Chapter 3 solutions to Stochastic Calculus for Finance I by Steven E. Shreve

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1. (i) Recall

$$\frac{1}{Z(\omega)} = \frac{\mathbf{P}(\omega)}{\tilde{\mathbf{P}}(\omega)}.$$

But all $\omega \in \Omega$, we assumed that both $\mathbf{P}(\omega)$ and $\tilde{\mathbf{P}}(\omega)$ are positive. Therefore, $\frac{1}{Z(\omega)} > 0$ for all $\omega \in \Omega$. It follows that

$$\tilde{\mathbf{P}}(\frac{1}{Z} > 0) = \tilde{\mathbf{P}}(\Omega) = 1.$$

(ii)

$$\tilde{\mathbf{E}}\frac{1}{Z} = \sum_{\omega\in\Omega} \frac{1}{Z(\omega)}\tilde{\mathbf{P}}(\omega) = \sum_{\omega\in\Omega} \frac{\mathbf{P}(\omega)}{\tilde{\mathbf{P}}(\omega)}\tilde{\mathbf{P}}(\omega) = \sum_{\omega\in\Omega} \mathbf{P}(\omega) = 1$$

(iii)

$$\mathbf{E}Y = \sum_{\omega\in\Omega} Y(\omega)\mathbf{P}(\omega) = \sum_{\omega\in\Omega} Y(\omega)\frac{\mathbf{P}(\omega)}{\mathbf{P}(\tilde{\omega})}\mathbf{P}(\tilde{\omega}) = \sum_{\omega\in\Omega} Y(\omega)\frac{1}{Z(\omega)}\mathbf{P}(\tilde{\omega}) = \tilde{\mathbf{E}}\frac{1}{Z}Y$$

2. (i)

$$\tilde{\mathbf{P}}(\Omega) = \sum_{\omega\in\Omega} \tilde{\mathbf{P}}(\omega) = \sum_{\omega\in\Omega} Z(\omega)\mathbf{P}(\omega) = \mathbf{E}Z = 1.$$

For any event $A \subset \Omega$,

$$\tilde{\mathbf{P}}(A) = \sum_{\omega\in A} \tilde{\mathbf{P}}(\omega) = \sum_{\omega\in A} Z(\omega)\mathbf{P}(\omega) \geq 0,$$

since $Z(\omega) \geq 0$.
Putting A^c into A in the above inequality, we have

$$\tilde{\mathbf{P}}(A) = 1 - \tilde{\mathbf{P}}(A^c) \leq 1.$$

If $A \subset \Omega$ and $B \subset \Omega$ are disjoint,

$$\tilde{\mathbf{P}}(A) + \tilde{\mathbf{P}}(B) = \sum_{\omega\in A} \tilde{\mathbf{P}}(\omega) + \sum_{\omega\in B} \tilde{\mathbf{P}}(\omega) = \sum_{\omega\in A\cup B} \tilde{\mathbf{P}}(\omega) = \tilde{\mathbf{P}}(A \cup B).$$

Therefore, $\tilde{\mathbf{P}}$ is a probability measure.

(ii)

$$\tilde{\mathbf{E}}Y = \sum_{\omega\in\Omega} Y(\omega)\tilde{\mathbf{P}}(\omega) = \sum_{\omega\in\Omega} Y(\omega)Z(\omega)\mathbf{P}(\omega) = \mathbf{E}[ZY]$$

(iii) If $\mathbf{P}(A) = 0$,

$$\begin{aligned}
\tilde{\mathbf{P}}(A) &= \sum_{\omega\in A} \tilde{\mathbf{P}}(\omega) \\
&= \sum_{\omega\in A} Z(\omega)\mathbf{P}(\omega) \\
&\leq \sum_{\omega\in A} (\max_{\omega'\in A} Z(\omega'))\mathbf{P}(\omega) \\
&= (\max_{\omega'\in A} Z(\omega'))(\sum_{\omega\in A} \mathbf{P}(\omega)) \\
&= (\max_{\omega'\in A} Z(\omega'))\mathbf{P}(A) = 0
\end{aligned}$$

(iv) Assuming $\mathbf{P}(Z>0)=1$ and $\tilde{\mathbf{P}}(A)=0$.

$$0=\tilde{\mathbf{P}}(A)=\sum_{\omega\in A}\tilde{\mathbf{P}}(\omega)=\sum_{\omega\in A}Z(\omega)\mathbf{P}(\omega)\geq\sum_{\omega\in A\cap\{Z>0\}}Z(\omega)\mathbf{P}(\omega)\geq\sum_{\omega\in A\cap\{Z>0\}}(\min_{\omega'\in\{Z>0\}}Z(\omega'))\mathbf{P}(\omega)$$
$$=(\min_{\omega'\in\{Z>0\}}Z(\omega'))(\sum_{\omega\in A\cap\{Z>0\}}\mathbf{P}(\omega))=(\min_{\omega'\in\{Z>0\}}Z(\omega'))\mathbf{P}(A\cap\{Z>0\}),$$

which implies $\mathbf{P}(A\cap\{Z>0\})=0$ since $\min_{\omega'\in\{Z>0\}}Z(\omega')>0$.
It follows that

$$\mathbf{P}(A)\leq\mathbf{P}(A\cap\{Z>0\})+\mathbf{P}(Z=0)=0+(1-\mathbf{P}(Z>0))=0$$

(v)

$$\mathbf{P}(A)=1\iff\mathbf{P}(A^c)=1-\mathbf{P}(A)=0\iff\tilde{\mathbf{P}}(A^c)=0\iff\tilde{\mathbf{P}}(A)=1-\tilde{\mathbf{P}}(A^c)=1,$$

where we used the assumption that $\mathbf{P}$ and $\tilde{\mathbf{P}}$ are equivalent in the second equivalence.

(vi) Let $\Omega=\{H,T\}$. Let

$$\mathbf{P}(H)=\mathbf{P}(T)=\frac{1}{2}.$$

Let

$$Z(H)=2;\quad Z(T)=0.$$

Then

$$\mathbf{P}(Z\geq 0)=1$$

and

$$\mathbf{E}Z=\mathbf{P}(H)Z(H)+\mathbf{P}(T)Z(T)=\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 0=1.$$

Define $\tilde{\mathbf{P}}(\omega):=Z(\omega)\mathbf{P}(\omega)$. Then $\tilde{\mathbf{P}}(T)=0\cdot\frac{1}{2}=0$ but $\mathbf{P}(T)=\frac{1}{2}$. So $\tilde{\mathbf{P}}$ and $\mathbf{P}$ are not equivalent.

3.

$$M_3=\mathbf{E}_3[S_3]=S_3$$

Thus,

$$M_3(HHH)=32;\quad M_3(HHT)=M_3(HTH)=M_3(THH)=8;$$
$$M_3(HTT)=M_3(THT)=M_3(TTH)=2;\quad M_3(TTT)=\frac{1}{2}.$$

$$M_2(HH)=\mathbf{E}_2[S_3](HH)=\frac{2}{3}32+\frac{1}{3}8=24$$
$$M_2(HT)=\mathbf{E}_2[S_3](HT)=\frac{2}{3}8+\frac{1}{3}2=6;$$
$$M_2(TH)=6;$$
$$M_2(TT)=\mathbf{E}_2[S_3](TT)=\frac{2}{3}2+\frac{1}{3}\frac{1}{2}=\frac{3}{2}.$$
$$M_1(H)=\mathbf{E}_1[S_3](H)=\frac{4}{9}32+\frac{2}{9}8+\frac{2}{9}8+\frac{1}{9}2=18$$
$$M_1(T)=\mathbf{E}_1[S_3](T)=\frac{4}{9}8+\frac{2}{9}2+\frac{2}{9}2+\frac{1}{9}\frac{1}{2}=\frac{9}{2}$$
$$M_0=\mathbf{E}[S_3]=\frac{8}{27}32+\frac{12}{27}8+\frac{6}{27}2+\frac{1}{27}\frac{1}{2}=\frac{27}{2}.$$

It follows immediately from our computation above that $M_2 = \mathbf{E}_2[M_3]$, since $M_3 = S_3$. Let us check that $M_1 = \mathbf{E}_1[M_2]$:

$$\mathbf{E}_1[M_2](H) = \frac{2}{3}M_2(HH) + \frac{1}{3}M_2(HT) = \frac{2}{3}24 + \frac{1}{3}6 = 18 = M_1(H).$$

$$\mathbf{E}_1[M_2](T) = \frac{2}{3}M_2(TH) + \frac{1}{3}M_2(TT) = \frac{2}{3}6 + \frac{1}{3}\frac{3}{2} = \frac{9}{2} = M_1(T).$$

Let us check that $M_0 = \mathbf{E}[M_1]$:

$$\mathbf{E}[M_1] = \frac{2}{3}M_1(H) + \frac{1}{3}M_1(T) = \frac{2}{3}18 + \frac{1}{3}\frac{9}{2} = \frac{27}{2} = M_0.$$

4. (i)

$$\xi_3(HHH) = \frac{Z(HHH)}{(1+r)^3} = \frac{64}{125}\frac{\tilde{\mathbf{P}}(HHH)}{\mathbf{P}(HHH)} = \frac{64}{125}\frac{\frac{1}{8}}{\frac{8}{27}} = \frac{27}{125}.$$

$$\xi_3(HHT) = \xi_3(HTH) = \xi_3(THH) = \frac{Z(THH)}{(1+r)^3} = \frac{64}{125}\frac{\tilde{\mathbf{P}}(THH)}{\mathbf{P}(THH)} = \frac{64}{125}\frac{\frac{1}{8}}{\frac{4}{27}} = \frac{54}{125}.$$

$$\xi_3(HTT) = \xi_3(THT) = \xi_3(TTH) = \frac{Z(TTH)}{(1+r)^3} = \frac{64}{125}\frac{\tilde{\mathbf{P}}(TTH)}{\mathbf{P}(TTH)} = \frac{64}{125}\frac{\frac{1}{8}}{\frac{2}{27}} = \frac{108}{125}.$$

$$\xi_3(TTT) = \frac{Z(TTT)}{(1+r)^3} = \frac{64}{125}\frac{\tilde{\mathbf{P}}(TTT)}{\mathbf{P}(TTT)} = \frac{64}{125}\frac{\frac{1}{8}}{\frac{1}{27}} = \frac{216}{125}.$$

(ii) Let us compute the time three payoff of the Asian option.

$$\begin{aligned}
V_3(HHH) &= (\tfrac{1}{4}(4+8+16+32)-4)^+ = 11; & V_3(HHT) &= (\tfrac{1}{4}(4+8+16+8)-4)^+ = 5;\\
V_3(HTH) &= (\tfrac{1}{4}(4+8+4+8)-4)^+ = 2; & V_3(THH) &= (\tfrac{1}{4}(4+2+4+8)-4)^+ = \tfrac{1}{2};\\
V_3(HTT) &= (\tfrac{1}{4}(4+8+4+2)-4)^+ = \tfrac{1}{2}; & V_3(THT) &= (\tfrac{1}{4}(4+2+4+2)-4)^+ = 0;\\
V_3(TTH) &= (\tfrac{1}{4}(4+2+1+2)-4)^+ = 0; & V_3(TTT) &= (\tfrac{1}{4}(4+2+1+.5)-4)^+ = 0.
\end{aligned}$$

The time zero value of the Asian option is:

$$\begin{aligned}
&V_0 = \mathbf{E}[\xi V_3]\\
&= V_3(HHH)\xi(HHH)\mathbf{P}(HHH) + V_3(HHT)\xi(HHT)\mathbf{P}(HHT) + V_3(HTH)\xi(HTH)\mathbf{P}(HTH)\\
&+ V_3(THH)\xi(THH)\mathbf{P}(THH) + V_3(HTT)\xi(HTT)\mathbf{P}(HTT)\\
&= 11\cdot\frac{27}{125}\cdot\frac{8}{27} + 5\cdot\frac{54}{125}\cdot\frac{4}{27} + 2\cdot\frac{54}{125}\cdot\frac{4}{27} + \frac{1}{2}\cdot\frac{54}{125}\cdot\frac{4}{27} + \frac{1}{2}\cdot\frac{108}{125}\cdot\frac{2}{27}\\
&= \frac{88}{125} + \frac{40}{125} + \frac{16}{125} + \frac{4}{125} + \frac{4}{125}\\
&= \frac{152}{125}
\end{aligned}$$

(iii)

$$\xi_2(HT) = \xi_2(TH) = \frac{Z(TH)}{(1+r)^2} = \frac{16}{25}\frac{\tilde{\mathbf{P}}(TH)}{\mathbf{P}(TH)} = \frac{16}{25}\frac{\frac{1}{4}}{\frac{2}{9}} = \frac{18}{25}.$$

(iv)

$$
\begin{aligned}
V_2(HT) &= \frac{1}{\xi_2(HT)}\mathbf{E}_2[\xi_3 V_3](HT) \\
&= \frac{25}{18}(\xi_3(HTH)V_3(HTH)\mathbf{P}(\omega_3 = H) + \xi_3(HTT)V_3(HTT)\mathbf{P}(\omega_3 = T)) \\
&= \frac{25}{18}(\frac{54}{125}\cdot 2\cdot\frac{2}{3} + \frac{108}{125}\cdot\frac{1}{2}\cdot\frac{1}{3}) = 1
\end{aligned}
$$

$$
\begin{aligned}
V_2(TH) &= \frac{1}{\xi_2(TH)}\mathbf{E}_2[\xi_3 V_3](TH) \\
&= \frac{25}{18}(\xi_3(THH)V_3(THH)\mathbf{P}(\omega_3 = H) + \xi_3(THT)V_3(THT)\mathbf{P}(\omega_3 = T)) \\
&= \frac{25}{18}(\frac{54}{125}\cdot\frac{1}{2}\cdot\frac{2}{3} + \frac{108}{125}\cdot 0\cdot\frac{1}{3}) = \frac{1}{5}
\end{aligned}
$$

5. (i)

$$
\begin{aligned}
Z(HH) &= \frac{\tilde{\mathbf{P}}(HH)}{\mathbf{P}(HH)} = \frac{\frac{1}{4}}{\frac{4}{9}} = \frac{9}{16}. \\
Z(HT) &= \frac{\tilde{\mathbf{P}}(HT)}{\mathbf{P}(HT)} = \frac{\frac{1}{4}}{\frac{2}{9}} = \frac{9}{8}. \\
Z(TH) &= \frac{\tilde{\mathbf{P}}(TH)}{\mathbf{P}(TH)} = \frac{\frac{1}{12}}{\frac{2}{9}} = \frac{3}{8}. \\
Z(TT) &= \frac{\tilde{\mathbf{P}}(TT)}{\mathbf{P}(TT)} = \frac{\frac{5}{12}}{\frac{1}{9}} = \frac{15}{4}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
Z_1(H) &= \mathbf{E}_1[Z](H) = Z(HH)\mathbf{P}(HH|H) + Z(HT)\mathbf{P}(HT|H) = \frac{9}{16}\cdot\frac{2}{3} + \frac{9}{8}\cdot\frac{1}{3} = \frac{3}{4}; \\
Z_1(T) &= \mathbf{E}_1[Z](T) = Z(TH)\mathbf{P}(TH|T) + Z(TT)\mathbf{P}(TT|T) = \frac{3}{8}\cdot\frac{2}{3} + \frac{15}{4}\cdot\frac{1}{3} = \frac{3}{2};
\end{aligned}
$$

$$
\begin{aligned}
Z_0 = \mathbf{E}Z &= Z(HH)\mathbf{P}(HH) + Z(HT)\mathbf{P}(HT) + Z(TH)\mathbf{P}(TH) + Z(TT)\mathbf{P}(TT) \\
&= \frac{9}{16}\cdot\frac{4}{9} + \frac{9}{8}\cdot\frac{2}{9} + \frac{3}{8}\cdot\frac{2}{9} + \frac{15}{4}\cdot\frac{1}{9} = \frac{1}{4} + \frac{1}{4} + \frac{1}{12} + \frac{5}{12} = 1.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
V_1(H) &= \frac{1}{Z_1(H)(1+r_1(H))}\mathbf{E}_1[Z_2V_2](H) \\
&= \frac{1}{\frac{3}{4}\cdot\frac{5}{4}}(Z_2(HH)V_2(HH)\mathbf{P}(HH|H) + Z_2(HT)V_2(HT)\mathbf{P}(HT|H)) \\
&= \frac{16}{15}(\frac{9}{16}\cdot 5\cdot\frac{2}{3} + \frac{9}{8}\cdot 1\cdot\frac{1}{3}) = \frac{16}{15}(\frac{15}{8}+\frac{3}{8}) = \frac{12}{5}.
\end{aligned}
$$

$$
\begin{aligned}
V_1(T) &= \frac{1}{Z_1(T)(1+r_1(T))}\mathbf{E}_1[Z_2V_2](T) \\
&= \frac{1}{\frac{3}{2}\cdot\frac{3}{2}}(Z_2(TH)V_2(TH)\mathbf{P}(TH|T) + Z_2(TT)V_2(TT)\mathbf{P}(TT|T)) \\
&= \frac{4}{9}(\frac{3}{8}\cdot 1\cdot\frac{2}{3} + \frac{15}{4}\cdot 0\cdot\frac{1}{3}) = \frac{16}{15}(\frac{15}{8}+\frac{3}{8}) = \frac{1}{9}.
\end{aligned}
$$

$$
\begin{aligned}
V_0 &= \mathbf{E}[\frac{Z_2}{(1+r_0)(1+r_1)}V_2] \\
&= \frac{Z_2(HH)V_2(HH)}{(1+r_0)(1+r_1(H))}\mathbf{P}(HH) + \frac{Z_2(HT)V_2(HT)}{(1+r_0)(1+r_1(H))}\mathbf{P}(HT) \\
&+ \frac{Z_2(TH)V_2(TH)}{(1+r_0)(1+r_1(T))}\mathbf{P}(TH) + \frac{Z_2(TT)V_2(TT)}{(1+r_0)(1+r_1(T))}\mathbf{P}(TT) \\
&= \frac{\frac{9}{16}\cdot 5}{\frac{5}{4}\cdot\frac{5}{4}}\cdot\frac{4}{9} + \frac{\frac{9}{8}\cdot 1}{\frac{5}{4}\cdot\frac{5}{4}}\cdot\frac{2}{9} + \frac{\frac{3}{8}\cdot 1}{\frac{5}{4}\cdot\frac{3}{2}}\cdot\frac{2}{9} + \frac{\frac{15}{4}\cdot 0}{\frac{5}{4}\cdot\frac{3}{2}}\cdot\frac{1}{9} \\
&= \frac{4}{5} + \frac{4}{25} + \frac{2}{45} + 0 = \frac{226}{225}.
\end{aligned}
$$

6. Differentiating the Langrangian, the optimal wealth process at $t = N$ satisfies

$$U'(X_N) = \lambda\zeta_N.$$

This is (3.3.24) from the book. We have $U'(x) = \frac{1}{x}$, so the inverse function of U' is $I(y) := U'^{-1}(y) = \frac{1}{y}$. Applying this to the above equation, we have

$$X_N = \frac{1}{\lambda\zeta_N}$$

The risk neutral pricing formula (3.3.26) becomes

$$\mathbf{E}[\zeta_N X_N] = \mathbf{E}[\zeta_N\frac{1}{\lambda\zeta_N}] = \mathbf{E}[\frac{1}{\lambda}] = X_0.$$

This implies $\lambda = \frac{1}{X_0}$, and

$$X_N = \frac{X_0}{\zeta_N}.$$

Using the risk-neurtral pricing formula (3.2.6), we have

$$X_n = \frac{1}{\zeta_n}\mathbf{E}_n[\zeta_N X_N] = \frac{1}{\zeta_n}\mathbf{E}_n[\zeta_N\frac{X_0}{\zeta_N}] = \frac{X_0}{\zeta_n}.$$

7. Note that for U to be increasing, p has to be > 0 in the definition of U. From differentiating the Langrangian, the optimal wealth at time N, X_N satisfies

$$U'(X_N) = \lambda \frac{Z}{(1+r)^N}.$$

Let us compute the derivative of U:

$$U'(x) = x^{p-1}.$$

Let us solve for x in terms of y:

$$\begin{aligned} y &= x^{p-1} \\ \iff \frac{1}{y} &= x^{1-p} \\ \iff y^{-\frac{1}{1-p}} &= x \end{aligned}$$

Therefore, the inverse of U' is:

$$I(y) := U'^{-1}(y) = y^{\frac{1}{(p-1)}}.$$

It follows that the optimal wealth X_N can be expressed as:

$$X_N = (\lambda \frac{Z}{(1+r)^N})^{\frac{1}{p-1}}. \tag{1}$$

Plugging this into the risk-neutral pricing formula, we have

$$X_0 = \mathbf{E}[\frac{Z}{(1+r)^N}(\lambda \frac{Z}{(1+r)^N})^{\frac{1}{p-1}}] = \lambda^{\frac{1}{p-1}} \cdot \frac{1}{(1+r)^{N \cdot \frac{p}{p-1}}} \mathbf{E}[Z^{\frac{p}{p-1}}]$$

Solving for λ, we have

$$\lambda = \frac{X_0^{p-1}(1+r)^{Np}}{\mathbf{E}[Z^{\frac{p}{p-1}}]^{p-1}}$$

Plugging the above equation on λ into the (1), we have

$$X_N = (\frac{X_0^{p-1}(1+r)^{Np}}{\mathbf{E}[Z^{\frac{p}{p-1}}]^{p-1}} \cdot \frac{Z}{(1+r)^N})^{\frac{1}{p-1}} = \frac{X_0(1+r)^N Z^{\frac{1}{p-1}}}{\mathbf{E}[Z^{\frac{p}{p-1}}]}$$

8. (i) Let $g(x) = U(x) - yx$. Let us solve for $g'(x) = 0$.

$$\begin{aligned} g'(x) &= 0 \\ \iff U'(x) - y &= 0 \\ \iff x &= I(y). \end{aligned}$$

Since $g''(x) = U''(x) < 0$ for all x using concavity of U, $x^* = I(y)$ is a global maximum of g.

(ii) Let X_N be a wealth process at time N that satisfies the risk neutral pricing formula

$$\mathbf{E}[\frac{Z}{(1+r)^N}X_N] = X_0.$$

After putting the random variable X_N into x and $\frac{\lambda Z}{(1+r)^N}$ into y, (3.6.3) becomes

$$U(X_N) - \frac{\lambda Z}{(1+r)^N}X_N \le U(I(\frac{\lambda Z}{(1+r)^N})) - \frac{\lambda Z}{(1+r)^N}I(\frac{\lambda Z}{(1+r)^N}).$$

We let $X_N^* = I(\frac{\lambda Z}{(1+r)^N})$. The above equation becomes

$$U(X_N) - \frac{\lambda Z}{(1+r)^N}X_N \le U(X_N^*) - \frac{\lambda Z}{(1+r)^N}X_N^*.$$

Taking expectation, we have

$$\mathbf{E}[U(X_N)] - \lambda \cdot \mathbf{E}[\frac{Z}{(1+r)^N}X_N] \le \mathbf{E}[U(X_N^*)] - \lambda \cdot \mathbf{E}[\frac{Z}{(1+r)^N}X_N^*].$$

By the risk neutral pricing formula,

$$\mathbf{E}[\frac{Z}{(1+r)^N}X_N] = X_0; \quad \mathbf{E}[\frac{Z}{(1+r)^N}X_N^*] = X_0.$$

Thus,

$$\mathbf{E}[U(X_N)] \le \mathbf{E}[U(X_N^*)].$$

9. (i) This follows from the risk neutral pricing formula:

$$X_n = \tilde{\mathbf{E}}_n[\frac{X_N}{(1+r)^{N-n}}].$$

The term we are taking expectation over is always non-negative, therefore $X_n \ge 0$.

(ii) Let $g(x) = U(x) - yx$. We would like to find x that maximizes g. Note that g can be expressed as follows:

$$g(x) = \begin{cases} -yx & 0 \le x < \gamma \\ 1 - yx & \gamma \le x. \end{cases}$$

Over the interval $[0, \gamma)$, g achieves its maximum at $x = 0$; over the interval $[\gamma, \infty)$, g achieves its maximum at $x = \gamma$. It follows that g achieves its maximum at $x = \gamma$ if and only if $0 = g(0) \le g(\gamma) = 1 - y\gamma \iff y \le \frac{1}{\gamma}$. Therefore, g achieves its maximum at

$$I(y)^* = \begin{cases} 0 & (y > \frac{1}{\gamma}) \\ \gamma & (0 \le y \le \frac{1}{\gamma}). \end{cases}$$

Note that U is the utility function here that measures the probability that $X_N \ge \gamma$. Indeed,

$$\mathbf{E}[U(X_N)] = 0 \cdot \mathbf{P}(X_N < \gamma) + 1 \cdot \mathbf{P}(X_N \ge \gamma) = \mathbf{P}(X_N \ge \gamma).$$

(iii) Let $X_N^* = I(\frac{\lambda Z}{(1+r)^N})$ and let X_N be a wealth process at time N that satisfies the risk neutral pricing formula

$$\mathbf{E}[\frac{Z}{(1+r)^N}X_N] = X_0.$$

Following the argument of Exercise 3.8, we have

$$\mathbf{E}[U(X_N)] - \lambda \cdot \mathbf{E}[\frac{Z}{(1+r)^N}X_N] \leq \mathbf{E}[U(X_N^*)] - \lambda \cdot \mathbf{E}[\frac{Z}{(1+r)^N}X_N^*].$$

The assumption made in the question says

$$\mathbf{E}[\frac{Z}{(1+r)^N}X_N^*] = X_0.$$

This implies

$$\mathbf{E}[U(X_N)] \leq \mathbf{E}[U(X_N^*)].$$

(iv) Assume that (3.6.4) holds. Then

$$X_0 = \mathbf{E}[\zeta I(\lambda\zeta)] = \sum_{m=1}^{M} \zeta_m I(\lambda\zeta_m) p_m \tag{2}$$

If $\zeta_1 > \frac{1}{\lambda^2}$, then $I(\lambda\zeta_m) = 0$ for all $1 \leq m \leq M$, which implies $X_0 = 0$ from the above equation. A contradiction.

Since we have shown that $\zeta_1 \leq \frac{1}{\lambda^2}$, we can let K be the largest $1 \leq m \leq M$ such that $\zeta_m \leq \frac{1}{\lambda^2}$. Then the above displayed equation can be rewritten as

$$\begin{aligned}
\sum_{m=1}^{K} \zeta_m I(\lambda\zeta_m) p_m + \sum_{m=K+1}^{M} \zeta_m I(\lambda\zeta_m) p_m &= X_0 \\
\sum_{m=1}^{K} \zeta_m \gamma p_m + 0 &= X_0 \qquad\qquad (\lambda\zeta_m \leq \frac{1}{\lambda} \iff m \leq K) \\
\sum_{m=1}^{K} \zeta_m p_m &= \frac{X_0}{\gamma}.
\end{aligned}$$

If $K = M$, then (2) becomes

$$\sum_{m=1}^{M} \zeta_m \gamma p_m = \gamma \mathbf{E}[\zeta] = \frac{\gamma}{(1+r)^N} = X_0.$$

Then we can just invest all of our initial wealth X_0 into the money market and get γ at time N. Therefore, for the problem to make sense, K has to be strictly less than M. This proves the $\Rightarrow$ direction.

Suppose there exists $1 \leq K < M$ such that $\zeta_K < \zeta_{K+1}$ and

$$\gamma \sum_{m=1}^{K} \zeta_m p_m = X_0. \tag{3}$$

We let $\lambda = \frac{1}{\sqrt{\zeta_K}}$. Then $\zeta_m \leq \frac{1}{\lambda^2} \iff m \leq K$. The equation (3) can be rewritten now as

$$X_0 = \sum_{m=1}^{K} \zeta_m \gamma p_m = \sum_{m=1}^{K} \zeta_m I(\gamma \zeta_m) p_m = \sum_{m=1}^{M} \zeta_m I(\gamma \zeta_m) p_m = \mathbf{E}[\zeta I(\lambda \zeta)].$$

The third equality above holds because $\lambda \zeta_m > \frac{1}{\lambda}$ for $m > K$.

(v) This follows from part three and the argument we gave in part four.