The Basic Physics of the Binary Blackhole Merger GW150914

PH821 Project Report

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Abstract

The first direct gravitational-wave detection was made by the Advanced Laser Interferometer Gravitational Wave Observatory on September 14, 2015. The GW150914 signal was strong enough to be apparent, without using any waveform model, in the filtered detector strain data. Here, features of the signal visible in the data are analyzed using concepts from Newtonian physics and general relativity, accessible to anyone with a general physics background. The simple analysis presented here is consistent with the fully general-relativistic analyses published elsewhere, in showing that the signal was produced by the inspiral and subsequent merger of two black holes. The black holes were each of approximately $35 M_{\odot}$, still orbited each other as close as ~ 350 km apart and subsequently merged to form a single black hole. Similar reasoning, directly from the data, is used to roughly estimate how far these black holes were from the Earth, and the energy that they radiated in gravitational waves.

Contents

1 Introduction

Advanced LIGO made the first observation of a gravitational wave (GW) signal, GW150914, on September 14th, 2015, a successful confirmation of a prediction by Einstein's theory of general relativity (GR). The signal was clearly seen by the two LIGO detectors located in Hanford, WA and Livingston, LA.

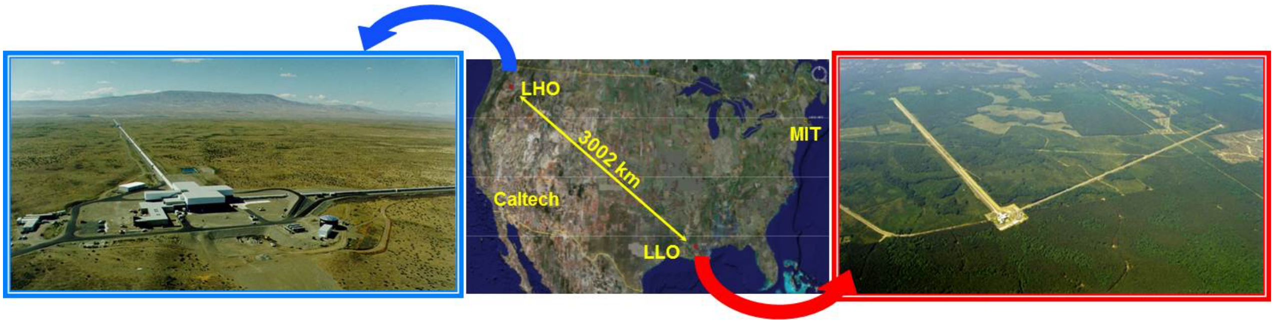

Figure 1: LIGO Hanford and LIGO Livingston that detected GW150914. Image sourced from Caltech/MIT/LIGO Lab

Extracting the full information about the source of the signal requires detailed analytical and computational methods. However, we can infer a lot about the source by direct inspection of the detector data and some basic physics. Here, the result that GW150914 was emitted by the inspiral and merger of two black holes follows from:

- the strain data visible at the instrument output
- dimensional and scaling arguments
- primarily Newtonian orbital dynamics and
- the Einstein quadrupole formula for the luminosity of a gravitational wave source.

2 Analysis

Our starting point is shown below: the instrumentally observed strain data $h(t)$, after applying a band-pass filter to the LIGO sensitive frequency band (35-350 Hz), and a band-reject filter around known instrumental noise frequencies.

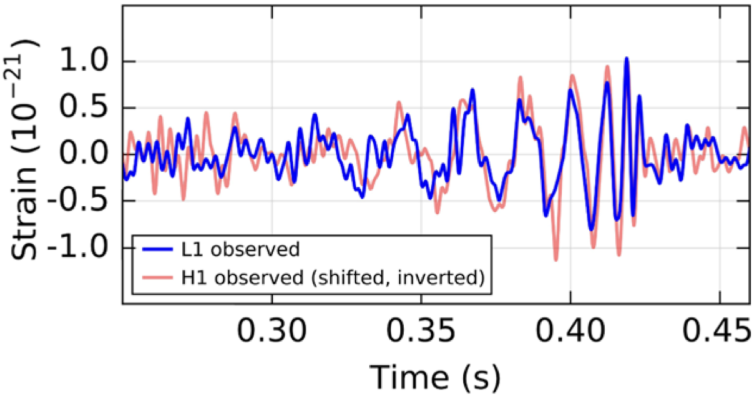

Figure 2: The instrumental strain data in the Livingston detector (blue) and Hanford detector (red). Both have been bandpass- and notch-filtered. The Hanford strain has been shifted back in time by 6.9 ms and inverted. [1]

The signal is dominated by several cycles of a wave pattern whose amplitude is initially increasing, starting from around the time mark 0.30 s. In this region the gravitational wave period is decreasing, thus the frequency is increasing. After a time around 0.42 s, the amplitude drops rapidly, and the frequency appears to stabilize. The last clearly visible cycles indicate that the final instantaneous frequency is above 200 Hz. The entire visible part of the signal lasts for around 0.15 s.

We know that accelerating masses produce GWs. Since the waveform clearly shows at least eight oscillations, we know that a mass or masses were oscillating. Also notice that frequency and amplitude increase initially. So this can't be due to a perturbed system returning to stable oscillations since oscillations around equilibrium are characterized by roughly constant frequencies and decaying amplitudes. Thus the data indicates that the masses must be inspiraling.

The time-frequency behavior of the signal is depicted below.

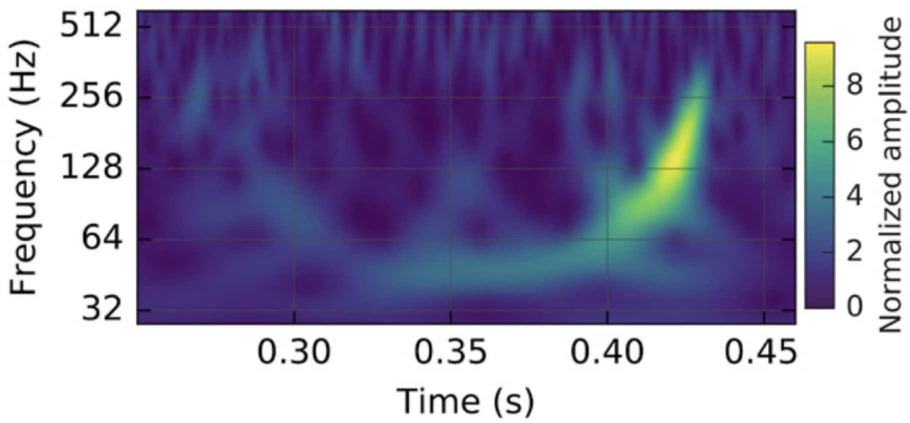

Figure 3: A representation of the strain-data as a time-frequency plot, where the increase in signal frequency ("chirp") can be traced over time [1]

Here we can see that the amplitude peaks at around the 150 Hz mark, and the final frequency goes to about 250 Hz. In order to analyse this data, we will use Newtonian Dynamics. Orbital motion is approximated by Newtonian Mechanics to a high precision for sufficiently large separations and for $v << c$. Note that we will be using ND until quite later in the orbital motion stage, but we will revisit this assumption under a later point of time.

2.1 Newtonian Calculation

Let's outline the calculations for a binary system of two masses m_1 and m_2

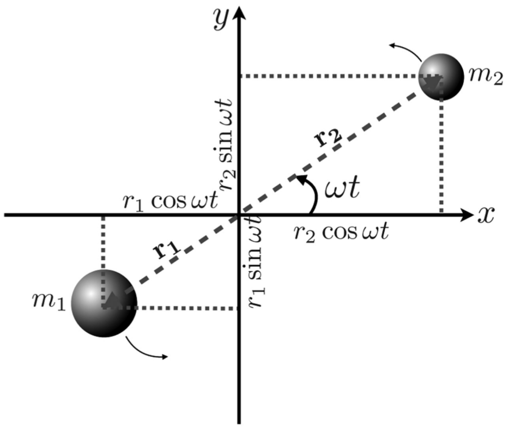

Figure 4: A two-body system, m_1 and m_2 orbiting in the xy-plane around their C.O.M. [1]

Compute the quadrupole moment Q_{ij}.

We use a Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ whose origin is the center-of-mass, with r the radial distance from the origin. $\delta_{ij} = \text{diag}(1, 1, 1)$ is the Kronecker-delta and $\rho(\mathbf{x})$ denotes the mass density. For the two masses $A \in \{1, 2\}$ in the xy-plane.

$$\begin{aligned} Q_{ij} &= \int \mathrm{d}^3 x \rho(\mathbf{x}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right) \\ &= \sum_{A \in \{1,2\}} m_A \begin{pmatrix} \frac{2}{3}x_A^2 - \frac{1}{3}y_A^2 & x_A y_A & 0 \\ x_A y_A & \frac{2}{3}y_A^2 - \frac{1}{3}x_A^2 & 0 \\ 0 & 0 & -\frac{1}{3}r_A^2 \end{pmatrix}, \end{aligned} \tag{1}$$

Assume that the orbit is circular, so at separation $r = r_1 + r_2$ and frequency $f = \frac{\omega}{2\pi}$

$$\begin{aligned} Q_{ij}^A(t) =& \frac{m_A r_A^2}{2} I_{ij} \\ =& \frac{m_A r_A^2}{2} \begin{pmatrix} \cos(2\omega t) + \frac{1}{3} & \sin(2\omega t) & 0 \\ \sin(2\omega t) & \frac{1}{3} - \cos(2\omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \end{aligned} \tag{2}$$

Doing the summation, we find $Q_{ij}(t) = \frac{1}{2}\mu r^2 I_{ij}$

Now, the gravitational wave strain h at a (luminosity) distance d_L from a system whose traceless mass quadrupole moment is Q_{ij} is

$$h_{ij} = \frac{2G}{c^4 d_L} \frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}$$

The rate at which energy is carried away by these gravitational waves is given by the quadrupole formula

$$\begin{aligned} \frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}t} &= \frac{c^3}{16\pi G} \iint |\dot{h}|^2 \,\mathrm{d}S = \frac{1}{5}\frac{G}{c^5} \sum_{i,j=1}^{3} \frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} \frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}, \\ \text{where } |\dot{h}|^2 &= \sum_{i,j=1}^{3} \frac{\mathrm{d}h_{ij}}{\mathrm{d}t} \frac{\mathrm{d}h_{ij}}{\mathrm{d}t} \end{aligned} \tag{3}$$

The integral is over a sphere at radius d_L (contributing a factor $4\pi d_L^2$), and the quantity on the right-hand side must be averaged over one orbit. In our case, Eq. 3 gives the rate of loss of orbital energy to gravitational waves, when the $v << c$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{GW}} = \frac{32}{5} \frac{G}{c^5} \mu^2 r^4 \omega^6 \tag{4}$$

This energy loss drains the orbital energy

$$E_{\mathrm{orb}} = -\frac{GM\mu}{2r} \tag{5}$$

thus $\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{orb}} = \frac{GM\mu}{2r^2}\dot{r} = -\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{GW}}$. We assume that the energy radiated away over each orbit is small compared to E_{orb} , in order to describe each orbit as approximately Keplerian.

Define:

$$\begin{aligned} &M = m_1 + m_2 : \text{ total mass} \\ &\mu = m_1 m_2 / M : \text{ reduced mass} \\ &q = m_1 / m_2 : \text{ mass ratio} \\ &\mathscr{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} : \text{ chirp mass} \end{aligned}$$

Now using Kepler's third law $r^3 = GM/\omega^2$ and its derivative $\dot{r} = -\frac{2}{3} r \dot{\omega}/\omega$ we can substitute for all the r 's and obtain

$$\dot{\omega}^3 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} G^5 \mu^3 M^2 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} (G\mathscr{M})^5 \tag{6}$$

having defined the chirp mass $\mathscr{M} = (\mu^3 M^2)^{1/5}$.

Because the quadrupole moment (defined above) is symmetric under rotations by π about the orbital axis, the radiation has a frequency twice that of the orbital frequency. So we can express Eq. 4.5 in terms of frequency of GWs ($\omega = \pi f_{GW}$)

$$\mathscr{M} = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \pi^{-8} \left(f_{\rm GW}\right)^{-11} \left(\dot{f}_{\rm GW}\right)^3\right)^{1/5} \tag{7}$$

The equation can be integrated to obtain

$$f_{\rm GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{G\mathscr{M}}{c^3}\right)^{5/3}(t_c - t) \tag{8}$$

We can also find the time-frequency behavior of the signal directly from the strain data in Fig. 2 by measuring the time differences Δt between successive zero-crossings and estimating $f_{\rm GW} = 1/(2\Delta t)$,

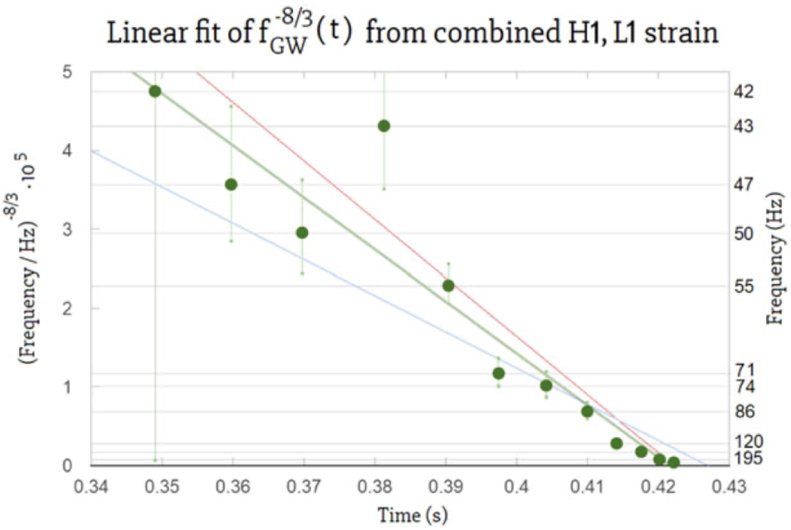

Figure 5: A linear fit (green) of $f_{\rm GW}^{-8/3}(t)$. The slope of this fitted line gives an estimate of the chirp mass of $\sim 37\mathrm{M}_\odot$. The blue and red lines indicate $\mathcal{M}$ of $30\mathrm{M}_\odot$ and $40\mathrm{M}_\odot$, respectively [1]

3 Evidence for compactness in the simplest case

For simplicity, suppose that the two bodies have equal masses, $m_1 = m_2$. The value of the chirp mass then implies that $m_1 = m_2 = 2^{1/5}\mathscr{M} = 35\mathrm{M}_\odot$, so that the total mass would be $M = m_1 + m_2 = 70\mathrm{M}_\odot$. We also assume for now that the objects are not spinning, and that their orbits remain Keplerian and essentially circular until the point of peak amplitude.

Around the time of peak amplitude the bodies therefore had an orbital separation given by $R = \left(\frac{GM}{\omega_{\rm Kep}^2\big|_{\rm max}}\right)^{1/3} = 350$ km

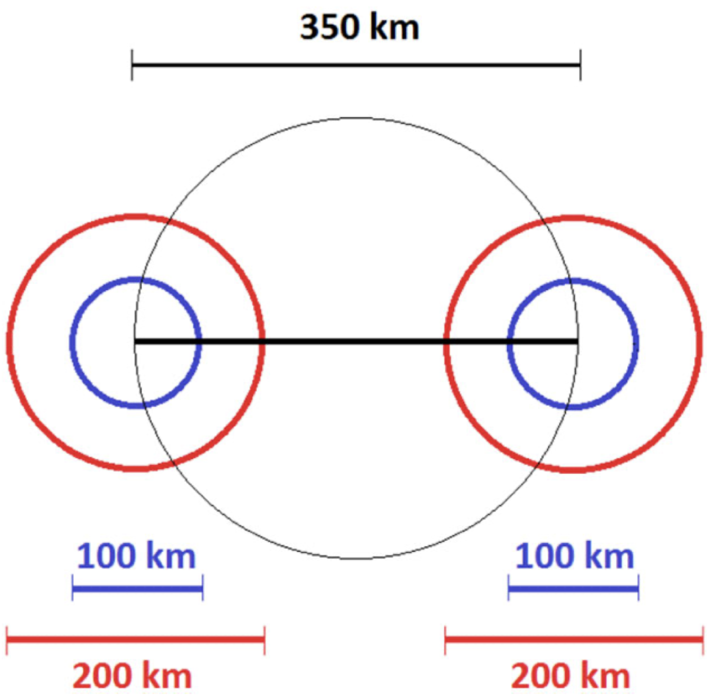

Figure 6: A demonstration of the scale of the orbit at minimal separation (black, 350 km) vs. the scale of the compact radii: Schwarzschild (red, diameter 200 km) and extremal Kerr (blue, di- ameter 100 km) [1]

We can compute the compactness ratio:

$$\mathscr{R} = \frac{\text{Orbital Separation}}{\text{Sum of smallest possible radii}} \tag{9}$$

The following table summarises $\mathscr{R}$ for various systems:

System	$\mathscr{R}$
Mercury and Sun	2×10^7
Cyg X-1 (Binary orbit of SBH)	3×10^5
HM Cancri (WD Binary system of highest ω_{orbit})	2×10^4
Sgr A* and S2 (Supermassive BH and star)	10^3
Two neutron stars just touching	2 to 5
GW150914	1.7

This indicates that GW150914 merger is much more compact compared to even two neutron stars which are almost touching, this indicates that what we are looking at is probably not a NS-NS merger. The fact that ND breaks down when $\mathscr{R} \sim 1$ also indicates that the two objects are highly compact.

4 Revisiting the assumptions

Our initial assumptions only took circular orbit, equal masses, non-spinning objects into consideration. Let us look at what happens when we deviate from these assumptions

4.1 Orbital Eccentricity

For $e > 0$ the minimum separation becomes the semi-major axis$(1-e)R$ instead. We thus see that the compactness bound imposed by eccentric orbits is even tighter (the compactness ratio $\mathscr{R}$ is smaller)

4.2 Unequal Masses

We express the component masses and total mass in terms of the chirp mass $\mathcal{M}$ and the mass ratio q, as $m_1 = \mathcal{M}(1+q)^{1/5}q^{2/5}, m_2 = \mathscr{M}(1+q)^{1/5}q^{-3/5}$, and

$$M = m_1 + m_2 = \mathscr{M}(1+q)^{6/5}q^{-3/5} \tag{10}$$

The compactness ratio $\mathscr{R}$ is the ratio of the orbital separation R to the sum of the Schwarzschild radii of the two component masses, $r_{\text{Schwarz}}(M) = r_{\text{Schwarz}}(m_1) + r_{\text{Schwarz}}(m_2)$, giving

$$\begin{aligned} \mathscr{R} &= \frac{R}{r_{\text{Schwarz}}(M)} = \frac{c^2}{2\left(\omega_{\text{Kep}}|_{\max} GM\right)^{2/3}} \\ &= \frac{c^2}{2\left(\pi f_{\text{GW}}|_{\max} G\mathscr{M}\right)^{2/3}} \frac{q^{2/5}}{(1+q)^{4/5}} \approx \frac{3.0q^{2/5}}{(1+q)^{4/5}}. \end{aligned} \tag{11}$$

The compactness ratio $\mathscr{R}$ also gets smaller with increasing mass-ratio, as that implies a higher total mass for the observed value of the Newtonian order chirp mass. Thus, for a given chirp mass and orbital frequency, a system composed of unequal masses is more compact than one composed of equal masses.

The following figure summarises the dependence of e and q on the compactness ratio

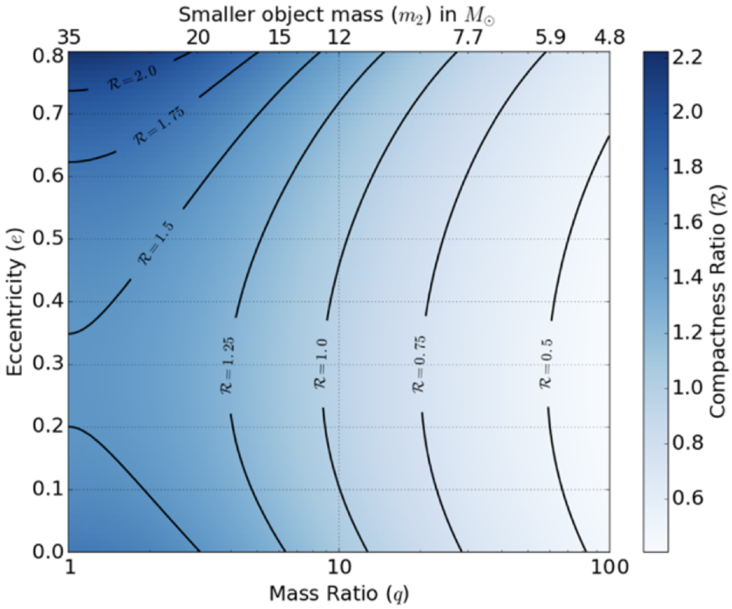

Figure 7: This figure shows the compactness ratio constraints imposed on the binary system by $\mathscr{M} = 30\mathrm{M}_\odot$ and $f_{\mathrm{GW}}|_{\max} = 150$ Hz. It plots the compactness ratio (the ratio of the separation between the two objects to the sum of their Schwarzschild radii) as a function of mass ratio and eccentricity from $e = 0$ to the very high (arbitrary) value of $e = 0.8$ [1]

We can put a constrain on the max q. For $q \sim 13, \mathscr{R} \sim 1$ giving the limit for the smaller mass as $m_2 \geq 11M_\odot$, which is much higher than neutron star masses.

4.3 Spins

Ror a mass m with spin angular momentum S we define the dimensionless spin parameter

$$\chi = \frac{c}{G}\frac{S}{m^2}.$$

Allowing the objects to have angular momentum (spin) pushes the limit of radius (Schwarzchild radius) down by a factor of two, to the radius of an extremal Kerr black hole (for which $\chi = 1$), $r_{\mathrm{EK}}(m) = \frac{1}{2}r_{\mathrm{Schwarz}}(m) = Gm/c^2$.

A lower limit on the Newtonian separation of two adjacent non-black hole bodies of total mass M is

$$\begin{aligned} r_{\mathrm{EK}}(m_1) + r_{\mathrm{EK}}(m_2) &= \frac{1}{2}r_{\mathrm{Schwarz}}(M) \\ &= \frac{GM}{c^2} \approx 1.5\left(\frac{M}{\mathrm{M}_\odot}\right)\mathrm{km} \end{aligned}$$

Thus,

$$
\begin{aligned}
\mathscr{R} = \frac{r_{\rm sep}(M)}{r_{\rm EK}(M)} \leq \frac{R(M)}{r_{\rm EK}(M)} &= \frac{c^2}{\left(GM\omega_{\rm Kep}\right)^{2/3}} \\
&\leq \frac{c^2}{\left(2^{6/5}G\mathscr{M}\omega_{\rm Kep}\right)^{2/3}} \\
&= \frac{c^2}{\left(2^{6/5}\pi G\mathscr{M}f_{\rm GW}|_{\rm max}\right)^{2/3}} \simeq 3.4,
\end{aligned}
\tag{12}
$$

The constrain $\mathscr{R} \sim 1$ also constrains $M_{max} \simeq 432M_{\odot}(\text{and}q_{max} \simeq 83)$ and $m_2 > 5M_{\odot}$. The conclusion is the same as in the equal-mass or non-spinning case: both objects must be black holes.

4.4 Post Newtonian Corrections

The validity of ND breaks when $v \to c$ or gravitational energy is comparable to rest mass energy. If we call the Post-Newtonian parameter $x = (v/c)^2 = GM/\left(c^2 r_{\rm sep}\right)$, then the corrections to Newtonian Dynamics can be expanded in powers of x. $x \sim (2\mathscr{R})^{-1}$ so the PN corrections are not very significant until the orbits are very close to the black hole

4.5 Chirp Mass

In GR, close to a merger allowed trajectories must be non-circular and plunge inwards and no circular orbits are allowed.

If $m_2 \leq 4.76M_{\odot}$ and if the same $\mathscr{M}$ (implying $q \geq 100$) were to be used, then such a high mass ratio suggests a treatment of the system as an extremal mass ratio inspiral (EMRI), where the smaller mass approximately follows a geodesic orbit around the larger mass ($m_1 \sim M$).

The orbital frequency $\omega_{\rm orb}$ as measured at infinity of a circular, equatorial orbit at radius r (in Boyer-Lindquist coordinates) is given by [2]

$$
\omega_{\rm orb} = \frac{\sqrt{GM}}{r^{3/2} + \chi(\sqrt{GM}/c)^3} = \frac{c^3}{GM}\left(\chi + \left(\frac{c^2 r}{GM}\right)^{3/2}\right)^{-1}.
$$

The maximum GW frequency from the final plunge $\sim$ frequency from the light ring. The distance of the light ring is given by

$$
r_{\rm LR} = \frac{2GM}{c^2}\left(1 + \cos\left(\frac{2}{3}\cos^{-1}(-\chi)\right)\right) \tag{13}
$$

which is $3GM/c^2$ for a Schwarzschild black hole, and GM/c^2 for a Kerr black hole. Eq. 4.5 then gives us maximal frequency as 67 Hz, whereas the observed is close to 250 Hz.

All this suggests that $\mathscr{M}$ has to be calculated by NR and also, q is not that high and the masses correspond to black holes.

5 Luminosity and Distance

Recall Eq. 3

$$\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}t} = \frac{c^3}{16\pi G}\iint |\dot{h}|^2 \,\mathrm{d}S \tag{14}$$

We can write

$$L \sim \frac{c^3 d_L^2}{4G}|\dot{h}|^2 \sim \frac{c^5}{4G}\left(\frac{\omega_{\mathrm{GW}} d_L h}{c}\right)^2 \tag{15}$$

Through dimensional analysis of the quadrupole formula 4, we can write luminosity $L \sim \frac{2}{5}\frac{G}{c^5}M^2 r^4 \omega^6$. For an object falling into a Schwarzschild black hole, $M \sim \frac{1}{6}c^2 r_{\mathrm{ISCO}}/G$ and $\omega r \sim 0.5c$. So we can write:

$$L \sim 0.2 \times 10^{-3} c^5/G \tag{16}$$

The Planck Luminosity (proposed as the upper limit on the luminosity of any physical system) would be

$$L_{\mathrm{Planck}} = c^5/G = 3.6 \times 10^{52}\ \mathrm{W} \tag{17}$$

Thus we have

$$\frac{L_{\mathrm{peak}}}{L_{\mathrm{Planck}}} \equiv \frac{L|_{\mathrm{max}}}{L_{\mathrm{Planck}}} \sim 0.2 \times 10^{-3} \sim \left(\frac{\omega_{\mathrm{GW}} d_L h|_{\mathrm{max}}}{c}\right)^2 \tag{18}$$

and we estimate the distance from the change of the measured strain in time over the cycle at peak amplitude, as

$$d_L \sim 45\mathrm{Gpc}\left(\frac{\mathrm{Hz}}{f_{\mathrm{GW}}|_{\mathrm{max}}}\right)\left(\frac{10^{-21}}{h|_{\mathrm{max}}}\right),$$

which for GW150914 gives $d_L \sim 300$Mpc. This distance corresponds to a redshift of $z \leq 0.1$, and so does not substantially affect any of the conclusions.

Using Eq. 5 we may also estimate the total energy radiated as gravitational waves during the system's evolution from a very large initial separation (where $E^{\mathrm{i}}_{\mathrm{orb}} \to 0$) down to a separation r. For GW150914, using $m_1 \sim m_2 \sim 35\mathrm{M}_\odot$ and $r \sim R = 350$ km (Eq. 9),

$$E_{\mathrm{GW}} = E^{\mathrm{i}}_{\mathrm{orb}} - E^{\mathrm{f}}_{\mathrm{orb}} = 0 - \left(-\frac{GM\mu}{2R}\right) \sim 3\mathrm{M}_\odot c^2.$$

This quantity should be considered an estimate for a lower bound on the total emitted energy (as some energy is emitted in the merger and ringdown).

6 Ringdown

After the merger, the BH horizon is very distorted and it begins to settle down into the final state of a Kerr BH. In the ringdown stage, the remaining perturbations start to linearize and the GW have characteristic quasi-normal-mode (QNMs). A detailed analysis of these modes reveals that the final mass is $\sim 65M_\odot$

7 Conclusions

Till now we have argued that the gravitational waves are due to inspiraling masses and that the chirp mass is around $30M_\odot$ through mainly Newtonian Dynamics and taking non-ideal assumptions into account too. We conclude that the masses must be blackholes and the mass ratio is not very high (by Newtonian Dynamics it is ~ 13). We cannot calculate q from the data because it only appears in PN calculations, and hence can only be calculated through them.

A full NR calculation [3] of the parameters of this merger reveal that the

- Final BH Mass $62^{+4}_{-3}M_\odot$
- Component masses are $35^{+5}_{-3}\mathrm{M}_\odot$ and $30^{+3}_{-4}\mathrm{M}_\odot$
- $d_L = 440^{+160}_{-180}\mathrm{Mpc}$

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