Non-linear protocols for optimal distributed consensus in networks of dynamic agents

March 15, 2021

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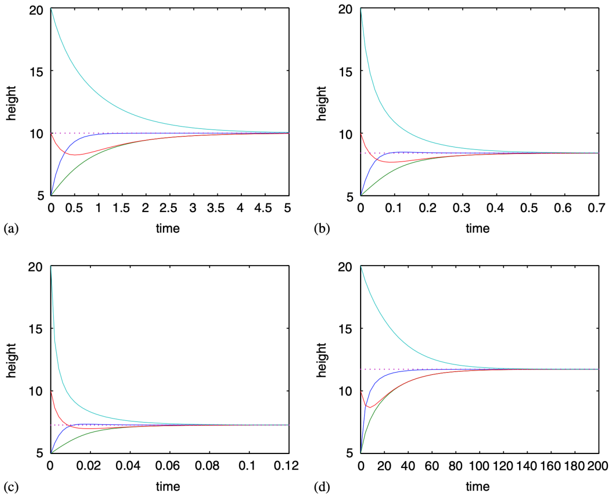
TLDR...



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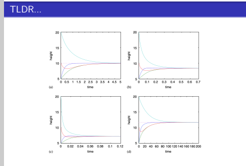

Outline

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1. Definitions
2. The consensus problem
    - Time-invariance of χ
    - Convergence of χ
3. Mechanism design problem

Definitions

Definitions

Definition 1 (Agents)

$\Gamma = 1, \ldots, n$ is a set of *agents/players/nodes/vertices* and $G = (\Gamma, E)$ is a fixed (in time) undirected, connected, network describing the connections between vertices $i \in \Gamma$, where $E \subset \Gamma \times \Gamma$ is the edge set.

Definition 2 (Neighborhood)

A *neighborhood* of a vertex i is the set of all vertices j for which there is a single edge connecting i, j, that is to say $N_i := \{j \mid (i,j) \in E\}$.

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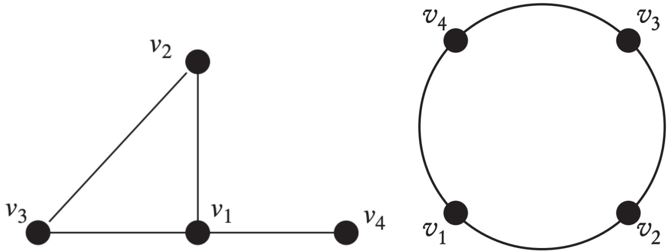

Figure: Neighborhoods

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└─Definitions

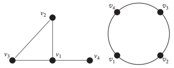

Figure: Neighborhoods

Definition 3 (Control policy)

Let $x_i(t)$ be the state of the agent i at time t, then x_i evolves according to a *distributed* and *stationary* control policy u_i, if $\dot{x}_i = u_i(x_i, \mathbf{x}_{N_i})$, where $\mathbf{x}_{N_i}$ are the states of x_i's neighbors.

Definition 4 (Protocol)

The *protocol* of the network is the collection of controls $\vec{u}(\vec{x}) := (u_i(x_i, \mathbf{x}_{N_i}))$ for all $i \in \Gamma$.

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Definition 5 (Agreement function)

The *agreement function* $\chi : \mathbb{R}^n \to \mathbb{R}$ is any continuous, differentiable function which is permutation invariant, i.e.

$$\chi(x_1, \ldots, x_n) = \chi(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

Definition 6 (Consensus)

To *reach consensus* on *consensus value* $\chi(\mathbf{x}(0))$ means

$$\lim_{t \to \infty} \mathbf{x}(t) = \chi(\mathbf{x}(0))\mathbf{1}$$

where $\mathbf{1} := (1, 1, \ldots, 1)$.

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Time-invariance of χ

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Lemma 8 (Time invariancy)

Let $\mathbf{u}$ *be a stationary consensus protocol. Then* $\chi(\mathbf{x}(t))$ *is stationary, i.e.* $\chi(\mathbf{x}(t)) = \chi(\mathbf{x}(0))$ *for all* $t > 0$*.*

Proof.

By assumption $\mathbf{x}(t) \to \chi(\mathbf{x}(0))\mathbf{1}$. Stationary $\mathbf{u}$ is equivalent to autonomous and therefore, if $\mathbf{x}(t)$ is a solution, then $\mathbf{y}_s(t) := \mathbf{x}(t+s)$ (with $\mathbf{y}_s(0) := \mathbf{x}(s)$) is also a solution. For such $\mathbf{y}_s$ we also have $\mathbf{y}_s(t) \to \chi(\mathbf{y}_s(0))\mathbf{1}$ i.e.

$$\lim_{t\to\infty} \mathbf{y}_s(t) = \chi(\mathbf{y}_s(0)\mathbf{1}) = \chi(\mathbf{x}(s))$$

But since both $\mathbf{y}_s, \mathbf{x}$ converge to the same limit we must have $\chi(\mathbf{x}(s)) = \chi(\mathbf{x}(0))$ for all s. $\square$

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└─The consensus problem

  └─Time-invariance of χ

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Note

$$\frac{\mathrm{d}\chi(\mathbf{x}(t))}{\mathrm{d}t} = \sum_{i\in\Gamma} \frac{\partial\chi}{\partial x_i}\dot{x}_i = \sum_{i\in\Gamma} \frac{\partial\chi}{\partial x_i} u_i = 0$$

Thus, a consensus protocol must satisfy $\nabla\chi \cdot \mathbf{u} = 0$.

Example 9

For $\chi(\mathbf{x}) = \min_{i\in\Gamma}(x_i)$, a suitable $\mathbf{u}$ is

$$u_i = h\left(x_i, \min_{j\in N_i} x_j\right)$$

where $h(x, y) = 0$ when $x = y$.

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Henceforth, we assume further structure for the agreement function:

$$\chi(\mathbf{x}) := f\left(\sum_{i\in\Gamma} g(x_i)\right) \tag{1}$$

with $f, g : \mathbb{R} \to \mathbb{R}$ and $g' \neq 0$.

Fact 10

Means of order psatisfy the assumptions

Mean	$\chi(\mathbf{x})$	$f(y)$	$g(z)$
Arithmetic	$\frac{1}{\lvert\Gamma\rvert}\sum_{i\in\Gamma} x_i$	$\frac{y}{\lvert\Gamma\rvert}$	z
Geometric	$\left(\prod_{i\in\Gamma} x_i\right)^{1/\lvert\Gamma\rvert}$	$e^{y/\lvert\Gamma\rvert}$	$\log(z)$
Harmonic	$\frac{\lvert\Gamma\rvert}{\sum_{i\in\Gamma} x_i^{-1}}$	$\frac{\lvert\Gamma\rvert}{y}$	$\frac{1}{z}$
p−mean	$\left(\frac{1}{\lvert\Gamma\rvert}\sum_{i\in\Gamma} x_i^p\right)^{1/p}$	$\left(\frac{y}{\lvert\Gamma\rvert}\right)^{1/p}$	z^p

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Theorem 11 (Protocol design rule)

The following protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{1}{g'} \sum_{j \in N_i} \phi(x_j, x_i) \tag{2}$$

with $g' \neq 0$, induces time-invariance in χ if ϕ is antisymmetric, i.e.

$$\phi(x_j, x_i) = -\phi(x_i, x_j)$$

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Proof.

χ is time-invariant iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\Gamma} g(x_i) = \sum_{i\in\Gamma}\frac{\mathrm{d}g(x_i(t))}{\mathrm{d}t} = \sum_{i\in\Gamma}\frac{\mathrm{d}g(x_i)}{\mathrm{d}x_i}\dot{x}_i = \sum_{i\in\Gamma} g' u_i = 0$$

Finally, since ϕ is antisymmetric and the graph defining the network is undirected, we have that

$$\sum_{i\in\Gamma} g' u_i = \frac{1}{g'}\sum_{i\in\Gamma} g' \sum_{j\in N_i}\phi(x_j, x_i) = 0$$

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Consider $\phi(x_j, x_i) := \alpha \cdot (x_j - x_i)$. Then the $p-$mean is time invariant under protocol

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Convergence of χ

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Owing to time-invariance of χ, if the system converges, it will converge to $\chi(\mathbf{x}(0))\mathbf{1}$. But it does not necessarily converge. when g is strictly increasing and the function ϕ is defined

$$\phi(x_j, x_i) := \alpha\phi(\theta(x_j) - \theta(x_i))$$

where $\alpha > 0$ and ϕ is continuous, locally Lipschitz, odd and strictly increasing, and $\theta : \mathbb{R} \to \mathbb{R}$ is differentiable with θ' locally Lipschitz and strictly positive. Thus, the protocol (2) becomes

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{\alpha}{g'} \sum_{j \in N_i} \phi\left(\theta(x_j) - \theta(x_i)\right) \tag{3}$$

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Lemma 13

Let G be a network and $\mathbf{u}$ be a protocol with components defined by eqn. (3). Then all equilibria $\mathbf{x}^*$ of the network have the following properties:

1. $\mathbf{x}^* = \lambda \mathbf{1}$ *for some* λ
2. *if* $\mathbf{x}(t)$ *converges to the equilibrium* $\lambda_0 \mathbf{1}$, *then* $\lambda_0 = \chi(\mathbf{x}(0))$, *for any initial state* $\mathbf{x}(0)$

Proof (sketch).

Sufficiency 1 $x_i = \lambda$, then $\phi(\theta(\lambda) - \theta(\lambda)) = 0$ since ϕ is odd and continuous.

Necessity 1 $\mathbf{x}^* \neq \lambda$, then there exists $u_i < 0$ (contradicts equilibrium).

Convergence if $\lambda \neq \chi(\mathbf{x}(0))$ then χ is not time-invariant. □

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└The consensus problem
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Theorem 14

Let G be a network of agents that implement a distributed and stationary protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{\alpha}{g'} \sum_{j \in N_i} \phi\left(\theta(x_j) - \theta(x_i)\right)$$

against agreement function of the form (1) with $g' > 0$. Then the agents asymptotically reach consensus on $\chi\left(\mathbf{x}(0)\right)$ for any initial state $\mathbf{x}(0)$.

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Proof (idea).

Define

$$\eta_i := g(x_i) - g\left(\chi\left(\mathbf{x}\left(0\right)\right)\right)$$

and note that, since $\boldsymbol{\eta}$ is strictly increasing (since g is) and $\boldsymbol{\eta} = 0$ iff $\mathbf{x} = \chi\left(\mathbf{x}\left(0\right)\right)$, consensus corresponds to asymptotic stability of $\boldsymbol{\eta}$ around 0. Introduce a candidate Lyapunov function

$$V\left(\boldsymbol{\eta}\right) := \frac{1}{2}\sum_{i\in\Gamma}\eta_i^2$$

Then $V\left(\boldsymbol{\eta}\right) = 0$ iff $\boldsymbol{\eta} = 0$, $V\left(\boldsymbol{\eta}\right) > 0$ if $\boldsymbol{\eta} \neq 0$, and $\dot{V}\left(\boldsymbol{\eta}\right) < 0$ if $\boldsymbol{\eta} \neq 0$ proves stability. $\square$

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Mechanism design problem

Mechanism design problem

Definition 15 (Individual objective function)

Define an *individual objective function* for an agent i

$$J_i(x_i, \mathbf{x}_{N_i}, u_i) := \lim_{T\to\infty} \int_0^T \left(F(x_i, \mathbf{x}_{N_i}) + \rho u_i^2\right) \mathrm{d}t$$

where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative *penalty function* that measures the deviation of agent i's state x_i from its neighbors' states. A protocol is *optimal* if each u_i optimizes an agent's corresponding individual objective.

Mechanism design problem

Consider a network of agents. The *mechanism design problem* is, for any agreement function χ, determine a penalty F such that there exists an optimal consensus protocol $\mathbf{u}$ with respect to $\chi(\mathbf{x}(0))$ for any initial state $x(0)$.

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where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative *penalty function* that measures the deviation of agent i's state x_i from its neighbors' states. A protocol is *optimal* if each u_i optimizes an agent's corresponding individual objective.
Mechanism design problem
Consider a network of agents. The *mechanism design problem* is, for any agreement function χ, determine a penalty F such that there exists an optimal consensus protocol $\mathbf{u}$ with respect to $\chi(\mathbf{x}(0))$ for any initial state $x(0)$.

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