LWE and Encryption

Indian Workshop on Post-Quantum Cryptography

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Outline

LWE

LWE and Lattices

Variants

LWE Encryption

CCA Security

Practical Performance

LWE

1-dim LWE (even easier than RSA)

KeyGen

- Pick an integer $q \approx 2^{10000}$
- Pick a random integer $s \in \mathbb{Z}_q$
- Pick about $t = 20000$ random $a_i \in \mathbb{Z}_q$ and $e_i \approx 2^{9990}$
- Publish pairs $a_i, c_i = a_i \cdot s + e_i \bmod \mathbb{Z}_q$

Encrypt $m \in \{0,1\}$

- Pick $b_i \in \{-1, 0, 1\}$
- $d_0 = \sum_{i=0}^{t-1} b_i \cdot a_i$
- $d_1 = q/2 \cdot m + \sum_{i=0}^{t-1} b_i \cdot c_i$
- Return d_0, d_1

Decrypt

- Compute $d = d_1 - d_0 \cdot s$

$$
\begin{aligned}
&= q/2 \cdot m + \sum_{i=0}^{t-1} b_i \cdot c_i - \sum_{i=0}^{t-1} b_i \cdot a_i \cdot s \\
&= q/2 \cdot m + \sum_{i=0}^{t-1} b_i \cdot (a_i \cdot s + e_i) - \sum_{i=0}^{t-1} b_i \cdot a_i \cdot s \\
&= q/2 \cdot m + \sum_{i=0}^{t-1} b_i \cdot e_i
\end{aligned}
$$

- Return 1 if d is closer to $q/2$ than zero and 0 otherwise.

The Learning with Errors Problem (LWE)

Given $(\mathbf{A}, \mathbf{c})$ with $\mathbf{c} \in \mathbb{Z}_q^m$, $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$, $\mathbf{s} \in \mathbb{Z}_q^n$ and small $\mathbf{e} \in \mathbb{Z}^m$ is

$$\begin{pmatrix} \\ \\ \mathbf{c} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \leftarrow & n & \rightarrow \\ & & \\ & \mathbf{A} & \\ & & \\ & & \end{pmatrix} \times \begin{pmatrix} \mathbf{s} \end{pmatrix} + \begin{pmatrix} \\ \\ \mathbf{e} \\ \\ \\ \end{pmatrix}$$

or $\mathbf{c} \leftarrow_{\$} \mathcal{U}\left(\mathbb{Z}_q^m\right)$.

The Learning with Errors Problem (LWE)

Definition

Let n, q be positive integers, χ be a probability distribution on $\mathbb{Z}$ and $\mathbf{s}$ be a uniformly random vector in $\mathbb{Z}_q^n$. We denote by $\mathcal{L}_{\mathbf{s},\chi}$ the probability distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$ obtained by choosing $\mathbf{a} \in \mathbb{Z}_q^n$ uniformly at random, choosing $e \in \mathbb{Z}$ according to χ and considering it in $\mathbb{Z}_q$, and returning $(\mathbf{a}, c) = (\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$.

Decision-LWE is the problem of deciding whether pairs $(\mathbf{a}, c) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ are sampled according to $\mathcal{L}_{\mathbf{s},\chi}$ or the uniform distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$.

Search-LWE is the problem of recovering $\mathbf{s}$ from pairs $(\mathbf{a}, c) = (\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ sampled according to $\mathcal{L}_{\mathbf{s},\chi}$.

Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. In: *Journal of the ACM* 56.6 (Sept. 2009), 34:1–34:40. ISSN: 0004-5411 (print), 1557-735X (electronic). DOI: `http://doi.acm.org/10.1145/1568318.1568324`

A Fair Warning

Gaussian Distributions

In this talk I am ignoring the specifics of the distribution χ. That is, the only slide with the phrase "Discrete Gaussian distribution" is this slide.

In practice, **for encryption** the shape of the error does not seem to matter much.

Also, ignoring the distribution allows to brutally simply proof sketches: almost all technical difficulty in these proofs derives from arguing about two distributions being close.

Normal Form LWE

- Consider $\mathbf{A} \in \mathbb{Z}_q^{2n \times n}$, with $\mathbf{A}^T = [\mathbf{A}_0^T \mid \mathbf{A}_1^T]$, $\mathbf{s} \in \mathbb{Z}_q^n$, $\mathbf{e} \leftarrow_{\$} \chi^m$ with $\mathbf{e}^T = (\mathbf{e}_0^T \mid \mathbf{e}_1^T)$
- We have $\mathbf{c}_0 = \mathbf{A}_0 \cdot \mathbf{s} + \mathbf{e}_0$ and $\mathbf{c}_1 = \mathbf{A}_1 \cdot \mathbf{s} + \mathbf{e}_1$
- We also have

$$\begin{aligned}
\mathbf{c}' &= \mathbf{c}_1 - \mathbf{A}_1 \cdot \mathbf{A}_0^{-1} \cdot \mathbf{c}_0 \\
&= \mathbf{A}_1 \cdot \mathbf{s} + \mathbf{e}_1 - \mathbf{A}_1 \cdot \mathbf{A}_0^{-1}(\mathbf{A}_0 \cdot \mathbf{s} + \mathbf{e}_0) \\
&= \mathbf{A}_1 \cdot \mathbf{s} + \mathbf{e}_1 - \mathbf{A}_1 \cdot \mathbf{s} - \mathbf{A}_1 \cdot \mathbf{A}_0^{-1} \cdot \mathbf{e}_0 \\
&= -\mathbf{A}_1 \cdot \mathbf{A}_0^{-1} \cdot \mathbf{e}_0 + \mathbf{e}_1 \\
&= \mathbf{A}' \cdot \mathbf{e}_0 + \mathbf{e}_1
\end{aligned}$$

[App+09]

We might as well assume that our secret is also sampled from χ.

Dimension/Modulus Trade-Off

Consider $\mathbf{a}, \mathbf{s} \in \mathbb{Z}_q^d$ where $\mathbf{s}$ is small, then

$$q^{d-1} \cdot \langle \mathbf{a}, \mathbf{s} \rangle \approx \left(\sum_{i=0}^{d-1} q^i \cdot a_i \right) \cdot \left(\sum_{i=0}^{d-1} q^{d-i-1} \cdot s_i \right) \bmod q^d = \tilde{a} \cdot \tilde{s} \bmod q^d.$$

Thus, if there exists an efficient algorithm solving the problem in $\mathbb{Z}_{q^d}$, we can use it to solve the problem in $\mathbb{Z}_q^d$.

Example ($\mathbb{Z}_{q^2}$)

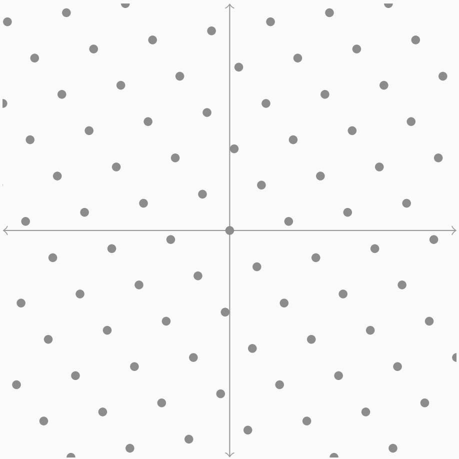
$$q \cdot (a_0 \cdot s_0 + a_1 \cdot s_1) + a_0 \cdot s_1 + q^2 \cdot a_1 \cdot s_0 \bmod q = (a_0 + q \cdot a_1) \cdot (q \cdot s_0 + s_1)$$

Zvika Brakerski et al. Classical hardness of learning with errors. In: *45th ACM STOC*. ed. by Dan Boneh, Tim Roughgarden, and Joan Feigenbaum. ACM Press, June 2013, pp. 575–584. DOI: 10.1145/2488608.2488680

LWE and Lattices

Lattices

- A lattice is a discrete subgroup of $\mathbb{R}^d$
- It can be written as $\Lambda = \{\sum_{i=0}^{d-1} v_i \cdot \mathbf{b}_i \mid v_i \in \mathbb{Z}\}$ for some basis vectors $\mathbf{b}_i$.
- We write $\Lambda(\mathbf{L})$ for the lattices spanned by the columns of $\mathbf{L}$.
- A lattice is q-ary if it contains $q\,\mathbb{Z}^d$, e.g. $\{\mathbf{x} \in \mathbb{Z}_q^d \mid \mathbf{x} \cdot \mathbf{A} \equiv \mathbf{0}\}$ for some $\mathbf{A} \in \mathbb{Z}^{d \times d'}$.



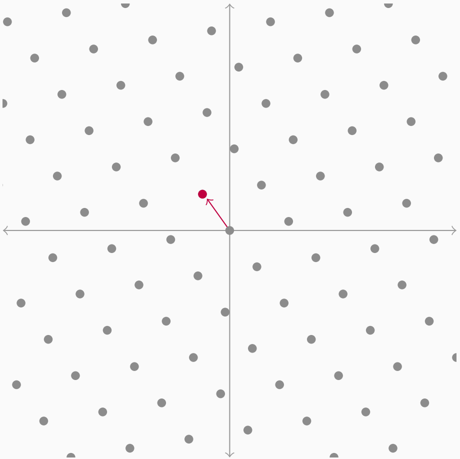
Picture credit: David Wong

Shortest Vector Problem

Definition

Given a lattice basis $\mathbf{B}$, find a shortest non-zero vector in $\Lambda(\mathbf{B})$.

- The most natural problem on lattices
- We write $\lambda_1(\Lambda)$ for the Euclidean norm of a shortest vector.
- NP-hard to solve exactly
- Cryptography relies on approximate variants without such a reduction



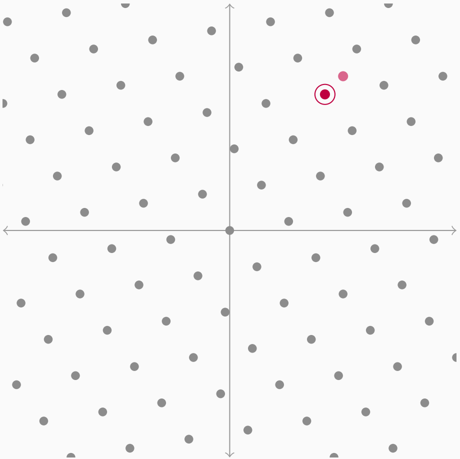
Picture credit: David Wong

Bounded Distance Decoding

Definition

Given a lattice basis $\mathbf{B}$, a vector $\mathbf{t}$, and a parameter $0 < \alpha$ such that the Euclidean distance $\mathrm{dist}(\mathbf{t}, \mathbf{B}) < \alpha \cdot \lambda_1(\Lambda(\mathbf{B}))$, find the lattice vector $\mathbf{v} \in \Lambda(\mathbf{B})$ which is closest to $\mathbf{t}$.

- When $\alpha < 1/2$ unique decoding is guaranteed but for $\alpha < 1$ we typically still expect unique decoding.
- BDD is a special case of the Closest Vector Problem where there is no bound on the distance to the lattice.



Picture credit: David Wong

LWE is Bounded Distance Decoding (BDD) on Random q-ary Lattices

Let

$$\mathbf{L} = \begin{pmatrix} q\mathbf{I} & \mathbf{A} \\ 0 & \mathbf{I} \end{pmatrix}$$

We can reformulate the matrix form of the LWE equation $\mathbf{A} \cdot \mathbf{s} + \mathbf{e} \equiv \mathbf{c} \bmod q$ as a linear system over the Integers as:

$$\mathbf{L} \cdot \begin{pmatrix} * \\ \mathbf{s} \end{pmatrix} + \begin{pmatrix} \mathbf{e} \\ -\mathbf{s} \end{pmatrix} = \begin{pmatrix} q\mathbf{I} & -\mathbf{A} \\ 0 & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} * \\ \mathbf{s} \end{pmatrix} + \begin{pmatrix} \mathbf{e} \\ -\mathbf{s} \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0} \end{pmatrix}$$

The vector $(\mathbf{c}^T, \mathbf{0}^T)^T$ is close to the lattice $\Lambda(\mathbf{L})$ with offset $(\mathbf{e}^T, -\mathbf{s}^T)^T$.

Is that a Good Choice?

- Maybe BDD on random q-ary lattices is easier than BDD in general?
- Maybe BDD is easier than SVP?

Sketch: BDD on Random q-ary Lattices solves BDD on any Lattice

- We are given some basis $\mathbf{B} \in \mathbb{Z}^{d \times d}$ and some target $\mathbf{t}$ s.t. $\mathbf{t} = \mathbf{B} \cdot \mathbf{s} + \mathbf{e}$ with $\mathbf{e}$ small
- Pick some large $q \geq 2^{2d}$
- Sample some $\mathbf{U}$ (see below)
- Set $\mathbf{A} = \mathbf{U} \cdot \mathbf{B} \bmod q$ and consider $\mathbf{c} = \mathbf{U} \cdot \mathbf{t} + \mathbf{e}'$ with $\mathbf{e}'$ small

$$\mathbf{c} = \mathbf{U} \cdot \mathbf{t} + \mathbf{e}' = \mathbf{U} \cdot (\mathbf{B} \cdot \mathbf{s} + \mathbf{e}) + \mathbf{e}' = \mathbf{U} \cdot \mathbf{B} \cdot \mathbf{s} + \mathbf{U} \cdot \mathbf{e} + \mathbf{e}' = \mathbf{A} \cdot \mathbf{s} + \mathbf{e}''$$

- We can pick $\mathbf{U}$
  - large enough to make $\mathbf{A}$ uniform mod q and
  - small enough to make $\mathbf{U} \cdot \mathbf{e} + \mathbf{e}'$ small and well distributed

  using "smoothing parameter" arguments on $\Lambda(\mathbf{B}^{-T})$

Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. In: *Journal of the ACM* 56.6 (Sept. 2009), 34:1–34:40. ISSN: 0004-5411 (print), 1557-735X (electronic). DOI: `http://doi.acm.org/10.1145/1568318.1568324`

Sketch: Solving BDD on any Lattice implies solving GapSVP

Say we want to decide if $\lambda_1(\Lambda) \leq 1$ or $\lambda_1(\Lambda) > \gamma$ and we have a BDD solver with $\alpha = c \cdot \gamma$.

- Pick a random $\mathbf{z} \in \Lambda$, add a small error $\mathbf{e}$ of norm $c \cdot \gamma$
- Run the BDD solver.
- If it returns $\mathbf{z}$ then output $\lambda_1(\Lambda) > \gamma$, else output $\lambda_1(\Lambda) \leq 1$.

Chris Peikert. Public-key cryptosystems from the worst-case shortest vector problem: extended abstract. In: *41st ACM STOC*. ed. by Michael Mitzenmacher. ACM Press, 2009, pp. 333–342. DOI: 10.1145/1536414.1536461

Regev showed: If you have a BDD solver you can find a short basis on a quantum computer

Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. In: *Journal of the ACM* 56.6 (Sept. 2009), 34:1–34:40. ISSN: 0004-5411 (print), 1557-735X (electronic). DOI: http://doi.acm.org/10.1145/1568318.1568324

Concrete Hardness: Cryptanalysis

- This tells us random q-ary lattices are not a terrible choice
- To establish how long it actually takes to solve LWE, we rely on cryptanalysis

```
load("estimator.py")
primal_usvp(n=768, q=2^13, alpha=2^-11, reduction_cost_model=BKZ.ADPS16)
```

(rop: $2^{183.4}$, red: $2^{183.4}$, delta_0: 1.002888, beta: 628, d: 1504, m: 735)

Martin R. Albrecht, Rachel Player, and Sam Scott. On the concrete hardness of Learning with Errors. In: *Journal of Mathematical Cryptology* 9.3 (2015), pp. 169–203

Variants

LWE

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} & a_{0,5} & a_{0,6} & a_{0,7} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} \\ a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} \\ a_{5,0} & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} \\ a_{6,0} & a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} \\ a_{7,0} & a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix}$$

Performance

Storage: $\mathcal{O}(n^2)$; Computation $\mathcal{O}(n^2)$

Ring-LWE/Polynomial-LWE

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \begin{pmatrix} a_0 & -a_7 & -a_6 & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \\ a_1 & a_0 & -a_7 & -a_6 & -a_5 & -a_4 & -a_3 & -a_2 \\ a_2 & a_1 & a_0 & -a_7 & -a_6 & -a_5 & -a_4 & -a_3 \\ a_3 & a_2 & a_1 & a_0 & -a_7 & -a_6 & -a_5 & -a_4 \\ a_4 & a_3 & a_2 & a_1 & a_0 & -a_7 & -a_6 & -a_5 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & -a_7 & -a_6 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & -a_7 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix}$$

Ring-LWE / Polynomial-LWE

$$\sum_{i=0}^{n-1} c_i \cdot X^i = \left(\sum_{i=0}^{n-1} a_i \cdot X^i\right) \cdot \left(\sum_{i=0}^{n-1} s_i \cdot X^i\right) + \sum_{i=0}^{8} e_i \cdot X^i \bmod X^n + 1$$

$$c(X) = a(X) \cdot s(X) + e(X) \bmod \phi(X)$$

Performance (n is a power of two)

Storage: $\mathcal{O}(n)$; Computation $\mathcal{O}(n \log n)$

Vadim Lyubashevsky, Chris Peikert, and Oded Regev. On Ideal Lattices and Learning with Errors over Rings. In: *EUROCRYPT 2010*. Ed. by Henri Gilbert. Vol. 6110. LNCS. Springer, Heidelberg, 2010, pp. 1–23. DOI: 10.1007/978-3-642-13190-5_1

$$
\begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \\ c_{0,3} \\ c_{1,0} \\ c_{1,1} \\ c_{1,2} \\ c_{1,3} \end{pmatrix}
=
\left(\begin{array}{cccc|cccc}
a_{0,0} & -a_{0,3} & -a_{0,2} & -a_{0,1} & a_{1,0} & -a_{1,3} & -a_{1,2} & -a_{1,1} \\
a_{0,1} & a_{0,0} & -a_{0,3} & -a_{0,2} & a_{1,1} & a_{1,0} & -a_{1,3} & -a_{1,2} \\
a_{0,2} & a_{0,1} & a_{0,0} & -a_{0,3} & a_{1,2} & a_{1,1} & a_{1,0} & -a_{1,3} \\
a_{0,3} & a_{0,2} & a_{0,1} & a_{0,0} & a_{1,3} & a_{1,2} & a_{1,1} & a_{1,0} \\
\hline
a_{2,0} & -a_{2,3} & -a_{2,2} & -a_{2,1} & a_{3,0} & -a_{3,3} & -a_{3,2} & -a_{3,1} \\
a_{2,1} & a_{2,0} & -a_{2,3} & -a_{2,2} & a_{3,1} & a_{3,0} & -a_{3,3} & -a_{3,2} \\
a_{2,2} & a_{2,1} & a_{2,0} & -a_{2,3} & a_{3,2} & a_{3,1} & a_{3,0} & -a_{3,3} \\
a_{2,3} & a_{2,2} & a_{2,1} & a_{2,0} & a_{3,3} & a_{3,2} & a_{3,1} & a_{3,0}
\end{array}\right)
\cdot
\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix}
+
\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix}
$$

Module-LWE

$$\begin{pmatrix} c_0(X) \\ c_1(X) \end{pmatrix} = \begin{pmatrix} a_0(X) & a_1(X) \\ a_2(X) & a_3(X) \end{pmatrix} \cdot \begin{pmatrix} s_0(X) \\ s_1(X) \end{pmatrix} + \begin{pmatrix} e_0(X) \\ e_1(X) \end{pmatrix}$$

Performance (n is a power of two)

Storage: $\mathcal{O}(k^2 \cdot n)$; Computation $\mathcal{O}(k^2 \cdot n \log n)$

Adeline Langlois and Damien Stehlé. Worst-case to average-case reductions for module lattices. In: *Designs, Codes, and Cryptography* 75.3 (June 2015), pp. 565–599. ISSN: 0925-1022 (print), 1573-7586 (electronic). DOI: `http://dx.doi.org/10.1007/s10623-014-9938-4`. URL: `http://link.springer.com/article/10.1007/s10623-014-9938-4`

LWR

Instead of "wiping" the lower-order bits of $\mathbf{c}_i = \mathbf{A} \cdot \mathbf{s}$ by adding $\mathbf{e}_i$, throw them away

- More formally, output

$$\left\lfloor \frac{p}{q} \cdot (\mathbf{A} \cdot \mathbf{s}) \right\rceil$$

  for some $p < q$.

- This is no easier than LWE if

$$\left\lfloor \frac{p}{q} \cdot (\mathbf{A} \cdot \mathbf{s}) \right\rceil = \left\lfloor \frac{p}{q} \cdot (\mathbf{A} \cdot \mathbf{s} + \mathbf{e}) \right\rceil$$

- Can be quite fast if p, q are powers of two, saves bandwidth

Abhishek Banerjee, Chris Peikert, and Alon Rosen. Pseudorandom Functions and Lattices. In: *EUROCRYPT 2012*. Ed. by David Pointcheval and Thomas Johansson. Vol. 7237. LNCS. Springer, Heidelberg, Apr. 2012, pp. 719–737. DOI: 10.1007/978-3-642-29011-4_42

LWE Encryption

Convention

- I am going to use the Ring-LWE formulation

$$c_i(X) = a_i(X) \cdot s(X) + e_i(X)$$

  Thus, each sample corresponds to ”n LWE samples”
- I will suppress the ”(X)” in ”$a(X)$” etc.
- I will assume s is ”small” and that the product of two ”small” things is ”small”.
- I will write e_i to emphasise that e_i is small.

TL;DR: I will write

$$c_i = a_i \cdot s + e_i$$

DH to Ring-LWE Dictionary

DH Land	Ring-LWE Land
g	a
g^x	$a \cdot s + e$
$g^x \cdot g^y = g^{x+y}$	$(a \cdot s + e_0) + (a \cdot t + e_1) = a \cdot (s + t) + e'$
$(g^a)^b = (g^b)^a$	$(a \cdot s + e) \cdot t = (a \cdot s \cdot t + e \cdot t)$ $\approx a \cdot s \cdot t \approx (a \cdot t + e) \cdot s$
(g, g^a, g^b, g^{ab}) $\approx_c (g, g^a, g^b, u)$	$(a,\ a \cdot s + e,\ a \cdot t + d,\ a \cdot s \cdot t + e')$ $\approx_c (a,\ a \cdot s + e,\ a \cdot t + d,\ u)$

Regev

You have already seen it.

KeyGen Publish $c_i = a_i \cdot s + e_i$ for $i = 0, \ldots, \lceil 2\, n \log q \rceil$

Encrypt

$$d_0 = \sum b_i \cdot a_i, \quad d_1 = \left(\sum b_i \cdot c_i\right) + q/2 \cdot m \text{ with } b_i \in \{0,1\}, m \in \{0,1\}^n$$

Decrypt

$$\begin{aligned}
\left\lfloor \frac{2}{q} \cdot (d_1 - d_0 \cdot s) \right\rceil &= \left\lfloor \frac{2}{q} \cdot \left(\left(\sum b_i \cdot c_i \right) + \frac{q}{2} \cdot m - \sum b_i \cdot a_i \cdot s \right) \right\rceil \\
&= \left\lfloor \frac{2}{q} \cdot \left(\left(\sum b_i \cdot (a_i \cdot s + e_i) \right) + \frac{q}{2} \cdot m - \sum b_i \cdot a_i \cdot s \right) \right\rceil \\
&= \left\lfloor \frac{2}{q} \cdot \left(\left(\sum b_i \cdot e_i \right) + \frac{q}{2} \cdot m \right) \right\rceil = m
\end{aligned}$$

The public key is indistinguishable from uniform by the LWE assumption and $\sum b_i \cdot a_i$ is statistically close to uniformly random by the Leftover Hash Lemma (LHL).

ElGamal & LPR10

ElGamal

KeyGen $h = g^x$

Encrypt $d_0, \ d_1 = (g^r, \ m \cdot h^r)$ for some random r

Decrypt $d_1 / d_0^x = m \cdot (g^x)^r / (g^r)^x = m$

[LPR10][1]

KeyGen $c = a \cdot s + e$

Encrypt $d_0, \ d_1 = v \cdot a + e', \ v \cdot c + e'' + q/2 \cdot m$

Decrypt

$$\left\lfloor \frac{2}{q} \cdot (d_1 - d_0 \cdot s) \right\rceil = \left\lfloor \frac{2}{q} \cdot \left(v \cdot (a \cdot s + e) + e'' + \frac{q}{2} \cdot m - (v \cdot a + e') \cdot s \right) \right\rceil$$

$$= \left\lfloor \frac{2}{q} \cdot \left(v \cdot e + e'' + \frac{q}{2} \cdot m - e' \cdot s \right) \right\rceil = m$$

---

[1] All NIST PQC candidates based on (Ring-/Module-)LWE encrypt like this

Proof Sketch

KeyGen $c = a \cdot s + e$

- The public key (a, c) is indistinguishable from uniform (u', u'') by the (Ring-)LWE assumption

Encrypt $d_0,\ d_1 = v \cdot a + e',\ v \cdot c + e'' + q/2 \cdot m$

- Then $v \cdot u' + e'',\ v \cdot u'' + e''$ is indistinguishable from uniform by the (Ring)-LWE assumption

Once you have ElGamal, recovering Diffie-Hellman is straight forward.

Common a

Alice $c_0 = s \cdot a + e_0$

Bob $c_1 = a \cdot t + e_1$

Shared

$$c_0 \cdot t = (s \cdot a + e_0) \cdot t \approx s \cdot a \cdot t \approx s \cdot (a \cdot t + e_1) = s \cdot c_1$$

Reconciliation II

$$c_0 \cdot t = (s \cdot a + e_0) \cdot t \approx s \cdot a \cdot t \approx s \cdot (a \cdot t + e_1) = s \cdot c_1$$

- The problem with this construction is that "$\approx$" $\neq$ "$=$"
- Need to send a "hint" how to round correctly (2nd most significant bit)[2]
- Cannot have efficient Non-interactive Key Exchange (NIKE) without new ideas
- Here be dragons patents
- NIST asked for "key exchange" but meant "key encapsulation", can build former generically from latter

[2] Jintai Ding, Xiang Xie, and Xiaodong Lin. A Simple Provably Secure Key Exchange Scheme Based on the Learning with Errors Problem. Cryptology ePrint Archive, Report 2012/688. `http://eprint.iacr.org/2012/688`. 2012.

CCA Security

Active Attacks

- Recall decryption

$$\left\lfloor \frac{2}{q} \cdot (d_1 - d_0 \cdot s) \right\rceil = \left\lfloor \frac{2}{q} \cdot \left(\frac{q}{2} \cdot m + v \cdot e - e' \cdot s + e'' \right) \right\rceil = m$$

- When the result of the rounding $\neq m$ this contains information about

$$v \cdot e - e' \cdot s + e''$$

where the attacker/encrypter controls v, e'', e' and would like to learn s, e.

FO Transform (KEM Variant)

Encrypt $v, e', e'' \leftarrow \mathsf{H}(\text{seed})$ and $m = \text{seed}$ for some hash function H.

Decrypt After decryption

- compute $v, e', e'' \leftarrow \mathsf{H}(m')$ and
- check $c_0 \stackrel{?}{=} v \cdot a + e'$ and $c_1 \stackrel{?}{=} v \cdot c + e'' + q/2 \cdot m'$.

Eiichiro Fujisaki and Tatsuaki Okamoto. Secure Integration of Asymmetric and Symmetric Encryption Schemes. In: *Journal of Cryptology* 26.1 (Jan. 2013), pp. 80–101. DOI: `10.1007/s00145-011-9114-1`

Dennis Hofheinz, Kathrin Hövelmanns, and Eike Kiltz. A Modular Analysis of the Fujisaki-Okamoto Transformation. In: *TCC 2017, Part I*. ed. by Yael Kalai and Leonid Reyzin. Vol. 10677. LNCS. Springer, Heidelberg, Nov. 2017, pp. 341–371. DOI: `10.1007/978-3-319-70500-2_12`

(Q)ROM

- The FO transform was originally proven secure when modelling the hash function as a Random Oracle (RO)
- Hash functions are public functions and thus can be implemented on a quantum computer
- We must model the hash function as a Quantum Random Oracle (QRO), accepting superposition queries

Tsunekazu Saito, Keita Xagawa, and Takashi Yamakawa. Tightly-Secure Key-Encapsulation Mechanism in the Quantum Random Oracle Model. In: *EUROCRYPT 2018, Part III*. ed. by Jesper Buus Nielsen and Vincent Rijmen. Vol. 10822. LNCS. Springer, Heidelberg, 2018, pp. 520–551. DOI: 10.1007/978-3-319-78372-7_17

Practical Performance

Baseline: Pre Quantum Cryptography

RSA 2048

Key generation	$\approx$ 130,000,000 cycles
Encapsulation	$\approx$ 20,000 cycles
Decapsulation	$\approx$ 2,700,000 cycles
Ciphertext	256 bytes
Public key	256 bytes

`https://bench.cr.yp.to/results-kem.html`

Curve25519

Key generation	$\approx$ 60,000 cycles
Key agreement	$\approx$ 160,000 cycles
Public key	32 bytes
Key Share	32 bytes

`https://eprint.iacr.org/2015/943`

Curve25519

Key generation	$\approx$ 60,000 cycles
Key agreement	$\approx$ 160,000 cycles
Public key	32 bytes
Key Share	32 bytes

`https://eprint.iacr.org/2015/943`

Kyber-768 NIST PQC Round 2 submission:

Key generation	$\approx$ 42,000 cycles
Encapsulation	$\approx$ 60,000 cycles
Decapsulation	$\approx$ 52,000 cycles
Ciphertext	1,088 bytes
Public key	1,184 bytes

`https://bench.cr.yp.to/results-kem.html`

Interpretation

- An Ethernet frame takes 1,500 bytes
- Your laptop does about $2 \cdot 10^9$ cycles per second

Thank You

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