Chapter 1 solutions to Stochastic Calculus for Finance I by Steven E. Shreve

Chung Ching Lau

1. Let us rewrite the equation for X_1 while noting that $X_0 = 0$.

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = \Delta_0 S_0(\frac{S_1}{S_0} - (1+r)), \tag{1}$$

where $\frac{S_1}{S_0}$ depends on the results of the first coin toss:

$$\frac{S_1}{S_0} = \begin{cases} u & \text{if } \omega_1 = H \\ d & \text{if } \omega_1 = T. \end{cases}$$

Therefore the term $\frac{S_1}{S_0} - (1+r)$ that appears in (1) satisfies

$$\begin{aligned} \frac{S_1}{S_0} - (1+r) > 0 \quad & \text{if } \omega_1 = H \\ \frac{S_1}{S_0} - (1+r) < 0 \quad & \text{if } \omega_1 = T. \end{aligned}$$

It follows that if $\Delta_0 > 0$,

$$X_1(H) > 0; \quad X_1(T) < 0.$$

If $\Delta_0 < 0$,

$$X_1(H) < 0; \quad X_1(T) > 0.$$

And if $\Delta_0 = 0$,

$$X_1(H) = X_1(T) = 0.$$

In all three cases, $\mathbf{P}(X_1 > 0) > 0$ holds if and only if $\mathbf{P}(X_1 < 0) > 0$ also holds.

2.

$$X_1(H) = \Delta_0 S_1(H) + \Gamma_0(S_1(H) - 5)^+ - \frac{5}{4}(4\Delta_0 + 1.2\Gamma_0) = \Delta_0 \cdot (8-5) + \Gamma_0 \cdot (3 - \frac{3}{2}) = 3\Delta_0 + \frac{3}{2}\Gamma_0;$$

$$X_1(T) = \Delta_0 S_1(T) + \Gamma_0(S_1(T) - 5)^+ - \frac{5}{4}(4\Delta_0 + 1.2\Gamma_0) = \Delta_0 \cdot (2-5) + \Gamma_0 \cdot (0 - \frac{3}{2}) = -3\Delta_0 - \frac{3}{2}\Gamma_0.$$

In particular, $X_1(H) = -X_1(T)$. Therefore, if one of $X_1(H)$ or $X_1(T)$ is positive, the other one must be negative.

3. By the risk neutral pricing formula,

$$\begin{aligned} V_0 &= \frac{1}{1+r}\big(\tilde{p}V_1(H) + \tilde{q}V_1(T)\big) \\ &= \frac{1}{1+r}\Big(\frac{1+r-d}{u-d}S_1(H) + \frac{u-1-r}{u-d}S_1(T)\Big) \\ &= \frac{1}{1+r}\Big(\frac{1+r-d}{u-d}uS_0 + \frac{u-1-r}{u-d}dS_0\Big) \\ &= \frac{1}{1+r} \cdot S_0\Big(\frac{u+ur-ud+ud-d-dr}{u-d}\Big) \end{aligned}$$

4. Let us analyze $X_{n+1}(T)$:

$$X_{n+1}(T) = \Delta_n d S_n + (1+r)(X_n - \Delta_n S_n).$$

Plugging in $\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}$ and $X_n = V_n$ (by induction), we have

$$\begin{aligned} X_{n+1}(T) &= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} \cdot dS_n + (1+r)(V_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n) \\ &= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)} \cdot \big(d - (1+r)\big) + (1+r)V_n \\ &= -\tilde{p}(V_{n+1}(H) - V_{n+1}(T)) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\ &= (\tilde{p} + \tilde{q})V_{n+1}(T) = V_{n+1}(T). \end{aligned}$$

In the third equality, we used the definition of V_n to express it in terms of $V_{n+1}(H)$ and $V_{n+1}(T)$:

$$V_n = \frac{1}{1+r}\big(\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\big).$$

5. At time 1, with $\omega_1 = H$, the agent bought

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.2 - 2.4}{16 - 4} = \frac{1}{15}$$

unit of stock. The portfolio value at time 2 if $\omega_2 = H$ is:

$$\begin{aligned} X_2(HH) &= \Delta_1(H)S_2(HH) + (1+r)(X_1(H) - \Delta_1(H)S_1(H)) \\ &= \frac{1}{15} \cdot 16 + \frac{5}{4}(2.24 - \frac{1}{15} \cdot 8) = 3.2 \end{aligned}$$

The portfolio value at time 2 if $\omega_2 = T$ is:

$$\begin{aligned} X_2(HT) &= \Delta_1(H)S_2(HT) + (1+r)(X_1(H) - \Delta_1(H)S_1(H)) \\ &= \frac{1}{15} \cdot 4 + \frac{5}{4}(2.24 - \frac{1}{15} \cdot 8) = 2.4. \end{aligned}$$

Now assume the $\omega_1 = H$ and $\omega_2 = T$. The agent takes a position of

$$\Delta_2(HT) = \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{0 - 6}{8 - 2} = -1$$

in the stock. If $\omega_3 = H$, the value of the portfolio is:

$$\begin{aligned} X_3(HTH) &= \Delta_2(HT)S_3(HTH) + (1+r)\big(X_2(HT) - \Delta_2(HT)S_2(HT)\big) \\ &= -1 \cdot 8 + \frac{5}{4}\big(2.4 - (-1) \cdot 4\big) = 0. \end{aligned}$$

If $\omega_3 = T$, the value of the portfolio is:

$$\begin{aligned} X_3(HTT) &= \Delta_2(HT)S_3(HTT) + (1+r)\big(X_2(HT) - \Delta_2(HT)S_2(HT)\big) \\ &= -1 \cdot 2 + \frac{5}{4}\big(2.4 - (-1) \cdot 4\big) = 6. \end{aligned}$$

6. The hedging portfolio should have time zero value $X_0 = 0$ and satisfies

$$X_1 + V_1 = 1.5$$

regardless of the result of coin toss at time one. In other words, the time one value of the hedging portfolio should be

$$X_1(H) = 1.5 - V_1(H) = 1.5 - (S_1(H) - 5)^+ = 1.5 - 3 = -1.5;$$
$$X_1(T) = 1.5 - V_1(T) = 1.5 - (S_1(T) - 5)^+ = 1.5 - 0 = 1.5;$$

Applying the delta hedging formula, the agent should take a position of

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{-1.5 - 1.5}{8 - 2} = -0.5$$

in the stock.

The portfolio value at time 1 if $\omega_1 = H$ is:

$$X_1(H) = \Delta_0 S_1(H) + (1+r)\big(X_0 - \Delta_0 S_0\big) = -0.5 \cdot 8 + \frac{5}{4}\big(0 - (-0.5) \cdot 4\big) = -1.5$$

The payoff of the long position from the call is $(S_1(H) - 5)^+ = 8 - 5 = 3$. They add up to 1.5.

The portfolio value at time 1 if $\omega_1 = T$ is:

$$X_1(T) = \Delta_0 S_1(T) + (1+r)\big(X_0 - \Delta_0 S_0\big) = -0.5 \cdot 2 + \frac{5}{4}\big(0 - (-0.5) \cdot 4\big) = 1.5$$

The payoff of the long position from the call is $(S_1(T) - 5)^+ = (2-5)^+ = 0$. They add up to 1.5.

7. The hedging portfolio should have time zero value $X_0 = 0$ and satisfies

$$X_3 + V_3 = 2.6875$$

regardless of the result of coin toss at time one, two and three. In other words, the time three value of the hedging portfolio should be

$$\begin{array}{llll} X_3(HHH) = 2.6875; & X_3(HHT) = -5.3125; & X_3(HTH) = 2.6875; & X_3(HTT) = -3.3125; \\ X_3(THH) = 2.6875; & X_3(THT) \quad = 0.6875; & X_3(TTH) = 0.6875; & X_3(TTT) \ \ = -0.8125 \end{array}$$

We then use the risk neutral pricing formula to find the value of the hedging portfolio at time two:

$$X_2(HH) = \frac{4}{5}(\frac{1}{2}X_3(HHH) + \frac{1}{2}X_3(HHT)) = -1.05$$
$$X_2(HT) = \frac{4}{5}(\frac{1}{2}X_3(HTH) + \frac{1}{2}X_3(HTT)) = -0.25$$
$$X_2(TH) = \frac{4}{5}(\frac{1}{2}X_3(THH) + \frac{1}{2}X_3(THT)) = 1.35$$
$$X_2(TT) = \frac{4}{5}(\frac{1}{2}X_3(TTH) + \frac{1}{2}X_3(TTT)) = -0.05.$$

Applying the risk neutral pricing formula at time one:

$$X_1(H) = \frac{4}{5}(\frac{1}{2}X_2(HH) + \frac{1}{2}X_2(HT)) = -0.52$$
$$X_1(T) = \frac{4}{5}(\frac{1}{2}X_2(TH) + \frac{1}{2}X_2(TT)) = 0.52.$$

Applying the risk neutral pricing formula once again,

$$X_0 = \frac{4}{5}(\frac{1}{2}X_1(H) + \frac{1}{2}X_1(T)) = 0,$$

which verifies our assumption at the beginning.

The investment strategy is then determined by the delta hedging formula:

$$\begin{aligned}
\Delta_0 &= \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} &&= -\frac{13}{75} \\
\Delta_1(H) &= \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} &&= \frac{-1}{15} \\
\Delta_1(T) &= \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2(TT)} &&= \frac{7}{15} \\
\Delta_2(HH) &= \frac{X_3(HHH) - X_3(HHT)}{S_3(HHH) - S_3(HHT)} &&= \frac{1}{3} \\
\Delta_2(HT) &= \frac{X_3(HTH) - X_3(HTT)}{S_3(HTH) - S_3(HTT)} &&= 1 \\
\Delta_2(TH) &= \frac{X_3(THH) - X_3(THT)}{S_3(THH) - S_3(THT)} &&= \frac{1}{3} \\
\Delta_2(TT) &= \frac{X_3(TTH) - X_3(TTT)}{S_3(TTH) - S_3(TTT)} &&= 1.
\end{aligned}$$

Let us use the above portfolio process Δ's, with starting wealth $X_0 = 0$ and verify that the hedge is successful for $\omega_1 = T$, $\omega_2 = H$, $\omega_3 = T$. The value of the portfolio at time 1 is:

$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = \frac{-13}{75} \cdot 2 + \frac{5}{4}(\frac{13}{75} \cdot 4) = \frac{13}{25}.$$

The value of the portfolio at time 2 is:

$$X_2(TH) = \Delta_1(T)S_2(TH) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) = \frac{7}{15} \cdot 4 + \frac{5}{4}(\frac{13}{25} - \frac{7}{15} \cdot 2) = 1.35.$$

The value of the portfolio at time 3 is:

$$X_3(THT) = \Delta_2(TH)S_3(THT) + (1+r)(X_2(TH) - \Delta_2(TH)S_2(TH)) = \frac{1}{3} \cdot 2 + \frac{5}{4}(1.35 - \frac{1}{3} \cdot 4) = 0.6875.$$

The payoff from the option is $V_3(THT) = 2$. They add up to 2.6875. The hedge is successful in this case.

8. (i)

$$v_n(s,y) = \frac{1}{1+r}(\tilde{p} \cdot v_{n+1}(us, y+us) + \tilde{q} \cdot v_{n+1}(ds, y+ds)) = \frac{2}{5}(v_{n+1}(us, y+us) + v_{n+1}(ds, y+ds))$$

(ii) Let us compute v_3 from all the possible paths starting from $S_0 = 4$ and $Y_0 = 4$.

$$\begin{aligned}
v_3(32,60) &= (\frac{60}{4} - 4)^+ = 11; & v_3(8,36) &= (\frac{36}{4} - 4)^+ = 5; \\
v_3(8,24) &= (\frac{24}{4} - 4)^+ = 2; & v_3(2,18) &= (\frac{18}{4} - 4)^+ = \frac{1}{2}; \\
v_3(8,18) &= (\frac{18}{4} - 4)^+ = \frac{1}{2}; & v_3(2,12) &= (\frac{12}{4} - 4)^+ = 0; \\
v_3(2,9) &= (\frac{9}{4} - 4)^+ = 0; & v_3(\frac{1}{2}, \frac{15}{2}) &= (\frac{\frac{15}{2}}{4} - 4)^+ = 0.
\end{aligned}$$

We then apply the algorithm in part one to compute the price at $t = 2$.

$$\begin{aligned}
v_2(16,28) &= \frac{2}{5}(v_3(32,60)+v_3(8,36)) = \frac{2}{5}(11+5) = \frac{32}{5} \\
v_2(4,16) &= \frac{2}{5}(v_3(8,24)+v_3(2,18)) \quad = \frac{2}{5}(2+\frac{1}{2}) = 1 \\
v_2(4,10) &= \frac{2}{5}(v_3(8,18)+v_3(2,12)) \quad = \frac{2}{5}(\frac{1}{2}+0) = \frac{1}{5} \\
v_2(1,7) &= \frac{2}{5}(v_3(2,9)+v_3(\frac{1}{2},\frac{15}{2})) \qquad = 0.
\end{aligned}$$

We then compute the time one price:

$$\begin{aligned}
v_1(8,12) &= \frac{2}{5}(v_2(16,28)+v_2(4,16)) = \frac{2}{5}(\frac{32}{5}+1) = \frac{74}{25} \\
v_1(2,6) &= \frac{2}{5}(v_2(4,10)+v_2(1,7)) = \frac{2}{5}\frac{1}{5} = \frac{2}{25}.
\end{aligned}$$

Finally, for the time zero price:

$$v_0(4,4) = \frac{2}{5}(v_1(8,12)+v_1(2,6)) = \frac{2}{5}(\frac{74}{25}+\frac{2}{25}) = \frac{152}{125}.$$

(iii)

$$\delta_n(s,y) := \frac{v_{n+1}(us, y+us) - v_{n+1}(ds, y+ds)}{(u-d)s}. \tag{2}$$

Indeed, assume that $\omega_{n+1} = H$, the value of the hedging portfolio with the number of stock held at time n given by (2) is:

$$\begin{aligned}
X_{n+1}(H) &= \delta_n(s,y)us + (1+r)(v_n(s,y) - \delta_n(s,y)s) \\
&= \frac{v_{n+1}(us,y+us) - v_{n+1}(ds,y+ds)}{(u-d)s} \cdot s(u-1-r) + (1+r)v_n(s,y) \\
&= \tilde{q}v_{n+1}(us,y+us) - \tilde{q}v_{n+1}(ds,y+ds) + \tilde{p}v_{n+1}(us,y+us) + \tilde{q}v_{n+1}(ds,y+ds) \\
&= v_{n+1}(us,y+us) = V_{n+1}(H).
\end{aligned}$$

Here we let $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-1-r}{u-d}$. In the third equality, we have used the algorithm given in part one.

Assume that $\omega_{n+1} = T$, the value of the hedging portfolio with the number of stock held at time n given by (2) is:

$$\begin{aligned}
X_{n+1}(T) &= \delta_n(s,y)ds + (1+r)(v_n(s,y) - \delta_n(s,y)s) \\
&= \frac{v_{n+1}(us,y+us) - v_{n+1}(ds,y+ds)}{(u-d)s} \cdot s(d-1-r) + (1+r)v_n(s,y) \\
&= -\tilde{p}v_{n+1}(us,y+us) + \tilde{p}v_{n+1}(ds,y+ds) + \tilde{p}v_{n+1}(us,y+us) + \tilde{q}v_{n+1}(ds,y+ds) \\
&= v_{n+1}(ds,y+ds) = V_{n+1}(T).
\end{aligned}$$

We have verified that (2) is the hedging portfolio process.

9. (i) Recursively compute

$$V_n = \frac{1}{1+r_n}\left(\frac{1+r_n-d_n}{u_n-d_n}V_{n+1}(H) + \frac{u_n-1-r_n}{u_n-d_n}V_{n+1}(T)\right),$$

where $n = N-1, \cdots, 0$.

(ii) This is given by the delta hedging formula:

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}.$$

(iii) We note that the interest rate r_n is constant (in fact, it equals 0). The up factor u_n and down factor d_n depend only on the stock price. Indeed, $u_n(s) = \frac{s+10}{s}$ and $d_n(s) = \frac{s-10}{s}$. Moreover, the price of the call at time 5 depends only the price of the stock at time 5. Applying the recursive algorithm in part one, we see that the price of the call at time n depends only on the price of the stock at time n, for $0 \le n \le N$. Therefore, we will write $v_n(s)$ instead of $V_n(\omega_1, \cdots, \omega_n)$.

Let us compute the risk neutral probability:

$$\tilde{p_n} = \frac{1+r_n-d_n}{u_n-d_n} = \frac{1-\frac{s-10}{s}}{\frac{s+10}{s}-\frac{s-10}{s}} = \frac{1}{2}$$

$$\tilde{q_n} = \frac{u_n-1-r_n}{u_n-d_n} = \frac{\frac{s+10}{s}-1}{\frac{s+10}{s}-\frac{s-10}{s}} = \frac{1}{2}.$$

Therefore, the formula in part one can be simplified as

$$v_n(s) = \frac{1}{2}v_{n+1}(s+10) + \frac{1}{2}v_{n+1}(s-10) \tag{3}$$

Let us compute the time 5 value of this call, which is just its payoff.

$$v_5(130) = 50; \quad v_5(110) = 30; \quad v_5(90) = 10; \quad v_5(70) = v_5(50) = v_5(30) = 0.$$

We apply (3) to compute the time four value of this call.

$$\begin{aligned}
v_4(120) &= \frac{1}{2}(v_5(130) + v_5(110)) = 40;\\
v_4(100) &= \frac{1}{2}(v_5(110) + v_5(90)) = 20;\\
v_4(80) &= \frac{1}{2}(v_5(90) + v_5(70)) = 5;\\
v_4(60) &= 0;\\
v_4(40) &= 0.
\end{aligned}$$

We then apply (3) to compute the time three value of this call.

$$\begin{aligned}
v_3(110) &= \frac{1}{2}(v_4(120) + v_4(100)) = 30;\\
v_3(90) &= \frac{1}{2}(v_4(100) + v_4(80)) = \frac{25}{2};\\
v_3(70) &= \frac{1}{2}(v_4(80) + v_4(60)) = \frac{5}{2};\\
v_3(50) &= 0;
\end{aligned}$$

We then apply (3) to compute the time two value of this call.

$$
\begin{aligned}
v_2(100) &= \frac{1}{2}(v_3(110) + v_3(90)) = \frac{85}{4}; \\
v_2(80) &= \frac{1}{2}(v_3(90) + v_3(70)) = \frac{15}{2}; \\
v_2(60) &= \frac{1}{2}(v_3(70) + v_3(50)) = \frac{5}{4};
\end{aligned}
$$

We then apply (3) to compute the time one value of this call.

$$
\begin{aligned}
v_1(90) &= \frac{1}{2}(v_2(100) + v_2(80)) = \frac{115}{8}; \\
v_1(70) &= \frac{1}{2}(v_2(80) + v_2(60)) = \frac{35}{8};
\end{aligned}
$$

Finally, for the time 0 price,

$$
v_0(80) = \frac{1}{2}(v_1(90) + v_1(70)) = \frac{75}{8}.
$$

The price of this call at time zero is $\frac{75}{8}$.