№2

$u_{tt} = \Delta u, \quad 0 < x < 9, \quad t > 0$

$u|_{x=0} = u|_{x=2} = 0$

$u|_{y=0} = u|_{y=5} = 0$

$u|_{t=0} = \sin\left(\frac{\pi}{2}x\right)\sin\left(\frac{7}{5}\pi y\right)$

$u_t|_{t=0} = \sin\left(\frac{5\pi}{2}x\right)\sin\left(\frac{3\pi}{5}y\right)$

$u(x,y,t) = T(t)\cdot V(x,y), \; T \not\equiv 0, \; V \not\equiv 0$

$T''V = \Delta V \cdot T$

$$\frac{\Delta V(x,y)}{V(x,y)} = \frac{T''(t)}{T(t)} = -\lambda^2$$

Получена задача Штурма-Лиувилля.

$$\begin{cases} \Delta V + \lambda^2 V = 0 \\ V|_{x=0} = V|_{x=2} = 0 \\ V|_{y=0} = V|_{y=5} = 0 \end{cases}$$

Пусть $V(x,y) = X(x)\cdot Y(y); \; X \not\equiv 0, \; Y \not\equiv 0$

$X''Y + XY'' + \lambda^2 XY = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''(y)}{Y(y)} - \lambda^2 = -\mu^2$

$\lambda^2 = \mu^2 + \vartheta^2$

$$\begin{cases} X''(x) + \mu^2 X(x) = 0 \\ X'(0) = 0 \\ X(2) = 0. \end{cases} \quad (1)$$

$$\begin{cases} Y''(y) + \vartheta^2 Y(y) = 0 \\ Y'(0) = 0 \\ Y(5) = 0 \end{cases} \quad (2)$$

(1): $X(x) = A\cos\mu x + B\sin\mu x$

$X(0) = A; \quad A = 0.$

$X(2) = B\sin 2\mu = 0; \quad \mu_n = \frac{\pi n}{2}, n \in \mathbb{N}$

($B \neq 0$, т.к. $A = 0$ и при $B = 0$ $X(x) = 0$, что противоречит условию).

$X_n(x) = \sin\frac{\pi n}{2}x; \quad n \in \mathbb{N}$

(2): $Y(y) = A_1\cos\vartheta y + B_1\sin\vartheta y$

$Y(0) = A_1; \quad A_1 = 0$

$Y(5) = B_1\sin 5\vartheta = 0; \quad \vartheta_m = \frac{\pi m}{5}, m \in \mathbb{N}$

$Y_m(y) = \sin\frac{\pi m}{5}y$

$\lambda_{nm}^2 = \left(\frac{\pi n}{2}\right)^2 + \left(\frac{\pi m}{5}\right)^2$

$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}(t)\sin\frac{\pi n}{2}x\sin\frac{\pi m}{5}y$.

$T''_{nm}(t) + \lambda_{nm}^2 T_{nm}(t) = 0$

$T_{nm}(t) = A_{nm}\cos\lambda_{nm}t + B_{nm}\sin\lambda_{nm}t$

$u\big|_{t=0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}(0)\sin\frac{\pi n}{2}x\sin\frac{\pi m}{5}y =$

$= \sin\frac{\pi}{2}x\sin\frac{7}{5}\pi y;$

$T_{nm}(0) = \begin{cases} 1, & (n,m)=(1,7) \\ 0, & else \end{cases}$

$u_t\big|_{t=0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T'_{nm}(0)\sin\frac{\pi n}{2}x\sin\frac{\pi m}{5}y =$

$= \sin\frac{5\pi}{2}x\sin\frac{3\pi}{5}y \Rightarrow T'_{nm}(0) = \begin{cases} 1, & (n,m)=(5,3) \\ 0, & else \end{cases}$

$\Rightarrow T_{nm}(0) = A_{nm} = \begin{cases} 1, & (n,m)=(1,7) \\ 0, & else \end{cases}$ и

$T'_{nm}(0) = \lambda_{nm}B_{nm} = \begin{cases} 1, & (n,m)=(5,3) \\ 0, & else \end{cases}$

$\Rightarrow u(x,y,t) = \cos\lambda_{17}t\sin\frac{\pi}{2}x\sin\frac{7}{5}\pi y + \frac{1}{\lambda_{53}}\sin\lambda_{53}t$

$x \cdot \cancel{4} \sin\frac{5}{2}\pi x \sin\frac{3}{5}\pi y$, где

$$\lambda_{17}^2 = \left(\frac{\pi}{2}\right)^2 + \left(\frac{7}{5}\pi\right)^2, \quad \lambda_{17} = \frac{\pi\sqrt{221}}{10}$$

$$\lambda_{53}^2 = \frac{25}{4}\pi^2 + \frac{9}{25}\pi^2; \quad \lambda_{53} = \frac{\pi\sqrt{661}}{10}$$