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Problem 1 - Sampling theorem:

Sampling theorem can be defined as the conversion of a continues signal into discreet signal under the condition of tacking the sampling frequency as more than twice the input signal frequency, this condition is to guarantee the retrieving of the original signal. Let the maximum frequency of signal $x(t)$ in time domain be f_{max} [Hz], and f_s be the sampling frequency, the sampling theorem states that $f_s > 2f_{max}$.

If $f_s = 2f_{max}$ then it is called **Nyquist** criteria of sampling and when $f_s < 2f_{max}$, this condition causes **Aliasing** effect where the sampled signals will overlap with each other resulting distortion in the output signal. For this reason, the sampled frequency should be more than the Nyquist rate ($2f_{max}$).

Consider the following system, where x(t) is the input signal, p(t) is the pulse train function and $x_p(t)$ is the output signal.

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X\big(j(\omega - k\omega_s)\big)$$

As it is shown in fig.1, whenever $w_s > 2w_m$ there is no aliasing occurs or overlapping, where $w_s = 2\pi fs$. When $w_s < 2w_m$ we can see the overlapping occurs.

As it is known, in order to overcome the original signal, the cut-off frequency in the impulse response filter is set as $w_s > w_c > (w_s - w_m)$, in this case if the sampling condition is satisfied as $w_s > 2w_m$, the original signal can be recovered without any distortion. Otherwise, the recovered signal will be distorted.

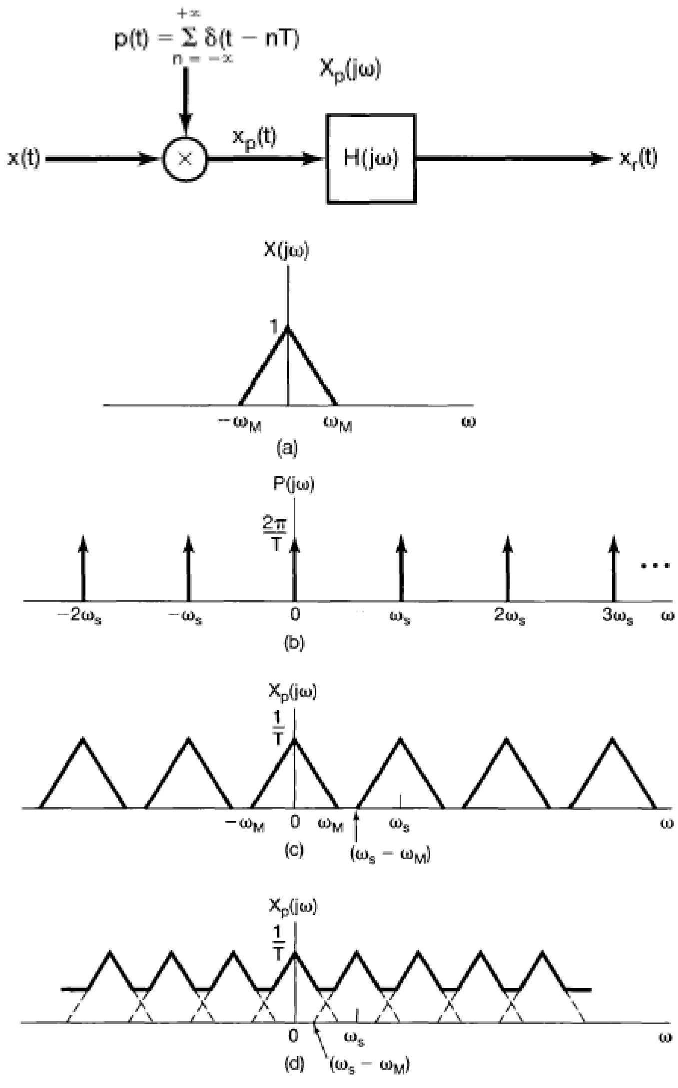

Figure 1: a) input sig. b) train pulse c) sampled sig. when Ws>2Wm d) sampled sig. when Ws<2Wm

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Problem 2 - Discrete Fourier Transform:

(i) The discreet time Fourier transform can be found by tacking the continues time Fourier transform of a sampled signal

$$x_S(t) = x(t)\sum_{k=-\infty}^{\infty} \delta\,(t - \mathrm{kT}_S)$$

The continues time Fourier transform of $x_s(t)$ can be found as following:

$X_s(w) = F[x_s(t)]$, by using the FT property of product

$$X_s(w) = X(w) * F\left[\sum_{k=-\infty}^{\infty} \delta(t - \mathrm{kT}_S)\right]$$

$$X_s(w) = X(w) * \left[\frac{1}{\mathrm{T}_S}\sum_{k=-\infty}^{\infty} \delta\left(w - \mathrm{k}\frac{2\pi}{\mathrm{T}_S}\right)\right] = \frac{1}{\mathrm{T}_S}\sum_{k=-\infty}^{\infty} X\left(w - k\frac{2\pi}{\mathrm{T}_S}\right)$$

Or

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty} \delta(t - \mathrm{kT}_S) = \sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S)\,\delta(t - \mathrm{kT}_S)$$

$by\ tacking\ FT$

$$X_s(w) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S)\int_{-\infty}^{\infty} \delta(t - \mathrm{kT}_S)e^{-j\omega t}\mathrm{dt} = \sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S)e^{-j\omega kT_S}$$

(ii) We multiply both sides with $e^{j\omega T_S}$

$$X_s(w) * e^{j\omega T_S} = \sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S)e^{-j\omega kT_S} * e^{j\omega T_S}$$

Integrate over one period

$$\int_{-\pi}^{\pi} X_s(w) * e^{j\omega T_S}\,dw = \int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S)e^{j\omega(T_S - kT_S)}\,dw$$

$$= \sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S)\int_{-\pi}^{\pi} e^{j\omega(T_S - kT_S)}$$

$$= \sum_{k=-\infty}^{\infty} x(\mathrm{kT}_S) * 2\pi\,\delta(\mathrm{T}_S - \mathrm{kT}_S)$$

$$\int_{-\pi}^{\pi} X_s(w) * e^{j\omega T_S}\,dw = 2\pi\, x_s(t)$$

$$\boldsymbol{x_s(t) = \frac{1}{2\pi}\int_{\frac{1}{T_S}} X_s(w) * e^{j\omega T_S}\,dw}$$

(iii)

$$X_S(f) = \sum_{k=0}^{\mathrm{N}-1} x(\mathrm{kT}_S)e^{\frac{-j2\pi\mathrm{n}k}{N}}$$

By multiplying both sides with $e^{\frac{j2\pi\mathrm{n}m}{N}}$

$$X_S(f) * e^{\frac{j2\pi\mathrm{n}m}{N}} = \sum_{k=0}^{\mathrm{N}-1} x(\mathrm{kT}_S)e^{\frac{-j2\pi\mathrm{n}k}{N}} * e^{\frac{j2\pi\mathrm{n}m}{N}}$$

By applying summation from m=0 to N-1 on both sides

$$\sum_{m=0}^{\mathrm{N}-1} X_S(f) * e^{\frac{j2\pi\mathrm{n}m}{N}} = \sum_{m=0}^{\mathrm{N}-1} \left[\sum_{k=0}^{\mathrm{N}-1} x(\mathrm{kT}_S)e^{\frac{-j2\pi\mathrm{n}k}{N}} * e^{\frac{j2\pi\mathrm{n}m}{N}}\right]$$

$$= \sum_{m=0}^{\mathrm{N}-1} x(\mathrm{kT}_S) \sum_{k=0}^{\mathrm{N}-1} e^{\frac{j2\pi\mathrm{n}(\mathrm{m}-k)}{N}}$$

$$= \sum_{m=0}^{\mathrm{N}-1} x(\mathrm{kT}_S)\ (N * \delta(\mathrm{m}-\mathrm{k}))$$

$\delta(\mathrm{m}-\mathrm{k}) = 1$ unless m=k

$$\boldsymbol{x(\mathrm{kT}_S) = \frac{1}{N}\sum_{n=0}^{\mathrm{N}-1} X_S(f) * e^{\frac{j2\pi\mathrm{n}k}{N}}}$$

Problem 3 - Practice DFT:

We calculate DFT of $x(nT_S) = \cos(2\pi f_1 nT_S)$. Note that the fundamental frequency is assumed to $f_1 = 1000$ Hz, sampling frequency is $f_{\text{samp}} = 8000$ Hz. And the number of samples is $N = 8$.

(i) First of all, calculate the DFT manually.

$$x(nT_s) = \cos\,(2\pi f_1 nT_s)$$

$x(nT_s) = \cos\,\left(2\pi n\frac{1}{8}\right)$
$x(0) = \cos\,(0) = 1$
$x(1) = \cos\,(\pi/4) = 0.707$
$x(2) = \cos\,(\pi/2) = 0$
$x(7) = \cos\,\left(\frac{3\pi}{4}\right) = -0.707$
$x(4) = \cos\,(\pi) = -1$
$x(5) = \cos\,\left(\frac{5\pi}{4}\right) = -0.707$
$x(6) = \cos\,\left(\frac{3\pi}{2}\right) = 0$
$x(7) = \cos\,\left(\frac{7\pi}{4}\right) = 0.707$

The DFT of $x(nT_s)$ can be evaluated as following:

$$x(k) = \sum_{n=0}^{7} x(n)e^{-\frac{j2\pi nk}{8}} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi nk}{4}}$$

$\boldsymbol{x(0)} = 1 * 1 = \mathbf{1}$

$$\boldsymbol{x(1)} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi n}{4}} \quad = x(0) + x(1)e^{-j\frac{\pi}{4}} + x(2)\cdot e^{-j\frac{\pi}{2}} + x(3)e^{-j\pi\frac{3}{4}} + x(4)e^{-j\pi} + x(5)e^{-j\pi\frac{5}{4}} + x(6)e^{-j\pi\frac{3}{2}} + x(7)e^{-j\pi\frac{7}{4}}$$

$$Using\ that\ (e^{jx} = \cos(x) + j\sin(x))$$

$$= 1 + 0.707(0.707 - j\,0.707) + 0 + 0.707(0.707 + j\,0.707) + (1 - 0) + 0.707\,(0.707 - j\,0.707) + 0 + 0.707(0.707 + j\,0.707)$$

$$= 2 + 0.5 - j0.5 + 0.5 + j0.5 + 0.5 - j0.5 + 0.5 + j0.5$$

$$= 2 + 2 = \mathbf{4}$$

$$\boldsymbol{x(2)} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi n}{2}} \quad = x(0) + x(1)e^{-j\frac{\pi}{2}} + x(2)\cdot e^{-j\pi} + x(3)e^{-j\pi\frac{3}{2}} + x(4)e^{-j2\pi} + x(5)e^{-j\pi\frac{5}{2}} + x(6)e^{-j3\pi} + x(7)e^{-j\pi\frac{7}{2}}$$

$$Using\ that\ (e^{jx} = \cos(x) + j\sin(x))$$

$$= 1 + 0.707(0 - j\,) + 0 - 0.707(0 + j\,) - (1 - 0) - 0.707\,(0 - j\,) + 0 + 0.707(0 + j\,)$$

$$= \mathbf{0}$$

$$\boldsymbol{x(3)} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi n3}{4}} \quad = x(0) + x(1)e^{-j\frac{\pi 3}{4}} + x(2)\cdot e^{-j\frac{\pi 3}{2}} + x(3)e^{-j\pi\frac{9}{4}} + x(4)e^{-j3\pi} + x(5)e^{-j\pi\frac{15}{4}} + x(6)e^{-j\pi\frac{9}{2}} + x(7)e^{-j\pi\frac{21}{4}}$$

$Using\ that\ (e^{j\mathrm{x}} = \cos(x) + j\sin(x))$

$$= 1 + 0.707(-0.707 - j\,0.707) + 0 - 0.707(0.707 - j\,0.707) - (-1 - 0) - 0.707\,(0.707 + j\,0.707) + 0 + 0.707(-0.707 + j\,0.707)$$

$$= 2 - 0.5 - j0.5 - 0.5 + j0.5 - 0.5 - j0.5 - 0.5 + j0.5$$

$$= 2 - 2 = \mathbf{0}$$

$$\boldsymbol{x(4)} = \sum_{n=0}^{7} x(n)e^{-j\pi n} \quad = x(0) + x(1)e^{-j\pi} + x(2)\cdot e^{-j2\pi} + x(3)e^{-j\pi 3} + x(4)e^{-j4\pi} + x(5)e^{-j5\pi} + x(6)e^{-j6\pi} + x(7)e^{-j7\pi}$$

$Using\ that\ (e^{j\mathrm{x}} = \cos(x) + j\sin(x))$

$$= 1 + 0.707(-1 + 0j\,) + 0 - 0.707(-1 + j\,0) - (1 - 0) - 0.707\,(-1 + 0j\,) + 0 + 0.707(-1 + 0j\,)$$

$$= \mathbf{0}$$

$$\boldsymbol{x(5)} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi n5}{4}} \quad = x(0) + x(1)e^{-j\frac{\pi 5}{4}} + x(2)\cdot e^{-j\frac{\pi 5}{2}} + x(3)e^{-j\pi\frac{15}{4}} + x(4)e^{-j5\pi} + x(5)e^{-j\pi\frac{25}{4}} + x(6)e^{-j\pi\frac{30}{4}} + x(7)e^{-j\pi\frac{35}{4}}$$

$Using\ that\ (e^{j\mathrm{x}} = \cos(x) + j\sin(x))$

$$= 1 + 0.707(-0.707 + j\,0.707) + 0 - 0.707(0.707 + j\,0.707) - (-1 + 0) - 0.707\,(0.707 - j\,0.707) + 0 + 0.707(-0.707 - j\,0.707)$$

$$= 2 - 0.5 + j0.5 - 0.5 - j0.5 - 0.5 + j0.5 - 0.5 - j0.5$$

$$= 2 - 2 = \mathbf{0}$$

$$\boldsymbol{x(6)} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi n3}{2}} \quad = x(0) + x(1)e^{-j\frac{\pi 3}{2}} + x(2)\cdot e^{-j3\pi} + x(3)e^{-j\pi\frac{9}{2}} + x(4)e^{-j6\pi} + x(5)e^{-j\pi\frac{15}{2}} + x(6)e^{-j9\pi} + x(7)e^{-j\pi\frac{21}{2}}$$

$Using\ that\ (e^{j\mathrm{x}} = \cos(x) + j\sin(x))$

$$= 1 + 0.707(0 + j\,) + 0 - 0.707(0 - j) - (1 - 0\mathrm{j}) - 0.707\,(0 + j\,) + 0 + 0.707(0 - j\,)$$

$$= \mathbf{0}$$

$$\boldsymbol{x(7)} = \sum_{n=0}^{7} x(n)e^{-\frac{j\pi n7}{4}} \quad = x(0) + x(1)e^{-j\frac{\pi 7}{4}} + x(2)\cdot e^{-j\frac{\pi 14}{4}} + x(3)e^{-j\pi\frac{21}{4}} + x(4)e^{-j7\pi} + x(5)e^{-j\pi\frac{35}{4}} + x(6)e^{-j\pi\frac{42}{4}} + x(7)e^{-j\pi\frac{49}{4}}$$

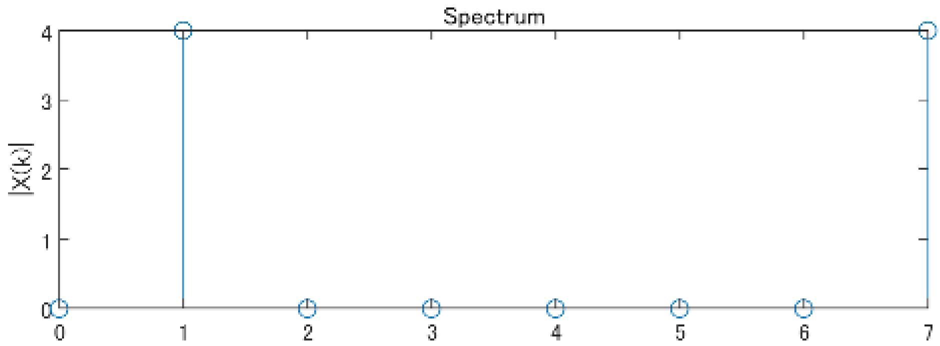
$Using\ that\ (e^{j\mathrm{x}} = \cos(x) + j\sin(x))$

$$= 1 + 0.707(0.707 + j\,0.707) + 0 - 0.707(-0.707 + j\,0.707) - (-1 - j0) - 0.707\,(-0.707 - j\,0.707) + 0 + 0.707(0.707 - j\,0.707)$$

$= 2 + 0.5 + j0.5 + 0.5 - j0.5 + 0.5 + j0.5 + 0.5 - j0.5$

$= 2 + 2 = \mathbf{4}$

$\boldsymbol{x}(\mathbf{0}) = 1,\quad \boldsymbol{x}(\mathbf{1}) = 0,\quad \boldsymbol{x}(\mathbf{2}) = 0,\quad \boldsymbol{x}(\mathbf{3}) = 0,\quad \boldsymbol{x}(\mathbf{4}) = 0,\quad \boldsymbol{x}(\mathbf{5}) = 0,\quad \boldsymbol{x}(\mathbf{6}) = 0,\quad \boldsymbol{x}(\mathbf{7}) = 1$



(ii)

(2) *Explain with equations, why the negative frequency is copied and appears as frequency components larger than the maximum frequency.*

The pulse that appears on k=7 is the one it should appear on the negative frequency as it is periodic it appears in the positive side.
The signal is not centered around zero as it should be, this is related to the matlab algorithm where it does not show negative frequency, if fftshift is used instead of fft, the negative part will appear.

(3) *Explain the relationship among sampling interval (time resolution)* T_S, *sampling frequency* f_S, *and frequency resolution* Δf.

Let us consider an array of time domain waveform samples and fft is applied to obtain frequency spectrum of waveform. The waveform is sampled in time domain N samples, let Ts be the **sampling interval** that can be given as $T_s = \frac{T_0}{N}$, where T0 is the total time interval of the waveform. Then the first sample at n = 0 would correspond to 0Ts and the second sample at n=1 would correspond to 1Ts and so on.

First sample n=0 → 0
Second sample n=1 → Ts
Third sample n=2 → 2Ts
:
:
Nth sample n=N-1 → (N-1)Ts

Ts is termed as sampling interval. The counter of sampling interval is the **sampling frequency** which equal to $f_s = \frac{1}{T_s} = \frac{N}{T_0}$

Similar to sampling interval in time domain, we have the sampling frequency in frequency domain which is termed as **frequency resolution** which equal to

$$\Delta f = \frac{B_b}{N} = \frac{f_s}{N}$$

(4) *Calulate the DFT of* $x(nT_S) = \cos(2\pi f_1 nT_S) + 0.7\cos(2\pi f_2 nT_S)$ *where* $f_1 = 1000$ *Hz,* $f_2 = 500$ *Hz,* $f_{\text{samp}} = 8000$ *Hz,* $N = 8$*. Fix the following code that includes errors. The frequency of 500 Hz is not output.*

The error in the code is N = 8, to fix it we should set N = 16. In case of summation of two sinusoidal functions, it is necessary to find each functions period (N1, N2), then we find the least common multiple (LCM). Follow the following process:

$$x(nT_S) = \cos(2\pi f_1 nT_S) + 0.7\cos(2\pi f_2 nT_S)$$

$$N_1 = \frac{2\pi}{2\pi f_1 T_s} k = \frac{f_s}{f_1} k = \frac{8000}{1000} = 8k, \qquad k = 1,2,3,..$$

$$\boldsymbol{N_1 = 8}$$

$$N_1 = \frac{2\pi}{2\pi f_2 T_s} k = \frac{f_s}{f_2} k = \frac{8000}{500} = 16k, \qquad k = 1,2,3,..$$

$$\boldsymbol{N_1 = 16}$$

The LCM (16,8) = 16