A Proof of Hadamard's Three Circle Theorem

by Andy Palan

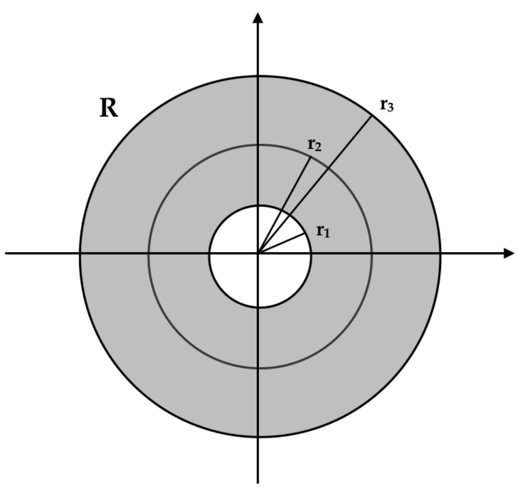
I FIRST CAME across Hadamard's Three Circle Theorem in Professor Michael Klass' offering of Complex Analysis (Math 185) here at Berkeley, when I was asked to provide a proof of the theorem in a homework assignment.

While the result itself does seem rather arbitrary and perhaps, one might argue, not of significant practical importance, I was left entirely stunned by the sheer outrageousness of the proof itself and felt compelled to share this proof.

A FAIR WARNING, someone unfamiliar with basic topics in complex analysis will like find reading this proof to be a fairly frustrating exercise as this proof uses many of the fundamental results of complex analysis (especially the Maximum Modulus Principle) without proof.

The Theorem

Let f be analytic on a region containing the set $R = \{z | r_1 \leq |z| \leq r_3\}$ (see image below), where $0 < r_1 < r_2 < r_3$.



Let M_1, M_2, M_3 be the maxima of $|f|$ on the circles $|z| = r_1, r_2, r_3$ respectively. Then, we have the inequality[1]

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}$$

[1] We state the theorem as it is stated in: Marsden, Jerrold E., and Michael J. Hoffman. *Basic complex Analysis*. San Francisco, Calif: W. H. Freeman, 1985. Print.

The Proof

WE DEFINE THE function $g(z) = z^\lambda f(z)$ for some $\lambda \in \mathbb{C}$. We have that z^λ is an entire function and f if analytic on the set $R = \{z | r_1 \leq |z| \leq r_3\}$. Hence, g is similarly analytic on the set $R = \{z | r_1 \leq |z| \leq r_3\}$.

Note that R is an open, connected, and bounded set in $\mathbb{C}$ and that g is analytic on the interior of R[2] and continuous on all of R (since it is analytic on all of R).

[2] In fact, as prior mentioned, it is analytic on all of R, but to apply the Maximum Modulus Principle, we only require it be analytic over the interior of R.

HENCE, BY THE Maximum Modulus Principle, $|g|$ has a maximum value on R, which it much attain on the boundary of R. Since g is clearly not a constant function, we must have that this maximum is not attained anywhere in the interior of R. Hence, we can write[3]

[3] Do you see why?

$$|g(z)| \leq \max\{r_1^\lambda M_1, r_3^\lambda M_3\}$$

To rid this inequality of the max term on the right hand side, we set both terms in the max term equal to each other, letting λ equal the value that satisfies such a constraint[4].

[4] This is the primary reason we defined g as we did; in doing so, we created a framework within which we could eliminate this max function, which as the reader should have noticed, would have inevitably appeared as a result of the definition of the set R.

$$\begin{aligned} r_1^\lambda M_1 &= r_3^\lambda M_3 \\ \lambda \log r_1 + \log M_1 &= \lambda \log r_3 + \log M_3 \\ \lambda &= -\frac{\log(M_3/M_1)}{log(r_3/r_1)} \end{aligned}$$

Hence, letting $\lambda = -\frac{\log(M_1/M_3)}{log(r_1/r_3)}$, we have that

$$|g(z)| \leq r_1^{-\frac{\log(M_3/M_1)}{log(r_3/r_1)}} M_1$$

Now, let $z = \xi$ where ξ is the point on the circle of radius r_2 that attains the maximum value on this circle; i.e. $g(z) = r_2^\lambda M_2$, with λ defined as above. We therefore have that

$$\begin{aligned} r_2^{-\frac{\log(M_3/M_1)}{log(r_3/r_1)}} M_2 &\leq r_1^{-\frac{\log(M_3/M_1)}{log(r_3/r_1)}} M_1 \\ -\frac{\log(M_3/M_1)}{log(r_3/r_1)} \log(r_2) + \log(M_2) &\leq -\frac{\log(M_3/M_1)}{log(r_3/r_1)} \log(r_1) + \log(M_1) \\ -\log(M_3/M_1)\log(r_2) + \log(r_3/r_1)\log(M_2) &\leq -\log(M_3/M_1)\log(r_1) + log(r_3/r_1)\log(M_1) \\ -\log(M_3)^{\log(r_2)} + \log(M_1)^{\log(r_2)} + \log(M_2)^{\log(r_3/r_1)} &\leq -\log(M_3)^{\log(r_1)} + \log(M_1)^{\log(r_1)} + \log(M_1)^{\log(r_3)} - \log(M_1)^{\log(r_1)} \\ \log M_2^{\log(r_3/r_1)} &\leq \log M_1^{\log(r_3/r_2)} + \log M_3^{\log(r_2/r_1)} \\ M_2^{\log(r_3/r_1)} &\leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)} \end{aligned}$$

Hence we have, as desired,

$$M_2^{\log(r_3/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}$$