Chapter 6 solutions to Stochastic Calculus for Finance I by Steven E. Shreve

Chung Ching Lau

1. (i)

$$
\begin{aligned}
&\mathbf{E}_n[c_1X+c_2Y](\bar{\omega_1},\cdots,\bar{\omega_n})\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}(c_1X+c_2Y)(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&=c_1\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}X(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&+c_2\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}Y(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&=c_1\mathbf{E}_n[X](\bar{\omega_1},\cdots,\bar{\omega_n})+c_2\mathbf{E}_n[Y](\bar{\omega_1},\cdots,\bar{\omega_n}).
\end{aligned}
$$

(ii) If X only depends on the first n coin tosses, i.e.

$$X(\bar{\omega_1},\cdots,\bar{\omega}_n,\bar{\omega}'_{n+1},\cdots,\bar{\omega}'_N)=X(\bar{\omega_1},\cdots,\bar{\omega}_n,\bar{\omega}''_{n+1},\cdots,\bar{\omega}''_N)$$

for any $\bar{\omega}'_{n+1},\cdots,\bar{\omega}'_N$ and $\bar{\omega}''_{n+1},\cdots,\bar{\omega}''_N$. Therefore, we may abuse notation and define

$$X(\bar{\omega_1},\cdots,\bar{\omega}_n):=X(\bar{\omega_1},\cdots,\bar{\omega}_n,\bar{\omega}'_{n+1},\cdots,\bar{\omega}'_N)$$

where the choice of $\bar{\omega}'_{n+1},\cdots,\bar{\omega}'_N$ is arbitrary.

$$
\begin{aligned}
&\mathbf{E}_n[XY](\bar{\omega_1},\cdots,\bar{\omega_n})\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}(XY)(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}X(\bar{\omega_1},\cdots,\bar{\omega}_N)Y(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&=X(\bar{\omega_1},\cdots,\bar{\omega}_n)\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}Y(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&=X\cdot\mathbf{E}_n[Y](\bar{\omega_1},\cdots,\bar{\omega_n}).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&\mathbf{E}_n[\mathbf{E}_m[X]](\bar{\omega_1},\cdots,\bar{\omega_n})\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_m}\mathbf{E}_m[X](\bar{\omega_1},\cdots,\bar{\omega}_m)\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_m=\bar{\omega}_m|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_m}\Big((\sum_{\bar{\omega}_{m+1},\cdots,\bar{\omega}_N}X(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\mathbf{P}(\omega_{m+1}=\bar{\omega}_{m+1},\cdots,\omega_N=\bar{\omega}_N|\omega_1=\bar{\omega}_1,\cdots,\omega_m=\bar{\omega}_m))\\
&\cdot\mathbf{P}(\omega_{n+1}=\bar{\omega}_{n+1},\cdots,\omega_m=\bar{\omega}_m|\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)\Big)\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}X(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\frac{\mathbf{P}(\omega_1=\bar{\omega}_1,\cdots,\omega_N=\bar{\omega}_N)}{\mathbf{P}(\omega_1=\bar{\omega}_1,\cdots,\omega_m=\bar{\omega}_m)}\cdot\frac{\mathbf{P}(\omega_1=\bar{\omega}_1,\cdots,\omega_m=\bar{\omega}_m)}{\mathbf{P}(\omega_1=\bar{\omega}_1,\cdots,\omega_n=\bar{\omega}_n)}\\
&=\sum_{\bar{\omega}_{n+1},\cdots,\bar{\omega}_N}X(\bar{\omega_1},\cdots,\bar{\omega}_N)\cdot\frac{\mathbf{P}(\omega_1=\bar{\omega}_1,\cdots,\omega_N=\bar{\omega}_N)}{\mathbf{P}(\omega_1=\bar{\omega}_1,\cdots,\omega_m=\bar{\omega}_n)}\\
&=\mathbf{E}_n[X](\bar{\omega_1},\cdots,\bar{\omega_n}).
\end{aligned}
$$

(iv) This is not true! The results of the coin tosses in $t = n+1, \cdots, N$ may depend on the results on $t = 1, \cdots, n$. If that is the case, them $\mathbf{E}_n[X]$ may not be a constant.

Let me give an example. Let $n = 1$ and $N = 2$. Let X be a random variable that depends on the result of the second toss, with

$$X(\omega_2 = H) = 1; \quad X(\omega_2 = T) = -1.$$

Let

$$\mathbf{P}(\omega_1 = H) = \mathbf{P}(\omega_1 = T) = \frac{1}{2};$$
$$\mathbf{P}(\omega_1 = H, \omega_2 = H) = \mathbf{P}(\omega_1 = T, \omega_2 = T) = 0.9;$$
$$\mathbf{P}(\omega_1 = H, \omega_2 = T) = \mathbf{P}(\omega_1 = T, \omega_2 = H) = 0.1.$$

Then

$$\mathbf{E}_1[X](H) = 0.9 \cdot 1 + 0.1 \cdot (-1) = 0.8.$$
$$\mathbf{E}_1[X](T) = 0.1 \cdot 1 + 0.9 \cdot (-1) = -0.8.$$

In other words, $\mathbf{E}_1[X]$ is not a constant at $t = 0$. The claim is only true if the result of the coin tosses at $t = n+1, \cdots, N$ are independent of the results at $t = 1, \cdots, n$, i.e.

$$\mathbf{P}(\omega_{n+1} = \bar{\omega_{n+1}}, \cdots \omega_N = \bar{\omega_N}) = \mathbf{P}(\omega_{n+1} = \bar{\omega_{n+1}}, \cdots \omega_N = \bar{\omega_N} | \omega_1 = \bar{\omega_1}, \cdots \omega_n = \bar{\omega_n}) \tag{1}$$

for any $\bar{\omega_1}, \cdots, \bar{\omega_N}$. This holds for example when all coin tosses are i.i.d. Assuming (1),

$$\begin{aligned}
&\mathbf{E}_n[X](\bar{\omega_1}', \cdots, \bar{\omega}_n') \\
&= \sum_{\bar{\omega}_{n+1}, \cdots, \bar{\omega}_N} X(\bar{\omega_1}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N | \omega_1 = \bar{\omega'}_1, \cdots, \omega_n = \bar{\omega'}_n) \\
&= \sum_{\bar{\omega}_{n+1}, \cdots, \bar{\omega}_N} X(\bar{\omega_{n+1}}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N) \\
&= \sum_{\bar{\omega}_{n+1}, \cdots, \bar{\omega}_N} X(\bar{\omega_{n+1}}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N) \cdot \sum_{\bar{\omega}_1, \cdots, \bar{\omega}_n} \mathbf{P}(\omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \\
&= \sum_{\bar{\omega}_1, \cdots, \bar{\omega}_N} X(\bar{\omega_{n+1}}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N) \cdot \mathbf{P}(\omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \\
&= \sum_{\bar{\omega}_1, \cdots, \bar{\omega}_N} \big(X(\bar{\omega_{n+1}}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N | \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \\
&\cdot \mathbf{P}(\omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \big) \\
&= \sum_{\bar{\omega}_1, \cdots, \bar{\omega}_N} \big(X(\bar{\omega_1}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N | \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \\
&\cdot \mathbf{P}(\omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \big) \\
&= \sum_{\bar{\omega}_1, \cdots, \bar{\omega}_N} X(\bar{\omega}_1, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_1 = \bar{\omega}_1, \cdots, \omega_N = \bar{\omega}_N) \\
&= \mathbf{E}[X],
\end{aligned}$$

where we used (1) in the second and the fifth equality; we also used the fact $\sum_{\bar{\omega}_1, \cdots, \bar{\omega}_n} \mathbf{P}(\omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) = 1$ in the third equality.

(v) Note that we have the following finite form of Jensen's inequality:

$$\varphi(\sum_i p_i x_i) \leq \sum_i p_i \varphi(x_i) \tag{2}$$

for φ convex and $\sum_i p_i = 1$.

$$\begin{aligned}
&\mathbf{E}_n[\varphi(X)](\bar{\omega_1}, \cdots \bar{\omega_n}) \\
&= \sum_{\bar{\omega}_{n+1}, \cdots, \bar{\omega}_N} \varphi(X)(\bar{\omega_1}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N | \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n) \\
&\geq \varphi\Big(\sum_{\bar{\omega}_{n+1}, \cdots, \bar{\omega}_N} X(\bar{\omega_1}, \cdots, \bar{\omega}_N) \cdot \mathbf{P}(\omega_{n+1} = \bar{\omega}_{n+1}, \cdots, \omega_N = \bar{\omega}_N | \omega_1 = \bar{\omega}_1, \cdots, \omega_n = \bar{\omega}_n)\Big) \\
&= \varphi(\mathbf{E}_n[X])(\bar{\omega_1}, \cdots \bar{\omega_n})
\end{aligned}$$

2. The hedging portfolio consists of a short position in $\frac{S_n}{B_{n,m}}$ unit of the zero coupon bond with maturity m and one unit of the asset itself. Thus, the value of it at $t = k$, where $n \leq k \leq m$ is:

$$\frac{S_n}{B_n, m} \cdot B_{k,m} + S_k;$$

and the discounted value is:

$$D_k \frac{S_n}{B_n, m} \cdot B_{k,m} + D_k S_k;$$

We claim that each of these terms is a martingale. Let us prove that for the first term:

$$\begin{aligned}
&\tilde{\mathbf{E}}_k[D_{k+1} \frac{S_n}{B_n, m} \cdot B_{k+1,m}] \\
&= \frac{S_n}{B_n, m} \tilde{\mathbf{E}}_k[D_{k+1} \cdot B_{k+1,m}] && \text{(Take out what is known)} \\
&= \frac{S_n}{B_n, m} \tilde{\mathbf{E}}_k[D_{k+1} \cdot \tilde{\mathbf{E}}_{k+1}[\frac{D_m}{D_{k+1}}]] \\
&= \frac{S_n}{B_n, m} \tilde{\mathbf{E}}_k[\tilde{\mathbf{E}}_{k+1}[D_m]] && \text{(Take out what is known)} \\
&= \frac{S_n}{B_n, m} \tilde{\mathbf{E}}_k[D_m] && \text{(Iterated conditioning)} \\
&= \frac{S_n}{B_n, m} D_k \tilde{\mathbf{E}}_k[\frac{D_m}{D_k}] && \text{(Take out what is known)} \\
&= \frac{S_n}{B_n, m} D_k B_{k,m}.
\end{aligned}$$

For the second term, it follows from the definition of risk neutral measure, which is chosen to make the discounted asset price a martingale.

3.

$$\begin{aligned}
&\frac{1}{D_n} \tilde{\mathbf{E}}_n[D_{m+1} R_m] \\
&= \frac{1}{D_n} \tilde{\mathbf{E}}_n[D_{m+1}(1 + R_m) - D_{m+1}] \\
&= \frac{1}{D_n} \tilde{\mathbf{E}}_n[D_m - D_{m+1}] && (\because D_{m+1} = \frac{D_m}{1 + R_m}) \\
&= \tilde{\mathbf{E}}_n[\frac{D_m}{D_n} - \frac{D_{m+1}}{D_n}] && \text{(Take out what is known)} \\
&= B_{n,m} - B_{n,m+1}.
\end{aligned}$$

4. (i)

$$\begin{aligned}
V_1(H) &= \frac{1}{1+R_1(H)}(\tilde{\mathbf{P}}(HH|H)V_2(HH) + \tilde{\mathbf{P}}(HT|H)V_2(HT)) \\
&= \frac{1}{1+\frac{1}{6}}(\frac{2}{3}\cdot\frac{1}{3}+0) \\
&= \frac{4}{21}.
\end{aligned}$$

Since $V_2(TH) = V_2(TT) = 0$, $V_1(T) = 0$.

(ii) Let $\Delta_{0,2}$ be the amount of 2-maturity zero coupon bond held in the portfolio at $t = 0$. We already knew that the value of the portfolio at $t = 0$ is $X_0 = \frac{2}{21}$; the time zero value of the 2-maturity bond is $B_{0,2} = \frac{11}{14}$ and $R_0 = 0$. Then the value of the portfolio at $t = 1$ is:

$$\begin{aligned}
X_1 &= \Delta_{0,2}B_{1,2} + (1+R_0)(X_0 - \Delta_{0,2}\cdot B_{0,2}) \\
&= \Delta_{0,2}B_{1,2} + (\frac{2}{21} - \Delta_{0,2}\cdot\frac{11}{14})
\end{aligned}$$

Recall that $B_{1,2}(H) = \frac{6}{7}$ and $B_{1,2}(T) = \frac{5}{7}$. We let $\omega_1 = H$ and solve for $\Delta_{0,2}$ in the equation $V_1(H) = X_1(H)$:

$$\begin{aligned}
\frac{4}{21} &= V_1(H) \\
&= \Delta_{0,2}B_{1,2}(H) + (\frac{2}{21} - \Delta_{0,2}\cdot\frac{11}{14}) \\
&= \Delta_{0,2}\frac{6}{7} + (\frac{2}{21} - \Delta_{0,2}\cdot\frac{11}{14}) \\
&= \frac{1}{14}\Delta_{0,2} + \frac{2}{21}.
\end{aligned}$$

The above equation holds if and only if $\Delta_{0,2} = \frac{4}{3}$. Let us check that $X_1(T) = V_1(T) = 0$ with $\Delta_{0,2}$ set to $\frac{4}{3}$:

$$\begin{aligned}
X_1(T) &= \Delta_{0,2}B_{1,2}(T) + (\frac{2}{21} - \Delta_{0,2}\cdot\frac{11}{14}) \\
&= \frac{4}{3}\cdot\frac{5}{7} + (\frac{2}{21} - \frac{4}{3}\cdot\frac{11}{14}) \\
&= \frac{20}{21} + \frac{2}{21} - \frac{22}{21} = 0.
\end{aligned}$$

This verifies that holding $\frac{4}{3}$ unit of 2-maturity zero coupon bond and investing in the money market hedges for the short position in the caplet at time one.

Since $B_{1,3}(H) = B_{1,3}(T) = \frac{4}{7}$, if we invest in the 3-maturity zero coupon bond instead, the portfolio value at $t = 1$, X_1, will be constant no matter the result of the first coin toss, while $V_1(H) \neq V_1(T)$.

(iii) Since $V_1(T) = V_2(TH) = V_2(TT) = 0$, there is no need to hedge when $\omega_1 = T$.

For the case $\omega_1 = H$, recall that $X_1(H) = \frac{4}{21}$; $V_2(HH) = \frac{1}{3}$ and $V_2(HT) = 0$. Say we invest in $\Delta_{1,3}(H)$ unit of 3-maturity zero coupon bond. Then

$$\begin{aligned} X_2 &= \Delta_{1,3}(H)B_{2,3} + (1 + R_1(H))(X_1(H) - \Delta_{1,3}(H) \cdot B_{1,3}(H)) \\ &= \Delta_{1,3}(H)B_{2,3} + \frac{7}{6}(\frac{4}{21} - \Delta_{1,3}(H) \cdot \frac{4}{7}). \end{aligned}$$

We let $\omega_2 = H$ and solve for $\Delta_{1,3}(H)$ in the equation $V_2(HH) = X_2(HH)$:

$$\begin{aligned} \frac{1}{3} &= V_2(HH) \\ &= \Delta_{1,3}(H)B_{2,3}(HH) + \frac{7}{6}(\frac{4}{21} - \Delta_{1,3}(H) \cdot \frac{4}{7}) \\ &= \Delta_{1,3}(H) \cdot \frac{1}{2} + \frac{7}{6}(\frac{4}{21} - \Delta_{1,3}(H) \cdot \frac{4}{7}) \\ &= \frac{-1}{6}\Delta_{1,3}(H) + \frac{2}{9}. \end{aligned}$$

This holds if and only if $\Delta_{1,3}(H) = \frac{-2}{3}$. Let us check that $X_2(HT) = V_2(HT) = 0$ with $\Delta_{1,3}(H)$ set to $\frac{-2}{3}$:

$$\begin{aligned} X_2(HT) &= \Delta_{1,3}(H)B_{2,3}(HT) + \frac{7}{6}(\frac{4}{21} - \Delta_{1,3}(H) \cdot \frac{4}{7}) \\ &= \frac{-2}{3} \cdot 1 + \frac{7}{6}(\frac{4}{21} - \frac{-2}{3} \cdot \frac{4}{7}) \\ &= \frac{-2}{3} + \frac{2}{9} + \frac{4}{9} = 0. \end{aligned}$$

This verifies that shorting $\frac{2}{3}$ unit of 3-maturity zero coupon bond at time one and investing in the money market hedges for the short position in the caplet at time two.

The value of the 2-maturity zero coupon bond at time two is just the value of its payoff - 1. This does not change regardless of the result of the second coin toss. If we invest in the 2-maturity zero coupon bond instead, the portfolio value at $t = 2$, X_2, will be constant no matter the result of the second coin toss, while $V_2(HH) \neq V_2(HT)$.

5. (i)

$$
\begin{aligned}
&\tilde{\mathbf{E}}_n^{m+1}[F_{n+1,m}] \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1} F_{n+1,m}] \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}(\frac{B_{n+1,m}}{B_{n+1,m+1}} - 1)] \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}(\frac{B_{n+1,m}}{B_{n+1,m+1}})] - \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}] \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[\tilde{\mathbf{E}}_{n+1}[D_{m+1}(\frac{B_{n+1,m}}{B_{n+1,m+1}})]] - \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}] && \text{(Iterated conditioning)} \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[\tilde{\mathbf{E}}_{n+1}[D_{m+1}](\frac{B_{n+1,m}}{B_{n+1,m+1}})] - \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}] && (\frac{B_{n+1,m}}{B_{n+1,m+1}} \text{ known at } t = n+1) \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[\tilde{\mathbf{E}}_{n+1}[D_{m+1}](\frac{\tilde{\mathbf{E}}_{n+1}[D_m]}{\tilde{\mathbf{E}}_{n+1}[D_{m+1}]})] - \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}] && (D_{n+1} B_{n+1,m} = \tilde{\mathbf{E}}_{n+1}[D_m]) \\
&= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_m] - \frac{1}{D_n B_{n,m+1}} \tilde{\mathbf{E}}_n[D_{m+1}] \\
&= \frac{B_{n,m} - B_{n,m+1}}{B_{n,m+1}} = F_{n,m}.
\end{aligned}
$$

(ii)

$$F_{0,2} = \frac{B_{0,2} - B_{0,3}}{B_{0,3}} = \frac{0.9071 - 0.8639}{0.8639} = 0.0500057877$$

$$F_{1,2}(H) = \frac{B_{1,2}(H) - B_{1,3}(H)}{B_{1,3}(H)} = \frac{0.9479 - 0.8985}{0.8985} = 0.05498052309$$

$$F_{1,2}(T) = \frac{B_{1,2}(T) - B_{1,3}(T)}{B_{1,3}(T)} = \frac{0.9569 - 0.9158}{0.9158} = 0.04487879449$$

$$
\begin{aligned}
\tilde{\mathbf{E}}^3[F_{1,2}] &= \tilde{\mathbf{P}}^3(H) \cdot F_{1,2}(H) + \tilde{\mathbf{P}}^3(T) F_{1,2}(T) \\
&= 0.4952 \cdot 0.05498052309 + 0.5048 \cdot 0.04487879449 \\
&= 0.04988117049 \approx 0.0500057877 = F_{0,2}
\end{aligned}
$$

6. (i) Let us consider a more general problem:

For a forward contract with delivery date m and delivery price K. Find the price of the forward contract at $t = n$. Let the value of this contract be V_n. An agent sells this contract at time $t = n$, receiving V_n. He then buys one unit of the asset, paying S_n. He then buys $\frac{V_n - S_n}{B_{n,m}}$-unit of m-maturity zero coupon bond with the money left in the account.

At time $t = m$, she delivers the asset, receiving K. She also receives $\frac{V_n - S_n}{B_{n,m}}$ from the investment in the zero coupon bond. If there is no arbitrage, she must now have 0, i.e.

$$K + \frac{V_n - S_n}{B_{n,m}} = 0.$$

Solving, we have

$$V_n = S_n - K B_{n,m}. \tag{3}$$

For the problem in the book, the delivery price of the forward contract bought at time $t = n$ is $\frac{S_n}{B_{n,m}}$. Therefore, the price of the forward contract at $t = n+1$ is $S_{n+1} - (\frac{S_n}{B_{n,m}}) B_{n+1,m}$, by (3).

(ii) By the previous part, the cash flow at $t = n+1$ is

$$(1+r)^{m-n-1}S_{n+1} - (1+r)^{m-n-1}\frac{S_n B_{n+1,m}}{B_{n,m}}. \tag{4}$$

We know that the discounted asset prices form a martingale. In particular,

$$\tilde{\mathbf{E}}_{n+1}[D_m S_m] = D_{n+1}S_{n+1}.$$

Since the interest rate r is a constant, $D_m = \frac{1}{(1+r)^m}$ and $D_{n+1} = \frac{1}{(1+r)^{n+1}}$. The above equality can then be written as

$$\tilde{\mathbf{E}}_{n+1}[S_m] = (1+r)^{m-n-1}S_{n+1}. \tag{5}$$

Similarly,

$$\tilde{\mathbf{E}}_n[S_m] = (1+r)^{m-n}S_n. \tag{6}$$

Under constant interest rate,

$$B_{n+1,m} = (1+r)^{-m+n+1}; \quad B_{n,m} = (1+r)^{-m+n}. \tag{7}$$

Putting (5), (6) and (7) into (4), the cash flow at $t = n+1$ can be written as

$$\tilde{\mathbf{E}}_{n+1}[S_m] - \tilde{\mathbf{E}}_n[S_m].$$

7. Let us analyze the payoff function $V_{n+1}(k)$ at $t = n+1$. Note that we can express the event in interest as follows.

$$\{\#H(\omega_1,\cdots,\omega_{n+1}) = k\} = (\{\#H(\omega_1,\cdots,\omega_n) = k, \omega_{n+1} = T\})\bigsqcup(\{\#H(\omega_1,\cdots,\omega_n) = k-1, \omega_{n+1} = H\})$$

$$\begin{aligned}
V_{n+1}(k) &= \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_{n+1})=k\}} \\
&= \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_n)=k,\omega_{n+1}=T\}} + \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_n)=k-1,\omega_{n+1}=H\}} \\
&= \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_n)=k\}} \cdot \mathbf{1}_{\{\omega_{n+1}=T\}} + \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_n)=k-1\}} \cdot \mathbf{1}_{\{\omega_{n+1}=H\}} \\
&= V_n(k) \cdot \mathbf{1}_{\{\omega_{n+1}=T\}} + V_n(k-1) \cdot \mathbf{1}_{\{\omega_{n+1}=H\}}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{1+R_n} \cdot V_n(k) &= \frac{1}{1+R_n} \cdot \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_n)=k\}} \\
&= \frac{1}{1+r_n(k)} \cdot \mathbf{1}_{\{\#H(\omega_1,\cdots,\omega_n)=k\}} \qquad \text{(Whole term vanishes if } \#H(\omega_1,\cdots,\omega_n) \neq k) \\
&= \frac{1}{1+r_n(k)} \cdot V_n(k).
\end{aligned}$$

Applying these two equations,

$$
\begin{aligned}
&\phi_{n+1}(k) \\
&= \tilde{\mathbf{E}}[D_{n+1}V_{n+1}(k)] \\
&= \tilde{\mathbf{E}}[D_{n+1}(V_n(k)\cdot \mathbf{1}_{\{\omega_{n+1}=T\}} + V_n(k-1)\cdot \mathbf{1}_{\{\omega_{n+1}=H\}})] && \text{(Using the first equation)} \\
&= \tilde{\mathbf{E}}[\tilde{\mathbf{E}}_n[D_{n+1}(V_n(k)\cdot \mathbf{1}_{\{\omega_{n+1}=T\}} + V_n(k-1)\cdot \mathbf{1}_{\{\omega_{n+1}=H\}})]] && \text{(Iterated conditioning)} \\
&= \tilde{\mathbf{E}}[D_{n+1}(V_n(k)\cdot \tilde{\mathbf{E}}_n[\mathbf{1}_{\{\omega_{n+1}=T\}}] + V_n(k-1)\cdot \tilde{\mathbf{E}}_n[\mathbf{1}_{\{\omega_{n+1}=H\}}])]] && (D_{n+1}, V_n(k), V_n(k-1) \text{ known at } t=n) \\
&= \tilde{\mathbf{E}}[D_{n+1}(V_n(k)\cdot \frac{1}{2} + V_n(k-1)\cdot \frac{1}{2}] \\
&= \tilde{\mathbf{E}}[\frac{D_n}{1+R_n}(V_n(k)\cdot \frac{1}{2} + V_n(k-1)\cdot \frac{1}{2}] \\
&= \tilde{\mathbf{E}}[\frac{D_nV_n(k)\cdot \frac{1}{2}}{1+r_n(k)} + \frac{D_nV_n(k-1)\cdot \frac{1}{2}}{1+r_n(k-1)}] && \text{(Using the second equation)} \\
&= \frac{\phi_n(k)}{2(1+r_n(k))} + \frac{\phi_n(k-1)}{2(1+r_n(k-1))}.
\end{aligned}
$$