Warp drives

Course Project for PHY 442: General Relativity

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Abstract

A general overview of warp drive spacetimes is provided. A classification of warp drive metrics based on the degree of warping in the warp bubble region is also given. The Alcubierre drive is analyzed in detail. A modification to the Alcubierre drive, namely the "Flattened" Alcubierre Drive is studied in passing. A subluminal solution in the form of the Lorentz Drive is also mentioned. Physical feasibility of general warp drives is discussed in the context of energy conditions and causality considerations.

Contents

1 Introduction

Warp drive spacetimes are of interest due to the possibility of covering vast distances in space while experiencing minimal relativistic effects (such as time dilation), or almost no such effects altogether in the case of extreme warp drives such as the Alcubierre drive. Aside from such practical considerations, warp drives are also very interesting as gedankenexperiments - scenarios that push our understanding of the underlying theory to its limits so that arguments can be made about what is physically possible and what isn't.

In addition, certain "mild" warp drive spacetimes reduce (in the weak field limit) to solutions corresponding to classical (non-exotic) matter moving at subluminal speeds while weakly changing spacetime within its cofines. These are physically realizable scenarios that deserve attention in their own right.

2 General warp drive spacetimes

Warp drive spacetimes are generally split into three physically interesting regions (Fig 1):

1. **Warping Region**: In literature this is usually referred to as the "warp bubble". This corresponds to a shell of some thickness in space. This region - in non-trivial cases - always has non-zero curvature.
2. **Passenger Region**: The warp bubble encloses a region of spacetime that is known as the "passenger region", this is where the hypothetical spacecraft is placed. This region in the most ideal scenario is flat. However, one can also get away with having "little" curvature here. Though at the expense of stronger relativistic effects inside.
3. **Outer Region**: This refers to the entirety of spacetime outside of the bubble which in almost all cases of interest is flat. This means that in the limit of infinite distance from the bubble, the metric reduces to the Minkowski metric.

All of the mass-energy - exotic or otherwise - producing the curvature is confined to the warping region.

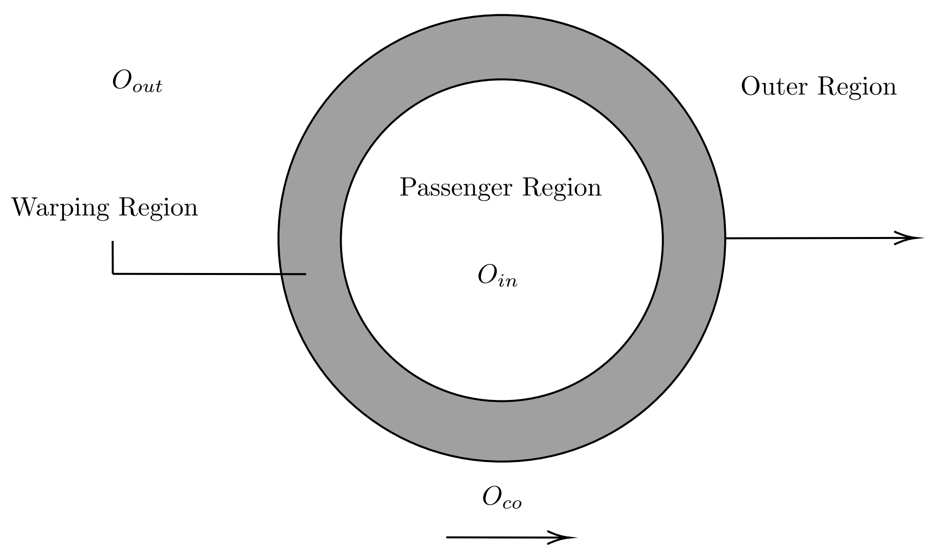

Figure 1: A general warp drive spacetime alongwith the observers/frames of interest. The arrows denote the direction in which the warp bubble and hence also O_{co} are moving. The bubble is assumed spherical for diagramatic simplicity

2.1 Classifying warp drives[2]

In order to gauge the decrease in relativistic effects experienced by passengers in the bubble it is usually helpful to place an observer in the flat spacetime outside of the warping region that is "co-moving" with the bubble. We shall call this observer/frame O_{co}. Any observers/frames in the passenger region will henceforth be referred to as O_{in}.

In addition to these two frames another observer/frame sitting far away from the bubble (effectively at infinity) is also considered. This observer will be referred to as O_{out}. The co-moving observer O_{co} has some non-zero three-velocity with respect to O_{out}. Based on the velocity that is chosen for the warp bubble through the metric - subluminal, luminal or superliminal - the co-moving observer's motion may become timelike, null or spacelike. While massive objects cannot physically move along null or spacelike trajectories, the misnomer "spacelike observer" may be assumed to be referring to a spacelike "frame".

Since we want to consider spacetimes that are stationary with respect to passengers inside the bubble, it is helpful to define a Killing vector field [1] ξ aligned with the four-velocity of the inner boundary of the warp bubble. This will define a "global rest frame" with respect to the warping region.

With these observers and the Killing vector field set up, we can now provide a classification of warp drive spacetimes based on the relations between these three frames.

2.1.1 Mild subluminal warp drives

The word "mild" refers to the degree of warping or curvature in the warp bubble region (the particulars of which will be made more concrete later on when we consider specific cases). Subluminal refers to the speed of the warp bubble i.e if v is the velocity of the bubble then $v \leq c$. In the case of such spacetimes the vector field ξ is timelike everywhere.

In the limit of extremely mild warping, the warping region essentially vanishes and the observers O_{co} and O_{in} reduce to a pair of timelike moving observers - the metric essentially reduces to Minkowski spacetime.

As mentioned before, this class also contains some interesting physically realizable spacetimes i.e ones that neither require exotic nor superluminal matter, but that weakly "flatten" the passenger region so that O_{co} and O_{in} experience different rates of time flow.

2.1.2 Mild superluminal warp drives

The warp bubbles in this category have $v \geq c$. Such warp drive solutions correspond to the Killing vector field being null or spacelike everywhere depending on the speed of the warp bubble. Of course in the limit of no warping at all, the observers O_{in} and O_{co} reduce to a pair of spacelike or lightlike observers. This is of course completely unphysical. Again, by introducing increased warping, one can introduce physically significant differences between the time ticked off by the clocks of the two observers O_{co} and O_{in}. However, primarily because null and spacelike frames do not correspond to physical observers, such metrics are of little to no interest.

2.1.3 Extreme superluminal warp drives

As the name suggests, such metrics corresponds to bubbles moving at superluminal speeds. However the Killing vector field ξ is in fact timelike in the passenger region whereas it becomes null and then eventually spacelike as one moves through the warp bubble to the outer region. Since the field should have a smooth

[1] I will make use of the concept of Killing vectors in the following classification. For a "very brief" introduction (which should suffice for our purposes) see Appendix A

transition from being timelike to spacelike, there must be a surface of null Killing vectors around the bubble. This is called the Killing horizon. Any mass - energy outside of this bubble would have to be moving superluminally (fig 2).

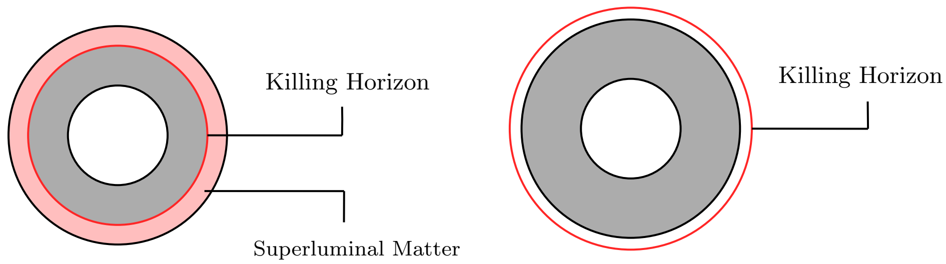

Figure 2: The killing horizon for extreme subluminal warp drives. If the Killing Horizon were inside the bubble [left], the matter shaded in red would have to be superluminal. If the Killing horizon were instead outside of the warp bubble [right], one could avoid violating the dominant energy condition.

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A necessary condition then, is that the Killing horizon at least be outside of the warping region, since if it wasn't, some of the matter generating the warping would have to be superluminal, violating the dominant energy condition.

Since the bubble is superluminal, the co-moving observer O_{co} would have to be spacelike, or at least null in order to keep up with the bubble.

2.1.4 Extreme subluminal warp drives

Such spacetimes are characterized by the Killing vector being null or spacelike in the passenger region and transitioning to being timelike in the outer region. Clearly the killing vector field being spacelike in the interior region means that timelike observers inside the bubble would have to be moving with respect to the inner boundary of the bubble.

However, we note that the condition is now reversed compared to the previous case. In order for the matter in the warping region to not be superluminal, one would require that the Killing horizon correspond to the inner boundary of the bubble, as can be seen in figure 3.

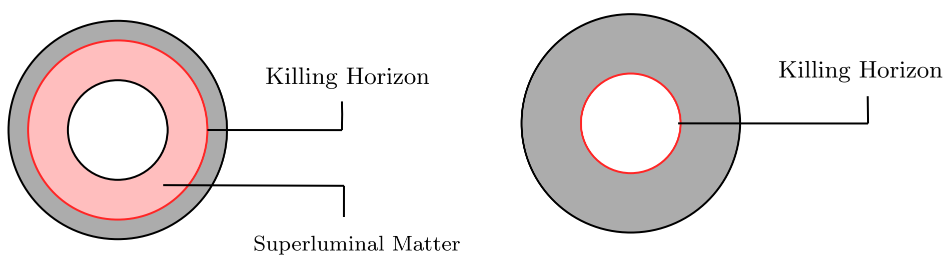

Figure 3: The Killing horizon for extreme superluminal warp drives. In order to prevent the dominant energy condition from being violated, one now requires the Killing horizon to coincide with the inner boundary of the bubble.

3 The Alcubierre Drive[1]

3.1 Introduction

Miguel Alcubierre came up with a metric for a warp drive for the first time in 1994, without much consideration into the kind of stress energy tensor that would be required to generate the metric. In alcubierre's metric, the velocity of the bubble can be any arbitrary function of time, and depending on whether the velocity is subluminal or superluminal, the metric falls into the categories of "mild subluminal" and "extreme superluminal" warp drives respectively as we will see.

3.2 Deriving the metric

Alcubierre used the ADM formalism[2] of general relativity to devise his metric. The following three parameters were chosen:

1. The induced metric: $h_{ij} = \delta_{ij}$
2. The lapse function: $\alpha = 1$
3. The shift vector: $\beta^i = (-v_s(t)f(r_s(t)), 0, 0)$

Since the shift vector decides how the spatial coordinates change when moving from one hypersurface[3] to another, it is evident that the "shifting" is only happening in the x-direction. The meaning of this will be evident once we write down the final metric.

Since the lapse function is defined as the dot product (with respect to the metric) of the normal vector field to the family of hypersurfaces and the time coordinate, a lapse function of 1 means that the normal vectors are purely timelike or the proper time (when moving normal to the hypersurfaces) is equal to the co-ordinate time.

Finally, since we have spacelike hypersurfaces and the induced metric chosen is Euclidean, we say that in this case "space is flat", though spacetime itself is not as we shall see.

v_s is some function of co-ordinate time t and is defined as the co-ordinate time derivative of x_s which is the x coordinate of the center of the bubble. Additionally the forms of the functions $f(r_s(t))$ and $r_s(t)$ are as follows:

$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}$$

$$f(r_s(t)) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)}$$

As is evident, since $x_s(t)$ is a function of time, $r_s(t)$ defines a point in cartesian coordinates that moves (with some arbitrary velocity $v_s(t)$) in the positive x direction. On the other hand, the function $f(r_s(t))$ defines this sort of "hump", with an "effective radius" R, and "rate of transition" σ from 0 to 1. That is, σ defines how quickly the function asymptotes to 1 as the argument $r_s(t)$ (which is the distance from the center of the bubble) approaches 0. Conversely, it also defines how quickly the function asymptotes to zero

[2] For a brief introduction to the ADM formalism (again only just enough for our purposes) see Appendix C: The ADM Formalism

[3] For a brief foray into hypersurfaces, you can read Appendix B: Hypersurfaces

as the argument approaches infinity. In fact, as can be seen from the plots of the function for different σ values, the plot quickly approaches a "top hat" or "bottle cap" shape as sigma grows larger. This is what we called the "degree of warping" when categorizing general warp drive spacetimes.

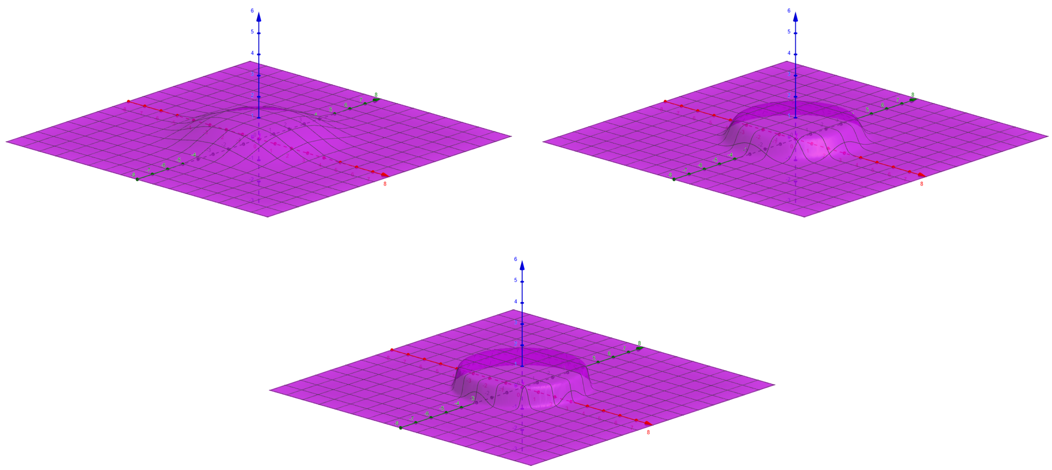

Figure 4: Plots of the function f from the Alcubierre metric for increasing values of σ restricted to two dimensions. The plots are for $\sigma = 1$,4 and 10 respectively (starting from the top left). Notice how for $\sigma = 10$ the plot resembles that of the 2-d step function.

How one would reach the form of the function written above is unimportant. One couldn't! It is just a cleverly chosen function that defines a spacetime which "flattens" both at the center of the bubble, and at an infinite distance away from it.

Ok, lets finally look at the metric in these co-ordinates. To do so we need to use the ADM formula that defines a metric based on the three parameters (h_{ij}, α, β^i) that we just defined. In cartesian coordinates this is:

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + h_{ij} dx^i dx^j \\ &= -(1 - \beta_x \beta^x) dt^2 + (-2\beta_x dx dt) + \delta_{ij} dx^i dx^j \\ &= -(1 - (v_s f(r_s(t))^2) dt^2 - 2v_s(t) f(r_s(t)) dx dt + dx^2 + dy^2 + dz^2 \end{aligned}$$

Where in the third equality we have simply used the fact that β^x is the only non-zero component of our shift vector. In the fourth equality the identity removes the non-diagonal entries from the $dx^i dx^j$ term and we substitute the value of our non-zero shift vector component. Finally we pack up the square and cross terms involving dx and dt neatly into the square of a sum and write the metric in its final form:

$$ds^2 = -dt^2 + (dx - v_s f(r_s(t))dt)^2 + dy^2 + dz^2 \tag{3.1}$$

Notice how there is mixing between the x and t components of the metric. This describes the propagation of the bubble in the x-direction. This is what we meant when we said space is flat, though "spacetime" is curved.

3.3 Passengers move along geodesics

We can now quickly show that the observers sitting at the center of the bubble are not only timelike, but move along geodesics. This amounts to proving that their proper time is maximized, or that proper time is equal to the co-ordinate time for such observers. Note that in this proof we imagine either that the warping factor σ is sufficiently large so that space time is effectively flat around the center, or conversely that the spaceship is small enough so that it does not extend towards the boundary of the bubble where strong tidal effects may be experienced.

Let us change our co-ordinate system by applying the transformation $x \to x_s(t)$. That is, we are essentially "following" the center of the bubble in our new co-ordinate system. Then:

$$r_s(t) = \sqrt{(x_s(t) - x_s(t))^2 + y^2 + z^2}$$
$$r_s(t) = \sqrt{y^2 + z^2}$$

That is $r_s(t)$ is now a constant function. This in turns make $f(r_s(t))$ a constant function. Lets call it C_o for now. Our metric has now become:

$$ds^2 = -dt^2 + (dx_s - \frac{dx_s}{dt} C_o dt)^2 + dy^2 + dz^2 \tag{3.2}$$
$$ds^2 = -dt^2 + (dx_s - dx_s C_o)^2 + dy^2 + dz^2 \tag{3.3}$$
$$ds^2 = -dt^2 + (1 - C_o)^2 dx_s^2 + dy^2 + dz^2 \tag{3.4}$$

In order to calculate the proper time, we set the spatial components equal to zero. Clearly by doing so the only term that survives is the dt^2 one. We have therefore shown that:

$$dt^2 = d\tau^2 \tag{3.5}$$

The coordinate time is equal to the proper time for observers at the center of the bubble. The passengers in our spaceship are indeed moving along geodesics. In fact, the space around them is essentially indistinguishable from the space around an observer sitting at infinity (unless ofcourse they move towards the boundaries of the bubble, in which case the tidal effects will make the two cases distinguishable).

Notice also that this motion along geodesics implies that the spaceship essentially does not experience any relativistic effects whatsoever as the bubble moves. There is no time dilation, the co-moving observer O_{co} (at least for subluminal speeds) sees the clocks for O_{in} and O_{out} to be running at exactly the same rate. This is such an important fact that it needs reiteration. Passengers inside the bubble experience **no relativistic effects** that usually result from moving at large velocities relative to other observers!

3.4 Expansion of the normal volume elements:

In what follows we will calculate the expansion of the volume elements in the case of "Eulerian observers", this just means observers that are static in space and only moving in time. An equivalent way of defining such observers is by requiring that their 4-velocity be normal to our hypersurfaces.

As mentioned earlier, the hypersurfaces have the induced metric δ_{ij} and are therefore completely flat. Any information about the curvature of "spacetime" is therefore contained in the extrinsic curvature tensor i.e how the spacelike hypersurfaces are embedded in 4-D spacetime. The extrinsic curvature tensor K_{ij} has the formula:

$$K_{ij} = \frac{1}{2\alpha}\left(\nabla_i \beta_j + \nabla_j \beta_i - \frac{\partial g_{ij}}{\partial t}\right) \tag{3.6}$$

We have to be careful here: ∇_k describes the covariant derivative with respect to the **induced metric on our hypersurfaces**. Additionally we note that after the co-ordinate transformation we performed in the previous section, the metric in its new form (3.4) is no longer dependent on time. This means that our spacetime has the Killing vector ∂_t. In other words, the partial derivative of our spacetime metric g_{ij} in the direction of coordinate time t is in fact zero. Moreover, since the hypersurfaces are flat the induced connection is zero, which means that the covariant derivative is in fact just the partial derivative. Finally we can substitute $\alpha = 1$. Our expression for the extrinsic curvature reduces to:

$$K_{ij} = \frac{1}{2\alpha}\left(\partial_i \beta_j + \partial_j \beta_i\right) \tag{3.7}$$

A small aside on expansion θ (you can skip to the next paragraph if you are familiar with the concept): The trace of the extrinsic curvature tensor of a hypersurface is identical to the "divergence" of the vectors normal to it. Therefore if the normal vectors define a family of geodesics, the expansion θ essentially determines whether the congruence of geodesics diverges or converges as one moves away normal to said hypersurfaces and how quickly it does so. In particular:

$$\begin{aligned} &\text{if} \quad \theta > 0: \quad \text{Geodesics Diverge} \\ &\text{if} \quad \theta < 0: \quad \text{Geodesics Converge} \end{aligned}$$

The formula for the expansion of volume elements is:

$$\theta = -\alpha(\mathrm{Tr}(K))$$

So if we substitude $\alpha = 1$ and calculate the trace in terms of the induced metric, we get:

$$\begin{aligned} \mathrm{Tr}(K) &= \frac{1}{2}\delta^{ij}\left(\partial_i \beta_j + \partial_j \beta_i\right) \\ &= \frac{1}{2}\left(\partial_i \beta^i + \partial_i \beta^i\right) \\ &= \partial_x \beta^x \end{aligned}$$

Computing this one term explicitly we find:

$$
\begin{aligned}
\frac{\partial \beta^x}{\partial x} &= \frac{\partial}{\partial x}\big(-v_s(t) f(r_s(t))\big) \\
&= -\left(f(r_s(t)) \cancelto{0}{\frac{dv_s(t)}{dx}}\right) - \left(v_s(t)\frac{df(r_s(t))}{dx}\right) \\
&= -\left(v_s(t)\frac{df(r_s(t))}{dr_s}\frac{dr_s}{dx}\right)
\end{aligned}
$$

Where:

$$
\begin{aligned}
\frac{dr_s}{dx} &= \frac{d}{dx}\big(\sqrt{(x-x_s(t))^2+y^2+z^2}\big) \\
&= \frac{1}{\cancel{2}}\frac{1}{\sqrt{(x-x_s(t))^2+y^2+z^2}}\cancel{2}(x-x_s) \\
&= \frac{(x-x_s)}{r_s}
\end{aligned}
$$

Substituting this back into the expression for $\frac{\partial \beta^x}{\partial x}$ we get:

$$
\boxed{\theta = v_s \frac{(x_s - x)}{r_s}\frac{df}{dr_s}}
$$

What does this expression for the expansion actually mean? The dependence on the derivative of the function f tells us that the expansion is greatest at the seams of the passenger region, i.e in the warping region, where the function suddenly starts asymptoting towards zero. A larger σ therefore, "constrains" the expansion to a thinner region, while simultaneously making the expansion more drastic (the function falls quickly, so has a larger slope). Additionally we note that the expansion increases if we increase the velocity of the warp bubble v_s. Finally the dependence on x_s signifies the following:

1. At the boundary of the bubble behind the passenger: $\frac{df}{dr_s} < 0$ and $(x - x_s) < 0$ so $\theta > 0$ (volume elements expand)
2. At the boundary of the bubble ahead of the passenger: $\frac{df}{dr_s} < 0$ and $(x - x_s) > 0$ so $\theta < 0$ (volume elements contract)

This particular feature is what makes such a spacetime resemble the science fiction concept of a "Warp Drive". Whatever contraption is generating this spacetime is simultaneously expanding spacetime behind it and contracting spacetime in front of it (Figure 5). "Pulling its destination towards itself" in some sense. This effect was not introduced by construction rather it seems to be a side-effect of the metric chosen by Alcubierre. Indeed Jose Natario shows in his paper [3] that by modifying the geometry of Alcubierre's solution one can get away with creating a warp drive that does not do this expanding and contracting at all.

3.5 A quick look at a space voyage

Well, what would a journey involving the use of this warp drive we just created look like (Figure 4)? One imagines that the spacecraft either carries the warp drive generating engine or the engine is in some parking orbit around the earth. Whatever the case, one must generate the warp drive spacetime sufficiently far away from the planet, lest it or some portion of it should be completely annihilated by the tidal forces at the warp

Figure 5: Expansion of volume elements around the warp bubble. Here $\rho = \sqrt{y^2 + z^2}$. Notice how space seems to be contracting in front of the bubble and expanding behind it. This is where the pop-science interpretation originates from.

bubble. The spacecraft therefore first escapes the earth's gravity by conventional means.

Say the parking orbit is some distance d away from where it was launched. We imagine another such parking orbit around the destination planet which for simplicity is also assumed to be a distance d away from where the spacecraft will eventually land. What remains now is the intermediate stretch of the journey, lets assume the total distance between the planets is D (with the parking orbit distances accounted for). We also make the valid assumption that:

$$d << D$$

Once the spacecraft is in Earth's parking orbit, the warp bubble is generated around it (Alcubierre does not explain the means by which the bubble would be created). Then, the bubble is quickly accelerated to some constant acceleration a. Once the bubble reaches the mid-point of the journey, the acceleration is reversed to $-a$. This eventually brings the spacecraft to a stop near the parking orbit of the destination planet, where again the bubble is diluted and the spacecraft lands on the planet by conventional means. Then assuming that the bubble attains its constant acceleration values very rapidly we can compute the total co-ordinate time elapsed using one of Newton's equations [4]. From Earth's parking orbit to the midpoint of the journey:

$$\frac{1}{2}D - d = \frac{1}{2}aT^2$$
$$T = \sqrt{\frac{D - 2d}{a}}$$

Accounting for the portion of the journey from earth to the parking orbit:

$$T_{\text{half}} = \frac{d}{v} + \sqrt{\frac{D - 2d}{a}}$$

Where v is assumed to be some strictly subluminal velocity ($v < 1$) since this portion of the journey is covered by conventional means. Finally accounting for the other half of the journey, we get:

[4] While this may seem hand wavy, the argument is in fact airtight. The bubble moves irrespective of any observers,. It is **a disturbance in spacetime itself**, not an object that moves on spacetime. That is the crux of the argument.

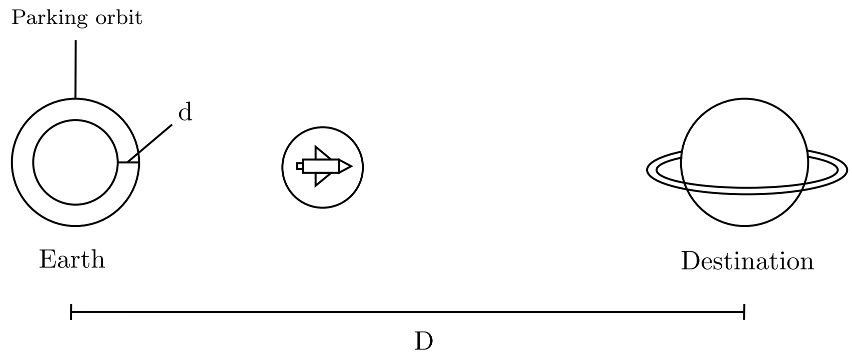

Figure 6: A prototypical warp drive journey. For the longest and most significant stretch of the journey ($D - 2d$) spacetime is flat for all three observers, i.e the ones on the planets and the ones inside the bubble. The time elapsed is therefore equal to the co-ordinate time for all three

$$T_{\text{tot}} = 2\left(\frac{d}{v} + \sqrt{\frac{D-2d}{a}}\right)$$

We already know that during the warping phase of the journey, the planets were far enough away from the bubble to be in flat spacetime. Therefore they experience co-ordinate time. However, the spaceship **almost** experiences the same amount of time. The only portions that create a difference in the time recorded are those in which the spacecraft moves "through" spacetime with velocity v. In the warp drive phase of the journey, the spacecraft is in flat spacetime and is moving along a geodesic, so the time elapsed is the same as the co-ordinate time. In other words:

$$T_{\text{spacecraft}} = 2\left(\frac{d}{\gamma v} + \sqrt{\frac{D-2d}{a}}\right)$$

Where $\gamma = \sqrt{(1-v^2)}$. Since the acceleration of the bubble can be made arbitrarily large (in theory) one could make the time elapsed for both the planets and the spacecraft arbitrarily small.

3.6 Features in favour of the Drive

3.6.1 Global Hyperbolicity

An upside to defining the metric using the ADM formalism is that the hypersurfaces considered in the Alcubierre spacetime are Cauchy surfaces and since any spacetime that contains a Cauchy surface as a submanifold is globally hyperbolic[5], so is the Alcubierre spacetime. While global hyperbolicity imposes other stronger conditions on the spacetime one of these is this: a globally hyperbolic spacetime is causal. There are therefore no closed causal (timelike or null) curves in the Alcubierre spacetime.

3.6.2 Passengers move along time-like trajectories

While this has been mentioned before it is important enough to reiterate, even when the warp bubble moves superluminally passengers at the center of the bubble stay on timelike trajectories.

[5] A definition of both Cauchy surfaces and global hyperbolicity is given in Global hyperbolicity in ADM

3.7 Problems

3.7.1 Energy Condition Violations

Solving the Einstein equation for the T_{00} component of the stress-energy tensor gives us the energy density. In $G = c = 1$ coordinates this is (with the assumption that the cosmological constant is zero):

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

Where the Einstein tensor can be computed from the Ricci tensor, the Ricci scalar and metric tensor as follows:

$$R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

In particular what we want is the energy density for our so-called "Eulerian Observers". That is observers only moving in time, not in space. This requirement amounts to contracting our stress-energy tensor with vectors normal to the hypersurfaces:

$$T^{00} n_0 n_0 = \frac{1}{8\pi} G^{00}$$

and our normal vectors are just the unit time vectors (due to our choice of the lapse function). The equation therefore reduces to:

$$T^{00} = \frac{1}{8\pi} G^{00}$$

Computing the Christoffel symbols and then the Ricci tensor and scalar from them finally gives us the following expression:

$$T^{00} = -\frac{1}{8\pi} \frac{v_s^2 \rho^2}{4 r_s^2} \left(\frac{df}{dr_s} \right)^2$$

This is the expression for the energy density as measured by Eulerian observers. There are a couple of important facts to note here. The most glaring issue is the negative sign. All quantities involved in the expression are squares, so the energy density is negative throughout. This is in violation of the Weak Energy Condition which requires that observers only observe positive energies in spacetime i.e for some timelike vector field X^a:

$$T^{ab} X^a X^b \geq 0$$

But as we just saw, our Eulerian observers have trajectories that violate this equation. On the other hand as we saw before $\frac{df}{dr_s}$ is more or less confined to the warp bubble since the derivative of our warping function quickly goes to zero both near the center of the bubble and at infinity. This proves that the exotic mass-energy should be confined to the warping region.

On the other hand, the energy density changes as the squares of both the strength of warping as well as the velocity of the warp bubble. One can imagine that the amount of negative energy required quickly blows up for large values of bubble velocity as well as the degree of warping. Additionally in the above expression $\rho = \sqrt{y^2 + z^2}$ which means the bigger the extent of the bubble in the y and z directions, the greater the energy requirements. This makes intuitive sense: a bigger bubble requires more energy to flatten the spacetime inside of it.

The Alcubierre Drive also suffers from another problem that most other warp drives in the category of "extreme superluminal" warp drives do. The Killing Horizon is at some point inside the warping region, which makes at least some of the matter in this region superluminal, thereby violating the Dominant energy condition.

Lastly, recall that when studying the expansion of normal volume elements around the warp bubble, we observed that the bubble propagates by "expanding" space behind it. This is in violation of the strong energy condition, which requires that matter gravitate matter. However this is not as problematic of a feature of the drive as the last two since we have our own inflationary universe as a physical example of a spacetime that violates the Strong Energy Condition.

3.7.2 Eternality of the Bubble

While Alcubierre mentions in passing that one could create the disturbance outside of the Earth, the metric he devised actually describes a bubble that has been an inherent part of spacetime since the dawn of time. This begs the question (all unrealistic energy requirements aside) how would one "create the bubble"? This problem is not addressed in Alcubierre's original paper.

3.7.3 Accelerating the Bubble

Another issue with Alcubierre's metric is that given some arbitrary choice of the velocity function $v_s(t)$ (with the assumption that the bubble starts from rest at some point in time), the energy required to accelerate the bubble seems to appear out of thin air (or thin space in this case). The field seems to give itself energy to accelerate. This is unphysical. Though the problem can be solved by imagining that some dynamical field providing the energy, Alcubierre does not address this.

4 A modification of the Alcubierre Geometry - The "flattened" Alcubierre Drive

In his original paper Alcubierre chose certain parameters rather arbitrarily. The form of the warping function $f(r_s)$ is the most important example. It is then natural to ask the question, could some of these parameters be modified so as to make at least some of the energy requirements more realistic? In this section, we address exactly this question.

While the original function $f(r_s)$ defines a spherically symmetric warp bubble, this is not strictly necessary. We consider a warp bubble which is only symmetric about the axis along which it moves (the x-axis in our case).

To analyze such a solution it is best to switch to cylindrical co-ordinates:

$$(x, y, z) \quad \rightarrow \quad (x, \rho, \phi)$$

In order to change the bubble's profile, we now change the function $f(r_s)$ to some arbitrary function $f((x - x_s), \rho)$. If one calculates the T^{00} component of the stress-energy tensor for this general function, one obtains:

$$T^{00} = -\frac{1}{8\pi}\frac{v_s^2}{4}\left(\frac{df}{d\rho}\right)^2$$

Note that this expression immediately gives way to an efficient way of reducing the Energy. The total energy can ofcourse be calculated by taking the integral of the energy density over space as follows:

$$E = \int_{-\infty}^{\infty} dx \int_0^{\infty} d\rho \int_0^{2\pi} d\phi \quad T^{00}\sqrt{-g}$$
$$E = -\frac{v_s^2}{16}\int_{-\infty}^{\infty} dx \int_0^{\infty} d\rho \int_0^{2\pi} d\phi \quad \left(\frac{df}{d\rho}\right)^2 \sqrt{-g}$$

Then we can see that if we ”flatten” the warp drive by introducing a factor $\alpha > 1$ into the argument of the function f as follows:

$$f((x - x_s), \rho) \quad \rightarrow \quad f(\alpha(x - x_s), \rho)$$

The new energy can be calculated by the following change of variables:

$$\alpha(x - x_s) \quad \rightarrow \quad x'$$

So that $dx = \frac{dx'}{\alpha}$. And since the new integral also runs over the same limits (all space), we see that the energy has been reduced by a factor of α:

$$E \quad \rightarrow \quad \frac{E}{\alpha}$$

Clearly through this ”flattening” of the drive (i.e by rescaling the warping function to be more spread out along the x-axis), the energy can be reduced by an arbitrary factor. Ofcourse for practical purposes there is a bound on this flattening based on the size of the object to be placed inside the bubble.

Another problem with the energy density of the original Alcubierre metric is that it is velocity dependent. As the drive is accelerated, the energy required increases as the **square of the velocity of the bubble**. One might therefore want to ”smooth out” this velocity dependence of the energy, or take it out altogether. This can be done by modifying the shape of the bubble en-route again using the parameter α. In fact one sees that if $\alpha = v_s^2$ at every instant of the journey, the velocity dependence of the energy disappears.

5 A subluminal solution - The Lorentz Drive[2]

5.1 General axisymmetric warp drives:

We will construct a warp drive spacetime that admits a bubble moving at subluminal speeds. While interesting as an example of a different category of warp drive spacetimes it also provides insight into features that are characteristic of such spacetimes irrespective of whether they move superluminally or subluminally.

To construct the metric we make use of a method devised by Bobrick and Martire. The method works for general axisymmetric (symmetric about the axis of motion) warp drives. The method makes use of the fact that there are two natural co-ordinate systems for warp-drive spacetimes:

1. The co-ordinates of any observers inside the bubble (i.e co-moving with its inner boundary). We call these co-ordinates x^α_{co}
2. The co-ordinates of the observers sitting at the asymptotic infinity of the bubble. We call these co-ordinates x^α

We then make use of the fact that these co-ordinates chart the entire manifold which means they overlap and therefore one set of co-ordinates can be mapped to the other. In particular we define a mapping $x^\alpha_{co}(x^\beta)$ between the co-ordinate systems. We then choose some arbitrary functions $f_x(x^i_{co})$, $f_y(x^i_{co})$, $f_z(x^i_{co})$, $f_t(x^i_{co})$. The constraint on these functions is that they should be 1 in the passenger region of the bubble and should be 0 in the outer region (much like Alcubierre's function f). These functions decide the form of the warp bubble.

Finally, we use the following general warp drive metric:

$$ds^2 = -c^2(dt(1-f_t) + f_t dt_{co})^2 + (dx(1-f_x) + f_x dx_{co})^2 + (dy(1-f_y) + f_y dy_{co})^2 + (dz(1-f_z) + f_z dz_{co})^2$$

A quicky intuitive way of checking whether this metric makes sense is to push it to its limits (literally). For example, consider the asymptotic infinity of the metric, in that case all of the functions go to zero and the metric reduces to:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

as it should, since this is the metric for the observer O_{out} and should indeed be the Minkowski metric. In a similar manner if we consider the origin then we should return to the flat metric experienced by the observer O_{in}. Setting the functions equal to 1:

$$ds^2 = -c^2dt^2_{co} + dx^2_{co} + dy^2_{co} + dz^2_{co}$$

We see that this is indeed what happens.

5.2 Constructing the drive:

Note that our method for creating axisymmetric warp drives hinges on the relation between the co-ordinates of the observers O_{in} and O_{out}. We ask the natural question: "What if we do not require the clocks of the passengers in the warp bubble and the observers at infinity to be synchronized?". In fact, what if we consider the completely opposite case i.e one in which O_{in} experiences the exact same time dilation effects as our co-moving observer O_{co} outside of the bubble (from Section 2)?

To do so, we make use of the Lorentz transformations. In particular we require:

$$\begin{aligned} dt_{co} &= \gamma\left(dt - \frac{v_s dx}{c^2}\right) \\ dx_{co} &= \gamma\big(dx - v_s dt\big) \\ dy_{co} &= dy \\ dz_{co} &= dz \end{aligned}$$

Using these in our generalized metric (and using $f_x = f_y = f$) we get:

$$
\begin{aligned}
ds^2 &= -c^2(dt(1-f) + f(\gamma(dt - \frac{v_s dx}{c^2}))))^2 + (dx(1-f) + f(\gamma(dx - v_s dt)))^2 + dy^2 + dz^2 \\
&= -c^2(dt - fdt + f\gamma dt - f\gamma\frac{v_s dx}{c^2})^2 + (dx - fdx + f\gamma dx - f\gamma v_s dt)^2 + dy^2 + dz^2 \\
&= -c^2((1 - f + f\gamma)dt - f\gamma\frac{v_s}{c^2}dx)^2 + ((1 - f + f\gamma)dx - f\gamma v_s dt)^2 + dy^2 + dz^2 \\
&= -c^2((1 - f(1+\gamma))dt - f\gamma\frac{v_s}{c^2}dx)^2 + ((1 - f(1+\gamma))dx - f\gamma v_s dt)^2 + dy^2 + dz^2
\end{aligned}
$$

Redefine:

$$
\begin{aligned}
\alpha &= (1 - f(1+\gamma)) \\
\beta &= f\gamma v_s
\end{aligned}
$$

We then get:

$$
\begin{aligned}
ds^2 &= -c^2(\alpha dt - \frac{\beta}{c^2}dx)^2 + (\alpha dx - \beta dt)^2 + dy^2 + dz^2 \\
ds^2 &= -c^2\alpha^2 dt^2 + \cancel{2\alpha\beta dx dt} - \frac{\beta^2}{c^2}dx^2 + \alpha^2 dx^2 - \cancel{2\alpha\beta dx dt} + \beta^2 dt^2 + dy^2 + dz^2
\end{aligned}
$$

Repackaging the terms we note that the metric reduces to:

$$ds^2 = -c^2 F^2 dt^2 + F^2 dx^2 + d\rho^2 + \rho^2 d\theta^2$$

Where:

$$F^2 = 1 + 2f(1-f)(\gamma - 1)$$

Computing the 00 component of the stress energy tensor one finds:

$$T^{00} = -\frac{1}{8\pi}\frac{(\rho F'_\rho)'_\rho}{\rho F}$$

Where the primes indicate derivatives with respect to ρ, the radial cylindrical co-ordinate. We will now "prove" (skewing the definition of this word to its very limits) that no matter what the form of the function f is the energy density T^{00} should always be negative at some point. Notice how the term in the numerator is very similar to the second derivative of the function F. Note also that as one moves towards the asymptotic infinity of the bubble, f goes to zero causing F to go to 1. Similarly at the center of the bubble (i.e on the axis) F is still zero. We require that function F smoothly vary between these two values. As the value of F varies (as indeed it should, for it is dependent on f) between these points, there **must be an inflection point in the intermediate region**. This inflection point corresponds to the sign of the second derivative of F changing. Therefore at least somewhere the stress energy tensor must be negative.

This then, is the crux of this section. It is not the superluminality of a warp drive that necessitates the requirement for negative energy. Instead, it is the "truncation" or "walling off" of spacetime around a certain region that does so.

6 Summary/Conclusions/Discussion

Warp Drive spacetimes have been studied extensively in literature ever since Alcubierre first introduced the concept. But while certain modifications of Alcubierre's original geometry certainly ease **some** of the ailments, Warp Drives that produce physically meaningful differences between the time recorded by passengers inside the bubble and observers co-moving with it in flat spacetime are nowhere close to being in the realm of possibility. The truncation of spacetime around the passenger region seems to necessitate large amounts of negative energy even at subluminal velocities of the bubble, which is in violation of the Weak Energy Condition. Even though quantum field theory allows for minor violations of this condition (for very brief periods of time such as in the Casimir effect), no violations at the macroscopic level to the extent required by most Warp Drives are even theoretically possible.

Finally we notice that even though Alcubierre's original metric is causal, one can easily imagine that a spacetime where the spaceship can eventually come out of the bubble (after the trip) in a reasonably finite amount of time is not causal (at least for superluminal velocities) since such a spacetime would allow for faster than light communication between spacelike separated events. Since spacelike separated events do not have a well-defined "chronology" (one cannot say with certainty that one of the events happened before or after the other), this can result in the creation of closed causal curves.

7 Appendix A: Killing Vectors

7.1 Definitions

A Killing vector field is a vector field on a manifold whose flow defines continuous isometries (distance preserving transformations) of said manifold. A Killing vector ξ is usually written multiplied with an infinitesimal as follows:

$$\xi = (0, 1, 0, 0)dx$$

The dx makes the following fact manifest: Points on the manifold **"infinitesimally"** moved in the direction of the killing vector do not expand or contract. Killing vectors can therefore also be thought of as the "generators" of continuous isometries. Discrete symmetries such as parity transformations are not generated by Killing Vectors.

Another way of enforcing the constraint that the flow of the vector field ξ define an isometry (that is to enforce that distances do not change in the direction of the flow) is naturally by requiring the metric to be unchanging in the direction of the flow. This leads us to the definition of Killing vectors in terms of the Lie derivative of the metric:

$$\mathcal{L}_\xi g = 0$$

Where $\mathcal{L}_\xi$ refers to the Lie derivative in the direction of the vector field ξ. This can be rephrased through some manipulation as the so-called "Killing equation":

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

Since this is a covariant equation, this assertion is co-ordinate independent. Indeed, one can talk about the isometries of a manifold without having to resort to a co-ordinate system to begin with.

7.2 Killing vectors for some common metrics

It is demonstrative to consider Killing vectors for some commonly encountered metrics.

1. Time independent metrics: An easily identifiable killing vector in the case of time independent metrics is the basis vector pointing in the direction of time ∂_t. Clearly moving forwards or backwards in time amounts to a trivial isometry (the identity).

2. The Schwarzschild metric: The Schwarzschild metric is both spherically symmetric and time invariant. Spherical symmetry implies that moving in the angular directions amounts to no change in the metric, hence in addition to ∂t, ∂_θ and ∂_ϕ are also Killing vectors.

8 Appendix B: Hypersurfaces

8.1 Definition

The hypersurfaces Σ of an n-dimensional manifold $\mathcal{M}$ are n-1 dimensional subsets of the manifold that are themselves manifolds (i.e they are submanifolds). There are two ways in which hypersurfaces can be defined:

1. In terms of **an embedding function** ϕ such that:

$$\phi : \quad \Sigma \rightarrow \mathcal{M}$$

   Where if we define some co-ordinates x^α on $\mathcal{M}$ and some co-ordinates y^a on Σ, then the embedding essentially defines a set of parametric equations on the manifold:

$$\phi : \quad x^\alpha(y^a)$$

2. As a subspace **embedded inside the manifold** $\mathcal{M}$:

$$\Sigma \subset \mathcal{M}$$

   Where one then defines some "constraint function" $S(x)$, such that points in Σ satisfy:

$$\Sigma = \{p \in \mathcal{M} | \quad S(p) = 0 \quad \}$$

8.2 Induced metric from pull-back:

In the embedding picture, one can define a Jacobian of the embedding function (that naturally takes one from Σ to $\mathcal{M}$) as:

$$E^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$$

Notice that this Jacobian is in fact an n x n-1 matrix, which means it has a maximum rank of n-1. Moreover being a rectangular matrix it does not have an inverse. Therefore any transformation between objects in $\mathcal{M}$ and Σ can only be done using E^α_a. For our purposes, it suffices to know that from this Jacobian, one can "pull-back" the metric from the manifold $\mathcal{M}$ to the hypersurface Σ turning it into the induced metric:

$$h_{ab} = E^\alpha_a E^\beta_b g_{\alpha\beta}$$

Notice that there is necessarily information loss. An object living in an n dimensional space is being represented in a space of dimension n-1. This is why we need the shift vector β^i and the lapse function α to reconstruct the total metric $g_{\alpha\beta}$.

8.3 Classifying hypersurfaces

Recall that an equivalent way of defining a hypersurface was by defining a constraint function $S(x)$ and requiring that it be zero for all of the points on the hypersurface. Since this is the only requirement, one can still impose:

$$\partial_\alpha S|_{S=0} \neq 0$$

That is even though the function vanishes, the gradient vector does not. However, we do have some information about this gradient vector. For one, it cannot have any component tangent to the hypersurface ΣD. This is because Σ is **by definition** a surface of constant S. That is if T^α is some vector tangent to the hypersurface, then its dot product with the gradient vector should be zero:

$$T^\alpha \partial_\alpha S(x) = 0 \quad \text{For points on } x \text{ on } \Sigma$$

Therefore the gradient vector $g_{\alpha\beta}\partial_\beta S$ has in fact no component tangent to the hypersurface and defines the normal vector to it!

If the normal vector field to the hypersurface has some property (timelike, spacelike, lightlike) throughout, then we put the hypersurface in any one of these three categories:

1. Timelike: If $g_{\alpha\beta}\partial_\alpha\partial_\beta S > 0$
2. Spacelike: If $g_{\alpha\beta}\partial_\alpha\partial_\beta S < 0$
3. Lightlike or Null: If $g_{\alpha\beta}\partial_\alpha\partial_\beta S = 0$

8.4 Extrinsic curvature for embedded hypersurfaces

Aside from the intrinsic curvature arising from the induced metric h_{ab} on the hypersurface, one can also talk about the curvature that arises from how the hypersurface Σ is "placed" inside the manifold $\mathcal{M}$, at least in the case of the embedded hypersurface picture.

A formula for the extrinsic curvature is:

$$K_{\alpha\beta} = h^\rho_\alpha h^\lambda_\beta \nabla_\rho n_\lambda$$

Without resorting to any elaborate derivations, I only give some intuition regarding the equation. The induced metrics with an index up and an index down essentially act as projectors tangent to the hypersurface. This equation answers the question: "As one moves away from the hypersurface, how is the normal vector field changing in the direction tangent to the hypersurface?"

9 Appendix C: The ADM Formalism

9.1 Determining the spacetime metric from the induced metric

The ADM formalism is an initial-value formulation of general relativity. This means that one starts with some initially defined parameters in a localized region of spacetime and then uses these to make predictions about the entire manifold. Also called the 3-1 formalism, in ADM one basically "foliates" spacetime into a set of spacelike hypersurfaces of constant time Σ_t with normal timelike vectors defining the flow of time. Lets call them t^a. In Appendix B we saw that we can "pull back" the total metric to a hypersurface. Another way of defining this pulled back or induced metric is as follows:

$$h_{ab} = g_{ab} + n_a n_b$$

Where the vectors n^a define the normal vector field to the hypersurfaces. We can then define two parameters, the so called "shift vector" β^a and the "lapse function" α as follows:

$$\alpha = g_{ab} n^a t^b$$
$$\beta^a = h^a_b t^b$$

The interpretations are quite straightforward. α decides how much proper time passes as one moves along the family of hypersurfaces in the direction of the normal timelike vectors. As has been mentioned previously, h^a_b "projects" spacetime vectors in the direction tangent to the hypersurface. Then β^a measures how much the purely spatial co-ordinates "shift" as one moves from one hypersurface to the next.

If one defines the "lapse **vector**" as αn^a, then clearly the time vector t^a can be decomposed into its components normal and tangent to the hypersurface as:

$$t^a = \beta^a + \alpha n^a$$

Now since the total metric can be determined from the normal vectors and the induced metric, using this decomposition we can write the normal vectors purely in terms of the lapse function, the shift vector and the time vectors as:

$$g^{ab} = h^{ab} + n^a n^b$$
$$g^{ab} = h^{ab} + (\frac{t^a - \beta^a}{\alpha})\frac{t^b - \beta^b}{\alpha})$$
$$g^{ab} = h^{ab} + \frac{(t^a - \beta^a)(t^b - \beta^b)}{\alpha^2}$$

Therefore, one can determine the entire spacetime metric from just h_{ab}, α and β^a. The following section defines Cauchy sufaces and intuitively explains why one can make predictions about the entire metric from just one hypersurface in ADM.

9.2 Global hyperbolicity in ADM

We need some preliminary definitions in order to define a Cauchy surface:

1. **Causal vector**: A vector is called causal if it is either timelike or null
2. **Causal curve**: A curve is said to be causal if its tangent vectors are causal throughout. In other words, the curve is never spacelike.
3. **Chronological future of a point**: The chronological future of a point p is the set of points in spacetime that can be reached by a future directed causal curve starting from p.
4. **Chronological past of a point**: The chronological past of a point p is the set of points in spacetime from which a causal curve can end at point p.

5. **Chronological future and past of a set of points**: The chronological future of a set of points is the union of their respective chronological futures. The definition for chronological past is analogous.

6. **Achronal set**: An achronal set has the property that none of the points in the set are causally connected. That is: the interesection of the set with its chronological future is zero.

7. **Domain of dependence**: The domain of dependence of a set S of points in spacetime is the set of points whose information can be gathered purely by information about the set S. In other words, it is the set of points where "only" causal curves intersecting S can reach.

8. **Slice**: A closed achronal set is called a slice.

9. **The edge of a slice**: A point p in a closed achronal set is said to be on its "edge" when any open neighbourhood around p has two points q and r such that one is in the chronological future of p, the other in its chronological past and they are connected by a timelike curve that **does not intersect** p.

10. **Cauchy surface**: A Cauchy surface is a closed achronal set whose domain of dependence is the entire manifold $\mathcal{M}$. Cauchy surfaces do not have an edge.

From this definition of a Cauchy surface, it is clear that they contain enough information to make predictions about the entire manifold as we claimed. The hypersurfaces used in the ADM formalism are Cauchy surfaces. Additionally, since a Cauchy surface is an achronal set, there is no causal curve that intersects two points on such a set. Therefore, closed causal (timelike or null) curves are impossible in a spacetime described by the ADM formalism. We give one final definition before concluding this section:

Globally hyperbolic spacetime: Any spacetime that has a Cauchy surface as a submanifold is said to be globally hyperbolic.

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