Non-linear protocols for optimal distributed consensus in networks of dynamic agents

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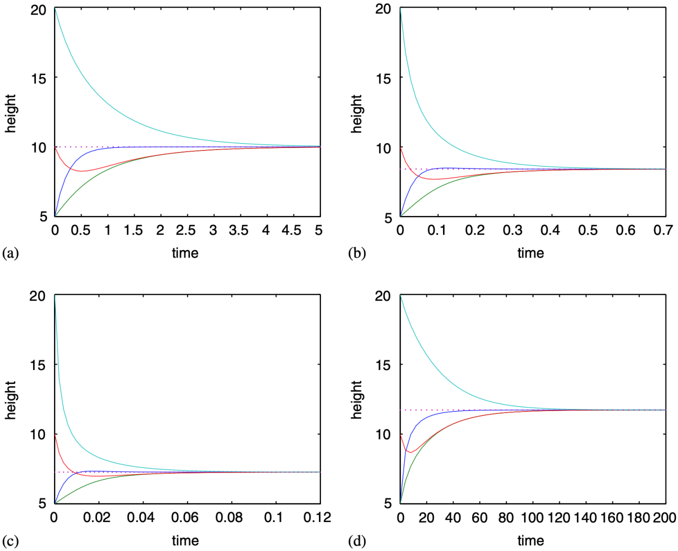
TLDR...



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(a)
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(b)
0 0.02 0.04 0.06 0.08 0.1 0.12
(c)
0 20 40 60 80 100 120 140 160 180 200
(d)

Outline

1. Definitions
2. The consensus problem
   - Time-invariance of χ
   - Convergence of χ
3. Mechanism design problem
4. Simulations
5. Bonus 1: Graph Laplacian
6. Bonus 2: Computational Power

Definitions

Definition 1 (Agents)

$\Gamma = 1, \ldots, n$ is a set of *agents/players/nodes/vertices* and $G = (\Gamma, E)$ is a fixed (in time) undirected, connected, network describing the connections between vertices $i \in \Gamma$, where $E \subset \Gamma \times \Gamma$ is the edge set.

Definition 2 (Neighborhood)

A *neighborhood* of a vertex i is the set of all vertices j for which there is a single edge connecting i, j, that is to say $N_i := \{j \mid (i, j) \in E\}$.

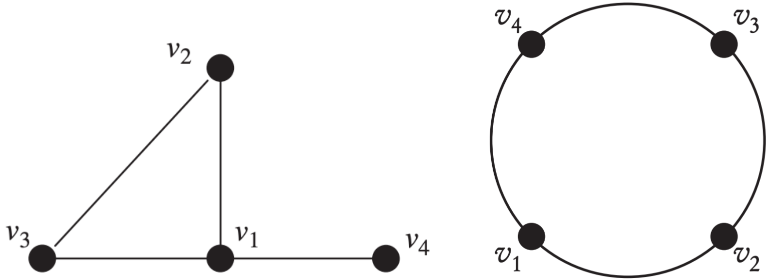


Figure: Neighborhoods

Definition 3 (Control policy)

Let $x_i(t)$ be the state of the agent i at time t, then x_i evolves according to a *distributed* and *stationary* control policy u_i, if $\dot{x}_i = u_i(x_i, \mathbf{x}_{N_i})$, where $\mathbf{x}_{N_i}$ are the states of x_i's neighbors.

Definition 4 (Protocol)

The *protocol* of the network is the collection of controls $\mathbf{u}(\mathbf{x}) := (u_i(x_i, \mathbf{x}_{N_i}))$ for all $i \in \Gamma$.

Definition 5 (Agreement function)

The *agreement function* $\chi : \mathbb{R}^n \to \mathbb{R}$ is any continuous, differentiable function which is permutation invariant, i.e.

$$\chi\left(\mathbf{x}\right) = \chi\left(x_1, \ldots, x_n\right) = \chi\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$$

Definition 6 (Consensus)

To *reach consensus* on *consensus value* $\chi\left(\mathbf{x}\left(0\right)\right)$ means

$$\lim_{t\to\infty} \mathbf{x}\left(t\right) = \chi\left(\mathbf{x}\left(0\right)\right)\mathbf{1}$$

where $\mathbf{1} := (1, 1, \ldots, 1)$.

The consensus problem

Definition 7 (Consensus problem)

Given a network G of agents and agreement function χ, the *consensus problem* is to design a protocol $\mathbf{u}$ such that consensus is reached for any consensus value $\chi\left(\mathbf{x}\left(0\right)\right)$.

Definition 8 (Consensus protocol)

A protocol is a *consensus protocol* if it is the solution to a consensus problem.

Time-invariance of χ

Lemma 9 (Time invariancy)

Let $\mathbf{u}$ *be a stationary consensus protocol. Then* $\chi(\mathbf{x}(t))$ *is stationary, i.e.* $\chi(\mathbf{x}(t)) = \chi(\mathbf{x}(0))$ *for all* $t > 0$.

Proof.

By assumption $\mathbf{x}(t) \to \chi(\mathbf{x}(0))\mathbf{1}$. Stationary $\mathbf{u} \implies$ if $\mathbf{x}(t)$ is a solution, then $\mathbf{y}_s(t) := \mathbf{x}(t+s)$, with $\mathbf{y}_s(0) := \mathbf{x}(s)$, is also a solution. For such $\mathbf{y}_s$ we also have $\mathbf{y}_s(t) \to \chi(\mathbf{y}_s(0))\mathbf{1}$, i.e.

$$\lim_{t\to\infty} \mathbf{y}_s(t) = \chi(\mathbf{y}_s(0)\mathbf{1}) = \chi(\mathbf{x}(s)\mathbf{1})$$

But since both $\mathbf{y}_s, \mathbf{x}$ converge to the same limit we must have $\chi(\mathbf{x}(s)) = \chi(\mathbf{x}(0))$ for all s. □

Note

$$\frac{\mathrm{d}\chi(\mathbf{x}(t))}{\mathrm{d}t} = \sum_{i\in\Gamma} \frac{\partial\chi}{\partial x_i}\dot{x}_i = \sum_{i\in\Gamma} \frac{\partial\chi}{\partial x_i} u_i = 0$$

Thus, a consensus protocol must satisfy $\nabla\chi \cdot \mathbf{u} = 0$.

Example 10

For $\chi(\mathbf{x}) = \min_{i\in\Gamma}(x_i)$, a suitable $\mathbf{u}$ is

$$u_i = h\left(x_i, \min_{j\in N_i} x_j\right)$$

where $h(x, y) = 0$ when $x = y$.

Henceforth, we assume further structure for the agreement function:

$$\chi(\mathbf{x}) := f\left(\sum_{i\in\Gamma} g(x_i)\right) \tag{1}$$

with $f, g : \mathbb{R} \to \mathbb{R}$ and $g' \neq 0$.

Fact 11

Means of order psatisfy the assumptions

Mean	$\chi(\mathbf{x})$	$f(y)$	$g(z)$
Arithmetic	$\frac{1}{\lvert\Gamma\rvert}\sum_{i\in\Gamma} x_i$	$\frac{y}{\lvert\Gamma\rvert}$	z
Geometric	$\left(\prod_{i\in\Gamma} x_i\right)^{1/\lvert\Gamma\rvert}$	$e^{y/\lvert\Gamma\rvert}$	$\log(z)$
Harmonic	$\frac{\lvert\Gamma\rvert}{\sum_{i\in\Gamma} x_i^{-1}}$	$\frac{\lvert\Gamma\rvert}{y}$	$\frac{1}{z}$
p−mean	$\left(\frac{1}{\lvert\Gamma\rvert}\sum_{i\in\Gamma} x_i^p\right)^{1/p}$	$\left(\frac{y}{\lvert\Gamma\rvert}\right)^{1/p}$	z^p

Theorem 12 (Protocol design rule)

The following protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{1}{g'} \sum_{j \in N_i} \phi(x_j, x_i) \tag{2}$$

with $g' \neq 0$, induces time-invariance in χ if ϕ is antisymmetric, i.e.

$$\phi(x_j, x_i) = -\phi(x_i, x_j)$$

Proof.

χ is time-invariant iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\Gamma} g(x_i) = \sum_{i\in\Gamma}\frac{\mathrm{d}g(x_i(t))}{\mathrm{d}t} = \sum_{i\in\Gamma}\frac{\mathrm{d}g(x_i)}{\mathrm{d}x_i}\dot{x}_i = \sum_{i\in\Gamma} g' u_i = 0$$

Finally, since ϕ is antisymmetric and the graph defining the network is undirected, we have that

$$\sum_{i\in\Gamma} g' u_i = \frac{1}{g'}\sum_{i\in\Gamma} g' \sum_{j\in N_i}\phi(x_j, x_i) = 0$$

$\square$

Example 13 (to be proved)

Consider $\phi(x_j, x_i) := \alpha \cdot (x_j - x_i)$. Then the $p-$mean is time invariant under protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{x_i^{1-p}}{p} \sum_{j \in N_i} \phi(x_j, x_i) = \alpha \cdot \frac{x_i^{1-p}}{p} \sum_{j \in N_i} (x_j - x_i)$$

Convergence of χ

- Owing to time-invariance of χ, if the system converges, it will converge to $\chi(\mathbf{x}(0))\mathbf{1}$.
- We prove convergence if

$$\phi(x_j, x_i) := \alpha\phi(\theta(x_j) - \theta(x_i))$$

with $\alpha > 0$ and ϕ is continuous, $\theta : \mathbb{R} \to \mathbb{R}$ is differentiable with θ' locally Lipschitz and strictly positive.

- Thus, the protocol (2) becomes

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{\alpha}{g'} \sum_{j \in N_i} \phi\left(\theta(x_j) - \theta(x_i)\right) \tag{3}$$

with g is strictly increasing.

Lemma 14

Let G be a network and $\mathbf{u}$ be a protocol with components defined by eqn. (3). Then all equilibria $\mathbf{x}^*$ of the network have the following properties:

1. $\mathbf{x}^* = \lambda \mathbf{1}$ *for some* λ
2. *if* $\mathbf{x}(t)$ *converges to the equilibrium* $\lambda_0 \mathbf{1}$, *then* $\lambda_0 = \chi(\mathbf{x}(0))$, *for any initial state* $\mathbf{x}(0)$

Proof (sketch).

Sufficiency 1 $\mathbf{x} = \lambda \mathbf{1}$, then $\phi(\theta(\lambda) - \theta(\lambda)) = 0$ since ϕ is odd and continuous.

Necessity 1 $\mathbf{x}^* \neq \lambda$, then there exists $u_i < 0$ (contradicts equilibrium).

Convergence if $\lambda \neq \chi(\mathbf{x}(0))$ then χ is not time-invariant.

$\square$

Theorem 15

Let G be a network of agents that implement a distributed and stationary protocol

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{\alpha}{g'} \sum_{j \in N_i} \phi\left(\theta(x_j) - \theta(x_i)\right)$$

against agreement function of the form (1) with $g' > 0$. Then the agents asymptotically reach consensus on $\chi\left(\mathbf{x}\left(0\right)\right)$ for any initial state $\mathbf{x}(0)$.

Proof (idea).

Define

$$\eta_i := g(x_i) - g\left(\chi\left(\mathbf{x}\left(0\right)\right)\right)$$

and note that, since $\boldsymbol{\eta}$ is strictly increasing (since g is) and $\boldsymbol{\eta} = 0$ iff $\mathbf{x} = \chi\left(\mathbf{x}\left(0\right)\right)$, consensus corresponds to asymptotic stability of $\boldsymbol{\eta}$ around 0. Introduce a candidate Lyapunov function

$$V\left(\boldsymbol{\eta}\right) := \frac{1}{2}\sum_{i\in\Gamma}\eta_i^2$$

Then $V\left(\boldsymbol{\eta}\right) = 0$ iff $\boldsymbol{\eta} = 0$, $V\left(\boldsymbol{\eta}\right) > 0$ if $\boldsymbol{\eta} \neq 0$, and $\dot{V}\left(\boldsymbol{\eta}\right) < 0$ if $\boldsymbol{\eta} \neq 0$ proves stability. □

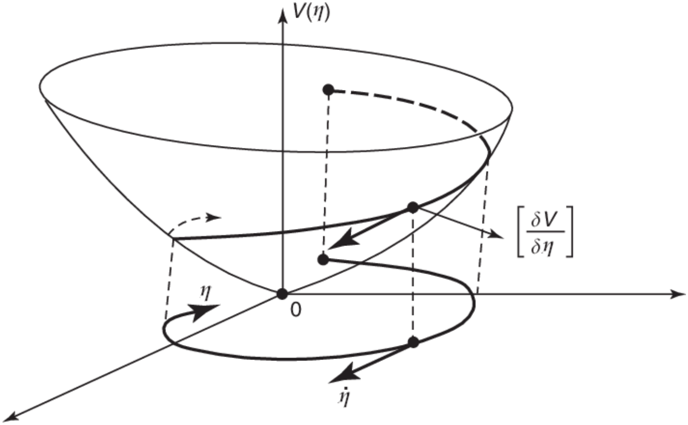


Figure: Lyapunov function

Mechanism design problem

Definition 16 (Individual objective function)

Define an *individual objective function* for an agent i

$$J_i(x_i, \mathbf{x}_{N_i}, u_i) := \lim_{T\to\infty} \int_0^T \left(F(x_i, \mathbf{x}_{N_i}) + \rho u_i^2\right) \mathrm{d}t$$

where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a non-negative *penalty function.* A protocol is *optimal* if each u_i optimizes an agent's corresponding individual objective.

Definition 17 (Mechanism design problem)

Consider a network of agents. The *mechanism design problem* is, for any agreement function χ, to determine a penalty F such that there exists an optimal consensus protocol $\mathbf{u}$ with respect to $\chi(\mathbf{x}(0))$ for any initial state $x(0)$.

Note that the mechanism design problem is over an infinite *planning horizon* $T \to \infty$. We need to discretize:

Definition 18

Let a one-step *action/planning period* be $\delta = t_{k+1} - t_k$. Further let $\hat{x}_i(\tau, t_k)$, $\hat{\mathbf{x}}_{N_i}(\tau, t_k)$, $\hat{u}_i(\tau, t_k)$ be agent and neighboring states and agent controls, for $\tau \geq t_k$.

Hence, we keep neighboring agents' states constant over a single planning period and ultimately let $\delta \to 0$ to get an approximation to the original problem.

Receding horizon problem

Define the *receding horizon objective function*

$$\hat{J}_i(\hat{x}_i, \hat{\mathbf{x}}_{N_i}, \hat{u}_i) := \lim_{T\to\infty} \int_{t_k}^{T} \left(\hat{F}\left(\hat{x}_i\left(\tau, t_k\right), \hat{\mathbf{x}}_{N_i}\left(\tau, t_k\right)\right) + \rho \hat{u}_i^2\right) \mathrm{d}\tau$$

Then for all agents $i \in \Gamma$ and discrete time steps t_k, given initial states $x_i(t_0), \mathbf{x}_{N_i}(t_0)$ find

$$\hat{u}_i^* := \operatorname{argmin} \hat{J}_i(\hat{x}_i, \hat{\mathbf{x}}_{N_i}, \hat{u}_i)$$

subject to

$$\begin{aligned}
\dot{\hat{x}}_i(\tau, t_k) &= \hat{u}_i(\tau, t_k) \\
\dot{\hat{x}}_j(\tau, t_k) &= \hat{u}_j(\tau, t_k) = 0 \quad \forall j \in N_i \\
\hat{x}_i(t_k, t_k) &= x_i(t_k) \\
\hat{x}_j(t_k, t_k) &= x_j(t_k) \quad \forall j \in N_i
\end{aligned}$$

Note that the assumption that all neighboring states are fixed during an action step implies

$$\hat{J}_i(\hat{x}_i, \hat{u}_i) := \lim_{T\to\infty} \int_{t_k}^{T} \left(\hat{F}(\hat{x}_i(\tau, t_k)) + \rho \hat{u}_i^2(\tau, t_k)\right) \mathrm{d}\tau$$

Thus, we use Pontryagin's minimum principle.

Definition 19 (Pontryagin's minimum principle)

Let *Hamiltonian* be

$$H(\hat{x}_i, \hat{u}_i, p_i) = L(\hat{x}_i, \hat{u}_i) + p_i \hat{u}_i$$

where the *Lagrangian* $L := F(\hat{x}_i + \rho \hat{u}_i^2)$. Then H abides by the Pontryagin necessary conditions at the optimum $(\hat{x}_i, \hat{u}_i, p_i)$:

$$\begin{aligned}
\frac{\partial H}{\partial \hat{u}_i} = 0 \Rightarrow p_i = -2\rho \hat{u}_i & \quad \text{optimality} \\
\dot{p}_i = -\frac{\partial H}{\partial x_i} & \quad \text{multiplier} \\
\dot{\hat{x}}_i = -\frac{\partial H}{\partial p_i} \Rightarrow \dot{\hat{x}}_i = \hat{u}_i & \quad \text{costate equation} \\
\left.\frac{\partial^2 H}{\partial \hat{u}_i^2}\right|_{\hat{x}_i = \hat{x}_i^*, \hat{u}_i = \hat{u}_i^*, p_i = p_i^*} \geq 0 \Rightarrow \rho \geq 0 & \quad \text{minimality equation} \\
H(\hat{x}_i^*, \hat{u}_i^*, p_i^*) = 0 & \quad \text{boundary}
\end{aligned}$$

Theorem 20

Consider the penalty function

$$F(\hat{x}_i(\tau, t_k)) := \rho \left(\frac{1}{g'} \sum_{j \in N_i} \theta(x_j(t_k)) - \theta(\hat{x}_i(\tau, t_k)) \right)^2$$

where g is increasing, θ is concave, and $(1/g')$ is convex. Then

$$\hat{u}_i^* := \frac{\alpha}{g'} \sum_{j \in N_i} \theta(x_j(t_k)) - \theta(x_i(\tau))$$

solves the mechanism design problem.

Proof (idea).

Check each of the conditions of Pontryagin's minimum principle. □

Corollary 21

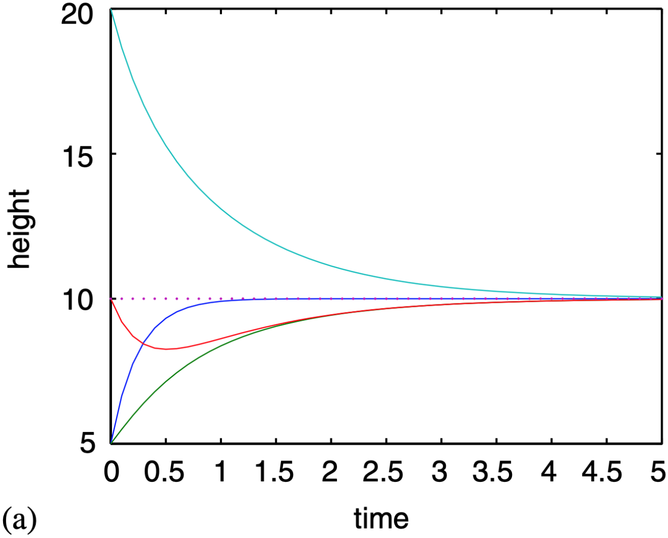
Taking $\delta \to 0$ we get that the penalty function

$$F(x_i, \mathbf{x}_{N_i}) := \rho \left(\frac{1}{g'} \sum_{j \in N_i} \theta(x_j) - \theta(x_i) \right)^2$$

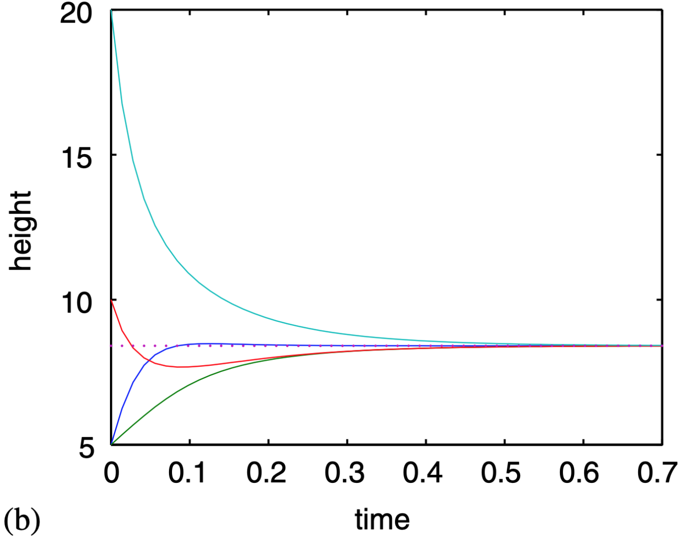
and the optimal control law

$$u_i(x_i, \mathbf{x}_{N_i}) := \frac{1}{g'} \sum_{j \in N_i} \theta(x_j) - \theta(x_i)$$

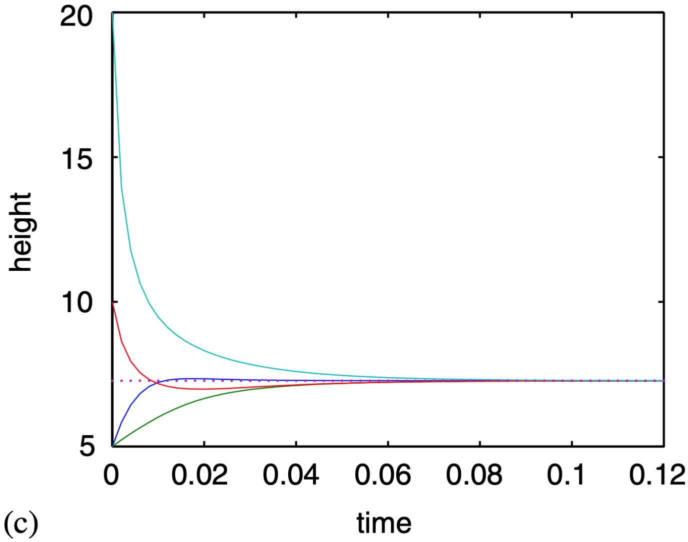
Simulations



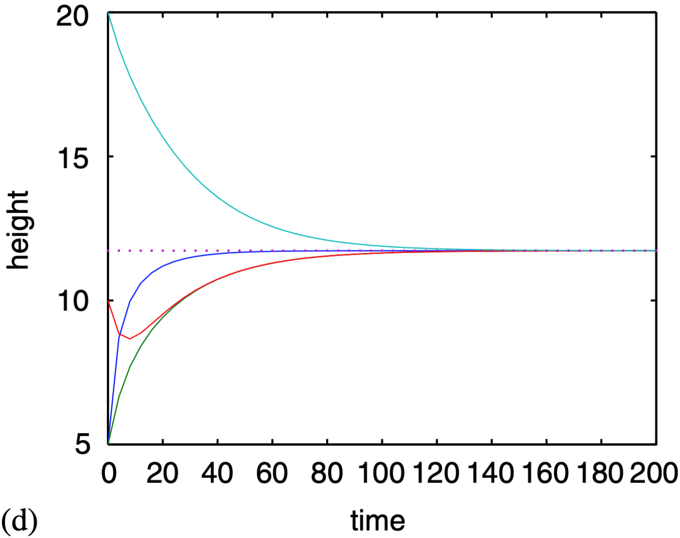
$$F := \left(\sum_{j\in N_i}(x_j - x_i)\right)^2 \Rightarrow u_i = \sum_{j\in N_i}(x_j(t) - x_i(t)) \Rightarrow \lim_{t\to\infty}\chi(\mathbf{x}(t)) \to \frac{1}{|\Gamma|}\sum_{i\in\Gamma} x_i(0)$$



$$F := \left(\sum_{j\in N_i} x_i(x_j - x_i)\right)^2 \Rightarrow u_i = x_i \sum_{j\in N_i}(x_j(t) - x_i(t)) \Rightarrow \lim_{t\to\infty} \chi(\mathbf{x}(t)) \to \left(\prod_{i\in\Gamma} x_i(0)\right)^{1/|\Gamma|}$$



$$F := \left(x_i^2 \sum_{j \in N_i} (x_j - x_i)\right)^2 \Rightarrow u_i = -x_i^2 \sum_{j \in N_i} (x_j(t) - x_i(t)) \Rightarrow \lim_{t \to \infty} \chi(\mathbf{x}(t)) \to \frac{|\Gamma|}{\sum_{i \in \Gamma} (x_i(0))^{-1}}$$



(d)

$$F := \frac{1}{2x_i}\left(\sum_{j\in N_i}(x_j - x_i)\right)^2 \Rightarrow u_i = \frac{1}{2x_i}\sum_{j\in N_i}(x_j(t) - x_i(t)) \Rightarrow \lim_{t\to\infty}\chi(\mathbf{x}(t)) \to \sqrt{\frac{1}{|\Gamma|}\sum_{i\in\Gamma}(x_i(0))^2}$$

In general

$$F\left(x_i, \mathbf{x}_{N_i}\right) := \left(\frac{x_i^{1-p}}{p} \sum_{j \in N_i} \left(x_j - x_i\right)\right)^2 \Rightarrow$$

$$u_i\left(x_i, \mathbf{x}_{N_i}\right) := \frac{x_i^{1-p}}{p} \sum_{j \in N_i} \left(x_j - x_i\right) \Rightarrow$$

$$\lim_{t \to \infty} \chi\left(\mathbf{x}\left(t\right)\right) \to \left(\frac{1}{|\Gamma|} \sum_{i \in \Gamma} \left(x_i\left(0\right)\right)^p\right)^{1/p}$$

Bonus 1: Graph Laplacian

Definitions 22

The *adjacency matrix* of a graph $G = (\Gamma, E)$ is defined

$$\Delta A_{ij}(G) := \begin{cases} 1 & (i,j) \in E \\ 1 & (j,i) \in E \\ 0 & \text{otherwise} \end{cases}$$

The *degree matrix* of a graph G is defined

$$\Delta_{ij}(G) := \begin{cases} \deg(v_i) & i = j \\ 0 & i \neq j \end{cases}$$

The *Laplacian* of the graph G is defined

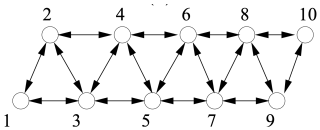
$$L(G) := \Delta(G) - A(G)$$

Theorem 23

Suppose that each node $v \in G$ of a connected graph G receives the information from its neighboring nodes after a fixed delay $\delta > 0$ and applies a linear protocol. Then for

$$\delta = \delta^* := \frac{\pi}{2\lambda_n} \quad \lambda_n = \lambda_{\max}(L)$$

where λ_i are eigenvalues of the Laplacian, the system has a globally asymptotically stable oscillatory solution with frequency $\omega = \lambda_n$.



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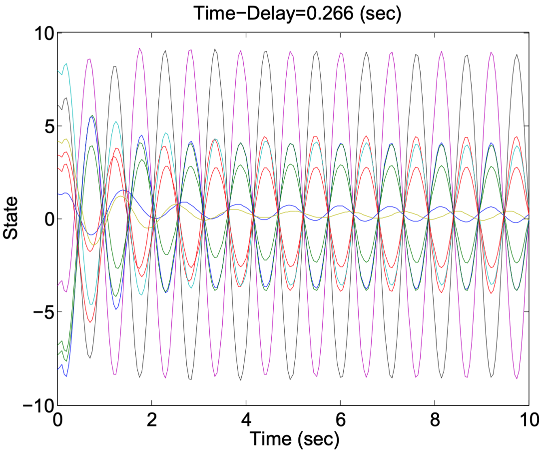


Time−Delay=0.266 (sec)
10
5
0
−5
−10
State
0
2
4
6
8
10
Time (sec)

Bonus 2: Computational Power

Definition 24

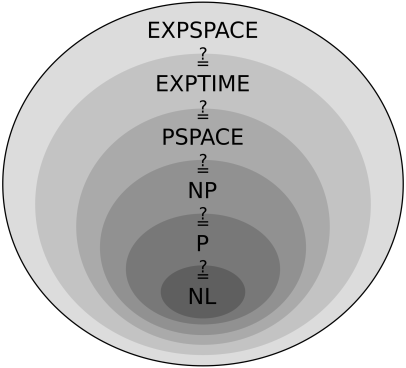
PSPACE is the set of all decision problems that can be solved by a Turing machine using a polynomial amount of space.

$\mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PH} \subseteq \mathrm{PSPACE}$

$\mathrm{PSPACE} \subseteq \mathrm{EXPTIME} \subseteq \mathrm{EXPSPACE}$

$\mathrm{NL} \subsetneq \mathrm{PSPACE} \subsetneq \mathrm{EXPSPACE}$

$\mathrm{P} \subsetneq \mathrm{EXPTIME}$



"The computational power of networks of small resource-limited mobile agents is explored. Two new models of computation based on pairwise interactions of finite-state agents in populations of finite but unbounded size are defined. With a fairness condition on interactions, the concept of stable computation of a function or predicate is defined. Protocols are given that stably compute any predicate in the class definable by formulas of Presburger arithmetic, which includes Boolean combinations of threshold-k, majority, and equivalence modulo m. All stably computable predicates are shown to be in $\mathsf{NL} := \mathsf{NSPACE}(\log n)$. Assuming uniform random sampling of interacting pairs yields the model of conjugating automata. Any counter machine with $O(1)$ counters of capacity $O(n)$ can be simulated with high probability by a conjugating automaton in a population of size n. All predicates computable with high probability in this model are shown to be in P; they can also be computed by a randomized logspace machine in exponential time."

"Population protocols are a formal model of computation by identical, anonymous mobile agents interacting in pairs. Their computational power is rather limited: Angluin et al. have shown that they can only compute the predicates over $\mathbb{N}^k$ expressible in Presburger arithmetic. For this reason, several extensions of the model have been proposed, including the addition of devices called cover-time services, absence detectors, and clocks. All these extensions increase the expressive power to the class of predicates over $\mathbb{N}^k$ lying in the complexity class NL when the input is given in unary. However, these devices are difficult to implement, since they require that an agent atomically receives messages from all other agents in a population of unknown size; moreover, the agent must know that they have all been received. Inspired by the work of the verification community on Emerson and Namjoshi's broadcast protocols, we show that NL-power is also achieved by extending population protocols with reliable broadcasts, a simpler, standard communication primitive."

"Broadcast consensus protocols (BCPs) are a model of computation, in which anonymous, identical, finite-state agents compute by sending/receiving global broadcasts. BCPs are known to compute all number predicates in $\mathsf{NL} = \mathsf{NSPACE}(\log n)$ where n is the number of agents. They can be considered an extension of the well-established model of population protocols. This paper investigates execution time characteristics of BCPs. We show that every predicate computable by population protocols is computable by a BCP with expected $O(n \log n)$ interactions, which is asymptotically optimal. We further show that every log-space, randomized Turing machine can be simulated by a BCP with $O(n \log n \cdot T)$ interactions in expectation, where T is the expected runtime of the Turing machine. This allows us to characterise polynomial-time BCPs as computing exactly the number predicates in ZPL, i.e. predicates decidable by log-space bounded randomised Turing machine with zero-error in expected polynomial time where the input is encoded as unary." unary."

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