Minicrypt with Algebraic Structure

Advanced Topics in ~~Cybersecurity~~Cryptography (7CCSMATC)

Martin R. Albrecht

Outline

Introduction

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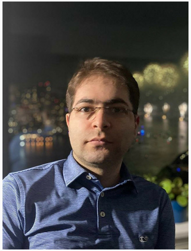
(Bounded) Input-Homomorphic Weak Pseudorandom Functions (IHwPRFs)

Introduction

Main Reference

Navid Alamati, Hart Montgomery, Sikhar Patranabis, and Arnab Roy. Minicrypt Primitives with Algebraic Structure and Applications. In: *Journal of Cryptology* 36.1 (Jan. 2023), p. 2.
DOI: 10.1007/s00145-022-09442-2




Minicrypt: One-Way Functions all the way down

- OWFs $\Rightarrow$ PRGs [BM84]; PRGs $\Rightarrow$ PRFs [GGM86]; PRFs $\Rightarrow$ PRPs [LR88]
- OWFs $\Rightarrow$ Digital Signatures [Lam79; Mer79; NY89]

[AMPR23]

"It is natural to ask: is there any sort of mathematical structure that is inherent to [Cryptomania] primitives as well?"

Minicrypt: One-Way Functions all the way down

- OWFs $\Rightarrow$ PRGs [BM84]; PRGs $\Rightarrow$ PRFs [GGM86]; PRFs $\Rightarrow$ PRPs [LR88]
- OWFs $\Rightarrow$ Digital Signatures [Lam79; Mer79; NY89]

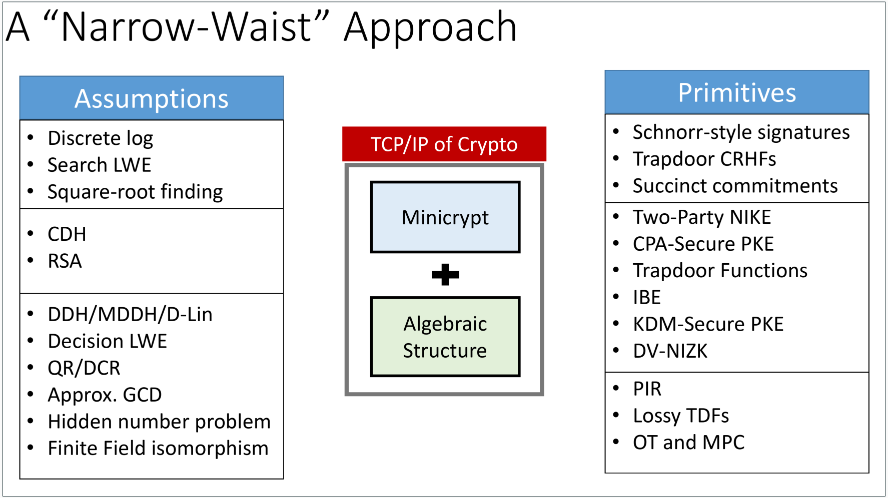
Question

Is it possible to construct Cryptomania primitives from simple Minicrypt primitives that are additionally equipped with some **algebraic structure**?

Alternative Pitch I

Picture credit: `https://drive.google.com/file/d/1amPSJROTuzPIyciW_WN9aat3m5BMnmY0/view`

Alternative Pitch II



Picture credit: `https://drive.google.com/file/d/1amPSJROTuzPIyciW_WN9aat3m5BMnmY0/view`

Computational Assumptions

Diffie-Hellman (DH)

Discrete Logarithms

Let p be a λ-bit prime and let g be a generator of the multiplicative group $\mathbb{Z}_p^*$. Given $g^a \bmod p$ find a.

DH

Let p be a λ-bit prime and let g be a generator of the multiplicative group $\mathbb{Z}_p^*$. Given

$$(g^a \bmod p, g^b \bmod p, u),$$

decide if $u \equiv g^{ab} \bmod p$ or random.

Learning with Errors (LWE)

Definition (LWE)

Let m, n, q be positive integers, χ be a probability distribution on $\mathbb{Z}$ and $\mathbf{s}$ be a uniformly random vector in $\mathbb{Z}_q^n$. Denote by $\mathcal{L}_{\mathbf{s},\chi}$ the probability distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$ obtained by choosing $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ uniformly at random, choosing every component of $\mathbf{e} \in \mathbb{Z}$ according to χ and returning

$$(\mathbf{A},\ \mathbf{c}) = (\mathbf{A},\ \mathbf{A} \cdot \mathbf{s} + \mathbf{e}) \in \mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m.$$

Search-LWE recover $\mathbf{s}$ from $(\mathbf{A}, \mathbf{c}) = (\mathbf{A}, \mathbf{A} \cdot \mathbf{s} + \mathbf{e}) \in \mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m$ sampled according to $\mathcal{L}_{\mathbf{s},\chi}$.

Decision-LWE decide if $(\mathbf{A}, \mathbf{c})$ is sampled from $\mathcal{L}_{\mathbf{s},\chi}$ or the uniform distribution.

Search-LWE is like DLOG, Decision-LWE is like DH

DH and LWE

DH Land	LWE Land
g	A
g^x	$\mathsf{A} \cdot \mathsf{s} + \mathsf{e}$
$g^x \cdot g^y = g^{x+y}$	$(\mathsf{A} \cdot \mathsf{s} + \mathsf{e}_0) + (\mathsf{A} \cdot \mathsf{t} + \mathsf{e}_1) = \mathsf{A} \cdot (\mathsf{s} + \mathsf{t}) + \mathsf{e}'$
$(g^a)^b = (g^b)^a$	$\mathsf{t}^T \cdot (\mathsf{A} \cdot \mathsf{s} + \mathsf{e}) = \mathsf{t}^T \cdot \mathsf{A} \cdot \mathsf{s} + \mathsf{t}^T \cdot \mathsf{e}$ $\approx \mathsf{t}^T \cdot \mathsf{A} \cdot \mathsf{s} + \mathsf{e} \cdot \mathsf{s}$
(g, g^a, g^b, g^{ab}) $\approx_c (g, g^a, g^b, u)$	$(\mathsf{A},\ \mathsf{A} \cdot \mathsf{s} + \mathsf{e},\ \mathsf{t}^T \cdot \mathsf{A} + \mathsf{d}^T,\ \mathsf{t}^T \cdot \mathsf{A} \cdot \mathsf{s} + e')$ $\approx_c (\mathsf{A},\ \mathsf{A} \cdot \mathsf{s} + \mathsf{e},\ \mathsf{t}^T \cdot \mathsf{A} + \mathsf{d}^T,\ u)$

(Bounded) Homomorphic One-way Functions (HOWFs)

One-way Functions (OWFs)

Definition (OWF)

A function $f : \{0,1\}^* \to \{0,1\}^*$ is **one-way** if the following two conditions hold:

1. **(Easy to compute)** There exists a polynomial-time algorithm $\mathcal{A}_f(x)$ computing f, i.e. $\mathcal{A}_f(x) = f(x)$ for all x.
2. **(Hard to invert)** for every probabilistic polynomial-time $\mathcal{B}$, we have:

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathcal{B}\left(1^n, f(x)\right) \in f^{-1}(f(x))\right] < 1/\text{poly}(n).$$

(Bounded) Homomorphic One-way Functions (HOWFs)

Definition (Homomorphic OWF)

A homomorphic one-way function (HOWF) over an input group $(\mathcal{X}, \oplus)$ and an output group $(\mathcal{Y}, \otimes)$ is a one-way function $f : \mathcal{X} \to \mathcal{Y}$ such that for any $x_0, x_1 \in \mathcal{X}$, we have $f(x_0 \oplus x_1 = f(x_0) \otimes f(x_1)$.

Definition (γ-Bounded Homomorphic OWF)

A γ-bounded homomorphic one-way function (γ-bounded HOWF) over an input group $(\mathcal{X}, \oplus)$ and an output group $(\mathcal{Y}, \otimes)$ is a one-way function $f : \mathcal{X} \to \mathcal{Y}$ such that

$$\mathcal{R}\left(f\left(\bigoplus_{j\in\{0,\ldots,L-1\}} x_j\right)\right) = \mathcal{R}\left(\bigotimes_{j\in\{0,\ldots,L-1\}} f\left(x_j\right)\right)$$

subject to the restriction $L \leq \gamma$ and some $\mathcal{R}$.

Instantiation

From DLOG/DH

$$\begin{aligned} f_g(x) &= g^x \\ f_g(x_0 + x_1) &= g^{x_0} \cdot g^{x_1} \\ &= f_g(x_0) \cdot f_g(x_1) \end{aligned}$$

From Search-LWE

$$\begin{aligned} \mathcal{R}\left(b \in \mathbb{Z}_q\right) &= \begin{cases} 0 & |b| \leq q/4 \\ 1 & |b| > q/4 \end{cases} \\ f_{\mathsf{A}}(\mathsf{s}, \mathsf{e}) &= \mathsf{A} \cdot \mathsf{s} + \mathsf{e} \\ \mathcal{R}(f_{\mathsf{A}}(\mathsf{s} + \mathsf{t}, \mathsf{e} + \mathsf{d})) &= \mathsf{A} \cdot \mathsf{s} + \mathsf{e} + \mathsf{A} \cdot \mathsf{t} + \mathsf{d} \\ &= \mathcal{R}(f_{\mathsf{A}}(\mathsf{s}, \mathsf{e}) + f_{\mathsf{A}}(\mathsf{t}, \mathsf{d})) \end{aligned}$$

Here be dragons!

Definition (Collision Resistance)

A function $f : \{0,1\}^{\mathrm{poly}(n)} \to \{0,1\}^n$ is called collision resistant if for every probabilistic polynomial-time $\mathcal{A}$, we have

$$\Pr\left[x_0 \neq x_1 \wedge f(x_0) = f(x_1) \mid x_0, x_1 \leftarrow \mathcal{A}(f)\right] < 1/\mathrm{poly}(n).$$

HOWF $\Rightarrow$ CRHF

Setup:

1. Sample $2n$ uniform elements in $\mathcal{X}$:

$$\mathsf{X} := \begin{pmatrix} x_{0,0} & \cdots & x_{0,n-1} \\ x_{1,0} & \cdots & x_{1,n-1} \end{pmatrix} \leftarrow\$ \mathcal{X}^{2\times n}$$

2. Compute

$$\mathsf{Y} := \begin{pmatrix} y_{0,0} & \cdots & y_{0,n-1} \\ y_{1,0} & \cdots & y_{1,n-1} \end{pmatrix} = \begin{pmatrix} f(x_{0,0}) & \cdots & f(x_{0,n-1}) \\ f(x_{1,0}) & \cdots & f(x_{1,n-1}) \end{pmatrix}$$

Call:

$$H_{\mathsf{y}}(\mathbf{r} = (r_0, \ldots, r_{n-1})) = \bigotimes_{0\leq j<n} y_{r_j,j}.$$

Proof I

We assume there exists some adversary $\mathcal{A}$ that breaks collision resistance, i.e. outputs $\mathbf{r}, \mathbf{r}'$ s.t. $\mathbf{r} \neq \mathbf{r}'$ but $H_{\mathbf{y}}(\mathbf{r}) = \mathcal{H}_{\mathbf{y}}(\mathbf{r}')$.

We construct an adversary $\mathcal{B}$ that breaks the one-wayness of $f(\cdot)$ using $\mathcal{A}$ as a subroutine.

1. $\mathcal{B}$ receives a challenge $y^* \in \mathcal{Y}$ and wants to find x^* s.t. $f(x^*) = y^*$.
2. $\mathcal{B}$ samples $2n$ uniform elements in $\mathcal{X}$: $\mathbf{x} := \{x_{b,j} \leftarrow\$ \mathcal{X}\}_{0 \leq j < n, b \in \{0,1\}}$ and sets $\mathbf{y} := \{y_{b,j} := f(x_{j,b})\}_{0 \leq j < n, b \in \{0,1\}}$
3. $\mathcal{B}$ picks a random $i \in \{0, \ldots, n-1\}$ and random $b^* \in \{0,1\}$ and sets $y_{b^*,i} := y^*$.

Proof II

4. $\mathcal{B}$ runs $\mathcal{A}$ on $\mathbf{y}$ and receives $\mathbf{r} \neq \mathbf{r}'$ back, s.t. $H_{\mathbf{y}}(\mathbf{r}) = H_{\mathbf{y}}(\mathbf{r}')$.
5. If $r_i = r_i'$ then $\mathcal{B}$ aborts (or outputs a random answer)

6. Assume $r_i = b^*$ (swap r_i and r'_i otherwise)
7. We have

$$\bigotimes_{0\le j<n} y_{r_j,j} = \bigotimes_{0\le j<n} y_{r'_j,j}$$

$$\left(\bigotimes_{0\le j<i} y_{r_j,j}\right)^{-1} \cdot \bigotimes_{0\le j<n} y_{r_j,j} \cdot \left(\bigotimes_{i+1\le j<n} y_{r_j,j}\right)^{-1} = \left(\bigotimes_{0\le j<i} y_{r_j,j}\right)^{-1} \cdot \bigotimes_{0\le j<n} y_{r'_j,j} \cdot \left(\bigotimes_{i+1\le j<n} y_{r_j,j}\right)^{-1}$$

$$y^* = \left(\bigotimes_{0\le j<i} y_{r_j,j}\right)^{-1} \cdot \bigotimes_{0\le j<n} y_{r'_j,j} \cdot \left(\bigotimes_{i+1\le j<n} y_{r_j,j}\right)^{-1}$$

8. Note that the RHS is independent of y^*

Proof IV

9. $\mathcal{B}$ outputs:

$$x^* = \left(\bigoplus_{0 \le j < i} x_{r_j,j}\right)^{-1} \cdot \bigoplus_{0 \le j < n} x_{r'_j,j} \cdot \left(\bigoplus_{i+1 \le j < n} x_{r_j,j}\right)^{-1}$$

10. By the input-homomorphism, we have $f(x^*) = y^*$.
11. $\mathcal{B}$ succeeds whenever $\mathcal{A}$ succeeds and $r_i \neq r'_i$, which happens with probability at least $1/n$ for randomly chosen i.

Schnorr-Style Signatures

Let

- $f : \mathcal{X} \to \mathcal{Y}$ be a HOWF
- n be some integer
- $H : \mathcal{Y} \times \{0,1\}^* \to \{0,1\}^n$ be a hash function

Gen

- $\mathsf{sk} := \mathbf{X} := \begin{pmatrix} x_{0,0} & \cdots & x_{0,n-1} \\ x_{1,0} & \cdots & x_{1,n-1} \end{pmatrix} \leftarrow\$ \mathcal{X}^{2\times n}$;
- $\mathsf{vk} := \mathbf{Y} := \begin{pmatrix} f(x_{0,0}) & \cdots & f(x_{0,n-1}) \\ f(x_{1,0}) & \cdots & f(x_{1,n-1}) \end{pmatrix}$

Sign(sk, m)

1. $x^* \leftarrow\$ \mathcal{X}$ and set $y^* \leftarrow f(x^*)$
2. $\mathbf{r} := H(y^*, m)$
3. $\hat{x} := x^* \oplus \left(\bigoplus_{0\le j<n} x_{r_j,j}\right)^{-1}$

$\sigma := (\mathbf{r}, \hat{x}, y^*)$

Verify(vk, m)

Check:

1. $\mathbf{r} \stackrel{?}{=} H(y^*, m)$
2. $y^* \stackrel{?}{=} f(\hat{x}) \otimes \left(\bigotimes_{0\le j<n} y_{r_j,j}\right)$

Schnorr-Style Signatures (DLog Instantiation: $f : x \to g^x \bmod p$)

Let

- n be some integer
- $H : \mathbb{Z}_p^* \times \{0,1\}^* \to \{0,1\}^n$ be a hash function

Gen

- $\mathsf{sk} := \mathbf{X} := \begin{pmatrix} x_{0,0} & \dots & x_{n-1,1} \\ x_{0,1} & \dots & x_{n-1,1} \end{pmatrix} \leftarrow\$ \mathbb{Z}_{p-1}^{2\times n};$
- $\mathsf{vk} := \mathbf{Y} := \begin{pmatrix} g^{x_{0,0}} & \dots & g^{x_{n-1,1}} \\ g^{x_{0,1}} & \dots & g^{x_{n-1,1}} \end{pmatrix}$

Sign(sk, m)

You will do this in Assignment 2.

Verify(vk, m)

You will do this in Assignment 2.

Actual Schnorr Signatures

Let

- $H : \mathbb{G} \times \{0,1\}^* \to \mathbb{Z}_p$ be a hash function

Gen

- $\mathsf{sk} := x \leftarrow\!\$\; \mathbb{Z}_p$;
- $\mathsf{vk} := Y := g^x$

Claus-Peter Schnorr. Efficient Identification and Signatures for Smart Cards. In: *CRYPTO'89*. Ed. by Gilles Brassard. Vol. 435. LNCS. Springer, New York, Aug. 1990, pp. 239–252. DOI: 10.1007/0-387-34805-0_22

Sign(sk, m)

1. $x^* \leftarrow\!\$\; \mathbb{Z}_p$ and set $y^* \leftarrow g^{x^*}$
2. $r := H(y^*, m)$
3. $\hat{x} := x^* - x \cdot r$

$\sigma := (r, \hat{x}, y^*)$

Verify(vk, m)

Check:

1. $r \stackrel{?}{=} H(y^*, m)$
2. $y^* \stackrel{?}{=} g^{\hat{x}} \cdot y^r = g^{\hat{x}} \cdot g^{xr}$

Proof

The proof of this (ubiquitous) construction relies on two proof techniques we have yet to cover:

- the Random Oracle Model
- Rewinding

Note 3.7. *The aforementioned signature scheme has an a priori bounded number of homomorphic operations, which allows it to be instantiated similarly using a γ-bounded HOWF, subject to the restriction that $2n < \gamma$. — [AMPR23]*

Here be dragons!

(Bounded) Input-Homomorphic Weak Pseudorandom Functions (IHwPRFs)

Statistical Distance (SD) aka Total Variation Distance

Definition (Statistical Distance)

Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be two distributions with support $\mathcal{X}$. The Statistical Distance (SD) between $\mathcal{D}_0$ and $\mathcal{D}_1$ is:

$$\mathrm{SD}(\mathcal{D}_0, \mathcal{D}_1) := 1/2 \cdot \sum_{x \in \mathcal{X}} |\mathcal{D}_0(x) - \mathcal{D}_1(x)|$$

Lemma (Distinguishing)

Given $\mathrm{poly}(\lambda)$ *samples, no adversary can distinguish* $\mathcal{D}_0, \mathcal{D}_1$ *with advantage* $\geq 1/\mathrm{poly}(\lambda)$ *when* $\mathrm{SD}(\mathcal{D}_0, \mathcal{D}_1) < 1/\mathrm{poly}(\lambda)$.

Leftover Hash Lemma (LHL)

Lemma (Leftover Hash Lemma)

Let $(\mathcal{X}, \oplus)$ *be a finite group of size* $|\mathcal{X}|$ *and let* n *be a positive integer. Denote by* $\mathcal{U}(\mathcal{X})$ *the uniform distribution on* $\mathcal{X}$*. For any fixed* $(2 \times n)$*-matrix of group elements* $\mathsf{X} := \{x_{b,j}\}_{b \in \{0,1\}, 0 \le j < n} \in \mathcal{X}^{2 \times n}$ *let:* $\mathcal{S}_{\mathsf{X}} = \left\{ \bigoplus_{0 \le j < n} x_{j,r_j} : \mathbf{r} \leftarrow \{0,1\}^n \right\}$*. For sufficiently large* n *it holds that*

$$\Pr_{\mathsf{X} \leftarrow \mathcal{X}^{2 \times n}} \left[\mathrm{SD}\left(\mathcal{S}_{\mathsf{X}}, \mathcal{U}(\mathcal{X})\right) > \sqrt[4]{\frac{|\mathcal{X}|}{2^n}} \right] \le \sqrt[4]{\frac{|\mathcal{X}|}{2^n}}.$$

If $n > \log(|\mathcal{X}|) + \omega(\log(\lambda))$ *and* X *uniform, then with overwhelming probability the statistical distance is negligible.*

PRF

Definition (PRF)

A PRF is a keyed function $F_k : \mathcal{X} \to \mathcal{Y}$ with $k \leftarrow\$ \mathcal{K}$. We say F_k is (t, ε)-secure **PRF** if for Game_0 and Game_1 defined below we have:

$$\forall \mathcal{D} \in t \text{ steps: } \mathsf{Adv}_F^{\mathrm{prf}}(\mathcal{D}) = \left|\Pr[\mathcal{D}^{\mathsf{Game}_1} = 1] - \Pr[\mathcal{D}^{\mathsf{Game}_0} = 1]\right| < \varepsilon$$

Game_0

$f \leftarrow \emptyset$

return $\mathcal{D}^{\mathsf{F}}$

Game_1

$f \leftarrow \emptyset;\ k \leftarrow\$ \mathcal{K}$

return $\mathcal{D}^{\mathsf{F}}$

$\mathsf{F}(x)$

if $x \in \pi.\mathsf{keys}$ **then** $y \leftarrow f[x]$

else

  $y \leftarrow\$ \mathcal{Y};\ f[x] \leftarrow y$

$y \leftarrow F_k(x)$ // Game_1

return y

Definition (wPRF)

A PRF is a keyed function $F_k : \mathcal{X} \to \mathcal{Y}$ with $k \leftarrow\$ \mathcal{K}$. We say F_k is (t, ε)-secure **wPRF** if for Game_0 and Game_1 defined below we have:

$$\forall \mathcal{D} \in t \text{ steps: } \mathsf{Adv}_F^{\mathrm{wprf}}(\mathcal{D}) = \left|\Pr[\mathcal{D}^{\mathsf{Game}_1} = 1] - \Pr[\mathcal{D}^{\mathsf{Game}_0} = 1]\right| < \varepsilon$$

Game_0

$f \leftarrow \emptyset$

return $\mathcal{D}^{\mathsf{F}}$

Game_1

$f \leftarrow \emptyset;\ k \leftarrow\$ \mathcal{K}$

return $\mathcal{D}^{\mathsf{F}}$

F()

$x \leftarrow\$ \mathcal{X}$

if $x \in \pi.\mathsf{keys}$ **then** $y \leftarrow f[x]$

else

$\quad y \leftarrow\$ \mathcal{Y};\ f[x] \leftarrow y$

$y \leftarrow F_k(x)$ // Game_1

return x, y

(Bounded) IHwPRF

Definition (Input-Homomorphic Weak PRF)

We call a keyed function $F_k : \mathcal{X} \to \mathcal{Y}$ an IHwPRF if the following conditions hold:

1. $F_k(\cdot)$ is a weak PRF.
2. Both $(\mathcal{X}, \oplus)$ and $(\mathcal{Y}, \otimes)$ are both efficiently samplable groups.
3. The group operations $\oplus$ and $\otimes$, and their inverses are efficiently computable.
4. For every $k \in \mathcal{K}$ and every $x_0, x_1 \in \mathcal{X}$, we have:

$$F_k(x_0 \oplus x_1) = F_k(x_0) \otimes F_k(x_1).$$

γ-Bounded Variant

$$\mathcal{R}\left(F_k\left(\bigoplus_{0 \le i < L} x_i\right)\right) = \mathcal{R}\left(\bigotimes_{0 \le i < L} F_k(x_i)\right) \text{ for } L \le \gamma \text{ and some } \mathcal{R}.$$

IHwPRF from DH

$$F_k(h) := h^k$$

Bounded IHwPRF From LWE

- Let $D_\chi()$ be an algorithm taking ℓ bits and outputting a sample following the distrbution χ
- Let $F' : \mathbb{Z}_q^n \to \{0,1\}^\ell$ be a weak PRF outputting ℓ bits
- Define the bounded IHwPRF as:

$$F_{(k,\mathbf{s})}(\mathbf{a}) = \langle \mathbf{a}, \mathbf{s} \rangle + D_\chi(F'_k(\mathbf{a})).$$

- Define the relation as

$$\mathcal{R}\left(b \in \mathbb{Z}_q\right) = \begin{cases} 0 & |b| \leq q/4 \\ 1 & |b| > q/4 \end{cases}$$

Here be dragons!

Construction:

Let $F_k : \mathcal{X} \to \mathcal{Y}$ be an IHwPRF. Fix $n > 3\log|\mathcal{X}|$ and sample

$$\mathsf{X} := \begin{pmatrix} x_{0,0} & \cdots & x_{n-1,1} \\ x_{0,1} & \cdots & x_{n-1,1} \end{pmatrix} \leftarrow\$ \mathcal{X}^{2\times n}$$

Define $OWF_{\mathsf{X}} : \{0,1\}^n \to \mathcal{X}$ as

$$OWF_{\mathsf{X}}(\mathbf{r}) = \bigoplus_{0\le j<n} x_{r_j,j}.$$

The construction **never** calls $F_k()$!

Proof I

We construct an adversary $\mathcal{B}$ against the IHwPRF security of F_k using a OWF adversary $\mathcal{A}$ against OWF.

1. $\mathcal{B}$ queries the wPRF oracle $2n$ times and obtains.

$$\mathbf{X} := \begin{pmatrix} x_{0,0} & \dots & x_{n-1,1} \\ x_{0,1} & \dots & x_{n-1,1} \end{pmatrix} \leftarrow\$ \mathcal{X}^{2\times n} \text{ and } \mathbf{Y} := \begin{pmatrix} y_{0,0} & \dots & y_{n-1,1} \\ y_{0,1} & \dots & y_{n-1,1} \end{pmatrix}$$

2. It queries the oracle one more time to obtain x^*, y^*. It wishes to decide if $y_{b,j} = F_k(x_{b,j})$ and $y^* = F_k(x^*$.
3. $\mathcal{B}$ sends $\mathbf{X}, x^*$ to the OWF adversary $\mathcal{A}$.
4. $\mathcal{A}$ outputs $\mathbf{r}$, if $x^* \neq \bigoplus_{0\leq j<n} x_{r_j,j}$ then $\mathcal{B}$ aborts.
5. $\mathcal{B}$ outputs $\bigotimes_{0\leq j<n} y_{r_j,j}$.

Proof II

6. If $y^*, y_{j,b}$ were sampled uniformly at random,
    - then $\Pr[y^* = \bigotimes_{0 \leq j < n} y_{r_j,j}] = 1/|\mathcal{Y}|$,
    - otherwise $\Pr[y^* = \bigotimes_{0 \leq j < n} y_{r_j,j}] = 1$.
7. By the LHL we know that for $\mathbf{X} \leftarrow\$ \mathcal{X}^{2\times n}$, $\mathbf{r} \leftarrow\$ \{0,1\}^n$, we have that $\oplus_{0 \leq j < n} x_{r_j,j}$ is statistically indistinguishable from x^*. Hence, $\mathcal{B}$ correctly simulates the one-wayness game for $\mathcal{B}$.
8. $\mathcal{B}$ succeeds when $\mathcal{A}$ succeeds.

KeyGen:

1. Sample $k \leftarrow\!\$\, \mathcal{K}$
2. Sample $2n$ uniform elements in $\mathcal{X}$:[1]

$$\mathsf{X} := \begin{pmatrix} x_{0,0} & \cdots & x_{0,n-1} \\ x_{1,0} & \cdots & x_{1,n-1} \end{pmatrix} \leftarrow\!\$\, \mathcal{X}^{2\times n}$$

3. Compute

$$\mathsf{Y} := \begin{pmatrix} y_{0,0} & \cdots & y_{0,n-1} \\ y_{1,0} & \cdots & y_{1,n-1} \end{pmatrix} = \begin{pmatrix} F_k(x_{0,0}) & \cdots & F_k(x_{0,n-1}) \\ F_k(x_{1,0}) & \cdots & F_k(x_{1,n-1}) \end{pmatrix}$$

4. $\mathsf{sk} := k$, $\mathsf{pk} := (\mathsf{X}, \mathsf{Y})$

CPA-Secure PKE II

Enc:

1. Sample $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2. Compute $c := \bigoplus_{0 \le j < n} x_{r_j, j}$
3. Compute $e := m \otimes \left(\bigotimes_{0 \le j < n} y_{r_j, j} \right)$
4. Output (c, e)

Dec:

1. Output $m' := e \otimes F_k(c)^{-1} = m \otimes \left(\bigotimes_{0 \le j < n} y_{r_j, j} \right) \otimes F_k \left(\bigoplus_{0 \le j < n} x_{r_j, j} \right)^{-1}$

---

[1] We let n be a function of λ.

Proof I

Game_0	$\mathsf{C}(m_0, m_1)$
$\mathsf{pk}, \mathsf{sk} \leftarrow\$ \mathsf{KeyGen}(1^\lambda)$	**if** $\lvert m_0 \rvert \neq \lvert m_1 \rvert$ **then**
$b \leftarrow\$ \{0, 1\}$	**return** $\perp$
$b' \leftarrow \mathcal{D}^{\mathsf{C}}$	$c \leftarrow\$ \mathsf{Enc}(\mathsf{pk}, m_b)$
return $b = b'$	**return** c

The definition of IND-CPA for PKE.

Proof II

Game_1	$\mathsf{C}(m_0, m_1)$
1 : $(\mathsf{X}, \mathsf{Y}), k \leftarrow\!\!\$\ \mathsf{KeyGen}(1^\lambda)$	1 : $\mathbf{r} \leftarrow\!\!\$\ \{0,1\}^n$
2 : $b \leftarrow\!\!\$\ \{0,1\}$	2 : $c \leftarrow \bigoplus_{0 \le j < n} x_{r_j, j}$
3 : $b' \leftarrow \mathcal{D}^{\mathsf{C}}$	3 : $e \leftarrow m_b \otimes \left(\bigotimes_{0 \le j < n} y_{r_j, j}\right)$
4 : **return** $b = b'$	4 : **return** (c, e)

We simply instantiated our scheme.

Proof III

Game_2	$\mathsf{C}(m_0, m_1)$
1 : $(\mathbf{X}, \mathbf{Y}), k \leftarrow\$ \mathsf{KeyGen}(1^\lambda)$	1 : $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2 : $b \leftarrow\$ \{0,1\}$	2 : $c \leftarrow \bigoplus_{0 \le j < n} x_{r_j, j}$
3 : $b' \leftarrow \mathcal{D}^{\mathsf{C}}$	3 : $e \leftarrow m_b \otimes F_k(c)$
4 : **return** $b = b'$	4 : **return** (c, e)

Correctness of IHwPRF

Proof IV

Game_3	$\mathsf{C}(m_0, m_1)$
1 : $(\mathsf{X}, \mathsf{Y}), k \leftarrow\$ (\mathcal{X}^{2\times n}, \mathcal{Y}^{2\times n}), \mathcal{K}$	1 : $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2 : $b \leftarrow\$ \{0,1\}$	2 : $c \leftarrow \bigoplus_{0\le j<n} x_{r_j,j}$
3 : $b' \leftarrow \mathcal{D}^{\mathsf{C}}$	3 : $e \leftarrow m_b \otimes F_k(c)$
4 : **return** $b = b'$	4 : **return** (c, e)

wPRF Security of $F_k()$

Proof v

Game_4	$\mathsf{C}(m_0, m_1)$
1 : $(\mathbf{X}, \mathbf{Y}), k \leftarrow\$ (\mathcal{X}^{2\times n}, \mathcal{Y}^{2\times n}), \mathcal{K}$	1 : $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2 : $b \leftarrow\$ \{0,1\}$	2 : $c \leftarrow\$ \mathcal{X}$
3 : $b' \leftarrow \mathcal{D}^{\mathsf{C}}$	3 : $e \leftarrow m_b \otimes F_k(c)$
4 : **return** $b = b'$	4 : **return** (c, e)

LHL

Game$_4$	C(m_0, m_1)
1 : $(\mathsf{X}, \mathsf{Y}), k \leftarrow\!\!\$\ (\mathcal{X}^{2\times n}, \mathcal{Y}^{2\times n}), \mathcal{K}$	1 : $\mathsf{r} \leftarrow\!\!\$\ \{0,1\}^n$
2 : $b \leftarrow\!\!\$\ \{0,1\}$	2 : $c \leftarrow\!\!\$\ \mathcal{X}$
3 : $b' \leftarrow \mathcal{D}^{\mathsf{C}}$	3 : $e \leftarrow\!\!\$\ \mathcal{Y}$
4 : **return** $b = b'$	4 : **return** (c, e)

wPRF security if $F_k()$

DH Instantiation I

KeyGen:

1. Sample $k \leftarrow\$ \mathcal{K}$
2. Sample $2n$ uniform elements in $\mathcal{X}$:

$$\mathsf{X} := \begin{pmatrix} x_{0,0} & \dots & x_{0,n-1} \\ x_{1,0} & \dots & x_{1,n-1} \end{pmatrix} \leftarrow\$ \mathcal{X}^{2\times n}$$

3. Compute

$$\mathsf{Y} := \begin{pmatrix} F_k(x_{0,0}) & \dots & F_k(x_{0,n-1}) \\ F_k(x_{1,0}) & \dots & F_k(x_{1,n-1}) \end{pmatrix}$$

4. $\mathsf{sk} := k$, $\mathsf{pk} := (\mathsf{X}, \mathsf{Y})$

KeyGen:

1. Sample $k \leftarrow\$ \mathbb{Z}_{p-1}$
2. Sample $2n$ uniform elements in $\mathbb{Z}_p^*$:

$$\mathsf{X} := \begin{pmatrix} x_{0,0} & \dots & x_{0,n-1} \\ x_{1,0} & \dots & x_{1,n-1} \end{pmatrix} \leftarrow\$ (\mathbb{Z}_p^*)^{2\times n}$$

3. Compute

$$\mathsf{Y} := \begin{pmatrix} (x_{0,0}^k & \dots & x_{0,n-1}^k \\ x_{1,0}^k & \dots & x_{1,n-1}^k \end{pmatrix}$$

4. $\mathsf{sk} := k$, $\mathsf{pk} := (\mathsf{X}, \mathsf{Y})$

DH Instantiation II

Enc:

1. Sample $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2. Compute $c := \bigoplus_{0 \le j < n} x_{r_j, j}$
3. Compute $e := m \otimes \left(\bigotimes_{0 \le j < n} y_{r_j, j}\right)$
4. Output (c, e)

Dec:

1. Output $m' := e \otimes F_k(c)^{-1}$

Enc:

1. Sample $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2. Compute $c := \prod_{0 \le j < n} x_{r_j, j}$
3. Compute $e := m \cdot \left(\prod_{0 \le j < n} y_{r_j, j}\right)$
4. Output (c, e)

Dec:

1. Output $m' := e \cdot c^{-k}$

Taher ElGamal. A Public Key Cryptosystem and a Signature Scheme Based on Discrete Logarithms. In: *CRYPTO'84*. Ed. by G. R. Blakley and David Chaum. Vol. 196. LNCS. Springer, Berlin, Heidelberg, Aug. 1984, pp. 10–18. DOI: 10.1007/3-540-39568-7_2

ElGamal II

KeyGen:

1. Sample $k \leftarrow\$ \mathbb{Z}_{p-1}$
2. Sample $\mathbf{X} \leftarrow\$ (\mathbb{Z}_p^*)^{2\times n}$
3. Compute $\mathbf{Y} := \{x_{b,j}^k\}_{b\in\{0,1\},0\leq j<n}$
4. $\mathsf{sk} := k$, $\mathsf{pk} := (\mathbf{X}, \mathbf{Y})$

Enc:

1. Sample $\mathbf{r} \leftarrow\$ \{0,1\}^n$
2. Compute $c := \prod_{0\leq j<n} x_{r_j,j}$
3. Compute $e := m \cdot \left(\prod_{0\leq j<n} y_{r_j,j}\right)$
4. Output (c, e)

Dec:

1. Output $m' := e \cdot c^{-k}$

KeyGen:

1. Sample $k \leftarrow\$ \mathbb{Z}_{p-1}$
2. $x = g$
3. Compute $y = g^k$
4. $\mathsf{sk} := k$, $\mathsf{pk} := (x, y)$

Enc:

1. Sample $r \leftarrow\$ \mathbb{Z}_{p-1}$
2. Compute $c := g^r$
3. Compute $e := m \cdot y^r$
4. Output (c, e)

Dec:

1. Output $m' := e \cdot c^{-k}$

Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. In: *Journal of the ACM* 56.6 (Sept. 2009), 34:1–34:40. ISSN: 0004-5411 (print), 1557-735X (electronic). DOI: http://doi.acm.org/10.1145/1568318.1568324

Ron Rothblum. Homomorphic Encryption: From Private-Key to Public-Key. In: *TCC 2011*. Ed. by Yuval Ishai. Vol. 6597. LNCS. Springer, Berlin, Heidelberg, Mar. 2011, pp. 219–234. DOI: 10.1007/978-3-642-19571-6_14

Regev Encryption II

KeyGen:

1. Sample $k \leftarrow\$ \mathcal{K}$
2. Sample $2n$ uniform elements in $\mathcal{X}$:

$$\mathsf{X} := \begin{pmatrix} x_{0,0} & \dots & x_{0,n-1} \\ x_{1,0} & \dots & x_{1,n-1} \end{pmatrix} \leftarrow\$ \mathcal{X}^{2\times n}$$

3. Compute

$$\mathsf{Y} := \begin{pmatrix} F_k(x_{0,0}) & \dots & F_k(x_{0,n-1}) \\ F_k(x_{1,0}) & \dots & F_k(x_{1,n-1}) \end{pmatrix}$$

4. $\mathsf{sk} := k$, $\mathsf{pk} := (\mathsf{X}, \mathsf{Y})$

KeyGen:

1. Sample $\mathbf{s} \leftarrow\$ \mathbb{Z}_q^k$
2. Sample $\mathsf{X} := \mathbf{A} \leftarrow\$ \mathbb{Z}_q^{(2\,n\,\log q)\times n}$
3. Sample $\mathbf{z} \leftarrow\$ \chi^m$ (can use wPRF construction as above).
4. Compute $\mathsf{Y} := \mathbf{b} = \mathbf{A} \cdot \mathbf{s} + \mathbf{z}$
5. $\mathsf{sk} := \mathbf{s}$, $\mathsf{pk} := (\mathbf{A}, \mathbf{b})$

Regev Encryption III

Enc:

1. Sample $\mathbf{r} \leftarrow\!\!\$ \{0,1\}^n$
2. Compute $c := \bigoplus_{0 \le j < n} x_{r_j, j}$
3. Compute $e := m \otimes \left(\bigotimes_{0 \le j < n} y_{r_j, j} \right)$
4. Output (c, e)

Dec:

1. Output $m' := e \otimes F_k(c)^{-1}$

Enc:

1. Sample $\mathbf{r} \leftarrow\!\!\$ \{0,1\}^{2\,n\,\log q}$
2. Compute $\mathbf{c} := \mathbf{r}^T \cdot \mathbf{A}$
3. Compute $e := \lfloor \frac{q}{2} \rfloor \cdot m + \mathbf{r}^T \cdot \mathbf{b}$
4. Output $(\mathbf{c}, e)$

Dec:

1. Output $\mathcal{R}(e - \langle \mathbf{c}, \mathbf{s} \rangle)$

DRAGONS!

Game_3	$\mathsf{C}(m_0, m_1)$
$(\mathbf{X}, \mathbf{Y}), \Bbbk \leftarrow\!\!\$\ \mathsf{KeyGen}()$	$\mathbf{r} \leftarrow\!\!\$\ \{0,1\}^n$
$b \leftarrow\!\!\$\ \{0,1\}$	$c \leftarrow \bigoplus_{0 \le j < n} x_{r_j, j}$
$b' \leftarrow \mathcal{D}^{\mathsf{C}}$	$e \leftarrow m_b \otimes F_{\Bbbk}(c)$
$\mathbf{return}\ b = b'$	$\mathbf{return}\ (c, e)$

Game_3	$\mathsf{C}(m_0, m_1)$
$(\mathbf{A}, \mathbf{b}), \mathbf{s} \leftarrow\!\!\$\ \mathsf{KeyGen}()$	$\mathbf{r} \leftarrow\!\!\$\ \{0,1\}^{2\,n \log q}$
$b \leftarrow\!\!\$\ \{0,1\}$	$\mathbf{c} \leftarrow \mathbf{r}^T \cdot \mathbf{A}$
$b' \leftarrow \mathcal{D}^{\mathsf{C}}$	$e \leftarrow \lfloor \frac{q}{2} \rfloor \cdot m + \mathbf{r}^T \cdot \mathbf{b}$
$\mathbf{return}\ b = b'$	$\mathbf{return}\ (\mathbf{c}, e)$

- We cannot rely on correctness here because

$$\begin{aligned} F_{(\Bbbk,\mathbf{s})}(\mathbf{a}_0) + F_{(\Bbbk,\mathbf{s})}(\mathbf{a}_1) &= \langle \mathbf{a}_0, \mathbf{s} \rangle + D_\chi(F'_{\Bbbk}(\mathbf{a}_0)) + \langle \mathbf{a}_1, \mathbf{s} \rangle + D_\chi(F'_{\Bbbk}(\mathbf{a}_1)) \\ &\neq \langle \mathbf{a}_0 + \mathbf{a}_1, \mathbf{s} \rangle + D_\chi(F'_{\Bbbk}(\mathbf{a}_0 + \mathbf{a}_1)) = F_{(\Bbbk,\mathbf{s})}(\mathbf{a}_0 + \mathbf{a}_1) \end{aligned}$$

- We only have $\mathcal{R}(F_{(\Bbbk,\mathbf{s})}(\mathbf{a}_0) + F_{(\Bbbk,\mathbf{s})}(\mathbf{a}_1)) = \mathcal{R}(F_{(\Bbbk,\mathbf{s})}(\mathbf{a}_0 + \mathbf{a}_1))$; could output $\mathcal{R}(e)$ instead.
- Regev solved this directly by appealing to the LHL on $(\mathbf{A}, \mathbf{b}) \in \mathbb{Z}_q^{(2\,n \log q) \times (n+1)}$

LP11 Encryption I

Richard Lindner and Chris Peikert. **Better Key Sizes (and Attacks) for LWE-Based Encryption**. In: *CT-RSA 2011*. Ed. by Aggelos Kiayias. Vol. 6558. LNCS. Springer, Berlin, Heidelberg, Feb. 2011, pp. 319–339. DOI: `10.1007/978-3-642-19074-2_21`

Key Idea: replace LHL with applying LWE again.

KeyGen:

1. Sample $\mathbf{s} \leftarrow\$ \mathbb{Z}_q^k$
2. Sample $\mathbf{X} := \mathbf{A} \leftarrow\$ \mathbb{Z}_q^{(2n\log q)\times n}$
3. Sample $\mathbf{e} \leftarrow\$ \chi^{2n\log q}$.
4. Compute $\mathbf{Y} := \mathbf{b} = \mathbf{A}\cdot\mathbf{s} + \mathbf{e}$
5. $\mathsf{sk} := \mathbf{s}$, $\mathsf{pk} := (\mathbf{A}, \mathbf{b})$

Enc:

1. Sample $\mathbf{r} \leftarrow\$ \{0,1\}^{2n\log q}$
2. Compute $\mathbf{c} := \mathbf{r}^T \cdot \mathbf{A}$
3. Compute $e := \lfloor \frac{q}{2} \rfloor \cdot m + \mathbf{r}^T \cdot \mathbf{b}$
4. Output $(\mathbf{c}, e)$

Dec:

1. Output $\mathcal{R}(e - \langle \mathbf{c}, \mathbf{s}\rangle)$

KeyGen:

1. Sample $\mathbf{s} \leftarrow\$ \mathbb{Z}_q^k$
2. Sample $\mathbf{X} := \mathbf{A} \leftarrow\$ \mathbb{Z}_q^{n\times n}$
3. Sample $\mathbf{e} \leftarrow\$ \chi^{n}$.
4. Compute $\mathbf{Y} := \mathbf{b} = \mathbf{A}\cdot\mathbf{s} + \mathbf{e}$
5. $\mathsf{sk} := \mathbf{s}$, $\mathsf{pk} := (\mathbf{A}, \mathbf{b})$

Enc:

1. Sample $\mathbf{r}, \mathbf{d}, d' \leftarrow\$ \chi^n, \chi^n, \chi$
2. Compute $\mathbf{c} := \mathbf{r}^T \cdot \mathbf{A} + \mathbf{d}$
3. Compute $e := \lfloor \frac{q}{2} \rfloor \cdot m + \mathbf{r}^T \cdot \mathbf{b} + d'$
4. Output $(\mathbf{c}, \mathbf{e})$

Dec:

1. Output $\mathcal{R}(e - \langle \mathbf{c}, \mathbf{s}\rangle)$

Fin

Read Minicrypt Primitives with Algebraic Structure and Applications!

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