Notes

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1 Differential Geometry

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1.1 Directional Derivative

Elements of the *tangent space* $T_p(\mathbb{R}^n)$ anchored at a point $p = (p^1, \ldots, p^n) \in \mathbb{R}^n$ can be visualized as arrows emanating from p. These arrows are called *tangent vectors* and represented by column vectors:

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \tag{1}$$

The line through a point p with direction $\boldsymbol{v}$ has parameterization

$$c(t) = \left(p^1 + tv^1, \ldots, p^n + tv^n\right) \tag{2}$$

If $f \in C^\infty$ in a neighborhood of p and $\boldsymbol{v}$ is a tangent vector at p, the *directional derivative* of f in the direction of $\boldsymbol{v}$ at p is defined

$$D_{\boldsymbol{v}} f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) \bigg|_{t=0} \tag{3}$$

By the chain rule

$$\begin{aligned} D_{\boldsymbol{v}} f &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\bigg|_p \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} && (4) \\ &= \sum_{i=1}^{n} \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f}{\partial x^i}\bigg|_p && (5) \\ &= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}\bigg|_p && (6) \\ & && (7) \end{aligned}$$

The directional derivative operator at p is defined

$$D_{\boldsymbol{v}} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{8}$$

The association $\boldsymbol{v} \mapsto D_{\boldsymbol{v}}$ offers a way to *isomorphically* identify tangent vectors with operators on functions. The following makes this rigorous.

1.2 Derivations

For each tangent vector $\boldsymbol{v}$ at a point $p \in \mathbb{R}^n$, the directional derivative at p gives a map of vector spaces

$$D_{\boldsymbol{v}} \colon C_p^\infty \to \mathbb{R}$$

$D_{\boldsymbol{v}}$ is a linear map that satisfies the *Leibniz rule*

$$D_{\boldsymbol{v}}(fg) = (D_{\boldsymbol{v}} f)g(p) + f(p)(D_{\boldsymbol{v}} g) \tag{9}$$

because the partial derivative satisfy the product rule. In general, any linear map $L \colon C_p^\infty \to \mathbb{R}$ that satisfies the Leibniz rule is called a *derivation* at p. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. **This set is also a real vector space**.

So far we know directional derivatives $D_{\boldsymbol{v}}$ at p are derivations at p. Thus, there is a map

$$\begin{aligned} \phi \colon T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ \boldsymbol{v} &\mapsto D_{\boldsymbol{v}} \end{aligned}$$

Theorem 1.1. The linear map ϕ is an isomorphism of vector spaces.

The implication is that we may identify tangent vectors at p with derivations at p (by way of directional derivatives against germs). Under this isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \dots, e_n\}$ for $T_p(\mathbb{R}^n)$ maps to

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \tag{10}$$

Therefore from now on we write a tangent vector as

$$\boldsymbol{v} = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p \tag{11}$$

The point being that, while not as geometrically intuitive as arrows, $\mathcal{D}_p(\mathbb{R}^n)$ generalizes to manifolds.

1.3 Vector Fields

A *vector field* X on an open $U \subset \mathbb{R}^n$ is function that assigns to $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Notice, carefully, that the vector field assigns at each point a vector in the tangent space anchored at that point. Using the tangent basis (eqn. (10))

$$X \colon p \mapsto \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \tag{12}$$

Note that both the coefficients **and** the partial derivatives are evaluated at p. Having said that, we often omit p in the specification of a vector field when it clear from context.

Example 1.1. On $\mathbb{R}^n - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$\begin{aligned} X &= \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}^T \end{aligned}$$

See figure 1

In general we can identify vector fields with parameterized column vectors

$$X = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \leftrightarrow \begin{bmatrix} a^1(p) \\ \vdots \\ a^n(p) \end{bmatrix} \tag{13}$$

2 Appendix

Contents

2.1 Definitions

2.1.1 Linear Operator

A map $L\colon V \to W$ between vector spaces over a field K is a *linear operator* if

1. **distributivity**: $L(u+v) = L(u) + L(v)$
2. **homogeneity**: $L(rv) = rL(v)$

To emphasize the field, L is said to be $K-$linear.

2.1.2 Germs

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f\colon U \to \mathbb{R}$ is a C^∞ function. We say that $(f, U) \sim (g, U')$ if there is an open W such that $p \in W \subset U \cap U'$ and $f = g$ when restricted to W. The equivalence class $[(f, U)]$ of (f, U) is the *germ* of f at p. We write

$$C_p^\infty(\mathbb{R}^n) \coloneqq \{[(f, U)]\} \tag{14}$$

for the set all germs of C^∞ functions on $\mathbb{R}^n$ at p.

2.1.3 Algebra

An *algebra* over *field* K is a vector space A over K with a multiplication map

$$\mu\colon A \times A \to A \tag{15}$$

usually written $\mu(a, b) = a \cdot b$, such that μ is associative, distributive, and homogeneous, where homogeneity is defined:

1. **associativity**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. **distributivity**: $(a+b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + a \cdot c$
3. **homogeneity**: $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$

If A, A' are algebras then an *algebra homomorphism* is a linear operator L that respects algebra multiplication $L(ab) = L(a)L(b)$. It's the case that addition and multiplication of functions **induces addition and multiplication on the set of germs** C_p^∞, making it into an algebra over $\mathbb{R}^n$.

2.1.4 Module

If R is a communicative ring with identity, then a (left) $R-$*module* is an abelian group A with a scalar multiplication map

$$\mu\colon R \times A \to A \tag{16}$$

such that μ is

1. **associative**: $(rs)a = r(sa)$ for $r, s \in R$
2. **identity**: $1 \in R \implies 1a = a$
3. **distributive**: $(r+s)a = ra+sa$ and $r(a+b) = ra+rb$

If R is a field, then an $R-$module is a vector space over R; in this sense modules generalize vector space to scalars from a ring rather than a field.

Let A, A' be $R-$modules. An $R-$*module homomorphism* $f\colon A \to A'$ is a map that preserves both addition and scalar multiplication.

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