Super Resolution for Automated Target Recognition

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Abstract

Super resolution is the process of producing high-resolution images from low-resolution images while preserving ground truth about the subject matter of the images and potentially inferring more such truth. Algorithms that successfully carry out such a process are broadly useful in all circumstances where high-resolution imagery is either difficult or impossible to obtain. In particular we look towards super resolving images collected using longwave infrared cameras since high resolution sensors for such cameras do not currently exist. We present an exposition of motivations and concepts of super resolution in general, and current techniques, with a qualitative comparison of such techniques. Finally we suggest directions for future research.

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1.1 Rayleigh Criterion

1.1.1 Wave Equation in a Vacuum

Starting with Maxwell's equations in a vacuum (in differential form) for the electric field $\boldsymbol{E}(x, y, z, t)$ and the magnetic field $\boldsymbol{B}(x, y, z, t)$:

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

Note that

$$\nabla \times (\nabla \times \boldsymbol{E}) = \frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{3}$$

and with the identity

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \tag{4}$$

we have the vector E-field *vector wave equation*:

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{5}$$

This decouples each components of the $\boldsymbol{E}$ field. We therefore arbitrarily choose the z component and solve the *scalar wave equation* for $E \coloneqq E_z$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = 0 \tag{6}$$

We proceed by *separation of variables*:

$$E(\boldsymbol{x}, t) = U(\boldsymbol{x})T(t) \tag{7}$$

Then U obeys the *Helmholtz equation*

$$(\nabla^2 + \beta^2)U = 0 \tag{8}$$

where β is separation constant. The solution for T is straightforward

$$T(t) = e^{-\mathrm{i}\omega t} \tag{9}$$

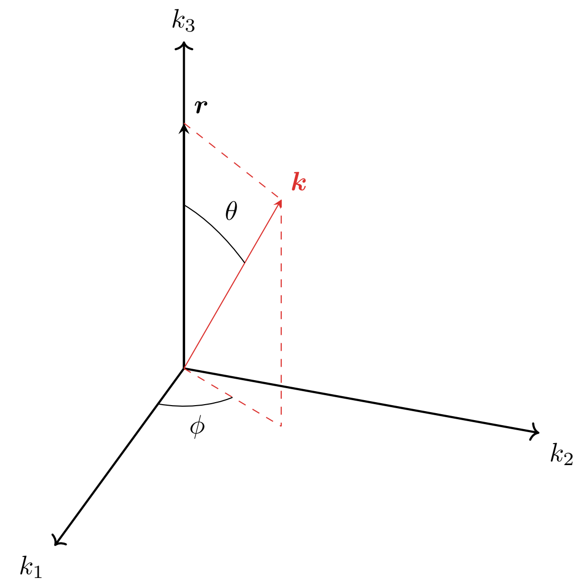

Figure 1: Rotated/aligned coordinate system.

where $\omega = \beta c$. For U we seek a *Green's function* $G(\boldsymbol{x}, \boldsymbol{x}')$ for eqn. (8)

$$\nabla^2 G + \beta^2 G = -\delta(\boldsymbol{r}) \tag{10}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. Substituting the Fourier transform $\tilde{G}(\boldsymbol{k})$ of G

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_{\mathbb{R}^3} \tilde{G} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}\boldsymbol{k} \tag{11}$$

where $\boldsymbol{k} = (k_1, k_2, k_3)$ and the Fourier representation of $\delta(\boldsymbol{r})$

$$\delta(\boldsymbol{r}) = \int\limits_{\mathbb{R}^3} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}\boldsymbol{k} \tag{12}$$

into eqn. (10)

$$\int\limits_{\mathbb{R}^3} \left(-k^2 + \beta^2\right) \tilde{G} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}\boldsymbol{k} = -\int\limits_{\mathbb{R}^3} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \, \mathrm{d}\boldsymbol{k} \tag{13}$$

where $k = \boldsymbol{k} \cdot \boldsymbol{k}$. Comparing both sides of eqn. (13) we conclude that

$$\tilde{G}(\boldsymbol{k}) = \frac{1}{k^2 - \beta^2} \tag{14}$$

and hence

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_{\mathbb{R}^3} \frac{e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2 - \beta^2} \, \mathrm{d}\boldsymbol{k} \tag{15}$$

Rotate the $\boldsymbol{k}$ coordinate system such that the k_3-axis and $\boldsymbol{r}$ are aligned and then transform to spherical coordinates (see figure 1). That implies

$$\boldsymbol{k} \cdot \boldsymbol{r} = k|\boldsymbol{r}| \cos(\theta)$$

and the differential volume element is

$$\mathrm{d}V = k^2 \sin(\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}k \tag{16}$$

Then

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_0^\infty \int\limits_0^\pi \int\limits_0^{2\pi} \frac{e^{\mathrm{i}k|\boldsymbol{r}|\cos(\theta)}}{k^2 - \beta^2} k \sin(\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}k = \frac{2\pi}{\mathrm{i}|\boldsymbol{r}|} \int\limits_0^\infty \frac{e^{\mathrm{i}k|\boldsymbol{r}|} - e^{-\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} k \, \mathrm{d}k \tag{17}$$

Since

$$\int\limits_0^\infty \frac{e^{\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} k \, \mathrm{d}k = \int\limits_{-\infty}^0 \frac{-e^{-\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} k \, \mathrm{d}k \tag{18}$$

equation (17) is

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{2\pi}{\mathrm{i}|\boldsymbol{r}|} \int\limits_{-\infty}^\infty \frac{k e^{\mathrm{i}k|\boldsymbol{r}|}}{(k-\beta)(k+\beta)} \, \mathrm{d}k \tag{19}$$

Note that there are there are two singularities or *poles* in the integrand in eqn. (17): the integrand goes to infinity as $k \to +\beta$ or $k \to -\beta$. To perform the integral, despite the poles, we need to use complex integration and residue theorem[1]: we perform a line integral (called a *contour integral*) in the k-complex plane such that its value along the real axis equals the integral in eqn. (19) and its value elsewhere is zero.

A critical requirement of the residue theorem is that the poles are wholly contained in the contour. To that end we perturb the poles $-\beta, +\beta$ by a pure imaginary component $+\mathrm{i}\epsilon$ and then take the limit as $\epsilon \to 0$. Consider the contour C in figure 2. It's composed of the portion C_1 along $\mathrm{Re}\{k\}$ and the semi-circle C_2 in the positive $\mathrm{Im}\{k\}$ half-plane. If we take the limit as $R \to \infty$ then the integral along C_1 agrees with the integrand in eqn. (19) and by *Jordan's lemma*[2], with $g(k) = 1/k^2 + \beta^2$, the integral along C_2 vanishes. Therefore

$$G(\boldsymbol{x}, \boldsymbol{x}') = \lim_{\epsilon \to 0} \frac{(2\pi)^2}{|\boldsymbol{r}|} \mathrm{Res}(f, \beta + \mathrm{i}\epsilon) \tag{20}$$

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[1] Consider a contour in the complex plane that encloses poles a_j. *Cauchy's residue theorem* dictates that for a *holomorphic* function f

$$\oint_C f(z)\, dz = 2\pi \mathrm{i} \sum \mathrm{Res}(f, a_j)$$

where $\mathrm{Res}(f, a_k)$ are the *residues* at the poles a_k:

$$\mathrm{Res}(f, a_j) = \frac{1}{(n-1)!} \lim_{a \to a_j} \left[\frac{d^{(n-1)}}{da^{(n-1)}} \Big((a - a_j)^n f(a) \Big) \right]$$

where n is the order of the pole.

[2] Consider a complex-valued, continuous function f, defined on a semicircular contour

$$C_R = \{ R e^{i\theta} \mid \theta \in [0, \pi] \}$$

of positive radius R lying in the upper half-plane, centered at the origin. If the function f is of the form

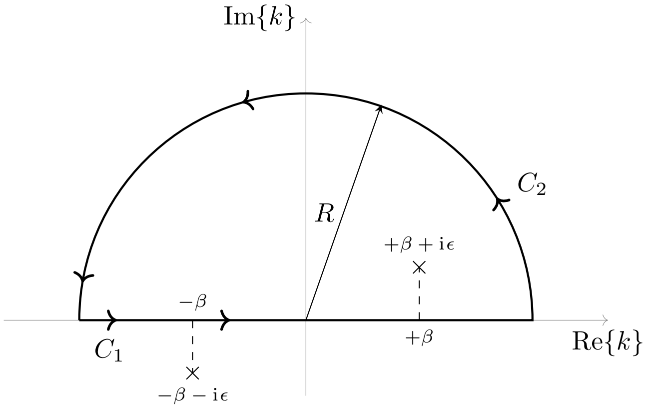

Figure 2: Contour $C = C_1 + C_2$ for the Helmholtz Green's function contour integral with poles $-\beta, +\beta$ and shifted poles $-\beta - \mathrm{i}\epsilon, +\beta + \mathrm{i}\epsilon$.

$$= \frac{(2\pi)^2}{|\boldsymbol{r}|} \lim_{\epsilon \to 0} \frac{\beta e^{\mathrm{i}\beta|\boldsymbol{r}|}}{2(\beta + \mathrm{i}\epsilon)} \tag{21}$$

$$= 2\pi^2 \frac{e^{\mathrm{i}\beta|\boldsymbol{r}|}}{|\boldsymbol{r}|} \tag{22}$$

Note we're off by some factors because our Fourier transform was missing components; the standard solution is

$$\frac{e^{\mathrm{i}k|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \tag{23}$$

with $k \equiv \beta$. Finally, recalling eqn. (7), and assumting we have that the *spherical wave* solution

$$E(r,t) = \frac{e^{\mathrm{i}(kr-\omega t)}}{4\pi r} \tag{24}$$

where $r = \left|\boldsymbol{x} - \boldsymbol{x}'\right|$.

$$f(z) = e^{iaz} g(z), \quad z \in C_R$$

with a positive parameter a, then Jordan's lemma states the following upper bound for the contour integral:

$$\left|\int_{C_R} f(z)\, dz\right| \leq \frac{\pi}{a} M_R \quad \text{where} \quad M_R := \max_{\theta \in [0,\pi]} \left|g\left(Re^{i\theta}\right)\right|$$

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