1 (2.1.1 → 2.1.3)

$$\dot{x} = \sin x$$

2.1.1 : Fixed points x_0 where $f(x_0) = \sin(x_0) = 0$ ✓ 2

$\Leftrightarrow x_0 = \arcsin(0)$

$= k\pi$ for $k \in \mathbb{Z}$

2.1.2 : Greatest velocity to the right means where $\dot{x}$ is greatest. ✓ 1

⇒ we know $\dot{x} = \sin x$ is greatest at $\dot{x} = 1$

⇒ $x = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$ is where

x has greatest velocity to the right

2.1.3 : a) $\ddot{x} = \frac{d\dot{x}}{dt} = \cos(x)\dot{x}$ ✓

$= \sin(x)\cos(x)$

b) we find max of $\ddot{x}$ is where

$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ maximum @ $\ddot{x} = \frac{1}{2}$ ✓ 2

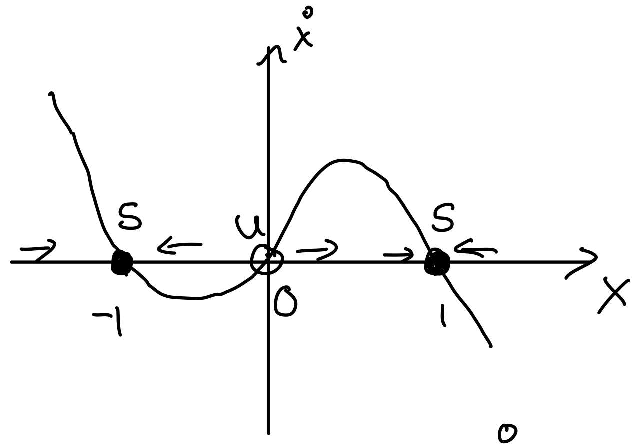
⇒ Find x so that $\sin(2x)$ reaches maximum @ $\sin(2x) = 1$

⇒ $x = k\pi + \frac{\pi}{4}$ for $k \in \mathbb{Z}$ is where acceleration is largest

② (2.2.3, 2.2.7)

2.2.3 : $\dot{x} = x - x^3$

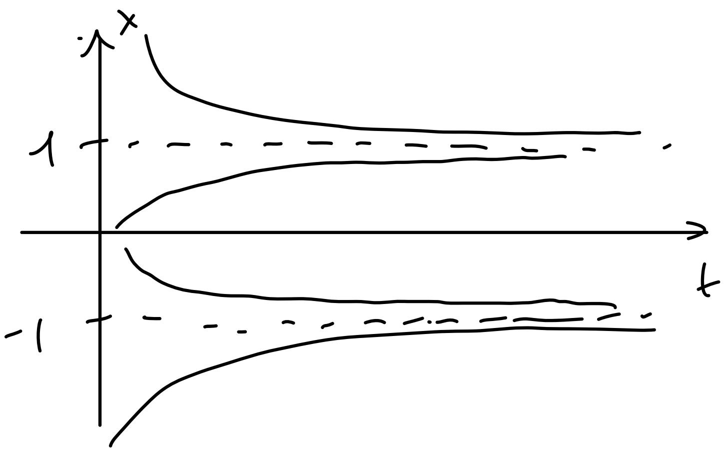
FP: $x = -1; 0; 1$



$\dot{x} = x - x^3 \rightsquigarrow \frac{\dot{x}}{x - x^3} = 1$ (separable)

$\to -\frac{\ln(-x^2+1)}{2} + \ln(x)$

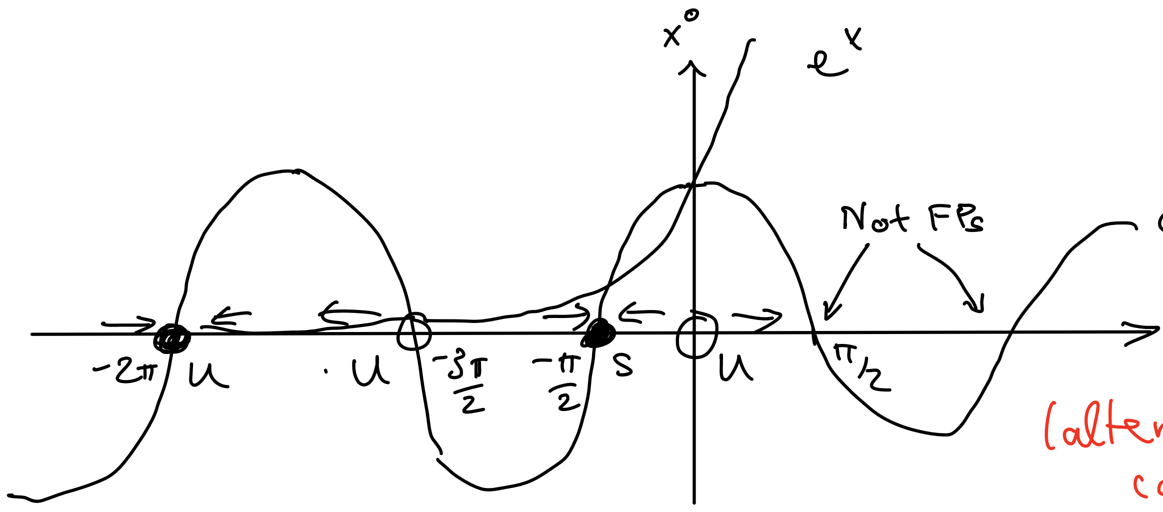
$\to x = \pm\sqrt{\frac{-e^{2t}}{c_1 - e^{2t}}}$



5

2.2.7 : $\dot{x} = e^x - \cos x$

As suggested, we'll sketch e^x, $\cos(x)$



✓

(alternating FP type is correct)

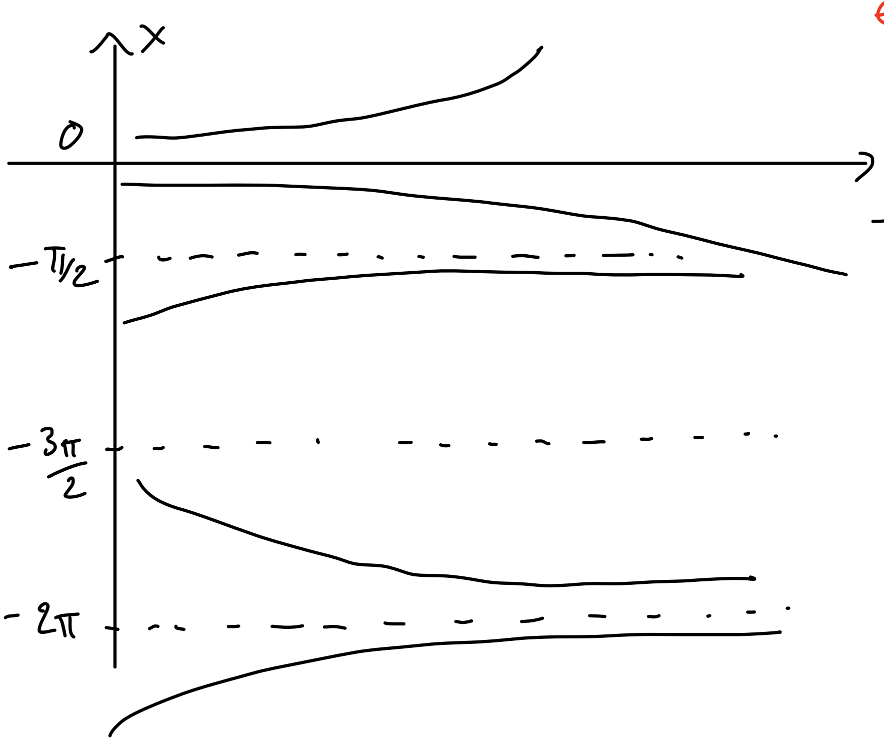
We can approximate the FP at the graphs' intersections and determine their vector field by compare the 2 graphs near the fixed points, categorizing the fixed points.

Could be more specific on comparing 2 graphs :

e^x above $\cos x$ from the right etc

$\Rightarrow$ unstable

$\dot{x} = e^x - \cos(x)$



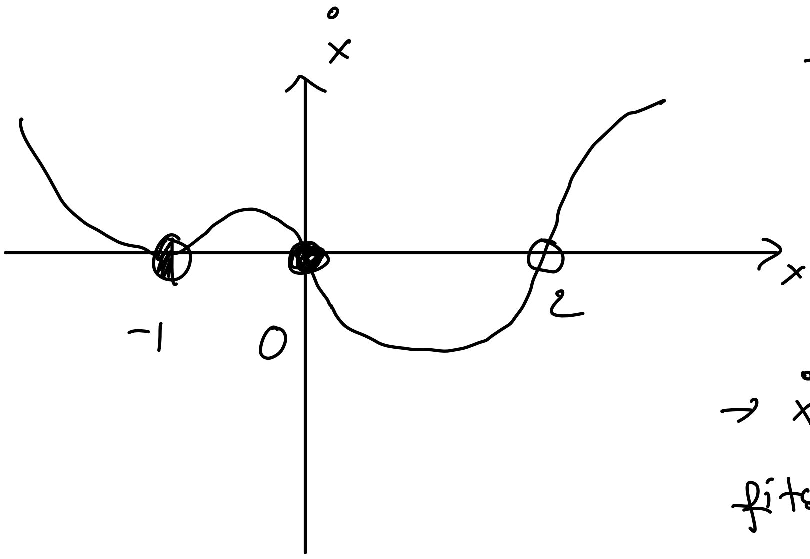
$\rightarrow -0.5$

(Can't find $x(t)$)

4.5

③ (2.2.8, 2.2.9)

2.2.8: We see for such phase portrait we need $\dot{x}$ that has the sketch



→ It has double root at $x = -1$

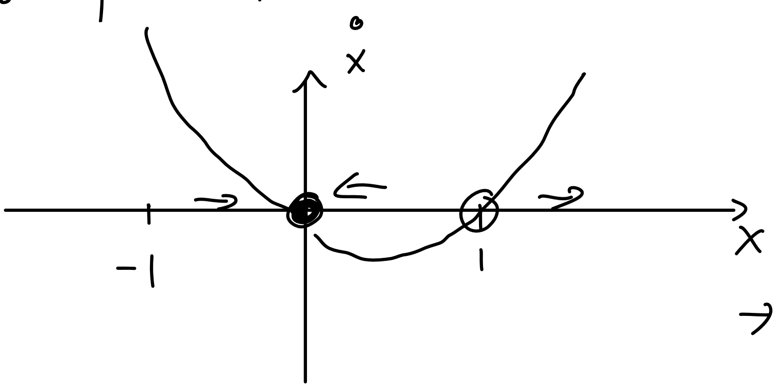
roots at $x = 0$ and $x = 2$

→ $\dot{x} = x(x-2)(x+1)^2$ ✓

fits such phase portrait

⑤

2.2.9: We see $\dot{x}$ with such graph $x(t)$ must have phase portrait



with root at

$x = 0$ and $x = 1$

→ $\dot{x} = x(x-1)$ ✓

fits such solution.

(The type of FP in the phase portrait in the given answer is wrong but the function is identical to this one so I assume there's no dispute)

⑤

④. 2.3.1 b) $\dot{N} = rN(1 - N/K)$

$$x = \frac{1}{N} \Rightarrow N = \frac{1}{x} \Rightarrow \dot{N} = \frac{-\dot{x}}{x^2} = rN(1 - N/K)$$

$$\checkmark = \frac{r}{x}\left(1 - \frac{1}{xK}\right)$$

$$\Rightarrow -\dot{x} = rx\left(1 - \frac{1}{xK}\right)$$

$$\checkmark$$

$$= rx - \frac{rx}{xK} = rx - \frac{r}{K}$$

$$\Rightarrow \dot{x} = \frac{r}{K} - rx$$

$$\Rightarrow x = \frac{1}{N} = Ce^{-rt} + \frac{1}{K}$$

$$\Rightarrow N = \frac{1}{Ce^{-rt} + \frac{1}{K}} \quad \checkmark$$

We have initial condition N_0:

$$\Rightarrow \frac{1}{Ce^{-r0} + \frac{1}{K}} = N_0$$

$$\Rightarrow \frac{1}{C + \frac{1}{K}} = N_0 \Rightarrow C = \frac{1}{N_0} - \frac{1}{K} \quad \times$$

$$\cancel{C = \frac{1}{N_0 - K}} \quad \times$$

$$\Rightarrow N(t) = \frac{1}{\frac{e^{-rt}}{N_0 - K} + \frac{1}{K}}$$

(I used a different method

(the answers are equivalent until C ⇒ I'll subtract 2pts

⇒ 3

5 (24.3, 2.4,7)

2.4.3 : $\dot{x} = \tan x$, FPs : $x_0 = k\pi$ with $k \in \mathbb{Z}$

$\rightarrow \ddot{x} = \sec^2 x$

3

$\ddot{x}(k\pi) = \sec^2(k\pi) = 1 \Rightarrow$ ~~Stable~~ ?? FPs

unstable (typo? −2)

2.4.7 : $\dot{x} = ax - x^3$, FPs : $\dot{x} = x(a - x^2) = x(\sqrt{a} - x)(\sqrt{a} + x)$

$\ddot{x} = a - 3x^2$ $\rightarrow$ FPs : $x = 0,\ x = \sqrt{a},\ x = -\sqrt{a}$

$\boxed{a < 0}$: $\ddot{x}(0) = a \rightarrow$ stable ✓

$\sqrt{a}$ doesn't exist in $\mathbb{R}$.

$\boxed{a = 0}$ $\ddot{x}(0) = a = 0 \Rightarrow$ stable

$\ddot{x}(-\sqrt{a}) = \ddot{x}(\sqrt{a}) = 0 \Rightarrow$ stable

$\}$ =0 is inconclusive, need graph. Correct answ −1.5

$\boxed{a > 0}$ $\ddot{x}(0) = a \Rightarrow$ unstable

$\ddot{x}(-\sqrt{a}) = -2a < 0 \Rightarrow$ stable ✓

$\ddot{x}(\sqrt{a}) = -2a < 0 \Rightarrow$ stable

3.5

6 2.5.1

$\dot{x} = -x^c$, $c \in \mathbb{R}$

a) $x = 0$ stable and $x \geq 0 \Rightarrow x = 0$ stable from the right side

$\dot{x} < 0 \Rightarrow -x^c < 0$

1.75

So we see $\forall x > 0$, $\forall c \in \mathbb{R}$, $-x^c < 0$. Thus all such values c makes $x = 0$ stable (Need $c > 0$) −0.25

b) $\dot{x} = -x^c \Rightarrow \dfrac{dx}{-x^c} = dt$ (separable)

$\Rightarrow \int \frac{1}{-x^c} dx = \int dt \Rightarrow \dfrac{x^{-(c-1)}}{c-1} = t + C$

Let the initial Condition be $x_0 = x(0)$

$$\Rightarrow \quad C = \frac{x_0^{-c+1}}{c-1}$$

$$\Rightarrow \quad \frac{x^{-c+1}}{c-1} - \frac{x_0^{-c+1}}{c-1} = t$$

Since $x_0 = 1$

$$\Rightarrow \quad \frac{x^{-c+1}}{c-1} - \frac{1}{c-1} = t$$

$$\Rightarrow \quad t = 0 = \frac{x^{-c+1} - 1}{c-1} \Rightarrow x = 1 \ ?$$

at $x = 0$ we have

$$t = \frac{0 - 1}{c-1} = \frac{1}{1-c}$$

But $t \geq 0 \Rightarrow$ c must be less than 1

$\Rightarrow$ The time for $x=1$ to $x=0$ is $t = \dfrac{1}{1-c}$ for $c < 1$

?

3

⑦ 2.8.2

$\dot{x} = 1 + x^{10}$. We know that for $\dot{x}' = 1 + x^2$ blow up to $+\infty$ in finite time. We consider all possible initial condition x_0 ← Need to prove. (-1)

Since if $x < 0$, $x^{10} > 0 = (-x)^{10}$ ⇒ we only need to consider $|x_0|$. ✓

$\boxed{|x_0| > 1}$: for $|x_0| > 1$, $x^{10} > x^2$ and x^2 blows up in finite time ⇒ $\dot{x}$ blows up in finite time

$\boxed{|x_0| \leq 1}$: Let $\dot{x}_0 = \dot{x}(0)$. Even if $|x_0| \leq 1$, ✓

$\dot{x} = 1 + x^{10}$ ⇒ $\dot{x}_0$ will alway be ≥ 1.
and thus will go to $+\infty$ in finite time by comparison with x^2.

4

⑧ 2.6.1

There's actually no paradox. The oscillation of the system happens in 1 dimension, but doesn't mean it is a 1-dimension system. $m\ddot{x}$ is actually a 2-dimensions system, the 2 dimensions being x and $\dot{x}$ since $m\ddot{x}$ is a 2nd order ODE, while 1-dimensional system reffered in the text is a 1st order ODE system. Thus, no contradiction. ✓

5

(9) 2.7.1

$$\dot{x} = x(1-x) \qquad (x^* = 0, 1)$$

$$= -V'(x) = x - x^2$$

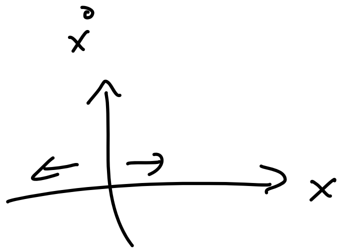
$$\Rightarrow V(x) = -\left(\frac{x^2}{2} - \frac{x^3}{3}\right)$$

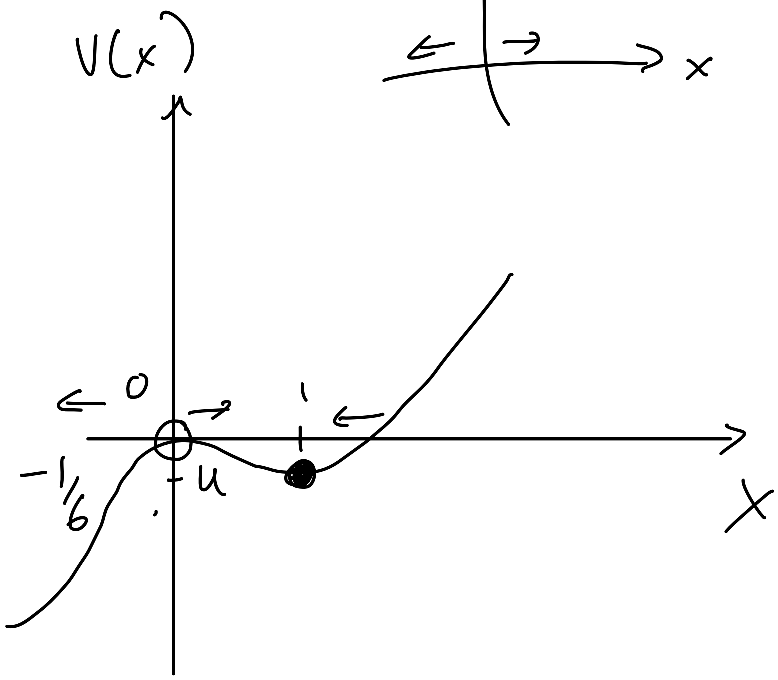
$$= -\frac{x^2}{2} + \frac{x^3}{3} \quad \Rightarrow \quad V(0) = 0$$

$$V(1) = \frac{-1}{2} + \frac{1}{3}$$

$$= \frac{-1}{6}$$

We have :





5

10 2.8.3 $\dot{x} = -x$, $x(0) = 1$

a) $\dot{x} = -x$

$\Rightarrow x = e^{-t} + C$, sub in $x(0) = 1$

$C = 1 - 1 = 0$

$\Rightarrow x = e^{-t}$

$\Rightarrow x(1) = e^{-1} = \frac{1}{e}$ ✓

2

b) Use $x_n = x_{n-1} + \Delta t(-x(t_{n-1}))$

$\Delta t = 10^{-1}$: $x_1 = 1 - 1/10 = 9/10$

$x_2 = \frac{9}{10} + \frac{1}{10}\left(\frac{-9}{10}\right) = 0.81$

$x_3 = 0.81 + \frac{1}{10}(-0.81) = 0.729$

$x_4 = 0.729 + \frac{1}{10}(-0.729) = 0.6561$

$x_5 = 0.6561 + \frac{1}{10}(-0.6561) = 059049$

$\vdots$

$\Rightarrow x_{10} = \hat{x}(1) = 0.3486784401$ ✓

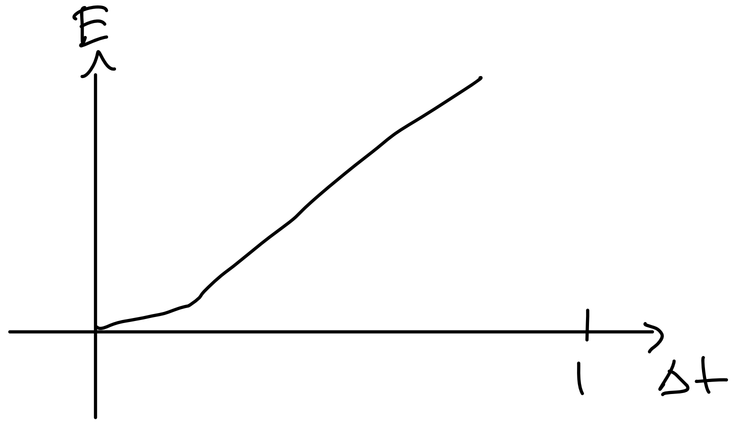
$\Delta t = 10^{-2} \Rightarrow \hat{x}(1) = 0.366032.3413$ ✓

$\Delta t = 10^{-3} \Rightarrow \hat{x}(1) = 0.3676954248$

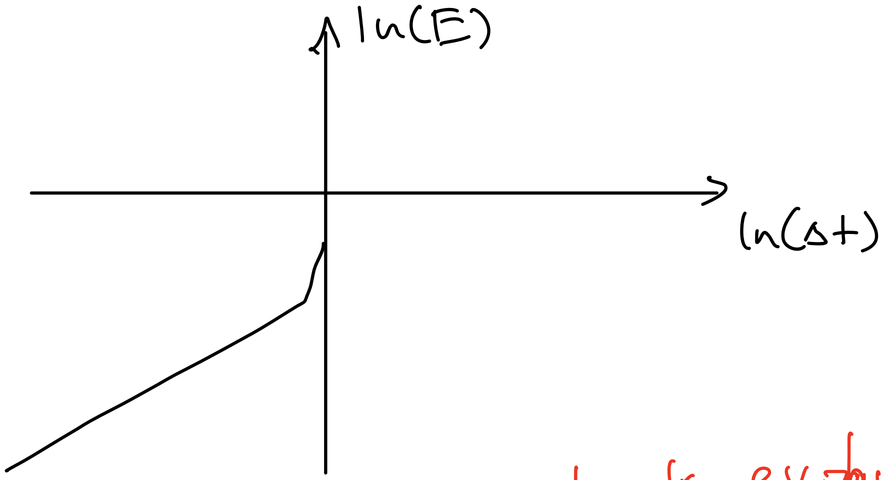
$\Delta t = 10^{-4} \Rightarrow \hat{x}(1) = 0.3678610464$

5

c)



This is because since the analytical answer is $\frac{1}{e}$ this proportion makes sense.



Lack explanation ??

1.5

Graphs are correct

$70 - 8.75 = \frac{61.25}{70}$