Analyzing The Dynamics Of A Network Of Sparsely Connected Excitatory And Inhibitory Neurons In The Asynchronous And Irregular State

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Theory

Introduction

After having studied models of individual neurons in the first half of the course, we now turn towards modelling networks of these neurons. Ofcourse in doing so, one has to impose some model on the individual neurons of the network. In this project, we will consider a network of Leaky Integrate and Fire neurons since they are relatively computationally inexpensive while also encapsulating the most interesting features of real neurons namely spiking and refractoriness.

The individual neurons in the system obey the following dynamical equations:

$$\tau \dot{V}_i(t) = -V_i(t) + RI_i(t)$$

This is Kirchoff's voltage law applied to the i^{th} LIF neuron. The terms on the right dictate how the voltage changes (scaled by the time constant τ on the left). The first term is the leak term which serves to decreases the voltage in the absence of any driving current. The second term on the other hand, encapsulates the effect one the voltage V_i of all of the currents I_i flowing into neuron i.

Ofcourse these currents can be either local (i.e from neurons within the network) or external (i.e currents from neurons outside of the system). Either way, the currents flows into the neuron through the synapses. The strength of these snyaptic connections is quantified by the PSP (post synaptic potential) amplitude matrix: J_{ij} (we will frequently also call the entries of this matrix the "efficacy" of a particular synapse). In fact, the equation governing the $RI_i(t)$ term is:

$$RI_i(t) = \tau \sum_j J_{ij} \sum_k \delta(t - (t_j^k + D))$$

Where the first sum is over all of the neurons j that might be connected to our neuron i. The second sum is over the k spikes coming from neuron j. As mentioned before J_{ij} quantifies the effect on neuron i of any spikes coming from the synapse connecting it to neuron j. Note also the D in the argument of the delta function. This is the delay in the transmission of the spike to neuron i. Clearly D is a constant. All synapses in the model therefore introduce the same amount of delay D. t_j^k is the time at which the k^{th} spike was emitted from neuron j.

What does "sparsely connected" mean?

This is a good point in the report to explain what the word "sparsely connected" means. In terms of the neuron picture what this means is that each neuron in the system is synaptically connected to only a small fraction of the total neurons in the system. This naturally translates to the following mathematical statement: **The PSP amplitude matrix J_{ij} has most of its entries equal to zero**.

In addition to this imposition on the matrix J_{ij}, we also choose the entries stochastically.

List of relevant parameters:

Before delving into the mathematics, it is helpful to know the meanings of the various variables used in the equations that will follow. Here is a comprehensive list that goes through all of the parameters and their significance (other than the ones that have been discussed in the previous sections):

1. θ is the threshold voltage of every LIF neuron in the model.
2. ν_{thr} is the threshold frequency. It is the minimum frequency required for a neuron to reach threshold in the absence of any feedback.
3. ν_{ext} is the frequency with which the external neurons fire.
4. C_{ext}, C_E and C_I are the numbers of synaptic connections that each neuron makes with the external neurons, the excitatory neurons and the inhibitory neurons respectively.
5. The strengths of the various connections between the two different neurons are quantified through the parameters: J_{EE}, J_{IE}, J_{EI} and J_{II}: the synaptic coupling strengths from excitatory to excitatory, excitatory to inhibitory, inhibitory to excitatory and inhibitory to inhibitory respectively.
6. N_E and N_I are the numbers of excitatory and inhibitory neurons respectively.

7. In what follows the inhibtory population N_I will always be related to the excitatory population N_E by a factor $\gamma < 1$, such that: $N_I = \gamma N_E$
8. Similarly the number of excitatory connections will also always be some fraction ϵ of the total number of excitatory neurons: $C_E = \epsilon N_E$
9. Aside from these, any quantities with subscripts E, I and ext will always correspond to the relevant quantity related to the excitatory, inhibtory and external populations respectively.

Solving the system analytically [2]

To solve for the model we have just written down, we consider a regime where a large number of inputs are received by each neuron per integration time. However, we also make the assumption that the individual inputs make contributions much smaller than the threshold voltage θ that is:

$$J << \theta$$

Under this assumption, it is quite intuitive to postulate that the input current takes the following form:

$$RI_i(t) = \mu(t) + \sigma\sqrt{\tau}\eta_i$$

Where the first term on the right corresponds to some mean value of the current, while the second term corresponds to "fluctuations" in the input current. The factor σ decides the "extent" of these deviations, whereas η_i is gaussian white noise with an autocorrelation function:

$$< \eta_i(t)\eta_i(t') >= \delta(t - t')$$

The autocorrelation being a delta function basically corresponds to the fact that the gaussian white noise at different times is not correlated at all.

The Fokker-Planck equation and stationary states

We now make the assumption that the correlation between the fluctuating part of the inputs of different neurons is sufficiently non-correlated that the system can be recast in terms of a distribution of "Depolorizations" of neurons:

$$P(V, t)$$

We note that this distribution simply gives us the probability of a random neuron having a membrane potential V at some time t. The time dependence will ofcourse be non-trivial (indeed that is what we are solving for in the first place!). It turns out that our required conditions for the network can be transformed into the Fokker-Planck equation:

$$\tau\frac{\partial P(V,t)}{\partial t} = \frac{\sigma^2(t)}{2}\frac{\partial^2 P(V,t)}{\partial V^2} + \frac{\partial}{\partial V}[(V - \mu(t))P(V,t)]$$

Where the first term on the right is the diffusive term (and corresponds to the synaptic fluctuations), whereas the second term is a driving or "drift" term corresponding to the average part of our synaptic input μ.

After applying the necessary boundary counditions to ensure that our solution function is continuous and acts as a distribution (i.e integrates to 1) and goes sufficiently quickly to zero at $-\infty$ we find that stationary states of the system (i.e states where the distribution settles to some time indepedent form) are of the following form:

$$P_o = 2\frac{\nu_o\tau}{\sigma_o}\exp\Big(-\frac{(V-\mu_o)^2}{\sigma_o^2}\Big) \times \int_{\frac{V-\mu_o}{\sigma_o}}^{\frac{\theta-\mu_o}{\sigma_o}} \Theta(u - V_r)e^{u^2}du$$

Where Θ is the Heaviside step function. Note that all of the quantities with an o in their subscripts correspond to the stationary constant value of the corresponding quantity e.g ν_o is the averarge firing rate of the neurons in the stationary state. In fact, the exact expression for ν_o is:

$$\frac{1}{\nu_o} = \tau_{rp} + 2\tau\sqrt{\pi}\int_{\frac{V-\mu_o}{\sigma_o}}^{\frac{\theta-\mu_o}{\sigma_o}} e^{u^2}(1 + \text{erf}(u))du$$

where:

$$\mu_o = C_E J\tau[\nu_{ext} + \nu_o(1 - g\gamma)]$$

$$\sigma_o^2 = C_E J^2\tau[\nu_{ext} + \nu_o(1 + g^2\gamma)]$$

Brief Aside - The relevant phase diagram from the Brunel paper:

Before we actually choose the parameters for our simulation we need to understand twhy we are choosing those particular parameters in the first place. Brunel (2000) numerically solved the aforementioned equations for different values of the parameters and plotted the phase space behaviour of the system. These phase diagrams are essentially plots between $\frac{\nu_{ext}}{\nu_{thr}}$ (the external frequency expressed in units of the threshold frequency) and g (the relative strength of the inhibitory neurons).

The phase diagrams demarkate different regimes for the system. The pertinent phase diagram is shown in Figure 1. What is important to note here is that the point marked (c) lies in the region labelled AI. In this region, the system has aysnchronous irregular behaviour. Therefore, our choice of parameters simply ensures that we are considering a point in the phase diagram that lies in this region.

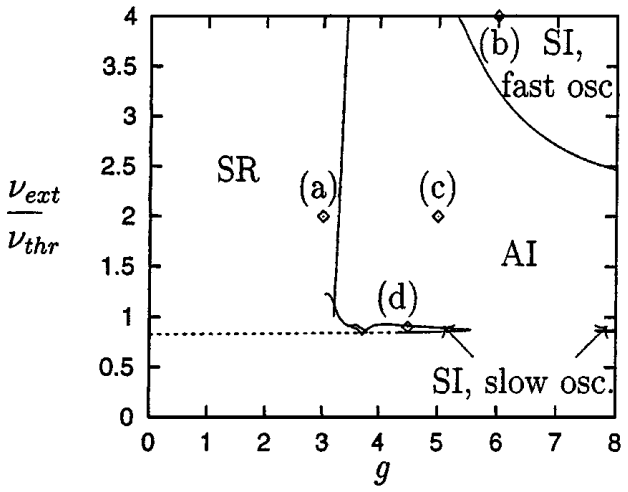

Figure 1: Phase diagram from the Brunel paper for the following parameters: $C_E = 1000, C_I = 250, J = 0.1mV, D = 1.5ms$. The region we are concerned with is the one labeled AI (asynchronous irregular).

Simulations:

Simulating the asynchronous and irregular state in Python:

We simulated a network of LIF neurons in python using primarily the brian2 library. This library is catered towards simulating both individual neurons and networks of neurons. In running the simulation for the first time we used the following parameters:

- $N_E = 10000$
- $\gamma = 0.25$
- $\epsilon = 0.1$
- $g = 5$
- $\tau = 20$ ms
- $\theta = 20$ mV
- $V_r = 10$ mV
- $\tau_{rp} = 2$ ms

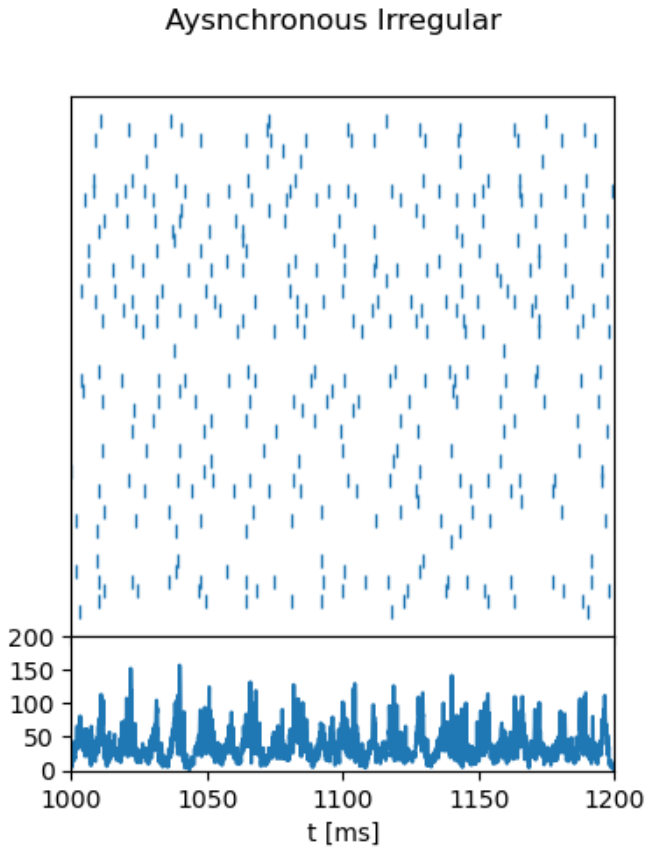

Figure 2: Raster plots of the spikes of 50 excitatory neurons followed by a plot of the average instantaneous firing rate against time (bottom) of the system in the asynchronous irregular state. Note that the individual neurons seem to behave as predicted by the theory but the global activity is still somewhat oscillatory

We ran the simulation for 1200 ms (1.2 seconds) and plotted the spikes of 50 individual neurons (recorded using the SpikeMonitor class from Brian2) as a raster plot. Each row in the attached figure corresponds to a neuron with a particular index. We note that the behaviour of the individual neurons is exactly as predicted by the theory, there is no correlation among the spikes of separate neurons. The lowest row in the figure corresponds to the global activity (the average instantaneous firing rate at different points in time). Note that this behaviour is certainly not as predicted by the theory. AI states have stationary global activity, however as can be seen in the figure the global activity seems to be somewhat oscillatory. The next section discusses this deviation from theory. and how one can minimize such effects.

Removing the oscillatory behaviour by accounting for finite-size effects [1]

It turns out that the reason why the global activity seems to have some remnants of oscillatory behaviour is because the sharp transitions predicted by the theory (from oscillatory to stationary global activity in our case) only occur in the limit

of $N \to \infty$. In our case, since the total number of neurons is finite, the transitions are "smoothed out" resulting in there being some oscillations even in the AI regime (as is evident from the bottom plot of Figure 2). These are known as "finite size effects".

If one considers the input to a single neuron in the system, then the fluctuations in said input can be accounted for by two separate random processes. One of these is the spike generation process $S(t)$ of the entire system and the other is the presence or absence of a synaptic connection with a neuron $\rho_i(t)$ (which is 1 with a probability of $\epsilon = \frac{C}{N}$ and 0 otherwise) that generated a spike.

Then the input to the i^{th} neuron can be written as:

$$RI_i(t) = -J\tau\rho_i(t)S(t-D)$$

where both $\rho_i(t)$ and $S(t)$ are random processes that can be decomposed into their mean and fluctation as:

$$\rho_i(t) = \frac{C}{N} + \delta\rho_i(t)$$

$$S(t) = N\nu(t) + \delta S(t)$$

Then we can rewrite the entire input as the following sum:

$$RI_i(t) = \mu(t) - J\tau N\nu(t)\delta\rho_i(t) - J\tau\frac{C}{N}\delta S(t)$$

Thus the synaptic input for the i^{th} neuron consists of a mean term $\mu(t)$ along with two fluctuating terms. It turns out that the last of these terms depends on $\frac{C}{N}$ and goes to zero in the limit of large N or instead as:

$$\epsilon = \frac{C}{N} \to \infty$$

The second term on the other hand is a Poisson process, which can be approximated as a Gaussian process in the limit that the network size becomes quite large, i.e we make a continuous approximation. This gives us the following expression for the mean synpatic input:

$$CJ\tau\nu(t) + J\sqrt{\epsilon C\nu_o\tau}\sqrt{\tau}\xi(t) + \mu_{ext}$$

Where the term ξ is a Gaussian white noise which satisfies:

$$\langle\xi(t)\rangle = 0 \qquad \langle\xi(t)\xi(t')\rangle = \delta(t-t')$$

If we substitute the final expression into the Fokker Planck equation for our system, the result is the appearance of a drift term. Brunel and Hakim (1998) [3] showed that the drift term has the following effect: **It results in the appearance of a strongly damped oscillatory component in the AI regime, which disappears as $\epsilon \to \infty$.**

Therefore, in order to account for these finite sized effects, we ran the simulation again for successively smaller values of ϵ. As expected, the oscillatory component in the global activity began to disappear!

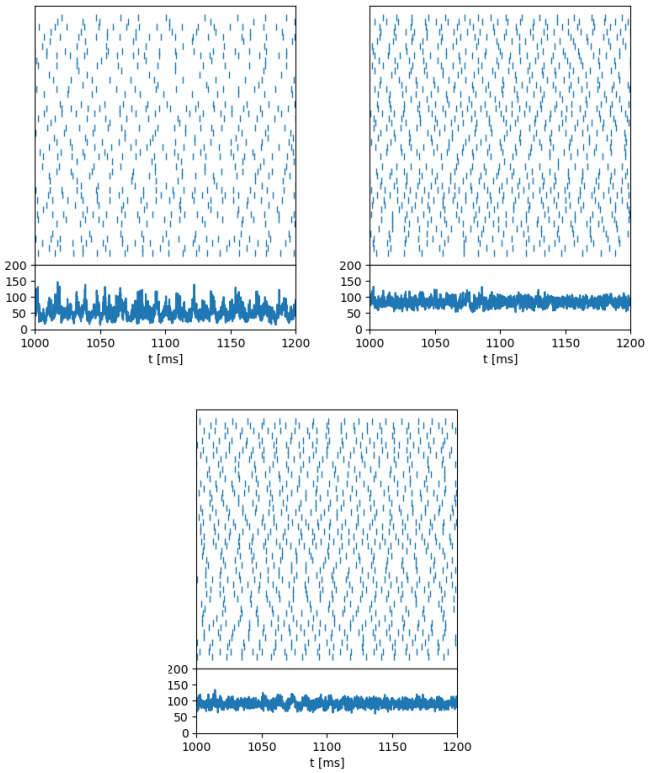

Figure 3: Plots for the AI regime following the same scheme as described in Figure 2. From top left to bottom, the ϵ value changes from 0.05 to 0.01 to 0.005. As is evident from the bottom row, the oscillatory component begins to disappear and stationarity is somewhat restored. Note also that the raster plots indicate that the firing is still irregular, meaning we are indeed in the AI regime.

Justifying the decrease in ν_o when the mean voltage is sufficiently smaller than the threshold:

Clearly the firing frequency seems to be decreasing upon increasing J. From the neuron system point of view, one can attribute this decrease to an increase in the strength of the inhibitory connections. From a purely mathematical stand point, we know from Brunel's paper that in the regime where

$$(\theta - \mu_o) >> \sigma_o$$

the expression for the population mean firing rate can be approximated as:

$$\nu_o\tau \approx \frac{(\theta-\mu_o)}{\sigma_o\sqrt{\pi}}\exp\Big(-\frac{(\theta-\mu_o)^2}{\sigma_o^2}\Big)$$

where, as mentioned before:

$$\mu_o = C_E J\tau[\nu_{ext} + \nu_o(1-g\gamma)]$$

$$\sigma_o^2 = C_E J^2\tau[\nu_{ext} + \nu_o(1+g^2\gamma)]$$

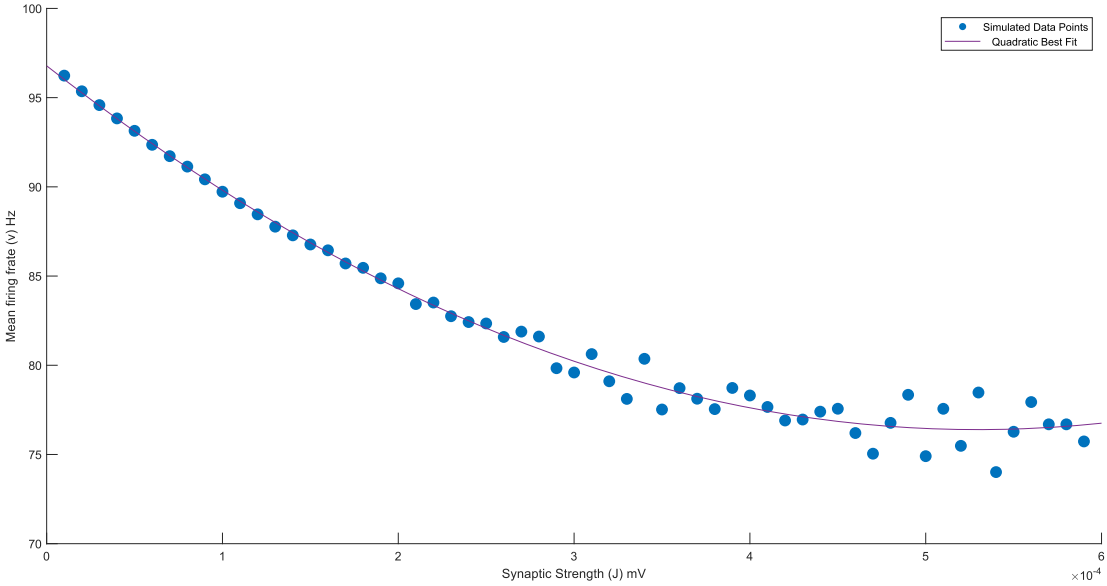

Figure 4: A plot of the average population mean firing rate ν_o against the synaptic strength J. The scatterplot points correspond to actual evaluations using the python simulations. The purple line is a quadratic best fit curve generated using MATLAB. Evidently, the general trend seems to be that the population mean firing rate decreases as the synaptic strength is increased. Note also that as we move towards higher values of J, fluctuations begin to appear in the data points, this corresponds to the theory "breaking down" due to the appearance of correlations.

And it is clear that σ_o goes as the square of the synaptic strength J whereas the relation of μ_o with J is linear. Therefore on increasing J, the σ_o term dominates causing an overall decrease in ν_o. The natural question to ask his, can we make this approximation? It turns out that we can:

At the start of the plot we preformed explicit calculations to find (J = 0.01 mV and $\nu_o \approx 95$):

$$(\theta - \mu_o) \approx 20 - (-197.5) = 217.5 \text{ mV}$$

whereas:

$$\sigma_o \approx \sqrt{69.275} \approx 8.323 \text{ mV}$$

Which indeed satisfies the requisite condition ($(\theta - \mu_o) >> \sigma_o$). The decrease technically isn't quadratic, it is hyperbolic and is modulated by a decaying exponential. But the quadratic fit is a good enough approximation for our interval of interest.

However for significantly large values of J (such as 0.6 mV), this approximation breaks down, which explains the fluctuations in the scatterplot towards the end.

Coefficient of variation

An exactly identical set of simulations was performed for calculating the dependence of the coefficient of variation σ of the firing rate on time . The plot obtained is shown in Figure 5.

Evidently the coefficient of variation of the interspike interval increases. One way of justifying this is the following: For weaker synaptic connections, the inhibition does not play a significant role, consequently the spiking frequency is mostly controlled by the refractory period (τ_{rp}) due to saturation. However as J increases, the dominant effect controlling the spiking frequency is the inhibition arising from the spikes generated by the inhibitory neurons.

Agreement with theory:

Let us explicitly see if the population mean firing rate computed through the simulations agrees with the theory. Upon running the simulation for the parameters in the Brunel paper ($\epsilon = 0.1$, $J = 0.1$ mV and $N = 10,000$ [1]) we see that the mean firing rates are:

Simulation:	38.4752 Hz
Theory:	38 Hz

which shows that there is certainly a great deal of agreement between the theory and the simulation (for large enough N). One can assume that as the limit $N \to \infty$ is taken, the error associated with finite sized effects completely disappears and one obtains complete agreement between theory and simulation.

[1] The Brian2 simulator crashed for any value of N larger than 10,000

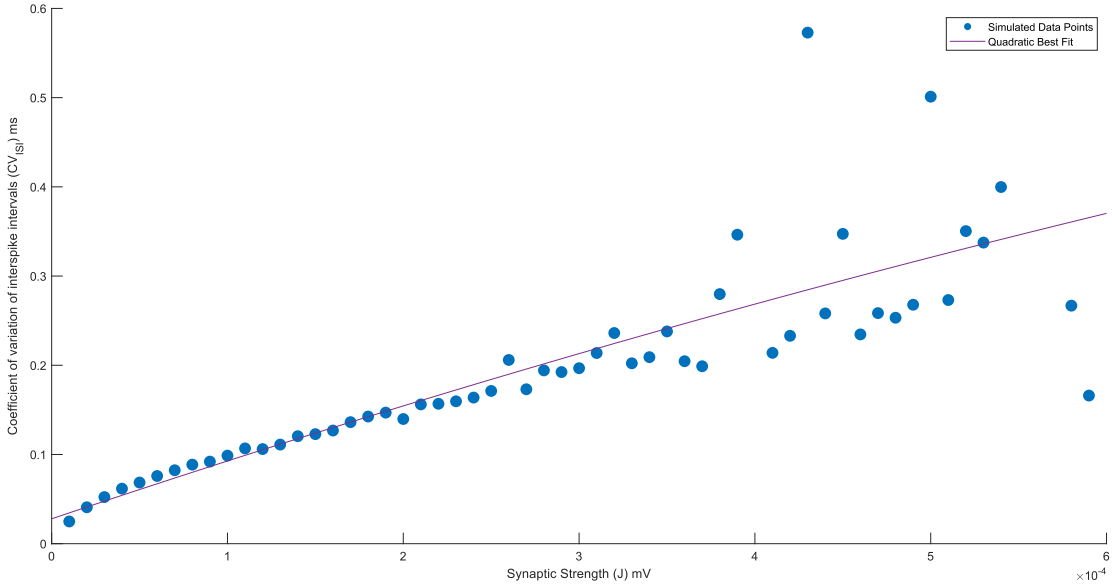

Figure 5: A plot of the coefficient of variation of the inter spike intervals (averaged over the first 50 excitatory neurons) against the synaptic strength J. Note again that the plot is quite well behaved for smaller values of J and begins to misbehave as J increases. This too can be attributed to the appearance of correlations in the inputs of different neurons.

Grading:

I have graded myself as follows:

References

[1] Github page with an ipynb file containing the code i used for the coefficient of variation (file name: Finalprojectcode). `https://github.com/malaikkabir/ComputationalNeuroscience`. Accessed: 2023-05-16.

[2] Nicolas Brunel. In *Dynamics of Sparsely Connected Networks of Excitatory and Inhibitory Spiking Neurons*, 2000.

[3] V. Hakim N. Brunel. In *Fast global oscillations in networks of integrate-and-fire neurons with low firing rates*, 1998.