Notes

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December 12, 2019

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1 Differential Geometry

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1.1 Directional Derivative

Elements of the *tangent space* $T_p(\mathbb{R}^n)$ anchored at a point $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ can be visualized as arrows emanating from p. These arrows are called *tangent vectors* and represented by column vectors:

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \tag{1}$$

The line through a point p with direction $\boldsymbol{v}$ has parameterization

$$c(t) = \left(p^1 + tv^1, \ldots, p^n + tv^n\right) \tag{2}$$

If $f \in C^\infty$ in a neighborhood of p and $\boldsymbol{v}$ is a tangent vector at p, the *directional derivative* of f in the direction of $\boldsymbol{v}$ at p is defined

$$D_{\boldsymbol{v}} f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t))\bigg|_{t=0} \tag{3}$$

By the chain rule

$$\begin{aligned} D_{\boldsymbol{v}} f &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\bigg|_p \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} && (4) \\ &= \sum_{i=1}^{n} \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f}{\partial x^i}\bigg|_p && (5) \\ &= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}\bigg|_p && (6) \\ & && (7) \end{aligned}$$

The directional derivative operator at p is defined

$$D_{\boldsymbol{v}} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{8}$$

The association $\boldsymbol{v} \mapsto D_{\boldsymbol{v}}$ offers a way to *isomorphically* identify tangent vectors with operators on functions. The following makes this rigorous.

1.2 Derivations

For each tangent vector $\boldsymbol{v}$ at a point $p \in \mathbb{R}^n$, the directional derivative at p gives a map of vector spaces

$$D_{\boldsymbol{v}} \colon C_p^\infty \to \mathbb{R}$$

$D_{\boldsymbol{v}}$ is a linear map that satisfies the *Leibniz rule*

$$D_{\boldsymbol{v}}(fg) = (D_{\boldsymbol{v}} f)g(p) + f(p)(D_{\boldsymbol{v}} g) \tag{9}$$

because the partial derivative satisfy the product rule. In general, any linear map $L \colon C_p^\infty \to \mathbb{R}$ that satisfies the Leibniz rule is called a *derivation* at p. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. **This set is also a real vector space**.

So far we know directional derivatives $D_{\boldsymbol{v}}$ at p are derivations at p. Thus, there is a map

$$\begin{aligned} \phi \colon T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ \boldsymbol{v} &\mapsto D_{\boldsymbol{v}} \end{aligned}$$

Theorem 1.1

The linear map ϕ is an isomorphism of vector spaces.

The implication is that we may identify tangent vectors at p with derivations at p (by way of directional derivatives against germs). Under this isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \ldots, e_n\}$ for $T_p(\mathbb{R}^n)$ maps to

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \tag{10}$$

Therefore from now on we write a tangent vector as

$$\boldsymbol{v} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{11}$$

The point being that, while not as geometrically intuitive as arrows, $\mathcal{D}_p(\mathbb{R}^n)$ generalizes to manifolds.

1.3 Vector Fields

A *vector field* X on an open $U \subset \mathbb{R}^n$ is function that assigns to $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Notice, carefully, that the vector field assigns at each point a vector in the tangent space anchored at that point. Using the tangent basis (eqn. (10))

$$X \colon p \mapsto \sum_i a^i(p) \frac{\partial}{\partial x^i}\bigg|_p \tag{12}$$

Note that both the coefficients **and** the partial derivatives are evaluated at p. Having said that, we often omit p in the specification of a vector field when it clear from context.

Example 1.1. On $\mathbb{R}^n - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$\begin{aligned} X &= \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}^T \end{aligned}$$

See figure **??**

In general we can identify vector fields with parameterized column vectors

$$X = \sum_i a^i(p) \frac{\partial}{\partial x^i}\bigg|_p \leftrightarrow \begin{bmatrix} a^1(p) \\ \vdots \\ a^n(p) \end{bmatrix} \tag{13}$$

1.4 Dual Space

The *dual space* $V^\wedge$ of V is the set of all real-valued linear functions on V i.e. all $f \colon V \to \mathbb{R}$. Elements of $V^\wedge$ are called *covectors*.

Assume V is finite dimensional and let $\{e_1, \ldots, e_n\}$ be a basis V. Recall that $e_i \coloneqq \partial_{x_i}$. Then $X = \sum a^i \partial_{x_i}$ for all

$X \in T_p$. Let $\alpha^i : V \to \mathbb{R}$ be the linear function that picks out the ith coordinate of a **vector**, i.e. $\alpha^i(X) = a^i(p)$. Note that

$$\alpha^i(\partial_j) = \alpha^i(1 \cdot \partial_j) \tag{14}$$
$$= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{15}$$
$$= \delta^i_j \tag{16}$$

Note that position of indices is important – upper indices are for covectors.

Proposition 1.1

$\{\alpha^i\}$ form a basis for $V^\wedge$.

Proof. We first prove that $\{\alpha^i\}$ span $V^\wedge$. If $f \in V^\wedge$ and $X = \sum a^i \partial_{x_i} \in V$, then

$$f(X) = \sum a^i f(\partial_{x_i}) \tag{17}$$
$$= \sum \alpha^i(X) f(\partial_{x_i}) \tag{18}$$
$$= \sum f(\partial_{x_i}) \alpha^i(X) \tag{19}$$

which shows that any f can be expanded as a linear sum of α^i. To show linear independence, suppose $\sum c_i \alpha^i = 0$ with at least one c_i non-zero. Applying this to an arbitrary ∂_{x_i} gives

$$0 = \left(\sum_i c_i \alpha^i \right) (\partial_{x_i}) = \sum_i c_i \alpha^i(\partial_{x_i}) = \sum_i c_i \delta^i_j = c_j \tag{20}$$

which is a contradiction. Hence α^i are linearly independent. □

This basis $\{\alpha^i\}$ for $V^\wedge$ is said to be *dual* to the basis $\{\partial_{x_i}\}$ for V.

Example 1.2. Coordinate functions With respect to a basis $\{\partial_{x_i}\}$ for V, every $X \in V$ can be written uniquely as a linear combination $X = \sum a^i \partial_{x_i}$ with $a^i \in \mathbb{R}$. Let $\{\alpha^i\}$ be the dual basis (i.e. the basis for $V^\wedge$). Then

$$\alpha^i(X) = \alpha^i \left(\sum_j a^j \partial_{x_j} \right)$$
$$= \sum_j a^j \alpha^i(\partial_{x_j})$$
$$= \sum_j a^j \delta^i_j$$
$$= a^i$$

Thus, the dual basis $\{\alpha^i\}$ to $\{\partial_{x_i}\}$ is the set of coordinate functions. The sense here is that since tangent vectors are directional derivatives (i.e. operators on functions) and the dual space is a mapping from those operators to $\mathbb{R}$, then a mapping from operators to scalars means hitting an operator with a function (or vice-versa). **And** the coordinate functions are constant with respect to each other coordinate (and hence partials wrt them are naturally zero).

1.5 Differential Forms on $\mathbb{R}^n$

A differential k-form assigns a k-covector from the dual space at each point p. The wedge product (alternation of tensor product) of differential forms is defined pointwise (as the wedge product of multi-covectors). Differential forms exist on an open set (why?) there is a notion of differentiation (called exterior derivative). Exterior derivative is coordinate independent and intrinsic to a manifold; it is the abstraction of gradient, curl, divergence to arbitrary manifolds. Differential forms extend Grassmann's exterior algebra (graded algebra of multi-covectors) from the tangent space at a point globally, i.e. to the entire manifold (how? bundles?).

1.5.1 Differential of a Function

Definition 1.1: Cotangent Space

The *cotangent space* to $\mathbb{R}^n$ at p, denoted $T^*_p(\mathbb{R}^n)$, is defined to be the dual space $(T_p(\mathbb{R}^n))^\vee$ of the tangent space $T_p(\mathbb{R}^n)$.

Thus, an element of the cotangent space $T^*_p(\mathbb{R}^n)$ is a **covector of linear functional on tangent space**.

Definition 1.2: Differential 1-form

A *covector field* or a *differential 1-form* on an open subset U of $\mathbb{R}^n$ is a function ω that assigns at each point p in U a covector $\omega_p \in T^*_p(\mathbb{R}^n)$

$$\begin{aligned} \omega \colon U &\to \bigcup_{p \in U} T^*_p(\mathbb{R}^n) \\ p &\mapsto \omega_p \in T^*_p(\mathbb{R}^n) \end{aligned} \tag{21}$$

We call a differential 1-form a *1-form* for short.

Definition 1.3: Differential

From any C^∞ function $f \colon U \to \mathbb{R}$, we can construct the 1-form $\mathrm{d}f$, called the *differential* of f, as follows: for $p \in U$ and $X_p \in T_p(U)$

$$(\mathrm{d}f)_p(X_p) := X_p f \tag{22}$$

In words the differential of f is the application of X_p to f or **the directional derivative of f in the direction of the tangent vector defined by the coefficients of X_p.**

Let $x^1, \dots, x^n$ be the standard coordinates on $\mathbb{R}$, $\{(\mathrm{d}x^1)_p, \dots, (\mathrm{d}x^n)_p\}$ their differentials defined

$$(\mathrm{d}x^i)_p(X_p) := (X_p)(x^i)$$

and

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

be the standard basis for $T_p(\mathbb{R}^n)$.

Proposition 1.2

$\{(\mathrm{d}x^1)_p, \dots, (\mathrm{d}x^n)_p\}$ is the basis for $T_p^*(\mathbb{R}^n)$ dual to the coordinate basis for $T_p(\mathbb{R}^n)$.

Proof. By definition,

$$(\mathrm{d}x^i)_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \frac{\partial x^i}{\partial x^j}\bigg|_p = \delta^i_j$$

□

If ω is a 1-form on $U \in \mathbb{R}^n$ then by proposition (1.2), at each point $p \in U$

$$\omega_p = \sum a_i(p)(dx^i)_p$$

Note the lower index on $a_i(p)$ as opposed to the upper index on $X_p = \sum a^i(p)\partial_{x_i}\big|_p$. If $x := x^1, y := x^2, z := x^3$, then $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$.

Proposition 1.3: $\mathrm{d}f$ in terms of coordinates

If $f\colon U \to \mathbb{R}$, then

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^i} \mathrm{d}x^i \tag{23}$$

Proof.

$$(\mathrm{d}f)_p = \sum a_i(p)(\mathrm{d}x^i)_p$$

for some real numbers $a_i(p)$ depending on p. Thus

$$\begin{aligned} \mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) &= \sum_i a_i \, \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) \\ &= \sum_i a_i \delta^i_j = a_j \end{aligned} \tag{24}$$

On the other hand, by the definition of the differential

$$\mathrm{d}f = \left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j} \tag{25}$$

Therefore

$$a_j = \frac{\partial f}{\partial x^j} \tag{26}$$

and hence

$$(\mathrm{d}f)_p = \frac{\partial f}{\partial x^i}\bigg|_p (\mathrm{d}x^i)_p$$

□

1.5.2 Differential k-forms

Definition 1.4: Differential k-forms

More generally, a *differential form ω of degree k* is a function that at each point assigns an alternating k-linear function on $T_p(\mathbb{R}^n)$, i.e. $\omega_p \in A^k(T_p(\mathbb{R}^n))$.

A basis for $A^k(T_p(\mathbb{R}^n))$ is

$$(\mathrm{d}x^I)_p := (\mathrm{d}x^{i_1})_p \wedge \cdots \wedge (\mathrm{d}x^{i_k})_p \tag{27}$$

where $1 \le i_1 < \cdots < i_k \le n$.

What is the nuance here?

Therefore, at each point $p \in U$, ω_p is a linear combination

$$\begin{gathered} \omega_p = \sum_I a_I(p)(\mathrm{d}x^I)_p \\ 1 \le i_1 < \cdots < i_k \le n \end{gathered} \tag{28}$$

and a k-form ω on open U is a linear combination

$$\omega = \sum_I a_I \, \mathrm{d}x^I \tag{29}$$

with function coefficients $a_I\colon U \to \mathbb{R}$. We say that a k-form ω is C^∞ on U if all of the coefficients a_I are C^∞ functions on U. Denote $\Omega^k(U)$ the vector space of k-forms on U. A 0-form on U assigns to each point p an element of $A^0(T_p(\mathbb{R}^n)) := \mathbb{R}$; thus, a 0-form on U is a constant function. Note there are no nonzero differential forms of degree $> n$ on U since if $\deg \mathrm{d}x^I > n$ then at least two of the component 1-forms of dx^I must be the same and therefore $dx^I = 0$.

Definition 1.5: Wedge product of forms

The *wedge product of a k-form ω and ℓ-form τ* is defined pointwise

$$\begin{gathered} (\omega \wedge \tau)_p := \omega_p \wedge \tau_p \\ \omega \wedge \tau = \sum_{I,J} (a_I b_J) \, \mathrm{d}x^I \wedge \mathrm{d}x^J \end{gathered} \tag{30}$$

where $I \cap J = \emptyset$.

Hence the wedge product is bilinear

$$\wedge\colon \Omega^k(U) \times \Omega^\ell \to \Omega^{k+\ell}(U) \tag{31}$$

The wedge product of forms is also anticommutative and associate (owing to the associativity and anticommutativity of the wedge product on multi-covectors) as therefore induces a graded algebra on $\Omega(U) := \bigoplus_k \Omega^k(U)$.

Example 1.3. In the case of

$$\wedge\colon \Omega^0(U) \times \Omega^\ell(U) \to \Omega^\ell$$

we have the pointwise multiplication of a C^∞ function and a C^∞ ℓ-form

$$(f \wedge \omega)_p = f(p) \wedge \omega_p = f(p)\omega_p$$

Let x, y, z be the coordinates on $\mathbb{R}^3$. Then, the 1-forms are

$$f \,\mathrm{d}x + g \,\mathrm{d}y + h \,\mathrm{d}z$$

the 2-forms are

$$f \,\mathrm{d}y \wedge \mathrm{d}z + g \,\mathrm{d}x \wedge \mathrm{d}z + h \,\mathrm{d}x \wedge \mathrm{d}y$$

and the 3-forms are

$$f \,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

1.5.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a 1-form and X is a vector field then

$$\begin{aligned} \omega(X)|_p &:= \omega_p(X_p) \\ \omega &= \sum a_i \,\mathrm{d}x^i \quad X = \sum b^i \frac{\partial}{\partial x^j} \\ \omega(X) &= \left(\sum a_i \,\mathrm{d}x^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right) \\ &= \sum a_i b^i \end{aligned} \tag{32}$$

1.5.4 Exterior Derivative

Definition 1.6: Exterior Derivative

The exterior derivative of a function $f \in C^\infty(U)$ is defined to be its differential $\mathrm{d}f$

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^i} \,\mathrm{d}x^i$$

For $k \geq 1$, if $\omega = \sum_I a_I \,\mathrm{d}x^I$ is a k-form, then

$$\begin{aligned} \mathrm{d}\omega &:= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \\ &= \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} \,\mathrm{d}x^j\right) \wedge \mathrm{d}x^I \end{aligned} \tag{33}$$

Example 1.4. Let ω be the 1-form $f \,\mathrm{d}x + g \,\mathrm{d}y$ on R^2. Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y && (34) \\ &= \left(\frac{\partial f}{\partial x} \,\mathrm{d}x + \frac{\partial f}{\partial y} \,\mathrm{d}y\right) \wedge \mathrm{d}x && (35) \\ &\quad + \left(\frac{\partial g}{\partial x} \,\mathrm{d}x + \frac{\partial g}{\partial y} \,\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y && (36) \end{aligned}$$

where we use that $\mathrm{d}x \wedge \mathrm{d}y = -\,\mathrm{d}y \wedge \mathrm{d}x$ and $\mathrm{d}x \wedge \mathrm{d}x = 0$.

Definition 1.7: Antiderivation

et $A = \bigoplus_k A^k$ be a graded algebra over a field K. An *antiderivation of the graded algebra* A is a k-linear map $D\colon A \to A$ such that for $a \in A^k, b \in A^\ell$

$$D(ab) = (Da)b + (-1)^k aDb \tag{37}$$

If there is an integer m such that D sends A^k to A^{k+m} for all k, then the antiderivation is of *degree* m.

Proposition 1.4: Properties of exterior differentiation

1. exterior differentiation is an antiderivation of degree 1:
$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\tau$$
2. $\mathrm{d}^2 = 0$

Proposition 1.5: Characterization of the exterior derivative

The properties of proposition (1.4) completely characterize exterior differentiation.

1.5.5 Closed and Exact Forms

A k-form ω is a *closed form* if $\mathrm{d}\omega = 0$. ω is an *exact form* if there is a $k-1$-form τ such that $\omega = \mathrm{d}\tau$. Since $\mathrm{d}(\mathrm{d}\tau) = 0$, every exact form is closed.

Example 1.5. A closed 1-form on the punctured plane Define ω on the manifold $\mathbb{R}^2 - \{\mathbf{0}\}$ by

$$\omega(x, y) = \frac{-y}{x^2 + y^2} \,\mathrm{d}x + \frac{x}{x^2 + y^2} \,\mathrm{d}y$$

To show that ω is closed we take the exterior derivative using example (36):

$$\begin{aligned} \mathrm{d}\omega &= \left(\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right) \mathrm{d}x \wedge \mathrm{d}y \\ &= 0 \end{aligned} \tag{38}$$

Definition 1.8: Differential Complex

A collection of vector spaces $\{V^0, V^1, \ldots\}$ with linear maps $d_k \colon V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *differential complex* or a *cochain complex*. For any open subset $U \subset \mathbb{R}^3$, the exterior derivative makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the *de Rham complex of* U:

$$0 \to \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots$$

The closed forms are the elements of the kernel of d and the exact forms are the elements of the image of d.

1.5.6 Applications to Vector Calculus

Recall the three operators grad, curl, div ($\boldsymbol{\nabla}, \boldsymbol{\nabla}\times, \boldsymbol{\nabla}\cdot$) on scalar fields and vector fields (i.e. scalar valued functions and vector valued functions) over $\mathbb{R}^3$:

$$\begin{aligned}
\boldsymbol{\nabla} f &= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \\
\boldsymbol{\nabla} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix} \\
\boldsymbol{\nabla} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\end{aligned} \tag{39}$$

Note we can identity 1-forms with vector fields:

$$P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Similarly 2-forms on $\mathbb{R}^3$

$$P\,\mathrm{d}x \wedge \mathrm{d}y + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

and 3-forms on U can be identified with functions U

$$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \longleftrightarrow f$$

In terms of these identifications the exterior derivative of 0-form f is the 1-form $\mathrm{d}f$ or $\boldsymbol{\nabla} f$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \longleftrightarrow \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \boldsymbol{\nabla} f$$

the exterior derivative of a vector field $\begin{bmatrix} P\ Q\ R \end{bmatrix}^T$ (i.e. 1-form) is the 2-form $\boldsymbol{\nabla} \times \begin{bmatrix} P\ Q\ R \end{bmatrix}^T$

$$\begin{aligned}
&\mathrm{d}(P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z) = \\
&\quad + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z \\
&\quad - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \\
&\quad \longleftrightarrow \\
&\quad \boldsymbol{\nabla} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}
\end{aligned} \tag{40}$$

and the exterior derivative of a 2-form is

$$\begin{aligned}
&\mathrm{d}(P\,\mathrm{d}x \wedge \mathrm{d}y + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y) = \\
&\left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial R}{\partial z}\,\mathrm{d}z\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\longleftrightarrow \boldsymbol{\nabla} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}
\end{aligned} \tag{41}$$

Thus, the exterior derivatives on 0-forms (functions) is the grad operator, on 1-forms is the curl operator, and on 2-forms is the divergence operator.

$$\begin{tikzcd}
\Omega^0(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^1(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^2(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^3(U) \arrow[d, "\cong"'] \\
C^\infty(U) \arrow[r, "\boldsymbol{\nabla}"'] & \mathfrak{X}(U) \arrow[r, "\boldsymbol{\nabla}\times"'] & \mathfrak{X}(U) \arrow[r, "\boldsymbol{\nabla}\cdot"'] & C^\infty(U)
\end{tikzcd}$$

where $\mathfrak{X}(U)$ is the *Lie algebra* of C^∞ vector fields on U.

Proposition 1.6: grad, curl, div properties

1. $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$
2. $\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \begin{bmatrix} P\ Q\ R \end{bmatrix}^T\right) = 0$
3. On $\mathbb{R}^3$, a vector field $\mathbf{F}$ is the gradient of some scalar function iff $\boldsymbol{\nabla} \times \mathbf{F} = 0$

Properties (1,2) express that $\mathrm{d}^2 = 0$ Property (3) expresses the fact that a 1-form on $\mathbb{R}^3$ is exact iff it is closed; it need not be true on a region other than $\mathbb{R}^3$. It turns out that whether proposition (3) is true depends on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k(U) \coloneqq \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}} \tag{42}$$

called the kth *de Rham cohomolgy* of U. The generalization of proposition (3) to any differential on $\mathbb{R}^n$ is called the *Poincare lemma*: for $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact. This is equivalent to the vanishing of the kth de Rham cohomology $H^k(\mathbb{R}^n)$ for $k \geq 1$.

1.5.7 Convention on Subscripts and Superscripts

Vector fields $e_1, e_2, \ldots$have subscripts and differential forms $\omega^1, \omega^2, \ldots$ have superscripts. Coordinate functions $x^1, x^2, \ldots,$ being 0-forms, have superscripts. Their differentials $\mathrm{d}x^i$ should also. Coordinate vector fields $\frac{\partial}{\partial x^i}$ are considered to have subscripts because the index is in the denominator. Coefficient functions have subscripts or subscripts depending on whether they're coefficients functions for vector fields or forms. This allows for "conservation of indices": if $X = \sum a^i \partial_{x^i}$ and $\omega = \sum b_j \, \mathrm{d}x^j$ then

$$\omega(X) = \left(\sum b_j \, \mathrm{d}x^j\right)\left(\sum a^i \frac{\partial}{\partial x^i}\right) = \sum b_i a^i$$

1.6 Manifolds

Definition 1.9: Locally Euclidean

A topological space M is *locally Euclidean of dimension n* if for every $p \in M$ there exists a neighborhood U such that there is a *homeomorphism*[1] ϕ from U **onto** an opensubset of $\mathbb{R}^n$. The pair $(U, \phi\colon U \to \mathbb{R}^n)$ is called a *chart*, with U being the *coordinate neighborhood* and ϕ the *coordinate system*.

Definition 1.10: Topological Manifold

A *topological manifold* is a *Hausdorff*[2], *second countable*[3], locally Euclidean space.

1.6.1 Compatible Charts

Suppose $(U, \phi\colon U \to \mathbb{R}^n)$ and $(V, \psi\colon V \to \mathbb{R}^n)$ are two charts of a topological manifold; since $U \cap V$ is open and ϕ is a homeomorphism onto an open subset, the image $\phi(U \cap V)$ is also an open subset (similarly $\psi(U \cap V)$).

Definition 1.11: Charts

Two charts $(U, \phi\colon U \to \mathbb{R}^n)$ and $(V, \psi\colon V \to \mathbb{R}^n)$ of a topological manifold are C^∞*-compatible* if the two composed maps

$$\begin{aligned} \phi \circ \psi^{-1}&\colon \psi(U \cap V) \to \phi(U \cap V) \\ \psi \circ \phi^{-1}&\colon \phi(U \cap V) \to \psi(U \cap V) \end{aligned} \tag{43}$$

See figure ??. These two maps $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are called the *transition functions* between the charts;

$$\begin{aligned} \phi \circ \psi^{-1}&\colon \mathbb{R}^n \to U \cap V \to \mathbb{R}^n \\ \psi \circ \phi^{-1}&\colon \mathbb{R}^n \to U \cap V \to \mathbb{R}^n \end{aligned} \tag{44}$$

Definition 1.12: Atlas

A C^∞ *atlas* on a topological manifold M is a collection $\mathfrak{U} \coloneqq \{(U_i, \phi_i)\}$ of pairwise C^∞-compatible charts that *cover* M, i.e. such that $M = \bigcup_i U_i$.

Example 1.6. A C^∞ atlas on a circle The unit circle S^1 in the complex plane $\mathbb{C}$ maybe described as

$$\{e^{\mathrm{i}} \in \mathbb{C} | 0 \leq t \leq 2\pi\}$$

Let U_1, U_2 be

$$\begin{aligned} U_1 &\coloneqq \{e^{\mathrm{i}t} \in \mathbb{C} | -\pi < t < \pi\} \\ U_2 &\coloneqq \{e^{\mathrm{i}t} \in \mathbb{C} | 0 < t < 2\pi\} \end{aligned} \tag{45}$$

and define

$$\begin{aligned} \phi_1(e^{\mathrm{i}t}) &= t \quad -\pi < t < \pi \\ \phi_2(e^{\mathrm{i}t}) &= t \quad 0 < t < 2\pi \end{aligned} \tag{46}$$

See figure ??. Both ϕ_1, ϕ_2 are homemorphisms onto their respective images; thus, U_1, ϕ_1 and U_2, ϕ_2 are charts on S^1. The intersection $U_1 \cap U_2$ consists of two disjoint connected components

$$\begin{aligned} A &\coloneqq \{e^{\mathrm{i}t} \in \mathbb{C} | -\pi < t < 0\} \\ B &\coloneqq \{e^{\mathrm{i}t} \in \mathbb{C} | 0 < t < \pi\} \end{aligned} \tag{47}$$

with

$$\begin{aligned} \phi_1(U_1 \cap U_2) = \phi_1(A \sqcup B) = \phi_1(A) \sqcup \phi_2(B) = (-\pi, 0) \sqcup (0, \pi) \\ \phi_2(U_1 \cap U_2) = \phi_1(B \sqcup A) = \phi_1(B) \sqcup \phi_2(A) = (0, \pi) \sqcup (\pi, 2\pi) \end{aligned} \tag{48}$$

where $\sqcup$ means disjoint union. The transition function

$$\left(\phi_2 \circ \phi_1^{-1}\right)(t) = \begin{cases} t + 2\pi & \text{for } t \in (-\pi, 0) \\ t & \text{for } t \in (0, \pi) \end{cases}$$

and similarly

$$\left(\phi_1 \circ \phi_2^{-1}\right)(t) = \begin{cases} t & \text{for } t \in (0, \pi) \\ t - 2\pi & \text{for } t \in (\pi, 2\pi) \end{cases}$$

[1] A homeomorphism is a continuous function between topological spaces that has a continuous inverse function.

[2] A Hausdorff space, separated space space is a topological space where for any two distinct points there exists a neighbourhood of each which is disjoint from the neighbourhood of the other.

[3] A topological space T is second-countable if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of T such that any open subset of T can be written as a union of elements of some subfamily of $\mathcal{U}$.

Therefore, since the transition functions are C^∞, the charts are C^∞ compatible and form an atlas on S^1.

Although compatibility is reflexive and symmetric it is not transitive because we don't know anything about mutual intersections. On the otherhand

Lemma 1.1

Let $\{(U_i, \phi_i)\}$ be an atlas for a topological manifold. If two other charts (not in the atlas) $(V, \psi), (U, \phi)$ are both compatible with the atlas (i.e. with all charts in the atlas), then they are mutually compatible.

1.6.2 Smooth Manifolds

An atlas $\mathfrak{M}$ on a topological manifold is said to be *maximal* if it is not strictly contained in another atlas.

Definition 1.13: Smooth Manifold

A *smooth C^∞ manifold* is a topological manifold M together with a maximal atlas. The maximal atlas is called a *differentiable structure* on M. M is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is called a *curve*, a 2-dimensional manifold is called a *surface*, and an n-dimensional manifold is called an n-manifold.

In the context of manifolds, we denote the standard coordinates on $\mathbb{R}^n$ by $r^1, \dots, r^n$. If (U, ϕ) is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = (x^1, \dots, x^n)$. Thus, $x^i(p) := (r^i \circ \phi)(p)$ is a point in $\mathbb{R}^n$. The functions x^i are called the *local coordinates on* U. The sense here is that x^i tell you where you are on the manifold in a standardized way, given some intrinsic description of where you are on the manifold. Think of how lat/lon tell you where you are on the globe given some intrinsic identification of where you are on the globe. This understanding gives sense to the words chart and atlas. Abusing notation, sometimes $(x^1, \dots, x^n)$ (sans p) stands for both the coordinates on U or for a point in $\mathbb{R}^n$.

1.6.3 Examples of Smooth Manifolds

Example 1.7. Graph of a Smooth Function For a subset $A \subset \mathbb{R}^n$ and a function $f\colon A \to \mathbb{R}^m$ the *graph of f* is defined

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^m\}$$

If U is an open subset $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^n$ is C^∞, then the two maps

$$\begin{aligned} \phi\colon \Gamma(f) \to U \quad & (x, f(x)) \mapsto x \\ (1, f)\colon \to \Gamma(f) \quad & x \mapsto (x, f(x)) \end{aligned}$$

constitute a homemorphism. Hence, $\Gamma(f)$ has an atlas with a single $(\Gamma(f), \phi)$ and is therefore a C^∞ manifold.

Example 1.8. General Linear group Let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is defined

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &:= \{A \in \mathbb{R}^{n\times n} \,|\, \det(A) \neq 0\} \\ &= \det{}^{-1}(\mathbb{R} - \{0\}) \end{aligned} \tag{49}$$

Since the determinant function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous, $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ is therefore a manifold.

Example 1.9. Unit circle in the xy-plane Take S^1 as the unit circle in the real plane $\mathbb{R}^2$ defined by $x^2 + y^2 = 1$. We can cover S^1 by four open sets: the upper and lower semicircles U_1, U_2 and the left and right semicircles U_3, U_4 (see figure **??**). On U_1, U_2 the coordinate function x is a homeomorphism onto the open interval $(-1, 1)$ on the x-axis. Thus, $\phi_i(x, y) := x$. Similarly, on U_3, U_4, y is a homeomorphism onto the open interval $(-1, 1)$ on the y-axis, and so $\phi_i(x, y) := y$. You can check that on $U_i \cap U_j$, the transition function $\phi_j \circ \phi_i^{-1}$ is C^∞. For example, on $U_1 \cap U_3$

$$(\phi_3 \circ \phi_1^{-1})(x) = \phi_3(x, \sqrt{1 - x^2}) = \sqrt{1 - x^2}$$

while on $U_2 \cap U_4$

$$(\phi_4 \circ \phi_2^{-1})(x) = \phi_4(x, -\sqrt{1 - x^2}) = -\sqrt{1 - x^2}$$

Thus, $\{(U_i, \phi_i)\}_{i=1}^4$ is a C^∞ atlas on S^1.

Proposition 1.7: An atlas for a product manifold

If $\{(U_i, \phi_i)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M, N of dimensions, respectively, then the collection

$$\{(U_i \times V_j, (\phi_i, \phi_j))\colon U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n\} \tag{50}$$

is a C^∞ atlas on $M \times N$. Therefore $M \times N$ is a C^∞ manifold of dimension $m + n$.

There n-dimensional torus $S^1 \times \cdots \times S^1$ is a manifold.

1.6.4 Smooths Maps on a Manifold

Definition 1.14: Smooth at a point to $\mathbb{R}$

Let M be a smooth manifold of dimension n. A function $f\colon M \to \mathbb{R}$ is said to be C^∞ or *smooth a point p* in M if there is a chart (U, ϕ) containing p such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U) \subset \mathbb{R}^n$, is C^∞ at $\phi(p)$. To summarize f is C^∞ if

$$f \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}$$

See figure **??**.

Definition 1.15: Pullback

Let $F\colon N \to M$ be a map and h a function M. The *pullback* of h by F, denoted F^*h, is the composite function $h \circ F$.

Thus, a function f on M is C^∞ on a chart (U, ϕ) iff $(\phi^{-1})^* f \equiv f \circ \phi^{-1}$ is C^∞ on $\phi(U)$.

Definition 1.16: Smooth at a point

Let M, N be manifolds of dimension m, n. A continuous map $F\colon N \to M$ is C^∞ at a point $p \in N$ if there are charts $(V, \psi), (U, \phi)$ about $F(p) \in M$ and $p \in N$ such that $\psi \circ F \psi \phi^{-1}$ is C^∞ at $\phi(p)$. To summarize F is C^∞ if

$$\psi \circ F \circ \phi^{-1}\colon \phi(F^{-1}(V) \cap U) \subset \mathbb{R}^n \to \mathbb{R}^m$$

See figure **??**.

Proposition 1.8: Composition of C^∞ maps

If $F\colon N \to M$ and $G\colon M \to P$ are both C^∞ maps of manifolds, then the composite $G \circ F\colon N \to P$ is C^∞.

Definition 1.17: Diffeomorphism

A *diffeomorphism* of manifolds is a bjiective C^∞ map $F\colon N \to M$ whose inverse F^{-1} is also C^∞.

Proposition 1.9

If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi\colon U \to \phi(U)$ is a diffeomorphism.

Definition 1.18: Lie group

A *Lie group* is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu\colon G \times G \to G$$

and the inverse map

$$\iota\colon G \to G \quad \iota(x) := x^{-1}$$

are both C^∞. Similarly, a *topological group* is a topological space having a group structure such that multiplication and inverse maps are both continuous.

Example 1.10. Recall the definition of $\mathrm{GL}(n, \mathbb{R})$. As an open subset of $\mathbb{R}^{n\times n}$, it is a manifold. Since the (i, j)-entry of the product of two matrices A, B in $\mathrm{GL}(n, \mathbb{R})$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

is a polynomial in the coordinates of A and B. Therefore matrix multiplication

$$\mu\colon \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

is a C^∞ map. Furthermore, by Cramer's rule

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det A}((j, i) - \text{minor of } A)$$

which is a C^∞ function of the a_{ij}s provided $\det A \neq 0$. Therefore, $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

1.6.5 Partial Derivatives

Let (U, ϕ) be a chart and f a C^∞ function on the manifold. As a function into $\mathbb{R}^n$, ϕ has n components $x^1, \ldots, x^n$. Therefore, if $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then

$$x^i = r^i \circ \phi$$

What this means is ϕ maps to some point $(\phi^1(p), \ldots, \phi^n(p))$ in R^n. The projection to the ith standard coordinate on $\mathbb{R}^n$ is $r^i \circ \phi$, i.e. just pick out the component of $(\phi^1(p), \ldots, \phi^n(p))$. This direct path from U to the ith standard coordinate of $\mathbb{R}^n$ is called a local coordinate for U (since it's only valid in a small neighborhood of p). Then the collection $(x^1, \ldots, x^n)$ are the local coordinates on U (but don't forget that the coordinates map from $U \to \mathbb{R}^n$).

Definition 1.19: Partial derivative

For $p \in U$, we define the *partial derivative* $\frac{\partial f}{\partial x^i}$ *of f with respect to x^i at p*

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p f := \left.\frac{\partial f}{\partial x^i}\right|_p \\ := \left.\frac{\partial (f \circ \phi^{-1})}{\partial r^i}\right|_{\phi(p)} \\ := \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}) \end{aligned} \tag{51}$$

Thus, as functions on $\phi(U)$ (and in terms of the local coordinates on U)

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial f \circ \phi^{-1}}{\partial r^i} \tag{52}$$

and therefore the partial derivative $\partial f / \partial x^i$ is C^∞ on U because its pullback

$$(\phi^{-1})^* \frac{\partial f}{\partial x^i}$$

is C^∞.

Proposition 1.10: Partial derivatives are dual

Suppose $U, x^1, \dots, x^n$ (with local coordinates $x^1, \dots, x^n$) is a chart on a manifold. Then $\partial x^i / \partial x^j = \delta^i_j$.

Proof. At a point $p \in U$, by the above definition of $\partial/\partial x^j\big|_p$

$$\begin{aligned}\frac{\partial x^i}{\partial x^j}\bigg|_p &= \frac{\partial x^i \circ \phi^{-1}}{\partial r^j}\bigg|_{\phi(p)} \\ &= \frac{\partial(r^i \circ \phi \circ \phi^{-1})}{\partial r^j}\bigg|_{\phi(p)} \\ &= \frac{\partial r^i}{\partial r^j}\bigg|_{\phi(p)} = \delta^i_j\end{aligned} \tag{53}$$

□

Definition 1.20: Jacobian

Let $F\colon N \to M$ be a smooth map, and let $(U, \phi) \equiv (U, x^1, \dots, x^n)$ and $(V, \psi) \equiv (U, y^1, \dots, y^n)$ be charts on N, M respectively such that $F(U) \subset V$. Denote by

$$(F_i \coloneqq y^i \circ F \equiv r^i \circ \psi \circ F)\colon U \to \mathbb{R}$$

the ith component of F in the chart V, ψ. Then the matrix $\left[\frac{\partial F^i}{\partial x^j}\right]$ is called the *Jacobian matrix of* F relative to the charts U, ϕ and (V, ψ).

A diffeomorphism $F\colon U \to F(U) \subset \mathbb{R}^n$ may be thought of as coordinate system on U. We say that a C^∞ map $F\colon N \to M$ is *locally invertible* or a *local diffeomorphism* at $p \in N$ if p has a neighborhood U on which F is a diffeomorphism. Given n smooth functions $F^1, \dots, F^n$ in a neighborhood of p one would like to know whether they form a coordinate system. This is equivalent to whether $F = (F^1, \dots, F^n)$ is a local diffeomorphism at p.

Theorem 1.2: Inverse function theorem $\mathbb{R}^n$

Let $F\colon W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset $W \subset \mathbb{R}^n$. For any point $p \in W$, the map F is locally invertible iff the determinant of the Jacobian

$$\det\left[\frac{\partial F^i}{\partial r^j}\bigg|_p\right]$$

is not zero.

Theorem 1.3: Inverse function theorem for manifolds

Let $F\colon N \to M$ be a C^∞ map between manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = (U, x^1, \dots, x^n)$, $(V, \psi) = (U, y^1, \dots, y^n)$ about p and $F(p)$ respectively and $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p iff

$$\det\left[\frac{\partial F^i}{\partial x^j}\bigg|_p\right]$$

is not zero.

Corollary 1.1

Let N be a manifold of dimension n. A set of smooth functions $F^1, \dots, F^n$ defined on a coordinate neighborhood $U, x^1, \dots, x^n$ of a point $p \in N$ forms a coordinate system about p iff the Jacobian determinant

$$\det\left[\frac{\partial F^i}{\partial x^j}\bigg|_p\right]$$

is not zero.

1.7 Tangent Space

The tangent space to a manifold at a point is the vector space of derivations (germs/directional derivatives) at the point. A smooth map of manifolds induces a linear map, called its differential, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian. In this sense, the differential of a map is the generalization of the derivative between Euclidean spaces.

A basic theme in manifold theory is linearization, according to which a manifold can be approximated by its tangent space and a smooth map can be approximated by the differential of that map. The differential further categorizes maps as either immersions or submersions (depending on whether the differential is injective or surjective).

The collection of tangent spaces to a manifold can be given the structure of a *vector bundle*; it is thus called the *tangent bundle*. Vector fields, which manifest themselves in the physical world as velocity, force, electricity, and magnetism, maybe viewed as sections of the tangent bundle.

1.7.1 The Tangent Space at a Point

Definition 1.21: Tangent vector

The germ of a C^∞ function $p \in M$ to be an equivalence class of C^∞ functions defined in a neighborhood of $p \in M$, with equivalence defined as agreement on a (possibly smaller) neighborhood of p. The set of germs of C^∞ real-valued functions is denoted $C_p^\infty(M)$. Addition and multiplication of functions induces a ring structure on $C_p^\infty(M)$; with scalar multiplication by real numbers $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$. A derivation at a point in M is a linear map $D\colon C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g + f(p)Dg$$

A tangent vector at a point p is a derivation at p.

Given a coordinate neighborhood $(U, \phi) = (U, x^1, \dots, x^n)$ and $(r^1, \dots, r^n)$ the standard coordinates on $\mathbb{R}^n$ and

$$x^i = r^i \circ \phi \colon U \to \mathbb{R}$$

If f is a smooth function in a neighborhood of p

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1})$$

The partial derivatives satisfy the derivation property and therefore qualify as tangent vectors.

1.7.2 The Differential of a Map

Let $F\colon N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces, called its *differential at p*

$$F_*\colon T_pN \to T_{F(p)}M$$

defined as follows: if $X_p \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ according to

$$(F_*(X_p))f = X_p(f \circ F)$$

for $f \in C^\infty_{F(p)}(M)$ a germ (or representative of the germ). Vectors *pushforward* through F.

Example 1.11. Differential of a map between Euclidean spaces Suppose $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $p \in \mathbb{R}^n$. Let $x^1, \dots, x^n$ be coordinates on $\mathbb{R}^n$ and $y^1, \dots, y^m$ be coordinates on $\mathbb{R}^m$. Then

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

form a basis for the tangent space $T_p(\mathbb{R}^n)$ and

$$\left\{ \left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \dots, \left.\frac{\partial}{\partial y^m}\right|_{F(p)} \right\}$$

form a basis for the tangent space $T_{F(p)}(\mathbb{R}^m)$. The linear map

$$F_*\colon T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$$

(the differential of F) is described by a matrix $\left[a^i_j\right]$ relative to these two bases:

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}$$

Let $F^i = y^\circ F$ be the ith component of F; we can find a^i_j by evaluating the right-hand side and left-hand side on y^i

$$\begin{aligned} \text{RHS} &= \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i = \sum_k a^k_j \delta^i_k = a^i_j \\ \text{LHS} &= F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p \right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p (y^i \circ F) = \left.\frac{\partial F^i}{\partial x^j}\right|_p \end{aligned} \tag{54}$$

where we've used the fact that $F^i = y^i \circ F$. Thus, F_* relative to the bases is the Jacobian

$$\left[\left.\frac{\partial F^i}{\partial x^j}\right|_p \right]$$

and hence the differential generalizes the derivative of a map between Euclidean spaces (because in this instance the Jacobian is the derivative and in abstract manifolds we use the same F_*). Note this means

$$F_* \iff \text{Jacobian} \iff \text{differential} \iff \text{pushforward}$$

Example 1.12. Define a diffeomorphism F such that

$$F\colon (r, \theta) \mapsto (r\cos\theta, r\sin\theta) \tag{55}$$

The Jacobian of F

$$\mathrm{J} = \begin{bmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \tag{56}$$

where $(r, \theta) \equiv (x_1, x_2)$ and

$$(x, y) \equiv (y_1, y_2) \equiv (F_1, F_2) \equiv (r\cos\theta, r\sin\theta)$$

Note that $\det \mathrm{J} = r$ and so F is a diffeomorphism iff $r \neq 0$. Given a vector field

$$X_p = a(r,\theta)\partial_r + b(r,\theta)\partial_\theta := a(r,\theta)\frac{\partial}{\partial r} + b(r,\theta)\frac{\partial}{\partial \theta} \tag{57}$$

we can compute the pushforward F_* wrt the ∂_x, ∂_y basis

$$F_*(X_p) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\cos(\theta) - br\sin(\theta) \\ a\sin(\theta) + br\cos(\theta) \end{pmatrix} \tag{58}$$

Hence, explicitly

$$F_*(X_p) = (a\cos(\theta) - br\sin(\theta))\partial_x + (a\sin(\theta) + br\cos(\theta))\partial_y \tag{59}$$

Since J is invertible we can investigate which vector fields map to ∂_x

$$F_* X_p = \partial_x \iff X_p = F_*^{-1} \partial_x \tag{60}$$

Let $X_p = a\partial_r + b\partial_\theta$. Then

$$X_p = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ -\frac{\sin(\theta)}{r} \end{pmatrix} \tag{61}$$

However we need to write r, θ in terms of x, y

$$F_*^{-1} \partial_x = \frac{x}{\sqrt{x^2+y^2}} \partial_r + \frac{y}{x^2+y^2} \partial_\theta \tag{62}$$

F_*^{-1} is called the pullback F^* of the vector field ∂_x along F.

Let $F\colon N \to M$ and $G\colon M \to P$ be smooth maps of manifolds, and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps

$$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{*,F(p)}} T_{G(F(p))}P$$

Theorem 1.4: Chain rule

If $F\colon N \to M$ and $G\colon M \to P$ are smooth maps of and $p \in N$, then

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$$

Proof. Let $X_p \in T_pN$ and let f be C^∞ at $G(F(p)) \in P$. Then

$$((G \circ F)_* X_p) f = X_p (f \circ G \circ F)$$

and

$$\begin{aligned} ((G_* \circ F_*) X_p) f &= (G_*(F_* X_p)) f \\ &= (F_* X_p)(f \circ G) \\ &= X_p (f \circ G \circ F) \end{aligned} \tag{63}$$

□

Example 1.13. Chain rule in Calculus notation Suppose $w = G(x,y,z)$ is a C^∞ function $\mathbb{R}^3 \to \mathbb{R}$ and $(x,y,z) = F(t)$ is a C^∞ function $\mathbb{R} \to \mathbb{R}^3$. Under composition

$$w = (G \circ F)(t) = G(x(t), y(t), z(t))$$

becomes a C^∞ function $\mathbb{R} \to \mathbb{R}$. The differentials

$$\begin{aligned} F_* &= \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \\ G_* &= \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \\ (G \circ F)_* &= \frac{dw}{dt} \end{aligned} \tag{64}$$

and since composition of linear maps is matrix multiplication

$$\begin{aligned} (G \circ F)_* &= G_* \circ F_* \\ &= \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \\ &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \end{aligned} \tag{65}$$

Proposition 1.11

If $F\colon N \to M$ is a diffeomorphism of manifolds then

$$F_*\colon T_pN \to T_{F(p)}M$$

is an isomorphism of vector spaces.

Recall r^i the standard coordinates on R^n, (U, ϕ) a chart about $p \in M$ (M dimension n), and $x^i = r^i \circ \phi$. Since ϕ is a diffeomorphism onto its image, the differential (push-forward) $\phi*$ is a vector space isomorphism and

Proposition 1.12

Let $(U, \phi) = (U, x^1, \ldots, x^n)$. Then

$$\phi_* \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)} \tag{66}$$

Proposition 1.13

Let $(U, \phi) = (U, x^1, \ldots, x^n)$. Then T_pM has basis

$$\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \ldots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$$

Proposition 1.14: Transition matrix for coordinate vectors

Suppose $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (U, y^1, \ldots, y^n)$. Then

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^n \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

on $U \cap V$.

Given $F\colon N \to M$ and $G\colon M \to P$ are smooth maps of and $p \in N, F(p) \in M$. We will find a local expression for the differential

$$F_{*,p}\colon T_pN \to T_{F(p)}M$$

relative to the two charts. Using the local coordinate bases (induced by the charts) $F_* \equiv F_{*,p}$, is completely determined by a^i_j such that

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{k=1}^{m} a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)} \tag{67}$$

Applying y^i, we find that

$$\begin{aligned} a^i_j &= \left(\sum_{k=1}^{m} a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}\right) y^i \\ &= F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i \\ &= \left.\frac{\partial}{\partial x^j}\right|_p (y^i \circ F) \\ &= \left.\frac{\partial F^i}{\partial x^j}\right|_p \end{aligned} \tag{68}$$

We restate as a proposition

Proposition 1.15

Given $F\colon N \to M$ and $G\colon M \to P$ are smooth maps of and $p \in N, F(p) \in M$. Relative to bases $\partial/\partial x^j\big|_p$ for T_pN and $\partial/\partial y^i\big|_{F(p)}$ for $T_{F(p)}M$ the differential $F_{*,p}\colon T_p \to T_{F(p)}M$ is represented by the matrix

$$\begin{bmatrix} \left.\frac{\partial F^1}{\partial x^1}\right|_p & \cdots & \left.\frac{\partial F^1}{\partial x^n}\right|_p \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial F^m}{\partial x^1}\right|_p & \cdots & \left.\frac{\partial F^m}{\partial x^n}\right|_p \end{bmatrix} \tag{69}$$

1.7.3 Curves in a Manifold

A *smooth curve* is a smooth map $c\colon (a,b) \to M$ with $0 \in (a,b)$ and we say that c is a curve starting at p if $c(0) = p$. The *velocity vector* $c'(t_0)$ at time t_0 is

$$c'(t_0) \coloneqq c_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \in T_{c(t_0)}M$$

Proposition 1.16: Velocity of a curve in local coordinates

Let $c\colon (a,b) \to M$ be a smooth curve and let $U, x^1, \ldots, x^n$ be a coordinate chart about $c(t)$. Write $c^i = x^i \circ c$ for the ith component of c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}$$

where $\dot{c}$ is the scalar deriviatve of the ith component. Thus, relative to the basis $\{\partial/\partial x^j\big|_p\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} \dot{c}^1(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix}$$

Every smooth curve c at p in a manifold M gives rise to a tangent vector $c'(0)$ in T_pM. Conversely, one can show that every tangent vector $X_p \in T_pM$ is the velocity vector of some curve at p

Proposition 1.17: Existence of a curve with a given initial vector

For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\varepsilon > 0$ and a smooth curve $(-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Proof. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart centered at p i.e. $\phi(p) = \mathbf{0} \in \mathbb{R}^n$. Suppose $X_p = \sum a^i \partial/\partial x^i\big|_p$ and let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$, with $x^i = r^i \circ \phi$. To find a curve c at p with $c'(0) = X_p$, start with a curve $\alpha \in \mathbb{R}^n$ with $\alpha(0) = \mathbf{0}$ and $\alpha'(0) = \sum a^i \partial/\partial r^i\big|_{\mathbf{0}}$. We then map α to M via ϕ^{-1} (see figure 1). Let

$$\alpha(t) \coloneqq (a^1 t, \ldots, a^n t) \quad t \in (-\varepsilon, \varepsilon)$$

with ε small enough such that $\alpha(t) \in \phi(U)$. Then define $c = \phi^{-1} \circ \alpha\colon (-\varepsilon, \varepsilon) \to M$. Then $c(0) = \phi^{-1}(\alpha(0)) = \phi^{-1}(\mathbf{0}) = p$ and

$$\begin{aligned} c'(0) &= (\phi^{-1})_* \alpha_* \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\right) \\ &= (\phi^{-1})_* \left(\sum_i a^i \left.\frac{\partial}{\partial r^i}\right|_{\mathbf{0}}\right) \\ &= \sum_i a^i \left.\frac{\partial}{\partial x^i}\right|_p = X_p \end{aligned} \tag{70}$$

$\square$

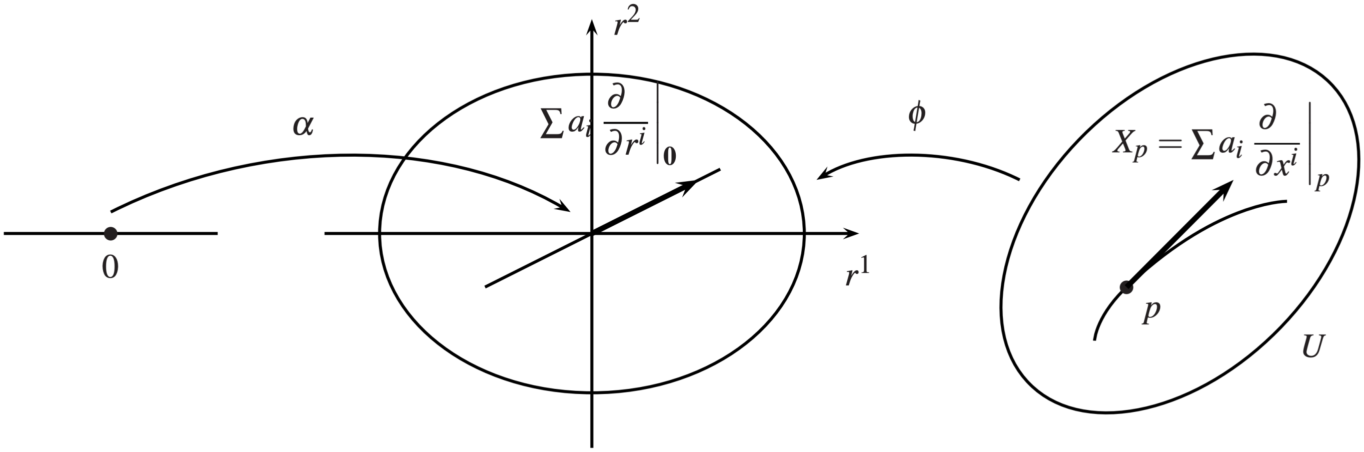

Figure 1: Existence of a curve through a point with a given initial vector.

This gives us a geometrical perspective on tangent vectors as directional derivatives

Proposition 1.18: Directional Derivatives

Suppose X_p is a tangent vector at a point $p \in M$ and $f \in C_p^\infty(M)$. If $c \in (-\varepsilon, \varepsilon) \to M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (f \circ c)$$

Proposition 1.19

Let $F\colon N \to M$ be a smooth map of manifolds, $p \in N, X_p \in T_pN$. If c is a smooth curve starting at $p \in N$ with velocity X_p at p,

$$F_{*,p}(X_p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (F \circ c)(t) \tag{71}$$

Example 1.14. Differential of left multiplication If $g \in \mathrm{GL}(n, \mathbb{R})$, let $\ell_g\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be left multiplication by g i.e. $\ell_g(B) = gB$. Since $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ the tangent space $T_g(\mathrm{GL}(n, \mathbb{R}))$ can be identified with $\mathbb{R}^{n\times n}$. We show that with this identification the differential at the identity

$$(\ell_g)_{*,\iota}\colon T_\iota(\mathrm{GL}(n, \mathbb{R})) \to T_g(\mathrm{GL}(n, \mathbb{R}))$$

is also left multiplication by g.

Let $X \in T_\iota(\mathrm{GL}(n, \mathbb{R})) = \mathbb{R}^{n\times n}$. To compute $(\ell_g)_{*,\iota}(X)$, choose a curve $c(t) \in \mathrm{GL}(n, \mathbb{R})$ with $c(0) = \iota$ and $c'(0) = X$. Then $\ell_g(c(t)) = g \cdot c(t)$ is simply matrix multiplication. Then

$$\begin{aligned}(\ell_g)_{*,\iota}(X) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \ell_g(c(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} gc(t) \\ &= gc'(0) = gX\end{aligned} \tag{72}$$

1.8 Tangent Bundle

A *smooth vector bundle* over a smooth manifold M is a smooth varying family of vector spaces, parameterized by M, that locally looks like a product (of the manifold (base space) and the tangent spaces (fibers)). The collection of tangent spaces to a manifold has the structure of a vector bundle, called the *tangent bundle*. A smooth map between manifolds induces, via its differential at each point, a bundle map of the corresponding tangent bundles. The tangent bundle is canonically associated to a manifold, hence invariants of the tangent bundle give rise to invariants of the manifold. For example *Chern-Weil theory of characteristic classes* uses differential geometry to construct invariants for vector bundles; applied to the tangent bundle, characteristic classes lead to numerical diffeomorphism invariants of a manifold called *characteristic numbers* (which generalize the *Euler characteristic*). A *section* of a vector bundle is a map that maps from each point of M into the *fiber* of the bundle over the point; both vector fields and differential forms on a manifold are sections of vector bundles.

1.8.1 Topology of the Tangent Bundle

Let M be a smooth manifold. Recall that at each point $p \in M$, the tangent space T_pM is the vector space of all point-derivations of $C_p^\infty(M)$ (itself the algebra of germs of C^∞ functions at p). The *tangent bundle* of M is the union of all the tangent spaces of M

$$TM \coloneqq \bigsqcup_{p \in M} T_pM$$

There is a natural map (in the sense that it does not depend on choice of atlas or local coordinates) $pi\colon TM \to M$ given by $\pi(v) = p$ if $v \in T_pM$. As a matter of notation, we sometimes write a tangent vector $v \in T_pM$ as a pair (p, v).

As defined, TM is a set, with no topology or manifold structure. We make it into a smooth manifold and show that it is a C^∞ vector bundle over M. The first step is the topology. If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a coordinate chart on M, let

$$TU \coloneqq \bigsqcup_{p\in U} T_pU = \bigsqcup_{p\in U} T_pM$$

where we've the fact that the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$ (since germ equivalence classes are determined by agreement in a neighborhood of p) and therefore $T_pU = T_pM$. At a point $p \in U$ a tangent vector $v \in T_pM$

$$v = \sum_{i=1}^{n} c^i \frac{\partial}{\partial x^i}\bigg|_p$$

In this expression, $c^i \equiv c^i(v)$ depend on v and so **therefore end up being functions on** TU. Define $\tilde{\phi}\colon TU \to \phi(U) \times \mathbb{R}^n$ by

$$v \mapsto (x^1(p), \ldots, x^n(p), c^1(v), \ldots, c^n(v)) \tag{73}$$

$$\iff \tag{74}$$

$$\tilde{\phi}(v) = (\bar{x}^1(v), \ldots, \bar{x}^n(v), c^1(v), \ldots, c^n(v)) \tag{75}$$

Then $\bar{\phi}$ has an inverse (for a point $(x^1, \ldots, x^n) = \phi(p)$)

$$\tilde{\phi}^{-1}(x^1(p), \ldots, x^n(p), c^1(v), \ldots, c^n(v)) = \sum c^i \frac{\partial}{\partial x^i}\bigg|_p$$

is therefore a bijection. Therefore we use topology of $\phi(U) \times \mathbb{R}^n$ to induce a topology on TU: a set $A \subset TU$ is open iff $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$ (where $\phi(U) \times \mathbb{R}^n$ has the standard topology as an open subset of $\mathbb{R}^{2n}$). With this identification TU becomes homeomorphic to $\phi(U) \times \mathbb{R}^n$.

1.8.2 Manifold Structure on the Tangent Bundle

Next we show that if $\{(U_i, \phi_i)\}$ is a C^∞ atlas for M, then $\{(TU_i, \tilde{\phi}_i)\}$ is a C^∞ atlas for the tangent bundle TM, where $\tilde{\phi}_i$ is the map on TU_i induced by ϕ_i as in (75). It's immediately clear that $TM = \bigcup_i TU_i$; remains to check that on $TU_i \cap TU_j$, $\tilde{\phi}_i, \tilde{\phi}_j$ are C^∞ compatible.

Recall that if $(U, x^1, \ldots, x^n), (V, y^1, \ldots, y^n)$ are two charts on M, then for any $p \in U \cap V$ there are two bases and so any vector $v \in T_pM$ has two representations

$$v = \sum_j a^j \frac{\partial}{\partial x^j}\bigg|_p = \sum_i b^i \frac{\partial}{\partial y^i}\bigg|_p$$

Applying v to either of x^k, y^k we get that

$$\begin{aligned} a^k &= \sum_i b^i \frac{\partial x^k}{\partial y^i} \\ b^k &= \sum_j a^j \frac{\partial y^k}{\partial x^j} \end{aligned} \tag{76}$$

Returning to the atlas $\{(U_i, \phi_i)\}$ and let $\phi_\alpha = (x^1, \ldots, x^n)$, $\phi_\beta = (y^1, \ldots, y^n)$. Then

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by (with $\phi_\alpha(p) = (x^1(p), \ldots, x^n(p))$)

$$\begin{aligned} (\phi_\alpha(p), a^1(v), \ldots, a^n(v)) &\mapsto \left(p, \sum_j a^j \frac{\partial}{\partial x^j}\bigg|_p\right) \\ &\mapsto ((\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p)), b^1, \ldots, b^n) \end{aligned} \tag{77}$$

where by (76) and the Jacobian matrix of a transition map

$$\begin{aligned} b^i &= \sum_j a^j \frac{\partial y^i}{\partial x^j}\bigg|_p \\ &= \sum_j a^j \frac{\partial(\phi_\beta \circ \phi_\alpha^{-1})}{\partial r^j}\bigg|_{\phi_\alpha(p)} \end{aligned} \tag{78}$$

By definition $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞ and therefore $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ is C^∞ and thefore TM is a C^∞ manifold, with $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ as a C^∞ atlas.

1.8.3 Vector Bundles

On the tangent bundle TM of a smooth manifold M, the natural projection map $\pi\colon TM \to M$

$$\pi(p, v) = p$$

makes TM into a C^∞ *vector bundle* over M which we now define. Given any map $\pi\colon E \to M$, where E is called the *total space* and $M$is called the *base space*, we call the inverse image $\pi^{-1}(p) \coloneqq \pi^{-1}(\{p\})$ of a point $p \in M$ the *fiber at* p. The fiber at p is often written E_p. For any two maps

$$\begin{aligned} \pi &\colon E \to M \\ \pi' &\colon E' \to M \end{aligned} \tag{79}$$

with the same target space M, a map $\phi\colon E \to E'$ is said to be *fiber-preserving* if $\phi(E_p) \subset E'_p$ for all $p \in M$.

Example 1.15. Fiber-preserving maps Given two maps π, π', the map ϕ is fiber-preserving iff the diagram

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes.

1.8.4 Smooth Sections

A *section* of a vector bundle $\pi\colon E \to M$ is a map $s\colon M \to E$ such that $\pi \circ s = \mathbf{1}_M$. This just means that s maps p into the fiber E_p above p (see figure **??**).

Definition 1.22: Vector field

A *vector field* X *on a manifold* M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi\colon TM \to M$ and the vector field is smooth if it is a smooth as a map from M to TM.

Example 1.16. The formula

$$X_{(x,y)} = \frac{1}{\sqrt{x^2+y^2}}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = \frac{1}{\sqrt{x^2+y^2}}\begin{bmatrix}-y\\ x\end{bmatrix}$$

See figure **??**.

1.8.5 Smooth Frames

A *frame* for a vector bundle $\pi\colon E \to M$ over an open set U is a collection of sections $s_1, \ldots, s_r$ of E over U such that at each point $p \in U$, the elements $s_1(p), dots, s_r(p)$ form a basis for the fiber $E_p \coloneqq \pi^{-1}(p)$. A frame for the tangent bundle $TM \to M$ over an open set U is simply called a *frame on* U.

Example 1.17. The collection of vector fields $\partial/\partial x, \partial/\partial y, \partial/\partial z$ is a smooth frame on $\mathbb{R}^3$.

Example 1.18. Let M be a manifold and $e_1, \ldots, e_r$ be the standard basis for $\mathbb{R}^n$. Define $\bar{e}_i\colon M \to M \times \mathbb{R}^r$ by $\bar{e}_i(p) \coloneqq (p, e_i)$. Then $\bar{e}_1, \ldots \bar{e}_r$ is a C^∞ frame for the product bundle $M \times \mathbb{R}^r \to M$.

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