Lattice Coding & Crypto Meeting

Antonio Campello

Sampling Algorithms for Lattice Gaussian Codes

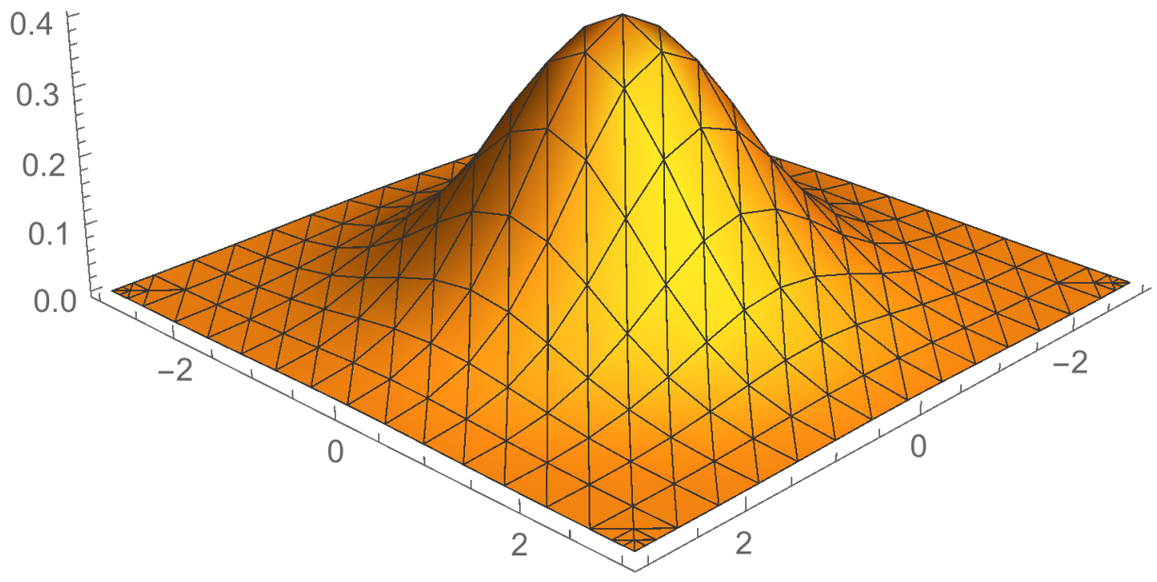
based on joint work with
J.-C. Belfiore (Huawei Technologies France)

Discrete Gaussian Measures

$$f : \mathbb{R}^n \to \mathbb{R}^+$$

$$f(\mathbf{x}) \propto e^{\frac{-\|\mathbf{x}\|^2}{2\sigma^2}}$$

$$f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi\sigma^2})^n} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}$$

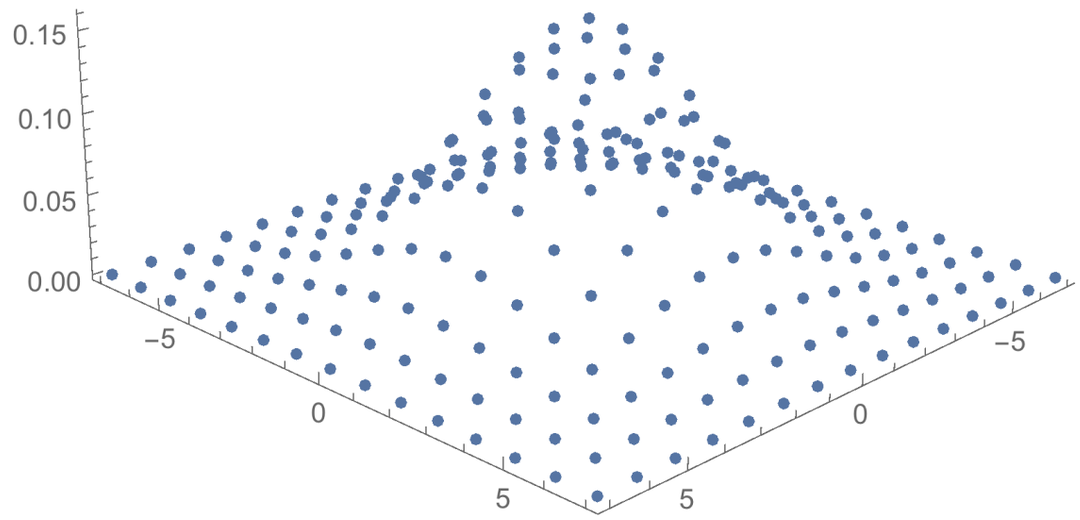


$$D : \Lambda \to [0, 1]$$

Λ is a discrete set

$$D(\mathbf{x}) \propto e^{\frac{-\|\mathbf{x}\|^2}{2\sigma^2}}$$

$$D(\mathbf{x}) = \frac{e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}}{\sum_{\mathbf{x}\in\Lambda} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}}$$



Discrete Gaussian Measures

- In Computer Science (lattice-based crypto): decoding algorithms [Klein ‘2000], homomorphic encryption, identity-based encryption [Regev ’05], complexity reductions

- In Mathematics: discrete Fourier analysis, transference theorems ([Banaszczyk ’92], [Cai ’03]), theta series,…

- In Communications: non-uniform signaling [Kschischang and Pasupathy ’93], semantically secure codes [Ling et al. ’15], capacity achieving in the AWGN [Ling and Belfiore ’15], compound and ergodic fading channels , [Campello, Ling and Belfiore ’16]

- In Mechanical Statistics: Maxwell-Boltzmann distribution

- …

Discrete Gaussian Measures

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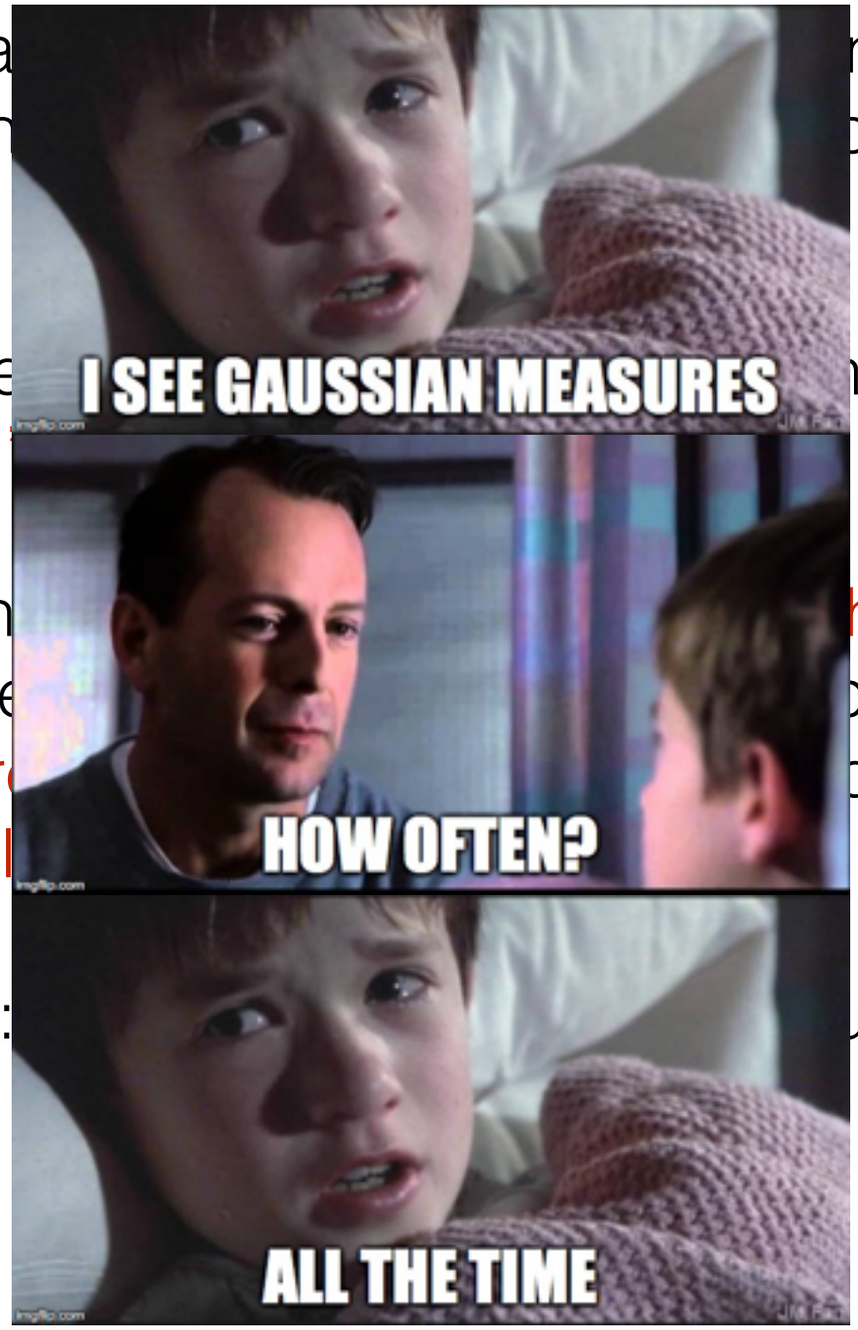


Lattice Gaussian Sampling Problem

- A lattice is a discrete *subgroup* of $\mathbb{R}^n$.

Sampling Algorithm

Given a lattice Λ and a parameter $\sigma > 0$,outputs a point $\mathbf{x} \in \Lambda$ with probability

$$D_{\Lambda,\sigma}(\mathbf{x}) = \frac{e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}}{\sum_{\mathbf{x}\in\Lambda} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}}$$

- Non-centered version:

$$D_{\Lambda+\mathbf{c},\sigma}(\mathbf{x}) = \frac{e^{-\frac{\|\mathbf{x}+\mathbf{c}\|^2}{2\sigma^2}}}{\sum_{\mathbf{x}\in\Lambda} e^{-\frac{\|\mathbf{x}+\mathbf{c}\|^2}{2\sigma^2}}}$$

Motivation: Simulating Probabilistic Shaping

- Lattice codes for the Gaussian channel:
  - Transmitter maps a « message » to a lattice point $\mathbf{x} \in \Lambda$

  - Receiver observes a distorted version $\rightarrow \mathcal{N}(0, \sigma_c^2)$

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

  and guesses $\hat{\mathbf{x}}$ in order to minimize error probability $P(\hat{\mathbf{x}} \neq \mathbf{x})$

- Messages are constrained (power-constraint)

$$\frac{1}{n} E\left[\|\mathbf{x}\|^2\right] \leq P$$

Motivation: Simulating Probabilistic Shaping

- Messages are constrained (power-constraint)

$$\frac{1}{n} E\left[\|\mathbf{x}\|^2\right] \leq P$$

- Deterministic Shaping: Choose a *shaping region* $\mathcal{S} \subset \mathbb{R}^n$ and a code $\mathcal{S} \cap \Lambda$ - e.g. cube, ball, or Voronoi region of sub-lattice

- Probabilistic Shaping: Pick $\mathbf{x} \sim D_{\Lambda,\sigma}$ (and adjust variance)

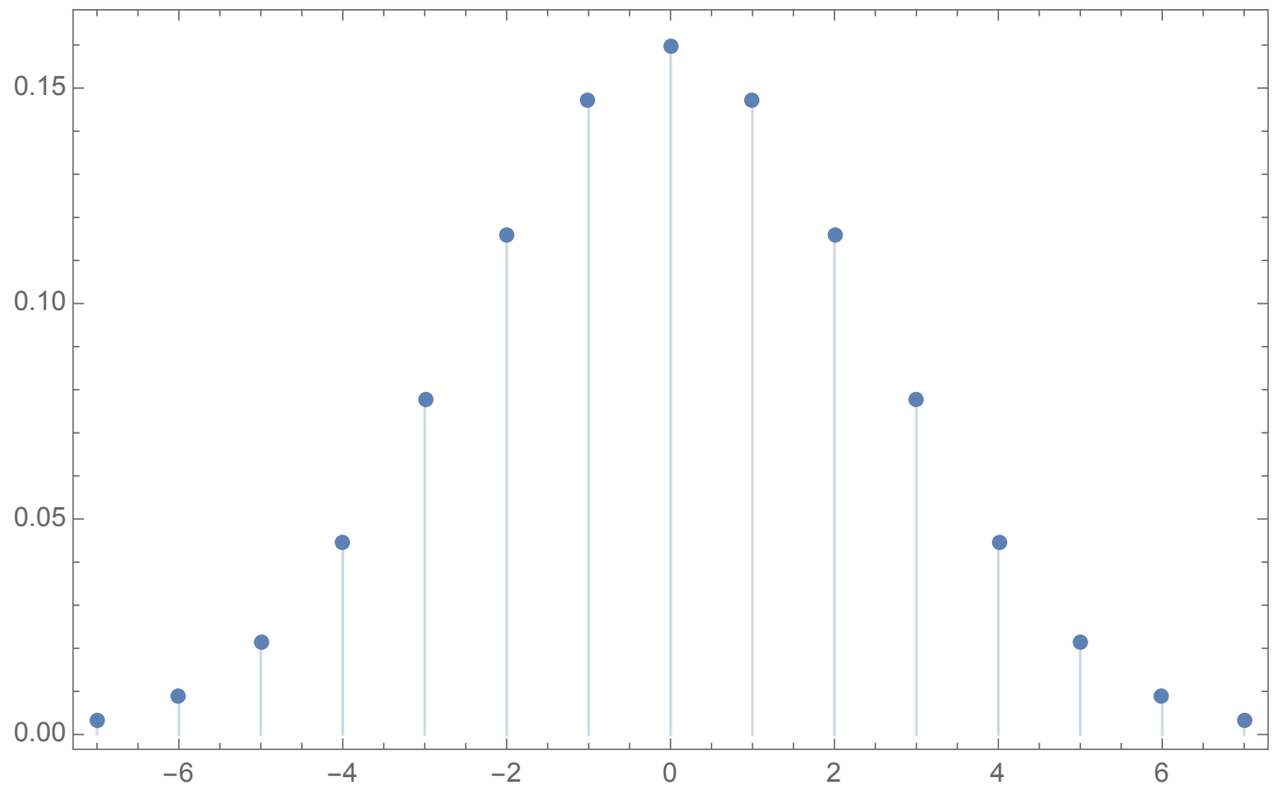
- [Forney '89] Coding gain *versus* shaping gain

- How to sample the lattices with best *coding gain* ? (known in low dimensions)

Lattice Gaussian Sampling Problem

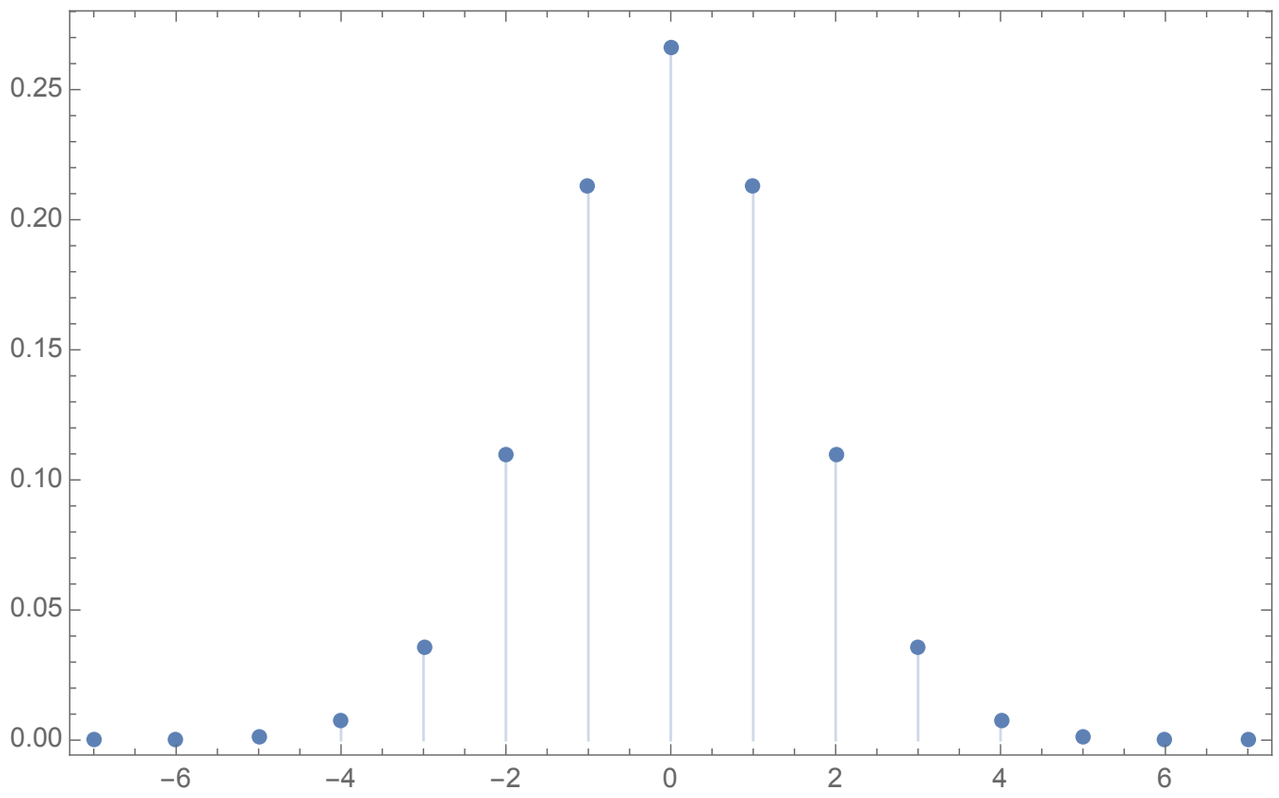
- Hardness: In general, as hard as finding the shortest vector in a lattice [Aggarwal et al ’14] and [Stephens-Davidowitz ’15].
- Universal algorithms (the Metropolis-Hastings-Klein algorithm) perform slow over specific lattices. E.g.: 24-dim Leech lattice and $\sigma = 1/\sqrt{2\pi}$ requires $24 \times 13434 = 322416$ calls of an uni-dimensional sampler [Wang, Ling ’14]
- In Communications: sampling from *special* lattices (constructed from error correcting codes, having decomposition as union of cosets, etc…).
- Applications: towards Gaussian shaping, lattice decoding.
- Insights between lattice Gaussian codes and theta series

Gaussian Measures: One Dimensional



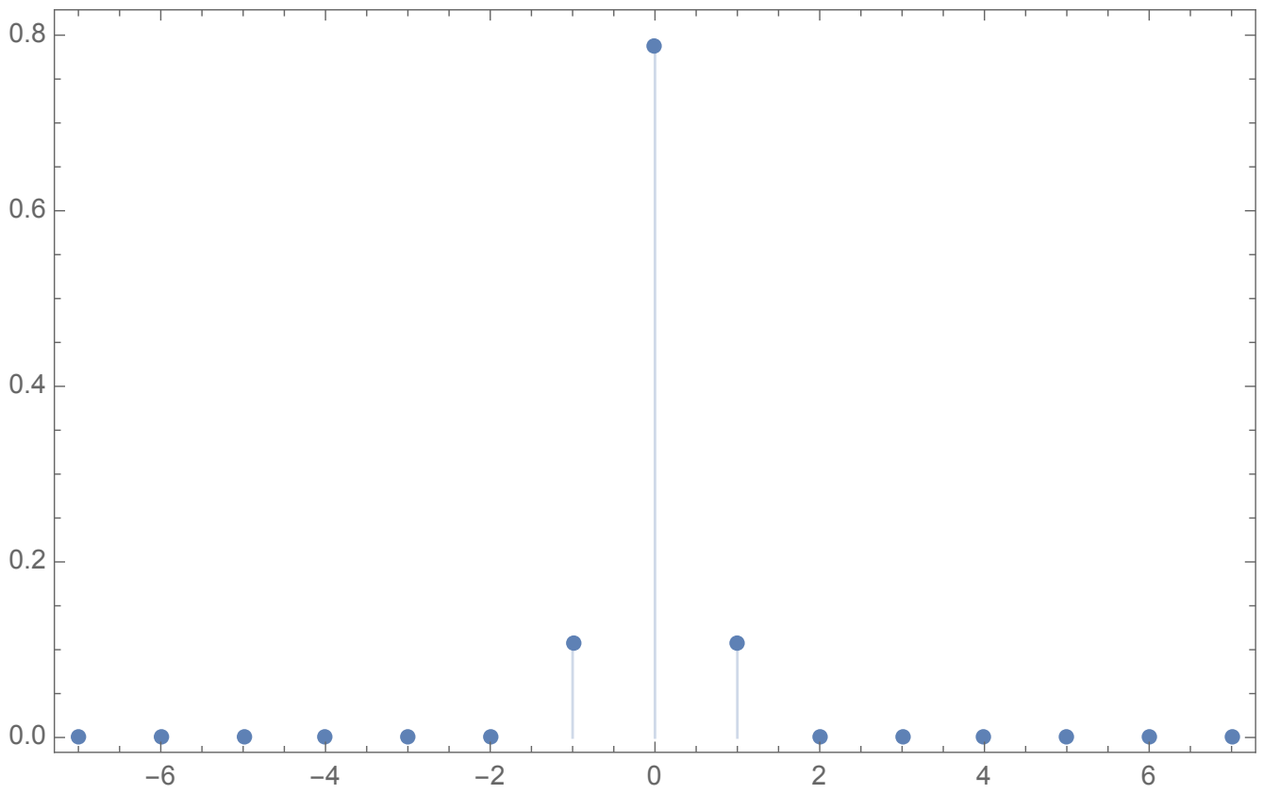
$$\sigma = 2.5$$

Gaussian Measures: One Dimensional



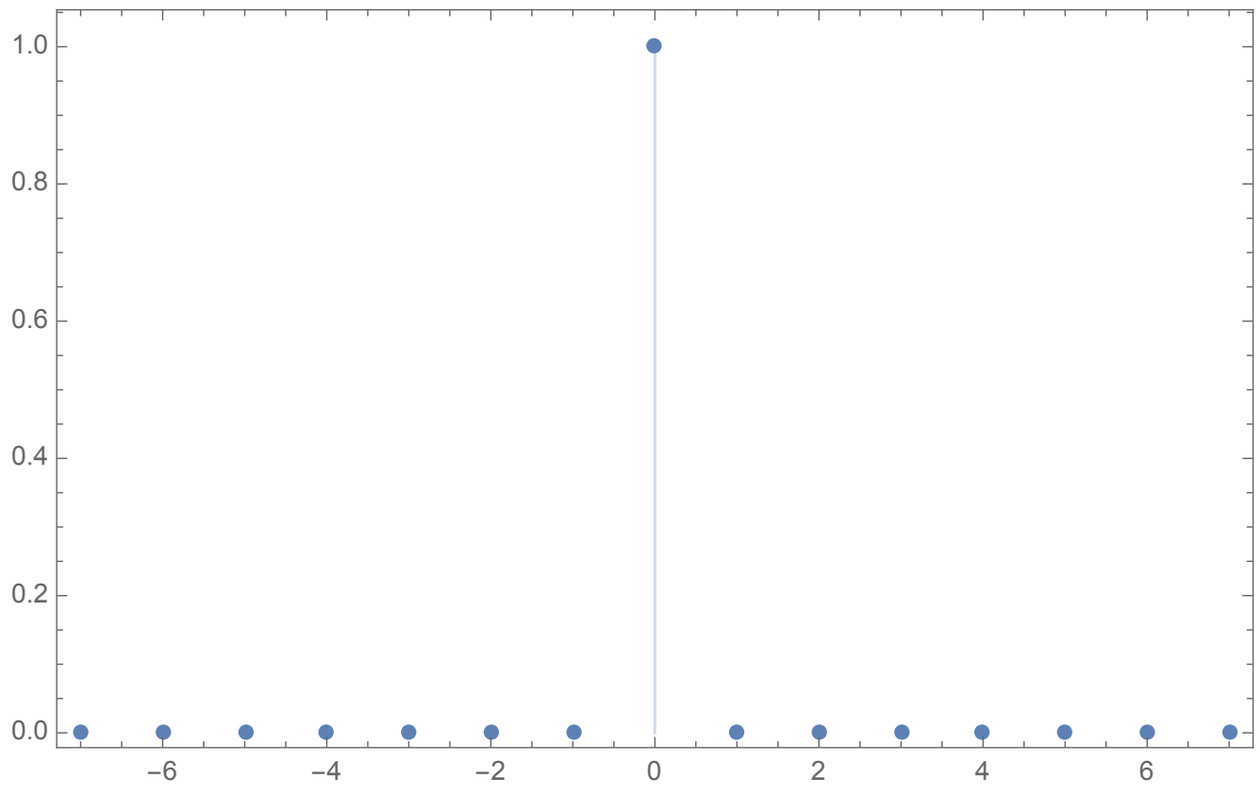
$$\sigma = 1.5$$

Gaussian Measures: One Dimensional



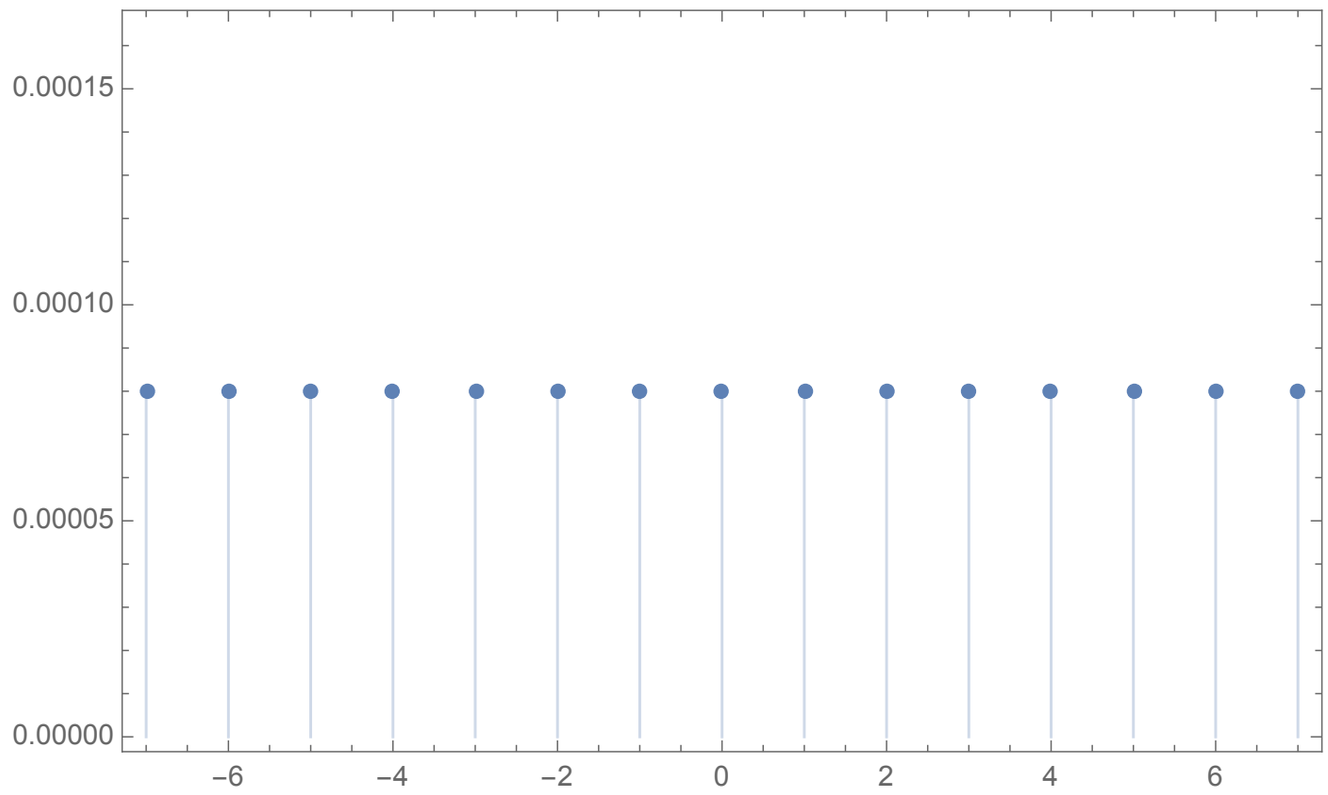
$$\sigma = 0.5$$

Gaussian Measures: One Dimensional



$$\sigma = 0.1$$

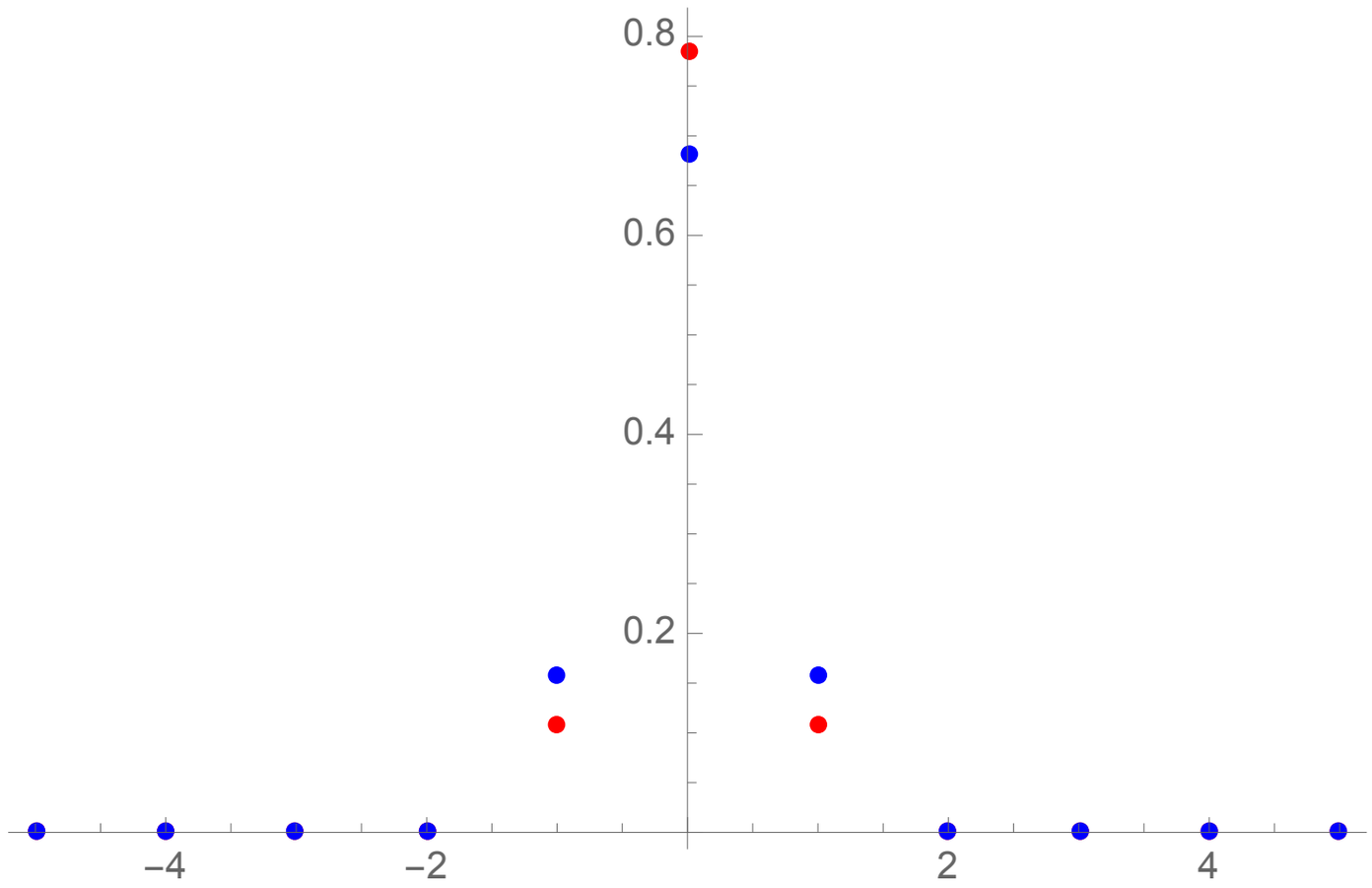
Gaussian Measures: One Dimensional



$$\sigma = 5 \times 10^3$$

One Dimensional Sampler (not so fast…)

- Wrong Idea: Generate $x \sim \mathcal{N}(0, \sigma^2)$ and output $\lfloor x \rceil$



One Dimensional Sampler

- Rejection Algorithm [Brakerski et al. '13]

Set $\mathcal{I} = \{c-l, c-(l-1), \ldots, 1-c, c, \ldots, c+l\}$ and calculate

$$p_{\mathcal{I}} = D_{\sigma^2,\mathbb{Z}+c}(\mathcal{I})$$

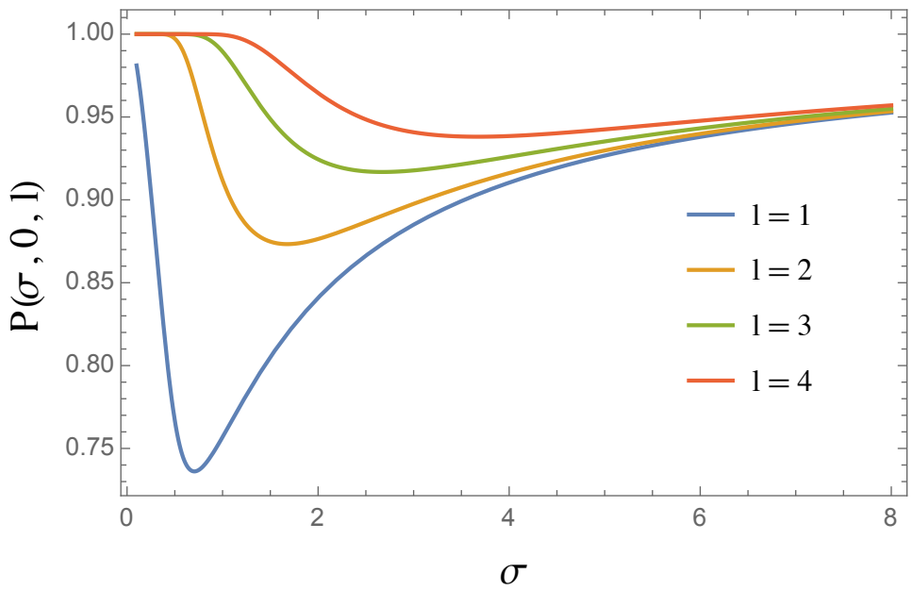
$$p'(i) = D_{\sigma^2,\mathbb{Z}+c}(i)/p_{\mathcal{I}}, i \in \mathcal{I}$$

1) With probability $p_{\mathcal{I}}$ sample on the *finite* distribution in $\mathcal{I}$
2) With probability $(1-p_{\mathcal{I}})$ sample on $\mathcal{I}^c$ by a rejection principle:

Sampling on $\mathcal{I}^c$:
Choose between positive or negative side. Ex:
(+) Generate y continuous Gaussian in $[c+l, +\infty]$
Output $x = \lceil y-c \rceil + c$ with prob.

$$\frac{e^{-x^2/2\sigma^2}}{e^{-y^2/2\sigma^2}}$$

Otherwise Repeat



Lattices and Theta Series

- Definition:

$$\Theta_{\Lambda+\mathbf{c}}(q) := \sum_{\mathbf{y}\in\Lambda+\mathbf{c}} q^{\|\mathbf{y}\|^2}$$

$$\Theta_{\Lambda+\mathbf{c}}(\tau) := \sum_{\mathbf{y}\in\Lambda+\mathbf{c}} e^{-\pi\tau\|\mathbf{y}\|^2} = \sum_{\mathbf{x}\in\Lambda} e^{-\pi\tau\|\mathbf{x}+\mathbf{c}\|^2}.$$

- Important easily numerically calculated one-dimensional theta series:

$$\theta_2(\tau) := \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2}, \theta_3(\tau) := \sum_{m=-\infty}^{\infty} q^{m^2}.$$

$$\Theta_{\mathbb{Z}+c}(\tau) = \sum_{m=-\infty}^{\infty} e^{-\pi\tau(m+c)^2} = \tau^{-2} \sum_{m=-\infty}^{\infty} e^{2\pi imc-\pi m^2/\tau} = \tau^{-2}\theta_3(\pi c|i\tau^{-1})$$

Lattices and Theta Series

- Important properties:

$$\Theta_{\Lambda_1 \oplus \Lambda_2}(\tau) = \Theta_{\Lambda_1}(\tau)\Theta_{\Lambda_2}(\tau)$$

$$\Theta_{\alpha\Lambda}(\tau) = \Theta_{\Lambda}(\alpha^2\tau)$$

$$\Theta_{\Lambda_1 \cup \Lambda_2}(\tau) = \Theta_{\Lambda_1}(\tau) + \Theta_{\Lambda_2}(\tau)$$

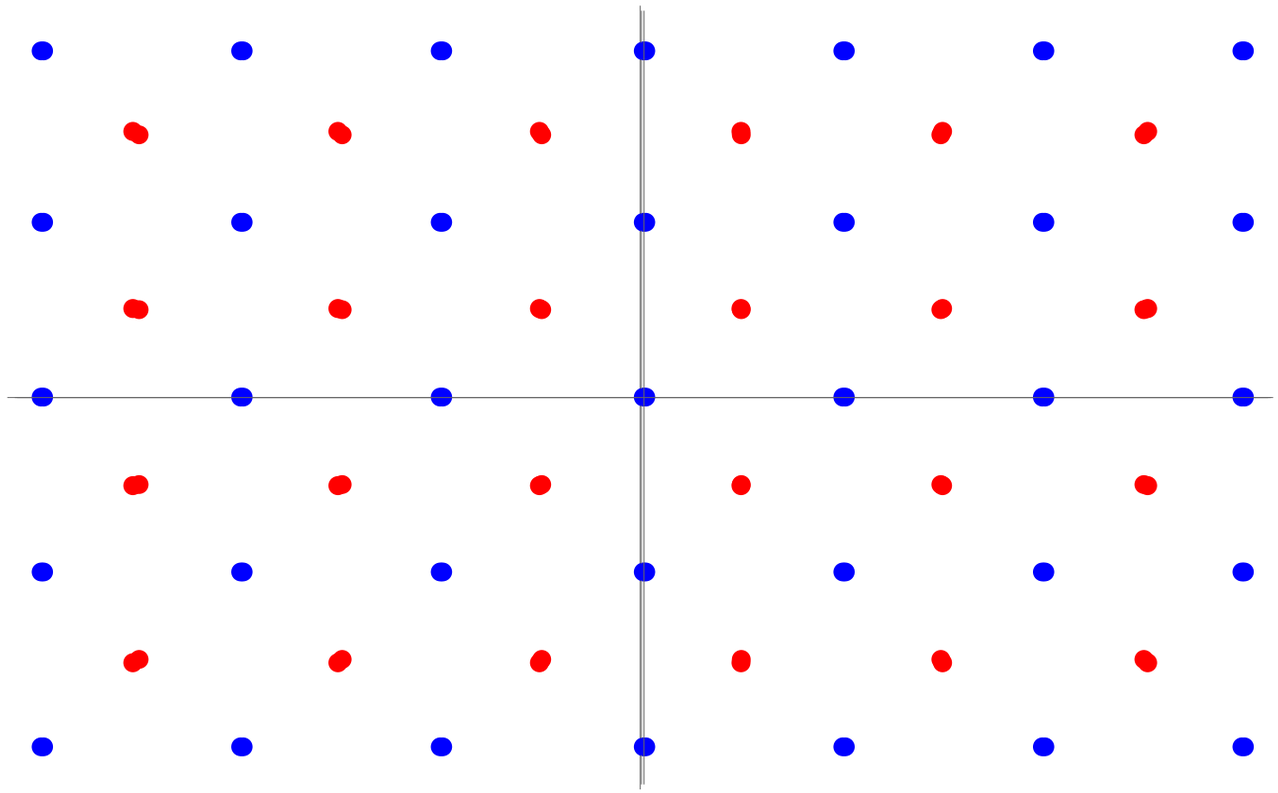
Example: Theta Series of $\mathbb{Z}^n$

$$\Theta_{\mathbb{Z}^n}(\tau) = \Theta_{\mathbb{Z}}(\tau)^n = \theta_3(\tau)^n$$

From Theta Series to Sampling

- Hexagonal lattice

$$A_2 = \left\{ (x_1, x_2) \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} : x_1, x_2 \in \mathbb{Z} \right\}$$

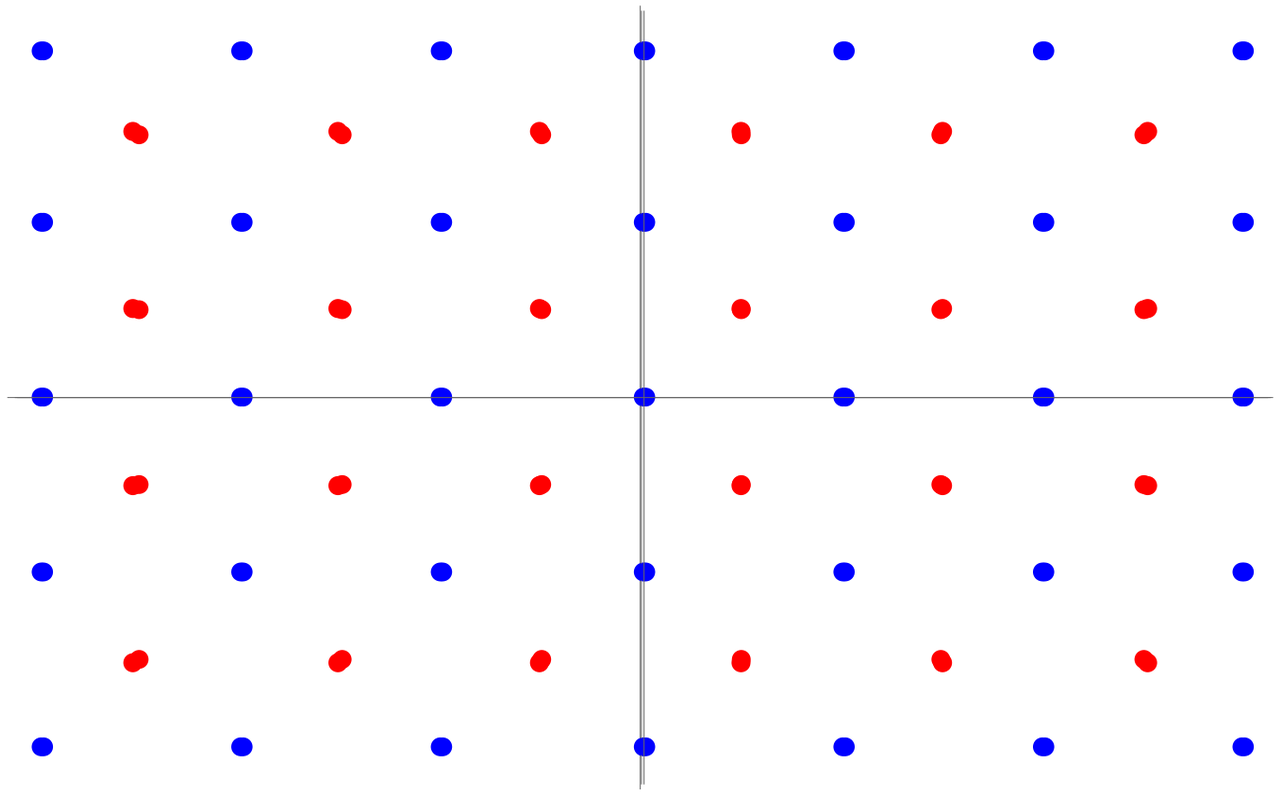


$$\Theta_{A_2}(\tau) = \theta_3(\tau)\theta_3(3\tau) + \theta_2(\tau)\theta_2(3\tau)$$

From Theta Series to Sampling

- Hexagonal lattice

$$A_2 = \left(\mathbb{Z} \oplus \sqrt{3}\mathbb{Z}\right) \bigcup \left(\mathbb{Z} \oplus \sqrt{3}\mathbb{Z} + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right)$$



$$\Theta_{A_2}(\tau) = \theta_3(\tau)\theta_3(3\tau) + \theta_2(\tau)\theta_2(3\tau)$$

From Theta Series to Sampling

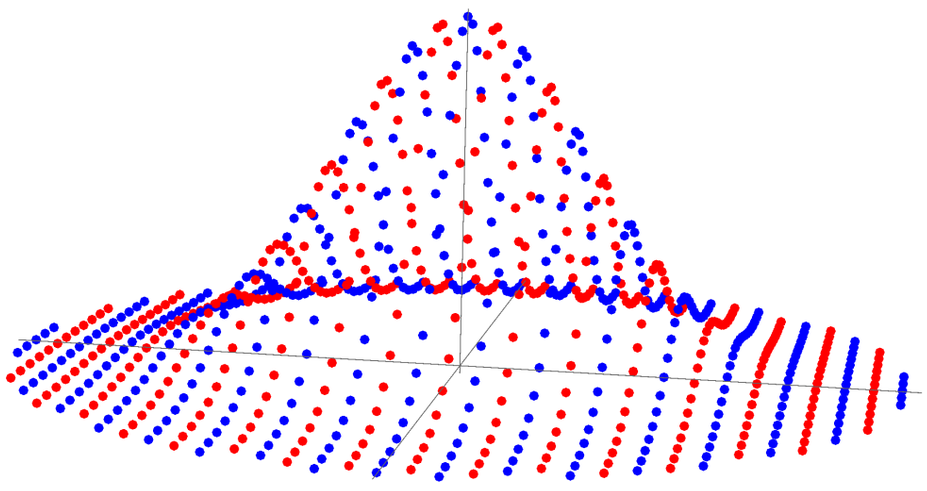
- Hexagonal lattice

$$p = D_{A_2,\sigma}(\mathbb{Z} \oplus \sqrt{3}\mathbb{Z}) = \frac{\theta_3(\frac{1}{2\pi\sigma^2})\theta_3(\frac{3}{2\pi\sigma^2})}{\theta_3(\frac{1}{2\pi\sigma^2})\theta_3(\frac{3}{2\pi\sigma^2}) + \theta_2(\frac{1}{2\pi\sigma^2})\theta_2(\frac{3}{2\pi\sigma^2})}$$

Algorithm

1) Throw a biased coin with probability *p* of heads
2) If heads, sample in the blue coset
3) If tails, sample in the red coset

- Sampling in each coset is possible by invoking the $\mathbb{Z}$-sampler twice.



Coset Decompositions

- Generalization to more general coset decompositions.

Construction A lattices

$$\Lambda = 2\mathbb{Z}^n + \mathcal{C}$$

the coset corresponding of a codeword of weight w has theta series

$$\theta_2(4\tau)^w \theta_3(4\tau)^{n-w}$$

Suppose there are A_w codewords of given weight w. The probability that a discrete distribution falls in some coset of a codeword of weight w is

$$A_w \frac{\theta_2(4\tau)^w \theta_3(4\tau)^{n-w}}{\Theta_\Lambda(\tau)}$$

General Idea

1) Pick a weight with probability p_w
2) Pick a word of weight w uniformity at random
3) Sample in the coset $2\mathbb{Z}^n + \mathbf{c}$

The lattice Dn

- Construction A lattices (best coding gains dimensions 3,4,5),

$$D_n = 2\mathbb{Z}^n + P_n$$

where P_n is a parity check code

$$\{(x_1, \ldots, x_n) \in \mathbb{F}_2^n : x_1 + \ldots + x_n \equiv 0 \bmod 2\}$$

There are $\binom{n}{2l}$ vectors of weight $2l$.

The probability of picking such a coset is

$$p_{2l} = \binom{n}{2l} \frac{\Theta_{\mathbb{Z}+\frac{1}{2}}(4\tau)^{2l}\Theta_{\mathbb{Z}+\frac{1}{2}}(4\tau)^{n-2l}}{\Theta_{D_n}(\tau)}$$

The lattice Dn

Algorithm

1) Pick a number $l \in \{1, \ldots, \lfloor n/2 \rfloor\}$ with probability p_{2l} .
2) Pick a subset $\mathcal{J} \subset \{1, \ldots, n\}$ with size $2l$
3) **For** $j \in \mathcal{J}$

$$x_j \leftarrow \mathrm{Sampler}_{\mathbb{Z}+\frac{1}{2}}(2\tau)$$

4) **For** $j \notin \mathcal{J}$

$$x_j \leftarrow \mathrm{Sampler}_{\mathbb{Z}}(2\tau)$$

- Generalizations to shifts by vectors of type $(\alpha, \beta, \beta, \ldots, \beta)$

Coset Decompositions

- Real Constructions (A and B)

$$\Lambda_A(\mathcal{C}) = 2\mathbb{Z}^n + \mathcal{C} \text{ and } \Lambda_B(\mathcal{C}) = 4\mathbb{Z}^n + 2P_n + \mathcal{C}.$$

$$\Lambda_B(\mathcal{C}) = 2D_n + \mathcal{C}, \text{ where } D_n = \Lambda_A(P_n)$$

- Complex Constructions (A and B)

$$\Lambda_A(\mathcal{C}) = \theta\mathbb{Z}[\omega]^n + \mathcal{C} \text{ and } \Lambda_B(\mathcal{C}) = \theta^2\mathbb{Z}[\omega]^n + \theta P_n + \mathcal{C},$$

where $\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}$, and θ is a prime of norm p

Coset Decompositions

- **Claim:** For the aforementioned constructions, the theta series of each coset depends only on the Hamming weight of each codeword.

Construction	Theta Series of a Coset	
A	$\theta_2(4\tau)^w\theta_3(4\tau)^{n-w}$	
B	$(1/2)\theta_2(4\tau)^w\theta_3(4\tau)^{n-w}$	$w \geq 1$
	$(1/2)\theta_3(4\tau)^n + (1/2)\theta_4(4\tau)^n$	$w = 0$
$\mathrm{A}_c, \theta = 2$	$\phi_1(4\tau)^w\phi_0(4\tau)^{n-w}$	
$\mathrm{A}_c, \theta = \sqrt{-3}$	$\phi_2(3\tau)^w\phi_0(3\tau)^{n-w}$	
$\mathrm{B}_c, \theta = \sqrt{-3}$	$(1/3)\phi_2(3\tau)^w\phi_0(3\tau)^{n-w}$	$w \geq 1$
	$(1/3)(\phi_0(3\tau)^n + 2(\phi_0(9\tau) - \phi_2(9\tau))^n)$	$w = 0$

TABLE I
THETA SERIES OF A COSET $\Lambda' + \boldsymbol{c}$, WT$(\boldsymbol{c}) = w$, FOR SEVERAL CONSTRUCTIONS

The Leech Lattice

- Extremal even unimodular lattice in dimension 24. Theta series:

$$\frac{1}{8}\left(\theta_2(0,\tau)^8+\theta_3(0,\tau)^8+\theta_4(0,\tau)^8\right)^3-\frac{45}{16}\theta_2(0,\tau)^8\theta_3(0,\tau)^8\theta_4(0,\tau)^8$$

The Leech Lattice

- Extremal even unimodular lattice in dimension 24. Theta series:

Density Doubling

Consider the construction B lattice $H_{24} = 2D_{24} + \mathcal{G}_{24}$, where $\mathcal{G}_{24}$ is the $(24, 12, 8)_{\mathbb{F}_2}$ Golay code. The Leech lattice is

$$\Lambda_{24} = H_{24} \cup (H_{24} + \mathbf{a})$$

where $\mathbf{a} = ((-3/2)^1, (1/2)^{23})$

- Theta series of « first » half is already known (Construction B)
  For the second half:
  All cosets $2D_{24} + \mathbf{c} + \mathbf{a}$ have *same theta series* given by

$$\frac{\beta(q^4)^{24} - \alpha(q^4)^{24}}{2}$$

The Leech Lattice

Algorithm

1) Throw a biased coin with probability $D_{\Lambda_{24},\sigma}(H_{24})$ of heads.

2) if the output is heads
   Sample $\mathbf{x} \in H_{24}$ from the Construction B sampler

3) else
   Choose $\mathbf{c} \in \mathcal{G}_{24}$ uniformly at random
   Draw $\mathbf{x} \in 2D_{24} + \mathbf{a} + \mathbf{c}$ using Dn sampler

Output $\mathbf{x}$

- Properties of Golay code for Cons. B sampler
- Uses 24 calls of a uni-dimensional sampler for any σ.

Simulating Probabilistic Shaping

- [Ling and Belfiore '15]. Gaussian Shaping.
  Choose a « good » lattice for coding
- Choose a point $\mathbf{x} \sim D_{\Lambda+c,\sigma^2}$ to be transmitted over a Gaussian channel.

Proposition (closed form power/rate)

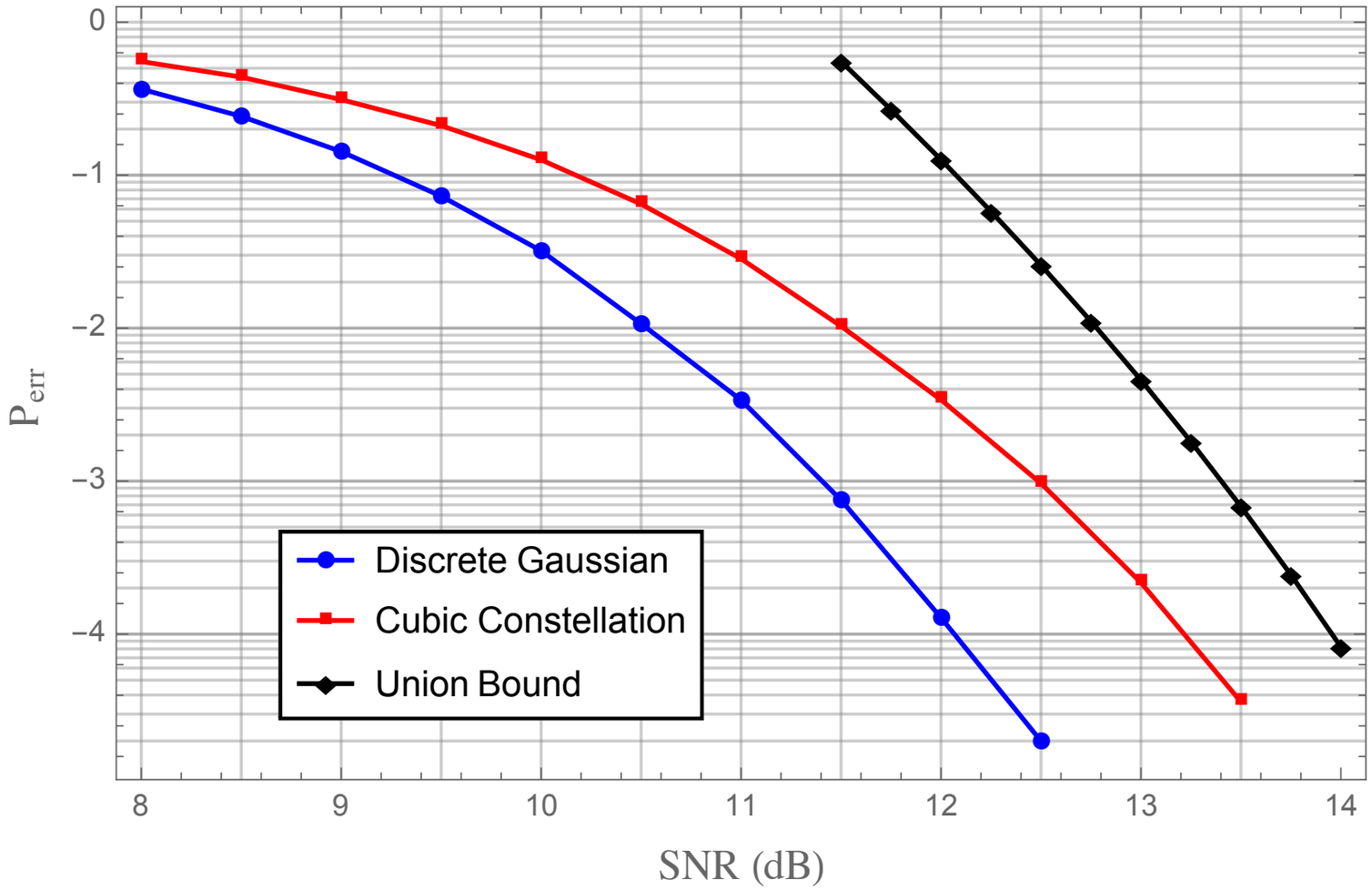
The power and rate of a lattice Gaussian code is

$$P = \frac{-1}{n\pi} \frac{\Theta'_{\Lambda+\mathbf{c}}(\tau)}{\Theta_{\Lambda+\mathbf{c}}(\tau)} \text{ and } R = -\frac{\tau}{n} \frac{\Theta'_{\Lambda+\mathbf{c}}(\tau)}{\Theta_{\Lambda+\mathbf{c}}(\tau)} + \frac{1}{n} \ln \Theta_{\Lambda+\mathbf{c}}(\tau)$$

Rate is maximized in the center distribution
Relations to modular forms

Probabilistic Shaping

- Leech Lattice Sampler: discrete Gaussian versus cubic constellation



What else?

- Sampling algebraic Construction A lattices (wireless channels):

E.g.: Ring of integers of $Q(\sqrt{d}), d \geq 5, d \equiv 1 \bmod 4$

Basis for ideal lattice:

$$\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{d}}{2} & \frac{1-\sqrt{d}}{2} \end{pmatrix}$$

Has rectangular sub-lattice of index 2 generated by embedding of $\mathbb{Z}[\sqrt{d}]$ and decomposes (up to rotation) as

$$\Lambda = \left(\sqrt{2}\mathbb{Z} \oplus \sqrt{2d}\mathbb{Z}\right) \bigcup \left(\sqrt{2}\mathbb{Z} \oplus \sqrt{2d}\mathbb{Z} + \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2d}}{2}\right)\right)$$

Final Remarks

- How to use symmetries between well-known lattices to deriving fast discrete sampling algorithms
- New insights between lattice Gaussian codes and theta series
- Open: sampling other algebraic lattices
- Probabilistic shaping models: from a sampler to an encoder

Thank you

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