

Decoherence: How Schrodinger's cat becomes an ordinary cat

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Introduction

Over the past century we have gained considerable insight into quantum mechanics, discovering bizarre phenomena such as superposition and entanglement that seem to be at odds with our knowledge of how the macroscopic world around us works. And yet, the legitimacy of said phenomena has stood the test of time. For all we know, quantum mechanics **is** the most complete description of the microscopic world we know.

What then of the macroscopic world? After all it arises - brick by microscopic brick - from the realm where quantum mechanics reigns supreme. And yet, the world around us seems to be manifestly classical and classical objects don't seem to possess the aforementioned "quantum" properties. Macroscopic objects, for example, never seem to communicate information about their state instantaneously to other objects across the galaxy, they appear localized in space, with no need to resort to the so-called wavefunction approach and the cat is never neither alive nor dead. Decoherence serves to answer these long-standing questions.

Decoherence (or environment-induced decoherence) is a process through which information about a quantum system leaks into the environment it is coupled with, essentially delocalizing its "coherence" (a term used interchangeably for the "quantum nature" of a system), to the level of the entire environment. This definition will be made more rigorous as we move along. For now, what is important to note is that decoherence generalizes the concept of measurement. Just as through measurement we collapse a wavefunction and gain information about a particular observable of a system, so through decoherence the "environment" gains information about a particular observable.

The problem

The problem that we are trying to solve for is broadly known as the measurement problem (or the problem of quantum-to-classical transition) and it can be divided into the following three sub-problems:

- **The problem of the preferred basis:** This asks the question. Why is it that nature singles out the eigenstates of certain observables (such as position and momentum) as the preferred "robust" states that are classically observed. In more mathematical terms one might say that a book placed on a table is classically seen to be at a particular position, which in quantum mechanical terms translates to:

$$|\psi\rangle_{\text{book}} = |x\rangle$$

But ofcourse a position basis vector is only the eigenvector of the position operator. Linear algebra allows one to decompose such a state into a linear superposition of the eigenvectors of some other observable (lets call it \mathcal{O}) with eigenvectors $|o\rangle_i$ such that:

$$|\psi\rangle_{\text{book}} = |x\rangle = \sum_{i=1}^{i_{\max}} c_i |o\rangle_i$$

The natural question then is: "Why does nature prefer to suppress superpositions of some basis ($|x\rangle$ in our case) but not others (such as the $|o\rangle$ basis)". This precisely is the problem of the preferred basis.

- **The problem of non-observability of interference:** Ever since the dawn of quantum mechanics, the interference experiment has been widely studied. The interference experiment makes manifest the fact that quantum mechanical superposition states are not "ignorance-interpretable" mixtures of their component states. That is a state such as:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$$

is not just an equal mixture of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ with the probabilities assigned due to our lack of knowledge of which is which. In order to demonstrate this we imagine an interference experiment setup as shown in Figure 1. Let us imagine that ψ_i corresponds to the arrival of an electron from slit i for $i = 1, 2$. If a beam of electrons is incident on a screen as shown in the left most part of Figure 1, by the time the electrons reach the recording screen (somewhere to the right of the first one), we do not have any information about which electron came from which slit (assuming we are carrying out no measurements). The density observed on the screen, therefore, must arise from the aforementioned superposition state:

$$\begin{aligned}\rho(x) &= \frac{1}{2}|\psi_1(x) + \psi_2(x)|^2 \\ &= \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \text{Re}\{\psi_1(x)\psi_2(x)^*\}\end{aligned}$$

It is precisely the cross term at the end that contributes towards the interference pattern. This is because the position basis wavefunction itself simply gives the "probability amplitude" of the component states not the probability itself. Squaring the amplitude gives the actual probability and we know that the square of a sum is not the sum of the squares. The cross terms resulting from this non-linearity distinguish the superposition state from the sum of the individual states.

The interference experiment has been conducted on numerous microscopic objects such as electrons, photons and even molecules of carbon (so-called Bucky balls). However, interference is yet to be observed on macroscopic objects (say tennis balls). The problem of non-observability of interference asks exactly this question. Why isn't interference observed in macroscopic objects. As we will see, the answer is actually closely tied to the middle frame of Figure 1. Knowing where the particles came from collapses their superpositions to one of the states $|\psi_i\rangle$ resulting in the disappearance of the interference pattern. It turns out that this which-path information leaks away into the environment very quickly for macroscopic objects. We will discuss this more thoroughly as we go.

- **The problem of outcomes:** This question is one whose resolution remains up to debate. The problem of outcomes asks "Why do measurements have outcomes at all, and what selects a particular outcome among the different possibilities described by the quantum probability distribution?" [1]. It turns out that this problem is inextricably linked to the particular interpretation of quantum mechanics one chooses to believe in. In this report we will mostly discuss the answers decoherence provides to the first two questions.

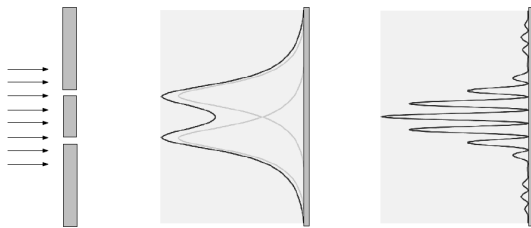


Figure 1: The interference experiment: The middle case arises when we have "which-path" information, i.e the particles have been collapsed into one of the states $|\psi_i\rangle$ ($i = 1, 2$). The right most case arises when there is no which-path information and the particles are in superposition states of $|\psi_i\rangle$.

Density Matrices

In quantum mechanics, all of the information that can be known about the system is encapsulated in its wavefunction. The dynamical evolution then follows from the time dependent

Schrodinger equation. The most one can talk about is "probabilities" of certain outcomes, determinism is out of the window. The classical analog of this is knowing the position and momentum of a particular system. All else follows directly from Newton's equations.

Wavefunctions in quantum mechanics correspond to what are known as "pure states". An example of a pure state would be our aforementioned superposition wavefunction:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$$

However, in most practical situations (such as in the lab), one usually does not know with complete certainty what state a particular system has been prepared in. This ignorance could be due to any number of reasons such as an imperfect state preparation mechanism etc. What matters is that there is now an added probabilistic element in the state. This, however, is not inherent to the system. It represents our lack of knowledge about the system.

Such a situation requires the use of a mathematical object known as a density operator $\hat{\rho}$ or equivalently a density **matrix** ρ (if one talks about the matrix representation of the operator).

Pure state density matrices

A pure state can also be represented in density matrix form:

$$|\psi\rangle \rightarrow |\psi\rangle \langle\psi|$$

These are both equivalent ways of writing the state $|\psi\rangle$. The right hand side can be interpreted as saying "We know with 100 percent certainty that the system has been prepared in the state $|\psi\rangle$ ".

Mixed state density matrices

As discussed, if the system is imperfectly prepared, one might be ignorant of the particular state the system is in. One then writes such a system in density matrix form as:

$$\sum_{i=1}^n p_i |\psi_i\rangle \langle\psi_i|$$

Where p_i is the probability of finding the system in the state $|\psi_i\rangle$. Such a density matrix therefore represents a "proper mixture" of the states $|\psi_i\rangle$.

It should be noted however, that the interpretation of a mixed state density matrix as a proper mixture of states whose probabilities represent our ignorance only works in so far as the context allows it. This is because basis transformations on density matrices in general change their form, and the interpretation of the matrix in the new basis may not be the same as in the original one.

Extracting information from density matrices- The Trace Rule

It turns out that simply from the Born Rule ¹ one can quickly prove that the expectation value of a given observable corresponding to some operator \hat{O} with eigenstates $|o_i\rangle$ can be extracted from the density operator as:

$$\langle \hat{O} \rangle = \text{Tr}(\hat{\rho} \hat{O})$$

This is known as the Trace Rule. Here's the (almost) one-liner proof:

$$\begin{aligned} \text{Tr}(\hat{\rho}(\hat{O})) &= \sum_i \langle o_i | \left(\sum_j p_j |\psi_j\rangle \langle \psi_j| \right) \hat{O} | o_i \rangle \\ &= \sum_i \sum_j o_i p_j |\langle o_i | \psi_j \rangle|^2 \end{aligned}$$

The expectation value is therefore a sum over the expectation values of the different states $|\psi_j\rangle$ weighted by the probabilities p_j corresponding to each of them.

Reduced Density Matrices

It turns out that when considering systems whose Hilbert spaces are tensor products of the Hilbert spaces of their constituent subsystems:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots$$

One can talk about measurements performed purely on one of the subsystems by essentially "tracing out" the contributions from the other subsystems. This results in what is known as the "Reduced density matrix" corresponding to the subsystem that has **not** been traced out. Say for example in a system where:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

One can talk about measurements performed purely on \mathcal{A} in terms of the reduced density matrix corresponding to the subsystem \mathcal{A} :

$$\hat{\rho}_A = \text{Tr}_B(\hat{\rho})$$

Then, if one wants to calculate the expectation value of a particular observable \hat{O} corresponding only to \mathcal{A} , that is:

$$\hat{O} = \hat{O}_A \otimes \hat{I}_B$$

(\hat{I}_B simply represents the identity operator applied on the subsystem \mathcal{B} , signifying that the operator has no effect on this subsystem) one can use the Trace Rule on the **reduced** density matrix of \mathcal{A} :

$$\langle \hat{O} \rangle = \text{Tr}(\hat{\rho}_A \hat{O}_A)$$

¹The Born Rule states that the probability of finding a state $|\psi\rangle$ in the state $|a\rangle$ upon measurement, is given by $|\langle \psi | a \rangle|^2$

without having to care about the other subsystem. The reduced density matrix of a particular subsystem in essence "averages out" the effect of the remaining subsystems.

Entangled States

Before we begin addressing the problem of the quantum to classical transition. We must first briefly look at entangled states. Entangled states are generally defined using "Non-entangled states" as a stepping stone.

"An non-entangled state is one which can be written as a tensor product of the states of the constituent subsystems"

An example of a non-entangled state would be:

$$|\psi\rangle = |a\rangle \otimes |b\rangle$$

Where the state $|a\rangle$ can be thought of as a basis vector of an operator of the subsystem \mathcal{A} and $|b\rangle$ as one of an operator of the subsystem \mathcal{B} . The state above is indeed a non-entangled state: it **has** been written in tensor product form!

The natural definition for entangled states is then:

"Any state which is not non-entangled, is an entangled state"

This non-definition may seem strange so an example should be illuminating. Consider the state:

$$\frac{1}{\sqrt{2}}(|a_1\rangle |b_1\rangle + |a_2\rangle |b_2\rangle)$$

Where we have disposed of the tensor product signs for brevity. It is not quite difficult to prove that such a state indeed has no decomposition into a tensor product of states from \mathcal{A} and \mathcal{B} .

In fact, this is what is known as a maximally entangled state. We will see that the degree of entanglement has in fact a lot to do with decoherence. For now, it suffices to note that measuring the state of the subsystem \mathcal{A} for this system immediately gives you information about the state of the subsystem \mathcal{B} (with the assumption of course that $|a_i\rangle$ mutually orthogonal and the same holds true for $|b_i\rangle$).

Brief Aside - Coherences

The presence of off-diagonal terms in a density matrix written in a particular basis signifies the existence of "coherences" in that basis. This means that interference should, at least in theory, be observable among the states in that basis.

In essence "coherences" or non-zero off-diagonal entries, signify the existence of superpositions in the basis in which the density matrix is written. Here is a quick example. Consider our original double slit experiment super position state. Then the pure state density matrix corresponding to it will be written as:

$$\begin{aligned} |\psi\rangle \langle \psi| &= \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) \frac{1}{\sqrt{2}}(\langle \psi_1| + \langle \psi_2|) \\ &= \frac{1}{2}(|\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2| + |\psi_1\rangle \langle \psi_2| + |\psi_2\rangle \langle \psi_1|) \end{aligned}$$

Evidently there are off-diagonal terms in the density matrix, signifying the presence of coherences. Indeed, it is these coherences that cause the interference pattern. In fact, we can show that if instead of an **actual** quantum superposition, we instead had an "equal mixture" of the two states then the density matrix would be given as:

$$\sum_{i=1}^n p_i |\psi_i\rangle \langle \psi_i| = \frac{1}{2} |\psi_1\rangle \langle \psi_1| + \frac{1}{2} |\psi_2\rangle \langle \psi_2|$$

The coherences went away. This is to be expected. The electrons are in the collapsed states (i.e either in $|\psi_1\rangle$ or $|\psi_2\rangle$) and so which way information is available, we are simply ignorant of said information. Thus no interference should be observed.

The Von Neumann Measurement Scheme

John Von Neumann came up with a theoretical scheme whereby quantum mechanical measurements may be performed. The scenario consists of a usually microscopic system \mathcal{S} and an apparatus \mathcal{A} . The difference from the Copenhagen state of affairs is that the apparatus \mathcal{A} here is also assumed to be quantum mechanical in nature.

The measurement goes as follows. Suppose the measurement apparatus has certain pointer states $|a_i\rangle$ (all mutually orthogonal so that the measurements can easily be distinguished) in one to one correspondence with certain states of the system $|s_i\rangle$. Then if the apparatus is in the state $|a_r\rangle$ which is usually termed the "ready state" where the apparatus is ready to perform a measurement. The dynamical evolution of the system-environment state via the Schrodinger equation is assumed to be as follows:

$$|s_i\rangle |a_r\rangle \rightarrow |s_i\rangle |a_i\rangle$$

That is the "pointer" of the apparatus reads off the " i^{th} " state. Since the evolution/measurement has no bearing on the state of the system $|s_i\rangle$ one calls such a measurement an "ideal" quantum measurement.

Now it turns out that entangled states evolve quite naturally through the Von Neumann measurement scheme. In fact, given that the system starts off initially in a superposition state:

$$\frac{1}{\sqrt{2}}(|s_1\rangle + |s_2\rangle)$$

The evolution dictates that the final form of the system environment state will in fact be:

$$\frac{1}{\sqrt{2}}(|s_1\rangle + |s_2\rangle) |a_r\rangle \rightarrow \frac{1}{\sqrt{2}}(|s_1\rangle |a_1\rangle + |s_2\rangle |a_2\rangle)$$

Which is indeed an entangled state. It is evident, however, that such a Von Neumann measurement does not correspond to an actual physical measurement. Afterall, the state has not collapsed yet and we are still in a superposition. Another rigorous way of showing that the Von Neumann "measurement"

does not actually correspond to a physical measurement is the following. Imagine that both our system and the apparatus are defined by a spin $\frac{1}{2}$ particle. If the systems starts in the entangled state:

$$\frac{1}{\sqrt{2}}(|0\rangle_z + |1\rangle_z)$$

and the apparatus is assumed to perform a measurement in the z-spin basis the dynamical evolution of the measurement system will be:

$$\frac{1}{\sqrt{2}}(|0\rangle_z + |1\rangle_z) |a_r\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_z |0\rangle_z + |1\rangle_z |0\rangle_z)$$

Where the states on the right are assumed to correspond to the apparatus. If we now change the bases of both the systems from $\{|0\rangle_z, |1\rangle_z\}$ to $\{|0\rangle_x, |1\rangle_x\}$. We find that the state is now:

$$\frac{1}{\sqrt{2}}(|0\rangle_x |0\rangle_x + |1\rangle_x |0\rangle_x)$$

If we were to take the original assumption of the final Von Neumann state as corresponding to a measurement at face value, it would appear as if simply by virtue of a measurement of spin in the z direction, we have simultaneously measured the spin in the x directions (and in fact in all spatial directions). Since the spins along different directions are non-commuting observables, this is in disagreement with quantum mechanics. The final system-apparatus state in the Von Neumann measuring scheme therefore corresponds to a sort of "premeasurement".

Compromise between coherence and which-path information:

When we briefly discussed interference, we saw that the knowledge of which-path information results in the disappearance of the interference pattern. A natural question then arises: "Is it possible to extract **some** which path information while also keeping **some** of the interference pattern intact". The answer is in the affirmative.

To tackle this question, we will make use of the Von Neumann measuring apparatus. Let us assume that we have our original superposition state. Then the Von Neumann measuring apparatus evolves as follows:

$$\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) |\text{"ready"}\rangle \rightarrow \frac{1}{\sqrt{2}}(|\psi_1\rangle |1\rangle + |\psi_2\rangle |2\rangle)$$

Where $|1\rangle$ and $|2\rangle$ indicate the "pointer" states of the system. For the moment, let us rid ourselves of the assumption that $|1\rangle$ and $|2\rangle$ are orthogonal (the reason will become clear in a moment).

Any measurements on the electron would then ofcourse be carried out through its reduced density matrix. This comes out to be:

$$\hat{\rho}_{\text{electron}} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2|\langle 2|1\rangle + |\psi_2\rangle\langle\psi_1|\langle 1|2\rangle)$$

Notice how the off-diagonal entries or "coherences" are scaled by the inner product of the two pointer states $|1\rangle$ and $|2\rangle$. It is precisely this inner product that signifies our degree of compromise between the visibility of the interference pattern and the acquisition of which-path information. This can be directly seen by computing the particle density on the screen as:

$$\begin{aligned}\rho(x) &= \langle x | \hat{\rho}_{\text{electron}} | x \rangle \\ &= \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \text{Re}\{\psi_1(x)\psi_2^*(x)\langle 2|1\rangle\}\end{aligned}$$

As discussed previously it is precisely this last term that causes the interference pattern to appear on the screen. How this term relates to the acquisition of which-path information is explained in the following paragraph.

It turns out that if the degree of overlap between the pointer states is significantly large it corresponds to a "greater degree of uncertainty" about what state the electron is actually in since the two pointer states $|1\rangle$ and $|2\rangle$ cannot be sufficiently differentiated. On the other hand, the limit of zero overlap corresponds to a situation where the pointer states are perfectly distinguishable, meaning that which path information is perfectly known. In this limit:

$$\hat{\rho}_{\text{electron}} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)$$

The reduced density matrix reduces to an "improper" mixture. The important thing to note here is that there are no coherences, meaning no interference will be observed.

Which-state monitoring by the environment

We are now in a position to address at least the second sub-problem that we listed: namely that of the "non-observability of interference" (on macroscopic scales).

In order to tackle this problem we consider a macroscopic object such as a pencil (I chose a pencil because I didn't want to right out copy the book and use the example of a book, but a pencil should suffice for the argument). With the assumption that the pencil, being made out of microscopic objects, should also behave quantum mechanically we consider two states that this quantum mechanically behaving pencil could be in a superposition of. Let these states represent the two orientations that this pencil could be lying on the table in. We call these $|\theta_1\rangle$ and $|\theta_2\rangle$.

The question we are then posing is: "Why don't we see this pencil in a superposition of said states?". This is an equivalent way of phrasing the problem of the non-observability of interference. After all, any interference effects result from superpositions of states.

Let us now introduce an environment \mathcal{E} into the picture. This environment contains all that an environment could contain: it is a fiesta of photons, neutrinos, thermal radiation and the like. In other words, it is continually interacting with the pencil.

Let us consider some time $t = 0$ where we switch on this system environment model. All of the particles in the environment interacting with the pencil collectively behave as a Von Neumann measuring apparatus in this scenario. The idea is that as time goes on, the state of the particles interacting with the pencil evolves differently for different orientation states of the pencil. Let us say that if $|E_o\rangle$ is the initial environment state, then the evolution is dictated by:

$$\begin{aligned}|\theta_1\rangle|E_o\rangle &\rightarrow |\theta_1\rangle|E_1\rangle \\ |\theta_2\rangle|E_o\rangle &\rightarrow |\theta_2\rangle|E_2\rangle\end{aligned}$$

Where $|E_i\rangle$ represent final states after interaction with the system. We want to know why superposition states don't exist in the macroscopic world so we ask the question: "What would happen to a superposition state of the pencil, should we actually be able to create one?". That is what would the interaction with the environment entail for the evolution of such a superposition state. Well, we know that if we start with a superposition state it evolves into an entangled state as follows:

$$\frac{1}{\sqrt{2}}(|\theta_1\rangle + |\theta_2\rangle)|E_o\rangle \rightarrow \frac{1}{\sqrt{2}}(|\theta_1\rangle|E_1\rangle + |\theta_2\rangle|E_2\rangle)$$

If we now look at the reduced density matrix of the pencil in this state we find that the coherences are again scaled by the dot product $\langle E_1|E_2\rangle$:

$$\begin{aligned}\hat{\rho}_{\text{pencil}} &= \frac{1}{2}(|\theta_1\rangle\langle\theta_1| + |\theta_2\rangle\langle\theta_2| + |\theta_1\rangle\langle\theta_2|\langle E_2|E_1\rangle \\ &\quad + |\theta_2\rangle\langle\theta_1|\langle E_1|E_2\rangle)\end{aligned}$$

we can imagine that these final states of the environment would be time dependent (more and more interactions take place as time goes on). As more and more components of the environment interact with the pencil, they "chip away" more and more of the information from the pencil. The crux of the argument then, is that we can regard all of the particles etc. in the environment as being its subsystems. That is:

$$\mathcal{E} = \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \dots$$

With the i^{th} subsystem \mathcal{S}_i corresponding to the i^{th} particle in the environment. Now we could write each individual particle state in some appropriate basis after the interaction so that we have:

$$E_1 = |1\rangle^{(1)} \otimes |2\rangle^{(1)} \otimes \dots$$

$$E_2 = |1\rangle^{(2)} \otimes |2\rangle^{(2)} \otimes \dots$$

Now even if we imagine that the orientation states for the system are not very different (that is we are considering angles that are relatively close to each other):

$$\langle \theta_1 | \theta_2 \rangle \approx 1$$

Then even though the effect on the final states of the individual particles in the environment may not be significant:

$$(\langle i |^{(1)})(|i\rangle^{(2)}) \approx 1$$

The differences will accumulate since:

$$\langle E_1 | E_2 \rangle = (\langle 1 |^{(1)})(|1\rangle^{(2)}) \times (\langle 2 |^{(1)})(|2\rangle^{(2)}) \times \dots$$

Which causes the dot product to go to zero very quickly:

$$\langle E_1 | E_2 \rangle \rightarrow 0$$

What this means for our pencil's reduced density matrix is that the coherences go away and the system becomes an "improper mixture" of the two orientation states:

$$\hat{\rho}_{\text{pencil}} = \frac{1}{2}(|\theta_1\rangle \langle \theta_1| + |\theta_2\rangle \langle \theta_2|)$$

This, in a somewhat crude manner answers the question of why interference is not observed in the macroscopic realm:

"The time scales over which macroscopic systems decohere (lose coherences from their reduced density matrices) due to environmental interactions are so small that it is impossible to observe pure superposition states in practice for such systems"

Note also that this section has demystified our original statement of "delocalization of coherence to the level of the environment". As can be seen, upon interaction with the environment the system becomes an "entangled superposition". It is not the system anymore that is coherent, but the system-environment as a whole.

References

- [1] Maximilian A. Schlosshauer. In *Decoherence and the Quantum-to-Classical Transition*, 2007.