High Dimensional Probability and Statistics **2nd Semester, 2022-2023**

Homework 2 (Deadline: Apr 22)

1. (10 pts) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix, and let $g \sim \mathcal{N}(0, I_n)$. Try to develop a Bernstein type upper tail for the deviation of $X = g^T Ag$ from its mean.

2. (10 pts) In Lecture 2, we have shown that if
$$\mathrm{Ent}\left[e^{\lambda X}\right] \lesssim \lambda^2\nu^2\mathbb{E}\left[e^{\lambda X}\right] \quad \text{for all} \quad \lambda \in \mathbb{R}. \tag{1.1}$$
then X is sub-Gassuain with parameter ν. We have also given two examples (Gaussian and bounded random variables) such that (1.1) holds. This question asks you show that (1.1) holds for all the sub-Gaussian random variables. More precisely, show that if X is $\frac{\nu^2}{4}$-sub-Gaussian, then (1.1) holds.

3. (10 pts) Consider a random variable X taking values in $\mathbb{R}$ with pdf of the form
$$p_\theta(x) = h(x)e^{\langle\theta,T(x)\rangle-\phi(\theta)},$$
where $\theta \in \mathbb{R}^d$. Assume $\nabla\phi(\theta)$ is L-Lipschitz, i.e.,
$$\|\nabla\phi(\theta_1) - \nabla\phi(\theta_2)\|_2 \leq L\|\theta_1 - \theta_2\|_2.$$
For a fixed unit-norm vector $v \in \mathbb{R}^d$, show that the random variable $Z = \langle v, T(X)\rangle$ is sub-Gaussian.

4. (10 pts) Verify that the equality in the tensorization property of entropy (Theorem 2.11 of Lecture 2) holds for
$$g(X_1, \cdots, X_n) = \exp\left(\lambda\sum_{k=1}^{n} X_k\right),$$
where $X_1, \cdots, X_n$ are independent.

5. (10 pts) Compute the KL divergence between two multivariate Gaussian distributions $\mathcal{N}(\mu_1, \Sigma_1)$ and $\mathcal{N}(\mu_2, \Sigma_2)$, where $\mu_1, \mu \in \mathbb{R}^d$ are the means and $\Sigma_1, \Sigma_2 \in \mathbb{R}^{d\times d}$ are two covariance matrices which are symmetric and positive definite.

6. (10 pts) Let $A \in \mathbb{R}^{d\times d}$ be a random matrix with each entry being i.i.d bounded random variables in $[a, b]$. Show that the spectral norm of A (i.e., $\|A\|_2$) is sub-Gaussian.

7. (10 pts) Let $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ be two probability measures defines on the discrete space $\mathcal{X} = \{x_1, \cdots, x_n\}$. Assume the metric (transportation cost per unit) has the following form
$$d(x_i, x_j) = \begin{cases} 0 & \text{if } x_i = x_j, \\ 1 & \text{otherwise.} \end{cases}$$
Compute the Wasserstein distance $W_1(a, b)$.