Non-linear protocols for optimal distributed consensus in networks of dynamic agents

D. Bauso and L. Giarre and R. Pesenti

Maksim Levental

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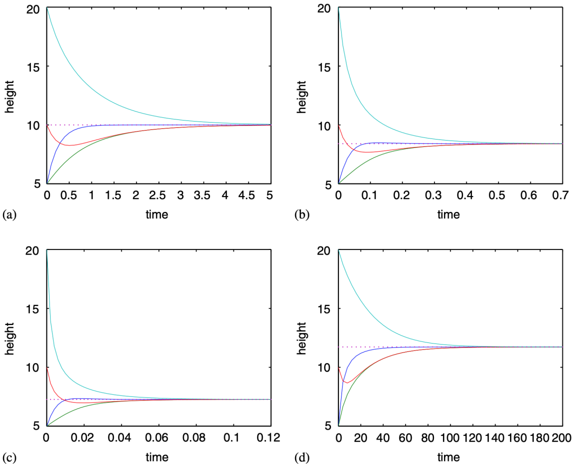
TLDR...



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Outline

1. Definitions
2. Consensus
   - Time-invariance of χ
   - Convergence of χ
3. Mechanism design
4. Simulations
   - Graph Laplacian
5. Epilogue: Computational Power

Definitions

Definition (Agents)

$\Gamma = 1, \ldots, n$ is a set of *agents/players/nodes/vertices* and $G = (\Gamma, E)$ is a fixed (in time) undirected, connected, network describing the connections between vertices $i \in \Gamma$, where $E \subset \Gamma \times \Gamma$ is the edge set.

Definition (Neighborhood)

A *neighborhood* of a vertex i is the set of all vertices j for which there is a single edge connecting i, j, that is to say $N_i := \{j \mid (i, j) \in E\}$.

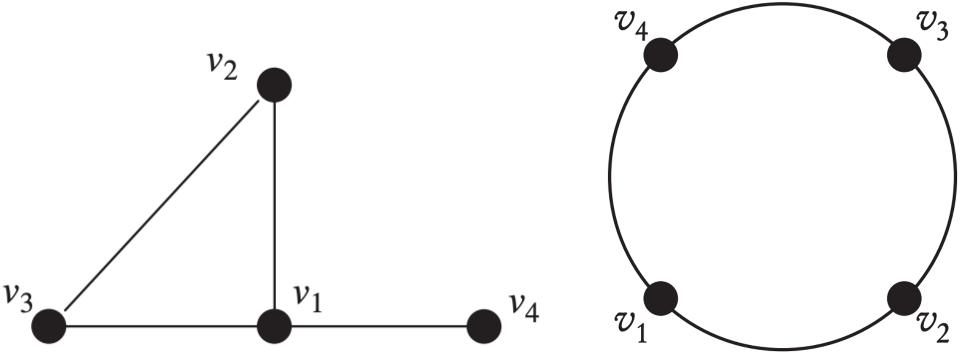


Figure: Example networks

Definition (Control policy)

Let $x_i(t)$ be the state of the agent i at time t, then x_i evolves according to a *distributed* and *stationary* control policy u_i, if $\dot{x}_i = u_i(x_i, \mathbf{x}_{N_i})$, where $\mathbf{x}_{N_i}$ are the states of x_i's neighbors.

Definition (Protocol)

The *protocol* of the network is the collection of controls $\mathbf{u}(\mathbf{x}) := (u_i(x_i, \mathbf{x}_{N_i}))$ for all $i \in \Gamma$.

Definition (Agreement function)

The *agreement function* $\chi : \mathbb{R}^n \to \mathbb{R}$ is any continuous, differentiable function which is permutation invariant, i.e.

$$\chi\left(\mathbf{x}\right) = \chi\left(x_1, \ldots, x_n\right) = \chi\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$$

Definition (Consensus)

To *reach consensus* on *consensus value* $\chi\left(\mathbf{x}\left(0\right)\right)$ means

$$\lim_{t\to\infty} \mathbf{x}\left(t\right) = \chi\left(\mathbf{x}\left(0\right)\right)\mathbf{1}$$

where $\mathbf{1} := \left(1, 1, \ldots, 1\right)$.

Consensus

Definition (Consensus problem)

Given a network G of agents and agreement function χ, the *consensus problem* is to design a protocol $\mathbf{u}$ such that consensus is reached for any consensus value $\chi\left(\mathbf{x}\left(0\right)\right)$.

Definition (Consensus protocol)

A protocol is a *consensus protocol* if it is the solution to a consensus problem.

Time-invariance of χ

Lemma (Time invariancy)

Let $\mathbf{u}$ *be a stationary consensus protocol. Then* $\chi(\mathbf{x}(t))$ *is stationary, i.e.* $\chi(\mathbf{x}(t)) = \chi(\mathbf{x}(0))$ *for all* $t > 0$.

Proof.

By assumption $\mathbf{x}(t) \to \chi(\mathbf{x}(0))\mathbf{1}$. Stationary $\mathbf{u} \implies$ if $\mathbf{x}(t)$ is a solution, then $\mathbf{y}_s(t) := \mathbf{x}(t+s)$, with $\mathbf{y}_s(0) := \mathbf{x}(s)$, is also a solution. For such $\mathbf{y}_s$ we also have $\mathbf{y}_s(t) \to \chi(\mathbf{y}_s(0))\mathbf{1}$, i.e.

$$\lim_{t\to\infty} \mathbf{y}_s(t) = \chi(\mathbf{y}_s(0)\mathbf{1}) = \chi(\mathbf{x}(s)\mathbf{1})$$

But since both $\mathbf{y}_s, \mathbf{x}$ converge to the same limit we must have $\chi(\mathbf{x}(s)) = \chi(\mathbf{x}(0))$ for all s. □

Note

$$\frac{\mathrm{d}\chi(\mathbf{x}(t))}{\mathrm{d}t} = \sum_{i\in\Gamma} \frac{\partial \chi}{\partial x_i}\dot{x}_i = \sum_{i\in\Gamma} \frac{\partial \chi}{\partial x_i} u_i = 0$$

Thus, a consensus protocol must satisfy $\nabla\chi \cdot \mathbf{u} = 0$.

Example

For $\chi(\mathbf{x}) = \min_{i\in\Gamma}(x_i)$, a suitable $\mathbf{u}$ is

$$u_i = h\left(x_i, \min_{j\in N_i} x_j\right)$$

where $h(x, y) = 0$ when $x = y$.

Henceforth, we assume further structure for the agreement function:

$$\chi(\mathbf{x}) := f\left(\sum_{i\in\Gamma} g(x_i)\right) \tag{1}$$

with $f, g : \mathbb{R} \to \mathbb{R}$ and $g' \neq 0$.

Fact

Means of order psatisfy the assumptions

Mean	$\chi(\mathbf{x})$	$f(y)$	$g(z)$
Arithmetic	$\frac{1}{\lvert\Gamma\rvert}\sum_{i\in\Gamma} x_i$	$\frac{y}{\lvert\Gamma\rvert}$	z
Geometric	$\left(\prod_{i\in\Gamma} x_i\right)^{1/\lvert\Gamma\rvert}$	$e^{y/\lvert\Gamma\rvert}$	$\log(z)$
Harmonic	$\frac{\lvert\Gamma\rvert}{\sum_{i\in\Gamma} x_i^{-1}}$	$\frac{\lvert\Gamma\rvert}{y}$	$\frac{1}{z}$
p−mean	$\left(\frac{1}{\lvert\Gamma\rvert}\sum_{i\in\Gamma} x_i^p\right)^{1/p}$	$\left(\frac{y}{\lvert\Gamma\rvert}\right)^{1/p}$	z^p

Theorem (Protocol design rule)

The following protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{1}{g'} \sum_{j \in N_i} \phi(x_j, x_i) \tag{2}$$

with $g' \neq 0$*, induces time-invariance in* χ *if* ϕ *is antisymmetric, i.e.*

$$\phi(x_j, x_i) = -\phi(x_i, x_j)$$

Proof.

χ is time-invariant iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\Gamma} g(x_i) = \sum_{i\in\Gamma}\frac{\mathrm{d}g(x_i(t))}{\mathrm{d}t} = \sum_{i\in\Gamma}\frac{\mathrm{d}g(x_i)}{\mathrm{d}x_i}\dot{x}_i = \sum_{i\in\Gamma} g' u_i = 0$$

Finally, since ϕ is antisymmetric and the graph defining the network is undirected, we have that

$$\sum_{i\in\Gamma} g' u_i = \frac{1}{g'}\sum_{i\in\Gamma} g' \sum_{j\in N_i}\phi(x_j, x_i) = 0$$

$\square$

Example (to be proved)

Consider $\phi(x_j, x_i) := \alpha \cdot (x_j - x_i)$. Then the $p-$mean is time invariant under protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{x_i^{1-p}}{p}\sum_{j\in N_i}\phi(x_j, x_i) = \alpha\cdot\frac{x_i^{1-p}}{p}\sum_{j\in N_i}(x_j - x_i)$$

Convergence of χ

- Owing to time-invariance of χ, if the system converges, it will converge to $\chi(\mathbf{x}(0))\mathbf{1}$.
- We prove convergence if

$$\phi(x_j, x_i) := \alpha\phi(\theta(x_j) - \theta(x_i))$$

  with $\alpha > 0$ and ϕ is continuous, $\theta : \mathbb{R} \to \mathbb{R}$ is differentiable with θ' locally Lipschitz and strictly positive.
- Thus, the protocol (2) becomes

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{\alpha}{g'} \sum_{j \in N_i} \phi\left(\theta(x_j) - \theta(x_i)\right) \tag{3}$$

  with g is strictly increasing.

Lemma

Let G be a network and $\mathbf{u}$ be a protocol with components defined by eqn. (3). Then all equilibria $\mathbf{x}^*$ of the network have the following properties:

1. $\mathbf{x}^* = \lambda\mathbf{1}$ *for some* λ
2. *if* $\mathbf{x}(t)$ *converges to the equilibrium* $\lambda_0\mathbf{1}$, *then* $\lambda_0 = \chi(\mathbf{x}(0))$, *for any initial state* $\mathbf{x}(0)$

Proof (sketch).

Sufficiency 1 $\mathbf{x} = \lambda\mathbf{1}$, then $\phi(\theta(\lambda) - \theta(\lambda)) = 0$ since ϕ is odd and continuous.

Necessity 1 $\mathbf{x}^* \neq \lambda$, then there exists $u_i < 0$ (contradicts equilibrium).

Convergence if $\lambda \neq \chi(\mathbf{x}(0))$ then χ is not time-invariant.

$\square$

Theorem

Let G be a network of agents that implement a distributed and stationary protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{\alpha}{g'} \sum_{j \in N_i} \phi\left(\theta(x_j) - \theta(x_i)\right)$$

against agreement function of the form (1) with $g' > 0$. Then the agents asymptotically reach consensus on $\chi\left(\mathbf{x}\left(0\right)\right)$ for any initial state $\mathbf{x}(0)$.

Proof (idea).

Define

$$\eta_i := g(x_i) - g\left(\chi\left(\mathbf{x}\left(0\right)\right)\right)$$

and note that, since $\boldsymbol{\eta}$ is strictly increasing (since g is) and $\boldsymbol{\eta} = 0$ iff $\mathbf{x} = \chi\left(\mathbf{x}\left(0\right)\right)$, consensus corresponds to asymptotic stability of $\boldsymbol{\eta}$ around 0. Introduce a candidate Lyapunov function

$$V\left(\boldsymbol{\eta}\right) := \frac{1}{2}\sum_{i\in\Gamma}\eta_i^2$$

Then $V\left(\boldsymbol{\eta}\right) = 0$ iff $\boldsymbol{\eta} = 0$, $V\left(\boldsymbol{\eta}\right) > 0$ if $\boldsymbol{\eta} \neq 0$, and $\dot{V}\left(\boldsymbol{\eta}\right) < 0$ if $\boldsymbol{\eta} \neq 0$ proves stability. □

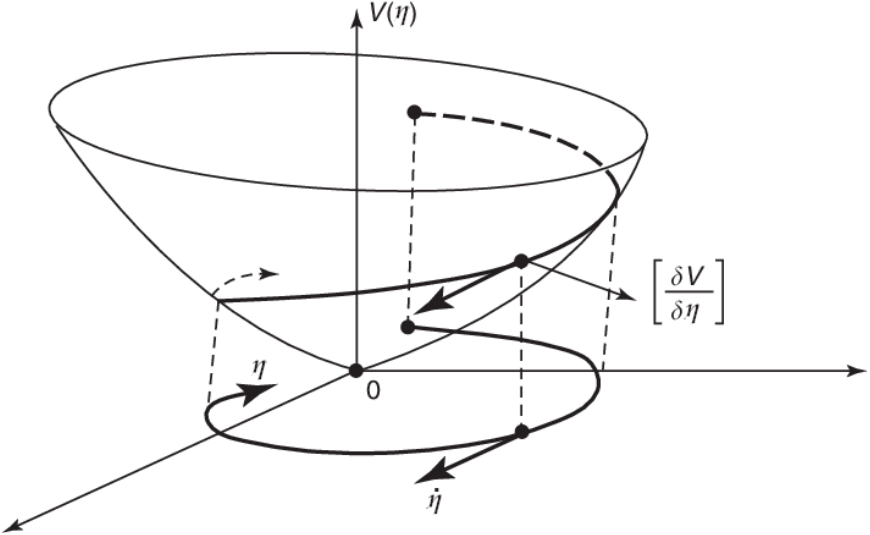


Figure: Lyapunov function

Mechanism design

Definition (Individual objective function)

Define an *individual objective function* for an agent i

$$J_i(x_i, \mathbf{x}_{N_i}, u_i) := \lim_{T\to\infty} \int_0^T \left(F(x_i, \mathbf{x}_{N_i}) + \rho u_i^2\right) \mathrm{d}t$$

where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative *penalty function.* A protocol is *optimal* if each u_i optimizes an agent's corresponding individual objective.

Definition (Mechanism design problem)

Consider a network of agents. The *mechanism design problem* is, for any agreement function χ, to determine a penalty F such that there exists an optimal consensus protocol $\mathbf{u}$ with respect to $\chi(\mathbf{x}(0))$ for any initial state $x(0)$.

Note that the mechanism design problem is over an infinite *planning horizon* $T \to \infty$.

Definition

Let a one-step *action/planning period* be $\delta = t_{k+1} - t_k$. Further let $\hat{x}_i(\tau, t_k)$, $\hat{\mathbf{x}}_{N_i}(\tau, t_k)$, $\hat{u}_i(\tau, t_k)$ be agent and neighboring states and agent controls, for $\tau \geq t_k$.

Hence, we keep neighboring agents' states constant over a single planning period and ultimately let $\delta \to 0$ to get an approximation to the original problem.

Receding horizon problem

Define the *receding horizon objective function*

$$\hat{J}_i(\hat{x}_i, \hat{\mathbf{x}}_{N_i}, \hat{u}_i) := \lim_{T\to\infty} \int_{t_k}^{T} \left(\hat{F}(\hat{x}_i\left(\tau, t_k\right), \hat{\mathbf{x}}_{N_i}\left(\tau, t_k\right)) + \rho \hat{u}_i^2 \right) \mathrm{d}\tau$$

Then for all agents $i \in \Gamma$ and discrete time steps t_k, given initial states $x_i(t_0), \mathbf{x}_{N_i}(t_0)$ find

$$\hat{u}_i^* := \operatorname{argmin} \hat{J}_i(\hat{x}_i, \hat{\mathbf{x}}_{N_i}, \hat{u}_i)$$

subject to

$$\begin{aligned}
\dot{\hat{x}}_i(\tau, t_k) &= \hat{u}_i(\tau, t_k) \\
\dot{\hat{x}}_j(\tau, t_k) &= \hat{u}_j(\tau, t_k) = 0 \quad \forall j \in N_i \\
\hat{x}_i(t_k, t_k) &= x_i(t_k) \\
\hat{x}_j(t_k, t_k) &= x_j(t_k) \quad \forall j \in N_i
\end{aligned}$$

Note that the assumption that all neighboring states are fixed during an action step implies

$$\hat{J}_i(\hat{x}_i, \hat{u}_i) := \lim_{T\to\infty} \int_{t_k}^{T} \left(\hat{F}(\hat{x}_i(\tau, t_k)) + \rho \hat{u}_i^2(\tau, t_k)\right) \mathrm{d}\tau$$

Thus, we use Pontryagin's minimum principle.

Definition (Pontryagin's minimum principle)

Let *Hamiltonian* be

$$H(\hat{x}_i, \hat{u}_i, p_i) = L(\hat{x}_i, \hat{u}_i) + p_i \hat{u}_i$$

where the *Lagrangian* $L := F(\hat{x}_i + \rho \hat{u}_i^2)$. Then H abides by the Pontryagin necessary conditions at the optimum $(\hat{x}_i, \hat{u}_i, p_i)$:

$$\frac{\partial H}{\partial \hat{u}_i} = 0 \Rightarrow p_i = -2\rho \hat{u}_i \quad \text{optimality}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} \quad \text{multiplier}$$

$$\dot{\hat{x}}_i = -\frac{\partial H}{\partial p_i} \Rightarrow \dot{\hat{x}}_i = \hat{u}_i \quad \text{costate equation}$$

$$\left.\frac{\partial^2 H}{\partial \hat{u}_i^2}\right|_{\hat{x}_i = \hat{x}_i^*, \hat{u}_i = \hat{u}_i^*, p_i = p_i^*} \geq 0 \Rightarrow \rho \geq 0 \quad \text{minimality equation}$$

$$H(\hat{x}_i^*, \hat{u}_i^*, p_i^*) = 0 \quad \text{boundary}$$

Theorem

Consider the penalty function

$$F(\hat{x}_i(\tau, t_k)) := \rho \left(\frac{1}{g'} \sum_{j \in N_i} \theta(x_j(t_k)) - \theta(\hat{x}_i(\tau, t_k)) \right)^2$$

where g is increasing, θ is concave, and $(1/g')$ is convex. Then

$$\hat{u}_i^* := \frac{\alpha}{g'} \sum_{j \in N_i} \theta(x_j(t_k)) - \theta(x_i(\tau))$$

solves the mechanism design problem.

Proof (idea).

Check each of the conditions of Pontryagin's minimum principle. □

Corollary

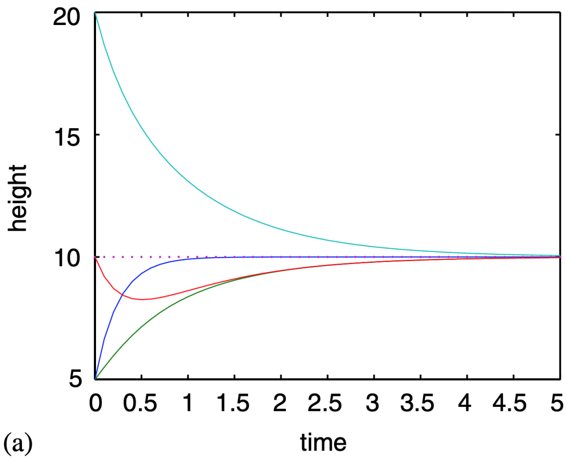
Taking $\delta \to 0$ we get that the penalty function

$$F(x_i, \mathbf{x}_{N_i}) := \rho \left(\frac{1}{g'} \sum_{j \in N_i} \theta(x_j) - \theta(x_i) \right)^2$$

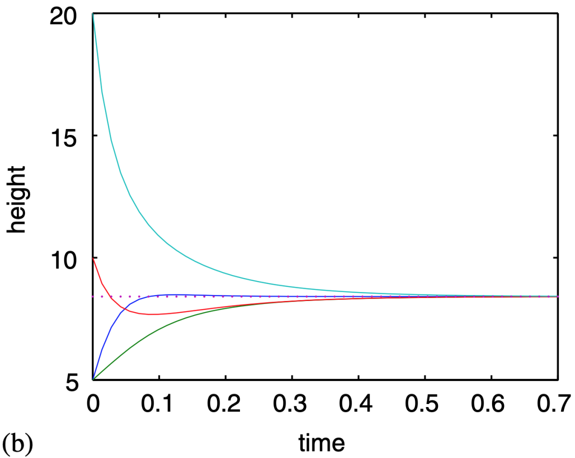
and the optimal control law

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{1}{g'} \sum_{j \in N_i} \theta(x_j) - \theta(x_i)$$

Simulations

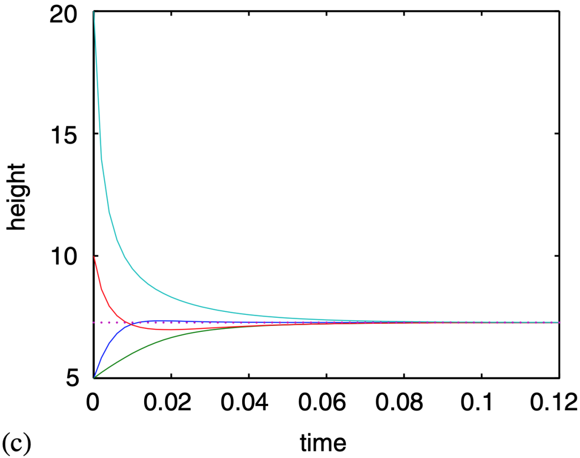


$$F := \left(\sum_{j \in N_i} (x_j - x_i)\right)^2 \Rightarrow u_i = \sum_{j \in N_i} (x_j(t) - x_i(t)) \Rightarrow \lim_{t \to \infty} \chi(\mathbf{x}(t)) \to \frac{1}{|\Gamma|} \sum_{i \in \Gamma} x_i(0)$$



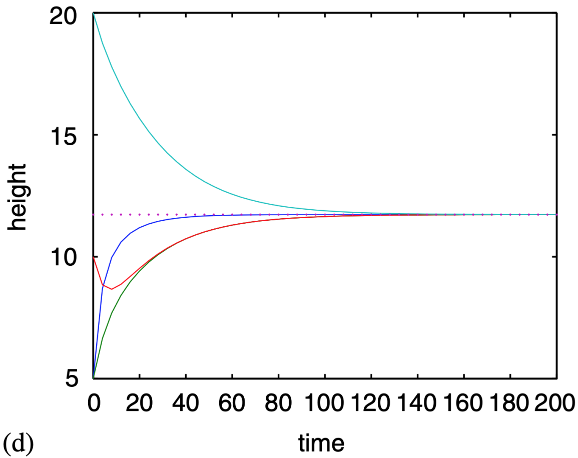
(b)

$$F{:=}\left(\sum_{j\in N_i} x_i(x_j-x_i)\right)^2 \Rightarrow u_i{=}x_i\sum_{j\in N_i}(x_j(t)-x_i(t)) \Rightarrow \lim_{t\to\infty}\chi(\mathbf{x}(t))\to\left(\prod_{i\in\Gamma} x_i(0)\right)^{1/|\Gamma|}$$



(c)

$$F := \left(x_i^2 \sum_{j \in N_i} (x_j - x_i)\right)^2 \Rightarrow u_i = -x_i^2 \sum_{j \in N_i} (x_j(t) - x_i(t)) \Rightarrow \lim_{t\to\infty} \chi(\mathbf{x}(t)) \to \frac{|\Gamma|}{\sum_{i\in\Gamma} (x_i(0))^{-1}}$$



(d)

$$F := \frac{1}{2x_i}\left(\sum_{j\in N_i}(x_j - x_i)\right)^2 \Rightarrow u_i = \frac{1}{2x_i}\sum_{j\in N_i}(x_j(t) - x_i(t)) \Rightarrow \lim_{t\to\infty}\chi(\mathbf{x}(t)) \to \sqrt{\frac{1}{|\Gamma|}\sum_{i\in\Gamma}(x_i(0))^2}$$

In general

$$F\left(x_i, \mathbf{x}_{N_i}\right) := \left(\frac{x_i^{1-p}}{p} \sum_{j \in N_i} \left(x_j - x_i\right)\right)^2 \Rightarrow$$

$$u_i\left(x_i, \mathbf{x}_{N_i}\right) := \frac{x_i^{1-p}}{p} \sum_{j \in N_i} \left(x_j - x_i\right) \Rightarrow$$

$$\lim_{t \to \infty} \chi\left(\mathbf{x}\left(t\right)\right) \to \left(\frac{1}{|\Gamma|} \sum_{i \in \Gamma} \left(x_i\left(0\right)\right)^p\right)^{1/p}$$

Graph Laplacian

Definitions

The *adjacency matrix* of a graph $G = (\Gamma, E)$ is defined

$$\Delta A_{ij}(G) := \begin{cases} 1 & (i,j) \in E \\ 1 & (j,i) \in E \\ 0 & \text{otherwise} \end{cases}$$

The *degree matrix* of a graph G is defined

$$\Delta_{ij}(G) := \begin{cases} \deg(v_i) & i = j \\ 0 & i \neq j \end{cases}$$

The *Laplacian* of the graph G is defined

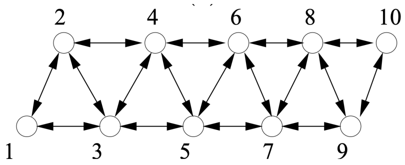
$$L(G) := \Delta(G) - A(G)$$

Theorem

Suppose that each node $v \in G$ of a connected graph G receives the information from its neighboring nodes after a fixed delay $\delta > 0$ and applies a linear protocol. Then for

$$\delta = \delta^* := \frac{\pi}{2\lambda_n} \quad \lambda_n = \lambda_{\max}(L)$$

where λ_i are eigenvalues of the Laplacian, the system has a globally asymptotically stable oscillatory solution with frequency $\omega = \lambda_n$ [6].



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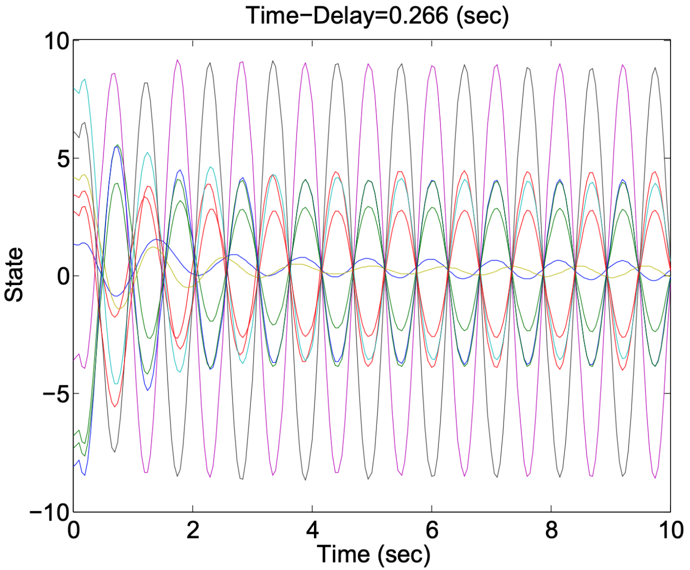


Time−Delay=0.266 (sec)
State
Time (sec)
10
5
0
−5
−10
0
2
4
6
8
10

Epilogue: Computational Power

Definition

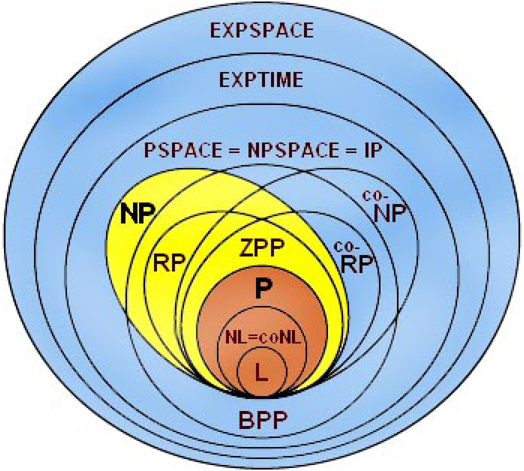
NL := NSPACE($\log n$) is the set of all decision problems that can be solved by a non-deterministic Turing machine using a logarithmic amount of space.

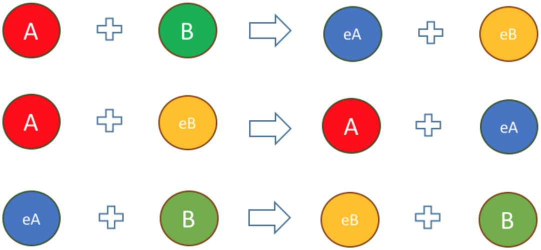
$\mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PH} \subseteq \mathrm{PSPACE}$

$\mathrm{PSPACE} \subseteq \mathrm{EXPTIME} \subseteq \mathrm{EXPSPACE}$

$\mathrm{NL} \subsetneq \mathrm{PSPACE} \subsetneq \mathrm{EXPSPACE}$

$\mathrm{P} \subsetneq \mathrm{EXPTIME}$





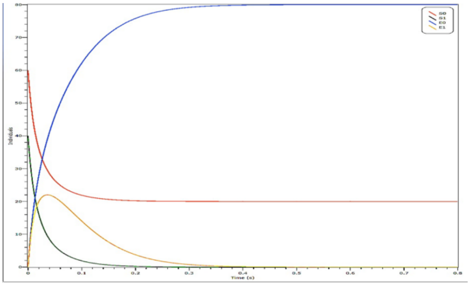


Figure: Four-state exact majority algorithm; nodes start in either A (55%) or B state (45%), and proceed to interact. The resulting population has converged to only A and eA node types, corresponding to the initial majority of A [1].

Population protocols stably compute any predicate in the class definable by formulas of Presburger arithmetic, which includes Boolean combinations of threshold-k, majority, and equivalence modulo m. All stably computable predicates are shown to be in NL [2].

- "...Inspired by the work of the verification community on Emerson and Namjoshi's broadcast protocols, we show that NL-power is also achieved by extending population protocols with reliable broadcasts, a simpler, standard communication primitive." [4]
- "...We show that every predicate computable by population protocols is computable by a BCP with expected $O(n \log n)$ interactions, which is asymptotically optimal. We further show that every log-space, randomized Turing machine can be simulated by a BCP with $O(n \log n \cdot T)$ interactions in expectation, where T is the expected runtime of the Turing machine. This allows us to characterise polynomial-time BCPs as computing exactly the number predicates in ZPL..." [5]

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