THE AUSTRALIAN NATIONAL UNIVERSITY

Midsemester Examination — March 2021

Macroeconomic Theory

ECON 4422/8022

⏲ Upload Time	:	30 Minutes
✍ Writing Time	:	120 Minutes
☑ Permitted Materials	:	Everything plus the kitchen sink
$\langle \cdot\% \rangle$	:	Mark allocation operator

— IMPORTANT —

There are **TWO Parts** to this examination with varying levels of difficulty:

Part A is a *partially guided* test of your ability to solve basic problem(s) related to and extending from class material, to reason logically and to be able to communicate your workings and economic insights in plain English. This part also tests your general comprehension of basic concepts or definitions, your ability to read, use and adapt `Python` code and to design algorithms related an economic problem. Completion of this section would enable you to attain up to 90% of the maximum examination mark.

- Handwrite your answers and scan them to a *single* PDF file. Upload these to the `WATTLE` exam portal.
- Where computation is required, submit your computational answers as a separate, executable `Jupyter Notebook` through the same `WATTLE` exam portal.
- Answers are expected to be succinct but complete. *Unreasonably long and irrelevant answers may be penalized.*
- Recommended time for completion: 100 minutes.

Part B is a `WATTLE` quiz. It tests you on your basic understanding of key concepts learned thus far in this course. This component is worth up to 10% of the maximum examination mark.

- Select or Enter your answers directly in the `WATTLE` quiz module.
- Recommended time for completion: 20 minutes.

Time management. You are wholly responsible for managing your total time. Ensure that you have ample time left to complete scanning and uploading documents to `WATTLE` within the additional thirty-minute window allocated after the official exam-writing time ends.

— Part A —

Question A (90%) Consider a version of the optimal growth (or capital accumulation) *planning problem.* The natural state variable is capital stock $k_t \in [0, 1]$, where $t \in \mathbb{N} := \{0, 1, ...\}$.

Production technologies. There are two production sectors in this model. There is a (pure) consumption good (c) production sector that uses services from capital (K_C) and labor (L_C) as inputs. The (Leontief) production function for c requires an input mixed in fixed proportions:

$$c_t = \min\left\{K_{C,t}, \frac{2}{\gamma}L_{C,t}\right\} \geq 0, \qquad \gamma \in (1, 2). \tag{A.1}$$

There is also a (pure) capital good (i) sector using a similar technology:

$$i_t = \min\{\gamma K_{Y,t}, L_{Y,t}\} \geq 0. \tag{A.2}$$

Total labor endowment is 1 each period and this constrains the labor inputs demanded by both sectors:

$$L_{C,t} + L_{Y,t} \leq 1. \tag{A.3}$$

Capital services used by the sectors are constrained by the each date's initial total capital stock, k_t:

$$K_{C,t} + K_{Y,t} \leq k_t. \tag{A.4}$$

Preferences and planning problem. The planner's per-period payoff for consuming c_t is explicitly state dependent. The flow-payoff function is $u(\cdot, k_t) : \mathbb{R}_+ \to \mathbb{R}$. The optimal planning problem, beginning from an initial state $k_0 \in (0, 1]$, is one of selecting an infinite sequence $\{c_t, k_{t+1}, L_{C,t}, K_{C,t}, L_{Y,t}, K_{Y,t}\}_{t=0}^{\infty}$ such that

$$V(k_0) = \sup\left\{\sum_{t=0}^{\infty} \delta^t u(c_t, k_t) : (\text{A.1}) - (\text{A.4}) \text{ and } k_{t+1} \leq i_t\right\}. \tag{A.5}$$

The parameter $0 < \delta < 1$ is a discount factor.

This program is actually quite manageable. We will break the choice problem down into a static component and a dynamic one.

1. *Static sub-program.* The state is given, say $k_t = k$, at the beginning of each date t. Then, for a given target level of capital good to be produced, $i_t = i$, we can solve for the optimal allocation of production inputs, and thus consumption, as a static program:

$$c^{\star}(k,i) = \max\{c_t : (\text{A.1}) - (\text{A.4})\}. \tag{A.6}$$

Show that for each date t, the solution is

$$c^{\star}(k_t,i_t) = \begin{cases} k_t - \frac{1}{\gamma}i_t, & \text{if } i_t \leq 2 - \gamma k_t \\ \frac{2}{\gamma} - \frac{2}{\gamma}i_t, & \text{if } i_t \geq 2 - \gamma k_t \end{cases}, \tag{A.7}$$

and, the optimizer's domain is $D = \{(k_t,i_t) \in \mathbb{R}^2_+ | i_t \leq \min\{\gamma k_t, 1\}\}$. [1]

⟨ 20% ⟩

Interpret in words what the optimizer, $c^{\star}$, represents.

⟨ 5% ⟩

[1] *Hints.* A picture paints a thousand words: It would be more efficient to visualize this static problem in an Edgeworth-box-style diagram in (L_C, K_C)-space. Label the "southwest corner" of this box as O_C which represents the coordinate $(L_C, K_C) = (0,0)$. Next, label the "northeast corner" of this box as O_K which represents the coordinate $(L_C, K_C) = (1,k) \equiv (L_Y, K_Y) = (0,0)$. Recall that the expansion path of a Leontief function is a set of input coordinates defined by some affine (or linear) function. The vertices of the "L"-shaped Leontief production level sets lie on the expansion path. You will see that there are two possible cases or regions of resource- and incentive-feasible production input mixes and output levels. Also, use the fact that the level sets and the expansion paths are increasing in particular directions.

There exists a function u such that we can work with a more convenient, reduced-form utility function: $U(k,i) = u\left[c^\star(k,i), k\right]$, where

$$U(k,i) = \begin{cases} \left(\beta - \frac{2}{\gamma}\right)k - (\alpha-1)k^2 - \left(\beta\delta - \frac{2}{\gamma^2}\right)i - \left(\frac{1}{\gamma^2} - \alpha\delta\right)i^2, & \text{if } i \leq 2-\gamma k \\ \begin{aligned}(4\gamma\eta+\beta)k - (\alpha+\gamma^2\eta)k^2 - (\beta\delta - 4\eta)i \\ - (\eta - \alpha\delta)i^2 - 2\gamma\eta k i - 4\eta\end{aligned}, & \text{if } i \geq 2-\gamma k \end{cases}, \tag{A.8}$$

and $(k,i) \in D$.

We will assume the parameters $(\alpha,\beta,\eta,\delta,\gamma)$ to be such that:

$$\begin{gathered} \gamma^2\delta < \frac{1}{\alpha} < 1, \qquad \beta > 2\max\left\{\frac{1}{\gamma^2\delta}, \alpha + \gamma\eta(\gamma-1)\right\}, \\ 1 < \gamma < 2, \qquad \eta > \frac{\alpha\delta}{1-\gamma^2\delta}, \qquad 0 < \delta < 1. \end{gathered} \tag{A.9}$$

2. *Dynamic program.* At an optimum, all of i produced is not wasted, i.e., $k_{t+1} = i_t$. The sequence problem (A.5) has a recursive representation,

$$V(k_t) = U(k_t, k_{t+1}) + \delta V(k_{t+1}), \tag{A.10}$$

and $V : [0,1] \to \mathbb{R}$ is induced by an optimal decision function g:

$$k_{t+1} = g(k_t), \tag{A.11}$$

for all $t \in \mathbb{N}$.

Claim: The optimal value function is

$$V(k) = \beta k - \alpha k^2, \tag{A.12}$$

and this is sustained by the optimal policy function,

$$g(k) = \begin{cases} \gamma k, & \text{if } k \in [0, 1/\gamma] \\ 2 - \gamma k, & \text{if } k \in [1/\gamma, 1] \end{cases}. \tag{A.13}$$

Verify this claim.

⟨ **20%** ⟩

3. Sketch the graph of the function that describes an optimal accumulation path of this model economy in (k_t, k_{t+1})-space. The graph must be technically accurate.

$\langle$ **15%** $\rangle$

4. Assume restrictions (A.9) above. Provide a description of the dynamic behavior of the optimal plan or trajectory of the model economy.

$\langle$ **10%** $\rangle$

What explains the peculiar behavior in this model, in contrast to the Ramsey-Cass-Koopmans model? Explain why.

$\langle$ **10%** $\rangle$

5. Suppose you have parameter values satisfying the restrictions (A.9) above. Outline your pseudocode or algorithm for simulating the optimal trajectory, beginning from a given state.

⟨ **10%** ⟩

Appendix: Useful definitions and results

Neumann expansion. Let A be a stable square matrix, L the lag operator, and I an identity matrix conformable to A:

$$(I - AL)^{-1} = I + AL + AL^2 + ...$$

Integration by parts.

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

Leibniz' rule. Let $\phi(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$ for $t \in [c,d]$ and f and f_t are continuous and a, b are differentiable on $[c,d]$. Then $\phi(t)$ is differentiable on $[c,d]$ and

$$\phi(t) = f(b(t), t)\, b'(t) - f(a(t), t)\, a'(t) + \int_{a(t)}^{b(t)} f_t(x,t)\,dx.$$

Independent random variable and geometric distribution. Let N be a geometrically distributed random waiting time until the arrival of a desired signal. Let $\lambda = \int_0^{\overline{x}} dF(x)$ be the probability that the desired signal x is not observed in one period. Then,

$$\Pr\{N = 1\} = 1 - \lambda$$

$$\Pr\{N = j\} = (1 - \lambda)\,\lambda^{j-1}.$$

The mean waiting time is then $\overline{N} = (1 - \lambda)^{-1}$.

Definition 1 *A correspondence $\Gamma : X \rightrightarrows Y$ is* lower semi-continuous (lsc) at x *if for every open set V that meets $\Gamma(x)$ – i.e. $V \cap \Gamma(x) \neq \emptyset$ – there is an open set $U(x) \ni x$ such that if $x' \in U(x)$, then V also meets $\Gamma(x')$ or $\Gamma(x') \cap V \neq \emptyset$. The correspondence Γ is said to be lsc if it is lsc at every $x \in X$.*

Definition 2 *A correspondence $\Gamma : X \rightrightarrows Y$ is* upper semi-continuous (usc) at x *if for every open set $V \supset \Gamma(x)$, there is an open set $U(x) \ni x$ such that if $x' \in U(x)$, then $V \supset \Gamma(x')$. A correspondence is said to be upper semi-continuous and compact-valued if it is usc at every $x \in X$.*

Definition 3 *A correspondence $\Gamma : X \rightrightarrows Y$ is* continuous at x *if it is both usc and lsc at x. Then we say Γ is* continuous *if it is both usc and lsc (i.e. usc and lsc at every $x \in X$).*

Definition 4 *Let (S, d) be a metric space and the map $T : S \to S$. Let $T(w) := Tw$ be the value of T at $w \in S$. T is a contraction with modulus $0 \leq \beta < 1$ if $d(Tw, Tv) \leq \beta d(w, v)$ for all $w, v \in S$.*

Theorem 1 (Banach Fixed Point Theorem) *If (S, d) is a complete metric space and $T : S \to S$ is a contraction, then there is a fixed point for T and it is unique.*

Theorem 2 *The following metric spaces are complete:*

1. $(\mathbb{R}, |\cdot|)$.
2. *$(C_b(X), d_\infty)$, where $C_b(X)$ is the set of continuous and bounded functions on X.*
3. *$(C_b'(X), d_\infty)$, where $C_b'(X)$ is the set of continuous, bounded and nondecreasing functions on X.*
4. *$(C_b''(X), d_\infty)$, where $C_b''(X)$ is the set of continuous, bounded and strictly increasing functions on X.*

Furthermore, $C_b''(X) \subset C_b'(X) \subset C_b(X)$, are closed subsets relative to $C_b(X)$.

Theorem 3 *Suppose (S, d) is a complete metric space, and $T : S \to S$ is a $\beta < 1$ contraction mapping with fixed point $v \in S$. If $S' \subset S$ is closed and $T(S') \subseteq S'$, then $v \in S'$. Furthermore, if $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$.*

Theorem 4 (Blackwell's sufficient conditions for a contraction) *Let $M : S \to S$ be any map satisfying*

1. *Monotonicity: For any $v, w \in S$ such that $w \geq v \Rightarrow Mw \geq Mv$.*
2. *Discounting: There exists a $0 \leq \beta < 1$ such that $M(w + c) = Mw + \beta c$, for all $w \in S$ and $c \in \mathbb{R}$. (Define $(f + c)(x) = f(x) + c$.)*

Then M is a contraction with modulus β.

Let (P, λ_0) be a Markov chain on a finite state space S.

Theorem 5 *If $P_{ij}^{(\tau)} > 0$ for all $i, j = 1, ..., n$, then there exists a unique invariant distribution $\lambda^* = \lim_{t\to\infty} \lambda_0 P^t$ satisfying $\lambda^* = \lambda^* P$.*

Theorem 6 *Let $h : S \to \mathbb{R}$. If $\{\varepsilon_t\}$ is a Markov chain (P, λ_0) on the finite set $S = \{s_1, ..., s_n\}$ such that it is asymptotically stable with stationary distribution λ^*, then as $T \to \infty$,*

$$\frac{1}{T}\sum_{t=0}^{T} h(\varepsilon_t) \to \sum_{j=1}^{n} h(s_j)\lambda^*(s_j)$$

with probability one.

Theorem 7 *The characteristic polynomial of a (2×2) matrix $\mathbf{F}$ is*

$$P(\lambda) = \lambda^2 - \text{trace}(\mathbf{F})\lambda + \det(\mathbf{F}).$$

The (at most two distinct) eigenvalues, λ, solve $P(\lambda) = 0$.

Theorem 8 *Let $\mathbf{A}$ be a $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then*

1. *$\lambda_1 + \lambda_2 + ... + \lambda_n = trace(\mathbf{A})$, and*
2. *$\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n = \det(\mathbf{A})$.*

Theorem 9 *$\mathbf{F}$ is a stable matrix, or all of its eigenvalues are such that $|\lambda_i| < 1$, if and only if*

1. *$|\det(\mathbf{F})| < 1$, and*
2. *$|-trace(\mathbf{F})| - \det(\mathbf{F}) < 1$.*

Note: The trace of a matrix $\mathbf{A}$ is the sum of its diagonal elements. The determinant of a (2×2) matrix $\mathbf{F}$ is given by $f_{11}f_{22} - f_{21}f_{12}$, where f_{ij} is the row-i and column-j element of the matrix.

Implicit differentiation (bivariate example). Consider a smooth, bivariate function $(x, y) \mapsto R(x, y)$. If $R(x, y) = 0$, the derivative of the implicit function $x \mapsto f(x) \equiv y$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial R/\partial x}{\partial R/\partial y} = -\frac{R_x}{R_y}$. (This formula is obtained from the generalized chain rule to obtain the total derivative with respect to x.)

——— *End of Examination* ———