Deuxième partie :

Question préliminaire :

Cette équation s'appelle ainsi car elle décrit le déplacement d'une quantité de matière dans un espace.

Existence :

Soit u / $u(t,x) = u_0(x - \beta t)$

$$\frac{\partial u}{\partial t} = -\beta\, u_0' = -\beta \frac{\partial u}{\partial x}$$

$\boxed{\text{Donc } u \text{ est solution de (1)}}$

Unicité : Soit u une solution de (1)

On considère $\frac{\partial \varphi}{\partial t}(t) = \frac{\partial u(t, x_0 + \beta t)}{\partial t}$

$$= \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x}$$

$= 0$ car u est solution de (1)

Donc $\boxed{\varphi \text{ est constant}}$ en 0

Il suffit ~~Ent~~ faibles de connaître la solution

lors de la solution

2.1

Méthodes numériques

Deuxième partie 2:

- soit : $u(t,x) = u_0(x-\beta t)$ on a: $\frac{\partial u}{\partial t}(t,x) = -\beta u_0'(x-\beta t)$

$\frac{\partial u(t,x)}{\partial x} = \frac{\partial u_0(x-\beta t)}{\partial x} = u_0'(x-\beta t)$

Alors: $\frac{\partial u(t,x)}{\partial t} + \beta \frac{\partial u(t,x)}{\partial x} = -\beta u_0'(x-\beta t) + \beta u_0'(x-\beta t) = 0$

Pour tt x_0 on a: $t \mapsto \varphi(t) = u(t, x_0 + \beta t)$

on a : $\frac{\partial \varphi}{\partial t} = \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = 0$

- on a: $U_{New} = \begin{pmatrix} U_1^{n+1} \\ \vdots \\ U_{M+1}^{n+1} \end{pmatrix}$ $\forall n$

$\forall m \in \{1, \dots, M+1\}, \forall n \in \{1, \dots, N-1\}$

$\frac{1}{k}(U_m^{n+1} - U_m^n) + \frac{\beta}{h}(U_m^n - U_{m-1}^n) = 0$

on a: $\frac{1}{k}(U_1^{n+1} - U_1^n) + \frac{\beta}{h}(U_1^n - U_0^n) = 0$

Donc $U_1^{n+1} = (1 - \frac{k\beta}{h}) U_1^n + \frac{k\beta}{h} U_0^n$

$U_2^{n+1} = (1 - k\frac{\beta}{h}) U_2^n + k\frac{\beta}{h} U_1^n$

$\vdots$

$U_{M+1}^{n+1} = (1 - k\frac{\beta}{h}) U_{M+1}^n + k\frac{\beta}{h} U_M^n$

$$U_{New} = \begin{pmatrix} (1-\frac{k}{h}\beta) U_1^n + k\frac{\beta}{h} U_0^n \\ (1-\frac{k}{h}\beta) U_2^n + k\frac{\beta}{h} U_1^n \\ \vdots \\ (1-k\frac{\beta}{h}) U_{M+1}^n + k\frac{\beta}{h} U_M^n \end{pmatrix} = (1 - k\frac{\beta}{h}) U_{old} + k\frac{\beta}{h} \underbrace{\begin{pmatrix} U_0^n \\ U_1^n \\ \vdots \\ U_M^n \end{pmatrix}}_{AU_{old}}$$

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$$\begin{pmatrix} U_1^{n+1} \\ \vdots \\ U_{M-1}^{n+1} \\ U_M^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta^2 k^2}{2h^2}\left(U_2^n - 2U_1^n + U_0^n\right) - \frac{\beta k}{2h}\left(U_2^n - U_0^n\right) - U_1^n \\ \vdots \\ \frac{\beta^2 k^2}{2h^2}\left(U_M^n - 2U_{M-1}^n + U_{M-2}^n\right) - \frac{k\beta}{2h}\left(U_M^n - U_{M-2}^n\right) - U_{M-1}^n \\ \frac{\beta^2 k^2}{2h^2}\left(U_{M+1}^n - 2U_M^n + U_{M-1}^n\right) - \frac{k\beta}{2h}\left(U_{M+1}^n - U_{M-1}^n\right) - U_M^n \end{pmatrix}$$

$$= \frac{\beta^2 k^2}{2h^2}\left[\begin{pmatrix} U_2^n \\ \vdots \\ U_{M+1}^n \end{pmatrix} - 2\begin{pmatrix} U_1^n \\ \vdots \\ U_M^n \end{pmatrix} + \begin{pmatrix} U_0^n \\ \vdots \\ U_{M-1}^n \end{pmatrix}\right] - \frac{\beta k}{2h}\left[\begin{pmatrix} U_2^n \\ \vdots \\ U_{M+1}^n \end{pmatrix} - \begin{pmatrix} U_0^n \\ \vdots \\ U_{M-1}^n \end{pmatrix}\right] - \begin{pmatrix} U_1^n \\ \vdots \\ U_M^n \end{pmatrix}$$

$\rightsquigarrow$

$$= \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ \\ \\ M \end{array} \begin{bmatrix} -\frac{\beta^2 k^2}{h^2}-1, & \frac{\beta^2 k^2}{2h^2}-\frac{\beta k}{2h}, & 0 \dots & & & (0) & 0 \\ \frac{\beta^2 k^2}{2h^2}+\frac{\beta k}{2h}, & -\frac{\beta^2 k^2}{h^2}-1, & \frac{\beta^2 k^2}{2h^2}-\frac{\beta k}{2h}, & 0 \dots & & & 0 \\ 0, & \frac{\beta^2 k^2}{2h^2}+\frac{\beta k}{2h}, & -\frac{\beta^2 k^2}{h^2}-1, & \frac{\beta^2 k^2}{2h^2}-\frac{\beta k}{2h}, & 0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & & \vdots \\ & (0) & & \ddots & \ddots & \ddots & 0 \\ 0, & \dots, & 0, & \frac{\beta^2 k^2}{2h^2}+\frac{\beta k}{2h}, & -\frac{\beta^2 k^2}{h^2}-1, & & \frac{\beta^2 k^2}{2h^2}-\frac{\beta k}{2h} \\ 0, & \dots, & 0, & 0, & \frac{\beta^2 k^2}{2h^2}+\frac{\beta k}{2h}, & & -\frac{\beta^2 k^2}{h^2}-1 \end{bmatrix} \begin{pmatrix} U_1^n \\ \vdots \\ \\ U_M^n \end{pmatrix}$$

$$
\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \left(\frac{\beta^2 k^2}{2h^2} - \frac{\beta k}{2h}\right) U_{M+1}^{n} \end{bmatrix}
$$

$$
\begin{cases} U_{M+1}^{n+1} = -\frac{\beta k}{h}\left(U_{M+1}^{n} - U_{M}^{n}\right) + U_{M+1}^{n} \\ U_{M+1}^{0} = e^{-5(5x-1)^2} \end{cases}
$$

2.3
$$\begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{M-1}^{n+1} \\ U_M^{n+1} \end{pmatrix} = k \begin{bmatrix} 0 & \frac{1}{2}-\frac{\beta k}{2h} & 0 & \cdots & 0 \\ \frac{\beta k}{2h}+\frac{1}{2} & 0 & \ddots & (0) & 0 \\ 0 & \ddots & \ddots & \ddots & \frac{1}{2}-\frac{\beta h}{2h} \\ \vdots & (0) & & \ddots & \\ 0 & \cdots & 0 & \frac{\beta h}{2h}+\frac{1}{2} & 0 \end{bmatrix} \begin{pmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{M-1}^n \\ U_M^n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \left(\frac{1}{2}-\frac{\beta h}{2h}\right) U_{M+1}^n \end{pmatrix}$$

2.4
$$U_m^{n+2} = \frac{k\beta}{h}\left(U_{m-1}^{n+1} - U_{m+1}^{n+1}\right) + U_m^n$$

$$\begin{pmatrix} U_1^{n+2} \\ \vdots \\ U_M^{n+2} \end{pmatrix} = \begin{bmatrix} 0 & -\frac{k\beta}{h} & 0 & \cdots & 0 \\ \frac{k\beta}{h} & \ddots & \ddots & (0) & \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & (0) & & \ddots & -\frac{k\beta}{h} \\ 0 & \cdots & 0 & \frac{k\beta}{h} & 0 \end{bmatrix} \begin{pmatrix} U_1^{n+1} \\ \vdots \\ U_M^{n+1} \end{pmatrix} + \begin{pmatrix} U_1^n \\ \vdots \\ U_M^n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{k\beta}{h} U_{M+1}^n \end{pmatrix}$$

2.5
$$U_{m+1}^{n+1} = \frac{kh}{\beta h}\left(U_m^n - U_m^{n+1}\right) + U_{m-1}^{n+1} - U_{m+1}^n + U_{m-1}^n$$

U^n

$$
\begin{pmatrix} U_1^{n+1} \\ \vdots \\ \vdots \\ U_M^{n+1} \end{pmatrix}
=
\begin{bmatrix}
1 & -\frac{\beta k}{4h} & & (0) \\
\frac{\beta k}{4h} & \ddots & \ddots & \\
& \ddots & \ddots & -\frac{\beta k}{4h} \\
(0) & & \frac{\beta k}{4h} & 1
\end{bmatrix}
\begin{bmatrix} U_1^{n} \\ \vdots \\ \vdots \\ U_M^{n} \end{bmatrix}
$$

$$
\frac{1}{k} U_m^{n+1} + \frac{\beta}{4h}\left(U_{m+1}^{n+1} - U_{m-1}^{n+1}\right) = \frac{1}{h} U_m^{n} - \frac{\beta}{4h}\left(U_{m+1}^{n} - U_{m-1}^{n}\right)
$$

$$
A\, U_{New} = B\, U_{Old}
$$

$$
\begin{bmatrix}
\frac{1}{k}, & \frac{\beta}{4h}, & 0 & \cdots & 0 \\
-\frac{\beta}{4h} & \ddots & \ddots & \ddots & \vdots \\
& \ddots & \ddots & \ddots & 0 \\
& & \ddots & \ddots & \frac{\beta}{4h} \\
& & & -\frac{\beta}{4h} & \frac{1}{k}
\end{bmatrix}
\begin{bmatrix} U_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ U_M^{n+1} \end{bmatrix}
=
\begin{bmatrix}
- & \ddots & & (0) \\
+ & \ddots & \ddots & - \\
& \ddots & \ddots & \\
(0) & & + &
\end{bmatrix}
\begin{bmatrix} U_1^{n} \\ \vdots \\ \vdots \\ \vdots \\ U_M^{n} \end{bmatrix}
$$

$1 \quad U_{M_1}$

$\frac{\beta k}{h} \qquad 1 - \frac{\beta k}{h}$

<u>Laplace:</u>

On a $-u''(x) = f(x) \quad \forall x \in]a,b[$

$a c \quad u(a) = u(b) = 0$

ⓐ $\begin{cases} a = 0 \\ b = \pi \\ f = \sin \end{cases}$ soit $u'(x) = \cos x + C_1$

$u(x) = \sin(x) + C_1 x + C_2 \qquad u(0) = C_2 = 0$

$\longrightarrow \boxed{u = \sin} \qquad u(\pi) = C_1\pi = 0$

ⓑ $\begin{cases} a = -1 \\ b = 2 \\ f(x) = (x-a)(x-b) \end{cases}$ soit $-u'(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C_1$

$-u(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - x^2 + C_1 x + C_2$

$u(-1) = u(2) = 0$

$$\begin{cases} -\frac{3}{4} - C_1 + C_2 = 0 \\ -4 + 2C_1 + C_2 = 0 \end{cases}$$

$$\boxed{\begin{cases} C_1 = 1{,}08\bar{3} \\ C_2 = 1{,}8\bar{3} \end{cases}}$$

ⓒ $-u''(x) = -e^x(x^2 + 4)$

$u'(x) = e^x(x^2 + 2x - 2) + C_1$

$u(x) = e^x(x^2 - 2) + C_1 x + C_2$

$u(\sqrt{2}) = C_1\sqrt{2} + C_2 = 0$

$u(-\sqrt{2}) = -C_1\sqrt{2} + C_2 = 0$

$2C_2 = 0 \longrightarrow \boxed{C_2 = C_1 = 0}$

$$f(x+h) = f(x) + f^{(1)}(x)\,h + \frac{f^{(2)}(x)}{2} h^2 + \frac{f^{(3)}(x)}{6} h^3 + \frac{f^{(4)}(x)}{24} h^4 + h^4 \varepsilon(h)$$

$$f(x-h) = f(x) - f^{(1)}(x)\,h + \frac{f^{(2)}(x)}{2} h^2 - \frac{f^{(3)}(x)}{6} h^3 + \frac{f^{(4)}(x)}{24} h^4 + h^4 \varepsilon(h)$$

$$f(x+h) + f(x-h) = 2f(x) + f^{(2)}(x)\,h^2 \underbrace{+ \frac{f^{(4)}(x)}{12} h^4 + h^4 \varepsilon(h)}_{h^2 \varepsilon(h)}$$

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f^{(2)}(x)$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i)$$

$$\forall i \in [\![1, n]\!]$$

$$-\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & (0) \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ (0) & & 1 & -2 \end{bmatrix}\begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ U_M \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \vdots \\ \vdots \\ f(x_M) \end{bmatrix}$$

5.1

$$\begin{bmatrix} U^n_1 \\ \vdots \\ \vdots \\ U^n_M \end{bmatrix} \vdots \begin{bmatrix} 1-\frac{2\alpha k}{h^2} & \frac{\alpha k}{h^2} & & (0) \\ \frac{\alpha k}{h^2} & \ddots & \ddots & \\ & \ddots & \ddots & \\ (0) & & & 1-\frac{2\alpha k}{h^2} \end{bmatrix} = \begin{bmatrix} U^{n+1}_1 \\ \\ \\ U^{n+1}_M \end{bmatrix}$$

5.2

$u_n = \sum_{i=1}^{n} \alpha_k \phi_k$ est solution de (Q8)

ssi $\int \sum_k \alpha_k \phi_k'(x) \phi_i'(x)\, dx = \int f(x) \phi_i(x)\, dx$

$$\underbrace{\int \begin{pmatrix} \phi_1'(x)\phi_1'(x) & \cdots & \phi_1'\phi_n' \\ \vdots & \ddots & \\ \phi_n'\phi_1' & & \cdot \end{pmatrix}}_{A} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \underbrace{\int \begin{pmatrix} \\ \\ \end{pmatrix}}_{B}$$

$\sum a_i \psi_i = 0 \quad \forall x$ dc $\forall x_i$ Dom ψ_i libre

Soit $y \in \mathbb{R}^n$ (les valeurs aux sommets) de c'est générat :

$$y_1 \psi_1 + y_2 \psi_2 + \dots + y_n \psi_n$$

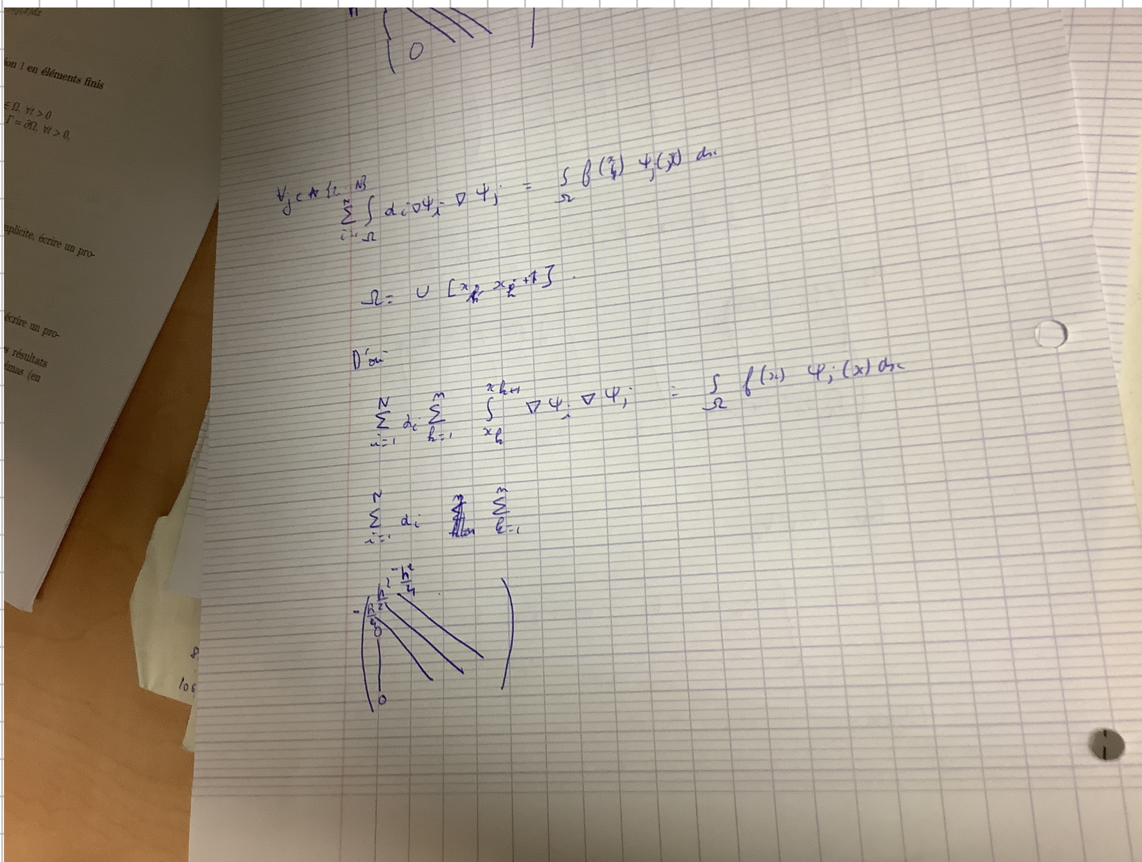
ψ_i

x_i

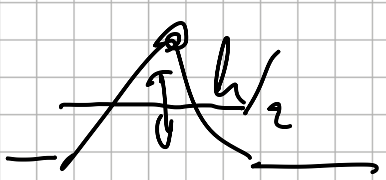
$$\int \psi_i' \psi_j' = \sum_{k=0}^{n} \int_{x_{k-1}}^{x_k} \psi_i' \psi_j' \int_{x_k}^{x_{k+1}} \psi_i' \psi_j'$$

$$\int_{x_{i-1}}^{x_i} \varphi_i' \varphi_j' \qquad \int_{x_i}^{x_{i+1}} \varphi_i' \varphi_j'$$

$$+ \int_{x_{j-1}}^{x_j} \varphi_j$$



$$B_j = \int f(x)\,\varphi_j(x)\,dx = \int_{x_{-1}}^{x_j} f\,\varphi_j + \int_{x_j}^{x_{j+1}} f\,\varphi_j$$



$$= f(x_j)\,\frac{h^2}{\cancel{2}} \times 2$$

$$= f(x_j)\,h^2$$