THE AUSTRALIAN NATIONAL UNIVERSITY

Mid-semester Examination — June 2020

Macroeconomic Theory

ECON 4422/8022

⏲ Upload Time	:	30 Minutes
✍ Writing Time	:	120 Minutes
☑ Permitted Materials	:	Everything plus the kitchen sink
$\langle \cdot\% \rangle$	:	Mark allocation operator

— IMPORTANT —

There are **TWO Parts** to this examination with varying levels of difficulty:

Part A is a *partially guided* test of your ability to solve basic problems related to class material, to reason logically and to be able to interpret resulting economic insights from your analyses. This part also tests your general comprehension of basic concepts or definitions, your ability to read `Python` code and to design algorithms related an economic problem. Completion of this section would enable you to attain up to 70% of the maximal final examination mark.

- Use the **Multiple Choice Form** (MCF) section on `WATTLE` to submit your answers to **Part A**.
- Pick the most accurate answer.
- **Warning**: Each sub-Question's *arabic* number correspond to the question number on the `WATTLE` MCF. You are responsible for correct entries.
- You may upload a PDF scan of your background or scratch work together with your Part B answers. Depending on how informative and clear these "scratch works" are, they may or may not be considered in grading your answers here.
- Recommended time for completion: 60-70 minutes

Part B examines the same attributes as Part A. In addition, in this part, you must be able to apply existing knowledge and skills to problems that go beyond familiar examples.

- You must hand-write your answers and scan them to a single PDF file. Upload these to the `WATTLE` exam portal.
- Answers are expected to be succinct but complete. *Unreasonably long and irrelevant answers may be penalized.*
- Recommended time for completion: 30-40 minutes

Time management. You are wholly responsible for managing your total time. Ensure that you have ample time left to complete scanning and uploading documents to `WATTLE` within the total two-hour allocation.

— Part A —

Question A (10%)

1. Which among the following papers provided the earliest account of the use of the Bellman functional equation in economic modelling and analysis? ⟨ **2%** ⟩

   (a) Frank Ramsey's "A Mathematical Theory of Saving" published in the *Economic Journal* in 1928.

   (b) David Cass' 1965 paper in the *Review of Economic Studies* entitled "Optimum Growth in an Aggregative Model of Capital Accumulation".

   (c) Abraham Wald's "Sequential Analysis" published in 1947.

   (d) Martin Beckmann and Richard Muth's 1954 Cowles Commission paper, "On the Solution to the 'Fundamental Equation' of Inventory Theory".

   (e) Buzz Brock and Len Mirman's 1972 *Journal of Economic Theory* paper "Optimal Economic Growth And Uncertainty: The Discounted Case".

2. Which of the following is the most accurate statement? ⟨ **2%** ⟩

   (a) Observed macroeconomic time-series data is often serially correlated. That is why economists tend to focus on Markovian or recursive equilibrium concepts for their models.

   (b) John Rawls recommended the use of ex-post heterogeneous agents models in order to understand the mechanics of wealth and consumption inequality.

   (c) Rabee Tourky is the current director of ANU's Research School of Economics.

   (d) Economists such as Emi Nakamura and Jón Steisson of UC Berkeley had previously documented that prices tend to be sticky at the microeconomic or at the universal-product-code level. This implies that price stickiness is economically inefficient and thus requires monetary policy correction.

   (e) If you can prove that a decision maker's Bellman equation is a contraction map then there is a unique value function solving that equation.

3. Pick the most accurate answer. We tend to refer to models with search and matching markets as Non-Walrasian because: ⟨ **2%** ⟩

   (a) Agents must play a bargaining game in these models.

   (b) The Hosio's condition does not always holds.

   (c) The decentralized solution may sometimes be efficient.

   (d) There is no notion of a centralized market.

   (e) The invisible hand only exists as a supercritical Hopf bifurcation.

4. Real Business Cycle models [...] ⟨ **2%** ⟩

   (a) are real because they explain how different sectors of the economy interact.

   (b) imply a system of linear logit regressions.

(c) gave economists a starting point to think about how microeconomic theory and stochastic processes can be combined to rationalize the time-series behavior of some macroeconomic data.

(d) are real because they were used to predict economic downturns.

(e) were invented because microeconomists were bored with studying axiomatic choice theory.

5. Which of the following is true or the most accurate? ⟨ **2%** ⟩

   (a) Robert E. Lucas, Jr. invented the concept of rational expectations.

   (b) Rational expectations is the product of neo-conservative political thinking among decision theorists.

   (c) Rational expectations is the idea that selfish, optimizing decision makers arrive a correct forecasts of future events.

   (d) Rational expectations is necessarily the consequence of evolutionary learning.

   (e) Rational expectations only requires that agents make forecasts based on probability models that are consistent with an equilibrium of their environment.

Question B (20%) Time is finite and indexed by $t \in \{0, 1, ..., T\}$. Let the optimal value of a policy maker beginning with resources k_0 be given by:

$$V_0(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \beta^t (c_t)^\alpha : k_{t+1} = \min\{k_t, 1\} - c_t, 0 \leq c_t \leq \min\{k_t, 1\}, k_{T+1} \geq 0 \right\}, \tag{B.1}$$

where $\alpha \in (0, 1)$; and $k = K/L$ and c, respectively, refer to per-worker capital stock and consumption. The state space $X \ni k_t$ is bounded.

6. Describe precisely what we mean by a strategy in this setting. ⟨ **4%** ⟩

   (a) A strategy is a date and state contingent plan $\{g_t(k_t)\}_{t=0}^{T}$ such that $c_t = g_t(k_t)$ at each date t and state k_t.

   (b) A strategy is an optimal date and state contingent plan $\{g_t(k_t)\}_{t=0}^{T}$ such that $c_t = g_t(k_t)$ at each date t and state k_t.

   (c) A strategy is the optimal date and state contingent plan $\{g_t(k_t)\}_{t=0}^{T}$ such that $c_t = g_t(k_t)$ at each date t and state k_t.

   (d) A strategy is a policy selection $c_t = g_t(k_t)$ at each date t and state k_t.

   (e) None of the above.

7. Now re-write the sequence problem (B.1) as a recursive one. We know $V_{T+1}(\min\{k_T, 1\} - c_T) = 0$. At each $t \in \{0, 1, ..., T\}$, the Bellman equation can be written as: ⟨ **4%** ⟩

   (a) $V_t(k_t) = \max_{c_t} \{\alpha(k_t) + \beta V_{t+1}(\min\{k_t, 1\} - k_{t+1}) : 0 \leq k_{t+1} \leq \min\{k_t, 1\}\}$.

   (b) $V_t(k_t) = \max_{k_{t+1}} \{(c_t)^\alpha + \beta V_{t+1}(\min\{k_t, 1\} - c_t) : 0 \leq k_{t+1} \leq \min\{k_t, 1\}\}$.

   (c) $V_t(k_t) = \max_{c_t} \{\alpha \ln(c_t) + \beta V_{t+1}(\min\{k_t, 1\} - c_t) : 0 \leq c_t \leq \min\{k_t, 1\}\}$.

   (d) $V(k_t) = \max_{c_t} \{(c_t)^\alpha + \beta V(\min\{k_t, 1\} - c_t) : 0 \leq c_t \leq \min\{k_t, 1\}\}$.

   (e) There is more than one correct answer.

8. There exists a unique solution to the recursive representation of problem (B.1) because ________ [✄]. This solution is ________ [↯]. The optimizer to problem (B.1) ________ [✂].

   ⟨ **6%** ⟩

   (a) ✄ — the Bellman operator is a β-contraction
   ↯ — a unique value function
   ✂ — is unique.

   (b) ✄ — at each date, the preference set is strictly convex and the production set is convex.
   ↯ — a non-singleton set of value functions
   ✂ — is unique.

   (c) ✄ — the Bellman operator is a β-contraction
   ↯ — a unique value function
   ✂ — not unique.

(d) ✃ — at each date, U and f are twice-continuously differentiable
↯ — a unique value function
✂ — is unique.

(e) None of the above.

9. Suppose you are supplied a number $k_0 = \texttt{a} > 0$, a time horizon `T`, pre-defined function `f` representing f, and, a solution contained as a `Python` list `g` of consumption-function optimizers to problems (B.1). Which of these code snippets are logically correct? ⟨ **6%** ⟩

   (a) This one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T-1)
for t in range(T-1):
  k[t+1] = g(k[t])
  y[t] = f(k[t])
```

   (b) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T-1)
for t in range(T-1):
  k[t+1] = g[t](k[t])
  y[t] = f[t](k[t])
```

   (c) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T)
for t in range(T-1):
  k[t+1] = g(k[t])
  y[t] = f(k[t])
```

   (d) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T)
for t in power.range.r(T-1):
  k[t+1] = g(k[t])
  y[t] = f(k[t])
```

   (e) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T-1)
for t in range(T-1):
  k[t+1] = f(k[t]) - g[t](k[t])
  y[t] = f(k[t])
```

Question C (20%)

10. Which of these statements is correct? ⟨ **5%** ⟩

    (a) The existence of complete markets for Arrow securities means that agents' consumption outcome is constant.

    (b) Aiyagari's model of complete markets relies on natural borrowing limits to generate a wealth distribution in equilibrium.

    (c) Ocassionally binding borrowing limits are necessary for generating a wealth distribution in equilibrium in Aiyagari's model.

    (d) Aiyagari's model of incomplete markets relies on natural borrowing limits to generate a wealth distribution in equilibrium.

    (e) None of the above.

11. In a competitive equilibrium of an economy with Arrow securities, ⟨ **5%** ⟩

    (a) equilibrium relative prices are always the same as a benevolent social planner's shadow prices of resources.

    (b) allocation of resources are necessarily efficient.

    (c) agents cannot renegotiate the terms of their securities contracts *ex post*.

    (d) agents do not renegotiate the terms of their securities contracts *ex post* if their preferences are dynamically consistent.

    (e) agent's *ex-post* choices are always consistent with their initial optimal plans.

12. Lucas' consumption-based asset-pricing model [...] ⟨ **5%** ⟩

    (a) taught us how to price assets the correct way.

    (b) was the first instance of dynamic programming used in Finance.

    (c) showed how asset pricing dynamics could depend on how one models preferences.

    (d) implies that consumption inherits the volatility of stock markets.

    (e) None of the above.

13. Consider the following approximate recursive competitive equilibrium conditions in a Real Business Cycle model:

$$\mathbb{E}_t \left\{ \hat{c}_{t+1} \right\} \approx \left[1 + (1-\alpha)\frac{\beta Y_{ss}}{K_{ss}} \right]^{-1} \left[\hat{c}_t + \beta \left(\frac{Y_{ss}}{K_{ss}} \right) \rho \hat{a}_t \right], \tag{LRCE-1}$$

and

$$\hat{k}_{t+1} \approx \left[(1-\delta) + \frac{Y_{ss}}{K_{ss}} \right] \hat{k}_t - \left[\frac{C_{ss}}{K_{ss}} + \frac{Y_{ss}}{K_{ss}} \left(\frac{1-\alpha}{\alpha} \right) \right] \hat{c}_t + \frac{Y_{ss}}{K_{ss}} \left(\frac{1}{\alpha} \right) \hat{a}_t. \tag{LRCE-2}$$

where $\hat{c}_t$ and $\hat{k}_t$, respectively, refer to consumption and capital stock expressed as percentage deviations from their respective constant reference points. All parameters are positive valued.

This implies a linear, expectational stochastic difference equation system of the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ \mathbb{E}_t \hat{c}_{t+1} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \mathbf{L} \hat{a}_t,$$

given an exogenous specification for the random variable $\hat{a}_t$.

What are $\mathbf{M}$ and $\mathbf{L}$? ⟨ **5%** ⟩

(a) $\mathbf{M} = \begin{bmatrix} 1 - \delta + \frac{Y_{ss}}{K_{ss}} & -\frac{C_{ss}}{K_{ss}} - \frac{Y_{ss}(1-\alpha)}{K_{ss}} \\ 0 & \frac{1}{1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}} \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \frac{Y_{ss}}{K_{ss}\alpha} \\ \frac{Y_{ss}\beta\rho}{K_{ss}\left(1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}\right)} \end{bmatrix}$

(b) $\mathbf{M} = \begin{bmatrix} 1 - \delta + \frac{Y_{ss}}{K_{ss}} & -\frac{C_{ss}}{K_{ss}} - \frac{Y_{ss}(1-\alpha)}{K_{ss}\alpha} \\ 0 & \frac{1}{1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}} \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \frac{Y_{ss}}{K_{ss}\alpha} \\ \frac{Y_{ss}\beta\rho}{K_{ss}\left(1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}\right)} \end{bmatrix}$

(c) $\mathbf{M} = \begin{bmatrix} 1 - \delta \frac{Y_{ss}}{K_{ss}} & -\frac{C_{ss}}{K_{ss}} - \frac{Y_{ss}(1-\alpha)}{K_{ss}\alpha} \\ 0 & \frac{1}{1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}} \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \frac{Y_{ss}}{K_{ss}\alpha} \\ \frac{Y_{ss}\beta\rho}{K_{ss}\left(1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}\right)} \end{bmatrix}$

(d) There are two correct answers

(e) None of the above

Question D (20%) *Chairbear Pooh P/L* is a company that produces a product called `iRuleUrEeyore`. Assume units of `iRuleUrEeyore` are divisible goods, and are representable as non-negative real numbers. To produce y units of `iRuleUrEeyore` requires capital input $k \in \mathbb{R}_+$. This production technology is linear. But installing capital for use in each subsequent period is costly in terms of output. For each unit of capital investment expended, i_t, the cost is $i_t(1 + i_t/(2k_t))$.

Chairbear Pooh P/L is a price taker in both the capital market and in the market for its generic product. Let the price of a unit of `iRuleUrEeyore` be 1. *Chairbear Pooh P/L* discounts future profit flows at a constant discount factor $\beta \in (0, 1)$. Time is denumerable, $t \in \mathbb{N}$. Suppose that capital must be purchased one period ahead for production next period, and during production, it depreciates at rate $\delta \in (0, 1)$ per period. Denote the per-period new investment demand of *Chairbear Pooh P/L* as i_t, and,

$$i_t = k_{t+1} - (1 - \delta)k_t, \tag{D.1}$$

where k_t is predetermined by the end of $t - 1$. Given initial capital stock k_0, *Chairbear Pooh P/L* has a value given by

$$\pi(k_0) = \max_{(i_0,k_1,i_1,k_2,\dots)\in\mathbb{R}_+^\infty} \sum_{t=0}^{\infty} \beta^t \left\{ f(k_t, i_t) - q_t[k_{t+1} - i_t - (1 - \delta)k_t] \right\}.$$

The variable $q_t \geq 0$ is the Lagrange multiplier on the capital accummulation constraint (D.1) for every $t \in \mathbb{N}$. Assume that $f(k, i) = Ak - \kappa[i + (i)^2/(2k)]$, where $\kappa \geq 0$. Let a variable $x := x_t$ denote its current value, and $x_+ := x_{t+1}$ its next-period value.

14. The function $f(k, i)$ can be interpreted as ⟨ **2%** ⟩

    (a) the firm's revenue net of capital investment adjustment cost.
    (b) the firm's profit net of capital investment adjustment cost.
    (c) the firm's revenue net of capital investment production.
    (d) the firm's profit net of capital investment production.
    (e) the firm's capital production function.

15. The associated Bellman equation is ⟨ **2%** ⟩

    (a) $\pi(k) = \max_{k_+\in\Gamma(k)} \left\{ f(k, i) - q[k_+ - (1 - \delta)k] + \beta\pi(k_+) \right\}$.
    (b) $\pi(k) = \max_{(k_+,i)\in\Gamma(k)} \left\{ f(k, i) - q[k_+ - (1 - \delta)k - i] + \beta\pi(k_+) \right\}$.
    (c) $\pi(k) = \max_{i\in\Gamma(k)} \left\{ f(k, i) - q[k_+ - (1 - \delta)k - i] + \beta\pi(k_+) \right\}$.
    (d) $\pi(k) = \max_{k_+,i\in\Gamma(k)} \left\{ f[k, k_+ - (1 - \delta)k] - q[k_+ - (1 - \delta)k] + \beta\pi(k_+) \right\}$.
    (e) $\pi(k) = \max_{k_+\in\Gamma(k)} \left\{ f[k, k_+ - (1 - \delta)k] - q[k_+ - (1 - \delta)k] + \beta\pi((1 - \delta)k + i) \right\}$.

    where $\Gamma(k)$ for all $k \in X$, describes the firm's feasible choice correspondence.

16. Which of these is true or the most accurate? ⟨ **2%** ⟩

    (a) There is a unique optimal strategy.
    (b) There is a unique, interior optimal strategy.
    (c) There is a unique optimal mixed strategy.
    (d) There is a unique, interior optimal mixed strategy.
    (e) The optimal strategy is to install capital immediately at the steady state level.

17. At an optimal $i \equiv i^{\star}(k)$, the marginal value of capital to *Chairbear Pooh P/L* , $\pi'(k)$, is __________ ⟨ **4%** ⟩

    (a) q.
    (b) A.
    (c) $A + \frac{\kappa}{2}\left(\frac{i}{k}\right)^2$.
    (d) $A + \frac{\kappa}{2}\left(\frac{i}{k}\right)^2 + q$.
    (e) $A + \frac{\kappa}{2}\left(\frac{i}{k}\right)^2 + q(1-\delta)$.

18. Optimal investment $i^{\star}(k)$ must satisfy these conditions:

$$q = 1 + \kappa \frac{i^{\star}(k)}{k},$$

and,

$$q = \beta \left[A + \frac{\kappa}{2} \left(\frac{i^{\star}(k_+)}{k_+} \right)^2 \right] + q_+ \beta (1-\delta),$$

for every state k.

Since the program has a quadratic objective with linear constraints, we can guess that the optimal solution must be a linear one: $i^{\star}(k) = B \times k$, where B is an undetermined coefficient. Using this guess verify the solution to the Euler equation above.

Assume $\kappa = 1$. The optimal stationary investment policy is such that B equals __________ [⚒].

Hint: We will require the investment-capital ratio to be uniformly bounded above: $i/k < 1/\beta - (1-\delta)$. Also, the marginal product of capital cannot be too large: $A < \frac{1}{2}[((1+\beta\delta)/\beta)^2 - 1]$ for a well-defined solution.

⟨ **6%** ⟩

    (a) ⚒ $B \equiv A + \beta(1-\delta)$.
    (b) ⚒ $B \equiv \beta^{-1}[1-\beta(1-\delta)] - \sqrt{(1+\beta\delta)^2 - \beta^2(2A+1)}$.
    (c) ⚒ $B \equiv \delta$.
    (d) ⚒ $B \equiv \beta^{-1}[1-\beta(1-\delta)]$.
    (e) ⚒ None of the above.

19. From the last question, we showed that from any initial state $k_t \in X$, the firm's optimal investment strategy is to invest at a constant proportion of k_t. Why?

⟨ **2%** ⟩

(a) The marginal profit with respect to additional investment is diminishing. The marginal cost with respect to investing increases linearly. There is an optimal investment-to-capital-stock ratio that is constant.

(b) The firm has perfect foresight. Therefore maximizing its long run capital stock is optimal.

(c) The marginal profit for investing into an additional unit of capital productive in the next period is constant at q. The marginal cost is zero. Therefore it is optimal to invest k^* each period.

(d) There is no risk. As such the optimal investment strategy is to maximizing its flow per period.

(e) None of the above.

20. If $\kappa = 0$, then ... ⟨ **2%** ⟩

(a) the firm never invests.

(b) the firm invests an amount that is always constant.

(c) the firm invests in odd periods.

(d) the firm invests in even periods.

(e) none of the above.

— Part B —

A test of your ability to apply, to think creatively and to communicate clearly.
See further instructions at the start of this document.

Question E (30%) In 1807 the *Embargo Act* came into force in the United States. It was a general trade embargo on all foreign countries. As the Napoleonic Wars continued, the Act represented the United States' attempt to pressure Britain to stop forcing Americans into participating in their naval force. The United States also wanted Britain to respect American neutrality with regard to the Napoleonic Wars.

Around that time the United States Congress was also worried about the economic impacts of the war on international trade and reduced tax revenue to the government. In an 1807 *Report on Finances*, the then Secretary of Treasury stated that:

> [I]t appears necessary to provide a revenue at least equal to the annual expenses on a peace establishment, the interest of the existing debt, and the interest on the loans which may lie raised.
>
> ...
>
> [A]nd without inquiring whether a similar cause may not still more deeply and permanently affect a nation at war with the United States, it seems to follow that, so far as relates to America, the losses and privations caused by the war should not be aggravated by taxes beyond what is strictly necessary.
>
> An addition to the debt is doubtless an evil; but experience having now shown with what rapid progress the revenue of the Union increases in time of peace; with what facility the debt, formerly contracted, has in a few years been reduced; a hope may confidently be entertained that all the evils of the war will be temporary and easily repaired; and that the return of peace will, without any effort, afford ample resources for reimbursing whatever may have been borrowed during the war.
>
> (pp.360-361)

Suppose there is an exogenously given random stream of government expenditures $\{g_t\}_{t\geq 0}$ and an initial level of public debt b_0. Let τ_t denote total tax revenue at date $t \geq 0$, and, let $R > 1$ be a given one-period, gross real interest on a stock of issued government bonds, b_t. The government commits to a policy plan $\{\tau_t, b_{t+1}\}_{t\geq 0}$, where $\tau_t := \tau(b_t)$. It picks one such plan to minimize the total, discounted expected (social) cost of taxation:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} R^{-t} \left(\gamma \tau_t + \frac{\delta}{2} \tau_t^2 \right), \qquad \gamma, \delta > 0,$$

subject to a sequence of government budget constraints,

$$g_t - \tau_t = b_{t+1} - R b_t, \qquad t = 0, 1, 2, ...$$

We will also require that $\lim_{T \nearrow +\infty} \mathbb{E}_t R^{-T} b_{t+T} = 0$.

1. Write down the government's dynamic programming problem in terms of a Bellman equation. Hint: You may assume outright that b_t is the only state variable. $\langle$ **2%** $\rangle$

2. Show that the optimal tax plan is characterized by a random walk behavior for taxes,

$$\mathbb{E}_t \tau_{t+1} = \tau_t,$$

for all $t \geq 0$. $\langle$ **3%** $\rangle$

What consequence would this have for an econometrician attempting to forecast the trajectory of the government's tax revenues? $\langle$ **1%** $\rangle$

3. Show that the sequential government budget constraints can be re-written as

$$b_t = \frac{1}{R} \mathbb{E}_t \sum_{j=0}^{\infty} R^{-j} (\tau_{t+j} - g_{t+j}).$$

$\langle$ **2%** $\rangle$

Explain in words what this equation above says. $\langle$ **2%** $\rangle$

4. Suppose the stochastic process for government spending is

$$g_t = \bar{g} + \epsilon_t,$$

where $\bar{g} > 0$ is a constant, ϵ_t is independently and identically distributed, and, $\mathbb{E}_t \epsilon_{t+j} = 0$, for any $j \geq 1$. We can interpret the shock ϵ_t as an unanticipated, temporary war-time expense by the government. If this shock eventuates, it is known at the start of date t—i.e., $\mathbb{E}_t \epsilon_t = \epsilon_t$. The number $\bar{g}$ denotes government expenditure in normal times or, in the words of the Secretary, "`annual` [government] `expenses on a peace establishment` (*sic*)".

Show that you can further describe the *optimal policy* in terms of this behavioral response:

$$\tau_t = [\bar{g} + (R-1)b_t] + \left(\frac{R-1}{R} \right) \epsilon_t.$$

Hint: An optimal policy must also be feasible in an expected, present-value intertemporal budget sense.

$\langle$ **6%** $\rangle$

Interpret this optimal policy prescription in relation to a vital part of the Secretary's speech above—i.e., *decompose* this policy and *explain* using the speech. $\langle$ **6%** $\rangle$

5. Show that you can also derive this insight:

$$\mathbb{E}_t b_{t+1} = b_t + \frac{\epsilon_t}{R}.$$

$\langle$ **4%** $\rangle$

Interpret what this says about optimal public debt dynamics, and how this connects to the Secretary's speech above. (This goes hand in hand with the last question.)

$\langle$ **4%** $\rangle$

Appendix: Useful definitions and results

Neumann expansion. Let A be a stable square matrix, L the lag operator, and I an identity matrix conformable to A:

$$(I - AL)^{-1} = I + AL + AL^2 + ...$$

Integration by parts.

$$\int_a^b u dv = uv|_a^b - \int_a^b v du$$

Leibniz' rule. Let $\phi(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$ for $t \in [c,d]$ and f and f_t are continuous and a, b are differentiable on $[c,d]$. Then $\phi(t)$ is differentiable on $[c,d]$ and

$$\phi(t) = f(b(t), t)\, b'(t) - f(a(t), t)\, a'(t) + \int_{a(t)}^{b(t)} f_t(x,t)\,dx.$$

Independent random variable and geometric distribution. Let N be a geometrically distributed random waiting time until the arrival of a desired signal. Let $\lambda = \int_0^{\overline{x}} dF(x)$ be the probability that the desired signal x is not observed in one period. Then,

$$\Pr\{N = 1\} = 1 - \lambda$$

$$\Pr\{N = j\} = (1-\lambda)\,\lambda^{j-1}.$$

The mean waiting time is then $\overline{N} = (1-\lambda)^{-1}$.

Definition 1 *A correspondence $\Gamma : X \rightrightarrows Y$ is* lower semi-continuous (lsc) at x *if for every open set V that meets $\Gamma(x)$ – i.e. $V \cap \Gamma(x) \neq \emptyset$ – there is an open set $U(x) \ni x$ such that if $x' \in U(x)$, then V also meets $\Gamma(x')$ or $\Gamma(x') \cap V \neq \emptyset$. The correspondence Γ is said to be lsc if it is lsc at every $x \in X$.*

Definition 2 *A correspondence $\Gamma : X \rightrightarrows Y$ is* upper semi-continuous (usc) at x *if for every open set $V \supset \Gamma(x)$, there is an open set $U(x) \ni x$ such that if $x' \in U(x)$, then $V \supset \Gamma(x')$. A correspondence is said to be upper semi-continuous and compact-valued if it is usc at every $x \in X$.*

Definition 3 *A correspondence $\Gamma : X \rightrightarrows Y$ is* continuous at x *if it is both usc and lsc at x. Then we say Γ is* continuous *if it is both usc and lsc (i.e. usc and lsc at every $x \in X$).*

Definition 4 *Let (S,d) be a metric space and the map $T : S \to S$. Let $T(w) := Tw$ be the value of T at $w \in S$. T is a contraction with modulus $0 \leq \beta < 1$ if $d(Tw, Tv) \leq \beta d(w,v)$ for all $w, v \in S$.*

Theorem 1 (Banach Fixed Point Theorem) *If (S,d) is a complete metric space and $T : S \to S$ is a contraction, then there is a fixed point for T and it is unique.*

Theorem 2 *The following metric spaces are complete:*

1. $(\mathbb{R}, |\cdot|)$.
2. *$(C_b(X), d_\infty)$, where $C_b(X)$ is the set of continuous and bounded functions on X.*
3. *$(C_b'(X), d_\infty)$, where $C_b'(X)$ is the set of continuous, bounded and nondecreasing functions on X.*
4. *$(C_b''(X), d_\infty)$, where $C_b''(X)$ is the set of continuous, bounded and strictly increasing functions on X.*

Furthermore, $C_b''(X) \subset C_b'(X) \subset C_b(X)$, are closed subsets relative to $C_b(X)$.

Theorem 3 *Suppose (S,d) is a complete metric space, and $T : S \to S$ is a $\beta < 1$ contraction mapping with fixed point $v \in S$. If $S' \subset S$ is closed and $T(S') \subseteq S'$, then $v \in S'$. Furthermore, if $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$.*

Theorem 4 (Blackwell's sufficient conditions for a contraction) *Let $M : S \to S$ be any map satisfying*

1. *Monotonicity: For any $v, w \in S$ such that $w \geq v \Rightarrow Mw \geq Mv$.*
2. *Discounting: There exists a $0 \leq \beta < 1$ such that $M(w + c) = Mw + \beta c$, for all $w \in S$ and $c \in \mathbb{R}$. (Define $(f + c)(x) = f(x) + c$.)*

Then M is a contraction with modulus β.

Let (P, λ_0) be a Markov chain on a finite state space S.

Theorem 5 *If $P_{ij}^{(\tau)} > 0$ for all $i, j = 1, ..., n$, then there exists a unique invariant distribution $\lambda^* = \lim_{t\to\infty} \lambda_0 P^t$ satisfying $\lambda^* = \lambda^* P$.*

Theorem 6 *Let $h : S \to \mathbb{R}$. If $\{\varepsilon_t\}$ is a Markov chain (P, λ_0) on the finite set $S = \{s_1, ..., s_n\}$ such that it is asymptotically stable with stationary distribution λ^*, then as $T \to \infty$,*

$$\frac{1}{T}\sum_{t=0}^{T} h(\varepsilon_t) \to \sum_{j=1}^{n} h(s_j)\lambda^*(s_j)$$

with probability one.

Theorem 7 *The characteristic polynomial of a (2×2) matrix $\mathbf{F}$ is*

$$P(\lambda) = \lambda^2 - \text{trace}(\mathbf{F})\lambda + \det(\mathbf{F}).$$

The (at most two distinct) eigenvalues, λ, solve $P(\lambda) = 0$.

Theorem 8 *Let $\mathbf{A}$ be a $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then*

1. *$\lambda_1 + \lambda_2 + ... + \lambda_n = trace(\mathbf{A})$, and*
2. *$\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n = \det(\mathbf{A})$.*

Theorem 9 *$\mathbf{F}$ is a stable matrix, or all of its eigenvalues are such that $|\lambda_i| < 1$, if and only if*

1. *$|\det(\mathbf{F})| < 1$, and*
2. *$|- trace(\mathbf{F})| - \det(\mathbf{F}) < 1$.*

Note: The trace of a matrix $\mathbf{A}$ is the sum of its diagonal elements. The determinant of a (2×2) matrix $\mathbf{F}$ is given by $f_{11}f_{22} - f_{21}f_{12}$, where f_{ij} is the row-i and column-j element of the matrix.

Implicit differentiation (bivariate example). Consider a smooth, bivariate function $(x, y) \mapsto R(x, y)$. If $R(x, y) = 0$, the derivative of the implicit function $x \mapsto f(x) \equiv y$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial R/\partial x}{\partial R/\partial y} = -\frac{R_x}{R_y}$. (This formula is obtained from the generalized chain rule to obtain the total derivative with respect to x.)

——— *End of Examination* ———