GPV Signatures (Hash-then-Sign Signatures)

Advanced Topics in ~~Cybersecurity~~Cryptography (7CCSMATC)

Martin R. Albrecht

Falcon (signature scheme)

Falcon is a post-quantum signature scheme selected by the NIST at the fourth round of the post-quantum standardisation process. It was designed by Thomas Prest, Pierre-Alain Fouque, Jeffrey Hoffstein, Paul Kirchner, Vadim Lyubashevsky, Thomas Pornin, Thomas Ricosset, Gregor Seiler, William Whyte, and Zhenfei Zhang.[1][2][3] It relies on the hash-and-sign technique over the Gentry, Peikert, and Vaikuntanathan framework[4] over NTRU lattices. The name *Falcon* is an acronym for ***Fa**st Fourier **l**attice-based **co**mpact signatures over **N**TRU*.

Outline

Lattices

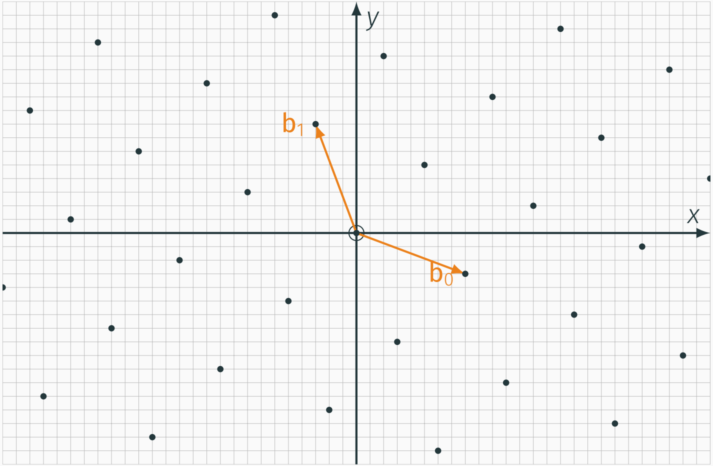
q-ary Lattices

GPV Signatures

Security Proof

Lattices

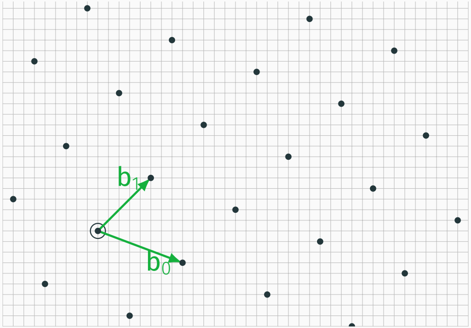
Lattices

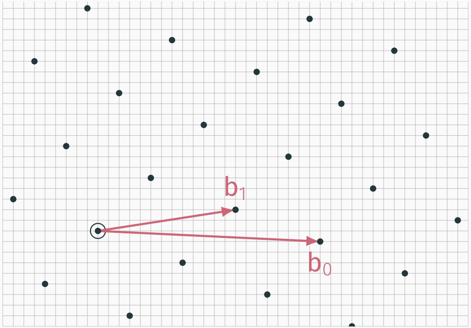


A lattice Λ is a discrete subgroup of $\mathbb{R}^d$.

Picture credit: Léo Ducas

Lattice Bases



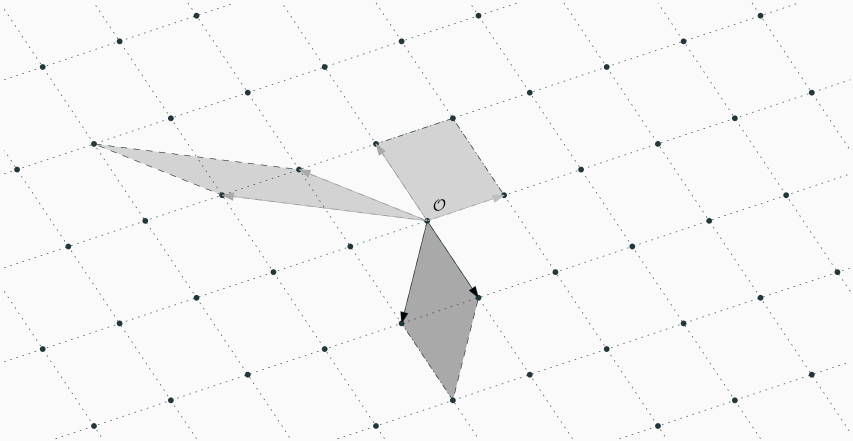


G	$\rightarrow$	B	: easy	(compute Hermite Normal form);
B	$\rightarrow$	G	: hard	(BKZ, lattice sieve ...).

Picture credit: Léo Ducas

Lattice Volume I

The volume of a lattice is the volume of its fundamental domain.



Picture credit: Joop van de Pol

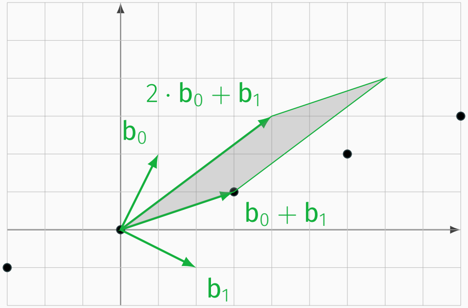
Lattice Volume II

For any two bases $\mathbf{G}$, $\mathbf{B}$ of the same lattice Λ:

$$\det(\mathbf{G} \cdot \mathbf{G}^T) = \det(\mathbf{B} \cdot \mathbf{B}^T).$$

We can therefore define:

$$\mathrm{Vol}(\Lambda) = \sqrt{\det(\mathbf{B} \cdot \mathbf{B}^T)}.$$



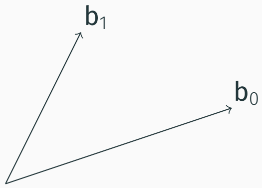
Picture credit: `https://tex.stackexchange.com/a/42559`

Length of Gram–Schmidt Vectors

It will be useful to consider the lengths of the Gram–Schmidt vectors.

The vector $\mathbf{b}_i^*$ is the orthogonal projection of $\mathbf{b}_i$ to the space spanned by the vectors $\mathbf{b}_0, \ldots, \mathbf{b}_{i-1}$.

Informally, this means taking out the contributions in the directions of previous vectors $\mathbf{b}_0, \ldots, \mathbf{b}_{i-1}$.



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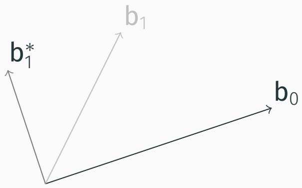


Gram–Schmidt Vectors and Lattice Volumes

Let $\mathbf{B}^\star$ be the Gram–Schmidt Orthogonalisation of $\mathbf{B}$. The matrix $\mathbf{B}^\star$ is **not** a basis for Λ, but

$$\mathrm{Vol}(\Lambda) = \sqrt{\det(\mathbf{B}^\star \cdot {\mathbf{B}^\star}^T)} = \prod \|\mathbf{b}_i^\star\|.$$

“Good” Bases

Recall that, independently of the basis $\mathbf{B}$ it hold that:

$$\mathrm{vol}(\Lambda) = \prod \|\mathbf{b}_i^\star\|.$$

Therefore, it is somehow equivalent that

- $\max_i \|\mathbf{b}_i^\star\|$ is small
- $\min_i \|\mathbf{b}_i^\star\|$ is large
- $\kappa(\mathbf{B}) = \min_i \|\mathbf{b}_i^\star\| / \max_i \|\mathbf{b}_i^\star\|$ is small

Good Basis

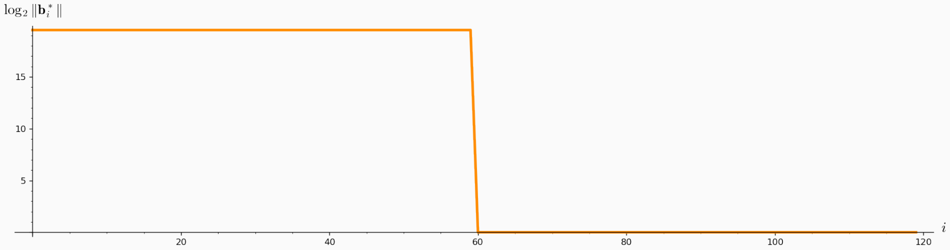
$$\kappa(\mathbf{G}) = \mathrm{poly}(d), \qquad \forall i : \|\mathbf{g}_i^\star\| = \mathrm{poly}(d) \cdot \mathrm{Vol}(\Lambda)^{1/d}.$$

LLL-reduced Basis

$$\kappa(\mathbf{G}) \approx (1.04)^d, \qquad \max_i \|\mathbf{g}_i^\star\| \approx (1.02)^d \cdot \mathrm{Vol}(\Lambda)^{1/d}.$$

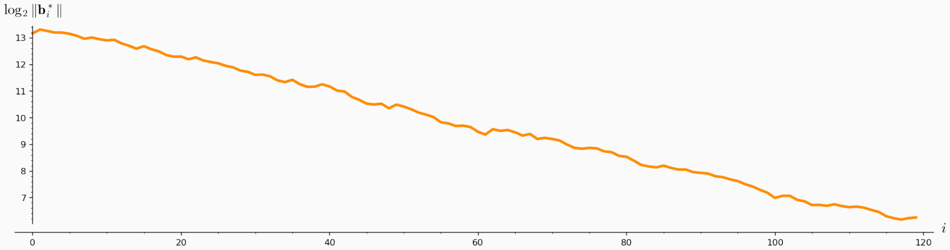
Example Gram–Schmidt "Shape"

```
A = IntegerMatrix.random(120, "qary", k=60, bits=20)[::-1]
M = GSO.Mat(A, update=True)
line([(i,log(r_, 2)/2) for i, r_ in enumerate(M.r())], **plot_kwds)
```



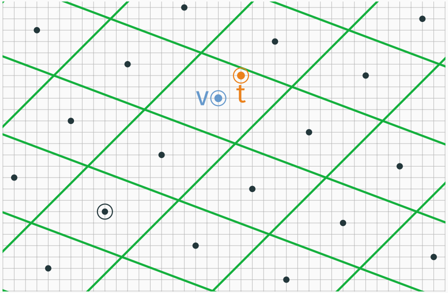
Example Gram–Schmidt "Shape" after LLL

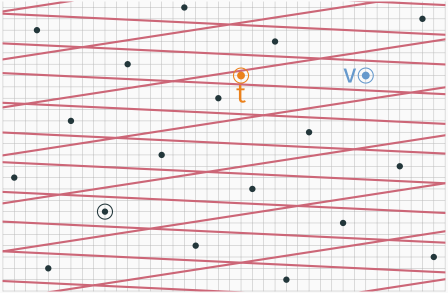
```
A = LLL.reduction(A)
M = GSO.Mat(A, update=True)
line([(i,log(r_, 2)/2) for i, r_ in enumerate(M.r())], **plot_kwds)
```



Bases and Fundamental Domains

Each basis defines a **tiling**.

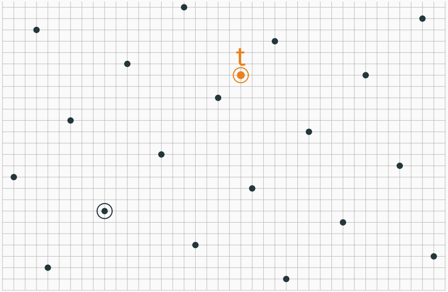




Round'off Algorithm [Lenstra, Babai]:

Given a target $\mathbf{t}$, find $\mathbf{v} \in \Lambda$ at the center the tile.

Round Off Algorithm

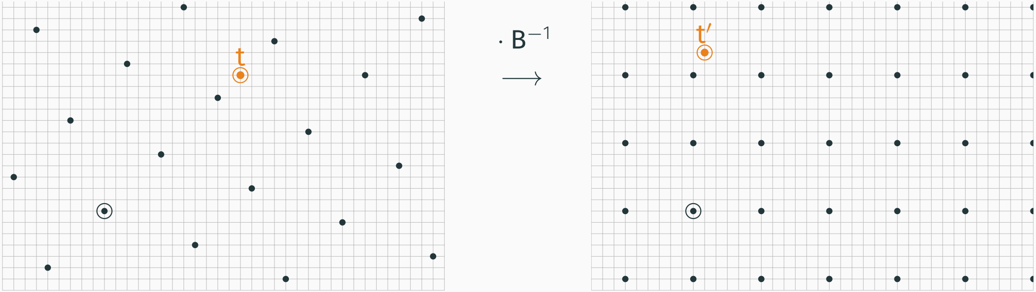


$\longrightarrow$

Round Off Algorithm [Lenstra,Babai]:

Credit: Léo Ducas

Round Off Algorithm

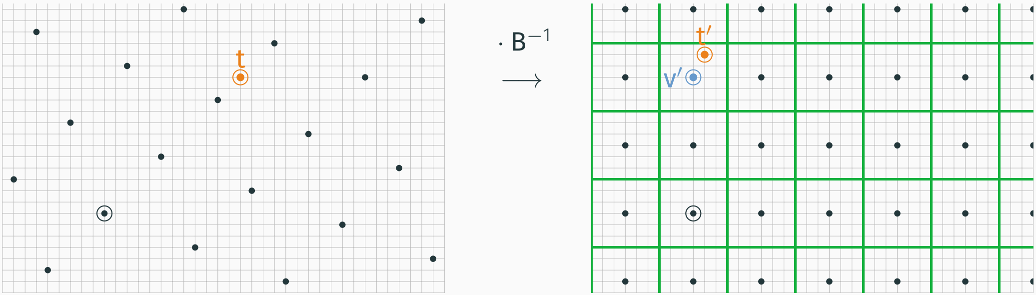


Round Off Algorithm [Lenstra,Babai]:

- Use $\mathbf{B}$ to switch to the lattice $\mathbb{Z}^d$ $\quad \mathbf{t}' = \mathbf{B}^{-1} \cdot \mathbf{t}$

Credit: Léo Ducas

Round Off Algorithm

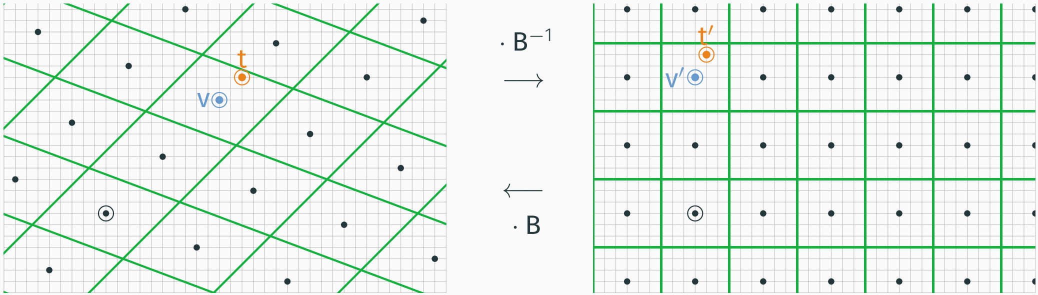


Round Off Algorithm [Lenstra,Babai]:

- Use $\mathbf{B}$ to switch to the lattice $\mathbb{Z}^d$ $\quad \mathbf{t}' = \mathbf{B}^{-1} \cdot \mathbf{t}$
- Round each coordinate $\quad \mathbf{v}' = \lfloor \mathbf{t}' \rceil$

Credit: Léo Ducas

Round Off Algorithm



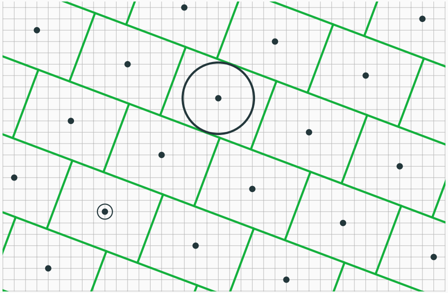
Round Off Algorithm [Lenstra,Babai]:

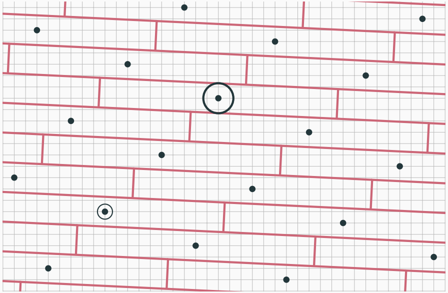
Credit: Léo Ducas

- Use $\mathbf{B}$ to switch to the lattice $\mathbb{Z}^d$ $\quad \mathbf{t}' = \mathbf{B}^{-1} \cdot \mathbf{t}$
- Round each coordinate $\quad \mathbf{v}' = \lfloor \mathbf{t}' \rceil$
- Switch back to Λ $\quad \mathbf{v} = \mathbf{B} \cdot \mathbf{v}'$

Nearest-Plane Algorithm

There is a better algorithm (Nearest Plane) based on Gram–Schmidt Orthogonalisation:



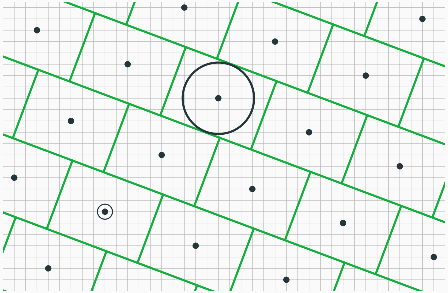


- Worst-case distance: $\frac{1}{2}\sqrt{\sum \|\mathbf{b}_i^\star\|^2}$ (Approx-CVP)
- Correct decoding of $\mathbf{t} = \mathbf{v} + \mathbf{e}$ for $\mathbf{v} \in \Lambda$ if $\|\mathbf{e}\| \leq \min \|\mathbf{b}_i^\star\|$ (BDD)

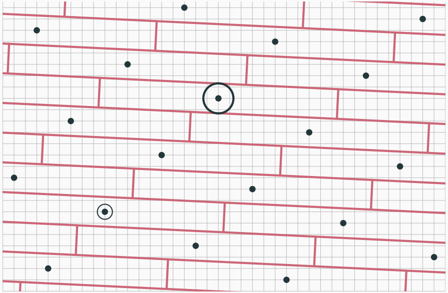
Credit: Léo Ducas

Trapdoors from Lattices ?

Good basis **G**: can solve Approx-CVP / BDD.



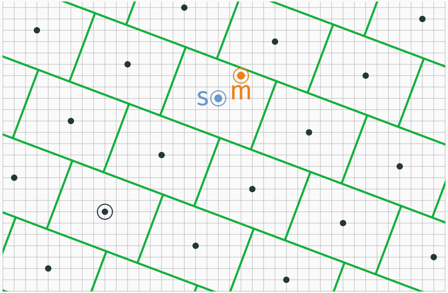
Bad basis **B**: solving CVP is **hard**.



Can we use this as a trapdoor?

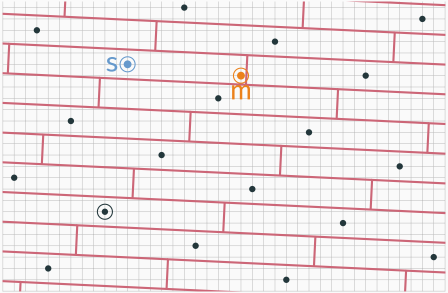
Sign:

1. Hash message to a random vector $\mathbf{m}$.
2. Apply Nearest Plane with a good basis $\mathbf{G}$: find $\mathbf{s} \in \Lambda$ close to $\mathbf{m}$.



Verify:

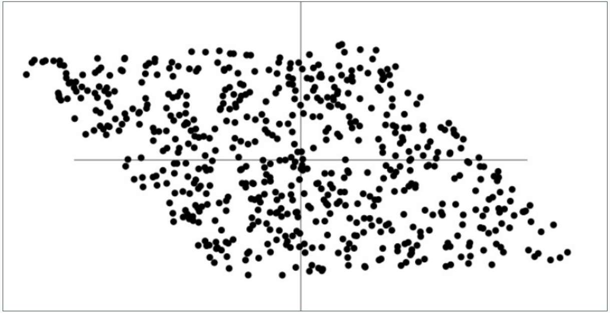
1. Check that $\mathbf{s} \in \Lambda$ using the bad basis $\mathbf{B}$
2. Check that $\mathbf{m}$ is close to $\mathbf{s}$.



Credit: Léo Ducas

A Statistical Attack

The difference $\mathbf{s} - \mathbf{m}$ is always inside the parallelepiped spanned by the good basis $\mathbf{G}$ or its Gram–Schmidt Orthogonalisation $\mathbf{G}^\star$:



- Each signatures $(\mathbf{s}, \mathbf{m})$ leaks information about $\mathbf{G}$.
- Learning a parallepiped from few signatures [NR06]
- Total break of original GGH and NTRUSign schemes

The distribution of $s - m$ can be made **independent** of G by randomising the above algorithms:

Seminal Work

Craig Gentry, Chris Peikert, and Vinod Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. In: *40th ACM STOC*. ed. by Richard E. Ladner and Cynthia Dwork. ACM Press, May 2008, pp. 197–206. DOI: 10.1145/1374376.1374407

q-ary Lattices

Construction of q-ary Lattices (Primal / Construction A)

- Let q be a prime integer[1] and $n < m$ two positive integers.
- The matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ spans the q-ary lattice:

$$\Lambda_q(\mathbf{A}) := \{\mathbf{x} \in \mathbb{Z}^m \mid \exists \mathbf{y} \in \mathbb{Z}_q^n, \mathbf{x} \equiv \mathbf{A} \cdot \mathbf{y} \bmod q\} = \mathbf{A} \cdot \mathbb{Z}_q^n + q\mathbb{Z}^m$$

Lattice Parameters

Assuming $\mathbf{A}$ is full-rank:

- $\dim(\Lambda_q(\mathbf{A})) = m$
- $\mathrm{Vol}(\Lambda_q(\mathbf{A})) = q^{m-n}$

[1] for simplicity

Example

```
q = 19; n = 3; m = 7; A = random_matrix(GF(q), n, m)
B = A.lift().stack(q * identity_matrix(m)); B.T
```

```
[1      4 19                  ]
[ 3  5      19                ]
[15  3 17      19             ]
[ 2  7            19          ]
[14  7               19       ]
[ 9  8 15               19    ]
[15  1 11                  19]
```

```
B.echelon_form()[:B.rank()].T
```

```
[ 1                    ]
[    1                 ]
[       1              ]
[18 14 17 19           ]
[ 1  4  2    19        ]
[ 2 17  6       19     ]
[ 5  7 14          19]
```

Row-Based: SageMath's convention is "row based", i.e. it considers combinations of rows not columns. The literature on lattices favours column-based notation, i.e. combinations of columns. I transposed the output to make it consistent with the slide before.

Construction of q-ary Lattices (Dual / Parity-Check)

- Let q be a prime integer[2] and $n < m$ two positive integers.
- The matrix $\mathbf{A} \in \mathbb{Z}_q^{\mathbf{n} \times \mathbf{m}}$ is the parity-check of the lattice:

$$\Lambda_q^{\perp}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{Z}^m \, | \mathbf{A} \cdot \mathbf{x} \equiv \mathbf{0} \bmod q\} = \ker(\mathbf{x} \mapsto \mathbf{A} \cdot \mathbf{x} \bmod q)$$

Lattice parameters

Assuming $\mathbf{A}$ is full-rank:

- $\dim(\Lambda_q^{\perp}(\mathbf{A})) = m$
- $\mathrm{Vol}(\Lambda_q^{\perp}(\mathbf{A})) = q^{\mathbf{n}}$

[2] for simplicity

Example

```
q = 19; n = 3; m = 7; A = random_matrix(GF(q), n, m)
K = A.right_kernel().basis_matrix()
B = K.lift().stack(q * identity_matrix(m)); B.T
```

```
[ 1          19                  ]
[    1          19               ]
[       1          19            ]
[          1          19         ]
[13  1  1 16             19      ]
[12 15  1 11                19   ]
[12 12  4  2                   19]
```

```
B.echelon_form()[:B.rank()].T
```

```
[ 1                    ]
[    1                 ]
[       1              ]
[          1           ]
[13  1  1 16 19        ]
[12 15  1 11    19     ]
[12 12  4  2       19]
```

The Short Integer Solution Problem (SIS)

Definition (SIS Assumption)

Given a random matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, finding a small non-zero $\mathbf{x} \in \mathbb{Z}^m$ such that $\mathbf{A} \cdot \mathbf{x} \equiv \mathbf{0} \bmod q$ is hard.[3]

Lattice Formulation

Solving Approx-SVP in $\Lambda_q^{\perp}(\mathbf{A})$ is hard.

In [Ajt98], Ajtai established that if solving SIS is easy then solving Approx-SVP for any lattice is also easy.

[3] Also on a quantum computer!

A Simple Application of SIS

Consider the function:

$$f_{\mathbf{A}} : \{0,1\}^m \to \mathbb{Z}_q^n, \qquad \mathbf{x} \mapsto \mathbf{A} \cdot \mathbf{x} \bmod q$$

SIS $\Rightarrow$ Collision Resistant Hashing and One-Way Function

- Finding collisions is as hard as SIS[4] (take the difference)

Moreover, if $m \gg n \log q$:

- $f_{\mathbf{A}}$ is highly surjective (many pre-images for any image)
- Finding pre-images is hard (show it!)

---

[4] Collisions must exist when $m > n \log q$.

GPV Signatures

Outline

KeyGen generate a random (looking) matrix $\mathbf{A}$ together with a short basis td for the lattice $\Lambda_q^{\perp}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{Z}^m \,|\mathbf{A} \cdot \mathbf{x} \equiv \mathbf{0} \bmod q\}$.

Sign Compute $H(m)$.

Find an arbitrary, not-necessarily short, preimage $\mathbf{u}$: $\mathbf{A} \cdot \mathbf{u} \equiv H(m) \bmod q$.

Use the trapdoor td to solve Approx-CVP for $\mathbf{u}$ on $\Lambda_q^{\perp}(\mathbf{A})$, call it $\mathbf{z}$. By construction $\mathbf{A} \cdot \mathbf{z} \equiv \mathbf{0} \bmod q$.

Output $\mathbf{y} := \mathbf{u} - \mathbf{z}$. By construction $\|\mathbf{y}\|$ is small.

Verify Check that $\mathbf{A} \cdot \mathbf{y} \equiv H(m) \bmod q$ and that $\|\mathbf{y}\|$ is small.

Scheme

$\underline{\mathsf{KeyGen}(1^\lambda)}$

$\mathbf{A}, \mathsf{td} \leftarrow\!\!\$\ \mathsf{TrapGen}(1^n, 1^m, q, \beta)$

return $\mathsf{vk} := \mathbf{A}, \mathsf{sk} := \mathsf{td}$

$\underline{\mathsf{Sign}(m, \mathsf{sk})}$

$\mathbf{r} \leftarrow\!\!\$\ \{0,1\}^\lambda$

$\mathbf{y} \leftarrow\!\!\$\ \mathsf{SampPre}(\mathsf{td}, H(m, \mathbf{r}), \beta')$

return $\sigma := (\mathbf{y}, \mathbf{r})$

$\underline{\mathsf{Verify}(\mathsf{vk}, \sigma, m)}$

return $\|\mathbf{y}\| \overset{?}{\leq} \beta' \wedge H(m, \mathbf{r}) \overset{?}{\equiv} \mathbf{A} \cdot \mathbf{y}$

- $(\mathbf{A}, \mathsf{td}) \leftarrow \mathsf{TrapGen}(1^n, 1^m, q, \beta)$ takes dimensions $n, m \in \mathbb{N}$, a modulus $q \in \mathbb{N}$ and a norm bound $\beta \in \mathbb{R}$ and generates a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ and a trapdoor td.
  When $\ell > 2m \log q$, the distribution of $\mathbf{A}$ is within $\mathsf{negl}(\lambda)$ statistical distance to the uniform distribution on $\mathbb{Z}_q^{n \times m}$.
- $\mathbf{u} \leftarrow \mathsf{SampD}(1^n, 1^m, \beta')$ outputs an element in $\mathbf{u} \in \mathbb{Z}^m$ with norm bound $\beta' \geq \beta$.
  $\mathbf{v} := \mathbf{A} \cdot \mathbf{u} \bmod q$ *is within* $\mathsf{negl}(\lambda)$ *statistical distance to the uniform distribution on $\mathbb{Z}_q^n$.*
- $\mathbf{u} \leftarrow \mathsf{SampPre}(\mathsf{td}, \mathbf{v}, \beta')$ takes a trapdoor td, a vector $\mathbf{v} \in \mathbb{Z}_q^n$ and a norm bound $\beta' \geq \beta$ and samples $\mathbf{u} \in \mathbb{Z}^\ell$ satisfying $\mathbf{A} \cdot \mathbf{u} \equiv \mathbf{v} \bmod q$ and $\|\mathbf{u}\| \leq \beta'$.
  $\mathbf{u}$ *is within* $\mathsf{negl}(\lambda)$ *statistical distance to* $\mathbf{u} \leftarrow \mathsf{SampD}(1^n, 1^m, \mathcal{R}, \beta')$ *conditioned on* $\mathbf{v} \equiv \mathbf{A} \cdot \mathbf{u} \bmod q$.

Security Proof

EUF-CMA in the ROM

EUF-CMA	$\mathsf{S}(m)$	$\mathsf{H}(m)$
$\mathcal{Q}, \mathcal{H} \leftarrow \emptyset, \emptyset;$	$\sigma \leftarrow\!\!\$\ \Sigma.\mathsf{Sign}(\mathsf{sk}, m)$	**if** $m \notin \mathcal{H}$ **then**
$\mathsf{vk}, \mathsf{sk} \leftarrow\!\!\$\ \Sigma.\mathsf{KeyGen}(1^\lambda);$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	$\quad \mathcal{H}[m] \leftarrow\!\!\$\ \{0,1\}^\lambda$
$(m^\star, \sigma^\star) \leftarrow\!\!\$\ \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	**return** σ	**return** $\mathcal{H}[m]$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$		
$b_1 := \Sigma.\mathsf{Verify}(\mathsf{vk}, \sigma^\star, m^\star) = 1$		
return $b_0 \wedge b_1$		

$$\mathsf{Adv}^{\text{euf-cma}}_{\mathcal{A},\Sigma}(\lambda) := \Pr[\mathsf{EUF\text{-}CMA}^{\mathcal{A}}_{\Sigma}(\lambda) \Rightarrow 1]$$

Inlining GPV Signatures

Game_0	$\mathsf{S}(m)$	$\mathsf{H}(m, \mathbf{r})$
$\mathcal{Q}, \mathcal{H} \leftarrow \emptyset, \emptyset;$	$\mathbf{r} \leftarrow\!\$\ \{0,1\}^{\lambda}$	**if** $(m, \mathbf{r}) \notin \mathcal{H}$ **then**
$\mathbf{A}, \mathsf{td} \leftarrow\!\$\ \mathsf{TrapGen}(\cdots)$	$\mathbf{y} \leftarrow\!\$\ \mathsf{SampPre}(\mathsf{td}, H(m, \mathbf{r}), \beta')$	$\quad \mathcal{H}[m, \mathbf{r}] \leftarrow\!\$\ \mathbb{Z}_q^n$
$(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star)) \leftarrow \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	$\sigma := (\mathbf{y}, \mathbf{r})$	**return** $\mathcal{H}[m, \mathbf{r}]$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	
$b_1 := \lVert \mathbf{y}^\star \rVert \leq \beta'$	**return** σ	
$b_2 := H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$		
return $b_0 \wedge b_1 \wedge b_2$		

Changing the RO

Game_1	$\mathsf{S}(m)$	$\mathsf{H}(m, \mathbf{r})$
$\mathcal{Q}, \mathcal{H}, \mathcal{P} \leftarrow \emptyset, \emptyset, \emptyset;$	$\mathbf{r} \leftarrow\!\!\$ \{0,1\}^\lambda$	**if** $(m, \mathbf{r}) \notin \mathcal{H}$ **then**
$\mathbf{A}, \mathsf{td} \leftarrow\!\!\$ \mathsf{TrapGen}(\cdots)$	$\mathbf{y} \leftarrow\!\!\$ \mathsf{SampPre}(\mathsf{td}, H(m, \mathbf{r}), \beta')$	$\quad \mathbf{y} \leftarrow\!\!\$ \mathsf{SampD}(\cdots, \beta')$
$(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star)) \leftarrow \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	$\sigma := (\mathbf{y}, \mathbf{r})$	$\quad \mathbf{t} := \mathbf{A} \cdot \mathbf{y} \bmod q$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	$\quad \mathcal{H}[m, \mathbf{r}] \leftarrow \mathbf{t}$
$b_1 := \|\mathbf{y}^\star\| \leq \beta'$	**return** σ	$\quad \mathcal{P}[m, \mathbf{r}] \leftarrow \mathbf{y}$
$b_2 := H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$		**return** $\mathcal{H}[m, \mathbf{r}]$
$b_0 \wedge b_1 \wedge b_2$		

By the Leftover Hash Lemma the distributions of Game_0 and Game_1 are statistically close.

Bookkeeping

Game_2	$\mathsf{S}(m)$	$\mathsf{H}(m, \mathbf{r})$
$\mathcal{Q}, \mathcal{H}, \mathcal{P} \leftarrow \emptyset, \emptyset, \emptyset;$	$\mathbf{r} \leftarrow\$ \{0,1\}^{\lambda}$	**if** $(m, \mathbf{r}) \notin \mathcal{H}$ **then**
$\mathsf{bad} \leftarrow \mathbf{false}$	**if** $(m, \mathbf{r}) \in \mathcal{H}$ **then**	$\quad \mathbf{y} \leftarrow\$ \mathsf{SampD}(\cdots, \beta')$
$\mathbf{A}, \mathsf{td} \leftarrow\$ \mathsf{TrapGen}(\cdots)$	$\quad \mathsf{bad} \leftarrow \mathbf{true}$	$\quad \mathbf{t} := \mathbf{A} \cdot \mathbf{y} \bmod q$
$(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star)) \leftarrow \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	$\mathbf{y} \leftarrow\$ \mathsf{SampPre}(\mathsf{td}, H(m, \mathbf{r}), \beta')$	$\quad \mathcal{H}[m, \mathbf{r}] \leftarrow \mathbf{t}$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$	$\sigma := (\mathbf{y}, \mathbf{r})$	$\quad \mathcal{P}[m, \mathbf{r}] \leftarrow \mathbf{y}$
$b_1 := \lVert \mathbf{y}^\star \rVert \leq \beta'$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	**return** $\mathcal{H}[m, \mathbf{r}]$
$b_2 := H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$	**return** σ	
$b_0 \wedge b_1 \wedge b_2$		

No change in behaviour.

Avoid Collisions

Game_3	$\mathsf{S}(m)$	$\mathsf{H}(m, \mathbf{r})$
$\mathcal{Q}, \mathcal{H}, \mathcal{P} \leftarrow \emptyset, \emptyset, \emptyset;$	$\mathbf{r} \leftarrow\!\!\$ \{0,1\}^{\lambda}$	**if** $(m, \mathbf{r}) \notin \mathcal{H}$ **then**
$\mathsf{bad} \leftarrow$ **false**	**if** $(m, \mathbf{r}) \in \mathcal{H}$ **then**	$\quad \mathbf{y} \leftarrow\!\!\$ \mathsf{SampD}(\cdots, \beta')$
$\mathbf{A}, \mathsf{td} \leftarrow\!\!\$ \mathsf{TrapGen}(\cdots)$	$\quad \mathsf{bad} \leftarrow$ **true**	$\quad \mathbf{t} := \mathbf{A} \cdot \mathbf{y} \bmod q$
$(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star)) \leftarrow \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	$\quad$ **abort**	$\quad \mathcal{H}[m, \mathbf{r}] \leftarrow \mathbf{t}$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$	$\mathbf{y} \leftarrow\!\!\$ \mathsf{SampPre}(\mathsf{td}, H(m, \mathbf{r}), \beta')$	$\quad \mathcal{P}[m, \mathbf{r}] \leftarrow \mathbf{y}$
$b_1 := \lVert \mathbf{y}^\star \rVert \leq \beta'$	$\sigma := (\mathbf{y}, \mathbf{r})$	**return** $\mathcal{H}[m, \mathbf{r}]$
$b_2 := H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	
$b_0 \wedge b_1 \wedge b_2$	**return** σ	

Apply Fundamental Lemma of Game Playing with $\Pr[\mathsf{bad}] \approx |\mathcal{Q}|/2^{\lambda/2}$.

Fake Signatures

Game_4	$\mathsf{S}(m)$	$\mathsf{H}(m, \mathbf{r})$
$\mathcal{Q}, \mathcal{H}, \mathcal{P} \leftarrow \emptyset, \emptyset, \emptyset;$	$\mathbf{r} \leftarrow\$ \{0,1\}^\lambda$	**if** $(m, \mathbf{r}) \notin \mathcal{H}$ **then**
$\mathsf{bad} \leftarrow \mathbf{false}$	**if** $(m, \mathbf{r}) \in \mathcal{H}$ **then**	$\quad \mathbf{y} \leftarrow\$ \mathsf{SampD}(\cdots, \beta')$
$\mathbf{A}, \mathsf{td} \leftarrow\$ \mathsf{TrapGen}(\cdots)$	$\quad \mathsf{bad} \leftarrow \mathbf{true}$	$\quad \mathbf{t} := \mathbf{A} \cdot \mathbf{y} \bmod q$
$(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star)) \leftarrow \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	$\quad$ **abort**	$\quad \mathcal{H}[m, \mathbf{r}] \leftarrow \mathbf{t}$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$	$H(m, \mathbf{r})$	$\quad \mathcal{P}[m, \mathbf{r}] \leftarrow \mathbf{y}$
$b_1 := \|\mathbf{y}^\star\| \leq \beta'$	$\mathbf{y} \leftarrow \mathcal{P}[m, \mathbf{r}]$	**return** $\mathcal{H}[m, \mathbf{r}]$
$b_2 := H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$	$\sigma := (\mathbf{y}, \mathbf{r})$	
$b_0 \wedge b_1 \wedge b_2$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	
	return σ	

Use SampPre $\approx$ SampD.

Throw Away Trapdoor

Game_5	$\mathsf{S}(m)$	$\mathsf{H}(m, \mathbf{r})$
$\mathcal{Q}, \mathcal{H}, \mathcal{P} \leftarrow \emptyset, \emptyset, \emptyset;$	$\mathbf{r} \leftarrow\!\!\$\ \{0,1\}^{\lambda}$	**if** $(m, \mathbf{r}) \notin \mathcal{H}$ **then**
$\mathsf{bad} \leftarrow \mathbf{false}$	**if** $(m, \mathbf{r}) \in \mathcal{H}$ **then**	$\quad \mathbf{y} \leftarrow\!\!\$\ \mathsf{SampD}(\cdots, \beta')$
$\mathbf{A} \leftarrow\!\!\$\ \mathbb{Z}_q^{n \times 2\,n\lceil \log q \rceil}$	$\quad \mathsf{bad} \leftarrow \mathbf{true}$	$\quad \mathbf{t} := \mathbf{A} \cdot \mathbf{y} \bmod q$
$(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star)) \leftarrow \mathcal{A}^{\mathsf{S},\mathsf{H}}(\mathsf{vk});$	$\quad$ **abort**	$\quad \mathcal{H}[m, \mathbf{r}] \leftarrow \mathbf{t}$
$b_0 := (m^\star, \cdot) \notin \mathcal{Q}$	$H(m, \mathbf{r})$	$\quad \mathcal{P}[m, \mathbf{r}] \leftarrow \mathbf{y}$
$b_1 := \lVert \mathbf{y}^\star \rVert \leq \beta'$	$\mathbf{y} \leftarrow \mathcal{P}[m, \mathbf{r}]$	**return** $\mathcal{H}[m, \mathbf{r}]$
$b_2 := H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$	$\sigma := (\mathbf{y}, \mathbf{r})$	
$b_0 \wedge b_1 \wedge b_2$	$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{(m, \sigma)\}$	
	return σ	

Use that TrapGen produces pseudorandom $\mathbf{A}$.

Invoke SIS

We turn an adversary $\mathcal{A}$ that wins Game_5 into an adversary that breaks SIS for $\mathbf{A}$.

- The adversary outputs $(m^\star, (\mathbf{y}^\star, \mathbf{r}^\star))$ s.t. $H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star$ and $\|\mathbf{y}^\star\| \leq \beta'$.
- If $\mathcal{A}$ has not called $H(m^\star, \mathbf{r}^\star)$ before producing its forgery, $\mathbf{t} := H(m^\star, \mathbf{r}^\star)$ is unknown and (close to) uniformly random: it succeeds with probability $1/q^n < 2^{-\lambda}$.
- If $\mathcal{A}$ has called $H(m^\star, \mathbf{r}^\star)$ then $\mathbf{y} \leftarrow \mathcal{P}[m^\star, \mathbf{r}^\star]$ with $\|\mathbf{y}\| \leq \beta'$ and with high probability $\mathbf{y} \neq \mathbf{y}^\star$.

$$H(m^\star, \mathbf{r}^\star) \equiv \mathbf{A} \cdot \mathbf{y}^\star \equiv \mathbf{A} \cdot \mathbf{y} \quad \Rightarrow \quad \mathbf{0} \equiv \mathbf{A} \cdot (\mathbf{y}^\star - \mathbf{y}) \text{ and } \|\mathbf{y}^\star - \mathbf{y}\| \leq 2\,\beta'$$

The Role of r

If the signing algorithm did not sample a fresh $\mathbf{r}$ for each signature, then

- Game_3 would output new, fresh $\mathbf{y}$ on each call, but
- Game_4 would output the same $\mathbf{y}$ on each call.

which is easy to distinguish by just calling the signing algorithm twice.

Corresponding Attack

Without $\mathbf{r}$, when $\mathcal{A}$ calls the signing oracle twice on the same m, it would get two different preimages $\mathbf{y}_0, \mathbf{y}_1$ for $H(m)$:

$$H(m) \equiv \mathbf{A} \cdot \mathbf{y}_0 \equiv \mathbf{A} \cdot \mathbf{y}_1 \quad \Rightarrow \quad \mathbf{0} \equiv \mathbf{A} \cdot (\mathbf{y}_0 - \mathbf{y}_1) \text{ and } \|\mathbf{y}_0 - \mathbf{y}_1\| \leq 2\,\beta'.$$

- This solves SIS, which is supposed to be hard for our security proof to work.

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- This solves SIS, which is supposed to be hard for our security proof to work.
- This gives a short vector in $\Lambda_q^\perp(\mathbf{A})$, such short vectors make up our trapdoor!

In the Random Oracle Model we can choose what the hash function outputs.

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