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3.1 Drive the eigenvalue equations from Maxwell's equations :-

$$\nabla \times \tilde{E} = -\mu_0 \frac{\partial \tilde{H}}{\partial t} \quad — ①$$

$$\nabla \times \tilde{H} = \varepsilon_0 n^2 \frac{\partial \tilde{E}}{\partial t} \quad — ②$$

$$\underbrace{\qquad\qquad\qquad}_{\text{Maxwell equation in time domain}}$$

$$\nabla \times \tilde{E} = j\omega \mu_0 \tilde{H} \quad — ③$$

$$\nabla \times \tilde{H} = -j\varepsilon_0 n^2 \tilde{E} \quad — ④$$

$$\underbrace{\qquad\qquad\qquad}_{\text{Maxwell equations in frequency domain}}$$

we are dealing with plane-wave propagation which is in form of :-

$$\tilde{E} = E(x,y)\, e^{j(\omega t - \beta z)} \quad — ⑤$$

$$\tilde{H} = H(x,y)\, e^{j(\omega t - \beta z)} \quad — ⑥$$

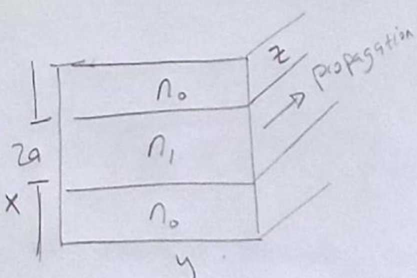
$$\nabla \times \tilde{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ E_x & E_y & E_z \end{vmatrix} \xrightarrow{\frac{d}{dz} \to j\beta} = \hat{a}_x\left(\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\cancel{\partial y}}\right)$$

$$\nabla \times \tilde{E} = (-j\beta E_y)\hat{a}_x - \left(j\beta E_x - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \frac{\partial E_y}{\partial x}\hat{a}_z \quad — ⑦$$

The same for $\nabla \times \hat{H} = (-j\beta H_y)\hat{a}_x - \left(j\beta H_x - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \frac{\partial H_y}{\partial x}\hat{a}_z$ — ⑧

Substituting ⑦, ⑧ into ③, ④

$$\left.\begin{aligned} -j\beta E_y &= j\omega\mu_0 H_x \\ j\beta E_x - \frac{\partial E_z}{\partial x} &= j\omega\mu_0 H_y \\ \frac{\partial E_y}{\partial x} &= j\omega\mu_0 H_z \end{aligned}\right\} ⑨$$



$$\left.\begin{aligned} -j\beta H_y &= -j\omega\varepsilon_0 n^2 E_x \\ j\beta H_x - \frac{\partial H_z}{\partial x} &= -j\omega\varepsilon_0 n^2 E_y \\ \frac{\partial H_y}{\partial x} &= -j\omega\varepsilon_0 n^2 E_z \end{aligned}\right\} ⑩$$

In the shown slab waveguide E, H don't have y-axis dependency so $\frac{\partial E}{\partial y} = 0$, $\frac{\partial H}{\partial y} = 0$

By extracting H_x and H_z equations in (9) and substituting it in their places in (10) we can obtain the TE mode combinations The same as following:-

$$\frac{\partial^2 E_y}{\partial x^2} + (k^2 n^2 - \beta^2) E_y = 0 \quad \text{— (11)}$$

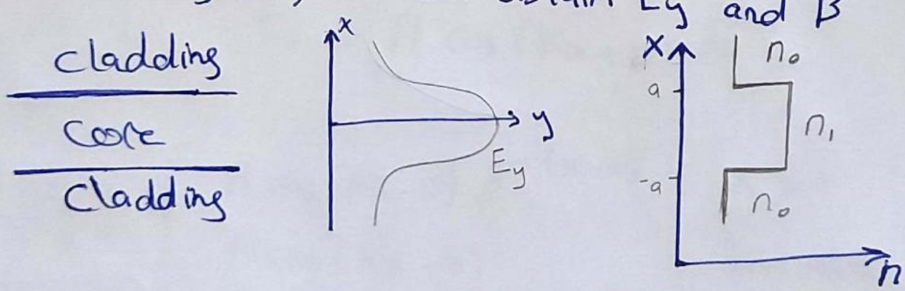
$$H_x = -\frac{\beta}{\omega \mu_0} E_y \quad \text{— (12)}$$

$$H_z = \frac{J}{\omega \mu_0} \frac{\partial E_y}{\partial x} \quad \text{— (13)}$$

TE Mode

$$E_x = E_z = H_y = 0 \quad , \quad K = \omega \sqrt{\varepsilon_0 \mu_0} = \frac{\omega}{c}$$

By solving equation (11) according to the waveguide we are given (slab waveguide) we can obtain E_y and β



for $x > a$

$$\frac{d^2 E_y}{\partial x^2} - \sigma^2 E_y = 0 \qquad \sigma = \sqrt{\beta^2 - k_0^2 n_0^2} \qquad (\sigma > 0)$$

$$E_y = C_1 e^{-\sigma x} + C_2 e^{\sigma x} \quad , \quad C_2 = 0 \text{ caus of boundary condition } (E_y = 0) \text{ for } x \to \infty$$

$$\boxed{E_y = C_1 e^{-\sigma x}}$$

for $x < -a$:. the same as before

$$E_y = \underbrace{C_3}_{C_3 = 0 \;\; E_y = 0 \;\; x \to -\infty} e^{-\sigma x} + C_4 e^{\sigma x} \Rightarrow \boxed{E_y = C_4 e^{\sigma x}}$$

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For $-a \leq x \leq a$:-

$$\frac{\partial^2 E_y}{\partial x^2} + K^2 E_y = 0 \quad — \quad K = \sqrt{k^2 n_1^2 - \beta^2}$$

$$E_y = C_5 \sin Kx + C_6 \cos Kx$$

$$\boxed{E_y = A \cos(KX - \phi)} \qquad A = \sqrt{C_5^2 + C_6^2}$$

$$\phi = -\sin^{-1} \frac{C_6}{\sqrt{C_5^2 + C_6^2}}$$

For $X = a$ $\rightarrow$ $C_1 e^{-\sigma a} = A \cos(Ka - \phi)$

$$C_1 = A \cos(Ka - \phi) e^{\sigma a}$$

For $X = -a$ $\rightarrow$ $C_4 e^{\sigma a} = A \cos(-Ka - \phi)$

$$C_4 = A \cos(Ka + \phi) e^{-\sigma a}$$

So

$$E_y = \begin{cases} A \cos(Ka - \phi) e^{-\sigma(x-a)} & x > a \\ A \cos(Kx - \phi) & |x| \leq a \\ A \cos(Ka + \phi) e^{\sigma(x+a)} & x < -a \end{cases}$$

where $\left.\begin{array}{l} K = \sqrt{k^2 n_1^2 - \beta} \\ \sigma = \sqrt{\beta^2 - k^2 n_0^2} \end{array}\right\}$ these are wave numbers

* H_z should be continous at x axis so $\frac{\partial E_y}{\partial x}$ should be continous...

$$\frac{\partial E_y}{\partial x} = \begin{cases} -\sigma A \cos(Ka-\phi)\, e^{-\sigma(x-a)} & x > a \\ -KA \sin(Kx-\phi) & |x| \leq a \\ \sigma A \cos(Ka+\phi)\, e^{\sigma(x+a)} & |x| < -a \end{cases}$$

when $x = a \rightarrow \sigma \cancel{A} \cos(Ka-\phi) = K\cancel{A} \sin(Ka-\phi)$ — (14)

$x = -a \rightarrow K\cancel{A} \sin(Ka+\phi) = \sigma \cancel{A} \cos(Ka+\phi)$ — (15)

(14) $\frac{\sigma}{K} = \tan(Ka-\phi)$

(15) $\frac{\sigma}{K} = \tan(Ka+\phi)$

$\left.\begin{matrix} u = Ka \\ w = \sigma a \end{matrix}\right\} \rightarrow \tan(u-\phi) = \frac{w}{u}$, $\rightarrow \tan(u+\phi) = \frac{w}{u}$

$$u + \phi = \tan^{-1}\left(\frac{w}{u}\right) + m\pi \qquad (m = 1, 2, 3 \ldots)$$

$$\tan\left(\tan^{-1}\left(\frac{w}{u}\right) + m\pi - 2\phi\right) = \tan(u-\phi) = \frac{w}{u}$$

$$\tan(2\phi) = 0$$

So

$$\boxed{\begin{aligned} \phi &= \frac{m\pi}{2} \\ u &= \frac{m\pi}{2} + \tan^{-1}\left(\frac{w}{u}\right) \end{aligned}}$$

Eigenvalue equations

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(3.2) Using k, σ, u, w equations, we can obtain

$$u^2 + w^2 = k^2 a^2 (n_1^2 + n_0^2) = v^2$$ (normalized frequency)

to consider dispersion:-

$$n_e = \frac{\beta}{k}$$ effective index $n_0 \leqslant n_e \leqslant n_1$

new normalized parameter ~~then~~ cand be intriduced

$$b = \frac{n_e^2 - n_0^2}{n_1^2 - n_0^2}$$ normalized propagation constant.

the eigenvalue equation can be written using v and b as following:-

$$\boxed{v\sqrt{1-b} = \frac{m\pi}{2} + \tan^{-1}\sqrt{\frac{b}{1-b}}}$$

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```
clear all; close all; clc;

%Draw dispersion curves for the TE mode
v = 0:0.01:10; % Normalized frequency grid
b0 = [0 1]; % Initial interval for

for m = 0:6 % Mode number
    for n = 1:length(v);
        if v(n) > m*pi/2
        Fun = @(x) v(n).*sqrt(1-x) - m*pi/2 - atan(sqrt(x/(1-x)));  % Find root of
f(m,v,b)=0
        b(m+1,n) = fzero(Fun,b0);   % Find b for f(m,v,b)=0
        else
        b(m+1,n) = NaN;
        end
    end
end

% Plot dispersion curves
figure('position', [200 200 600 350]);
plot(v,b)
hold on; grid on; box on;
ylim([0 1])
xlim([0 10])
xticks([0:1:10])
yticks([0:0.1:1])
xlabel('Normalized frequency (v)')
ylabel('Normalized propagation constant (b)')
legend('m=0','m=1','m=2',...
'm=3','m=4','m=5','m=6','Location','bestoutside')
title('Dispersion curve')
set(gca,'FontSize',12,'FontName','Times')


%Draw representative electric field distributions
a = 25; % Core size, micrometer
A = 1; % Amplitude
n1 = 1.444; % Core refractive index
n0 = 1.440; % Cladding refractive index
lambda = 1.55; % Wavelength, micrometer
k0 = 2*pi/lambda; % Wavenumber, rad/micrometer
x = -50:0.1:50; % Spatial grid, x, micrometer

for m = 0:1:2; % Mode number
    vn = k0*a*sqrt(n1^2 - n0^2);     % The normalized frequency v is determined.
    Fun = @(x) vn.*sqrt(1-x) - m*pi/2 - atan(sqrt(x/(1-x)));    %Dispersion equation
    bn = fzero(Fun,b0);   % Find the normalized propagation constant b form the
dispersion eq.
    ne = sqrt(bn*(n1^2 - n0^2) + n0^2);
    beta = ne*k0;
    K = sqrt(k0^2 * n1^2 - beta^2);  % The parameters κ, σ, and ϕ are determined.
    sigma = sqrt(beta^2 - k0^2 * n0^2);
    phi = m*pi/2;
% Calculate electric field distribution Ey(x)
    for n = 1:length(x)
```

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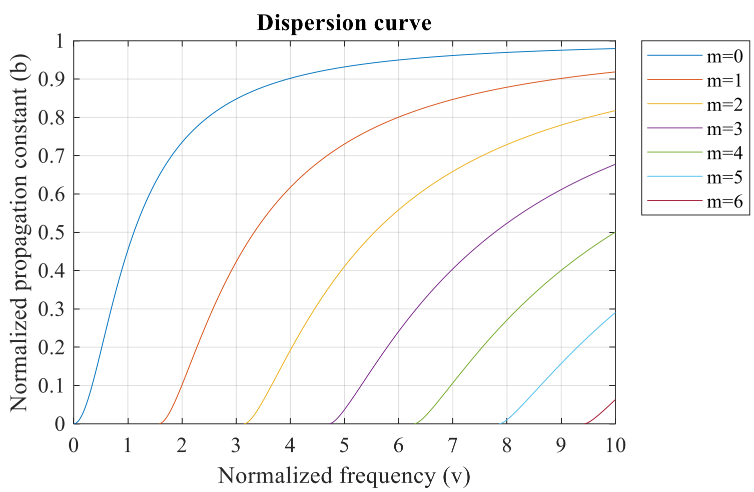
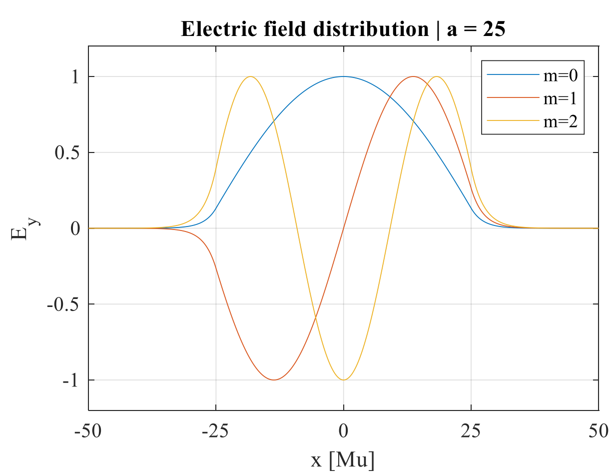
        if x(n) > a
            Ey(m+1,n) = A*cos(K*a - phi) * exp(-sigma*(x(n)-a));
        elseif x(n) < -a
            Ey(m+1,n) = A*cos(K*a + phi) * exp(sigma*(x(n)+a));
        else
            Ey(m+1,n) = A*cos(K*x(n) - phi);
        end
    end
end

% Plot electric field distribution
figure('position', [800 200 500 350]);
plot(x,Ey)
hold on; grid on; box on;
ylim([-1.2 1.2])
xticks([-50:25:50])
xlabel('x [Mu]')
ylabel('E_y')
legend('m=0','m=1','m=2')
title('Electric field distribution')
set(gca,'FontSize',12,'FontName','Times')
```





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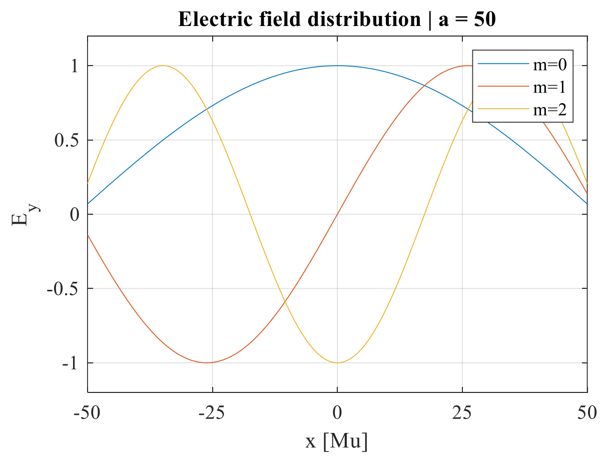

In this project, dispersion analysis on slab waveguide has been done for different modes from 1 to 6, the first figure shows the dispersion curve, where it can be seen that the propagation constant shows better performance for the fundamental mode m = 0.

Then the TE mode curves are shown, for the fundamental mode m = 0, it can be seen that there is just one peak over the core size, when m = 2 there are two peaks and so on, it is noticed that the electric field distribution modes expand over the thickness of the core when the core size increases, the change of the core size effects the intensity of the electric field.