Algorithmic and Theoretical Foundations of RL

Policy Optimization II

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Gradient Method over Distributions

It is clear that policy optimization for RL is a special case of optimization over probability distributions:

$$\max_{\theta} J(\theta) = \mathbb{E}_{X \sim P_\theta} [f(X)] .$$

The gradient ascent method for this problem is given by

$$\theta^+ = \theta + \eta \cdot \nabla J(\theta),$$

where the search direction $\Delta\theta = \nabla J(\theta)$ satisfies

$$\Delta\theta \propto \underset{\|d\|_2 \leq \alpha}{\operatorname{argmax}} \{J(\theta) + \langle \nabla J(\theta), d \rangle\}.$$

Question: Is it more natural to search over probability distribution space since $J(\theta)$ essentially relies on P_θ? YES –> Natural gradient method.

Natural Gradient over Distributions

Natural gradient method conducts search based on KL divergence between probability distributions ($F(\theta)^\dagger$ is pseudoinverse of $F(\theta)$):

$$\begin{aligned}\Delta\theta &\propto \underset{\mathrm{KL}\left(P_\theta \| P_{\theta+d}\right)\leq\alpha}{\operatorname{argmax}} \{J(\theta) + \langle \nabla J(\theta), d\rangle\} \\ &\propto F(\theta)^\dagger \nabla J(\theta),\end{aligned}$$

where $F(\theta)$ is the Fisher information matrix at θ, defined by

$$F(\theta) = \mathbb{E}_{X\sim P_\theta}\left[\nabla_\theta \log p_\theta(X)(\nabla_\theta \log p_\theta(X))^T\right].$$

This leads to natural gradient method:

$$\theta^+ = \theta + \eta \cdot F(\theta)^\dagger \nabla J(\theta),$$

which can also be viewed as preconditioned gradient method.

Derivation of Natural Gradient Direction

Given two probability distributions P and Q with pdf $p(x)$ and $q(x)$ respectively, the KL divergence is defined by

$$\mathrm{KL}(P\|Q) = \mathbb{E}_P\left[\log\frac{dP}{dQ}\right] = \mathbb{E}_P\left[\log\frac{p(X)}{q(X)}\right].$$

It follows that

$$\begin{aligned}
\mathrm{KL}(P_\theta\|P_{\theta+d}) &= \mathbb{E}_{P_\theta}\left[\log\frac{p_\theta(X)}{p_{\theta+d}(X)}\right] \\
&= -\mathbb{E}_{P_\theta}\left[\log p_{\theta+d}(X) - \log p_\theta(X)\right] \\
&\approx -d^T \underbrace{\mathbb{E}_{P_\theta}\left[\frac{\nabla_\theta p_\theta(X)}{p_\theta(X)}\right]}_{I_1 = \mathbb{E}_{P_\theta}[\nabla_\theta \log p_\theta(X)]} - \frac{1}{2} d^T \underbrace{\mathbb{E}_{P_\theta}\left[\frac{\nabla_\theta^2 p_\theta(X)}{p_\theta(X)} - \frac{\nabla_\theta p_\theta(X)(\nabla_\theta p_\theta(X))^T}{p_\theta(X)^2}\right]}_{I_2 = \mathbb{E}_{P_\theta}\left[\nabla_\theta^2 \log p_\theta(X)\right]} d.
\end{aligned}$$

Derivation of Natural Gradient Direction

For I_1, one has

$$\mathbb{E}_{P_\theta}\left[\frac{\nabla_\theta p_\theta(X)}{p_\theta(X)}\right] = \int \nabla_\theta p_\theta(X)dx = 0.$$

For I_2, one has

$$\mathbb{E}_{P_\theta}\left[\frac{\nabla_\theta^2 p_\theta(X)}{p_\theta(X)}\right] = \int \nabla_\theta^2 p_\theta(X)dx = 0$$

and

$$\mathbb{E}_{P_\theta}\left[\frac{\nabla_\theta p_\theta(X)(\nabla_\theta p_\theta(X))^T}{p_\theta(X)^2}\right] = \mathbb{E}_{P_\theta}\left[\nabla_\theta \log p_\theta(X)(\nabla_\theta \log p_\theta(X))^T\right] = F(\theta).$$

It follows that

$$\Delta\theta = \operatorname*{argmax}_{\mathrm{KL}(P_\theta \| P_{\theta+d}) \le 2\alpha} \{J(\theta) + \langle \nabla J(\theta), d\rangle\} \approx \operatorname*{argmax}_{d^T F(\theta) d \le 2\alpha} \{J(\theta) + \langle \nabla J(\theta), d\rangle\} \propto F(\theta)^\dagger \nabla J(\theta).$$

The pseudoinverse basically means that we won't consider the direction such $F(\theta)d = 0$ since in this case one has $\mathrm{KL}\left(P_\theta \| P_{\theta+d}\right) \approx d^T F(\theta) d = 0$ and the objective function roughly remains unchanged.

Natural Policy Gradient (NPG)

Natural policy gradient is natural gradient applied to RL optimization problem:

$$\max_{\theta} V^{\pi_\theta}(\mu) = \mathbb{E}_{s_0 \sim \mu}\left[V^{\pi_\theta}(s_0)\right] = \mathbb{E}_{\tau \sim P_{\mu}^{\pi_\theta}}\left[r(\tau)\right],$$

where given $\tau = (s_t, a_t, r_t)_{t=0}^{\infty}$,

$$P_{\mu}^{\pi_\theta}(\tau) = \mu(s_0)\prod_{t=0}^{\infty}\pi_\theta(a_t|s_t)P(s_{t+1}|s_t, a_t) \quad \text{and} \quad r(\tau) = \sum_{t=0}^{\infty}\gamma^t r_t.$$

Natural gradient search direction can be incorporated into different policy optimization methods (including REINFORCE, actor-critic) after MC evaluation of $F(\theta)$ (e.g., using data from an episode). We only focus on expression for $F(\theta)$.

By the definition of $F(\theta)$ and expression for $P_{\mu}^{\pi_\theta}$ (assuming $\pi_\theta(a|s) = 1$ for any θ),

$$\begin{aligned} F(\theta) =& \mathbb{E}_{\tau \sim P_{\mu}^{\pi_\theta}}\left[\left(\sum_{t=0}^{\infty}\nabla_\theta \log \pi_\theta(a_t|s_t)\right)\left(\sum_{t=0}^{\infty}\nabla_\theta \log \pi_\theta(a_t|s_t)\right)^{\mathsf{T}}\right] \\ =& \mathbb{E}_{\tau \sim P_{\mu}^{\pi_\theta}}\left[\sum_{t=0}^{\infty}\nabla_\theta \log \pi_\theta(a_t|s_t)(\nabla_\theta \log \pi_\theta(a_t|s_t))^{\mathsf{T}}\right]. \end{aligned}$$

Two Common Expressions of $F(\theta)$ to Avoid Divergence

- Average case:

$$
\begin{aligned}
F(\theta) &= \lim_{T\to\infty} \frac{1}{T} \mathbb{E}_{\tau\sim P_\mu^{\pi_\theta}} \left[\sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(a_t|s_t) \left(\nabla_\theta \log \pi_\theta(a_t|s_t)\right)^T \right] \\
&= \mathbb{E}_{s\sim d^{\pi_\theta}} \mathbb{E}_{a\sim\pi_\theta(\cdot|s)} \left[\nabla_\theta \log \pi_\theta(a|s) \left(\nabla_\theta \log \pi_\theta(a|s)\right)^T \right],
\end{aligned}
$$

  where $d^{\pi_\theta}(s) = \mathbb{E}_{s_0\sim\mu}\left[\lim_{t\to\infty} P(s_t = s|s_0, \pi_\theta)\right]$ is state stationary distribution.

- Discounted case:

$$
\begin{aligned}
F(\theta) &= (1-\gamma)\mathbb{E}_{\tau\sim P_\mu^{\pi_\theta}} \left[\sum_{t=0}^{+\infty} \gamma^t \nabla_\theta \log \pi_\theta(a_t|s_t) (\nabla_\theta \log \pi_\theta(a_t|s_t))^T \right] \\
&= \mathbb{E}_{s\sim d_\mu^{\pi_\theta}} \mathbb{E}_{a\sim\pi_\theta(\cdot|s)} \left[\nabla_\theta \log \pi_\theta(a|s) \left(\nabla_\theta \log \pi_\theta(a|s)\right)^T \right],
\end{aligned}
$$

  where $d_\mu^{\pi_\theta}(s) = \mathbb{E}_{s_0\sim\mu}\left[(1-\gamma)\sum_{t=0}^{\infty} \gamma^t P(s_t = s|s_0, \pi_\theta)\right]$ is discounted state visitation measure.

Remark

- For the discounted case, it is not difficult to verify that the natural gradient direction $F(\theta)^\dagger \nabla_\theta V^{\pi_\theta}(\mu)$ satisfies

$$F(\theta)^\dagger \nabla_\theta V^{\pi_\theta}(\mu) = \frac{1}{1-\gamma}\omega^*,$$

where ω^* is the (ℓ_2-minimal) solution to

$$\min_\omega L(\omega) = \mathbb{E}_{s\sim d_\mu^{\pi_\theta}, a\sim\pi_\theta(\cdot|s)}\left[\left((\nabla_\theta \log \pi_\theta(a|s))^T \omega - A^{\pi_\theta}(s,a)\right)^2\right].$$

See "On the theory of policy gradient methods: Optimality, approximation, and distribution shift" by Agarwal et al. 2021 for details.

Remark

- For the softmax parameterization (i.e., $\pi_\theta(a|s) = \exp(\theta_{s,a})/(\sum_{a'} \exp(\theta_{s,a'}))$), it can be verified all the solutions to $\min_\omega L(\omega)$ has the following general form:

$$\omega^*_{s,a} = A^{\pi_\theta}(s,a) + c_s,$$

where c_s is a constant relying on s. Thus NPG in policy space is given by

$$\pi_{\theta^+}(a|s) = \frac{\pi_\theta(a|s) \cdot \exp\left(\frac{\eta}{1-\gamma} A^{\pi_\theta}(s,a)\right)}{\sum_{a'} \pi_\theta(a'|s) \cdot \exp\left(\frac{\eta}{1-\gamma} A^{\pi_\theta}(s,a')\right)},$$

which coincides with EQA in Lecture 7 (a policy mirror ascent method).

See "On the theory of policy gradient methods: Optimality, approximation, and distribution shift" by Agarwal et al. 2021 for details.

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Trust Region Policy Optimization (TRPO)

Overall Idea

Given a policy π_{θ_t}, by performance difference lemma, we can rewrite $V^{\pi_\theta}(\mu)$ as

$$V^{\pi_\theta}(\mu) = V^{\pi_{\theta_t}}(\mu) + \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_\mu^{\pi_\theta}}\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[A^{\pi_{\theta_t}}(s,a)\right].$$

Since we do not have access to $d_\mu^{\pi_\theta}$, instead maximize the approximation:

$$\max_\theta\ V_t(\theta) = V^{\pi_{\theta_t}}(\mu) + \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[A^{\pi_{\theta_t}}(s,a)\right].$$

Two Facts

- Assume $\sum_a \pi_\theta(a|s) = 1$ for any θ. It is easy to see that $V^{\pi_\theta}(\mu)$ and $V_t(\theta)$ match at θ_t up to first derivative.
- It can be shown that

$$V^{\pi_\theta}(\mu) \geq V_t(\theta) - \frac{2\gamma\varepsilon_t}{(1-\gamma)^2} \max_s \mathrm{KL}(\pi_{\theta_t}(\cdot|s) \| \pi_\theta(\cdot|s)),$$

where $\varepsilon_t = \max_{s,a} |A^{\pi_{\theta_t}}(s,a)|$.

See “Trust region policy optimization” by Schulman et al. 2017 for derivation of second fact.

TRPO is Approximately NPG Plus Line Search

The second fact suggests that we may seek a new estimator by maximizing $V_t(\theta)$ in a small neighborhood of θ_t:

$$\max_{\theta} \; V_t(\theta) \quad \text{subject to} \quad \max_{s} \mathrm{KL}(\pi_{\theta_t}(\cdot|s) \| \pi_{\theta}(\cdot|s)) \leq \delta.$$

Moreover, replace constraint by the average version and instead solve

$$\max_{\theta} \; V_t(\theta) \quad \text{subject to} \quad \mathbb{E}_{s \sim d_{\mu}^{\pi_{\theta_t}}} \left[\mathrm{KL}(\pi_{\theta_t}(\cdot|s) \| \pi_{\theta}(\cdot|s))\right] \leq \delta.$$

Trust Region Policy Optimization (TRPO)

TRPO is Approximately NPG Plus Line Search

After linear approximation to $V_t(\theta)$ and quadratic approximation to KL at θ_t,

$$V_t(\theta) \approx (\nabla_\theta V^{\pi_{\theta_t}}(\mu))^T(\theta - \theta_t),\ \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}}[\mathrm{KL}(\pi_{\theta_t}(\cdot|s) \| \pi_\theta(\cdot|s))] \approx \frac{1}{2}(\theta - \theta_t)^T F(\theta_t)(\theta - \theta_t),$$

we arrive at the same problem as that for NPG,

$$\max_\theta (\nabla_\theta V^{\pi_{\theta_t}}(\mu))^T(\theta - \theta_t) \quad \text{subject to} \quad \frac{1}{2}(\theta - \theta_t)^T F(\theta_t)(\theta - \theta_t) \le \delta.$$

- TRPO is NPG with adaptive line search in implementations.

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Proximal Policy Optimization (PPO)

Recall from last section that

$$
\begin{aligned}
V_t(\theta) &\propto \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \mathbb{E}_{a\sim\pi_\theta(\cdot|s)} \left[A^{\pi_{\theta_t}}(s,a)\right] \\
&= \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}} \mathbb{E}_{a\sim\pi_{\theta_t}(\cdot|s)} \left[\frac{\pi_\theta(a|s)}{\pi_{\theta_t}(a|s)} A^{\pi_{\theta_t}}(s,a)\right],
\end{aligned}
$$

serves as a surrogate function of true target in small region around θ_t.

PPO keeps new policy close to old one through clipped objective.

PPO with Clipped Objective

Let $r(\theta) = \frac{\pi_\theta(a|s)}{\pi_{\theta_t}(a|s)}$. Then $r(\theta_t) = 1$. The clipped objective function is given by

$$V_t^{\mathrm{clip}}(\theta) = \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \mathbb{E}_{a \sim \pi_{\theta_t}(\cdot|s)} \left[\min \left(r(\theta) A^{\pi_{\theta_t}}(s,a), \mathrm{clip}\left(r(\theta), 1-\epsilon, 1+\epsilon\right) A^{\pi_{\theta_t}}(s,a) \right) \right],$$

where

$$\mathrm{clip}\left(r(\theta), 1-\epsilon, 1+\epsilon\right) = \begin{cases} 1+\epsilon, & r(\theta) > 1+\epsilon, \\ r(\theta), & r(\theta) \in [1-\epsilon, 1+\epsilon], \\ 1-\epsilon, & r(\theta) < 1-\epsilon. \end{cases}$$

- The $\min$ operation ensure $V_t^{\mathrm{clip}}(\theta)$ provides a lower bound. Since a maximal point will be computed subsequently, $\min$ will not cancel the effect of clip.
- PPO policy update (in expectation): $\theta_{t+1} = \mathrm{argmax}_\theta \, V_t^{\mathrm{clip}}(\theta)$.
- In flat region, gradient of $V_t^{\mathrm{clip}}(\theta)$ is zero, thus won't move far from θ_t is using policy gradient type method to solve the sub-problem.

See "Proximal policy optimization algorithms" by Schulman et al. 2017 for details.

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Deterministic Policy Parameterization

Consider the case where $\mathcal{S}$ and $\mathcal{A}$ are continuous, and use π_θ to denote a deterministic policy: $a = \pi_\theta(s)$ is an action.

- Average state value:

$$V^{\pi_\theta}(\mu) = \int_{\mathcal{S}} V^{\pi_\theta}(s_0)\mu(s_0)\mathrm{d}s_0 = \mathbb{E}_{\tau \sim p_\mu^{\pi_\theta}}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, \pi_\theta(s_t), s_{t+1})\right],$$

  where given trajectory $\tau = (s_t, \pi_\theta(s_t), s_{t+1})_{t=0}^{\infty}$,

$$p_\mu^{\pi_\theta}(\tau) = \mu(s_0)\prod_{t=0}^{\infty} p(s_{t+1}|s_t, \pi_\theta(s_t))$$

  is the probability density over τ. Note that there is no probability over action space since $\pi_\theta(s)$ selects a deterministic action.
- It is worth noting that $V^{\pi_\theta}(s) = Q^{\pi_\theta}(s, \pi_\theta(s))$.

Deterministic Policy Parameterization

- Similarly, we can express $V^{\pi_\theta}(\mu)$ over state space

$$
\begin{aligned}
V^{\pi_\theta}(\mu) &= \frac{1}{1-\gamma} \int_{\mathcal{S}} d_\mu^{\pi_\theta}(s) \mathrm{d}s \int_{\mathcal{S}} p(s'|s, \pi_\theta(s)) r(s, \pi_\theta(s), s') \mathrm{d}s' \\
&= \frac{1}{1-\gamma} \mathbb{E}_{s \sim d_\mu^{\pi_\theta}} \mathbb{E}_{s' \sim p(\cdot|s, \pi_\theta(s))} \left[r(s, \pi_\theta(s), s') \right],
\end{aligned}
$$

where $d_\mu^{\pi_\theta}(s) = \mathbb{E}_{s_0 \sim \mu} \left[(1-\gamma) \sum_{t=0}^{\infty} \gamma^t p_t(s|s_0, \pi_\theta) \right]$ is state visitation density, and $p_t(s|s_0, \pi_\theta)$ is the density over state space after transitioning t time steps. Note there is no expectation over action space since $\pi_\theta(s)$ is deterministic.

Deterministic Policy Gradient Theorem

Theorem 1 (Deterministic Policy Gradient Theorem)

Suppose that $\nabla_\theta \pi_\theta(s)$ *and* $\nabla_a Q^{\pi_\theta}(s,a)$ *exist. Then,*

$$\nabla_\theta V^{\pi_\theta}(\mu) = \frac{1}{1-\gamma} \mathbb{E}_{s \sim d_\mu^{\pi_\theta}(s)} \left[\nabla_\theta \pi_\theta(s) \nabla_a Q^{\pi_\theta}(s,a)|_{a=\pi_\theta(s)} \right].$$

Proof of Theorem 1

First note that

$$\begin{aligned} V^{\pi_\theta}(s_0) &= Q^{\pi_\theta}(s_0, \pi_\theta(s_0)) \\ &= \int_{\mathcal{S}} \big(r(s_0, \pi_\theta(s_0), s_1) + \gamma V^{\pi_\theta}(s_1)\big) p(s_1|s_0, \pi_\theta(s_0)) \mathrm{d}s_1. \end{aligned}$$

Therefore, one has

$$\begin{aligned} \nabla_\theta V^{\pi_\theta}(s_0) &= \int_{\mathcal{S}} \nabla_a r(s_0, a, s_1)|_{a=\pi_\theta(s_0)} \nabla_\theta \pi_\theta(s_0) p(s_1|s_0, \pi_\theta(s_0)) \mathrm{d}s_1 \\ &+ \int_{\mathcal{S}} r(s_0, \pi_\theta(s_0), s_1) \; \nabla p(s_1|s_0, a)|_{a=\pi_\theta(s_0)} \nabla_\theta \pi_\theta(s_0) \mathrm{d}s_1 \\ &+ \gamma \int_{\mathcal{S}} V^{\pi_\theta}(s_1) \; \nabla p(s_1|s_0, a)|_{a=\pi_\theta(s_0)} \nabla_\theta \pi_\theta(s_0) \mathrm{d}s_1 \\ &+ \gamma \int_{\mathcal{S}} \nabla_\theta V^{\pi_\theta}(s_1) p(s_1|s_0, \pi_\theta(s_0)) \mathrm{d}s_1. \end{aligned}$$

Proof of Theorem 1 (Cont'd)

Moreover, it is easy to verify that the sum of the first three terms is equal to

$$\nabla_\theta \pi_\theta(s_0)\ \nabla_a Q^{\pi_\theta}(s,a)\big|_{a=\pi_\theta(s_0)}\,.$$

Therefore,

$$\begin{aligned}
\nabla_\theta V^{\pi_\theta}(s_0) &= \nabla_\theta \pi_\theta(s_0)\ \nabla_a Q^{\pi_\theta}(s,a)\big|_{a=\pi_\theta(s_0)} + \gamma \int_{\mathcal{S}} \nabla_\theta V^{\pi_\theta}(s_1) p(s_1|s_0,\pi_\theta(s_0))\mathrm{d}s_1 \\
&= \dots \\
&= \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t \nabla_\theta \pi_\theta(s_t) \nabla_a Q^{\pi_\theta}(s_t,a)|_{a=\pi_\theta(s_t)} | s_0, \pi_\theta\Big] \\
&= \frac{1}{1-\gamma} \mathbb{E}_{s\sim d_{s_0}^{\pi_\theta}} \left[\nabla_\theta \pi_\theta(s) \nabla_a Q^{\pi_\theta}(s,a)|_{a=\pi_\theta(s)}\right].
\end{aligned}$$

Averaging over all s_0 completes the proof of Theorem 1.

Deep Deterministic Policy Gradient (DDPG)

- DDPG is a policy gradient method which learns a deterministic policy π_θ and an action value function $Q^\omega(s, a) \approx Q^{\pi_\theta}(s, a)$. It is an actor-critic algorithm.
- Policy of DDPG is deterministic, need to add random noisy when collecting data; experience replay buffer is also used to break statistical dependence.
- Update of ω for action value function is overall the same to Fitted Q-learning.

See "Continuous control with deep reinforcement learning" by Lillicrap et al. 2016 for details.

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Motivation: Enhance exploration by entropy regularization

Given a policy π, entropy regularized objective function is define by

$$V_\lambda^\pi(\mu) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_\mu^\pi}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}\mathbb{E}_{s'\sim P(\cdot|s,a)}\left[r(s,a,s')\right] + \lambda H(\pi(\cdot|s))\right]$$
$$= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_\mu^\pi}\mathbb{E}_{a\sim\pi(\cdot|s)}\mathbb{E}_{s'\sim P(\cdot|s,a)}\left[r(s,a,s') - \lambda\log\pi(a|s)\right],$$

where $H(\pi(\cdot|s))$ denotes the entropy of the probability distribution $\pi(\cdot|s)$:

$$H(\pi(\cdot|s)) = \mathbb{E}_{a\sim\pi(\cdot|s)}\left[\log\frac{1}{\pi(a|s)}\right].$$

We can rewrite $V_\lambda^\pi(\mu)$ in terms of state values based on a regularized reward

$$V_\lambda^\pi(\mu) = \mathbb{E}_{s\sim\mu}\left[V_\lambda^\pi(s)\right],$$

where $V_\lambda^\pi(s) = \mathbb{E}\left[\sum_{t=0}^\infty \gamma^t r_\lambda(s_t,a_t,s_{t+1})|s_0=s,\pi\right]$ with

$$r_\lambda(s,a,s') = r(s,a,s') - \lambda\log\pi(a|s).$$

- Note that $r_\lambda(s,a,s')$ is not a fixed reward but varies from π to π.

Soft Bellman Equation

- Soft state value V_λ^π :

$$V_\lambda^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_\lambda(s_t, a_t, s_{t+1}) | s_0 = s, \pi\right].$$

- Soft action value $Q_\lambda^\pi(s,a)$: [a_0 is chosen, thus entropy equal to 0]

$$Q_\lambda^\pi(s,a) = \mathbb{E}\left[r(s_0, a_0, s_1) + \sum_{t=1}^{\infty} \gamma^t r_\lambda(s_t, a_t, s_{t+1}) | s_0 = s, a_0 = a, \pi\right].$$

- Relation between Q_λ^π and V_λ^π:

$$Q_\lambda^\pi(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)}[r(s,a,s') + \gamma V_\lambda^\pi(s')],$$
$$V_\lambda^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)}[-\lambda \log \pi(a|s) + Q_\lambda^\pi(s,a)].$$

- Soft Bellman equation:

$$V_\lambda^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[r_\lambda(s,a,s') + \gamma V_\lambda^\pi(s')\right],$$
$$Q_\lambda^\pi(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[r(s,a,s') + \gamma \mathbb{E}_{a' \sim \pi(\cdot|s')}[Q_\lambda^\pi(s',a') - \lambda \log \pi(a'|s')]\right].$$

Soft Bellman Operator

- For state value, soft Bellman operator $\mathcal{T}_\lambda^\pi$ under a policy π is defined by

$$[\mathcal{T}_\lambda^\pi V_\lambda](s) = \mathbb{E}_{a\sim\pi(\cdot|s)}\mathbb{E}_{s'\sim P(\cdot|s,a)}[r_\lambda(s,a,s') + \gamma V_\lambda(s')].$$

  - $\mathcal{T}_\lambda^\pi$ is γ-contraction with respect to ℓ_∞-norm and V_λ^π is unique fixed point.
- For action value, soft Bellman operator $\mathcal{F}_\lambda^\pi$ under a policy π is defined by

$$[\mathcal{F}_\lambda^\pi Q_\lambda](s,a) = \mathbb{E}_{s'\sim P(\cdot|s,a)}\left[r(s,a,s') + \gamma\mathbb{E}_{a'\sim\pi(\cdot|s')}\left[Q_\lambda(s',a') - \lambda\log\pi(a'|s')\right]\right],$$

  - $\mathcal{F}_\lambda^\pi$ is γ-contraction with respect to ℓ_∞-norm and Q_λ^π is unique fixed point.

Soft Bellman Optimality Equation: State Value

For any $V_\lambda \in \mathbb{R}^{|\mathcal{S}|}$, the soft Bellman optimality operator $\mathcal{T}_\lambda$ is defined by

$$
\begin{aligned}
[\mathcal{T}_\lambda V_\lambda](s) &= \max_\pi \mathbb{E}_{a\sim\pi(\cdot|s)} \mathbb{E}_{s'\sim P(\cdot|s,a)} [r_\lambda(s,a,s') + \gamma V_\lambda(s')] \\
&= \max_\pi \mathbb{E}_{a\sim\pi(\cdot|s)} \Big[\underbrace{\mathbb{E}_{s'\sim P(\cdot|s,a)} [r(s,a,s') + \gamma V_\lambda(s')]}_{:=Q_\lambda(s,a)} - \lambda \log \pi(a|s) \Big] \\
&= \lambda \log \left(\left\| \exp\left(Q_\lambda(s,\cdot)/\lambda \right) \right\|_1 \right),
\end{aligned}
$$

where maximum value is attained at (i.e, extracted policy)

$$
\begin{aligned}
\pi_\lambda(a|s) &= \frac{\exp(Q_\lambda(s,a)/\lambda)}{\left\| \exp\left(Q_\lambda(s,\cdot)/\lambda \right) \right\|_1} \\
&= \frac{\exp(Q_\lambda(s,a)/\lambda)}{\exp([\mathcal{T}_\lambda V_\lambda](s)/\lambda)},
\end{aligned}
$$

following Lemma 5 of Lecture 7.

Remark

- Entropy regularization moves the maxima to the interior so that it has an explicit solution in terms of softmax representation.
- Also by Lemma 5 of Lecture 7, one has for any $a \neq a'$,

$$Q_\lambda(s,a) - \lambda \log \pi_\lambda(a|s) = Q_\lambda(s,a') - \lambda \log \pi_\lambda(a'|s)$$

  at optimal π (adding entropy tends to average something). Thus,

$$[\mathcal{T}_\lambda V_\lambda](s) = Q_\lambda(s,a) - \lambda \log \pi_\lambda(a|s), \quad \forall a.$$

- $\mathcal{T}_\lambda$ is γ-contraction with respect to ℓ_∞-norm.

Soft Bellman Optimality Equation: Action Value

For $Q_\lambda \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$, the soft Bellman optimality operator $\mathcal{F}_\lambda$ is defined by

$$
\begin{aligned}
[\mathcal{F}_\lambda Q_\lambda](s,a) &= \mathbb{E}_{s'\sim P(\cdot|s,a)}\left[r(s,a,s') + \gamma \max_\pi \mathbb{E}_{a'\sim\pi(\cdot|s')}\left[Q_\lambda(s',a') - \lambda \log \pi(a'|s')\right]\right] \\
&= \mathbb{E}_{s'\sim P(\cdot|s,a)}\left[r(s,a,s') + \gamma\left[\lambda \log\left(\left\|\exp\left(Q_\lambda\left(s',\cdot\right)/\lambda\right)\right\|_1\right)\right]\right],
\end{aligned}
$$

where maximum value is attained at $\pi_\lambda(\cdot|s') \propto \exp(Q_\lambda(s',\cdot)/\lambda)$.

- $\mathcal{F}_\lambda$ is γ-contraction with respect to ℓ_∞-norm.

Optimal Policy

Theorem 2
Let V^*_λ and Q^*_λ be the fixed points of $\mathcal{T}_\lambda$ and $\mathcal{F}_\lambda$, respectively. One has

$$V^*_\lambda(s) = \max_\pi V^\pi_\lambda(s),\ \forall s \quad \textit{and} \quad Q^*_\lambda(s,a) = \max_\pi Q^\pi_\lambda(s,a),\ \forall s, a.$$

The equality is achieved by the optimal policy given by

$$\pi^*_\lambda(a|s) = \frac{\exp(Q^*_\lambda(s,a)/\lambda)}{\|\exp(Q^*_\lambda(s,\cdot)/\lambda)\|_1}.$$

Moreover, $Q^*_\lambda(s,a) = \mathbb{E}_{s'\sim P(\cdot|s,a)}[r(s,a,s') + \gamma V^*_\lambda(s')]$ *and*

$$V^*_\lambda(s) = \lambda \log\left(\left\|\exp\left(Q^*_\lambda(s,\cdot)/\lambda\right)\right\|_1\right) = Q^*_\lambda(s,a) - \lambda \log \pi^*_\lambda(a|s), \quad \forall a.$$

See “Bridging the gap between value and policy based reinforcement learning” by Nachum et al. 2017 for details.

Remark

- Theorem 2 implies that optimal policy is unique with entropy regularization.
- It is evident that as $\lambda \to 0$, $\pi_\lambda^*(a|s) \to 0$ for $a \notin \operatorname{argmax} Q^*(s,a)$.
- Since one has

$$\max_a Q_\lambda^*(s,a) \le \lambda \log\left(\left\|\exp\left(Q_\lambda^*\left(s,\cdot\right)/\lambda\right)\right\|_1\right) \le \lambda \log|\mathcal{A}| + \max_a Q_\lambda^*(s,a),$$

it is easy to see that $V_\lambda^*(s) \to \max_a Q^*(s,a) = V^*(s)$ as $\lambda \to 0$.

Soft Policy Iteration

- Soft policy evaluation:

$$Q_\lambda^{\pi_k} = \mathcal{F}_\lambda^\pi Q_\lambda^{\pi_k} = \mathbb{E}_{s' \sim P(\cdot|s,a)}[r(s,a,s') + \gamma V_\lambda^{\pi_k}(s')].$$

- Soft policy improvement (soft greedy, use softmax to approximate max):

$$\pi_{k+1} = \frac{\exp(Q_\lambda^{\pi_k}(s,\cdot)/\lambda)}{\|\exp(Q_\lambda^{\pi_k}(s,\cdot)/\lambda\|_1}.$$

Theorem 3 (Informal)
It can be shown that π_{k+1} is an improved policy compared to π_k and the γ-rate convergence of soft PI can also be established.

See "Soft actor-critic: Off-policy maximum entropy deep reinforcement learning with a stochastic actor" by Haarnoja et al. 2018 for details.

Soft Actor Critic (SAC)

SAC is a policy based or actor-critic method for solving

$$\max_{\theta} V_{\lambda}^{\pi_\theta}(\mu) = \mathbb{E}_{s \sim \mu}\left[V_{\lambda}^{\pi_\theta}(s)\right].$$

In addition to typical ways for updating value function and policy parameters,

- Reparametrizarion trick is used in the computation of policy gradient;
- Both state and action values have been parametrized for stable training.

See "Soft actor-critic: Off-policy maximum entropy deep reinforcement learning with a stochastic actor" by Haarnoja et al. 2018 for details.

Questions?