One Variable Calculus-III

Lecture 3

11 October 2024

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Drawing the Graph of a Function

Theorem: First Derivative

Suppose that f is a C^1 at x_0. Then,

1. if $f'(x) > 0$, there is an open interval containing x_0 on which $f(x)$ is **increasing** function
2. if $f'(x) < 0$, there is an open interval containing x_0 on which $f(x)$ is **decreasing** function
3. if $f'(x) = 0$, $f(x)$ is constant or isn't defined. **critical points**

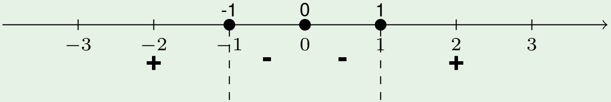
Drawing the Graph of a Function

Example 1: First Derivative

$f(x) = x^3 - 3x$. Find the critical points and draw the graph.

- First derivative of $f(x)$: $f'(x) = 3x^2 - 3$
- Factorizing: $f'(x) = 3(x-1)(x+1)$
- Critical points are found by setting $f'(x) = 0$: $3(x-1)(x+1) = 0 \quad \Rightarrow \quad x = 1$ or $x = -1$

Sign Chart:



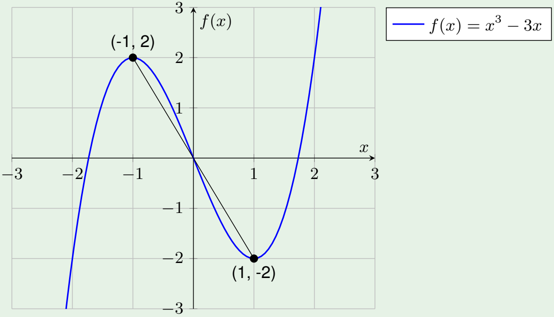
Example 1: First Derivative (cont.)

- For $x = -1$:

$$f(-1) = (-1)^3 - 3(-1) = 2 \quad \Rightarrow \quad (-1, 2)$$

- For $x = 1$:

$$f(1) = (1)^3 - 3(1) = -2 \quad \Rightarrow \quad (1, -2)$$



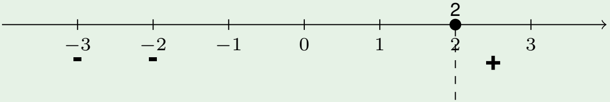
Drawing the Graph of a Function

Example 1: First Derivative

$g(x) = x^2 - 4x + 3$. Find the critical points and draw the graph.

- First derivative of $g(x)$: $\quad g'(x) = 2x - 4$
- Setting $g'(x) = 0$ gives: $\quad 2x - 4 = 0 \quad \Rightarrow \quad x = 2$
- Critical point: $x = 2$

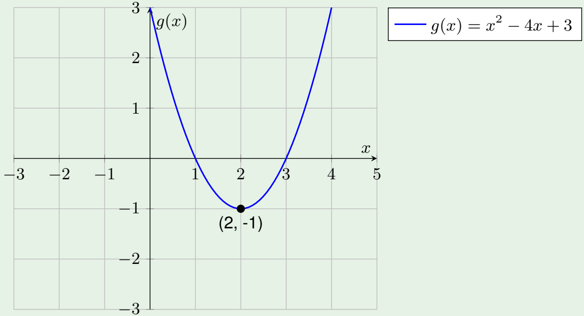
Sign Chart:



Example 1: First Derivative (cont.)

- For $x = 2$:

$$g(2) = (2)^2 - 4(2) + 3 = -1 \quad \Rightarrow \quad (2, -1)$$



Drawing the Graph of a Function

Theorem: Second Derivative and Convexity

1. A differentiable function f for which $f''(x) \geq 0$ on an *interval I* (so that f' is increasing on *I*) is said to be **concave up** on *I*
   - A function f is called **concave up** or simply **convex** on an interval *I* if an only if

     $$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$$

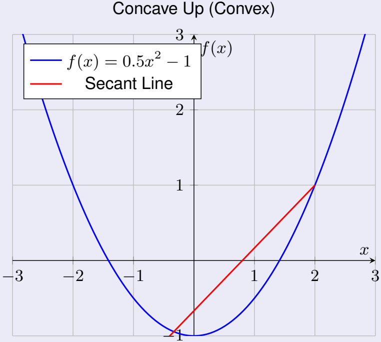
2. A differentiable function f for which $f''(x) \leq 0$ on an *interval I* (so that f' is decreasing on *I*) is said to be **concave down** on *I*
   - A function f is called **concave down** or simply **concave** on an interval *I* if an only if

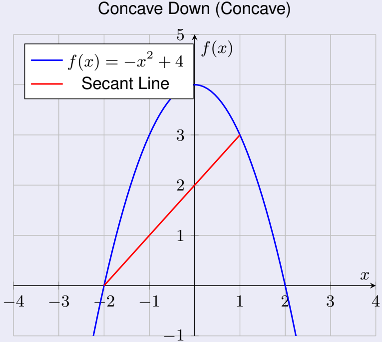
     $$f((1-t)a + tb) \geq (1-t)f(a) + tf(b)$$

Drawing the Graph of a Function

Theorem: Second Derivative and Convexity

- For a function which is concave up **(convex)**, the secant line always lies *above* the graph.
- For a function which is concave down **(concave)**, the secant line always lies *below* the graph.





Drawing the Graph of a Function

Theorem: Second Derivative and Convexity

- The test to see whether a function is convex or concave mimics the test used in determining whether a function is increasing or decreasing, but uses the second derivative instead of the first.
- These points are called the **second order critical points** of f, or if the second derivative actually changes sign there, **inflection points** of f.
- These points divide the domain of f into intervals on each of which f'' is always positive or negative.
- **The characteristics of function changes in Inflection Point**

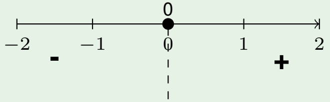
Drawing the Graph of a Function

Example 1: Second Derivative

Continuing from our analysis of $f(x) = x^3 - 3x$:

- Second derivative of $f(x)$:
$$f''(x) = 6x$$
- Setting $f''(x) = 0$ to find inflection points:
$$6x = 0 \quad \Rightarrow \quad x = 0$$
- The sign of $f''(x)$ determines concavity:
  - For $x < 0$: $f''(x) < 0$ (concave down)
  - For $x > 0$: $f''(x) > 0$ (concave up)

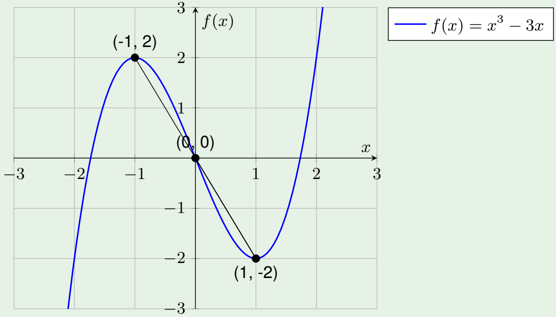
Sign Chart for the Second Derivative:



Drawing the Graph of a Function

Example 1: Second Derivative (cont.)

- The function $f(x)$ is:
  - Concave down on the interval $(-\infty, 0)$
  - Concave up on the interval $(0, \infty)$
- At $x = 0$, we have an inflection point.



Drawing the Graph of a Function

Example 2: Second Derivative

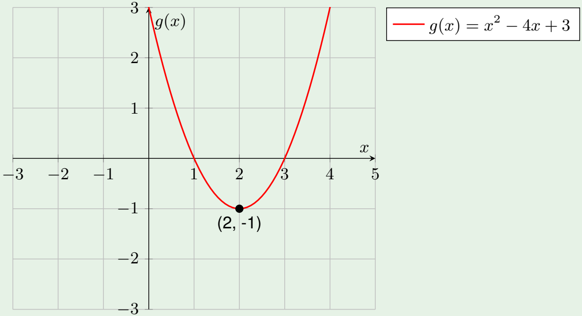
Continuing from our analysis of $g(x) = x^2 - 4x + 3$:

- Second derivative of $g(x)$:
$$g''(x) = 2$$
- Since $g''(x) > 0$ for all x, the function is **convex** everywhere.
- The positivity of the second derivative indicates that the slope of the tangent line is increasing.
- Therefore, the graph of $g(x)$ is concave up, confirming that $x = 2$ a minimum.

Drawing the Graph of a Function

Example 2: Second Derivative (cont.)

- Inflection point found at $x = 2$ where $g(2) = -1$ indicates a minimum.
- The function is convex for all x as confirmed by $g''(x) > 0$.



Drawing the Graph of a Function

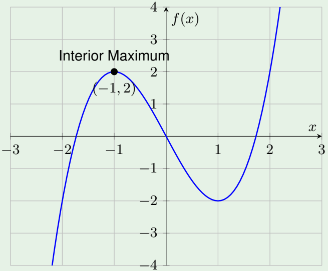
Maxima and Minima

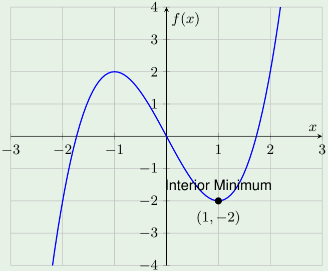
- One of the major uses of calculus in mathematical models is to find and characterize maximum and minimum of functions.
- For instance, economists are interested in maximizing utility and profit and in minimizing cost.
- A maximum or minimum of a function can occur at an endpoint of the domain of f or at a point which is not an endpoint - in the *interior* of the domain.

Drawing the Graph of a Function

Examples: Interior Maximum and Minimum

- **Example 1:** This function has an interior maximum at $x = -1$.
- **Example 2:** This function has an interior minimum at $x = 1$.





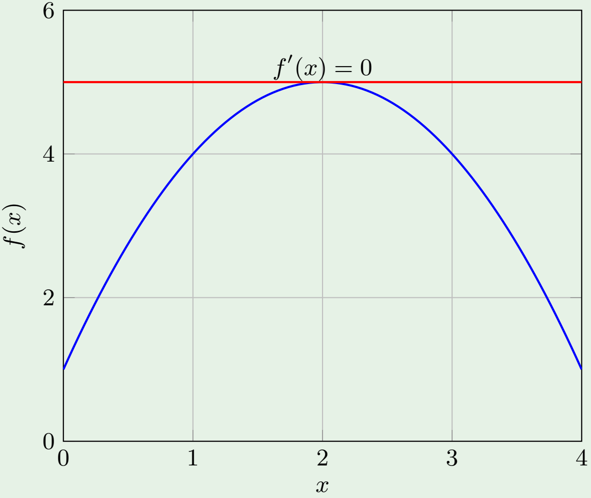
Drawing the Graph of a Function

Theorem: First Order Condition

- If x_0 is an interior maximum or minimum of f, then x_0 is a critical point of f. It is necessary condition but insufficient.
- **Proof**: A function is neither increasing nor decreasing on an interval about an interior maximum or minimum. The first derivative of this function cannot be positive or negative there; that is, $f'(x_0$ must be zero or undefined - x_0 is a critical point of f
- If the graph of f has a tangent line at a maximum or minimum, that tangent line must be horizontal since the graph of f turns around there. In other words, $f'(x_0)$ must be zero.
- **Note: Identifiy Critical Point (We may have a solution)**

Drawing the Graph of a Function

Example :The graph of f at max x_0



Drawing the Graph of a Function

Theorem: Second Order Condition

- If x_0 is a critical point of a function f, we can use second derivative to decide whether critical point x_0 is a maximum, minimum, or neither.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is *maximum* of f;
- If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is *minimum* of f;
- If $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 can be a *maximum, minimum, or neither*
- **Note: Identifiy Inflection Point is maximum, minimum, or neither**

Drawing the Graph of a Function

Theorem: Second Order Condition

Proof

- $f'(x_0) = 0$ means that the tangent line to the graph of f is horizontal at x_0, and $f''(x_0) < 0$ means that the graph curves downward.
- These two conditions together imply that f has a local maximum at x_0.
- $f''(x_0) < 0$ means that the first derivative of f' of f is a decreasing function in an interval about x_0.
- The facts that f' is decreasing and that $f'(x_0) = 0$ mean that f' is positive to the left of x_0 and negative to the right of x_0.
- These two derivative conditions imply that f is increasing to the left of x_0 and decreasing to the right of x_0. In other words, f has a local maximum at x_0.

Drawing the Graph of a Function

First Order Condition

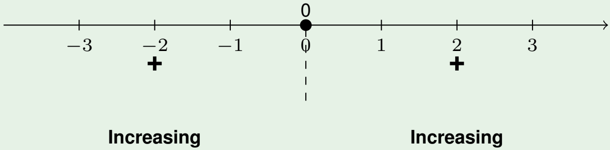
- Consider $f(x) = x^3$.
- Calculate the first derivative:

$$f'(x) = \frac{d}{dx}(x^3) = 3x^2$$

- Set the first derivative to zero to find critical points:

$$3x^2 = 0 \implies x = 0 \quad \text{(Critical Point)}$$

- **Sign Chart:**



Drawing the Graph of a Function

Second Order Condition

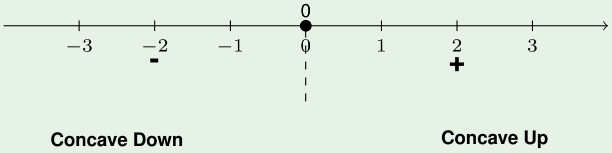
- Calculate the second derivative:

$$f''(x) = \frac{d}{dx}(3x^2) = 6x$$

- Evaluate at the critical point and identify the inflection point:

$$f''(0) = 6(0) = 0 \quad \text{(inconclusive, inflection point at } x = 0)$$

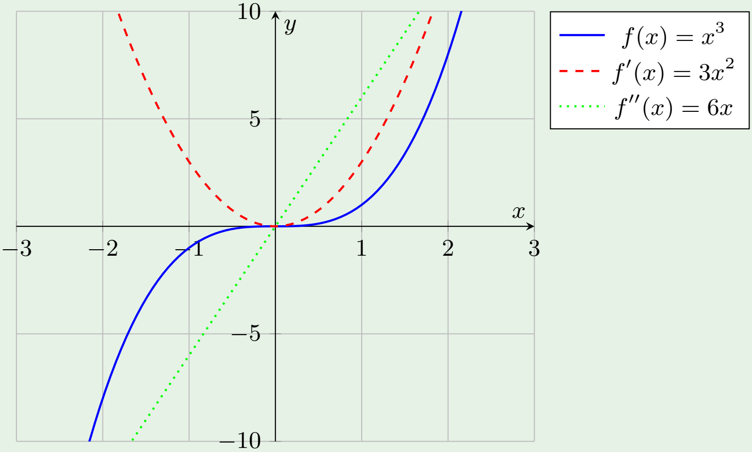
- **Sign Chart:**



Drawing the Graph of a Function

Graph of $f(x)$

Graphs of $f(x) = x^3$, $f'(x) = 3x^2$, and $f''(x) = 6x$



Drawing the Graph of a Function

First Order Condition

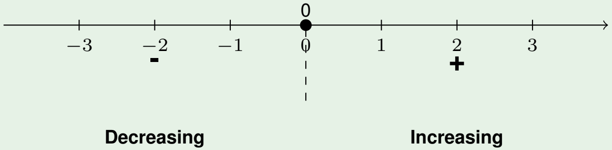
- Consider $f(x) = x^4$.
- Calculate the first derivative:

$$f'(x) = \frac{d}{dx}(x^4) = 4x^3$$

- Set the first derivative to zero to find critical points:

$$4x^3 = 0 \implies x = 0 \quad \text{(Critical Point)}$$

- **Sign Chart:**



Drawing the Graph of a Function

Second Order Condition

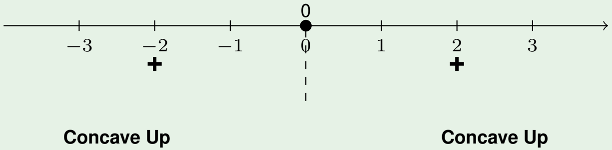
- Calculate the second derivative:

$$f''(x) = \frac{d}{dx}(4x^3) = 12x^2$$

- Evaluate at the critical point and identify the inflection point:

$$f''(0) = 12(0)^2 = 0 \quad \text{(inconclusive, inflection point at } x = 0\text{)}$$

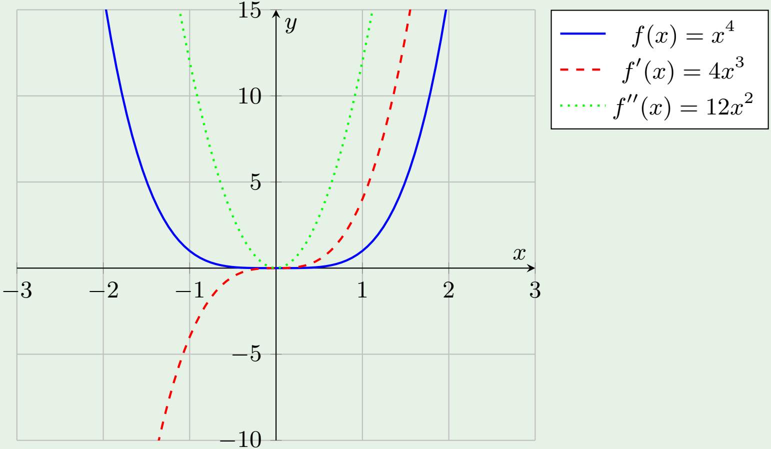
- **Sign Chart:**



Drawing the Graph of a Function

Graph of $f(x)$

Graphs of $f(x) = x^4$, $f'(x) = 4x^3$, and $f''(x) = 12x^2$



Drawing the Graph of a Function

Global Maxima or Minima

In general, it is difficult to find a global maximum of a function or even to prove that a given local maximum is a global maximum. There are, however, three situations in which this problem is somewhat easier:

1. when f has only one critical point in its domain
2. when $f'' > 0$ or $f'' < 0$ throughout the domain of f, and
3. when the domain of f is a closed finite interval.

Drawing the Graph of a Function

Theorem: Global Maxima

Suppose that

1. the domain of f is an interval I (finite or infinite) in $\mathbb{R}^1$,
2. x_0 is a local maximum of f, and
3. x_0 is the only critical point of f on I.

Then, x_0 is the global maximum of f on I.

Drawing the Graph of a Function

Theorem: Global Maxima or Minima

- If c^2 functions whose domain is an interval I and if f'' is never zero on I, then
- f has at most one critical point in I
- This critical point is a global minimum if $f'' > 0$ and a global maximum if $f'' < 0$

Drawing the Graph of a Function

Example 1

$f(x) = x^4 - 4x^3 + 4x^2 - 10$

- First derivative:

$$f'(x) = 4x^3 - 12x^2 + 8x$$

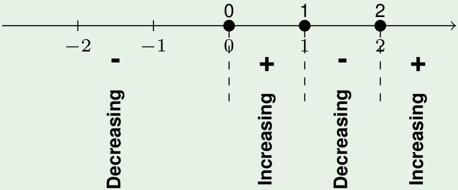
- Set the first derivative to zero:

$$4x^3 - 12x^2 + 8x = 0$$

- Factoring:

$$4x(x^2 - 3x + 2) = 0 \implies x = 0,\ x = 1,\ x = 2 \quad \text{(Critical Points)}$$

- **Sign Chart:**



Drawing the Graph of a Function

Example 1 (cont.)

- Calculate the second derivative:

$$f''(x) = \frac{d}{dx}(4x^3 - 12x^2 + 8x) = 12x^2 - 24x + 8$$

- Evaluate the second derivative at the critical points:

$$f''(0) = 8 \quad \text{(Local Minimum)}$$

$$f''(1) = -4 \quad \text{(Local Maximum)}$$

$$f''(2) = 8 \quad \text{(Local Minimum)}$$

Drawing the Graph of a Function

Example 1 (cont.)

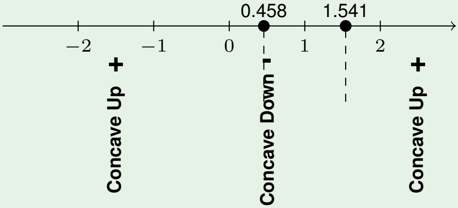
- Set the second derivative to zero:

$$12x^2 - 24x + 8 = 0$$

- Solving gives inflection points:

$$x \approx 0.458, \quad x \approx 1.541$$

Sign Chart



Drawing the Graph of a Function

Example 1 (cont.)

$$f(x) = x^4 - 4x^3 + 4x^2 - 10$$