Studying Nonlinear Dynamics with a Magnetic Pendulum

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Abstract

The transition from linearity to non-linearity was studied using a simple pendulum by gradually increasing the amplitude of oscillation and observing the effect on the time period. A magnet was then introduced some distance from the original mean position such that it repelled another magnet placed at the tip of the pendulum. Consequently, Chaos was studied by plotting phase portraits of the system for different distances between the magnets using PhysLogger.

1 Introduction

The simple pendulum has been widely used as a pedagogical tool to demonstrate simple harmonic motion. In its simplest form it is studied in the small angle regime, where Newton's equation of motion reduces to a linear differential equation in the angle θ. Modifications such as the addition of a damping term β due to friction or some time variable driving force $F(t)$ add further complexity to the system making it more interesting to study even in the small angle regime.

However "non-linear" arrangements of the simple pendulum have also been studied aplenty in the literature. An example of such cases is a situation where the initial amplitude is increased to a point where the small angle approximation breaks down and the equation of motion becomes inherently non-linear. Such non-linearity gives way to very interesting properties, one of which is Chaos (which we formally define below). The magnetic pendulum is one such example of a non-linear system that displays Chaotic behaviour.

2 Theory

2.1 The small angle approximation

The simple pendulum is described by the scenario shown in Figure 1.

Assuming there is no damping and solving Newton's force equation for the pendulum one obtains:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} sin(\theta) \tag{1}$$

If one makes the assumption that the angle θ is very small, the following approximation can be made:

$$sin(\theta) \approx \theta,$$

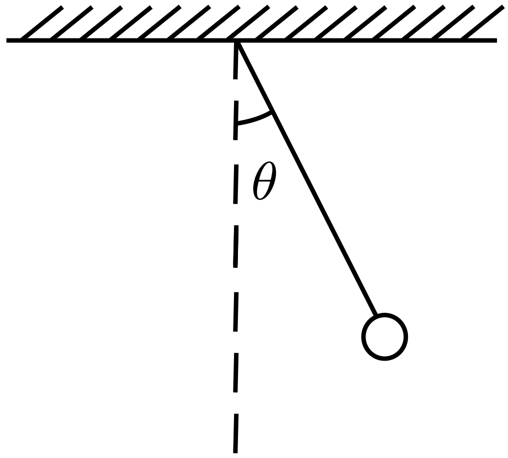

Figure 1: The simple pendulum. The variable θ measures the deviation from the equilibrium position

Where we considered the Taylor expansion of $sin(\theta)$ only upto linear order. This converts the non-linear equation we just wrote into a second order linear differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \tag{2}$$

which ofcourse admits oscillatory solutions of the form $Asin(\omega x) + Bcos(\omega x)$ with the parameter ω being $\sqrt{\frac{g}{L}}$.

The solutions to **this** differential equation have a time period independent of the initial amplitude. Indeed, solving (2) for any initial condition results in a sinusoidal solution (with some phase shift determined by the initial condition) which has a constant frequency determined by ω.

2.2 Introducing non-linearity

As one can imagine the small angle approximation is quite restrictive. If the entire range of possible angles is considered, one is faced with solving the non-linear differential equation (1), whose solutions in fact do not have a time period independent of the initial amplitude. Upon solving (1) we get the following formula for the time period:

$$T(\theta_0) = 4\sqrt{\frac{l}{g}}K(k) \tag{3}$$

where the variable K(k) is in fact an integral of the form:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2 sin^2\phi}} d\phi \tag{4}$$

which is a complete elliptic integral of the first kind. The only important point to note about this integral for our purposes is that it in fact depends on the amplitude, but the dependence is hidden in the variable k:

$$k = \sin\left(\frac{\theta_0}{2}\right) \tag{5}$$

where θ_o is the initial amplitude.

2.3 Phase portraits

When dealing with complicated differential equations, it is usually beneficial to consider the idea of the "Phase portrait" in order to understand the major qualitative features of the solutions without having to actually solve the equation (either analytically or numerically). The phase portrait, for our purposes, is a plot of the dynamical variable θ and its time derivative $\dot{\theta}$. Such a plot defines a family of curves or trajectories, each curve in the family representing a solution corresponding to a different set of initial conditions.

The phase portrait of a simple pendulum, for example, is a set of circles of increasing radii which eventually flatten as the small angle approximation breaks down. This is shown in figure 2.

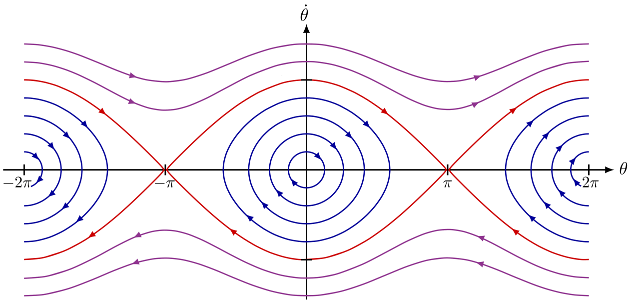

Figure 2: Phase portrait of a simple pendulum, notice how the solutions start off almost as circles of increasing radii before becoming oblong. This signifies the transition from nonlinearity to linearity

Finally note that the regions where the curves "cross-over" to the other side (called the separatrixes) correspond to the pendulum crossing the $\theta = \pi$ threshold and flipping over to the other side. All of these are examples of qualitative fatures of the solutions that can immediately be read off from the phase portrait.

2.3.1 The Magnetic Pendulum equation

Figure 3 shows the experimental setup for the magnetic pendulum:

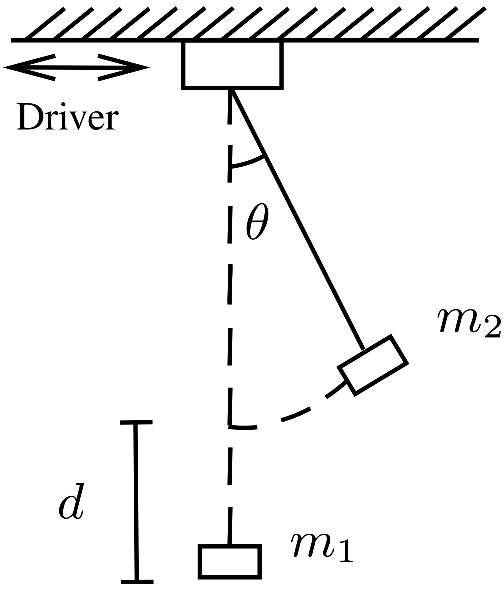

Figure 3: The magnetic pendulum. d signifies the distance between the magnets when the pendulum is at its equilibrium position. m_1 and m_2 are strengths of the magnetic dipoles of magnets 1 and 2 respectively. Notice how the top of the pendulum is connected to a driver which can move back and forth

The equation for this setup can understood more intuitively in terms of the sum of the torques acting on the pendulum. Namely, Newton's force equation turns into a "torque" equation of the form:

$$I\frac{d^2\theta}{dt^2} = \Sigma\tau_i$$

With I signifying the moment of inertia, which can be thought of as the angular analog of mass and $\frac{d^2\theta}{dt^2}$ signifying the angular acceleration. $\Sigma\tau_i$ runs over all of the torques acting on the system namely:

$$\Sigma\tau_i = \tau_{\text{gravity}} + \tau_{\text{driver}} + \tau_{\text{damping}} + \tau_{\text{magnetic}}$$

where the subscripts make the individual torques quite self-explanatory. If one puts in the expressions for these torques, the final equation becomes [1]:

$$\frac{ML^2}{3}\frac{d^2\theta}{dt^2} = -\frac{L}{2}Mgsin(\theta) + T_{\text{driver}}sin(\Omega t) - \gamma\frac{d\theta}{dt} + \frac{|\theta|}{\theta}L\frac{\mu_o m_1 m_2}{4\pi r_\theta^2}cos(|\theta| + tan^{-1}(-|\frac{h_\theta}{Lsin\theta}|))$$

Which is a manifestly non-linear second order differential equation. The most important terms in this equation can be interpreted physically as follows:

1. The first term on the right hand side is the original simple pendulum term.

2. In the second term T_{driver} is the maximum torque exerted by the driver. The total torque therefore varies sinusoidally with frequency Ω. One can imagine that if the first and the second sinusoidal terms are in sync, there is resonance.

3. The third term adds damping, with a damping factor of γ

4. The final term introduces an inverse square law which serves to quantify the magnetic force between the two magnets. m_1 and m_2 are the magnetic dipole moments of the two magnets and r_θ is the distance between them. Finally this force varies as the cosine of $|\theta|$ which makes sense since the magnitude of the force should be the same on either side of the equilibrium position

   Note that h_θ is the height of the lower end of the pendulum above the surface.

2.4 Chaos [2]

Chaos is a property possessed by certain deterministic non-linear systems which causes the dynamical solutions of such systems to be extremely sensitive to initial conditions. For a chaotic system, one cannot make accurate predictions beyond a certain point in time since the actual evolution starts to deviate quite drastically from the predicted solution.

2.4.1 Period Doubling and Halving Bifurcations

Chaotic systems usually exhibit a property whereby if the system has some stable orbit (i.e some non-divergent solution), then varying the parameters of the system slightly causes such stable trajectories to "bifurcate". That is, a new stable orbit emerges from the originally existing one. This new orbit can have either half or double the period of the original. These bifurcations are termed "Period Doubling" and "Period Halving" bifurcations.

Chaos emerges naturally in the Period doubling picture. This is because the "distances" between successive period doublings (i.e the amount by which the parameter needs to be varied to cause a bifurcation) decreases quite rapidly. This happens until the bifurcations converge to a single value of the parameter. It is beyond this point that the realm of Chaos begins. Figure 4 shows period doubling bifurcations for the discrete logistic map characterized by the equation:

$$x_{n+1} = rx_n(1 - x_n)$$

3 Running the simulation:

We first ran the simulation for the magnetic pendulum using the matlab script provided. The simulation solves equation (6) numerically. The equation is recast in the following form:

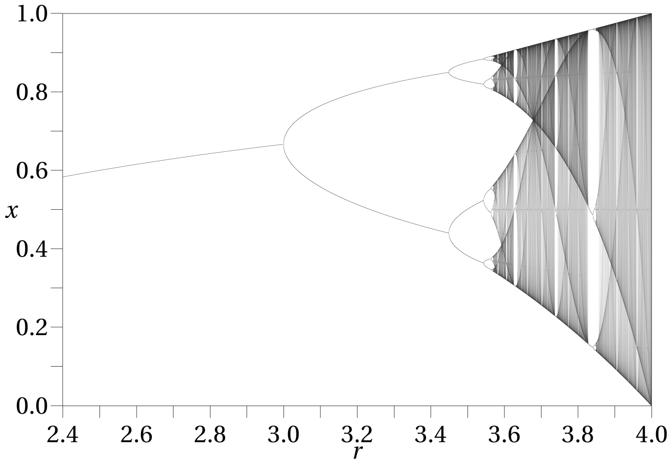

Figure 4: Period doubling bifurcations for the logistic map caused by variations in the parameter r. The shaded area corresponds to chaotic behaviour. This area can roughly be thought of as having "infinitely many stable orbits"

$$\begin{aligned}\frac{d^2\theta}{dt^2} =& - Asin(\theta) + Bsin(\Omega t) - C\frac{d\theta}{dt} \\ &+ \frac{|\theta|}{\theta} L\frac{E}{r_\theta^2} cos(|\theta| + tan^{-1}(-|\frac{h_\theta}{Lsin\theta}|))\end{aligned}$$

Here A, B, C and E are just some constants. We chose the following parameters for the simulation:

$$A = 110 \quad B = 0.01 \quad C = 0.001 \quad g = 9.8$$
$$L = \frac{3g}{2A} \quad E = 0.5$$

The plots for both the phase portraits as well as the time series data for different values of the distance d between the magnets are shown in figure 5. The general trend makes intuitive sense. Since the distance is decreasing downwards the first plot corresponds to a situation where the pendulum experiences almost no magnetic force at all, correspondingly the phase portrait is circular, which corresponds to an almost constant amplitude solution, the time series indeed shows this: θ is varying sinusoidally as a function of time. However at 10 cm as the magnets come closer, the phase portrait starts to get skewed in the horizontal directions. The magnetic is essentialy "spending more time" on either side of the equilibrium position. Indeed, this makes sense. The magnetic force is repulsive, so the magnet tends to stay away from the equilibrium position for longer. This trend continues and we see that at a distance of 9.7 cm, the phase portrait **almost** develops a separatrix. The magnetic force makes the "crossing over" from one side to another difficult. This trend eventually results in the extreme case at a distance of 9.5 cm, where the magnetic stops crossing over and instead stays to only one side of the equilibrium position (by reading off the x-axis we can see that the shape is no longer centered around 0 but is instead centered to the left of the original equilibrium position. In the last portrait we see that at a distance of 9.5 cm, the magnet actually ends up settling to the **right** side of the original equilibrium position instead of the left.

4 Experimental Setup

The pendulum was driven through a stepper motor controller (SMC) using slider-crank mechanism. As shown in figure 3 one of the magnets was attached to the tip of the pendulum while the other was placed on top of a vertically adjustable post. The poles of the magnets were such that they repelled each other.

All of the recordings were done in real time using PhysLogger. The angle readings were taken using PhysCompass. Additionally, the SMC was also connected to PhysLogger, this allowed us to change the frequency of the driver from within the PhysLogger application.

The PhysCompass was tared to zero at the mean position. This required removing the magnet on the post so that the repulsion did not interfere with the calibration process.

4.1 Dependence of time period on amplitude

We first conducted a fairly simple experiment to detect the dependence of the time period of a simple pendulum on the amplitude for large initial angles. In order to do so, the magnet was removed from the post. We then took readings of the time period for different angles using PhysLogger. The resulting data points were subjected to curve fitting. The curve obtained along with the data points and corresponding error bars is shown in Figure 6.

4.2 Non-linear Dynamics with the magnetic pendulum

4.3 Repulsive case

We put the magnet back on the post vertically below the pendulum's equilibrium position with the relative polarities adjusted such that the magnets repelled. Then, using the differentiation module in PhysLogger, we recorded the data needed to plot the phase portraits as well as the time series for different distances between the magnets. Since the Liveplots appeared noisy, we introduced a low-pass filter that "smoothened out" the noise. The plots obtained are shown in Figure 7.

4.4 Attractive case

We then switched the relative polarity of the magnets such that they attracted and then repeated the entire procedure from the previous section. The plots obtained are shown in Figure 6. It is evident from the phase portraits that the attractive case is physically quite uninteresting. As opposed to the repulsive case where bringing the magnets close together changed the form of the phase portrait , in the attractive case a decrease in distance only serves to decrease the amplitude of the oscillations more quickly. The remaining features of the solutions remain unaltered.

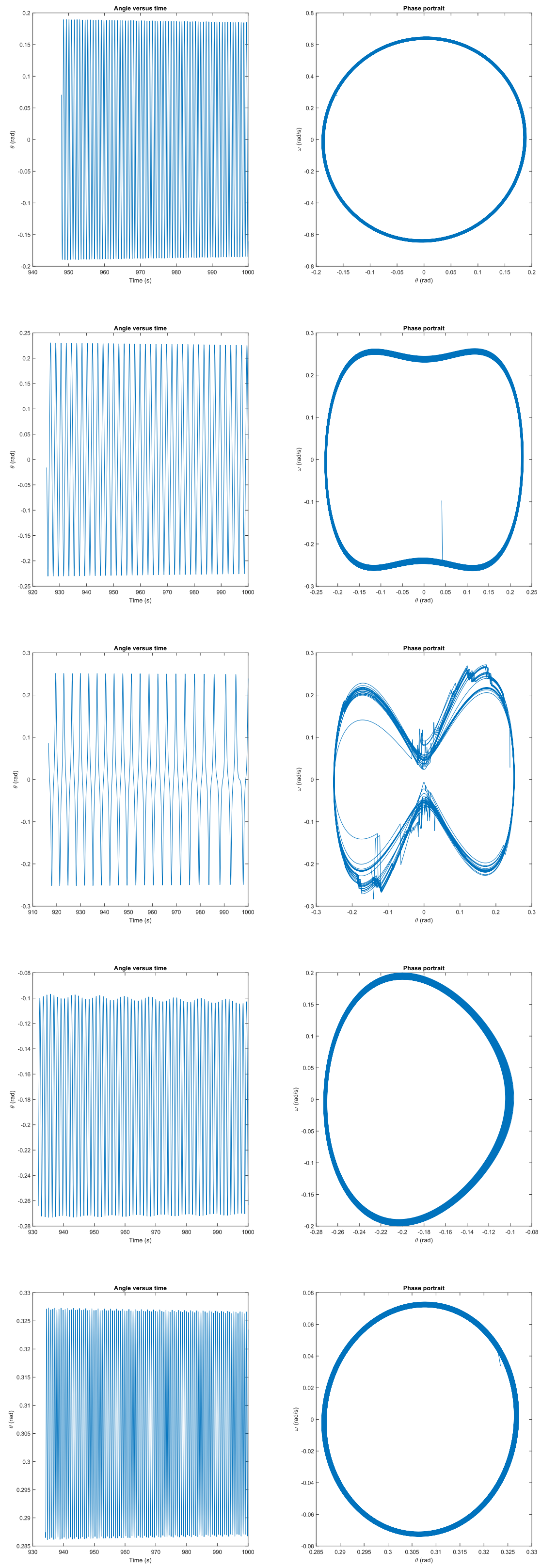

Figure 5: Time series data (left column) and phase portraits (right column) of the MATLAB simulation of the magnetic pendulum for different distances between the magnets. The distance is **decreasing** downwards. The plots are for d = 15, 10, 9.7, 9.5 and 8.5 cm

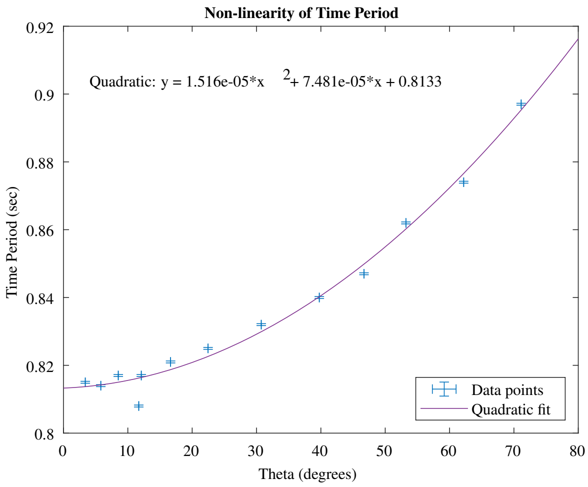

Figure 6: A quadratic best fit curve for the non-linear dependence of Time period on the amplitude θ_o. The blue points are the readings along with their corresponding error bars

5 Results and Discussion

In the repulsive case we noticed the same general trend as seen in the simulations. The phase portrait at the largest distance i.e 10 cm is almost exactly similar to that of a simple pendulum. Then as the distance is decreased, the trajectory spreads out due to the repulsion encountered close to the equilibrium position. A separatrix is once more observed at a distance of about 4 cm. The time series data is also interesting to note here. For distances of both 4 and 2 cm, the pendulum initially oscillates about the mean position, however as the damping kicks in, the effect of the repulsion becomes more pronounced, causing the magnet to starts oscillating on one side of the equilibrium position. This is characterized by the time series plot staying above zero beyond a certain point. In these cases the pendulum is oscillating to the right of the original equilibrium position.

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Finally we note that there are inherent uncertainties in the plots. These may have been caused by:

1. The sampling rate chosen in PhysLogger
2. The uncertainties in the θ readings from PhysCompass
3. The uncertainties and errors associated with numerical differentiation
4. The data lost due to the low pass filter

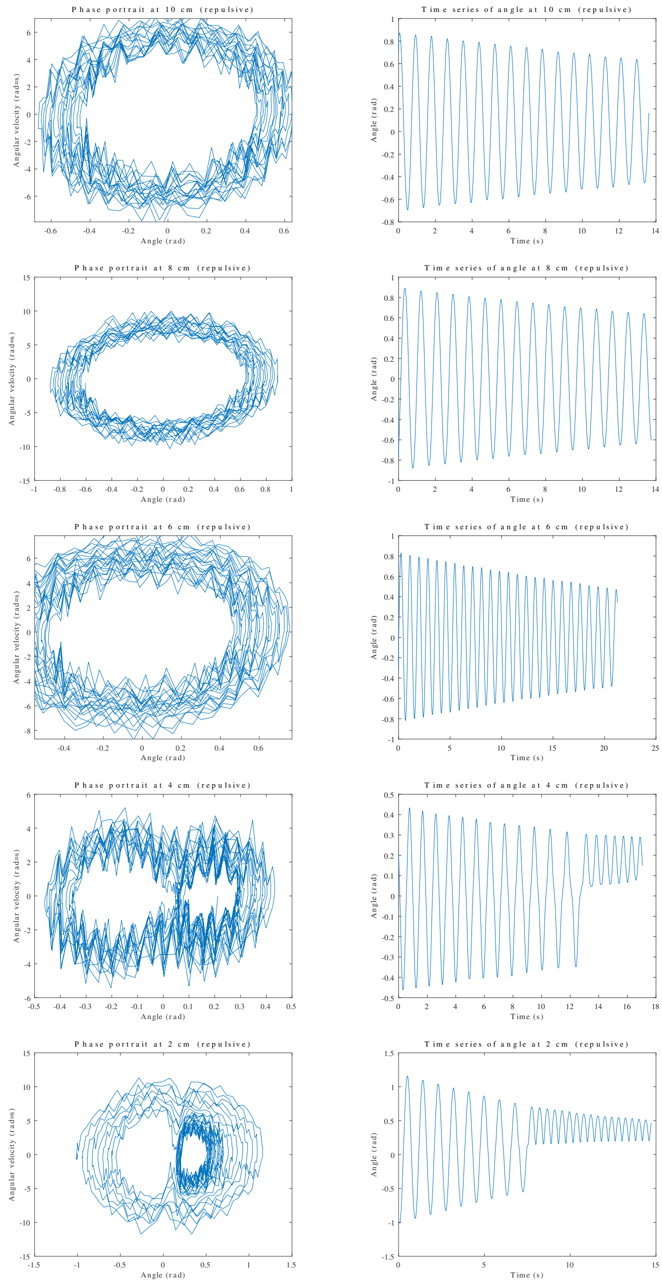

Figure 7: Plots of the phase portraits (left column) and the time series data (right column) for the magnetic pendulum determined experimentally for the repulsive case. The data was obtained through PhysLogger and plotted in MATLAB. The distance is decreasing downwards. The distances used were d = 2, 4, 6, 8, and 10 cm. Angles measured in radians and angular velocity in radians per second.

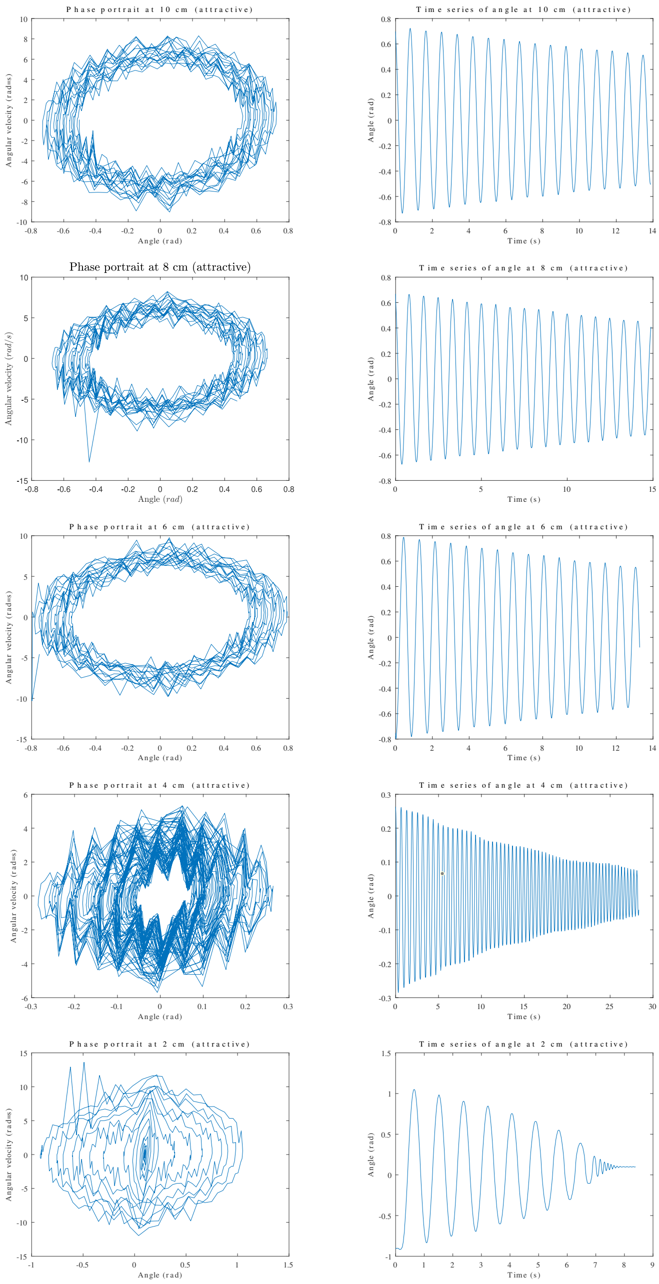

Figure 8: Plots of the phase portraits (left column) and the time series data (right column) for the magnetic pendulum determined experimentally for the attractive case. The data was obtained through PhysLogger and plotted in MATLAB. The distance is decreasing downwards. The distances used were d = 2, 4, 6, 8, and 10 cm. Angles measured in radians and angular velocity in radians per second.

References

[1] Jeffrey Patterson Azad Siahmakoun, Valentina A. French. In *Nonlinear dynamics of a sinusoidally driven pendulum in a repulsive magnetic field*, 1997.

[2] Josh Bevivino. In *The Path From the Simple Pendulum to Chaos*, 2009.