

Sample Questions and Answers-III

Q & A 3

11 October 2024

Example B1

Question:

Consider the function of

$$f(x) = (x^2 - 1)(x^2 - 5)$$

- a) find the critical and inflection points for the function
- b) use sign chart where the function increases and decreases
- c) use sign chart where the function is concave down and concave up
- d) draw the graph of function
- e) identify the local maximum and minimum
- f) investigate whether the optimal points are global or not

a) calculate the first derivative and find critical points

Step 1

First, compute the first derivative of the function using the product rule:

$$f'(x) = 4x(x^2 - 3)$$

Step 2

Critical points occur where $f'(x) = 0$. Solve:

$$4x(x^2 - 3) = 0$$

This gives:

$$x = 0, \quad x = \pm\sqrt{3}$$

Therefore, the critical points are:

$$x = 0, \pm\sqrt{3}$$

a) calculate the second derivative and find the inflection points

Step 3

Next, compute the second derivative $f''(x)$:

$$f''(x) = 12(x^2 - 1)$$

Step 4

Inflection points occur where $f''(x) = 0$. Solve:

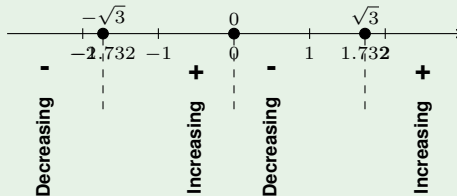
$$12(x^2 - 1) = 0 \quad \Rightarrow \quad x = \pm 1$$

Therefore, the inflection points are:

$$x = \pm 1$$

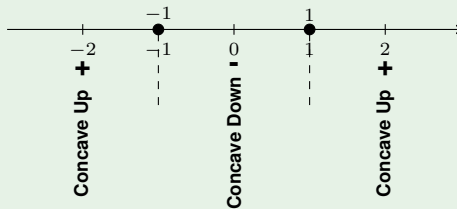
b) use sign chart where the function increases and decreases

Sign Chart for Increasing/Decreasing



c) use sign chart where the function is concave down and concave up

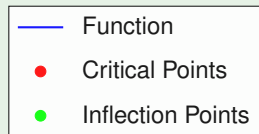
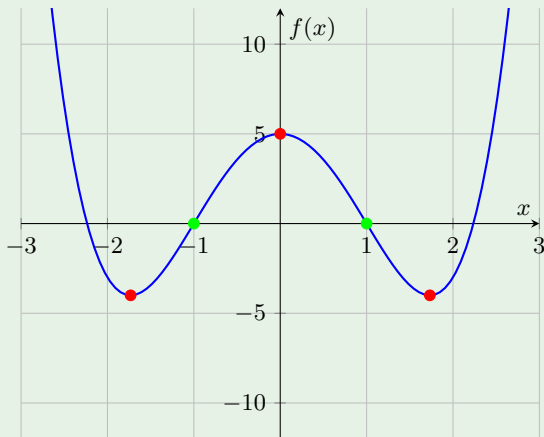
Sign Chart for Concavity



d) draw the graph of the function

Graph of $f(x) = (x^2 - 1)(x^2 - 5)$

$$f(x) = (x^2 - 1)(x^2 - 5)$$



e) identify the local maximum and minimum

Determining Local Maxima and Minima

We have found the critical points at $x = -\sqrt{3}, 0, \sqrt{3}$ by setting $f'(x) = 0$. Now we apply the first derivative test:

- At $x = -\sqrt{3}$: The derivative changes from negative to positive, so $x = -\sqrt{3}$ is a **local minimum**.
- At $x = 0$: The derivative changes from positive to negative, so $x = 0$ is a **local maximum**.
- At $x = \sqrt{3}$: The derivative changes from negative to positive, so $x = \sqrt{3}$ is a **local minimum**.

The function values at these critical points are:

$$f(0) = 5, \quad f(\pm\sqrt{3}) = -4$$

Therefore, the local maxima and minima are:

$$\text{Local maximum at } x = 0, f(0) = 5$$

$$\text{Local minima at } x = \pm\sqrt{3}, f(\pm\sqrt{3}) = -4$$

f) investigate whether the optimal points are global or not

Global Extrema of $f(x) = (x^2 - 1)(x^2 - 5)$

We already know that the function has local extrema at the critical points:

- Local maximum at $x = 0$, $f(0) = 5$.
- Local minima at $x = \pm\sqrt{3}$, $f(\pm\sqrt{3}) = -4$.

Now, we analyze the end behavior of the function:

$$f(x) = x^4 - 6x^2 + 5$$

As $x \rightarrow \infty$ or $x \rightarrow -\infty$, the leading term x^4 dominates, so:

$$f(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow \pm\infty$$

Conclusion:

- The function has no global maximum because $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$.
- The function has a global minimum at $x = -\sqrt{3}$ and $x = \sqrt{3}$, where $f(x) = -4$.

Example B2

Question:

Consider the function

$$f(x) = x^3 - 6x^2 + 9x + 1$$

- Find the critical and inflection points for the function.
- Use a sign chart to determine where the function is increasing and decreasing.
- Use a sign chart to determine where the function is concave down and concave up.
- Sketch the graph of the function showing the critical and inflection points.
- Identify the local maximum and minimum.
- Investigate whether the optimal points are global or not.

a) Calculate the first derivative and find critical points

Step 1

First, compute the first derivative of the function:

$$f'(x) = 3x^2 - 12x + 9$$

Step 2

Critical points occur where $f'(x) = 0$. Solve:

$$3x^2 - 12x + 9 = 0$$

This factors as:

$$3(x^2 - 4x + 3) = 0 \quad \Rightarrow \quad (x - 1)(x - 3) = 0$$

Therefore, the critical points are:

$$x = 1, x = 3$$

a) Calculate the second derivative and find inflection points

Step 3

Next, compute the second derivative $f''(x)$:

$$f''(x) = 6x - 12$$

Step 4

Inflection points occur where $f''(x) = 0$. Solve:

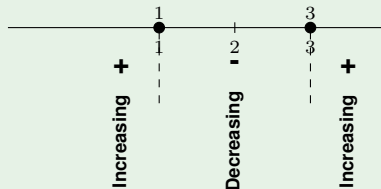
$$6x - 12 = 0 \quad \Rightarrow \quad x = 2$$

Therefore, the inflection point is:

$$x = 2$$

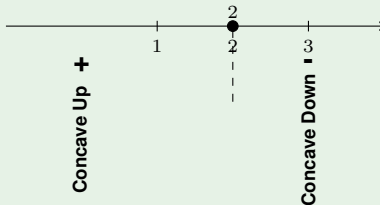
b) Use sign chart where the function increases and decreases

Sign Chart for Increasing/Decreasing



c) Use sign chart where the function is concave down and concave up

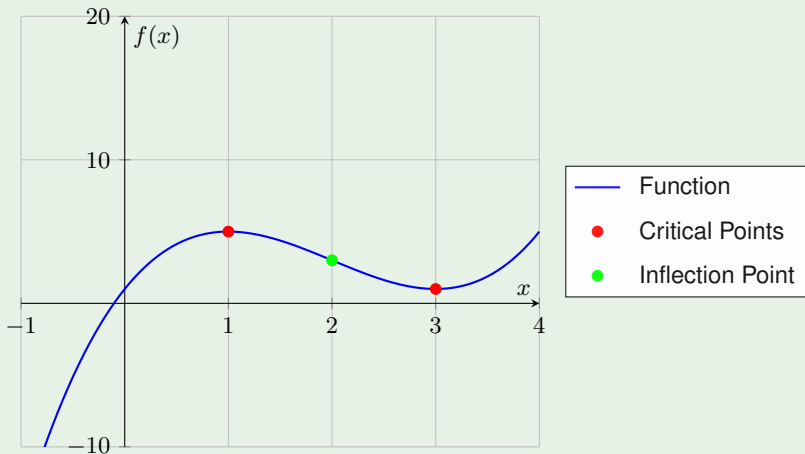
Sign Chart for Concavity



d) Sketch the graph of the function

Graph of $f(x) = x^3 - 6x^2 + 9x + 1$

$$f(x) = x^3 - 6x^2 + 9x + 1$$



e) Identify the local maximum and minimum

Local Maximum and Minimum

We have found the critical points at $x = 1$ and $x = 3$. Now, we apply the first derivative test:

- At $x = 1$: The function changes from increasing to decreasing, so $x = 1$ is a **local maximum**.
- At $x = 3$: The function changes from decreasing to increasing, so $x = 3$ is a **local minimum**.

The function values at these critical points are:

$$f(1) = 1, \quad f(3) = 1$$

Therefore, the local maximum and minimum are:

Local maximum at $x = 1, f(1) = 1$

Local minimum at $x = 3, f(3) = 1$

f) Investigate Global Extrema

Global Extrema of $f(x) = x^3 - 6x^2 + 9x + 1$

We already know:

- The function has a local maximum at $x = 1$, with $f(1) = 5$.
- The function has a local minimum at $x = 3$, with $f(3) = 1$.

- As $x \rightarrow \infty$,

$$f(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow \infty.$$

- As $x \rightarrow -\infty$,

$$f(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow -\infty.$$

Conclusion:

- There is no global maximum because the function increases without bound as $x \rightarrow \infty$.
- There is no global minimum because the function decreases without bound as $x \rightarrow -\infty$.

Example 3.5

Question:

Consider the function

$$f(x) = x^4 - 4x^3 + 4x^2 + 4$$

- Find the critical and inflection points for the function
- Use a sign chart to determine where the function increases and decreases
- Use a sign chart to determine where the function is concave up and concave down
- Draw the graph of the function
- Identify the local maximum and minimum
- Investigate whether the optimal points are global or not

a) Calculate the first derivative and find critical points

Step 1: First Derivative

Compute the first derivative of the function:

$$f'(x) = 4x^3 - 12x^2 + 8x$$

Step 2: Critical Points

Critical points occur when $f'(x) = 0$. Solve the equation:

$$4x(x^2 - 3x + 2) = 0$$

Factor the quadratic:

$$4x(x - 1)(x - 2) = 0$$

Therefore, the critical points are:

$$x = 0, 1, 2$$

a) Calculate the second derivative and find inflection points

Step 3: Second Derivative

Compute the second derivative of the function:

$$f''(x) = 12x^2 - 24x + 8$$

Step 4: Inflection Points

Inflection points occur when $f''(x) = 0$. Solve the equation:

$$12x^2 - 24x + 8 = 0$$

Divide by 4:

$$3x^2 - 6x + 2 = 0$$

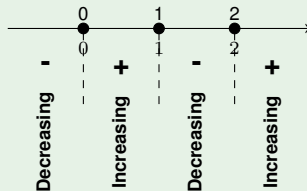
Using the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} = \frac{6 \pm \sqrt{12}}{6} = \frac{6 \pm 2\sqrt{3}}{6}$$

$$x = 1 + \frac{\sqrt{3}}{3}, \quad x = 1 - \frac{\sqrt{3}}{3}$$

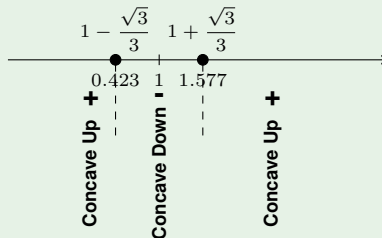
b) Use sign chart where the function increases and decreases

Sign Chart for Increasing/Decreasing

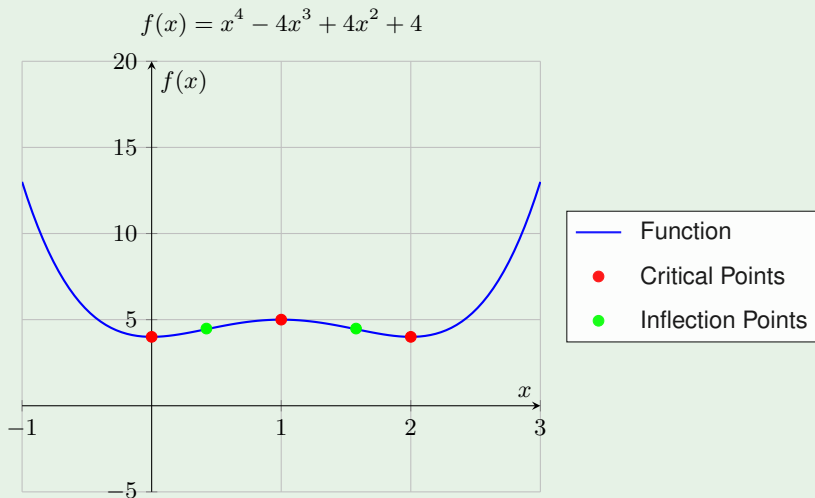


c) Use sign chart where the function is concave down and concave up

Sign Chart for Concavity



d) Sketch the graph of the function

Graph of $f(x) = x^4 - 4x^3 + 4x^2 + 4$ 

e) Local Maxima and Minima

Step 1: Apply the Second Derivative Test

To determine whether the critical points correspond to local maxima or minima, we apply the second derivative test:

$$f''(x) = 12x^2 - 24x + 8$$

At $x = 0$:

$$f''(0) = 12(0)^2 - 24(0) + 8 = 8 \quad (\text{positive, so local minimum at } x = 0)$$

At $x = 1$:

$$f''(1) = 12(1)^2 - 24(1) + 8 = -4 \quad (\text{negative, so local maximum at } x = 1)$$

At $x = 2$:

$$f''(2) = 12(2)^2 - 24(2) + 8 = 8 \quad (\text{positive, so local minimum at } x = 2)$$

If $f''(x) > 0$, it's a local minimum. If $f''(x) < 0$, it's a local maximum. If $f''(x) = 0$, the test is inconclusive.

f) Investigate whether the optimal points are global or not

Global Minima or Maxima

Note that $x = 0$ and $x = 2$ are global minimum of f . However, $x = 1$ is not definitely a global maximum, since f eventually takes on arbitrarily large values as $x \rightarrow \infty$

The function has global minimum at:

- **Global minimum at** $x = 0$ and $x = 2$ with $f(0) = f(2) = 4$

$f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$, the function has no global maximum.

The function increases without bound as $x \rightarrow \infty$ or $x \rightarrow -\infty$, so there are no global maximum.

Example B3

Question:

Consider the function

$$f(x) = x^3 - 3x^2 + 2x + 5$$

- Find the critical points and inflection points of the function
- Use a sign chart to determine where the function is increasing and decreasing
- Use a sign chart to determine where the function is concave up and concave down
- Sketch the graph of the function
- Identify the local maximum and minimum
- Investigate whether the optimal points are global or not

a) Calculate the first derivative and find critical points

Step 1: First Derivative

Compute the first derivative of the function:

$$f'(x) = 3x^2 - 6x + 2$$

Step 2: Critical Points

Critical points occur when $f'(x) = 0$. Solve the equation:

$$3x^2 - 6x + 2 = 0$$

Divide by 3:

$$x^2 - 2x + \frac{2}{3} = 0$$

Using the quadratic formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(\frac{2}{3})}}{2(1)} = \frac{2 \pm \sqrt{4 - \frac{8}{3}}}{2}$$

a) Calculate the first derivative and find critical points

Step 2: Critical Points

$$x = \frac{2 \pm \sqrt{\frac{12}{3} - \frac{8}{3}}}{2} = \frac{2 \pm \sqrt{\frac{4}{3}}}{2}$$
$$x = 1 \pm \frac{\sqrt{3}}{3}$$

Therefore, the critical points are:

$$x = 1 + \frac{\sqrt{3}}{3}, \quad x = 1 - \frac{\sqrt{3}}{3}$$

b) Calculate the second derivative and find inflection points

Step 3: Second Derivative

Compute the second derivative of the function:

$$f''(x) = 6x - 6$$

Step 4: Inflection Points

Inflection points occur when $f''(x) = 0$. Solve the equation:

$$6x - 6 = 0$$

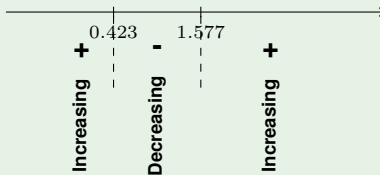
$$x = 1$$

Therefore, the inflection point is:

$$x = 1$$

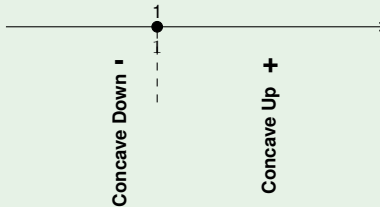
c) Use sign chart where the function is increasing and decreasing

Sign Chart for Increasing/Decreasing

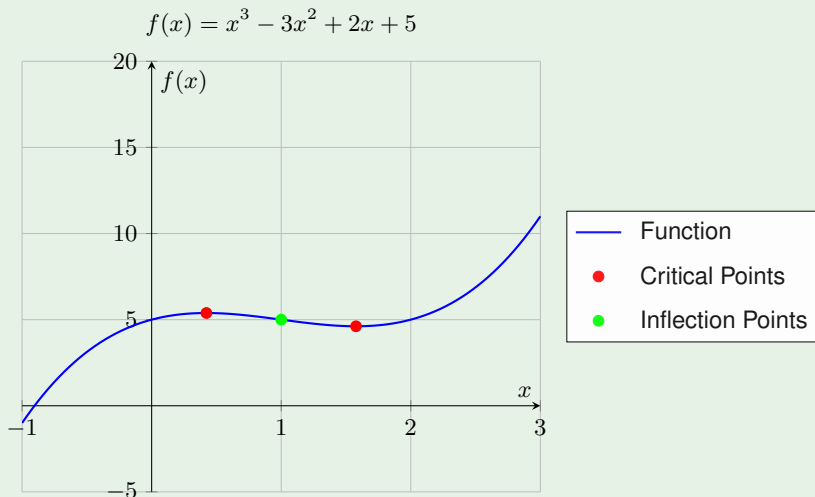


d) Use sign chart where the function is concave up and concave down

Sign Chart for Concavity



e) Sketch the graph of the function

Graph of $f(x) = x^3 - 3x^2 + 2x + 5$ 

f) Local Maxima and Minima

Apply the Second Order Condition

To determine whether the critical points correspond to local maxima or minima, we apply the second derivative test:

$$f''(x) = 6x - 6$$

At $x = 1 + \frac{\sqrt{3}}{3}$:

$$f''\left(1 + \frac{\sqrt{3}}{3}\right) = 6\left(1 + \frac{\sqrt{3}}{3}\right) - 6 = \text{positive value, so local minimum.}$$

At $x = 1 - \frac{\sqrt{3}}{3}$:

$$f''\left(1 - \frac{\sqrt{3}}{3}\right) = 6\left(1 - \frac{\sqrt{3}}{3}\right) - 6 = \text{negative value, so local maximum.}$$

g) Investigate whether the optimal points are global or not

Global Extrema

Any strictly increasing or strictly decreasing function whose domain is an *open interval* will not have a maximum or a minimum in its domain

- As $x \rightarrow +\infty$, $f(x) \rightarrow +\infty$.
- As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.

Conclusion:

- The function does **not have a global maximum** because as $x \rightarrow +\infty$, the function increases without bound.
- The function does **not have a global minimum** because as $x \rightarrow -\infty$, the function decreases without bound.

Therefore, while the function has a local maximum and a local minimum, it does not have any global extrema.

Example 3.6

Question:

The membership of the Association of Smart Statisticians is given by the function

$$f(x) = 2x^3 - 45x^2 + 300x + 500$$

where x is the number of years after 1960. Find the largest and smallest membership between 1960 and 1980, i.e., for $x \in [0, 20]$, and determine when these extreme values occur. Mathematically, this is the problem of maximizing

$$f(x) = 2x^3 - 45x^2 + 300x + 500 \text{ for } x \text{ in the closed interval } [0, 20].$$

First Derivative and Critical Points

First Derivative

To find the critical points, we first compute the first derivative of the function:

$$f'(x) = 6x^2 - 90x + 300$$

Critical Points

Critical points occur where $f'(x) = 0$. Solve the equation:

$$6x^2 - 90x + 300 = 0$$

Dividing through by 6:

$$x^2 - 15x + 50 = 0$$

Using the quadratic formula:

$$x = \frac{-(-15) \pm \sqrt{(-15)^2 - 4(1)(50)}}{2(1)} = \frac{15 \pm \sqrt{225 - 200}}{2} = \frac{15 \pm \sqrt{25}}{2} = \frac{15 \pm 5}{2}$$

The solutions are:

$$x = 10 \quad \text{or} \quad x = 5$$

Evaluate the Function

Evaluate at Critical Points and Endpoints

Next, we evaluate the function $f(x)$ at the critical points and the endpoints of the interval $[0, 20]$:

$$f(0) = 2(0)^3 - 45(0)^2 + 300(0) + 500 = 500$$

$$f(5) = 2(5)^3 - 45(5)^2 + 300(5) + 500 = 1125$$

$$f(10) = 2(10)^3 - 45(10)^2 + 300(10) + 500 = 0$$

$$f(20) = 2(20)^3 - 45(20)^2 + 300(20) + 500 = 500$$

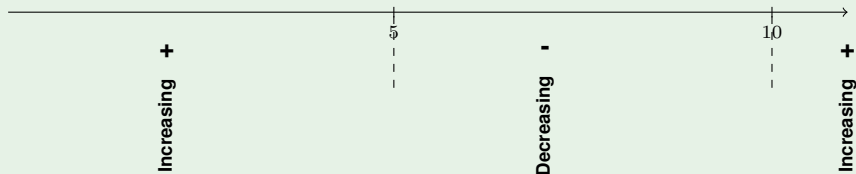
Sign Chart for Increasing/Decreasing

Sign Chart for Increasing/Decreasing

To determine where the function is increasing or decreasing, we use the first derivative:

$$f'(x) = 6x^2 - 90x + 300$$

Critical points are $x = 5$ and $x = 10$. We now test the sign of $f'(x)$ in the intervals $(-\infty, 5)$, $(5, 10)$, and $(10, \infty)$.



Second Derivative and Concavity

Second Derivative

To determine where the function is concave up or concave down, we compute the second derivative:

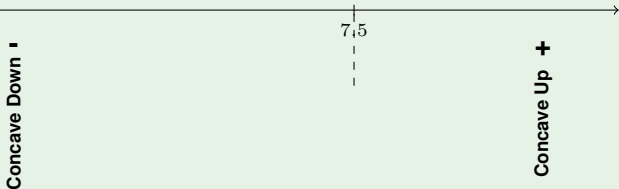
$$f''(x) = 12x - 90$$

Setting $f''(x) = 0$, we find the inflection point:

$$12x - 90 = 0 \Rightarrow x = 7.5$$

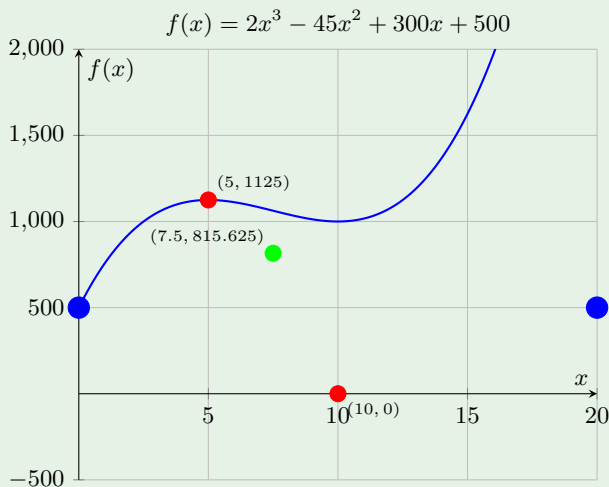
Sign Chart for Concavity

We now analyze the concavity by testing the sign of $f''(x)$ in the intervals $(-\infty, 7.5)$ and $(7.5, \infty)$.



Graph of the Function

Graph of $f(x) = 2x^3 - 45x^2 + 300x + 500$



Global Maximum and Minimum

Global Maximum and Minimum

The function values at the critical points and endpoints are:

$$f(0) = 500, \quad f(5) = 1125, \quad f(10) = 0, \quad f(20) = 500$$

- The global maximum occurs at $x = 5$ with $f(5) = 1125$. - The global minimum occurs at $x = 10$ with $f(10) = 0$.