High Dimensional Statistics **2nd Semester, 2022-2023**

Homework 3 (Deadline: Jun 6)

1. (10 pts) Given a set $T \subset K$. Recall the definition of the covering number: $N(T, d, \varepsilon)$ is the smallest number of points in T which form a ε-covering of T under the metric d. Suppose we are allowed to use the points outside of T to do covering. Then the smallest number of points that are needed to form a ε-covering of T is referred to as the exterior covering number, denoted $N^{ext}(T, d, \varepsilon)$. Show that
$$N^{ext}(T, d, \varepsilon) \leq N(T, d, \varepsilon) \leq N^{ext}(T, d, \varepsilon/2).$$

2. (10 pts) Suppose T is a finite set and $\log \mathbb{E}\left[e^{\lambda X_t}\right] \leq \psi(\lambda)$ for all $\lambda \geq 0$ and $t \in T$, where $\psi(\cdot)$ is a convex function and $\psi(0) = \psi'(0) = 0$. Show that
$$\mathbb{E}\left[\max_{t \in T} X_t\right] \leq \psi^{*-1}(\log |T|),$$
where $\psi^*(x) = \sup\limits_{\lambda \geq 0}\{\lambda x - \psi(\lambda)\}$.

3. (10pts) Define
$$T^n(s) = \{x \in \mathbb{R}^n : \ \|x\|_0 \leq s, \ \|x\|_2 \leq 1\},$$
where $\|x\|_0$ counts the number of non-zero entries in x. Show that the Gaussian complexity of $T^n(s)$, denoted $\mathcal{G}(T^n(s)) = \mathbb{E}\left[\sup_{x \in T^n(s)} \langle g, x\rangle\right]$, $g \sim \mathcal{N}(0, I_n)$, satisfies
$$\mathcal{G}(T^n(s)) \lesssim \sqrt{s \log\left(\frac{en}{s}\right)}.$$

4. (10 pts) Let $W \in \mathbb{R}^{m \times n}$ be a random matrix with i.i.d $\mathcal{N}(0,1)$ entries. We have established an upper bound of $\mathbb{E}\left[\|W\|_2\right]$ using the one-step discretization bound as well as the Gaussian comparison inequality. In this problem you are required to establish an upper bound of $\mathbb{E}\left[\|W\|_2\right]$ using the Dudley integral.

5. (20 pts) Let $B_1^n = \{x : \ \|x\|_1 \leq 1\}$ be the ℓ_1-norm unit ball. We have already seen that the Gaussian complexity of B_1^n satisfies
$$\mathcal{G}(B_1^n) = \mathbb{E}\left[\sup_{\|x\|_1 \leq 1} \langle g, x\rangle\right] \lesssim \sqrt{\log n}, \quad g \sim \mathcal{N}(0, I_n)$$
based on the duality between ℓ_1-norm and ℓ_∞-norm. In this problem, we attempt to provide a bound for $\mathcal{G}(B_1^n)$ based on the Dudley integral.

   - For $\varepsilon > 0$ being sufficiently small, show that the covering of B_1^n under the ℓ_2-norm satisfies
$$\sqrt{\log \mathcal{N}(B_1^n, \|\cdot\|_2, \varepsilon)} \lesssim \min\{\varepsilon^{-1}\sqrt{\log n}, \sqrt{n} \cdot \log(1/\varepsilon)\}.$$

   Hint: Volume argument as presented in Lecture 4 may be useful in providing one bound.

- Using the above result and the Dudley integral to provide a bound for $\mathcal{G}(B_1^n)$.

6. (10 pts) Assume $X, Y \in \mathbb{R}^n$ are finite centered Gaussian Processes. Suppose that there exist a pair of index sets $A, B \subset \{1, \cdots, n\}^2$ for which

$$\begin{aligned} \mathbb{E}[X_i X_j] &\leq \mathbb{E}[Y_i Y_j] \quad \text{for all} \quad (i,,j) \in A; \\ \mathbb{E}[X_i X_j] &\geq \mathbb{E}[Y_i Y_j] \quad \text{for all} \quad (i,,j) \in B; \\ \mathbb{E}[X_i X_j] &= \mathbb{E}[Y_i Y_j] \quad \text{for all} \quad (i,,j) \notin A \cup B. \end{aligned}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function whose second derivative satisfies

$$\begin{aligned} \partial_{ij} f &\geq 0 \quad \text{for all} \quad (i,,j) \in A; \\ \partial_{ij} f &\leq 0 \quad \text{for all} \quad (i,,j) \in B. \end{aligned}$$

Show that

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)].$$

7. (10 pts) Let $\phi_j : \mathbb{R} \to \mathbb{R}$ ($j = 1, \cdots, n$) be 1-Lipschitz (i.e., $|\phi_j(t) - \phi_j(s)| \leq |t - s|$). Let w_j ($j = 1, \cdots, n$) be i.i.d $\mathcal{N}(0, 1)$ random variables. For any $T \subset \mathbb{R}^n$, show that

$$\mathbb{E}\left[\sup_{t=(t_1,\cdots,t_n)\in T} \sum_{j=1}^n w_j \phi_j(t_j)\right] \leq \mathbb{E}\left[\sup_{t=(t_1,\cdots,t_n)\in T} \sum_{j=1}^n w_j t_j\right].$$

What does the above result mean in terms of the Gaussian complexity?