Notes

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1 Differential Geometry

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1.1 Directional Derivative

Elements of the *tangent space* $T_p(\mathbb{R}^n)$ anchored at a point $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ can be visualized as arrows emanating from p. These arrows are called *tangent vectors* and represented by column vectors:

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \tag{1}$$

The line through a point p with direction $\boldsymbol{v}$ has parameterization

$$c(t) = \left(p^1 + tv^1, \dots, p^n + tv^n\right) \tag{2}$$

If $f \in C^\infty$ in a neighborhood of p and $\boldsymbol{v}$ is a tangent vector at p, the *directional derivative* of f in the direction of $\boldsymbol{v}$ at p is defined

$$D_{\boldsymbol{v}} f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) \Big|_{t=0} \tag{3}$$

By the chain rule

$$D_{\boldsymbol{v}} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\bigg|_p \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \tag{4}$$

$$= \sum_{i=1}^{n} \frac{\mathrm{d}c^i}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f}{\partial x^i}\bigg|_p \tag{5}$$

$$= \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}\bigg|_p \tag{6}$$

$$\tag{7}$$

The directional derivative operator at p is defined

$$D_{\boldsymbol{v}} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{8}$$

The association $\boldsymbol{v} \mapsto D_{\boldsymbol{v}}$ offers a way to *isomorphically* identify tangent vectors with operators on functions. The following makes this rigorous.

1.2 Derivations

For each tangent vector $\boldsymbol{v}$ at a point $p \in \mathbb{R}^n$, the directional derivative at p gives a map of vector spaces

$$D_{\boldsymbol{v}} \colon C_p^\infty \to \mathbb{R}$$

$D_{\boldsymbol{v}}$ is a linear map that satisfies the *Leibniz rule*

$$D_{\boldsymbol{v}}(fg) = (D_{\boldsymbol{v}} f)g(p) + f(p)(D_{\boldsymbol{v}} g) \tag{9}$$

because the partial derivative satisfy the product rule. In general, any linear map $L \colon C_p^\infty \to \mathbb{R}$ that satisfies the Leibniz rule is called a *derivation* at p. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. **This set is also a real vector space**.

So far we know directional derivatives $D_{\boldsymbol{v}}$ at p are derivations at p. Thus, there is a map

$$\phi \colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$$
$$\boldsymbol{v} \mapsto D_{\boldsymbol{v}}$$

Theorem 1.1

The linear map ϕ is an isomorphism of vector spaces.

The implication is that we may identify tangent vectors at p with derivations at p (by way of directional derivatives against germs). Under this isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \ldots, e_n\}$ for $T_p(\mathbb{R}^n)$ maps to

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \tag{10}$$

Therefore from now on we write a tangent vector as

$$\boldsymbol{v} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \tag{11}$$

The point being that, while not as geometrically intuitive as arrows, $\mathcal{D}_p(\mathbb{R}^n)$ generalizes to manifolds.

1.3 Vector Fields

A *vector field* X on an open $U \subset \mathbb{R}^n$ is function that assigns to $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Notice, carefully, that the vector field assigns at each point a vector in the tangent space anchored at that point. Using the tangent basis (eqn. (10))

$$X \colon p \mapsto \sum_i a^i(p) \frac{\partial}{\partial x^i}\bigg|_p \tag{12}$$

Note that both the coefficients **and** the partial derivatives are evaluated at p. Having said that, we often omit p in the specification of a vector field when it clear from context.

Example 1.1. On $\mathbb{R}^n - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$X = \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y}$$
$$= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}^T$$

See figure **??**

In general we can identify vector fields with parameterized column vectors

$$X = \sum_i a^i(p) \frac{\partial}{\partial x^i}\bigg|_p \leftrightarrow \begin{bmatrix} a^1(p) \\ \vdots \\ a^n(p) \end{bmatrix} \tag{13}$$

1.4 Dual Space

The *dual space* $V^\wedge$ of V is the set of all real-valued linear functions on V i.e. all $f \colon V \to \mathbb{R}$. Elements of $V^\wedge$ are called *covectors*.

Assume V is finite dimensional and let $\{e_1, \ldots, e_n\}$ be a basis V. Recall that $e_i := \partial_{x_i}$. Then $X = \sum a^i \partial_{x_i}$ for all $X \in T_p$. Let $\alpha^i \colon V \to \mathbb{R}$ be the linear function that picks out the ith coordinate of a **vector**, i.e. $\alpha^i(X) = a^i(p)$. Note that

$$\alpha^i(\partial_j) = \alpha^i(1 \cdot \partial_j) \tag{14}$$

$$= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{15}$$

$$= \delta^i_j \tag{16}$$

Note that position of indices is important – upper indices are for covectors.

Proposition 1.1

$\{\alpha^i\}$ form a basis for $V^\wedge$.

Proof. We first prove that $\{\alpha^i\}$ span $V^\wedge$. If $f \in V^\wedge$ and $X = \sum a^i \partial_{x_i} \in V$, then

$$\begin{aligned} f(X) &= \sum a^i f(\partial_{x_i}) && (17) \\ &= \sum \alpha^i(X) f(\partial_{x_i}) && (18) \\ &= \sum f(\partial_{x_i}) \alpha^i(X) && (19) \end{aligned}$$

which shows that any f can be expanded as a linear sum of α^i. To show linear independence, suppose $\sum c_i \alpha^i = 0$ with at least one c_i non-zero. Applying this to an arbitrary ∂_{x_i} gives

$$0 = \left(\sum_i c_i \alpha^i\right)(\partial_{x_i}) = \sum_i c_i \alpha^i(\partial_{x_i}) = \sum_i c_i \delta^i_j = c_j \tag{20}$$

which is a contradiction. Hence α^i are linearly independent. ☐

This basis $\{\alpha^i\}$ for $V^\wedge$ is said to be *dual* to the basis $\{\partial_{x_i}\}$ for V.

Example 1.2. Coordinate functions With respect to a basis $\{\partial_{x_i}\}$ for V, every $X \in V$ can be written uniquely as a linear combination $X = \sum a^i \partial_{x_i}$ with $a^i \in \mathbb{R}$. Let $\{\alpha^i\}$ be the dual basis (i.e. the basis for $V^\wedge$). Then

$$\begin{aligned} \alpha^i(X) &= \alpha^i\left(\sum_j a^j \partial_{x_j}\right) \\ &= \sum_j a^j \alpha^i(\partial_{x_j}) \\ &= \sum_j a^j \delta^i_j \\ &= a^i \end{aligned}$$

Thus, the dual basis $\{\alpha^i\}$ to $\{\partial_{x_i}\}$ is the set of coordinate functions. The sense here is that since tangent vectors are directional derivatives (i.e. operators on functions) and the dual space is a mapping from those operators to $\mathbb{R}$, then a mapping from operators to scalars means hitting an operator with a function (or vice-versa). **And** the coordinate functions are constant with respect to each other coordinate (and hence partials wrt them are naturally zero).

1.5 Differential Forms on $\mathbb{R}^n$

A differential k-form assigns a k-covector from the dual space at each point p. The wedge product (alternation of tensor product) of differential forms is defined pointwise (as the wedge product of multi-covectors). Differential forms exist on an open set (why?) there is a notion of differentiation (called exterior derivative). Exterior derivative is coordinate independent and intrinsic to a manifold; it is the abstraction of gradient, curl, divergence to arbitrary manifolds. Differential forms extend Grassmann's exterior algebra (graded algebra of multi-covectors) from the tangent space at a point globally, i.e. to the entire manifold (how? bundles?).

1.5.1 Differential of a Function

Definition 1.1: Cotangent Space

The *cotangent space* to $\mathbb{R}^n$ at p, denoted $T^*_p(\mathbb{R}^n)$, is defined to be the dual space $(T_p(\mathbb{R}^n))^\vee$ of the tangent space $T_p(\mathbb{R}^n)$.

Thus, an element of the cotangent space $T^*_p(\mathbb{R}^n)$ is a **covector of linear functional on tangent space**.

Definition 1.2: Differential 1-form

A *covector field* or a *differential 1-form* on an open subset U of $\mathbb{R}^n$ is a function ω that assigns at each point p in U a covector $\omega_p \in T^*_p(\mathbb{R}^n)$

$$\begin{aligned} \omega \colon U &\to \bigcup_{p \in U} T^*_p(\mathbb{R}^n) \\ p &\mapsto \omega_p \in T^*_p(\mathbb{R}^n) \end{aligned} \tag{21}$$

We call a differential 1-form a *1-form* for short.

Definition 1.3: Differential

From any C^∞ function $f \colon U \to \mathbb{R}$, we can construct the 1-form $\mathrm{d}f$, called the *differential* of f, as follows: for $p \in U$ and $X_p \in T_p(U)$

$$(\mathrm{d}f)_p(X_p) \coloneqq X_p f \tag{22}$$

In words the differential of f is the application of X_p to f or **the directional derivative of f in the direction of the tangent vector defined by the coefficients of X_p**.

Let $x^1, \ldots, x^n$ be the standard coordinates on $\mathbb{R}$, $\{(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p\}$ their differentials defined

$$(\mathrm{d}x^i)_p(X_p) \coloneqq (X_p)(x^i)$$

and

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

be the standard basis for $T_p(\mathbb{R}^n)$.

Proposition 1.2

$\{(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^n)_p\}$ is the basis for $T^*_p(\mathbb{R}^n)$ dual to the coordinate basis for $T_p(\mathbb{R}^n)$.

Proof. By definition,

$$(\mathrm{d}x^i)_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \frac{\partial x^i}{\partial x^j}\bigg|_p = \delta^i_j$$

□

If ω is a 1-form on $U \in \mathbb{R}^n$ then by proposition (1.2), at each point $p \in U$

$$\omega_p = \sum a_i(p)(dx^i)_p$$

Note the lower index on $a_i(p)$ as opposed to the upper index on $X_p = \sum a^i(p)\partial_{x_i}\big|_p$. If $x \coloneqq x^1, y \coloneqq x^2, z \coloneqq x^3$, then $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$.

Proposition 1.3: $\mathrm{d}f$ in terms of coordinates

If $f\colon U \to \mathbb{R}$, then

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \tag{23}$$

Proof.

$$(\mathrm{d}f)_p = \sum a_i(p)(\mathrm{d}x^i)_p$$

for some real numbers $a_i(p)$ depending on p. Thus

$$\begin{aligned} \mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) &= \sum_i a_i\,\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) \\ &= \sum_i a_i \delta^i_j = a_j \end{aligned} \tag{24}$$

On the other hand, by the definition of the differential

$$\mathrm{d}f = \left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j} \tag{25}$$

Therefore

$$a_j = \frac{\partial f}{\partial x^j} \tag{26}$$

and hence

$$(\mathrm{d}f)_p = \frac{\partial f}{\partial x^i}\bigg|_p (\mathrm{d}x^i)_p$$

□

1.5.2 Differential k-forms

Definition 1.4: Differential k-forms

More generally, a *differential form ω of degree k* is a function that at each point assigns an alternating k-linear function on $T_p(\mathbb{R}^n)$, i.e. $\omega_p \in A^k(T_p(\mathbb{R}^n))$.

A basis for $A^k(T_p(\mathbb{R}^n))$ is

$$(\mathrm{d}x^I)_p \coloneqq (\mathrm{d}x^{i_1})_p \wedge \cdots \wedge (\mathrm{d}x^{i_k})_p \tag{27}$$

where $1 \le i_1 < \cdots < i_k \le n$.

What is the nuance here?

Therefore, at each point $p \in U$, ω_p is a linear combination

$$\begin{gathered} \omega_p = \sum_I a_I(p)(\mathrm{d}x^I)_p \\ 1 \le i_1 < \cdots < i_k \le n \end{gathered} \tag{28}$$

and a k-form ω on open U is a linear combination

$$\omega = \sum_I a_I\,\mathrm{d}x^I \tag{29}$$

with function coefficients $a_I\colon U \to \mathbb{R}$. We say that a k-form ω is C^∞ on U if all of the coefficients a_I are C^∞ functions on U. Denote $\Omega^k(U)$ the vector space of k-forms on U. A 0-form on U assigns to each point p an element of $A^0(T_p(\mathbb{R}^n)) \coloneqq \mathbb{R}$; thus, a 0-form on U is a constant function. Note there are no nonzero differential forms of degree $> n$ on U since if $\deg \mathrm{d}x^I > n$ then at least two of the component 1-forms of dx^I must be the same and therefore $dx^I = 0$.

Definition 1.5: Wedge product of forms

The *wedge product of a k-form ω and ℓ-form τ* is defined pointwise

$$\begin{aligned} (\omega \wedge \tau)_p &\coloneqq \omega_p \wedge \tau_p \\ \omega \wedge \tau &= \sum_{I,J} (a_I b_J)\,\mathrm{d}x^I \wedge \mathrm{d}x^J \end{aligned} \tag{30}$$

where $I \cap J = \emptyset$.

Hence the wedge product is bilinear

$$\wedge\colon \Omega^k(U) \times \Omega^\ell \to \Omega^{k+\ell}(U) \tag{31}$$

The wedge product of forms is also anticommutative and associate (owing to the associativity and anticommutativity of the wedge product on multi-covectors) as therefore induces a graded algebra on $\Omega(U) \coloneqq \bigoplus_k \Omega^k(U)$.

Example 1.3. In the case of

$$\wedge\colon \Omega^0(U) \times \Omega^\ell(U) \to \Omega^\ell$$

we have the pointwise multiplication of a C^∞ function and a C^∞ ℓ-form

$$(f \wedge \omega)_p = f(p) \wedge \omega_p = f(p)\omega_p$$

Let x, y, z be the coordinates on $\mathbb{R}^3$. Then, the 1-forms are

$$f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$$

the 2-forms are

$$f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}x \wedge \mathrm{d}z + h\,\mathrm{d}x \wedge \mathrm{d}y$$

and the 3-forms are

$$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

1.5.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a 1-form and X is a vector field then

$$\begin{aligned} \omega(X)|_p &:= \omega_p(X_p) \\ \omega &= \sum a_i \, \mathrm{d}x^i \quad X = \sum b^i \frac{\partial}{\partial x^j} \\ \omega(X) &= \left(\sum a_i \, \mathrm{d}x^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right) \\ &= \sum a_i b^i \end{aligned} \tag{32}$$

1.5.4 Exterior Derivative

Definition 1.6: Exterior Derivative

The exterior derivative of a function $f \in C^\infty(U)$ is defined to be its differential $\mathrm{d}f$

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i$$

For $k \geq 1$, if $\omega = \sum_I a_I \, \mathrm{d}x^I$ is a k-form, then

$$\begin{aligned} \mathrm{d}\omega &:= \sum_I \mathrm{d}a_I \wedge \mathrm{d}x^I \\ &= \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} \, \mathrm{d}x^j\right) \wedge \mathrm{d}x^I \end{aligned} \tag{33}$$

Example 1.4. Let ω be the 1-form $f \, \mathrm{d}x + g \, \mathrm{d}y$ on R^2. Then

$$\mathrm{d}\omega = \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \tag{34}$$

$$= \left(\frac{\partial f}{\partial x} \mathrm{d}x + \frac{\partial f}{\partial y} \mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial g}{\partial x} \mathrm{d}x + \frac{\partial g}{\partial y} \mathrm{d}y\right) \wedge \mathrm{d}y \tag{35}$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \tag{36}$$

where we use that $\mathrm{d}x \wedge \mathrm{d}y = -\,\mathrm{d}y \wedge \mathrm{d}x$ and $\mathrm{d}x \wedge \mathrm{d}x = 0$.

Definition 1.7: Antiderivation

et $A = \bigoplus_k A^k$ be a graded algebra over a field K. An *antiderivation of the graded algebra* A is a k-linear map $D \colon A \to A$ such that for $a \in A^k, b \in A^\ell$

$$D(ab) = (Da)b + (-1)^k a Db \tag{37}$$

If there is an integer m such that D sends A^k to A^{k+m} for all k, then the antiderivation is of *degree* m.

Proposition 1.4: Properties of exterior differentiation

1. exterior differentiation is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\tau$$

2. $\mathrm{d}^2 = 0$

Proposition 1.5: Characterization of the exterior derivative

The properties of proposition (1.4) completely characterize exterior differentiation.

1.5.5 Closed and Exact Forms

A k-form ω is a *closed form* if $\mathrm{d}\omega = 0$. ω is an *exact form* if there is a $k-1$-form τ such that $\omega = \mathrm{d}\tau$. Since $\mathrm{d}(\mathrm{d}\tau) = 0$, every exact form is closed.

Example 1.5. A closed 1-form on the punctured plane Define ω on the manifold $\mathbb{R}^2 - \{\mathbf{0}\}$ by

$$\omega(x, y) = \frac{-y}{x^2 + y^2} \, \mathrm{d}x + \frac{x}{x^2 + y^2} \, \mathrm{d}y$$

To show that ω is closed we take the exterior derivative using example (36):

$$\begin{aligned} \mathrm{d}\omega &= \left(\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right) \mathrm{d}x \wedge \mathrm{d}y \\ &= 0 \end{aligned} \tag{38}$$

Definition 1.8: Differential Complex

A collection of vector spaces $\{V^0, V^1, \dots\}$ with linear maps $d_k \colon V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *differential complex* or a *cochain complex*. For any open subset $U \subset \mathbb{R}^3$, the exterior derivative makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the *de Rham complex of* U:

$$0 \to \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots$$

The closed forms are the elements of the kernel of d and the exact forms are the elements of the image of d.

1.5.6 Applications to Vector Calculus

Recall the three operators grad, curl, div $(\boldsymbol{\nabla}, \boldsymbol{\nabla}\times, \boldsymbol{\nabla}\cdot)$ on scalar fields and vector fields (i.e. scalar valued functions and vector valued functions) over $\mathbb{R}^3$:

$$\boldsymbol{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

$$\boldsymbol{\nabla} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix} \tag{39}$$

$$\boldsymbol{\nabla} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Note we can identity 1-forms with vector fields:

$$P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Similarly 2-forms on $\mathbb{R}^3$

$$P\,\mathrm{d}x \wedge \mathrm{d}y + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

and 3-forms on U can be identified with functions U

$$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \longleftrightarrow f$$

In terms of these identifications the exterior derivative of 0-form f is the 1-form $\mathrm{d}f$ or $\boldsymbol{\nabla} f$:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \longleftrightarrow \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \boldsymbol{\nabla} f$$

the exterior derivative of a vector field $\begin{bmatrix} P & Q & R \end{bmatrix}^T$ (i.e. 1-form) is the 2-form $\boldsymbol{\nabla} \times \begin{bmatrix} P & Q & R \end{bmatrix}^T$

$$\begin{aligned} \mathrm{d}(P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z) &= \\ +\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) &\mathrm{d}y \wedge \mathrm{d}z \\ -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) &\mathrm{d}z \wedge \mathrm{d}x \\ +\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) &\mathrm{d}x \wedge \mathrm{d}y \\ &\longleftrightarrow \\ \boldsymbol{\nabla} \times &\begin{bmatrix} P \\ Q \\ R \end{bmatrix} \end{aligned} \tag{40}$$

and the exterior derivative of a 2-form is

$$\begin{aligned} &\mathrm{d}(P\,\mathrm{d}x \wedge \mathrm{d}y + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y) = \\ &\left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial R}{\partial z}\,\mathrm{d}z\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &\longleftrightarrow \boldsymbol{\nabla} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \end{aligned} \tag{41}$$

Thus, the exterior derivatives on 0-forms (functions) is the grad operator, on 1-forms is the curl operator, and on 2-forms is the divergence operator.

$$\begin{tikzcd} \Omega^0(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^1(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^2(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^3(U) \arrow[d, "\cong"'] \\ C^\infty(U) \arrow[r, "\boldsymbol{\nabla}"'] & \mathfrak{X}(U) \arrow[r, "\boldsymbol{\nabla}\times"'] & \mathfrak{X}(U) \arrow[r, "\boldsymbol{\nabla}\cdot"'] & C^\infty(U) \end{tikzcd}$$

where $\mathfrak{X}(U)$ is the *Lie algebra* of C^∞ vector fields on U.

Proposition 1.6: grad, curl, div properties

1. $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$
2. $\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \begin{bmatrix} P & Q & R \end{bmatrix}^T\right) = 0$
3. On $\mathbb{R}^3$, a vector field $\mathbf{F}$ is the gradient of some scalar function iff $\boldsymbol{\nabla} \times \mathbf{F} = 0$

Properties (1,2) express that $\mathrm{d}^2 = 0$ Property (3) expresses the fact that a 1-form on $\mathbb{R}^3$ is exact iff it is closed; it need not be true on a region other than $\mathbb{R}^3$. It turns out that whether proposition (3) is true depends on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k(U) := \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}} \tag{42}$$

called the kth *de Rham cohomolgy* of U. The generalization of proposition (3) to any differential on $\mathbb{R}^n$ is called the *Poincare lemma*: for $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact. This is equivalent to the vanishing of the kth de Rham cohomology $H^k(\mathbb{R}^n)$ for $k \geq 1$.

1.5.7 Convention on Subscripts and Superscripts

Vector fields $e_1, e_2, \ldots$have subscripts and differential forms $\omega^1, \omega^2, \ldots$ have superscripts. Coordinate functions $x^1, x^2, \ldots$, being 0-forms, have superscripts. Their differentials $\mathrm{d}x^i$ should also. Coordinate vector fields $\frac{\partial}{\partial x^i}$ are considered to have subscripts because the index is in the denominator. Coefficient functions have subscripts or subscripts depending on whether they're coefficients functions for vector fields or forms. This allows for "conservation of indices": if $X = \sum a^i \partial_{x^i}$ and $\omega = \sum b_j\,\mathrm{d}x^j$ then

$$\omega(X) = \left(\sum b_j\,\mathrm{d}x^j\right)\left(\sum a^i \frac{\partial}{\partial x^i}\right) = \sum b_i a^i$$

1.6 Manifolds

2 Appendix

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2.1 Definitions

2.1.1 Linear Operator

Definition 2.1: Linear Operator

A map $L\colon V \to W$ between vector spaces over a field K is a *linear operator* if

1. **distributivity**: $L(u+v) = L(u) + L(v)$
2. **homogeneity**: $L(rv) = rL(v)$

To emphasize the field, L is said to be $K-$linear.

2.1.2 Germs

Definition 2.2: Germs

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We say that $(f, U) \sim (g, U')$ if there is an open W such that $p \in W \subset U \cap U'$ and $f = g$ when restricted to W. The equivalence class $[(f, U)]$ of (f, U) is the *germ* of f at p. We write

$$C_p^\infty(\mathbb{R}^n) \coloneqq \{[(f, U)]\} \tag{43}$$

for the set all germs of C^∞ functions on $\mathbb{R}^n$ at p.

2.1.3 Algebra

Definition 2.3: Algebra

An *algebra* over *field* K is a vector space A over K with a multiplication map

$$\mu\colon A \times A \to A \tag{44}$$

usually written $\mu(a, b) = a \cdot b$, such that μ is associative, distributive, and homogeneous, where homogeneity is defined:

1. **associativity**: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. **distributivity**: $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot b + a \cdot c$
3. **homogeneity**: $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$

If A, A' are algebras then an *algebra homomorphism* is a linear operator L that respects algebra multiplication $L(ab) = L(a)L(b)$. It's the case that addition and multiplication of functions **induces addition and multiplication on the set of germs** C_p^∞, making it into an algebra over $\mathbb{R}^n$.

2.1.4 Module

Definition 2.4: Module

If R is a commutative ring with identity, then a (left) $R-$*module* is an abelian group A with a scalar multiplication map

$$\mu\colon R \times A \to A \tag{45}$$

such that μ is

1. **associative**: $(rs)a = r(sa)$ for $r, s \in R$
2. **identity**: $1 \in R \implies 1a = a$
3. **distributive**: $(r+s)a = ra+sa$ and $r(a+b) = ra + rb$

If R is a field, then an $R-$module is a vector space over R; in this sense modules generalize vector space to scalars from a ring rather than a field.

Let A, A' be $R-$modules. An $R-$*module homomorphism* $f\colon A \to A'$ is a map that preserves both addition and scalar multiplication.

2.1.5 Tensor Product

Definition 2.5: Tensor Product

Let f be k-linear function and g be an ℓ-linear function on a vector space V. Then, their *tensor product* is the $(k+\ell)$-linear function $f \otimes g$

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) \coloneqq f(v_1, \ldots, v_k)g(v_{k+1}, \ldots, v_{k+\ell}) \tag{46}$$

Example 2.1. (*Bilinear maps*) Let $\{e_i\}$ be a basis for a vector space V and $\{\alpha^j\}$ be the dual basis for $V^\vee$. Also let $\langle\, ,\rangle\colon V \times V \to \mathbb{R}$ be a bilinear map on V. Set $g_{ij} = \langle e_i, e_j \rangle \in \mathbb{R}$. If

$$v = \sum v^i e_i \tag{47}$$

$$w = \sum w^i e_i \tag{48}$$

then $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$, where α are the coordinate functions a^i, a^j. By bilinearity, we can express $\langle\, ,\rangle$ in terms of the tensor product

$$\begin{aligned} \langle v, w \rangle &= \sum_{ij} v^i w^j \langle e_i, e_j \rangle \\ &= \sum \alpha^i(v)\alpha^i(w) g_{ij} \\ &= \sum (\alpha^i \otimes \alpha^j)(v, w) \times g_{ij} \end{aligned} \tag{49}$$

Hence $\langle\,,\rangle = \sum g_{ij}\,\alpha^i \otimes \alpha^j$.

2.1.6 Wedge Product

Definition 2.6: Wedge/Exterior Product

Let f be k-linear function and g be an ℓ-linear function on a vector space V. If f, g are alternating[a] then we would like their product to be alternating as well: the *wedge product* or *exterior product* $f \wedge g$

$$f \wedge g := \frac{1}{k!l!} A(f \otimes g) \tag{50}$$

$$:= \frac{1}{k!l!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}(\sigma)) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \tag{51}$$

where $S_{k+\ell}$ is the *permutation group* on $k+\ell$ elements.

[a] An alternating function is one that changes signs if arguments are transposed (e.g. cross-product or determinant).

Note that the wedge product of three alternating functions f, g, h generalizes to

$$f \wedge g \wedge h = \frac{1}{k!\ell!m!} = A(f \otimes g \otimes h) \tag{52}$$

and any number of alternating functions.

Proposition 2.1: Wedge product of 1-covectors

If $\{\alpha^i\}$ are linear functions on V and $v_i \in V$ then

$$\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \ldots, v_k) = \det\Big([\alpha^i(v_j)]\Big) \tag{53}$$

where $[\alpha^i(v_j)]$ is the matrix where the i, j-entry is $\alpha^i(v_j)$.

Proof.

$$\alpha^1 \wedge \cdots \wedge \alpha^k(v_1, \ldots, v_k) = A(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) \tag{54}$$

$$= \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma)) \alpha^1(v_{\sigma(1)}) \wedge \cdots \wedge \alpha^k(v_{\sigma(k)}) \tag{55}$$

$$= \det\Big([\alpha^i(v_j)]\Big) \tag{56}$$

∎

Let $\{e_i\}$ be a basis for a vector space V and let α_j be the dual basis for $V^\vee$. Define *multi-index notation*

$$\begin{aligned} I &:= (i_1, \ldots, i_k) \\ e_I &:= (e_{i_1}, \ldots, e_{i_k}) \\ \alpha_I &:= \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \end{aligned} \tag{57}$$

A k-linear function on V is completely determined by its values on all k-tuples e_I. If f is alternating then you only need all increasing multi-indices $1 \leq i_1 \leq \cdots \leq i_k \leq n$.

Lemma 2.1

Let I, J be increasing multi-indices of length k, then

$$\alpha^I(e_J) = \delta^I_J := \begin{cases} 1 \text{ for } I = J \\ 0 \text{ for } I \neq J \end{cases} \tag{58}$$

Proposition 2.2

The alternating k-linear functions α^I, $I := (i_1 < \cdots < i_k)$ form a basis for the space $A^k(V)$ of alternating k-linear functions on V.

Corollary 2.1

If the vector space V has dimension n, then the vector space $A^k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Corollary 2.2

If $k > \dim(V)$, then $A^k(V) = 0$.

2.1.7 Graded Algebra

Definition 2.7: Graded Algebra

A *graded algebra* A over a field K is one that can be written as a *direct sum*[1]

$$A = \bigoplus_{n \in \mathbb{N}_0} A^n = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$$

such that the multiplication map μ

$$\mu \colon A^k \times A^\ell \to A^{k+\ell}$$

i.e. $A^k \cdot A^\ell \subset A^{k+\ell}$. Elements of any factor A^n of the decomposition are called homogeneous elements of degree n. $A = \bigoplus_k A^k$ is said to *graded commutative* or *anticommutative* if for all $a \in A^k, b \in A^\ell$

$$ab = (-1)^{k\ell} ba$$

Example 2.2. Polynomial Algebra The *polynomial algebra*[2] $A = \mathbb{R}[x, y]$ is *graded by degree*; A^k consists of all homogeneous polynomials of total degree k. It is a direct sum of A^i consisting of homogeneous polynomials of degree i e.g.

$$\left(x \in A^1\right) \times \left(x + y \in A^1\right) = \left(x^2 + xy \in A^2\right)$$

Definition 2.8: Grassmann/Exterior Algebra

For a finite dimensional vector space V define

$$\begin{aligned} A^*(V) &\coloneqq \bigoplus_{k=0}^{\infty} A^k(V) \\ &= \bigoplus_{k=0}^{n} A^k(V) \end{aligned} \tag{59}$$

where $A^k(V)$ is the set of all k-linear covectors on V. With the wedge product as multiplication A^* becomes and anticommutative algebra called the ***exterior algebra*** or ***Grassmann algebra*** of covectors.

[1] The direct sum of two abelian groups A and B is another abelian group $A \oplus B$ consisting of the ordered pairs (a, b) where $a \in A$ and $b \in B$. (Confusingly this ordered pair is also called the cartesian product of the two groups.) To add ordered pairs, we define the sum $(a, b) + (c, d)$ to be $(a + c, b + d)$; in other words addition is defined coordinate-wise. Note that $A^2 \coloneqq A \oplus A$. A similar process can be used to form the direct sum of any two algebraic structures, such as rings, modules, and vector spaces.

[2] The algebra of polynomials in two variables with coefficients and scalars in $\mathbb{R}$ where addition is degree-coefficient wise and "foil" multiplication

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