Notes

Maksim Levental

December 6, 2019

Contents

1 Differential Geometry **1**
- 1.1 Directional Derivative . . . 1
- 1.2 Derivations . . . 2
- 1.3 Vector Fields . . . 2
- 1.4 Dual Space . . . 2
- 1.5 Differential Forms on $\mathbb{R}^n$ . . . 3
  - 1.5.1 Differential of a Function . . . 3
  - 1.5.2 Differential k-forms . . . 4
  - 1.5.3 Differential Forms as Multilinear Functions on Vector Fields . . . 5
  - 1.5.4 Exterior Derivative . . . 5

2 Appendix **6**
- 2.1 Definitions . . . 6
  - 2.1.1 Linear Operator . . . 6
  - 2.1.2 Germs . . . 6
  - 2.1.3 Algebra . . . 6
  - 2.1.4 Module . . . 6
  - 2.1.5 Tensor Product . . . 6
  - 2.1.6 Wedge Product . . . 7
  - 2.1.7 Graded Algebra . . . 7

1 Differential Geometry

Contents

- 1.1 Directional Derivative . . . 1
- 1.2 Derivations . . . 2
- 1.3 Vector Fields . . . 2
- 1.4 Dual Space . . . 2
- 1.5 Differential Forms on $\mathbb{R}^n$ . . . 3
  - 1.5.1 Differential of a Function . . . 3
  - 1.5.2 Differential k-forms . . . 4
  - 1.5.3 Differential Forms as Multilinear Functions on Vector Fields . . . 5
  - 1.5.4 Exterior Derivative . . . 5

1.1 Directional Derivative

Elements of the *tangent space* $T_p(\mathbb{R}^n)$ anchored at a point $p = (p^1, \ldots, p^n) \in \mathbb{R}^n$ can be visualized as arrows emanating from p. These arrows are called *tangent vectors* and represented by column vectors:

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \tag{1}$$

The line through a point p with direction $\boldsymbol{v}$ has parameterization

$$c(t) = \left(p^1 + tv^1, \ldots, p^n + tv^n\right) \tag{2}$$

If $f \in C^\infty$ in a neighborhood of p and $\boldsymbol{v}$ is a tangent vector at p, the *directional derivative* of f in the direction of $\boldsymbol{v}$ at p is defined

$$D_{\boldsymbol{v}} f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) \bigg|_{t=0} \tag{3}$$

By the chain rule

$$\begin{aligned} D_{\boldsymbol{v}} f &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}\bigg|_p \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} && (4) \\ &= \sum_{i=1}^n \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f}{\partial x^i}\bigg|_p && (5) \\ &= \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}\bigg|_p && (6) \\ & && (7) \end{aligned}$$

The directional derivative operator at p is defined

$$D_{\boldsymbol{v}} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{8}$$

The association $\boldsymbol{v} \mapsto D_{\boldsymbol{v}}$ offers a way to *isomorphically* identify tangent vectors with operators on functions. The following makes this rigorous.

1.2 Derivations

For each tangent vector $\boldsymbol{v}$ at a point $p \in \mathbb{R}^n$, the directional derivative at p gives a map of vector spaces

$$D_{\boldsymbol{v}} \colon C_p^{\infty} \to \mathbb{R}$$

$D_{\boldsymbol{v}}$ is a linear map that satisfies the *Leibniz rule*

$$D_{\boldsymbol{v}}(fg) = (D_{\boldsymbol{v}} f)g(p) + f(p)(D_{\boldsymbol{v}} g) \tag{9}$$

because the partial derivative satisfy the product rule. In general, any linear map $L \colon C_p^{\infty} \to \mathbb{R}$ that satisfies the Leibniz rule is called a *derivation* at p. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. **This set is also a real vector space**.

So far we know directional derivatives $D_{\boldsymbol{v}}$ at p are derivations at p. Thus, there is a map

$$\begin{aligned} \phi \colon T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ \boldsymbol{v} &\mapsto D_{\boldsymbol{v}} \end{aligned}$$

Theorem 1.1

The linear map ϕ is an isomorphism of vector spaces.

The implication is that we may identify tangent vectors at p with derivations at p (by way of directional derivatives against germs). Under this isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \dots, e_n\}$ for $T_p(\mathbb{R}^n)$ maps to

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \tag{10}$$

Therefore from now on we write a tangent vector as

$$\boldsymbol{v} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{11}$$

The point being that, while not as geometrically intuitive as arrows, $\mathcal{D}_p(\mathbb{R}^n)$ generalizes to manifolds.

1.3 Vector Fields

A *vector field* X on an open $U \subset \mathbb{R}^n$ is function that assigns to $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Notice, carefully, that the vector field assigns at each point a vector in the tangent space anchored at that point. Using the tangent basis (eqn. (10))

$$X \colon p \mapsto \sum_i a^i(p) \frac{\partial}{\partial x^i}\bigg|_p \tag{12}$$

Note that both the coefficients **and** the partial derivatives are evaluated at p. Having said that, we often omit p in the specification of a vector field when it clear from context.

Example 1.1. On $\mathbb{R}^n - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$\begin{aligned} X &= \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}^T \end{aligned}$$

See figure **??**

In general we can identify vector fields with parameterized column vectors

$$X = \sum_i a^i(p) \frac{\partial}{\partial x^i}\bigg|_p \leftrightarrow \begin{bmatrix} a^1(p) \\ \vdots \\ a^n(p) \end{bmatrix} \tag{13}$$

1.4 Dual Space

The *dual space* $V^\wedge$ of V is the set of all real-valued linear functions on V i.e. all $f \colon V \to \mathbb{R}$. Elements of $V^\wedge$ are called *covectors*.

Assume V is finite dimensional and let $\{e_1, \dots, e_n\}$ be a basis V. Recall that $e_i \coloneqq \partial_{x_i}$. Then $X = \sum a^i \partial_{x_i}$ for all $X \in T_p$. Let $\alpha^i \colon V \to \mathbb{R}$ be the linear function that picks out the ith coordinate of a **vector**, i.e. $\alpha^i(X) = a^i(p)$. Note that

$$\alpha^i(\partial_j) = \alpha^i(1 \cdot \partial_j) \tag{14}$$

$$= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{15}$$

$$= \delta^i_j \tag{16}$$

Note that position of indices is important – upper indices are for covectors.

Proposition 1.1

$\{\alpha^i\}$ form a basis for $V^\wedge$.

Proof. We first prove that $\{\alpha^i\}$ span $V^\wedge$. If $f \in V^\wedge$ and $X = \sum a^i \partial_{x_i} \in V$, then

$$f(X) = \sum a^i f(\partial_{x_i}) \tag{17}$$
$$= \sum \alpha^i(X) f(\partial_{x_i}) \tag{18}$$
$$= \sum f(\partial_{x_i}) \alpha^i(X) \tag{19}$$

which shows that any f can be expanded as a linear sum of α^i. To show linear independence, suppose $\sum c_i \alpha^i = 0$ with at least one c_i non-zero. Applying this to an arbitrary ∂_{x_i} gives

$$0 = \left(\sum_i c_i \alpha^i \right) (\partial_{x_i}) = \sum_i c_i \alpha^i(\partial_{x_i}) = \sum_i c_i \delta^i_j = c_j \tag{20}$$

which is a contradiction. Hence α^i are linearly independent. □

This basis $\{\alpha^i\}$ for $V^\wedge$ is said to be *dual* to the basis $\{\partial_{x_i}\}$ for V.

Example 1.2. Coordinate functions With respect to a basis $\{\partial_{x_i}\}$ for V, every $X \in V$ can be written uniquely as a linear combination $X = \sum a^i \partial_{x_i}$ with $a^i \in \mathbb{R}$. Let $\{\alpha^i\}$ be the dual basis (i.e. the basis for $V^\wedge$). Then

$$\begin{aligned}
\alpha^i(X) &= \alpha^i \left(\sum_j a^j \partial_{x_j} \right) \\
&= \sum_j a^j \alpha^i(\partial_{x_j}) \\
&= \sum_j a^j \delta^i_j \\
&= a^i
\end{aligned}$$

Thus, the dual basis $\{\alpha^i\}$ to $\{\partial_{x_i}\}$ is the set of coordinate functions. The sense here is that since tangent vectors are directional derivatives (i.e. operators on functions) and the dual space is a mapping from those operators to $\mathbb{R}$, then a mapping from operators to scalars means hitting an operator with a function (or vice-versa). **And** the coordinate functions are constant with respect to each other coordinate (and hence partials wrt them are naturally zero).

1.5 Differential Forms on $\mathbb{R}^n$

A differential k-form assigns a k-covector from the dual space at each point p. The wedge product (alternation of tensor product) of differential forms is defined pointwise (as the wedge product of multi-covectors). Differential forms exist on an open set (why?) there is a notion of differentiation (called exterior derivative). Exterior derivative is coordinate independent and intrinsic to a manifold; it is the abstraction of gradient, curl, divergence to arbitrary manifolds. Differential forms extend Grassmann's exterior algebra (graded algebra of multi-covectors) from the tangent space at a point globally, i.e. to the entire manifold (how? bundles?).

1.5.1 Differential of a Function

Definition 1.1: Cotangent Space

The *cotangent space* to $\mathbb{R}^n$ at p, denoted $T^*_p(\mathbb{R}^n)$, is defined to be the dual space $(T_p(\mathbb{R}^n))^\vee$ of the tangent space $T_p(\mathbb{R}^n)$.

Thus, an element of the cotangent space $T^*_p(\mathbb{R}^n)$ is a **covector of linear functional on tangent space**.

Definition 1.2: Differential 1-form

A *covector field* or a *differential 1-form* on an open subset U of $\mathbb{R}^n$ is a function ω that assigns at each point p in U a covector $\omega_p \in T^*_p(\mathbb{R}^n)$

$$\begin{aligned}
\omega \colon U &\to \bigcup_{p \in U} T^*_p(\mathbb{R}^n) \\
p &\mapsto \omega_p \in T^*_p(\mathbb{R}^n)
\end{aligned} \tag{21}$$

We call a differential 1-form a *1-form* for short.

Definition 1.3: Differential

From any C^∞ function $f \colon U \to \mathbb{R}$, we can construct the 1-form $\mathrm{d}f$, called the *differential* of f, as follows: for $p \in U$ and $X_p \in T_p(U)$

$$(\mathrm{d}f)_p(X_p) \coloneqq X_p f \tag{22}$$

In words the differential of f is the application of X_p to f or **the directional derivative of f in the direction of the tangent vector defined by the coefficients of X_p**.

Let $x^1, \ldots, x^n$ be the standard coordinates on $\mathbb{R}$, $\{(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p\}$ their differentials defined

$$(\mathrm{d}x^i)_p(X_p) \coloneqq (X_p)(x^i)$$

and

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

be the standard basis for $T_p(\mathbb{R}^n)$.

Proposition 1.2

$\{(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p\}$ is the basis for $T^*_p(\mathbb{R}^n)$ dual to the coordinate basis for $T_p(\mathbb{R}^n)$.

Proof. By definition,

$$(\mathrm{d}x^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)=\left.\frac{\partial x^i}{\partial x^j}\right|_p=\delta^i_j$$

□

If ω is a 1-form on $U\in\mathbb{R}^n$ then by proposition (1.2), at each point $p\in U$

$$\omega_p=\sum a_i(p)(dx^i)_p$$

Note the lower index on $a_i(p)$ as opposed to the upper index on $X_p=\sum a^i(p)\partial_{x_i}\big|_p$. If $x\coloneqq x^1, y\coloneqq x^2, z\coloneqq x^3$, then $\mathrm{d}x,\mathrm{d}y,\mathrm{d}z$.

Proposition 1.3: $\mathrm{d}f$ in terms of coordinates

If $f\colon U\to\mathbb{R}$, then

$$\mathrm{d}f=\sum\frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \tag{23}$$

Proof.

$$(\mathrm{d}f)_p=\sum a_i(p)(\mathrm{d}x^i)_p$$

for some real numbers $a_i(p)$ depending on p. Thus

$$\begin{aligned}\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right)&=\sum_i a_i\,\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)\\&=\sum_i a_i\delta^i_j=a_j\end{aligned}\tag{24}$$

On the other hand, by the definition of the differential

$$\mathrm{d}f=\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial f}{\partial x^j}\tag{25}$$

Therefore

$$a_j=\frac{\partial f}{\partial x^j}\tag{26}$$

and hence

$$(\mathrm{d}f)_p=\left.\frac{\partial f}{\partial x^i}\right|_p(\mathrm{d}x^i)_p$$

□

1.5.2 Differential k-forms

Definition 1.4: Differential k-forms

More generally, a *differential form ω of degree k* is a function that at each point assigns an alternating k-linear function on $T_p(\mathbb{R}^n)$, i.e. $\omega_p\in A^k(T_p(\mathbb{R}^n))$.

A basis for $A^k(T_p(\mathbb{R}^n))$ is

$$(\mathrm{d}x^I)_p\coloneqq(\mathrm{d}x^{i_1})_p\wedge\cdots\wedge(\mathrm{d}x^{i_k})_p\tag{27}$$

where $1\le i_1<\cdots<i_k\le n$.

What is the nuance here?

Therefore, at each point $p\in U$, ω_p is a linear combination

$$\begin{gathered}\omega_p=\sum_I a_I(p)(\mathrm{d}x^I)_p\\1\le i_1<\cdots<i_k\le n\end{gathered}\tag{28}$$

and a k-form ω on open U is a linear combination

$$\omega=\sum_I a_I\,\mathrm{d}x^I\tag{29}$$

with function coefficients $a_I\colon U\to\mathbb{R}$. We say that a k-form ω is C^∞ on U if all of the coefficients a_I are C^∞ functions on U. Denote $\Omega^k(U)$ the vector space of k-forms on U. A 0-form on U assigns to each point p an element of $A^0(T_p(\mathbb{R}^n))\coloneqq\mathbb{R}$; thus, a 0-form on U is a constant function. Note there are no nonzero differential forms of degree $>n$ on U since if $\deg\mathrm{d}x^I>n$ then at least two of the component 1-forms of dx^I must be the same and therefore $dx^I=0$.

Definition 1.5: Wedge product of forms

The *wedge product of a k-form ω and ℓ-form τ* is defined pointwise

$$\begin{aligned}(\omega\wedge\tau)_p&\coloneqq\omega_p\wedge\tau_p\\\omega\wedge\tau&=\sum_{I,J}(a_Ib_J)\,\mathrm{d}x^I\wedge\mathrm{d}x^J\end{aligned}\tag{30}$$

where $I\cap J=\emptyset$.

Hence the wedge product is bilinear

$$\wedge\colon\Omega^k(U)\times\Omega^\ell\to\Omega^{k+\ell}(U)\tag{31}$$

The wedge product of forms is also anticommutative and associate (owing to the associativity and anticommutativity of the wedge product on multi-covectors) as therefore induces a graded algebra on $\Omega(U)\coloneqq\bigoplus_k\Omega^k(U)$.

Example 1.3. In the case of

$$\wedge\colon\Omega^0(U)\times\Omega^\ell(U)\to\Omega^\ell$$

we have the pointwise multiplication of a C^∞ function and a C^∞ ℓ-form

$$(f\wedge\omega)_p=f(p)\wedge\omega_p=f(p)\omega_p$$

Let x,y,z be the coordinates on $\mathbb{R}^3$. Then, the 1-forms are

$$f\,\mathrm{d}x+g\,\mathrm{d}y+h\,\mathrm{d}z$$

the 2-forms are

$$f\,\mathrm{d}y\wedge\mathrm{d}z+g\,\mathrm{d}x\wedge\mathrm{d}z+h\,\mathrm{d}x\wedge\mathrm{d}y$$

and the 3-forms are

$$f\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$$

1.5.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a 1-form and X is a vector field then

$$\begin{aligned} \omega(X)|_p &:= \omega_p(X_p) \\ \omega &= \sum a_i \, \mathrm{d}x^i \quad X = \sum b^i \frac{\partial}{\partial x^j} \\ \omega(X) &= \left(\sum a_i \, \mathrm{d}x^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right) \\ &= \sum a_i b^i \end{aligned} \tag{32}$$

1.5.4 Exterior Derivative

Definition 1.6: Exterior Derivative

The exterior derivative of a function $f \in C^\infty(U)$ is defined to be its differential $\mathrm{d}f$

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i$$

For $k \geq 1$, if $\omega = \sum_I a_I \, \mathrm{d}x^I$ is a k-form, then

$$\begin{aligned} \mathrm{d}\omega &:= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \\ &= \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} \, \mathrm{d}x^j \right) \wedge \mathrm{d}x^I \end{aligned} \tag{33}$$

Example 1.4. Let ω be the 1-form $f \, \mathrm{d}x + g \, \mathrm{d}y$ on R^2. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= \left(\frac{\partial f}{\partial x} \, \mathrm{d}x + \frac{\partial f}{\partial y} \, \mathrm{d}y \right) \wedge \mathrm{d}x \\ &\quad + \left(\frac{\partial g}{\partial x} \, \mathrm{d}x + \frac{\partial g}{\partial y} \, \mathrm{d}y \right) \wedge \mathrm{d}y \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

where we use that $\mathrm{d}x \wedge \mathrm{d}y = -\,\mathrm{d}y \wedge \mathrm{d}x$ and $\mathrm{d}x \wedge \mathrm{d}x = 0$.

Definition 1.7: Antiderivation

et $A = \bigoplus_k A^k$ be a graded algebra over a field K. An *antiderivation of the graded algebra* A is a k-linear map $D \colon A \to A$ such that for $a \in A^k, b \in A^\ell$

$$D(ab) = (Da)b + (-1)^k a Db \tag{34}$$

If there is an integer m such that D sends A^k to A^{k+m} for all k, then the antiderivation is of *degree* m.

Proposition 1.4: Properties of exterior differentiation

1. exterior differentiation is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\tau$$

2. $\mathrm{d}^2 = 0$

Proposition 1.5: Characterization of the exterior derivative

The properties of proposition (1.4) completely characterize exterior differentiation.

2 Appendix

Contents

2.1 Definitions 6
2.1.1 Linear Operator 6
2.1.2 Germs 6
2.1.3 Algebra 6
2.1.4 Module 6
2.1.5 Tensor Product 6
2.1.6 Wedge Product 7
2.1.7 Graded Algebra 7

2.1 Definitions

2.1.1 Linear Operator

A map $L\colon V \to W$ between vector spaces over a field K is a *linear operator* if

1. **distributivity**: $L(u+v) = L(u) + L(v)$
2. **homogeneity**: $L(rv) = rL(v)$

To emphasize the field, L is said to be $K-$linear.

2.1.2 Germs

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f\colon U \to \mathbb{R}$ is a C^∞ function. We say that $(f, U) \sim (g, U')$ if there is an open W such that $p \in W \subset U \cap U'$ and $f = g$ when restricted to W. The equivalence class $[(f, U)]$ of (f, U) is the *germ* of f at p. We write

$$C_p^\infty(\mathbb{R}^n) \coloneqq \{[(f, U)]\} \tag{35}$$

for the set all germs of C^∞ functions on $\mathbb{R}^n$ at p.

2.1.3 Algebra

An *algebra* over *field* K is a vector space A over K with a multiplication map

$$\mu\colon A \times A \to A \tag{36}$$

usually written $\mu(a, b) = a \cdot b$, such that μ is associative, distributive, and homogeneous, where homogeneity is defined:

1. **associativity**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. **distributivity**: $(a+b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + a \cdot c$
3. **homogeneity**: $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$

If A, A' are algebras then an *algebra homomorphism* is a linear operator L that respects algebra multiplication $L(ab) = L(a)L(b)$. It's the case that addition and multiplication of functions **induces addition and multiplication on the set of germs** C_p^∞, making it into an algebra over $\mathbb{R}^n$.

2.1.4 Module

If R is a commutative ring with identity, then a (left) $R-$*module* is an abelian group A with a scalar multiplication map

$$\mu\colon R \times A \to A \tag{37}$$

such that μ is

1. **associative**: $(rs)a = r(sa)$ for $r, s \in R$
2. **identity**: $1 \in R \implies 1a = a$
3. **distributive**: $(r+s)a = ra + sa$ and $r(a+b) = ra + rb$

If R is a field, then an $R-$module is a vector space over R; in this sense modules generalize vector space to scalars from a ring rather than a field.

Let A, A' be $R-$modules. An $R-$*module homomorphism* $f\colon A \to A'$ is a map that preserves both addition and scalar multiplication.

2.1.5 Tensor Product

Let f be k-linear function and g be an ℓ-linear function on a vector space V. Then, their *tensor product* is the $(k+\ell)$-linear function $f \otimes g$

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) \coloneqq f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+\ell}) \tag{38}$$

Example 2.1. (*Bilinear maps*) Let $\{e_i\}$ be a basis for a vector space V and $\{\alpha^j\}$ be the dual basis for $V^\vee$. Also let $\langle\,,\rangle\colon V \times V \to \mathbb{R}$ be a bilinear map on V. Set $g_{ij} = \langle e_i, e_j \rangle \in \mathbb{R}$. If

$$v = \sum v^i e_i \tag{39}$$

$$w = \sum w^i e_i \tag{40}$$

then $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$, where α are the coordinate functions a^i, a^j. By bilinearity, we can express $\langle\,,\rangle$ in terms of the tensor product

$$\begin{aligned} \langle v, w \rangle &= \sum_{ij} v^i w^j \langle e_i, e_j \rangle \\ &= \sum \alpha^i(v) \alpha^i(w) g_{ij} \\ &= \sum (\alpha^i \otimes \alpha^j)(v, w) \times g_{ij} \end{aligned} \tag{41}$$

Hence $\langle\,,\rangle = \sum g_{ij}\, \alpha^i \otimes \alpha^j$.

2.1.6 Wedge Product

Let f be k-linear function and g be an ℓ-linear function on a vector space V. If f, g are alternating[1] then we would like their product to be alternating as well: the *wedge product* or *exterior product* $f \wedge g$

$$
\begin{aligned}
f \wedge g &:= \frac{1}{k!l!} A(f \otimes g) && (42)\\
&:= \frac{1}{k!l!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}(\sigma)) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) && (43)\\
&\qquad\qquad g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})
\end{aligned}
$$

where $S_{k+\ell}$ is the *permutation group* on $k + \ell$ elements.

Note that the wedge product of three alternating functions f, g, h generalizes to

$$
f \wedge g \wedge h = \frac{1}{k!\ell!m!} = A(f \otimes g \otimes h) \tag{44}
$$

and any number of alternating functions.

Proposition 2.1

(*Wedge product of 1-covectors*) If $\{\alpha^i\}$ are linear functions on V and $v_i \in V$ then

$$
\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \ldots, v_k) = \det\Big([\alpha^i(v_j)]\Big) \tag{45}
$$

where $[\alpha^i(v_j)]$ is the matrix where the i, j-entry is $\alpha^i(v_j)$.

Proof.

$$
\begin{aligned}
&\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \ldots, v_k) = \\
&\quad A(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) && (46)\\
&= \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma)) \alpha^1(v_{\sigma(1)}) \wedge \cdots \wedge \alpha^k(v_{\sigma(k)}) && (47)\\
&= \det\Big([\alpha^i(v_j)]\Big) && (48)
\end{aligned}
$$

□

Let $\{e_i\}$ be a basis for a vector space V and let α_j be the dual basis for $V^\vee$. Define *multi-index notation*

$$
\begin{aligned}
I &:= (i_1, \ldots, i_k)\\
e_I &:= (e_{i_1}, \ldots, e_{i_k}) \qquad (49)\\
\alpha_I &:= \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}
\end{aligned}
$$

A k-linear function on V is completely determined by its values on all k-tuples e_I. If f is alternating then you only need all increasing multi-indices $1 \leq i_1 \leq \cdots \leq i_k \leq n$.

[1] An alternating function is one that changes signs if arguments are transposed (e.g. cross-product or determinant).

Lemma 2.1

Let I, J be increasing multi-indices of length k, then

$$
\alpha^I(e_J) = \delta^I_J := \begin{cases} 1 \text{ for } I = J \\ 0 \text{ for } I \neq J \end{cases} \tag{50}
$$

Proposition 2.2

The alternating k-linear functions α^I, $I := (i_1 < \cdots < i_k)$ form a basis for the space $A^k(V)$ of alternating k-linear functions on V.

Corollary 2.1

If the vector space V has dimension n, then the vector space $A^k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Corollary 2.2

If $k > \dim(V)$, then $A^k(V) = 0$.

2.1.7 Graded Algebra

Definition 2.1

A *graded algebra* A over a field K is one that can be written as a *direct sum*[2]

$$
A = \bigoplus_{n \in \mathbb{N}_0} A^n = A^0 \oplus A^1 \oplus A^2 \oplus \cdots
$$

such that the multiplication map μ

$$
\mu \colon A^k \times A^\ell \to A^{k+\ell}
$$

i.e. $A^k \cdot A^\ell \subset A^{k+\ell}$. Elements of any factor A^n of the decomposition are called homogeneous elements of degree n. $A = \bigoplus_k A^k$ is said to *graded commutative* or *anticommutative* if for all $a \in A^k, b \in A^\ell$

$$
ab = (-1)^{k\ell} ba
$$

Example 2.2. The *polynomial algebra*[3] $A = \mathbb{R}[x, y]$ is *graded by degree*; A^k consists of all homogeneous polyno-

[2] The direct sum of two abelian groups A and B is another abelian group $A \oplus B$ consisting of the ordered pairs (a, b) where $a \in A$ and $b \in B$. (Confusingly this ordered pair is also called the cartesian product of the two groups.) To add ordered pairs, we define the sum $(a, b) + (c, d)$ to be $(a + c, b + d)$; in other words addition is defined coordinate-wise. Note that $A^2 := A \oplus A$. A similar process can be used to form the direct sum of any two algebraic structures, such as rings, modules, and vector spaces.

mials of total degree k. It is a direct sum of A^i consisting of homogeneous polynomials of degree i e.g.

$$\left(x \in A^1\right) \times \left(x + y \in A^1\right) = \left(x^2 + xy \in A^2\right)$$

Definition 2.2

For a finite dimensional vector space V define

$$\begin{aligned} A^*(V) &\coloneqq \bigoplus_{k=0}^{\infty} A^k(V) \\ &= \bigoplus_{k=0}^{n} A^k(V) \end{aligned} \tag{51}$$

where $A^k(V)$ is the set of all k-linear covectors on V. With the wedge product as multiplication A^* becomes and anticommutative algebra called the ***exterior algebra*** or ***Grassmann algebra*** of covectors.

[3]The algebra of polynomials in two variables with coefficients and scalars in $\mathbb{R}$ where addition is degree-coefficient wise and "foil" multiplication

Index

$R-$module homomorphism, 6
Grassmann algebra, 8
exterior algebra, 8
1-form, 3

algebra, 6
algebra homomorphism, 6
antiderivation of the graded algebra, 5

Bilinear maps, 6

cotangent space, 3
covector field, 3
covectors, 2

derivation, 2
differential, 3
differential 1-form, 3
differential form ω of degree k, 4
direct sum, 7
directional derivative, 1
dual space, 2

exterior product, 7

field, 6

germ, 6
graded algebra, 7
graded by degree, 7
graded commutative, 7

isomorphically, 2

Leibniz rule, 2
linear operator, 6

module, 6
multi-index notation, 7

permutation group, 7
polynomial algebra, 7

tangent space, 1
tangent vectors, 1
tensor product, 6

vector field, 2

wedge product, 7
wedge product of a k-form ω and ℓ-form τ, 4