Quantum Computing

Advanced Topics in ~~Cybersecurity~~Cryptography (7CCSMATC)

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Outline

Qubits

Many Qubits

Grover's Algorithm

Shor's Algorithm

Commitment Schemes

Programme

Teach you just enough quantum computing to get a sense of how it affects cryptography.

References

- Thomas Debris-Alazard. *Lecture 1: Introduction to Quantum Computing.* INF587 Quantum computer science and applications `https://tdalazard.io/S1.pdf`
- Noson S Yanofsky and Mirco A Mannucci. Quantum Computing for Computer Scientists. Cambridge University Press, 2008, esp. Chapters 1, 2, 3 and 6
- Fermi Ma's talk Quantum Secure Commitments and Collapsing Hash Functions delivered as part of the *Quantum Cryptography for Dummies* reading group at the *Lattices: Algorithms, Complexity, and Cryptography* special semester at the Simons Institute, 2020

Qubits

Classical Bit

$$b \in \{0, 1\}$$

Probabilistic Bit

- Probabilistic bit: $\begin{pmatrix} p \\ q \end{pmatrix}$ where $p := \Pr(b = 0)$ and $q := \Pr(b = 1)$
- Computing on probabilistic bits

$$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} \text{ where } \begin{cases} a + c = 1 \\ b + d = 1 \end{cases} \text{ and } a, b, c, d \geq 0$$

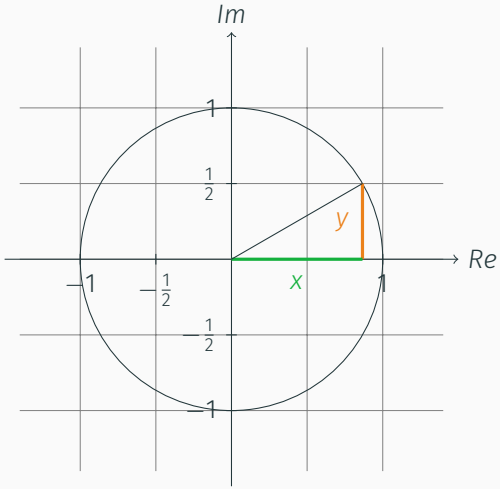
Example: $b \to b \oplus b$

$$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$$

Example: $b \to b \oplus 1$

$$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$$

Complex Numbers



- $i := \sqrt{-1}$
- $z := x + i\,y$
- $Re(z) := x$
- $Im(z) := y$
- $|z| = \sqrt{x^2 + y^2}$

Quantum Bit (Qubit): "Probabilistic Bits with Complex Probabilities"

- A qubit $|\psi\rangle$ is an element of $\mathbb{C}^2$ with Euclidean norm 1:

$$|\psi\rangle = \alpha \cdot |0\rangle + \beta \cdot |1\rangle \text{ with } \alpha, \beta \in \mathbb{C} \text{ (called amplitude) and } |\alpha|^2 + |\beta|^2 = 1$$

- $|0\rangle, |1\rangle$ is an orthonormal basis of $\mathbb{C}^2$. Usually defined as

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ which then implies } \alpha \cdot |0\rangle + \beta \cdot |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

- We call this a superposition of $|0\rangle$ and $|1\rangle$.

Measurements

We cannot “see” qubits, we can only measure in their classical states.

Measurement: probabilistic orthogonal projection. Given $|0\rangle, |1\rangle \in \mathbb{C}^2$:

$$|\psi\rangle = \alpha \cdot |0\rangle + \beta \cdot |1\rangle \xrightarrow{\text{measure}} \begin{cases} |0\rangle & \text{with probability } |\alpha|^2 \\ |1\rangle & \text{with probability } |\beta|^2 \end{cases}$$

Computation

- A **unitary matrix in $\mathbb{C}^{2\times 2}$** is any matrix such that $\mathsf{U} \cdot \mathsf{U}^\dagger = \mathsf{U}^\dagger \cdot \mathsf{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $\mathsf{U}^\dagger$ is the conjugate transpose of U:

$$\mathsf{U} = \begin{pmatrix} u_{00} + v_{00} \cdot i & u_{01} + v_{01} \cdot i \\ u_{10} + v_{10} \cdot i & u_{11} + v_{11} \cdot i \end{pmatrix}, \quad \mathsf{U}^\dagger := \begin{pmatrix} u_{00} - v_{00} \cdot i & u_{10} - v_{10} \cdot i \\ u_{01} - v_{01} \cdot i & u_{11} - v_{11} \cdot i \end{pmatrix}$$

- **Computation:** $|\psi\rangle \to \mathsf{U} \cdot |\psi\rangle$
  - All quantum computations are reversible:

$$|\psi\rangle \xrightarrow{\mathsf{U}} \mathsf{U} \cdot |\psi\rangle \xrightarrow{\mathsf{U}^\dagger} \mathsf{U}^\dagger \cdot \mathsf{U} \cdot |\psi\rangle = |\psi\rangle$$

- We call U **quantum gates**

Quantum Computers ...

... are linear-algebra machines.

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Mathematics is the art of reducing any problem to linear algebra.

— William Stein

Examples of Quantum Gates

Example NOT-gate $b \to$ "$b \oplus 1$"

$$|\psi\rangle = \alpha\,|0\rangle + \beta \cdot |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot |\psi\rangle$$

Examples of Quantum Gates

Example NOT-gate $b \to$ "$b \oplus 1$"

$$|\psi\rangle = \alpha\,|0\rangle + \beta \cdot |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot |\psi\rangle$$

Example $b \to b \oplus b$

Computation is not reversible!

Hadamard-gate H I

$$\mathsf{H} \cdot |\psi\rangle = \mathsf{H} \cdot (\alpha\,|0\rangle + \beta \cdot |1\rangle) = \mathsf{H} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

Hadamard-gate H I

$$\mathsf{H} \cdot |\psi\rangle = \mathsf{H} \cdot (\alpha\,|0\rangle + \beta \cdot |1\rangle) = \mathsf{H} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

- $\mathsf{H} \cdot |0\rangle = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1+0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
  - measure $|0\rangle$ or $|1\rangle$ with probability 1/2!

Hadamard-gate H I

$$\mathsf{H} \cdot |\psi\rangle = \mathsf{H} \cdot (\alpha\,|0\rangle + \beta \cdot |1\rangle) = \mathsf{H} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

- $\mathsf{H} \cdot |0\rangle = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1+0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
  - measure $|0\rangle$ or $|1\rangle$ with probability 1/2!
- $\mathsf{H} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1/\sqrt{2} + 1/\sqrt{2} \\ 1/\sqrt{2} - 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$
  - and we're back!

Hadamard-gate H II

The outputs of the Hadarmard gate applied to $|0\rangle$ and $|1\rangle$ are so important we give them names:

$$|+\rangle := \frac{1}{\sqrt{2}} \cdot \Big(|0\rangle + |1\rangle \Big) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\rangle := \frac{1}{\sqrt{2}} \cdot \Big(|0\rangle - |1\rangle \Big) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We cannot achieve the Hardmard gate with probabilistic bits $\Rightarrow$ quantum advantage

Many Qubits

Tensor Products

Let $\mathbf{v} \in \mathbb{C}^n$ and $\mathbf{w} \in \mathbb{C}^m$, their tensor product is $\mathbf{v} \otimes \mathbf{w} := (v_0 \cdot \mathbf{w}, \ldots, v_{n-1} \cdot \mathbf{w})$

- This is the same as the rows of:

$$\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \cdot \begin{pmatrix} w_0 & w_1 & \cdots & w_{m-1} \end{pmatrix} = \begin{pmatrix} v_0 \cdot w_0 & v_0 \cdot w_1 & \cdots & v_0 \cdot w_{m-1} \\ v_1 \cdot w_0 & v_1 \cdot w_1 & \cdots & v_1 \cdot w_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n-1} \cdot w_0 & v_{n-1} \cdot w_1 & \cdots & v_{n-1} \cdot w_{m-1} \end{pmatrix}$$

Tensor Products

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$$\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \cdot \begin{pmatrix} w_0 & w_1 & \cdots & w_{m-1} \end{pmatrix} = \begin{pmatrix} v_0 \cdot w_0 & v_0 \cdot w_1 & \cdots & v_0 \cdot w_{m-1} \\ v_1 \cdot w_0 & v_1 \cdot w_1 & \cdots & v_1 \cdot w_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n-1} \cdot w_0 & v_{n-1} \cdot w_1 & \cdots & v_{n-1} \cdot w_{m-1} \end{pmatrix}$$

- For any scalar z, we have $z \cdot (\mathbf{v} \otimes \mathbf{w}) = (z \cdot \mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (z \cdot \mathbf{w})$
- For any $\mathbf{v}_0, \mathbf{v}_1 \in \mathbb{C}^n$, we have $(\mathbf{v}_0 + \mathbf{v}_1) \otimes \mathbf{w} = \mathbf{v}_0 \otimes \mathbf{w} + \mathbf{v}_1 \otimes \mathbf{w}$
- For any $\mathbf{w}_0, \mathbf{w}_1 \in \mathbb{C}^m$, we have $\mathbf{v} \otimes (\mathbf{w}_0 + \mathbf{w}_1) = \mathbf{v} \otimes \mathbf{w}_0 + \mathbf{v} \otimes \mathbf{w}_1$

Tensor Products of Spaces

Consider

- $\mathbb{C}^n = \text{Span}_{\mathbb{C}}(\mathbf{v}_0, \ldots \mathbf{v}_{n-1})$ where e.g. $\mathbf{v}_0 \coloneqq (1, 0 \ldots, 0)$ etc, or some other basis of $\mathbb{C}^n$
- $\mathbb{C}^m = \text{Span}_{\mathbb{C}}(\mathbf{w}_0, \ldots \mathbf{w}_{m-1})$ where e.g. $\mathbf{w}_0 \coloneqq (1, 0 \ldots, 0)$ etc, or some other basis of $\mathbb{C}^m$

We have:

- $\mathbb{C}^n \otimes \mathbb{C}^m \coloneqq \text{Span}_{\mathbb{C}}(\mathbf{v}_i \otimes \mathbf{w}_j : 0 \leq i < n; 0 \leq j < m)$
- $\mathbb{C}^n \otimes \mathbb{C}^m$ has dimension $n \times m$
- $\mathbf{x} \in \mathbb{C}^n \otimes \mathbb{C}^m \iff \exists\, \alpha_{i,j} : \sum_{0 \leq i < n, 0 \leq j < m} \alpha_{i,j} \cdot \mathbf{v}_i \otimes \mathbf{w}_j$

Tensor Products of Matrices

$$\mathsf{A} \coloneqq \begin{pmatrix} a_{0,0} & \dots & a_{0,n-1} \\ \vdots & \ddots & \vdots \\ a_{m-1,0} & \dots & a_{m-1,n-1} \end{pmatrix}, \quad \mathsf{B} \coloneqq \begin{pmatrix} b_{0,0} & \dots & b_{0,q-1} \\ \vdots & \ddots & \vdots \\ b_{p-1,0} & \dots & b_{p-1,q-1} \end{pmatrix}$$

$$\mathsf{A} \otimes \mathsf{B} \coloneqq \begin{pmatrix} a_{0,0} \cdot \mathsf{B} & \cdots & a_{0,m-1} \cdot \mathsf{B} \\ \vdots & \ddots & \vdots \\ a_{n-1,0} \cdot \mathsf{B} & \cdots & a_{n-1,m-1} \cdot \mathsf{B} \end{pmatrix} \in \mathbb{C}^{np \times mq}$$

Examples

1. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 \\ 1 \cdot 3 \\ 2 \cdot 2 \\ 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}$

2. $\mathsf{X} \otimes \mathsf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

n Qubit States

- Recall that a qubit $|\psi\rangle$ is an element in $\mathbb{C}^2$ with norm 1.
- A register of n qubits $|\psi\rangle$ is an element in $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$.
- Let $|0\rangle, |1\rangle$ be an orthonormal basis of $\mathbb{C}^2$, then

  $$(|b_0\rangle \otimes |b_1\rangle \otimes \cdots \otimes |b_{n-1}\rangle : b_0, \ldots, b_{n-1} \in \{0,1\})$$

  is an orthormal basis of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = \mathbb{C}^{2^n}$.

```
ket0 = matrix(2, 1, [1,0])
ket1 = matrix(2, 1, [0,1])
ket0.tensor_product(ket0), \
ket0.tensor_product(ket1), \
ket1.tensor_product(ket0), \
ket1.tensor_product(ket1)
```

```
(
[1]  [0]  [0]  [0]
[0]  [1]  [0]  [0]
[-]  [-]  [-]  [-]
[0]  [0]  [1]  [0]
[0], [0], [0], [1]
)
```

Notation

- For $b_0, \ldots, b_{n-1} \in \{0,1\}$ we write

$$|b_0\, b_1\, \ldots\, b_{n-1}\rangle \coloneqq |b_0\rangle \otimes |b_1\rangle \otimes \ldots \otimes |b_{n-1}\rangle$$

- For $|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{n-1}\rangle \in \mathbb{C}^2$, we write

$$|\psi_0\rangle\, |\psi_1\rangle \ldots |\psi_{n-1}\rangle \coloneqq |\psi_0\rangle \otimes |\psi_1\rangle \otimes \ldots \otimes |\psi_{n-1}\rangle$$

- Any $|\psi\rangle \in \mathbb{C}^{2^n}$ of n qubits can be written as

$$|\psi\rangle = \sum_{\mathbf{x}\in\{0,1\}^n} \alpha_{\mathbf{x}} |\mathbf{x}\rangle \quad \text{where } \alpha_{\mathbf{x}} \in \mathbb{C} \text{ (called amplitude) and } \sum_{\mathbf{x}\in\{0,1\}^n} |\alpha_{\mathbf{x}}|^2 = 1$$

Separable States

Definition

An n-qubit state $|\psi\rangle$ is called separable if it can be decomposed as $|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle$.

Examples

- $|00\rangle = |0\rangle \otimes |0\rangle$
- $\frac{1}{2} \cdot (|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

```
ket0 = matrix(2, 1, [1,0]); ket1 = matrix(2, 1, [0,1])
(1/sqrt(2)*ket0 + 1/sqrt(2)*ket1).tensor_product(1/sqrt(2)*ket0 + 1/sqrt(2)*ket1)
```

```
[1/2]
[1/2]
[---]
[1/2]
[1/2]
```

Entangled States

Definition

An n-qubit state $|\psi\rangle$ is called entangled if it cannot be decomposed as $|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle$.

Example

$$\frac{1}{\sqrt{2}} \cdot (|00\rangle + |11\rangle)$$

```
ket0 = matrix(2, 1, [1,0]); ket1 = matrix(2, 1, [0,1])
(1/sqrt(2)) * (ket0.tensor_product(ket0) + ket1.tensor_product(ket1))
```

```
[1/2*sqrt(2)]
[          0]
[          0]
[1/2*sqrt(2)]
```

Measuring n Qubit States

Measuring the state:

$$|\psi\rangle = \sum_{i_0,\ldots,i_{n-1}\in\{0,1\}^n} \alpha_{i_0\ldots i_{-n}} \left|e_{i_0}\cdots e_{i_{n-1}}\right\rangle \xrightarrow{\text{measure}} \left|e_{j_0}\ldots e_{j_{n-1}}\right\rangle \text{ with probability } \left|\alpha_{j_0\ldots j_{n-1}}\right|^2$$

Measuring the first register:

$$|\psi\rangle = \alpha_0\cdot|e_0\rangle|\psi_0\rangle + \alpha_1\cdot|e_1\rangle|\psi_1\rangle \xrightarrow{\text{measure}} \begin{cases} |e_0\rangle|\psi_0\rangle & \text{with prob. } |\alpha_0|^2 \\ |e_1\rangle|\psi_1\rangle & \text{with prob. } |\alpha_1|^2 \end{cases}$$

We necessarily have $|\alpha_0|^2 + |\alpha_1|^2 = 1$

- A unitary matrix $\mathsf{U} \in \mathbb{C}^{2^n \times 2^n}$, i.e. $\mathsf{U}^\dagger \cdot \mathsf{U} = \mathsf{I}_{2^n}$ is called a quantum circuit
- Any classical circuit f on n-bits can be written as a unitary U_f on $2n$-qubits

$$\mathsf{U}_f \cdot |\psi\rangle |0\rangle = |\psi\rangle |f(\psi)\rangle$$

Measuring Entangled Registers

Consider $\mathbf{U}_f \cdot |\mathbf{x}, y\rangle = |\mathbf{x}, f(\mathbf{x}) \oplus y\rangle$ for some $f : \{0,1\}^n \to \{0,1\}$.

- Measure the last register $|f(\mathbf{x}) \oplus y\rangle$ to obtain v
- The first register $|\mathbf{x}\rangle$ collapses to those $\mathbf{x} \in \{0,1\}^n$ s.t. $f(\mathbf{x}) \oplus y = v$.
- The first n registers hold a superposition of the preimages of v under f.
- Measurements must stay consistent. Turns out this is quite powerful!

No-Cloning Theorem

Cannot copy a quantum state

- We can "cut" and "paste" a quantum state, but we cannot "copy" and "paste".
- "Move is possible. Copy is impossible."

Proof Sketch:

- There is no linear map C from $|\psi\rangle \otimes |0\rangle$ to $|\psi\rangle \otimes |\psi\rangle$.
- It would need to map $\frac{|x\rangle+|y\rangle}{\sqrt{2}} \otimes |0\rangle$ to $\frac{|x\rangle+|y\rangle}{\sqrt{2}} \otimes \frac{|x\rangle+|y\rangle}{\sqrt{2}}$
- By linearity, we'd need: $C\left(\frac{|x\rangle+|y\rangle}{\sqrt{2}} \otimes |0\rangle\right) =$
  - $\frac{1}{\sqrt{2}} \cdot C((|x\rangle + |y\rangle) \otimes |0\rangle)$
  - $\frac{1}{\sqrt{2}} \cdot C(|x\rangle \otimes |0\rangle) + C(|y\rangle \otimes |0\rangle)$
- This will not match $\frac{|x\rangle+|y\rangle}{\sqrt{2}} \otimes \frac{|x\rangle+|y\rangle}{\sqrt{2}}$

Grover's Algorithm

Problem Statement

Given some function $f : \{0,1\}^n \to \{0,1\}$, we want to find some special element x_0.

- For example, given an plaintext-ciphertext pair (p, c) for AES, we might write $f : k \to AES(k, p) \stackrel{?}{=} c$.
- Classically, we'd need to call f about 2^n times to find x_0
- Grover's algorithm only needs $\sqrt{2^n} = 2^{n/2}$ queries

Example

- $n = 2; x_0 = 10; \mathsf{U}_f \cdot |\mathbf{x}, y\rangle = |\mathbf{x}, f(\mathbf{x}) \oplus y\rangle$

$$\mathsf{U}_f := \begin{array}{c} \\ 00,0 \\ 00,1 \\ 01,0 \\ 01,1 \\ 10,0 \\ 10,1 \\ 11,0 \\ 11,1 \end{array} \begin{array}{c} \begin{array}{cccccccc} 00,0 & 00,1 & 01,0 & 01,1 & 10,0 & 10,1 & 11,0 & 11,1 \end{array} \\ \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \end{array}$$

First Attempt

Let's apply the Hadarmard gate on the first n registers, apply $\mathbf{U}_f$ and measure

$$|\psi_0\rangle = |\mathbf{0}, 0\rangle$$
$$|\psi_1\rangle = \left(\mathbf{H}^{\oplus n} \otimes \mathbf{I}\right) \cdot |\mathbf{0}, 0\rangle = \left[\frac{\sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle}{\sqrt{2^n}}\right] |0\rangle$$
$$|\psi_2\rangle = \mathbf{U}_f \cdot \left[\frac{\sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle}{\sqrt{2^n}}\right] |0\rangle = \frac{\sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x},\ f(\mathbf{x})\rangle}{\sqrt{2^n}}$$

Measuring the last qubit will produce 1 with probability $1/2^n$. If that event happens, then measuring the first n qubits will output the correct answer x_0.

Apply the Hadarmard gate on the last register and apply U_f

$$|\psi_0\rangle = |\mathsf{x}, 1\rangle$$

$$|\psi_1\rangle = (\mathsf{I}_n \otimes \mathsf{H}) \cdot |\mathsf{x}, 1\rangle = |\mathsf{x}\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = \left[\frac{|\mathsf{x}, 0\rangle - |\mathsf{x}, 1\rangle}{\sqrt{2}}\right]$$

$$|\psi_2\rangle = \mathsf{U}_f \cdot \left(|\mathsf{x}\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\right) = |\mathsf{x}\rangle \left[\frac{|f(\mathsf{x}) \oplus 0\rangle - |f(\mathsf{x}) \oplus 1\rangle}{\sqrt{2}}\right] = |\mathsf{x}\rangle \left[\frac{|f(\mathsf{x})\rangle - |\overline{f(\mathsf{x})}\rangle}{\sqrt{2}}\right]$$

$$= (-1)^{f(\mathsf{x})} \cdot |\mathsf{x}\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = \begin{cases} -1\,|\mathsf{x}\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right], & \text{if } \mathsf{x} = \mathsf{x}_0 \\ +1\,|\mathsf{x}\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right], & \text{if } \mathsf{x} \neq \mathsf{x}_0 \end{cases}$$

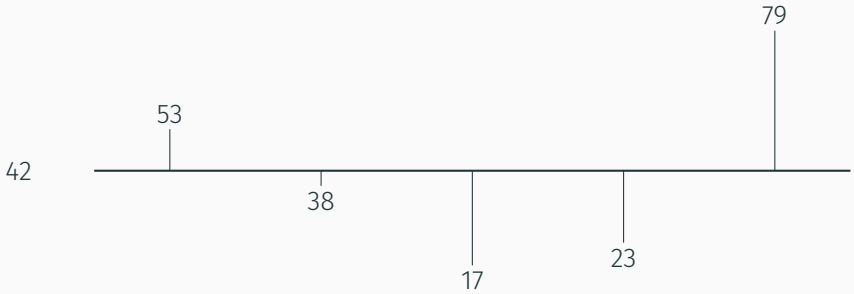
Applying:

$$U_f \cdot (I_n \otimes H) \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

Useless?

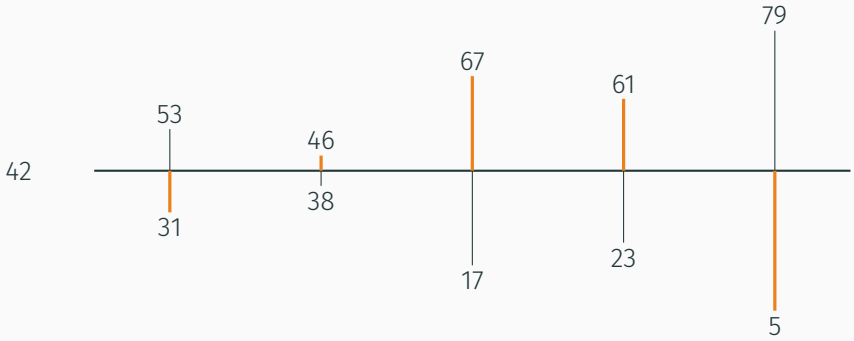
$$|(-1/2)|^2 = |1/2|^2$$

Trick 2: Inversion about the mean (Example)



$$y = \mu + (\mu - x) = -x + 2\mu$$

Trick 2: Inversion about the mean (Example)



$$y = \mu + (\mu - x) = -x + 2\mu$$

Trick 2: Inversion about the mean I

Computing the mean:

$$M = \begin{pmatrix} 1/2^n & 1/2^n & \dots & 1/2^n & 1/2^n \\ 1/2^n & 1/2^n & \dots & 1/2^n & 1/2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1/2^n & 1/2^n & \dots & 1/2^n & 1/2^n \\ 1/2^n & 1/2^n & \dots & 1/2^n & 1/2^n \end{pmatrix} \qquad M \cdot v = \begin{pmatrix} 1/2^n \cdot \sum_{i=0}^{2^n} v_i \\ 1/2^n \cdot \sum_{i=0}^{2^n} v_i \\ \vdots \\ 1/2^n \cdot \sum_{i=0}^{2^n} v_i \\ 1/2^n \cdot \sum_{i=0}^{2^n} v_i \end{pmatrix}$$

Trick 2: Inversion about the mean II

Inversion about the mean:

$$-\mathsf{I} + 2\,\mathsf{M} = \begin{pmatrix} (-1 + 2/2^n) & 2/2^n & \dots & 2/2^n & 2/2^n \\ 2/2^n & (-1 + 2/2^n) & \dots & 2/2^n & 2/2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2/2^n & \dots & 2/2^n & (-1 + 2/2^n) & 2/2^n \\ 2/2^n & \dots & 2/2^n & 2/2^n & (-1 + 2/2^n) \end{pmatrix}$$

$$(-\mathsf{I} + 2\,\mathsf{M}) \cdot \mathbf{v} = \begin{pmatrix} v_0 + 2/2^n \cdot \sum_{i=0}^{2^n-1} v_i \\ v_1 + 2/2^n \cdot \sum_{i=0}^{2^n-1} v_i \\ \vdots \\ v_{2^n-2} + 2/2^n \cdot \sum_{i=0}^{2^n-1} v_i \\ v_{2^n-1} + 2/2^n \cdot \sum_{i=0}^{2^n-1} v_i \end{pmatrix}$$

Trick 2: Inversion about the mean III

A straight-forward calculation shows that $-\mathbf{I} + 2\,\mathbf{M}$ is indeed unitary an thus a quantum circuit!

Combining the two tricks

```
def phasef(v, i):
    v = enumerate(v)
    v = [v_ * (-1)**int(i==j) for j,v_ in v]
    return vector(v)

phasef(vector(ZZ, 4, [1,1,1,1]), 2)
```

```
(1, 1, -1, 1)
```

```
def imeanf(v):
    N = ZZ(len(v))
    M = matrix(QQ, len(v), len(v), [1/N]*N**2)
    I = identity_matrix(N)
    return (-I + 2*M)*v

imeanf(vector(ZZ, 5, [53,38,17,23,79]))
```

```
(31, 46, 67, 61, 5)
```

Combining the two tricks

```
def phasef(v, i):
    v = enumerate(v)
    v = [v_ * (-1)**int(i==j) for j,v_ in v]
    return vector(v)

phasef(vector(ZZ, 4, [1,1,1,1]), 2)
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    M = matrix(QQ, len(v), len(v), [1/N]*N**2)
    I = identity_matrix(N)
    return (-I + 2*M)*v

imeanf(vector(ZZ, 5, [53,38,17,23,79]))
```

```
(31, 46, 67, 61, 5)
```

```
v = vector(RealField(prec=12), 5, [10]*5)
print(f"  input: {v}")
v = phasef(v, i=3); print(f"step 1a: {v}")
v = imeanf(v); print(f"step 1b: {v}, Δ: {max(v)-min(v)}")
v = phasef(v, i=3); print(f"step 2a: {v}")
v = imeanf(v); print(f"step 2b: {v}, Δ: {max(v)-min(v)}")
v = phasef(v, i=3); print(f"step 3a: {v}")
v = imeanf(v); print(f"step 3b: {v}, Δ: {max(v)-min(v)}")
```

```
  input: (10.0, 10.0, 10.0, 10.0, 10.0)
step 1a: (10.0, 10.0, 10.0, -10.0, 10.0)
step 1b: (2.00, 2.00, 2.00, 22.0, 2.00), Δ: 20.0
step 2a: (2.00, 2.00, 2.00, -22.0, 2.00)
step 2b: (-7.60, -7.60, -7.60, 16.4, -7.60), Δ: 24.0
step 3a: (-7.60, -7.60, -7.60, -16.4, -7.60)
step 3b: (-11.1, -11.1, -11.1, -2.33, -11.1), Δ: 8.80
```

Combining the two tricks

```
def phasef(v, i):
    v = enumerate(v)
    v = [v_ * (-1)**int(i==j) for j,v_ in v]
    return vector(v)

phasef(vector(ZZ, 4, [1,1,1,1]), 2)
```

```
(1, 1, -1, 1)
```

```
def imeanf(v):
    N = ZZ(len(v))
    M = matrix(QQ, len(v), len(v), [1/N]*N**2)
    I = identity_matrix(N)
    return (-I + 2*M)*v

imeanf(vector(ZZ, 5, [53,38,17,23,79]))
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```
(31, 46, 67, 61, 5)
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```
v = vector(RealField(prec=12), 5, [10]*5)
print(f"  input: {v}")
v = phasef(v, i=3); print(f"step 1a: {v}")
v = imeanf(v); print(f"step 1b: {v}, Δ: {max(v)-min(v)}")
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v = imeanf(v); print(f"step 3b: {v}, Δ: {max(v)-min(v)}")
```

```
  input: (10.0, 10.0, 10.0, 10.0, 10.0)
step 1a: (10.0, 10.0, 10.0, -10.0, 10.0)
step 1b: (2.00, 2.00, 2.00, 22.0, 2.00), Δ: 20.0
step 2a: (2.00, 2.00, 2.00, -22.0, 2.00)
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step 3b: (-11.1, -11.1, -11.1, -2.33, -11.1), Δ: 8.80
```

The optimal number of repetitions is $\sqrt{\dim(V)}$

Grover's Algorithm

1. Start with $|\mathbf{0}\rangle$
2. Apply $\mathbf{H}^{\otimes n}$
3. Repeat $\sqrt{2^n}$ times:
   3.1 Apply phase inversion $\mathbf{U}_f \cdot (\mathbf{I} \otimes \mathbf{H})$
   3.2 Apply inversion about the mean $-\mathbf{I} + 2\mathbf{M}$
4. Measure the qubits

Recap: Grover vs AES

Best known quantum algorithms for attacking symmetric cryptography are based on Grover's algorithm.

- Search key space of size 2^n in $2^{n/2}$ operations: AES-256 $\rightarrow$ 128 "quantum bits of security".
- Taking all costs into account: $> 2^{152}$ classical operations for AES-256.[1]
- Assuming a max depth of 2^{96} for a quantum circuit: overall AES-256 cost is $\approx 2^{190}$.
- Does not parallelise: have to wait for 2^X steps, cannot buy 2^{32} quantum computers and wait $2^X/2^{32}$ steps.

[1] Samuel Jaques, Michael Naehrig, Martin Roetteler, and Fernando Virdia. Implementing Grover Oracles for Quantum Key Search on AES and LowMC. In: *EUROCRYPT 2020, Part II*. ed. by Anne Canteaut and Yuval Ishai. Vol. 12106. LNCS. Springer, Cham, May 2020, pp. 280–310. DOI: 10.1007/978-3-030-45724-2_10.

Shor's Algorithm

Task

Given $N = p \cdot q$ for p, q prime find p or q.

A Magical New Operation

Consider a function $f_{a,N}(x)$ for any $0 < a < N$, which computes $f_{a,N}(x) \coloneqq a^x \bmod N$

Example:

```
p, q = 13, 15
N = p*q
a = 2

def f(x):
    return power_mod(a, x, N)

f(13)
```

```
2
```

```
p, q = 3, 5
N = p*q
a = 2

[list(range(N)), None, [f(i) for i in range(N)]]
```

0	1	2	3	4	5	6	7	8	9	10	11	12	13	1
1	2	4	8	1	2	4	8	1	2	4	8	1	2	

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	4	8	1	2	4	8	1	2	4	8	1	2	4

A Magical New Operation

Theorem (Euler's Theorem)

For any modulus N and any coprime integer a, it holds that

$$a^{\phi(n)} \equiv 1 \bmod N$$

where $\phi(n)$, Euler's totient function, counts the integers up to n relatively prime to n.

- So $f_{a,N}(\cdot)$ should have some period r: $f_{a,N}(x) \equiv f_{a,N}(x + r)$.
- We can implement $f_{a,N}(\cdot)$ efficiently on classical and on quantum computers
- On a quantum computer, we can find this period efficiently but this assumed hard on classical computers.

A Magical New Operation

Let $\mathcal{P}(a, N)$ be an oracle that outputs r s.t. $f_{a,N}(x) \equiv f_{a,N}(x + r)$.

Factoring with that Magical New Operation

1. Pick a random $2 \leq a < N$.
2. If $\gcd(a, N) \neq 1$, output a as a factor of N.
3. Call $\mathcal{P}(a, N)$ and retrieve r.
4. If r is not even, start over.
5. We have $a^r \equiv 1 \bmod N$ and thus $N \mid (a^r - 1)$.
6. Write $a^r - 1 = (\sqrt{a^r} + 1) \cdot (\sqrt{a^r} - 1)$.[2]
7. So we get $N \mid (a^{r/2} - 1) \cdot (a^{r/2} + 1)$, i.e. any factor of N is a factor of $(a^{r/2} - 1)$, $(a^{r/2} + 1)$ or both
   - 7.1 It can't be that $N \mid a^{r/2} - 1$ because the period is r and not $r/2$
   - 7.2 It could be that $N \mid a^{r/2} + 1$ and then the algorithm fails
8. Compute $d \coloneqq \gcd(N, a^{r/2} + 1)$

[2] $x^2 - y^2 = (x - y) \cdot (x + y)$

The Magical New Operation

1. We can implement $f_{a,N}(\cdot)$ as a quantum circuit $\mathsf{U}_{f_{a,N}(\cdot)}$ acting on $m \coloneqq \lceil \log N^2 \rceil$ qubits
2. We can apply Hadamard gates on the inputs before applying $\mathsf{U}_{f_{a,N}(\cdot)}$
3. This gives us a state[3]

$$|\phi_2\rangle \coloneqq \frac{\sum_{\mathsf{x}\in\{0,1\}^m} |\mathsf{x},\, f_{a,N}(\mathsf{x})\rangle}{\sqrt{2^m}} = \frac{\sum_{\mathsf{x}\in\{0,1\}^m} |\mathsf{x},\, a^{\mathsf{x}} \bmod N\rangle}{\sqrt{2^m}}.$$

4. The final ingredient is a Quantum Fourier Transform (QFT) which more or less extracts the period from such a state.[4]

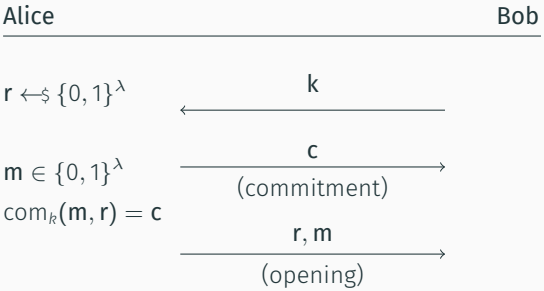
[3] I'm identifying the binary representation x of x with x here.

[4] I have yet to find a simple way of explaining it :(

Recap: Shor vs RSA, DH, ...

Commitment Schemes

Commitment Schemes



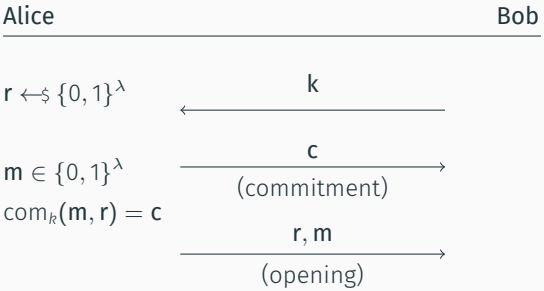
Statistically Hiding:

$$\Pr\left[b' = b \;\middle|\; \begin{array}{r} (\mathsf{m}_0, \mathsf{m}_1) \leftarrow \mathcal{A}(\mathsf{k}) \\ b \leftarrow \{0,1\} \\ \mathsf{r} \leftarrow\!\$\ \{0,1\}^\lambda \\ \mathsf{c} \leftarrow \mathrm{com}_k(\mathsf{m}_b, \mathsf{r}) \\ b' \leftarrow \mathcal{A}(\mathsf{c}) \end{array} \right] = \frac{1}{2}$$

for any $\mathcal{A}$.

Computationally Binding: "PPT adversary cannot change its mind after sending c"

Commitment Schemes



Statistically Hiding:

$$\Pr\left[b' = b \;\middle|\; \begin{array}{r} (\mathsf{m}_0, \mathsf{m}_1) \leftarrow \mathcal{A}(\mathsf{k}) \\ b \leftarrow \{0,1\} \\ \mathsf{r} \leftarrow\!\$\ \{0,1\}^\lambda \\ \mathsf{c} \leftarrow \mathrm{com}_k(\mathsf{m}_b, \mathsf{r}) \\ b' \leftarrow \mathcal{A}(\mathsf{c}) \end{array} \right] = \frac{1}{2}$$

for any $\mathcal{A}$.

Computationally Binding: "PPT adversary cannot change its mind after sending c"

How should we formalise this?

Classical Definition

"PPT adversary cannot change its mind after sending c"

Classical Definition

PPT $\mathcal{A}$ cannot find $(\mathsf{m}, \mathsf{r}, \mathsf{m}', \mathsf{r}')$ where $\mathsf{m} \neq \mathsf{m}'$ and

$$\mathrm{com}_k(\mathsf{m}, \mathsf{r}) = \mathrm{com}_k(\mathsf{m}', \mathsf{r}').$$

In particular, any collision-resistant hash function implies a binding commitment scheme.

Interlude

- A commitment scheme cannot be statistically hiding and statistically binding at the same time
  - If it is statistically hiding this means that for any $\mathbf{c} = \mathrm{com}_k(\mathbf{m}, \mathbf{r})$ there exists some $\mathbf{r}'$ such that $\mathbf{c} = \mathrm{com}_k(\mathbf{m}', \mathbf{r}')$ for any $\mathbf{m}'$.
  - If it is statistically binding this means that for any $\mathbf{c} = \mathrm{com}_k(\mathbf{m}, \mathbf{r})$ there exists no $\mathbf{r}'$ such that $\mathbf{c} = \mathrm{com}_k(\mathbf{m}', \mathbf{r})$ for any $\mathbf{m}' \neq \mathbf{m}$.
- Any IND-CPA secure encryption scheme is a hiding commitment scheme
- Any perfectly-correct encryption scheme is a binding commitment scheme, otherwise decryption might fail

Classical Definition

"PPT adversary cannot change its mind after sending c"

Classical Definition

PPT $\mathcal{A}$ cannot find (m, r, m', r') where $m \neq m'$ and

$$\mathrm{com}_k(m, r) = \mathrm{com}_k(m', r').$$

In particular, any collision-resistant hash function implies a binding commitment scheme.

Punchline

This is not true if $\mathcal{A}$ is a quantum adversary.

Attack on Classical Definition I

There exists a quantum-secure collision-resistant hash function H where $\mathcal{A}$ can open $\mathsf{com}_k(m, r) := H(m, r)$ to any m.[5]

- Quantum adversary cannot find two pairs (m, r), (m', r') that agree on $\mathsf{com}_k(m, r) = \mathsf{com}_k(m, r)$
- But it can open to some message m even if it learns it after sending c.

Caveat

The attack depends on an oracle that we do not know how to build. But even with this oracle collision resistance holds.

[5] Andris Ambainis, Ansis Rosmanis, and Dominique Unruh. Quantum Attacks on Classical Proof Systems: The Hardness of Quantum Rewinding. In: *55th FOCS*. IEEE Computer Society Press, Oct. 2014, pp. 474–483. DOI: 10.1109/FOCS.2014.57; Dominique Unruh. Computationally Binding Quantum Commitments. In: *EUROCRYPT 2016, Part II*. ed. by Marc Fischlin and Jean-Sébastien Coron. Vol. 9666. LNCS. Springer, Berlin, Heidelberg, May 2016, pp. 497–527. DOI: 10.1007/978-3-662-49896-5_18.

Attack on Classical Definition II

1. Prepare a quantum state

$$|\phi\rangle := \left[\frac{\sum_{\mathsf{m},\mathsf{r}\in\{0,1\}^\lambda\times\{0,1\}^\lambda} |\mathsf{m}\rangle\,|\mathsf{r}\rangle}{\sqrt{2^{2\,\lambda}}}\right] \quad |0\rangle$$

2. Apply H on the first two registers and add result to the third

$$|\phi\rangle := \frac{\sum_{\mathsf{m},\mathsf{r}\in\{0,1\}^\lambda\times\{0,1\}^\lambda} |\mathsf{m}\rangle\,|\mathsf{r}\rangle\,|H(\mathsf{m},\mathsf{r})\rangle}{\sqrt{2^{2\,\lambda}}}$$

3. Measure the third register to obtain some value h

$$|\phi\rangle := \frac{\sum_{(\mathsf{m},\mathsf{r})\,|\,\mathsf{h}=H(\mathsf{m},\mathsf{r})} |\mathsf{m}\rangle\,|\mathsf{r}\rangle}{\sqrt{|\,\{(\mathsf{m},\mathsf{r})\mid h = H(\mathsf{m},\mathsf{r})\}\,|}}\,|\mathsf{h}\rangle$$

The first register now contains all preimages of h.

Attack on Classical Definition III

4. Use the magic oracle[6] to filter $\left\{(\mathsf{m}, \mathsf{r}) \mid \mathsf{h} = H(\mathsf{m}, \mathsf{r})\right\}$ to

$$\left\{(\mathsf{m}, \mathsf{r}) \mid \mathsf{h} = H(\mathsf{m}, \mathsf{r}) \wedge \mathsf{m} = \mathsf{m}_0\right\}$$

   for any chosen m_0.
5. Measure the first register to obtain $(\mathsf{m}_0, \mathsf{r})$ and submit as an opening.

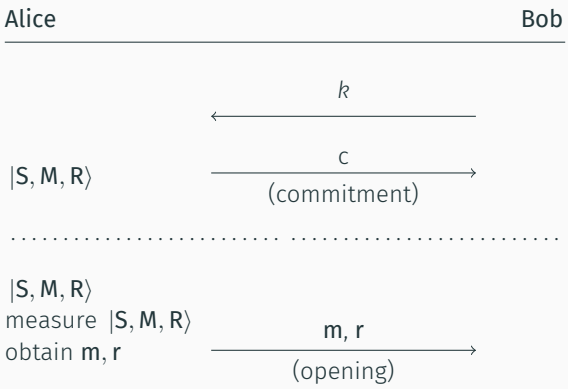
Collision Resistance

This does not violate collision resistance because we are "using up" our state, i.e. we can only measure once, still.

[6] This is a variant of Grover's algorithm but we don't know how to implement the required steps.

Corrected Definition: Formalising the Attacker

Can write down our attacker like this:



Corrected Definition: What does ideal look like?

Collapse-binding Commitment

1 : $b \leftarrow\!\$\ \{0,1\}$; $k \leftarrow\!\$\ \{0,1\}^\lambda$

2 : $\mathsf{c}, |\mathsf{S},\mathsf{M},\mathsf{R}\rangle \leftarrow \mathcal{A}(k)$

3 : compute $|\mathsf{S},\mathsf{M},\mathsf{R},V_\mathsf{c}(\mathsf{M},\mathsf{R})\rangle$ // $V_\mathsf{c}(\mathsf{M},\mathsf{R}) = 1$ iff $\mathrm{com}_k(\mathsf{M},\mathsf{R}) = \mathsf{c}$

4 : measure $|V_\mathsf{c}(\mathsf{M},\mathsf{R})\rangle = v$

5 : // measurement has no effect if $|\mathsf{M}\rangle = |\mathsf{m}\rangle$, i.e. "collapsed"

6 : **if** $v = 1 \wedge b = 0$ **then** measure $|\mathsf{M}\rangle$

7 : $b' \leftarrow \mathcal{A}\left(|\mathsf{S},\mathsf{M},\mathsf{R},V_\mathsf{c}(\mathsf{M},\mathsf{R})\rangle\right)$

8 : **return** $b = b'$

Dominique Unruh. Computationally Binding Quantum Commitments. In: *EUROCRYPT 2016, Part II*. ed. by Marc Fischlin and Jean-Sébastien Coron. Vol. 9666. LNCS. Springer, Berlin, Heidelberg, May 2016, pp. 497–527. DOI: 10.1007/978-3-662-49896-5_18

Collapsing Hash Functions

Collapsing Hash Function H

1: $b \leftarrow\$ \{0,1\}$
2: $|\psi\rangle_0 := |\mathsf{S}\rangle \sum_{\mathsf{x}} |\mathsf{x}, 0\rangle \leftarrow \mathcal{A}(H)$
3: $|\psi\rangle_1 := |\mathsf{S}\rangle \sum_{\mathsf{x}} |\mathsf{x}, H(\mathsf{x})\rangle$
4: **if** $b = 0$ **then**
5: measure $|\mathsf{x}\rangle \in |\psi\rangle_1 \rightarrow |\psi\rangle_2$
6: **else**
7: measure $|H(\mathsf{x})\rangle \in |\psi\rangle_1 \rightarrow |\psi\rangle_2$
8: $b' \leftarrow \mathcal{A}\left(|\psi\rangle_2\right)$
9: **return** $b = b'$

Figure 1: Collapsing Hash Function

Game indeed differs:

- $b = 0$: collapses to a single input-output pair
- $b = 1$: collapses to all preimages of measured value $H(\mathsf{x})$

[Unr16]: This implies collapse-binding commitments.

A New Classical Definition

Any somewhere statistically binding hash function is collapsing.

Somewhere Statistically Binding (SSB)

- Consider $H(\mathsf{x}_0 \mid \mathsf{x}_1 \mid \ldots \mid \mathsf{x}_{\ell-1})$
- There are "modes" $H^{(i)}(\mathsf{x}_0 \mid \mathsf{x}_1)$ that are **statistically binding** to block x_i
- We also have "index hiding": $H \approx_c H^{(i)} \approx_c H^{(j)}$ for any i, j.

- Since $H()$ is compressing it it cannot be statistically binding to its input
- But it can be be statistically binding for one small block
- If cannot tell which block it is statistically binding to, have an SSB hash function
- Can build this from a perfectly correct fully-homomorphic encryption scheme

Fin

If you take nothing else from this lecture: quantum computers won't solve hard problems instantly by just trying all solutions in parallel.

Credit: https://scottaaronson.blog/

References I

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[Unr16] Dominique Unruh. Computationally Binding Quantum Commitments. In: *EUROCRYPT 2016, Part II*. Ed. by Marc Fischlin and Jean-Sébastien Coron. Vol. 9666. LNCS. Springer, Berlin, Heidelberg, May 2016, pp. 497–527. DOI: 10.1007/978-3-662-49896-5_18.

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