High Dimensional Probability and Statistics **2nd Semester, 2022-2023**

Homework 1 (Deadline: Mar 25)

1. (5 pts) Let $g \sim \mathcal{N}(0, \sigma^2)$. Show that

$$\mathbb{P}\left[g \geq t\right] \leq \frac{1}{2} e^{-\frac{t^2}{2\sigma^2}}, \quad \text{for } t \geq 0.$$

2. (10 pts) Assume variance of X exists. Show that

$$\mathrm{Var}\left[X\right] = \min_a \mathbb{E}\left[(X-a)^2\right].$$

Moreover, letting X' be an independent copy of X, show that

$$\mathrm{Var}\left[X\right] = \frac{1}{2}\mathbb{E}\left[(X-X')^2\right].$$

3. (5 pts) Assume X is ν^2-sub-Gaussian. Show that

$$\mathrm{Var}\left[X\right] \leq \nu^2.$$

4. (5 pts) Let X be sub-Gaussian. Shown X^2 is sub-exponential. (You may use alternative characterizations for sub-Gaussian and sub-exponential random variables).

5. (10 pts) Let X be sub-Gaussian with parameter $\sigma > 0$. Let $f(x)$ be a Lipschitz function with constant $L > 0$, i.e., $|f(x) - f(y)| \leq L|x - y|$ for all x, y. Show that there exists a numerical constant $c > 0$ (which does not depend on any parameter, i.e., universal or absolute constant) such that $f(X)$ is sub-Gaussian with parameter $cL\sigma$.

6. (10 pts) Solve Exercise 1.29 (i.e., show (1.6)) in Lecture 1.

7. (10 pts) Let $X_1, \cdots, X_n$ be i.i.d samples drawn from a pdf $f(x)$ on the real line. A standard way to estimate f from the samples is the kernel density estimator,

$$\widehat{f}_n(x) = \frac{1}{nh}\sum_{k=1}^{n} K\left(\frac{x - X_k}{h}\right),$$

where $K : \mathbb{R} \to [0, \infty)$ is a kernel function satisfying $\int_{-\infty}^{\infty} K(x) = 1$, and $h > 0$ is a bandwidth parameter. Suppose we evaluate the quality of $\widehat{f}_n(x)$ using the L_1 norm

$$\|\widehat{f}_n - f\|_1 := \int_{-\infty}^{\infty} |\widehat{f}_n(x) - f(x)| dx.$$

Show that

$$\mathbb{P}\left[\left|\|\widehat{f}_n - f\|_1 - \mathbb{E}\left[\|\widehat{f}_n - f\|_1\right]\right| \geq t\right] \leq 2e^{-\frac{nt^2}{2}}.$$

8. (15 pts) Let $X \geq 0$ and assume the variance of X exists.

- Show that

$$\mathbb{P}\left[X \geq (1-t)\mathbb{E}\left[X\right]\right] \geq \frac{t^2(\mathbb{E}\left[X\right])^2}{\mathbb{E}\left[X^2\right]} \quad \text{for } t \in (0,1].$$

- Show that

$$\mathbb{E}\left[\exp(-\lambda(X - \mathbb{E}\left[X\right]))\right] \leq \exp(\lambda^2 \mathbb{E}\left[X^2\right]/2) \quad \text{for } \lambda \geq 0.$$

- Show that

$$\mathbb{P}\left[X \leq (1-t)\mathbb{E}\left[X\right]\right] \leq \exp\left(-\frac{t^2(\mathbb{E}\left[X\right])^2}{2\mathbb{E}\left[X^2\right]}\right) \quad \text{for } t \in (0,1].$$