

Notes

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1 Differential Geometry

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1.1 Directional Derivative

Elements of the *tangent space* $T_p(\mathbb{R}^n)$ anchored at a point $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ can be visualized as arrows emanating from p . These arrows are called *tangent vectors* and represented by column vectors:

$$\mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad (1)$$

The line through a point p with direction \mathbf{v} has parameterization

$$c(t) = (p^1 + tv^1, \dots, p^n + tv^n) \quad (2)$$

If $f \in C^\infty$ in a neighborhood of p and \mathbf{v} is a tangent vector at p , the *directional derivative* of f in the direction of \mathbf{v} at p is defined

$$D_{\mathbf{v}}f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} f(c(t)) \right|_{t=0} \quad (3)$$

By the chain rule

$$D_{\mathbf{v}}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \bigg|_p \frac{dc^i}{dt} \bigg|_{t=0} \quad (4)$$

$$= \sum_{i=1}^n \frac{dc^i}{dt} \bigg|_{t=0} \frac{\partial f}{\partial x^i} \bigg|_p \quad (5)$$

$$= \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i} \bigg|_p \quad (6)$$

$$(7)$$

The directional derivative operator at p is defined

$$D_{\mathbf{v}} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \bigg|_p \quad (8)$$

The association $\mathbf{v} \mapsto D_{\mathbf{v}}$ offers a way to *isomorphically* identify tangent vectors with operators on functions. The following makes this rigorous.

1.2 Derivations

For each tangent vector \mathbf{v} at a point $p \in \mathbb{R}^n$, the directional derivative at p gives a map of vector spaces

$$D_{\mathbf{v}}: C_p^\infty \rightarrow \mathbb{R}$$

$D_{\mathbf{v}}$ is a linear map that satisfies the *Leibniz rule*

$$D_{\mathbf{v}}(fg) = (D_{\mathbf{v}}f)g(p) + f(p)(D_{\mathbf{v}}g) \quad (9)$$

because the partial derivative satisfy the product rule. In general, any linear map $L: C_p^\infty \rightarrow \mathbb{R}$ that satisfies the Leibniz rule is called a *derivation* at p . Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. **This set is also a real vector space.**

So far we know directional derivatives $D_{\mathbf{v}}$ at p are derivations at p . Thus, there is a map

$$\begin{aligned} \phi: T_p(\mathbb{R}^n) &\rightarrow \mathcal{D}_p(\mathbb{R}^n) \\ \mathbf{v} &\mapsto D_{\mathbf{v}} \end{aligned}$$

Theorem 1.1. *The linear map ϕ is an isomorphism of vector spaces.*

The implication is that we may identify tangent vectors at p with derivations at p (by way of directional derivatives against germs). Under this isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \dots, e_n\}$ for $T_p(\mathbb{R}^n)$ maps to

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\} \quad (10)$$

Therefore from now on we write a tangent vector as

$$\mathbf{v} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \quad (11)$$

The point being that, while not as geometrically intuitive as arrows, $\mathcal{D}_p(\mathbb{R}^n)$ generalizes to manifolds.

1.3 Vector Fields

A *vector field* X on an open $U \subset \mathbb{R}^n$ is function that assigns to $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Notice, carefully, that the vector field assigns at each point a vector in the tangent space anchored at that point. Using the tangent basis (eqn. (10))

$$X: p \mapsto \sum_i a^i(p) \frac{\partial}{\partial x^i} \Big|_p \quad (12)$$

Note that both the coefficients **and** the partial derivatives are evaluated at p . Having said that, we often omit p in the specification of a vector field when it clear from context.

Example 1.1. On $\mathbb{R}^n - \{0\}$, let $p = (x, y)$. Then

$$\begin{aligned} X &= \frac{-y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2 + y^2}} \\ \frac{x}{\sqrt{x^2 + y^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2 + y^2}} & \frac{x}{\sqrt{x^2 + y^2}} \end{bmatrix}^T \end{aligned}$$

See figure ??

In general we can identify vector fields with parameterized column vectors

$$X = \sum_i a^i(p) \frac{\partial}{\partial x^i} \Big|_p \leftrightarrow \begin{bmatrix} a^1(p) \\ \vdots \\ a^n(p) \end{bmatrix} \quad (13)$$

1.4 Dual Space

The *dual space* V^\wedge of V is the set of all real-valued linear functions on V i.e. all $f: V \rightarrow \mathbb{R}$. Elements of V^\wedge are called *covectors*.

Assume V is finite dimensional and let $\{e_1, \dots, e_n\}$ be a basis V . Recall that $e_i := \partial_{x_i}$. Then $X = \sum a^i \partial_{x_i}$ for all $X \in T_p$. Let $\alpha^i: V \rightarrow \mathbb{R}$ be the linear function that picks out the i th coordinate of a **vector**, i.e. $\alpha^i(X) = a^i(p)$. Note that

$$\alpha^i(\partial_j) = \alpha^i(1 \cdot \partial_j) \quad (14)$$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (15)$$

$$= \delta_j^i \quad (16)$$

Note that position of indices is important – upper indices are for covectors.

Proposition 1.0.1. $\{\alpha^i\}$ form a basis for V^\wedge .

Proof. We first prove that $\{\alpha^i\}$ span V^\wedge . If $f \in V^\wedge$ and $X = \sum a^i \partial_{x_i} \in V$, then

$$f(X) = \sum a^i f(\partial_{x_i}) \quad (17)$$

$$= \sum \alpha^i(X) f(\partial_{x_i}) \quad (18)$$

$$= \sum f(\partial_{x_i}) \alpha^i(X) \quad (19)$$

which shows that any f can be expanded as a linear sum of α^i . To show linear independence, suppose $\sum c_i \alpha^i = 0$ with at least one c_i non-zero. Applying this to an arbitrary ∂_{x_i} gives

$$0 = \left(\sum_i c_i \alpha^i \right) (\partial_{x_i}) = \sum_i c_i \alpha^i(\partial_{x_i}) = \sum_i c_i \delta_j^i = c_j \quad (20)$$

which is a contradiction. Hence α^i are linearly independent. \square

This basis $\{\alpha^i\}$ for V^\wedge is said to be *dual* to the basis $\{\partial_{x_i}\}$ for V .

Example 1.2. (*Coordinate functions*) With respect to a basis $\{\partial_{x_i}\}$ for V , every $X \in V$ can be written uniquely as a linear combination $X = \sum a^i \partial_{x_i}$ with $a^i \in \mathbb{R}$. Let $\{\alpha^i\}$ be the dual basis (i.e. the basis for V^\wedge). Then

$$\begin{aligned}\alpha^i(X) &= \alpha^i\left(\sum_j a^j \partial_{x_j}\right) \\ &= \sum_j a^j \alpha^i(\partial_{x_j}) \\ &= \sum_j a^j \delta_j^i \\ &= a^i\end{aligned}$$

Thus, the dual basis $\{\alpha^i\}$ to $\{\partial_{x_i}\}$ is the set of coordinate functions. The sense here is that since tangent vectors are directional derivatives (i.e. operators on functions) and the dual space is a mapping from those operators to \mathbb{R} , then a mapping from operators to scalars means hitting an operator with a function (or vice-versa). **And** the coordinate functions are constant with respect to each other coordinate (and hence partials wrt them are naturally zero).

2 Appendix

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2.1 Definitions

2.1.1 Linear Operator

A map $L: V \rightarrow W$ between vector spaces over a field K is a *linear operator* if

1. **distributivity:** $L(u + v) = L(u) + L(v)$
2. **homogeneity:** $L(rv) = rL(v)$

To emphasize the field, L is said to be K -linear.

2.1.2 Germes

Consider the set of all pairs (f, U) , where U is a neighborhood of p and $f: U \rightarrow \mathbb{R}$ is a C^∞ function. We say that $(f, U) \sim (g, U')$ if there is an open W such that $p \in W \subset U \cap U'$ and $f = g$ when restricted to W . The equivalence class $[(f, U)]$ of (f, U) is the *germ* of f at p . We write

$$C_p^\infty(\mathbb{R}^n) := \{[(f, U)]\} \quad (21)$$

for the set all germs of C^∞ functions on \mathbb{R}^n at p .

2.1.3 Algebra

An *algebra* over field K is a vector space A over K with a multiplication map

$$\mu: A \times A \rightarrow A \quad (22)$$

usually written $\mu(a, b) = a \cdot b$, such that μ is associative, distributive, and homogeneous, where homogeneity is defined:

1. **associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. **distributivity:** $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + a \cdot c$
3. **homogeneity:** $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$

If A, A' are algebras then an *algebra homomorphism* is a linear operator L that respects algebra multiplication $L(ab) = L(a)L(b)$. It's the case that addition and multiplication of functions **induces addition and multiplication on the set of germs** C_p^∞ , making it into an algebra over \mathbb{R}^n .

2.1.4 Module

If R is a commutative ring with identity, then a (left) R -module is an abelian group A with a scalar multiplication map

$$\mu: R \times A \rightarrow A \quad (23)$$

such that μ is

1. **associative:** $(rs)a = r(sa)$ for $r, s \in R$
2. **identity:** $1 \in R \implies 1a = a$
3. **distributive:** $(r+s)a = ra+sa$ and $r(a+b) = ra+rb$

If R is a field, then an R -module is a vector space over R ; in this sense modules generalize vector space to scalars from a ring rather than a field.

Let A, A' be R -modules. An R -module homomorphism $f: A \rightarrow A'$ is a map that preserves both addition and scalar multiplication.

2.2 Tensor Product

Let f be k -linear function and g be an ℓ -linear function on a vector space V . Then, their *tensor product* is the $(k+\ell)$ -linear function $f \otimes g$

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) := f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}) \quad (24)$$

Example 2.1. (*Bilinear maps*) Let $\{e_i\}$ be a basis for a vector space V and $\{\alpha^j\}$ be the dual basis for V^\vee . Also let $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ be a bilinear map on V . Set $g_{ij} = \langle e_i, e_j \rangle \in \mathbb{R}$. If

$$v = \sum v^i e_i \quad (25)$$

$$w = \sum w^j e_j \quad (26)$$

then $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$, where α are the coordinate functions α^i, α^j . By bilinearity, we can express $\langle \cdot, \cdot \rangle$ in terms of the tensor product

$$\begin{aligned} \langle v, w \rangle &= \sum_{ij} v^i w^j \langle e_i, e_j \rangle \\ &= \sum \alpha^i(v) \alpha^j(w) g_{ij} \\ &= \sum (\alpha^i \otimes \alpha^j)(v, w) \times g_{ij} \end{aligned} \quad (27)$$

$$\text{Hence } \langle \cdot, \cdot \rangle = \sum g_{ij} \alpha^i \otimes \alpha^j$$

2.3 Wedge Product

Let f be k -linear function and g be an ℓ -linear function on a vector space V . If f, g are alternating¹ then we would

like their product to be alternating as well: the *wedge product* or *exterior product* $f \wedge g$

$$f \wedge g := \frac{1}{k!\ell!} A(f \otimes g) \quad (28)$$

$$:= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\text{sgn}(\sigma)) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \quad (29)$$

where $S_{k+\ell}$ is the *permutation group* on $k+\ell$ elements.

Note that the wedge product of three alternating functions f, g, h generalizes to

$$f \wedge g \wedge h = \frac{1}{k!\ell!m!} A(f \otimes g \otimes h) \quad (30)$$

and any number of alternating functions.

Proposition 2.0.1. (*Wedge product of 1-covectors*) If $\{\alpha^i\}$ are linear functions on V and $v_i \in V$ then

$$\alpha^1 \wedge \dots \wedge \alpha^k(v_1, \dots, v_k) = \det([\alpha^i(v_j)]) \quad (31)$$

where $[\alpha^i(v_j)]$ is the matrix where the i, j -entry is $\alpha^i(v_j)$.

Proof.

$$\alpha^1 \wedge \dots \wedge \alpha^k(v_1, \dots, v_k) = A(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) \quad (32)$$

$$= \sum_{\sigma \in S_k} (\text{sgn}(\sigma)) \alpha^1(v_{\sigma(1)}) \wedge \dots \wedge \alpha^k(v_{\sigma(k)}) \quad (33)$$

$$= \det([\alpha^i(v_j)]) \quad (34)$$

□

¹An alternating function is one that changes signs if arguments are transposed (e.g. cross-product or determinant).

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