpha^j\}$ be the dual basis for $V^\vee$. Also let $\langle\,,\rangle\colon V \times V \to \mathbb{R}$ be a bilinear map on V. Set $g_{ij} = \langle e_i, e_j\rangle \in \mathbb{R}$. If

$$v = \sum v^i e_i \tag{25}$$

$$w = \sum w^i e_i \tag{26}$$

then $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$, where α are the coordinate functions a^i, a^j. By bilinearity, we can express $\langle\,,\rangle$ in terms of the tensor product

$$\begin{aligned}\langle v, w\rangle &= \sum_{ij} v^i w^j \langle e_i, e_j\rangle \\ &= \sum \alpha^i(v)\alpha^i(w) g_{ij} \\ &= \sum (\alpha^i \otimes \alpha^j)(v, w) \times g_{ij}\end{aligned} \tag{27}$$

Hence $\langle\,,\rangle = \sum g_{ij}\, \alpha^i \otimes \alpha^j$.

2.1.6 Wedge Product

Let f be k-linear function and g be an ℓ-linear function on a vector space V. If f, g are alternating[1] then we would like their product to be alternating as well: the *wedge product* or *exterior product* $f \wedge g$

$$f \wedge g \coloneqq \frac{1}{k!l!} A(f \otimes g) \tag{28}$$

$$\coloneqq \frac{1}{k!l!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}(\sigma)) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \tag{29}$$

$$g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

where $S_{k+\ell}$ is the *permutation group* on $k+\ell$ elements.

Note that the wedge product of three alternating functions f, g, h generalizes to

$$f \wedge g \wedge h = \frac{1}{k!\ell!m!} = A(f \otimes g \otimes h) \tag{30}$$

and any number of alternating functions.

Proposition 2.1. *(Wedge product of 1-covectors) If $\{\alpha^i\}$ are linear functions on V and $v_i \in V$ then*

$$\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \dots, v_k) = \det\bigl([\alpha^i(v_j)]\bigr) \tag{31}$$

where $[\alpha^i(v_j)]$ is the matrix where the i,j-entry is $\alpha^i(v_j)$.

Proof.

$$\begin{aligned}&\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \dots, v_k) = \\ &\quad A(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k)\end{aligned} \tag{32}$$

$$= \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma)) \alpha^1(v_{\sigma(1)}) \wedge \cdots \wedge \alpha^k(v_{\sigma(k)}) \tag{33}$$

$$= \det\bigl([\alpha^i(v_j)]\bigr) \tag{34}$$

□

Let $\{e_i\}$ be a basis for a vector space V and let α_j be the dual basis for $V^\vee$. Define *multi-index notation*

$$\begin{aligned}I &\coloneqq (i_1, \dots, i_k) \\ e_I &\coloneqq (e_{i_1}, \dots, e_{i_k}) \\ \alpha_I &\coloneqq \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}\end{aligned} \tag{35}$$

A k-linear function on V is completely determined by its values on all k-tuples e_I. If f is alternating then you only need all increasing multi-indices $1 \leq i_1 \leq \cdots \leq i_k \leq n$.

2.1.7 Graded Algebra

[1] An alternating function is one that changes signs if arguments are transposed (e.g. cross-product or determinant).

Definition 2.1. A *graded algebra* A over a field K is one that can be written as a *direct sum*[2]

$$A = \bigoplus_{n \in \mathbb{N}_0} A_n = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that the multiplication map μ

$$\mu \colon A_k \times A_\ell \to A_{k+\ell}$$

i.e. $A_k \cdot A_\ell \subset A_{k+\ell}$. Elements of any factor A_n of the decomposition are called homogeneous elements of degree n. $A = \bigoplus\limits_k A_k$ is said to *graded commutative* or *anticommutative* if for all $a \in A_k, b \in A_\ell$

$$ab = (-1)^{k\ell} ba$$

Example 2.2. The *polynomial algebra*[3] $A = \mathbb{R}[x, y]$ is *graded by degree*; A_k consists of all homogeneous polynomials of total degree k. It is a direct sum of A_i consisting of homogeneous polynomials of degree i e.g.

$$(x \in A_1) \times (x + y \in A_1) = \left(x^2 + xy \in A_2\right)$$

Definition 2.2. For a finite dimensional vector space V define

$$\begin{aligned} A_*(V) &\coloneqq \bigoplus_{k=0}^{\infty} A_k(V) \\ &= \bigoplus_{k=0}^{n} A_k(V) \end{aligned} \tag{36}$$

where $A_k(V)$ is the set of all k-linear covectors on V. With the wedge product as multiplication A_* becomes and anticommutative algebra called the ***exterior algebra*** or ***Grassmann algebra*** of covectors.

[2] The direct sum of two abelian groups A and B is another abelian group $A \oplus B$ consisting of the ordered pairs (a, b) where $a \in A$ and $b \in B$. (Confusingly this ordered pair is also called the cartesian product of the two groups.) To add ordered pairs, we define the sum $(a, b) + (c, d)$ to be $(a + c, b + d)$; in other words addition is defined coordinate-wise. Note that $A^2 \coloneqq A \oplus A$. A similar process can be used to form the direct sum of any two algebraic structures, such as rings, modules, and vector spaces.

[3] The algebra of polynomials in two variables with coefficients and scalars in $\mathbb{R}$ where addition is degree-coefficient wise and "foil" multiplication

Index