Chapter 4 solutions to Stochastic Calculus for Finance II by Steven E. Shreve

Chung Ching Lau

1. We assume that $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. Then

$$\begin{aligned} I(t) = &\sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_l)[M(t_{l+1}) - M(t_l)] \\ &+ \sum_{j=l+1}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)] \end{aligned}$$

Taking conditional expectation for the first term, we have:

$$\mathbf{E}[\sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)]|\mathcal{F}(s)] = \sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] \tag{1}$$

since each term in the summation is $\mathcal{F}(s)$ measurable. For the second term,

$$\mathbf{E}[\Delta(t_l)[M(t_{l+1}) - M(t_l)]|\mathcal{F}(s)] = \Delta(t_l)(\mathbf{E}[M(t_{l+1})|\mathcal{F}(s)] - M(t_l)) = \Delta(t_l)(M(s) - M(t_l)). \tag{2}$$

Note that we used the fact that M is a martingale in the last equality.

We claim that the conditional expectation for the third and fourth term vanishes, i.e.

$$\mathbf{E}[\sum_{j=l+1}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)]|\mathcal{F}(s)] = 0. \tag{3}$$

To prove this, it suffices to show that for any pair $s_2 > s_1 > s$,

$$\mathbf{E}[\Delta(s_1)[M(s_2) - M(s_1)]|\mathcal{F}(s)] = 0.$$

Indeed,

$$\begin{aligned} \mathbf{E}[\Delta(s_1)[M(s_2) - M(s_1)]|\mathcal{F}(s)] &= \mathbf{E}[\mathbf{E}[\Delta(s_1)[M(s_2) - M(s_1)]|\mathcal{F}(s_1)]|\mathcal{F}(s)] \\ &= \mathbf{E}[\Delta(s_1)\mathbf{E}[M(s_2) - M(s_1)|\mathcal{F}(s_1)]|\mathcal{F}(s)] \\ &= \mathbf{E}[\Delta(s_1) \cdot 0|\mathcal{F}(s)] = 0, \end{aligned}$$

where we took iterated expectation in the first equality; we took out what was known in the second equality and we used the fact that M is a martingale in the last equality. This proves (3).

Adding (1), (2) and (3), we showed that

$$\mathbf{E}[I(t)|\mathcal{F}(s)] = I(s).$$

2. (i)

$$I(t_k) - I(t_l) = \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] \tag{4}$$

Note that the increment $W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_j)$, by the definition of Brownian motion (Definition 3.3.3). Furthermore, if $t_l \le t_j$, then $\mathcal{F}(t_l) \subseteq \mathcal{F}(t_l)$ (information accumulates). It follows that $W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_l)$. Since $\Delta(t_j)$ is nonrandom, $\Delta(t_j)[W(t_{j+1}) - W(t_j)]$ is independent of $\mathcal{F}(t_l)$. We have shown that each term in the summation (4) is independent of $\mathcal{F}(t_l)$, therefore $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$.

(ii) We again assume $s = t_l$ and $t = t_k$ are two partition points. Let us compute the moment generating function of $I(t_k) - I(t_l)$.

$$\begin{aligned}
\varphi(u) &= \mathbf{E}[\exp\{u\big(I(t_k) - I(t_l)\big)\}] \\
&= \mathbf{E}[\exp\{u \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]\}] \\
&= \prod_{j=l}^{k-1} \mathbf{E}[\exp\{u\Delta(t_j)[W(t_{j+1}) - W(t_j)]\}] \\
&= \prod_{j=l}^{k-1} \exp\{\frac{1}{2} u \Delta^2(t_j) \cdot (t_{j+1} - t_j)\} \\
&= \exp\{\frac{1}{2} u \int_{t_l}^{t_k} \Delta^2(r) dr\}.
\end{aligned}$$

We used the result in part (i) which says that $\Delta(t_{j'})[W(t_{j'+1}) - W(t_{j'})]$ is independent of $\mathcal{F}(t'_j)$ and the fact that $\sum_{j=l}^{j'-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$ is $\mathcal{F}(t'_j)$-measurable to prove the third equality inductively. For the fourth equality, note that $\Delta(t_j)[W(t_{j+1}) - W(t_j)]$ is a normally distributed random variable with variance $\Delta(t_j)^2(t_{j+1} - t_j)$.

(iii)

$$\begin{aligned}
\mathbf{E}[I(t)|\mathcal{F}(s)] &= \mathbf{E}[I(t) - I(s)|\mathcal{F}(s)] + I(s) && \text{(Take out what is known.)} \\
&= \mathbf{E}[I(t) - I(s)] + I(s) && (I(t) - I(s) \text{ is independent of } \mathcal{F}(s).) \\
&= 0 + I(s) && (I(t) - I(s) \text{ has mean 0.})
\end{aligned}$$

(iv) Let us compute the conditional expectation of $I^2(t) - \int_0^t \Delta^2(u) du$.

$$\begin{aligned}
&\mathbf{E}[I^2(t) - \int_0^t \Delta^2(u) du | \mathcal{F}(s)] \\
&= \mathbf{E}[(I(t) - I(s) + I(s))^2 - \int_0^t \Delta^2(u) du | \mathcal{F}(s)] \\
&= I^2(s) - \int_0^t \Delta^2(u) du + \mathbf{E}[(I(t) - I(s))^2 | \mathcal{F}(s)] + 2I(s)\mathbf{E}[I(t) - I(s) | \mathcal{F}(s)] \\
&= I^2(s) - \int_0^t \Delta^2(u) du + \mathbf{E}[(I(t) - I(s))^2] + 2I(s)\mathbf{E}[I(t) - I(s)] \\
&= I^2(s) - \int_0^t \Delta^2(u) du + \int_s^t \Delta^2(u) du + 2I(s) \cdot 0 \\
&= I^2(s) - \int_0^s \Delta^2(u) du.
\end{aligned}$$

For the third equality, we took out what was known; for the fourth equality, we used the fact that $I(t)-I(s)$ is independent of $\mathcal{F}(s)$; for the fifth equality, we used the fact that $I(t)-I(s)$ has mean 0 and variance $\int_s^t \Delta^2(u)du$.

3. (i) Consider the expectation

$$\begin{aligned}
&\mathbf{E}[\big(I(t)-I(s)\big)^2|\mathcal{F}(s)] \\
&= \mathbf{E}[W(s)^2(W(t)-W(s))^2|\mathcal{F}(s)] \\
&= W(s)^2\mathbf{E}[(W(t)-W(s))^2|\mathcal{F}(s)] && \text{(Take out what is known.)} \\
&= W(s)^2\mathbf{E}[(W(t)-W(s))^2] && \text{(The increment } W(t)-W(s) \text{ is independent of } \mathcal{F}(s). \\
&= W(s)^2\cdot(t-s),
\end{aligned}$$

which depends on the Brownian motion at time s, so this cannot be equal to the unconditional expectation. This shows that $I(t)-I(s)$ is not independent of $\mathcal{F}(s)$.

(ii) First, we note that the expectation $\mathbf{E}[I(t)-I(s)]=0$. Indeed,

$$\mathbf{E}[I(t)-I(s)] = \mathbf{E}[\mathbf{E}[W(s)(W(t)-W(s))|\mathcal{F}(s)]] = \mathbf{E}[W(s)\mathbf{E}[W(t)-W(s)|\mathcal{F}(s)]] = \mathbf{E}[W(s)\cdot 0] = 0.$$

Here, we used iterated conditioning in the second equality; we took out what was known in the third equality and we used the fact that $W(t)-W(s)$ is independent of $\mathcal{F}(s)$ in the last equality.

Let us compute the fourth central moment of $I(t)-I(s)$

$$\begin{aligned}
\mathbf{E}[(I(t)-I(s))^4] &= \mathbf{E}[W(s)^4(W(t)-W(s))^4] \\
&= \mathbf{E}[\mathbf{E}[W(s)^4(W(t)-W(s))^4|\mathcal{F}(s)]] \\
&= \mathbf{E}[W(s)^4\mathbf{E}[(W(t)-W(s))^4|\mathcal{F}(s)]] \\
&= \mathbf{E}[W(s)^4\cdot 3(t-s)^2] \\
&= 3\cdot s^2\cdot 3(t-s)^2.
\end{aligned}$$

Here, we used iterated conditioning in the second equality; we took out what was known in the third equality; for the fourth equality, we used the fact that $W(t)-W(s)$ is independent of $\mathcal{F}(s)$, that it is normally distributed with mean 0 and variance $t-s$ and the fourth central moment of a normally distributed random variable is equal to three times the square of the variance; for the last equality, we used similar facts applying instead to $W(s)$.

Now, let us compute the variance.

$$\begin{aligned}
\mathbf{E}[(I(t)-I(s))^2] &= \mathbf{E}[\mathbf{E}[(I(t)-I(s))^2|\mathcal{F}(s)]] \\
&= \mathbf{E}[W(s)^2(t-s)] \\
&= s(t-s)
\end{aligned}$$

Here, we used iterated conditioning in the first equality; we used the computation in part (i) in the second equality and we used the fact that $W(s)$ has mean 0 and variance s in the last equality.

We showed that $\mathbf{E}[(I(t)-I(s))^4] \neq 3\cdot\mathbf{E}[(I(t)-I(s))^2]^2$. Therefore, $I(t)-I(s)$ is not normally distributed.

(iii) This assertion is true.

$$\begin{aligned}\mathbf{E}[I(t)|\mathcal{F}(s)] &= \mathbf{E}[W(s)(W(t)-W(s))|\mathcal{F}(s)] + I(s)\\ &= W(s)\mathbf{E}[(W(t)-W(s))|\mathcal{F}(s)] + I(s)\\ &= W(s)\cdot 0 + I(s) = I(s).\end{aligned}$$

We used the fact that $W(t)-W(s)$ is independent of $\mathcal{F}(s)$ and has mean 0 in the second to last equality.

(iv) This equation holds. Let us analyze the left hand side of this equation.

$$\begin{aligned}&\mathbf{E}[I^2(t) - \int_0^t \Delta^2(u)du|\mathcal{F}(s)]\\ &= \mathbf{E}[(I(t)-I(s))^2 + 2I(s)(I(t)-I(s)) + I^2(s) - \int_0^s \Delta^2(u)du - \int_s^t \Delta^2(u)du|\mathcal{F}(s)]\\ &= I^2(s) - \int_0^s \Delta^2(u)du + \mathbf{E}[(I(t)-I(s))^2 - \int_s^t \Delta^2(u)du|\mathcal{F}(s)] + 2I(s)\mathbf{E}[(I(t)-I(s))|\mathcal{F}(s)]\\ &= I^2(s) - \int_0^s \Delta^2(u)du + W^2(s)\cdot(t-s) - W^2(s)(t-s) + 2I(s)W(s)\mathbf{E}[(W(t)-W(s))|\mathcal{F}(s)]\\ &= I^2(s) - \int_0^s \Delta^2(u)du.\end{aligned}$$

Note that we used the computation in part (i) for the third equality.

4. (i) Let us compute the expectation of $Q_{\frac{\Pi}{2}}$.

$$\mathbf{E}[Q_{\frac{\Pi}{2}}] = \mathbf{E}[\sum_{j=0}^{n-1}(W(t_j^*)-W(t_j))^2] = \sum_{j=0}^{n-1}(t_j^*-t_j) = \sum_{j=0}^{n-1}(\frac{t_{j+1}-t_j}{2}) = \frac{1}{2}T$$

For the variance,

$$\begin{aligned}\mathrm{Var}(Q_{\frac{\Pi}{2}}) &= \mathbf{E}[Q^2_{\frac{\Pi}{2}}] - (\frac{1}{2}T)^2\\ &= \sum_{j=0}^{n-1}\mathbf{E}[(W(t_j^*)-W(t_j))^4] + 2\sum_{0\le j<k\le n-1}\mathbf{E}[(W(t_j^*)-W(t_j))^2(W(t_k^*)-W(t_k))^2] - (\frac{1}{2}T)^2\\ &= \sum_{j=0}^{n-1}3(t_j^*-t_j)^2 + 2\sum_{0\le j<k\le n-1}(t_j^*-t_j)(t_k^*-t_k) - (\frac{1}{2}T)^2\\ &= 2\sum_{j=0}^{n-1}(t_j^*-t_j)^2 + (\sum_{j=0}^{n-1}(t_j^*-t_j))^2 - (\frac{1}{2}T)^2\\ &= 2\sum_{j=0}^{n-1}(t_j^*-t_j)^2\end{aligned}$$

For the third equality, note that the random variable $W(t_j^*)-W(t_j)$ is normally distributed, therefore its fourth central moment equals three times its variance, which is $t_j^*-t_j$.
Let us analyze the last term:

$$2\sum_{j=0}^{n-1}(t_j^*-t_j)^2 \le 2\max_{0\le j\le n-1}\{t_j^*-t_j\}\sum_{j=0}^{n-1}(t_j^*-t_j) = T\max_{0\le j\le n-1}\{t_j^*-t_j\}.$$

As $||\Pi||\to 0$, the last term goes to 0. Thus, $\lim_{||\Pi||\to 0}\mathrm{Var}(Q_{\frac{\Pi}{2}}) = 0$.

(ii) We can write the Stratonovich integral of $W(t)$ as a sum of an approximating sum of the Ito integral and $Q_{\frac{\Pi}{2}}$:

$$\begin{aligned}
&\sum_{j=0}^{n-1} W(t_j^*)(W(t_{j+1}) - W(t_j)) \\
&= \sum_{j=0}^{n-1} \big(W(t_j^*)(W(t_{j+1}) - W(t_j^*)) + W(t_j^*)(W(t_j^*) - W(t_j))\big) \\
&= \sum_{j=0}^{n-1} \big(W(t_j^*)(W(t_{j+1}) - W(t_j^*)) + W(t_j)(W(t_j^*) - W(t_j))\big) + \sum_{j=0}^{n-1}(W(t_j^*) - W(t_j))^2.
\end{aligned}$$

As $||\Pi|| \to 0$, the first term converges to the Ito integral $\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$, while the second term converges to $\frac{1}{2}T$ by part (i). Therefore, the Stratonovich integral

$$\int_0^T W(t) \circ dW(t) = \frac{1}{2}W^2(T).$$

5. (i) We let $f(x) = \log x$. Then $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$. We then apply Ito's formula to compute $df(S(t))$:

$$\begin{aligned}
df(S(t)) &= f'(S(t))dS(t) + \frac{1}{2}f''(S(t))dS(t)dS(t) \\
&= \frac{1}{S(t)}(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) - \frac{1}{2}\frac{1}{S^2(t)}\sigma^2(t)S^2(t)dt \\
&= \big(\alpha(t) - \frac{1}{2}\sigma^2(t)\big)dt + \sigma(t)dW(t)
\end{aligned}$$

(ii) Integrating both sides from 0 to t, we have

$$\log S(t) - \log S(0) = \int_0^t \alpha(s) - \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW(s).$$

Taking exponential, we have

$$\frac{S(t)}{S(0)} = \exp\{\int_0^t \alpha(s) - \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW(s)\}.$$

6. Recall that $S(t)$ satisfies the stochastic differential equation

$$d(S(t)) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t).$$

By Ito's formula,

$$\begin{aligned}
d(S^p(t)) &= pS^{p-1}(t)dS(t) + \frac{1}{2}p(p-1)S^{p-2}(t)dS(t)\cdot dS(t) \\
&= pS^{p-1}(t)\big(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\big) + \frac{1}{2}p(p-1)S^{p-2}(t)\sigma^2(t)S^2(t)dt \\
&= pS^p(t)(\alpha(t) - \frac{1}{2}(p-1)\sigma^2(t))dt + pS^p(t)\sigma(t)dW(t).
\end{aligned}$$

7. (i) Letting $f(x) = x^4$ and applying Ito's formula to $f(W(t))$, we have

$$dW^4(t) = 4W^3(t)dW(t) + 6W^2(t)dW(t)dW(t) = 4W^3(t)dW(t) + 6W^2(t)dt.$$

Integrating both sides, we get

$$W^4(T) = \int_0^T 4W^3(t)dW(t) + \int_0^T 6W^2(t)dt.$$

(ii) Taking expectation on both sides, we get

$$\mathbf{E}[W^4(T)] = \mathbf{E}[\int_0^T 4W^3(t)dW(t)] + \mathbf{E}[\int_0^T 6W^2(t)dt] = 6\int_0^T \mathbf{E}[W^2(t)]dt = 6\int_0^T tdt = 3T^2.$$

(iii)

$$dW^6(t) = 6W^5(t)dW(t) + 15W^4(t)dt.$$

Integrating, we have

$$W^6(T) = \int_0^T 6W^5(t)dW(t) + \int_0^T 15W^4(t)dt.$$

Taking expectation, we have

$$\mathbf{E}[W^6(T)] = \mathbf{E}[\int_0^T 6W^5(t)dW(t)] + \mathbf{E}[\int_0^T 15W^4(t)dt] = 15\int_0^T \mathbf{E}[W^4(t)]dt = 15\int_0^T 3t^2dt = 15T^3.$$

8. (i) Let $f(t,x) = e^{\beta t}x$. Then

$$f_t(t,x) = \beta e^{\beta t}x; \quad f_x(t,x) = e^{\beta t}; \quad f_{xx}(t,x) = 0.$$

Applying Ito's formula to $f(t, R(t)) = e^{\beta t}R(t)$, we have

$$\begin{aligned} d(e^{\beta t}R(t)) &= f_t(t,R(t))dt + f_x(t,R(t))dR(t) + \frac{1}{2}f_{xx}(t,R(t))dR(t)dR(t) \\ &= \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t) \\ &= \beta e^{\beta t}R(t)dt + e^{\beta t}((\alpha - \beta R(t))dt + \sigma dW(t)) \\ &= e^{\beta t}(\alpha dt + \sigma dW(t)) \end{aligned}$$

(ii) Integrating both sides on the above equality, we have

$$e^{\beta t}R(t) - R(0) = \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma\int_0^t e^{\beta s}dW(s).$$

This can also be written as

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s).$$

9. (i) Let us analyze the left hand side of this equation:

$$
\begin{aligned}
&Ke^{-r(T-t)}N'(d_-) \\
&= Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_-^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}K\exp\{-r(T-t) - \frac{(d_+ - \sigma\sqrt{T-t})^2}{2}\} \\
&= \frac{1}{\sqrt{2\pi}}K\exp\{-r(T-t) - \frac{d_+^2}{2} + d_+\sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2}\} \\
&= \frac{1}{\sqrt{2\pi}}K\exp\{-r(T-t) - \frac{d_+^2}{2} + \frac{1}{\sigma\sqrt{T-t}}[\log\frac{x}{K} + (r + \frac{1}{2}\sigma^2)(T-t)] \cdot \sigma\sqrt{T-t} - \frac{\sigma^2(T-t)}{2}\} \\
&= \frac{1}{\sqrt{2\pi}}K\exp\{-\frac{d_+^2}{2} + \log\frac{x}{K}\} \\
&= xN'(d_+).
\end{aligned}
$$

Here, we applied the first fundamental theorem of calculus in the first and last equalities; for the second equality, we plugged in the equation $d_- = d_+ - \sigma\sqrt{T-t}$ and for the fourth equality, we plugged in the definition of d_+.

(ii) The partial derivative of c with respect to x is:

$$
\begin{aligned}
c_x &= xN'(d_+)\frac{\partial d_+}{\partial x} + N(d_+) - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial x} \\
&= xN'(d_+)\frac{\partial d_+}{\partial x} + N(d_+) - Ke^{-r(T-t)}N'(d_-)\frac{\partial(d_+ - \sigma\sqrt{T-t})}{\partial x} \\
&= \frac{\partial d_+}{\partial x}(xN'(d_+) - Ke^{-r(T-t)}N'(d_-)) + N(d_+) \\
&= N(d_+).
\end{aligned}
$$

Note that we used the result in part (i) in the last step.

(iii) The partial derivative of c with respect to t is:

$$
\begin{aligned}
c_t &= xN'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial t} - rKe^{-r(T-t)}N(d_-) \\
&= xN'(d_+)\frac{\partial(d_+ - d_-)}{\partial t} - rKe^{-r(T-t)}N(d_-) \\
&= xN'(d_+)\frac{-\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_-).
\end{aligned}
$$

Note that we used the result in part (i) in the second equality and we plugged in the equation $d_- = d_+ - \sigma\sqrt{T-t}$ in the last equality.

(iv) The second order derivative of c with respect to x is:

$$c_{xx} = N'(d_+)\frac{\partial d_+}{\partial x} = N'(d_+)\frac{1}{\sigma x\sqrt{T-t}}.$$

Let us evaluate the left hand side of (4.10.3):

$$\begin{aligned}
&c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} \\
&= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) + \frac{1}{2}\sigma^2 x^2 N'(d_+)\frac{1}{\sigma x\sqrt{T-t}} \\
&= r\big(xN(d_+) - Ke^{-r(T-t)}N(d_-)\big) = rc.
\end{aligned}$$

This shows that the given formula for c satisfies the partial differential equation.

(v) Since $\lim_{t\to T^-}(d_+ - d_-) = \lim_{t\to T^-}\sigma\sqrt{T-t} = 0$. It suffices to show that

$$\lim_{t\to T^-} d_+ = \begin{cases} \infty & \text{for } K < x \\ -\infty & \text{for } 0 < x < K \end{cases}$$

Let us compute the limit of d_+.

$$\begin{aligned}
\lim_{t\to T^-} d_+ &= \lim_{t\to T^-}\frac{1}{\sigma\sqrt{T-t}}[\log\frac{x}{K} + (r + \frac{1}{2}\sigma^2)(T-t)] \\
&= \lim_{t\to T^-}\big(\frac{1}{\sigma\sqrt{T-t}}\log\frac{x}{K} + \frac{r+\frac{1}{2}\sigma^2}{\sigma}\sqrt{T-t}\big) \\
&= \lim_{t\to T^-}\frac{1}{\sigma\sqrt{T-t}}\log\frac{x}{K} + 0 \\
&= \begin{cases} \infty & \text{for } K < x \\ -\infty & \text{for } 0 < x < K. \end{cases}
\end{aligned}$$

In the last equality, we used the facts that $\sigma\sqrt{T-t}$ is positive for $t < T$ and goes to 0 as $t \to T$ and that $\log\frac{x}{K} > 0$ if $K < x$ and $\log\frac{x}{K} < 0$ if $0 < x < K$.

Note that $\lim_{t\to T^-} e^{-r(T-t)} = 1$; $\lim_{y\to\infty} N(y) = 1$ and $\lim_{y\to-\infty} N(y) = 0$. It follows that

$$\lim_{t\to T^-} c = \begin{cases} x\cdot 1 - K\cdot 1\cdot 1 = x - K & \text{for } x > K \\ x\cdot 0 - K\cdot 1\cdot 0 = 0 & \text{for } 0 < x < K. \end{cases}$$

(vi) Since $\lim_{x\to 0^-}(d_+ - d_-) = \sigma\sqrt{T-t}$ is finite, it suffices to show that $\lim_{x\to 0^-} d_+ = -\infty$.

$$\lim_{x\to 0^-} d_+ = \frac{r+\frac{1}{2}\sigma^2}{\sigma}\sqrt{T-t} + \frac{1}{\sigma\sqrt{T-t}}\lim_{x\to 0^-}\log\frac{x}{K} = -\infty.$$

Using this, together with the facts that $\lim_{y\to-\infty} N(y) = 0$, we get

$$\lim_{x\to 0^-} c = 0\cdot 0 - Ke^{-r(T-t)}\cdot 0 = 0.$$

(vii) Since $\lim_{x\to\infty}(d_+ - d_-) = \sigma\sqrt{T-t}$ is finite, it suffices to show that $\lim_{x\to\infty} d_+ = \infty$. Indeed,

$$\lim_{x\to\infty} d_+ = \frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{T-t} + \frac{1}{\sigma\sqrt{T-t}} \lim_{x\to\infty} \log\frac{x}{K} = \infty.$$

Let us compute the second boundary condition:

$$\begin{aligned}\lim_{x\to\infty}[c - (x - e^{-r(T-t)}K)] &= \lim_{x\to\infty}[x(N(d_+) - 1) - e^{-r(T-t)}K(N(d_-) - 1)] \\ &= \lim_{x\to\infty} \frac{N(d_+) - 1}{x^{-1}}.\end{aligned}$$

Note that we used $\lim_{x\to\infty} d_- = \infty$ to show that $\lim_{x\to\infty} N(d_-) - 1 = 0$ in the second equality. For the last term, as $x \to \infty$, both the numerator and the denominator go to 0. Therefore, we may apply L'Hopital's rule:

$$\begin{aligned}\lim_{x\to\infty}[c - (x - e^{-r(T-t)}K)] &= \lim_{x\to\infty} \frac{N'(d_+)\frac{\partial d_+}{\partial x}}{-x^{-2}} \\ &= \lim_{x\to\infty} \frac{N'(d_+)}{-x^{-2}\cdot x\sigma\sqrt{T-t}} \\ &= \lim_{x\to\infty} \frac{Ke^{-r(T-t)}N'(d_-)}{-\sigma\sqrt{T-t}} \\ &= \frac{Ke^{-r(T-t)}}{-\sigma\sqrt{T-t}} \lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}} = 0.\end{aligned}$$

We used the result in part (i) for the third equality; the first fundamental theorem of calculus for the fourth equality and the result $\lim_{x\to\infty} d_- = \infty$ that we proved earlier in the last equality.

10. (i) Applying Ito's product rule, we have

$$dX = \Delta dS + Sd\Delta + (dS)(d\Delta) + \Gamma dM + Md\Gamma + (dM)(d\Gamma)$$

Plugging (4.10.9) into the left hand side, while noting $X - \Delta S = \Gamma M$ we have

$$\Delta dS + r\Gamma M dt = \Delta dS + Sd\Delta + (dS)(d\Delta) + \Gamma dM + Md\Gamma + (dM)(d\Gamma)$$

Since $M(t) = e^{rt}$, $dM = re^{rt}dt = rMdt$. Plugging this into the above displayed equation, we have the continuous-time self-financing condition

$$Sd\Delta + (dS)(d\Delta) + Md\Gamma + (dM)(d\Gamma) = 0.$$

(ii) Writing $N(t) = \Gamma(t) \cdot M(t)$ and applying Ito's product rule to the left hand side of (4.10.21), we have

$$\Gamma dM + Md\Gamma + (d\Gamma)(dM) = c_t dt + c_x dS + \frac{1}{2}c_{xx}(dS)(dS) - \Delta dS - Sd\Delta - (d\Delta)(dS)$$

Applying the continuous-time self-financing condition, this becomes

$$\Gamma dM = c_t dt + c_x dS + \frac{1}{2}c_{xx}(dS)(dS) - \Delta dS$$

We then plugged in $\Gamma dM = r\Gamma M dt = rNdt$ and $(dS)(dS) = \sigma^2 S^2 dt$ to get

$$rNdt = (c_x - \Delta)dS + [c_t + \frac{1}{2}c_{xx}\sigma^2 S^2]dt$$

For this to be instantaneously riskless, the dS term must be killed, i.e. $c_x = \Delta$, then we are left with

$$rNdt = [c_t + \frac{1}{2}c_{xx}\sigma^2 S^2]dt$$

11. Applying Ito's formula to $c(t, S(t))$, we have

$$\begin{aligned} dc(t, S(t)) &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))dS(t)dS(t) \\ &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))\sigma_2^2 S^2(t)dt \end{aligned}$$

Plugging this into the given equation on $dX(t)$, we have

$$dX(t) = \Big(rX(t) + \big(c_t(t, S(t)) - rc(t, S(t)) + rS(t)c_x(t, S(t)) + \frac{1}{2}\sigma_1^2 S^2(t)c_{xx}(t, S(t))\big)\Big)dt$$

Since c satisfies the Black-Scholes partial differential equation:

$$c_t(t, S(t)) + rS(t)c_x(t, S(t)) + \frac{1}{2}\sigma_1^2 S^2(t)c_{xx}(t, S(t)) = rc(t, S(t))$$

The previous equation becomes

$$dX(t) = rX(t)dt \tag{5}$$

We now apply Ito's formula to compute $d(e^{-rt}X(t))$ by letting $f(t, x) = e^{-rt}x$:

$$d(e^{-rt}X(t)) = -re^{-rt}X(t)dt + e^{-rt}dX(t) = 0,$$

where we used (5) in the last equality.
Integrating, we have

$$e^{-rt}X(t) - X(0) = 0.$$

Since $X(0) = 0$ by assumption, $X(t) = 0$ as well.

12. (i) Let us first compute the partial derivatives of $f(t,x) = x - e^{-r(T-t)}K$.

$$f_t(t,x) = -rKe^{-r(T-t)}; \quad f_x(t,x) = 1; \quad f_{xx}(t,x) = 0.$$

By Put-Call parity,

$$p(t,x) = c(t,x) - f(t,x)$$

It follows that

$$p_t(t,x) = c_t(t,x) - f_t(t,x) = -rKe^{-r(T-t)}(N(d_-) - 1) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+);$$

$$p_x(t,x) = c_x(t,x) - f_x(t,x) = N(d_+) - 1;$$

$$p_{xx}(t,x) = c_{xx}(t,x) - f_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+).$$

(ii) This follows from the delta hedging rule. The number of shares held by the hedging portfolio is $p_x(t,x)$, which is negative by the above computation.
An alternative way to see this is to use put-call parity. A short position in the put is the same as a short position in the call and a long position in the forward contract. The static hedge for a long position in the forward is set up by shorting 1 unit of the stock at time zero. Therefore, the hedging portfolio for the short call and long forward position holds $c_x(t,S(t)) - 1 = N(d_+) - 1 < 0$ unit of the stock at time t.

(iii) Let us show that the forward contract satisfies the Black-Scholes partial differential equation.

$$f_t(t,x) + rxf_x(t,x) + \frac{1}{2}\sigma^2 f_{xx}(t,x) = -rKe^{-r(T-t)} + rx \cdot 1 + 0 = rf(t,x).$$

Now it follows by put-call parity that the put satisfies the Black-Scholes partial differential equation as well.

13. Note that $W_2(t)$ can be defined as $\int_0^t \frac{1}{\sqrt{1-\rho^2(s)}}dB_2(s) - \int_0^t \frac{\rho(s)}{\sqrt{1-\rho^2(s)}}dB_1(s)$. These two terms are both Ito integral, which implies that $W_2(t)$ is a martigale.

$$\begin{aligned}\rho(t)dt &= dB_1(t)dB_2(t)\\ &= dW_1(t)(\rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t))\\ &= \rho(t)dt + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t),\end{aligned}$$

which implies $dW_1(t)dW_2(t) = 0$, since $\sqrt{1-\rho^2(t)} > 0$.

$$\begin{aligned}dW_2(t)dW_2(t) &= \left(\frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)\right)\cdot\left(\frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)\right)\\ &= \frac{1}{1-\rho^2(t)}dB_2(t)dB_2(t) - 2\frac{\rho(t)}{1-\rho(t)}dB_2(t)dB_1(t) + \frac{\rho^2(t)}{1-\rho(t)}dB_1(t)dB_1(t)\\ &= \left(\frac{1}{1-\rho^2(t)} - 2\frac{\rho^2(t)}{1-\rho(t)} + \frac{\rho^2(t)}{1-\rho(t)}\right)dt = dt.\end{aligned}$$

We may now apply the two dimensional version of Levy's theorem to conclude that $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

14. (i) $W(t_{j+1})$ is $\mathcal{F}(t_{j+1})$-measurable by the adaptivity property of Brownian motion; $W(t_j)$ is $\mathcal{F}(t_{j+1})$-measurable because it is $\mathcal{F}(t_j)$-measurable (adaptivity) and $\mathcal{F}(t_j) \subset \mathcal{F}(t_{j+1})$ by the information accumulation property. Since all the random variables that appeared in the expression for Z_j are $\mathcal{F}(t_{j+1})$-measurable, Z_j is $\mathcal{F}(t_{j+1})$-measurable.

$$\begin{aligned}\mathbf{E}[Z_j|\mathcal{F}(t_j)] &= \mathbf{E}\big[f''(W(t_j)))[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)]|\mathcal{F}(t_j)\big] \\ &= f''(W(t_j)))\big(\mathbf{E}[(W(t_{j+1}) - W(t_j))^2|\mathcal{F}(t_j)] - (t_{j+1} - t_j)\big) \\ &= f''(W(t_j)))\big((t_{j+1} - t_j) - (t_{j+1} - t_j)\big) = 0\end{aligned}$$

In the third equality, we used the fact that the increment $W(t_{j+1}) - W(t_j)$ is normally distributed with variance $t_{j+1} - t_j$ and is independent of $\mathcal{F}(t_j)$.

$$\begin{aligned}&\mathbf{E}[Z_j^2|\mathcal{F}(t_j)] \\ &= [f''(W(t_j))]^2\mathbf{E}[(W(t_{j+1}) - W(t_j))^4 - 2(W(t_{j+1}) - W(t_j))^2(t_{j+1} - t_j) + (t_{j+1} - t_j)^2|\mathcal{F}(t_j)] \\ &= [f''(W(t_j))]^2\mathbf{E}[(W(t_{j+1}) - W(t_j))^4 - 2(W(t_{j+1}) - W(t_j))^2(t_{j+1} - t_j) + (t_{j+1} - t_j)^2] \\ &= [f''(W(t_j))]^2[3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)(t_{j+1} - t_j) + (t_{j+1} - t_j)^2] \\ &= 2[f''(W(t_j))]^2(t_{j+1} - t_j)^2.\end{aligned}$$

Note that in the third equality, we used the fact that for a normally distributed random variable, the fourth central moment equals to three time the square of its variance.

(ii)

$$\begin{aligned}\mathbf{E}[\sum_{j=0}^{n-1} Z_j] &= \sum_{j=0}^{n-1} \mathbf{E}[Z_j] \\ &= \sum_{j=0}^{n-1} \mathbf{E}[\mathbf{E}[Z_j|\mathcal{F}(t_j)]] = 0\end{aligned}$$

(iii)

$$\begin{aligned}\mathbf{E}[(\sum_{j=0}^{n-1} Z_j)^2] &= \sum_{j=0}^{n-1} \mathbf{E}[Z_j^2] + \sum_{0\le j<k\le n-1} \mathbf{E}[Z_j Z_k] \\ &= \sum_{j=0}^{n-1} \mathbf{E}[\mathbf{E}[Z_j^2|\mathcal{F}(t_j)]] + \sum_{0\le j<k\le n-1} \mathbf{E}[\mathbf{E}[Z_j Z_k|\mathcal{F}(t_j)]] \\ &= \mathbf{E}[\sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2[f''(W(t_j))]^2] + \sum_{0\le j<k\le n-1} \mathbf{E}[Z_j\mathbf{E}[Z_k|\mathcal{F}(t_j)]] \\ &= \mathbf{E}[\sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2[f''(W(t_j))]^2],\end{aligned}$$

where we used the fact that $\mathbf{E}[Z_k|\mathcal{F}(t_j)] = \mathbf{E}[\mathbf{E}[Z_k|\mathcal{F}(t_k)]|\mathcal{F}(t_j)] = 0$ in the last equality.

Therefore, we may bound

$$\mathbf{E}[(\sum_{j=0}^{n-1} Z_j)^2] \leq 2||\Pi|| \cdot \mathbf{E}[\sum_{j=0}^{n-1}[f''(W(t_j))]^2(t_{j+1}-t_j)],$$

which converges to $0 \cdot \mathbf{E}\int_0^T [f''(W(t))]^2 dt = 0$ as $||\Pi|| \to 0$. This implies that

$$\lim_{||\Pi||\to 0} \text{Var}[\sum_{j=0}^{n-1} Z_j] = 0.$$

15. (i) $B_i(t)$ is a martingale, since it is made up of a sum of Ito integrals. Let us compute its quadratic variation.

$$\begin{aligned} dB_i(t)dB_i(t) &= (\sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t))^2 \\ &= \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)} dW_j(t)dW_j(t) + 2\sum_{1\leq j<k\leq d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} \cdot \frac{\sigma_{ik}(t)}{\sigma_i(u)} dW_j(t)dW_k(t) \\ &= \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sum_{k=1}^{d} \sigma_{ik}^2(t)} dt = dt, \end{aligned}$$

where we used independence of W_j and W_k in the third equality. It follows from the one-dimensional version of Levy's theorem that $B_i(t)$ is a Brownian motion.

(ii) We have already taken care of the case where $i = k$, since we have shown that B_i is a Brownian motion and $\rho_{ii}(t) \equiv 1$ in this case. We may assume $i \neq k$.

$$\begin{aligned} dB_i(t)dB_k(t) &= (\sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} dW_j(t))(\sum_{l=1}^{d} \frac{\sigma_{kl}(t)}{\sigma_k(t)} dW_l(t)) \\ &= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} \frac{\sigma_{kj}(t)}{\sigma_k(t)} dW_j(t)dW_j(t) + 2\sum_{1\leq j<l\leq d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} \frac{\sigma_{kl}(t)}{\sigma_k(t)} dW_j(t)dW_l(t) \\ &= \sum_{j=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} \frac{\sigma_{kj}(t)}{\sigma_k(t)} dt = \rho_{ik}(t)dt. \end{aligned}$$

16. We define

$$W_i(t) := \sum_{j=1}^{m} \int_0^t \alpha_{ij}(u)dB_j(u).$$

Then $W_i(t)$'s are martingales. Let us compute the quadratic variations and cross variations of $W_i(t)$'s.

$$\begin{aligned}
dW_i(t)dW_k(t) &= (\sum_{j=1}^{m} \alpha_{ij}(t)dB_j(t))(\sum_{l=1}^{m} \alpha_{kl}(t)dB_l(t)) \\
&= \sum_{j=1}^{m}\sum_{l=1}^{m} \alpha_{ij}(t)\alpha_{kl}(t)\rho_{jl}(t)dt \\
&= (A^{-1}(t)C(t)(A^{-1}(t))^T)_{ik}dt = \delta_{ik}dt.
\end{aligned}$$

The multi-dimensional version of Levy's theorem now implies that $W_i(t)$'s are independent Brownian motions.

$$\begin{aligned}
\sum_{j=1}^{m} a_{ij}(t)dW_j(t) &= \sum_{j=1}^{m} a_{ij}(t)(\sum_{l=1}^{m} \alpha_{jl}(t)dB_l(t)) \\
&= \sum_{l=1}^{m}\sum_{j=1}^{m} a_{ij}(t)\alpha_{jl}(t)dB_l(t) \\
&= \sum_{l=1}^{m} \delta_{il}dB_l(t) = dB_i(t).
\end{aligned}$$

In other words, $W_1(t), \cdots, W_m(t)$ satisfy

$$B_i(t) = \sum_{j=1}^{m} \int_0^t a_{ij}(u)dW_j(u).$$

17. (i) By Ito's product rule,

$$d(B_1(t)B_2(t)) = B_1(t)dB_2(t) + B_2(t)dB_1(t) + dB_1(t)dB_2(t) = B_1(t)dB_2(t) + B_2(t)dB_1(t) + \rho dt.$$

Integrating from $t = t_0$ to $t = t_0 + \epsilon$ and taking conditional expectation, we have

$$\begin{aligned}
&\mathbf{E}[B_1(t_0+\epsilon)B_2(t_0+\epsilon) - B_1(t_0)B_2(t_0)|\mathcal{F}(t_0)] \\
&= \mathbf{E}[\int_{t_0}^{t_0+\epsilon} d(B_1(t)B_2(t))|\mathcal{F}(t_0)] \\
&= \mathbf{E}\Big[\int_{t_0}^{t_0+\epsilon} B_1(t)dB_2(t) + \int_{t_0}^{t_0+\epsilon} B_2(t)dB_1(t) + \int_{t_0}^{t_0+\epsilon} \rho dt|\mathcal{F}(t_0)\Big] \\
&= \mathbf{E}\Big[\int_{t_0}^{t_0+\epsilon} \rho dt|\mathcal{F}(t_0)\Big] = \rho\epsilon.
\end{aligned}$$

The second equality holds since expectation of Ito integrals are always zero.

$$
\begin{aligned}
&\mathbf{E}[\big(B_1(t_0+\epsilon)-B_1(t_0)\big)\big(B_2(t_0+\epsilon)-B_2(t_0)\big)|\mathcal{F}(t_0)]\\
&=\mathbf{E}[\big(B_1(t_0+\epsilon)B_2(t_0+\epsilon)-B_1(t_0)B_2(t_0)\big)\\
&-B_1(t_0)\big(B_2(t_0+\epsilon)-B_2(t_0)\big)-B_2(t_0)\big(B_1(t_0+\epsilon)-B_1(t_0)\big)|\mathcal{F}(t_0)]\\
&=\rho\epsilon-B_1(t_0)\mathbf{E}[B_2(t_0+\epsilon)-B_2(t_0)|\mathcal{F}(t_0)]-B_2(t_0)\mathbf{E}[B_1(t_0+\epsilon)-B_1(t_0)|\mathcal{F}(t_0)]=\rho\epsilon.
\end{aligned}
$$

Here, we used the previous result in the second equality and the fact that the increments $B_1(t_0+\epsilon)-B_1(t_0)$ and $B_2(t_0+\epsilon)-B_2(t_0)$ are independent of $\mathcal{F}(t_0)$ and have mean 0.

(ii)

$$M_i(\epsilon) := \mathbf{E}[X_i(t_0+\epsilon)-X_i(t_0)|\mathcal{F}(t_0)] = \mathbf{E}[\Theta_i\epsilon+\sigma_i\big(B_i(t_0+\epsilon)-B_i(t_0)\big)|\mathcal{F}(t_0)] = \Theta_i\epsilon.$$

$$
\begin{aligned}
V_i(\epsilon) &:= \mathbf{E}[\big(X_i(t_0+\epsilon)-X_i(t_0)\big)^2|\mathcal{F}(t_0)]-M_i^2(\epsilon)\\
&=\mathbf{E}[\Big(\Theta_i\epsilon+\sigma_i\big(B_i(t_0+\epsilon)-B_i(t_0)\big)\Big)^2|\mathcal{F}(t_0)]-\Theta_i^2\epsilon^2\\
&=\Theta_i^2\epsilon^2+\mathbf{E}[2\Theta_i\epsilon\sigma_i\big(B_i(t_0+\epsilon)-B_i(t_0)\big)+\sigma_i^2\big(B_i(t_0+\epsilon)-B_i(t_0)\big)^2|\mathcal{F}(t_0)]-\Theta_i^2\epsilon^2\\
&=\sigma_i^2\epsilon.
\end{aligned}
$$

$$
\begin{aligned}
C(\epsilon) &:= \mathbf{E}[\big(X_1(t_0+\epsilon)-X_1(t_0)\big)\big(X_2(t_0+\epsilon)-X_2(t_0)\big)|\mathcal{F}(t_0)]-M_1(\epsilon)M_2(\epsilon)\\
&=\mathbf{E}[\Big(\Theta_1\epsilon+\sigma_1\big(B_1(t_0+\epsilon)-B_1(t_0)\big)\Big)\Big(\Theta_2\epsilon+\sigma_2\big(B_2(t_0+\epsilon)-B_2(t_0)\big)\Big)|\mathcal{F}(t_0)]-\Theta_1\Theta_2\epsilon^2\\
&=\mathbf{E}[\sigma_1\sigma_2\big(B_1(t_0+\epsilon)-B_1(t_0)\big)\big(B_2(t_0+\epsilon)-B_2(t_0)\big)|\mathcal{F}(t_0)]\\
&+\mathbf{E}[\Theta_1\epsilon\sigma_2\big(B_2(t_0+\epsilon)-B_2(t_0)\big)+\Theta_2\epsilon\sigma_1\big(B_1(t_0+\epsilon)-B_1(t_0)\big)|\mathcal{F}(t_0)]\\
&=\sigma_1\sigma_2\rho\epsilon+0.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
M_i(\epsilon) &:= \mathbf{E}[X_i(t_0+\epsilon)-X_i(t_0)|\mathcal{F}(t_0)]\\
&=\mathbf{E}[\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du+\int_{t_0}^{t_0+\epsilon}\sigma_i(u)dB_i(u)|\mathcal{F}(t_0)]\\
&=\mathbf{E}[\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du|\mathcal{F}(t_0)]+0.
\end{aligned}
$$

The last equality holds since the expectation of an Ito integral is zero.
By the first fundamental theorem of calculus,

$$\frac{1}{\epsilon}\lim_{\epsilon\to0^+}\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du = \lim_{\epsilon\to0^+}\frac{\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du}{t_0+\epsilon-t_0} = \Theta_i(t_0).$$

Since $\frac{|\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du|}{\epsilon} \le \frac{M\epsilon}{\epsilon} = M$, we may apply the Dominated Convergence Theorem to get

$$\lim_{\epsilon\to0^+}\frac{M_i(\epsilon)}{\epsilon} = \lim_{\epsilon\to0^+}\mathbf{E}[\frac{\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du}{\epsilon}|\mathcal{F}(t_0)] = \mathbf{E}[\lim_{\epsilon\to0^+}\frac{\int_{t_0}^{t_0+\epsilon}\Theta_i(u)du}{\epsilon}|\mathcal{F}(t_0)] = \Theta_i(t_0).$$

(iv) Let $Y_i(t) = \int_0^t \sigma_i(u) dB_i(u)$. We assume that

$$\lim_{\epsilon \to 0^+} \mathbf{E}[|Y_i(t_0+\epsilon) - Y_i(t_0)| \big| \mathcal{F}(t_0)] = 0. \tag{6}$$

Using this, we have

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbf{E}[\big|Y_i(t_0+\epsilon) - Y_i(t_0)\big| \Big| \int_{t_0}^{t_0+\epsilon} \Theta_j(u) du \Big| \Big| \mathcal{F}(t_0)] \le \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbf{E}[\big|Y_i(t_0+\epsilon) - Y_i(t_0)\big| M\epsilon \big| \mathcal{F}(t_0)] = 0. \tag{7}$$

Similarly,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbf{E}[\big|Y_j(t_0+\epsilon) - Y_j(t_0)\big| \Big| \int_{t_0}^{t_0+\epsilon} \Theta_i(u) du \Big| \Big| \mathcal{F}(t_0)] = 0 \tag{8}$$

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbf{E}[\Big| \int_{t_0}^{t_0+\epsilon} \Theta_i(u) du \Big| \Big| \int_{t_0}^{t_0+\epsilon} \Theta_j(u) du \Big| \Big| \mathcal{F}(t_0)] \le \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbf{E}[(M\epsilon)(M\epsilon) \Big| \mathcal{F}(t_0)] = 0. \tag{9}$$

Likewise,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} |M_i(\epsilon)||M_j(\epsilon)| \le \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} |M_i(\epsilon)| \mathbf{E}[\int_{t_0}^{t_0+\epsilon} |\Theta_i(u)| du \Big| \mathcal{F}(t_0)] \le \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} |M_i(\epsilon)| \cdot M\epsilon = 0. \tag{10}$$

Using Ito's product rule, we have

$$d(Y_i(t)Y_j(t)) = Y_i(t) dY_j(t) + Y_j(t) dY_i(t) + dY_i(t) dY_j(t) = Y_i(t) dY_j(t) + Y_j(t) dY_i(t) + \rho_{ij}(t) \sigma_i(t) \sigma_j(t) dt.$$

Integrating, we have

$$\mathbf{E}[Y_i(t_0+\epsilon) Y_j(t_0+\epsilon) - Y_i(t_0) Y_j(t_0) \big| \mathcal{F}(t_0)] = \mathbf{E}[\int_{t_0}^{t_0+\epsilon} \rho_{ij}(t) \sigma_i(t) \sigma_j(t) dt \big| \mathcal{F}(t_0)].$$

Note that the expectations of the integrals for the cross terms vanish because Y_i and Y_j are martingales.

$$\begin{aligned}
&\mathbf{E}[\big(Y_i(t_0+\epsilon) - Y_i(t_0)\big)\big(Y_j(t_0+\epsilon) - Y_j(t_0)\big) \big| \mathcal{F}(t_0)] \\
&= \mathbf{E}[Y_i(t_0+\epsilon) Y_j(t_0+\epsilon) - Y_i(t_0) Y_j(t_0) \big| \mathcal{F}(t_0)] \\
&- Y_j(t_0) \mathbf{E}[Y_i(t_0+\epsilon) - Y_i(t_0) \big| \mathcal{F}(t_0)] - Y_i(t_0) \mathbf{E}[Y_j(t_0+\epsilon) - Y_j(t_0) \big| \mathcal{F}(t_0)] \\
&= \mathbf{E}[\int_{t_0}^{t_0+\epsilon} \rho_{ij}(t) \sigma_i(t) \sigma_j(t) dt \big| \mathcal{F}(t_0)],
\end{aligned}$$

where we used the fact that Y_i and Y_j are martingales and the previous displayed equation for the second equality. This implies

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbf{E}[\big(Y_i(t_0+\epsilon) - Y_i(t_0)\big)\big(Y_j(t_0+\epsilon) - Y_j(t_0)\big) \big| \mathcal{F}(t_0)] = \rho_{ij}(t_0) \sigma_i(t_0) \sigma_j(t_0) \tag{11}$$

using the same argument in part (iii).
Let us now expand $D_{ij}(\epsilon)$:

$$\begin{aligned}
D_{ij}(\epsilon) &= \mathbf{E}[\big(Y_i(t_0+\epsilon) - Y_i(t_0) + \int_{t_0}^{t_0+\epsilon} \Theta_i(u) du\big)\big(Y_j(t_0+\epsilon) - Y_j(t_0) + \int_{t_0}^{t_0+\epsilon} \Theta_j(u) du\big) | \mathcal{F}(t_0)] \\
&- M_i(\epsilon) M_j(\epsilon) \\
&= \mathbf{E}[\big(Y_i(t_0+\epsilon) - Y_i(t_0)\big)\big(Y_j(t_0+\epsilon) - Y_j(t_0)\big) | \mathcal{F}(t_0)] \\
&+ \mathbf{E}[\big(Y_i(t_0+\epsilon) - Y_i(t_0)\big) \int_{t_0}^{t_0+\epsilon} \Theta_j(u) du | \mathcal{F}(t_0)] + \mathbf{E}[\big(Y_j(t_0+\epsilon) - Y_j(t_0)\big) \int_{t_0}^{t_0+\epsilon} \Theta_i(u) du | \mathcal{F}(t_0)] \\
&+ \mathbf{E}[\big(\int_{t_0}^{t_0+\epsilon} \Theta_i(u) du\big)\big(\int_{t_0}^{t_0+\epsilon} \Theta_j(u) du\big) | \mathcal{F}(t_0)] - M_i(\epsilon) M_j(\epsilon).
\end{aligned}$$

Using the equations (7), (8), (9), (10) and (11), we showed that

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}D_{ij}(\epsilon)=\rho_{ij}(t_0)\sigma_i(t_0)\sigma_j(t_0).$$

(v) Letting $i=j$, we have $V_i(\epsilon)=D_{ii}(\epsilon)$ and

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}V_i(\epsilon)=\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}D_{ij}(\epsilon)=\rho_{ii}(t_0)\sigma_i^2(t_0)=1\cdot\sigma_i^2(t_0).$$

Letting $i=1$ and $j=2$, we have $C(\epsilon)=D_{12}(\epsilon)$ and

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}C(\epsilon)=\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}D_{12}(\epsilon)=\rho(t_0)\sigma_1(t_0)\sigma_2(t_0).$$

(vi)

$$\lim_{\epsilon\to 0^+}\frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}}=\lim_{\epsilon\to 0^+}\frac{C(\epsilon)}{\epsilon}\cdot\frac{1}{\sqrt{\frac{V_1(\epsilon)}{\epsilon}}}\cdot\frac{1}{\sqrt{\frac{V_2(\epsilon)}{\epsilon}}}=\rho(t_0)\sigma_1(t_0)\sigma_2(t_0)\cdot\frac{1}{\sigma_1(t_0)}\cdot\frac{1}{\sigma_2(t_0)}=\rho(t_0)$$

18. (i) Let $f(t,x)=\exp\{-\theta x-(r+\frac{1}{2}\theta^2)t\}$. Then

$$f_t(t,x)=-(r+\frac{1}{2}\theta^2)f(t,x)\};\quad f_x(t,x)=-\theta f(t,x);\quad f_x(t,x)=\theta^2 f(t,x).$$

Applying Ito's formula to $\eta(t)=f(t,W(t))$, we have

$$\begin{aligned}d\zeta(t)&=f_t(t,W(t))dt+f_x(t,W(t))dW(t)+\frac{1}{2}f_{xx}(t,W(t))dW(t)dW(t)\\&=-(r+\frac{1}{2}\theta^2)\zeta(t)dt-\theta\zeta(t)dW(t)+\frac{1}{2}\theta^2\zeta(t)dt\\&=-r\zeta(t)dt-\theta\zeta(t)dW(t).\end{aligned}$$

(ii) Let us apply Ito's product rule to compute $d(\zeta(t)X(t))$

$$\begin{aligned}d(\zeta(t)X(t))&=\zeta(t)dX(t)+X(t)d\zeta(t)+d\zeta(t)dX(t)\\&=\zeta(t)(rX(t)dt+\Delta(t)(\alpha-r)S(t)dt+\Delta(t)\sigma S(t)dW(t))\\&+X(t)(-r\zeta(t)dt-\theta\zeta(t)dW(t))\\&+(-r\zeta(t)dt-\theta\zeta(t)dW(t))(rX(t)dt+\Delta(t)(\alpha-r)S(t)dt+\Delta(t)\sigma S(t)dW(t))\\&=(\alpha-r)\zeta(t)\Delta(t)S(t)dt+\big(\zeta(t)\Delta(t)\sigma S(t)-\theta\zeta(t)X(t)\big)dW(t)+(-\theta\sigma\zeta(t)\Delta(t)S(t))dt\\&=\big(\zeta(t)\Delta(t)\sigma S(t)-\theta\zeta(t)X(t)\big)dW(t),\end{aligned}$$

where the last equality holds since $\theta=\frac{\alpha-r}{\sigma}$ by definition.

$$\begin{aligned}\mathbf{E}[\zeta(t)X(t)|\mathcal{F}(s)]&=\zeta(s)X(s)+\mathbf{E}[\zeta(t)X(t)-\zeta(s)X(s)|\mathcal{F}(s)]\\&=\zeta(s)X(s)+\mathbf{E}[\int_s^t d\big(\zeta(u)X(u)\big)|\mathcal{F}(s)]\\&=\zeta(s)X(s)+\mathbf{E}[\int_s^t\big(\zeta(t)\Delta(t)\sigma S(t)-\theta\zeta(t)X(t)\big)dW(t)|\mathcal{F}(s)]\\&=\zeta(s)X(s)+0.\end{aligned}$$

The last equality holds since the expectation of an Ito integral is zero.

(iii) The investor's portfolio at time T is $X(T) = V(T)$. Since $\zeta(t)X(t)$ is a martingale,

$$\zeta(0)X(0) = \mathbf{E}[\zeta(T)X(T)] = \mathbf{E}[\zeta(T)V(T)]$$

19. (i) $B(t)$ is clearly a martingale.

$$dB(t)dB(t) = \big(\text{sign}(W(t))\big)^2 dW(t)dW(t) = dt.$$

This implies $B(t)$ is a Brownian motion by the Levy's theorem.

(ii)

$$\begin{aligned} d[B(t)W(t)] &= B(t)dW(t) + W(t)dB(t) + dB(t)dW(t) \\ &= B(t)dW(t) + W(t)\text{sign}(W(t))dW(t) + \text{sign}(W(t))dW(t)dW(t) \\ &= B(t)dW(t) + W(t)\text{sign}(W(t))dW(t) + \text{sign}(W(t))dt \end{aligned}$$

Integrating both sides then taking expectation, we have

$$\begin{aligned} \mathbf{E}[B(t)W(t)] &= \mathbf{E}[\int_0^t d(B(u)W(u))] \\ &= \mathbf{E}[\int_0^t \big(B(u) + W(u)\text{sign}(W(u))\big)dW(u) + \int_0^t \text{sign}(W(u))du] \\ &= \mathbf{E}[\int_0^t \text{sign}(W(u))du]. \end{aligned}$$

Since $-W$ is also a Brownian motion,

$$\mathbf{E}[\int_0^t \text{sign}(W(u))du] = \mathbf{E}[\int_0^t \text{sign}(-W(u))du] = -\mathbf{E}[\int_0^t \text{sign}(W(u))du],$$

which implies $\mathbf{E}[\int_0^t \text{sign}(W(u))du] = 0$ and $\mathbf{E}[B(t)W(t)] = 0$.

(iii) Take $f(t,x) = x^2$. Then

$$f_t(t,x) = 0; \quad f_x(t,x) = 2x; \quad f_{xx}(t,x) = 2.$$

We now apply Ito's formula on $f(t, W(t)) = W^2(t)$:

$$dW^2(t) = f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dW(t)dW(t) = 2W(t)dW(t) + dt.$$

(iv) By Ito's product rule,

$$
\begin{aligned}
& d[B(t)W^2(t)] \\
&= B(t)dW^2(t) + W^2(t)dB(t) + dB(t)dW^2(t) \\
&= B(t)\big(2W(t)dW(t) + dt\big) + \operatorname{sign}(W(t))W^2(t)dW(t) + \operatorname{sign}(W(t))dW(t)\big(2W(t)dW(t) + dt\big) \\
&= \big(B(t) + 2|W(t)|\big)dt + \big(2W(t)B(t) + \operatorname{sign}(W(t))W^2(t)\big)dW(t).
\end{aligned}
$$

Integrating both sides and then taking expectation, we have

$$
\begin{aligned}
\mathbf{E}[B(t)W^2(t)] &= \mathbf{E}[\int_0^t d[B(u)W^2(u)]] \\
&= \mathbf{E}[\int_0^t \big(B(u) + 2|W(u)|\big)du + \int_0^t \big(2W(u)B(u) + \operatorname{sign}(W(u))W^2(u)\big)dW(u)] \\
&= \mathbf{E}[\int_0^t \big(B(u) + 2|W(u)|\big)du].
\end{aligned}
$$

Since $B(u)$ is a Brownian motion $-B(u)$ is also a Brownian motion. Therefore,

$$
\mathbf{E}[\int_0^t B(u)du] = \mathbf{E}[\int_0^t -B(u)du],
$$

which implies that

$$
\mathbf{E}[\int_0^t B(u)du] = 0.
$$

Using this and Tonelli's theorem, we have

$$
\mathbf{E}[B(t)W^2(t)] = \mathbf{E}[\int_0^t 2|W(u)|du] = \int_0^t \mathbf{E}[|W(u)|]du = \int_0^t \sqrt{\frac{2u}{\pi}}du = \sqrt{\frac{2}{\pi}}\frac{2}{3}t^{\frac{3}{2}}.
$$

On the other hand,

$$
\mathbf{E}[B(t)] = 0; \quad \mathbf{E}[W^2(t)] = t.
$$

Thus,

$$
\mathbf{E}[B(t)W^2(t)] \neq \mathbf{E}[B(t)] \cdot \mathbf{E}[W^2(t)].
$$

If $B(t)$ and $W(t)$ were independent, $B(t)$ and $W^2(t)$ must be independent, too. This would imply $\mathbf{E}[B(t)W^2(t)] = \mathbf{E}[B(t)] \cdot \mathbf{E}[W^2(t)]$. A contradiction.

20. (i)

$$
f'(x) = \begin{cases} 1 & \text{for } x > K \\ 0 & \text{for } x < K \\ \text{undefined} & \text{for } x = K. \end{cases}
$$

Differentiating once more, we have

$$
f''(x) = \begin{cases} 0 & \text{for } x \neq K \\ \text{undefined} & \text{for } x = K. \end{cases}
$$

(ii) If we substitute $f(x) = (x - K)^+$ into Ito's formula and replacing the $f''(W(t))$ term by zero, the left hand side is

$$(W(T) - K)^+$$

and the right hand side is

$$(W(0) - K)^+ + \int_0^T \mathbf{1}_{(K,\infty)}(W(u))dW(u) = \int_0^T \mathbf{1}_{(K,\infty)}(W(u))dW(u)$$

The expectation of the left hand side is:

$$\mathbf{E}[(W(T) - K)^+] = \frac{1}{\sqrt{2\pi T}} \int_K^\infty (u - K)e^{-\frac{u^2}{2T}} du > 0.$$

$$\mathbf{E}[\int_0^T \mathbf{1}_{(K,\infty)}(W(u))dW(u)] = 0.$$

This implies the two sides of Ito's formula are not equal for $f(x) = (x - K)^+$.

(iii) It is clear that

$$f_n'(x) = \begin{cases} 0 & \text{if } x < K - \frac{1}{2n} \\ n(x - K) + \frac{1}{2} & \text{if } K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ 1 & \text{if } x > K + \frac{1}{2n}. \end{cases}$$

Let us introduce the notation of $g'_+(a)$ and $g'_-(a)$ to denote the right and left derivative of a function g at $x = a$. Then

$$f'_{n,+}(K - \frac{1}{2n}) = n(K - \frac{1}{2n} - K) + \frac{1}{2} = 0; \quad f'_{n,-}(K - \frac{1}{2n}) = 0.$$

$$f'_{n,+}(K + \frac{1}{2n}) = 1; \quad f'_{n,-}(K + \frac{1}{2n}) = n(K + \frac{1}{2n} - K) + \frac{1}{2} = 1.$$

At both points $x = K - \frac{1}{2n}$ and $x = K + \frac{1}{2n}$, the right and left derivative agrees, which means f_n is differentiable at these points and has derivative equal to the right and left derivative, i.e.

$$f_n'(x) = \begin{cases} 0 & \text{if } x \le K - \frac{1}{2n} \\ n(x - K) + \frac{1}{2} & \text{if } K - \frac{1}{2n} \le x \le K + \frac{1}{2n} \\ 1 & \text{if } x \ge K + \frac{1}{2n}. \end{cases}$$

Differentiating f_n' on the open intervals, we have

$$f_n''(x) = \begin{cases} 0 & \text{if } x < K - \frac{1}{2n} \\ n & \text{if } K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ 0 & \text{if } x > K + \frac{1}{2n}. \end{cases}$$

It is clear that f_n'' is not defined at $x = K - \frac{1}{2n}$ and $x = K + \frac{1}{2n}$ since the right and left derivative do not agree there.

(iv) For any $x < K$ fixed, there is N such that for all $n \geq N$, $x \leq K - \frac{1}{2n}$. Therefore, $\lim_{n\to\infty} f_n(x) = 0$. Similarly, for any $x > K$ fixed, there is N such that for all $n \geq N$, $x \geq K + \frac{1}{2n}$. Therefore, $\lim_{n\to\infty} f_n(x) = x - K$.

On the other hand,

$$\lim_{n\to\infty} f_n(K) = \lim_{n\to\infty} \frac{n}{2}(K-K)^2 + \frac{1}{2}(K-K) + \frac{1}{8n} = 0.$$

Putting these together, we have shown that $\lim_{n\to\infty} f_n(x) = (x-K)^+$.

For any $x < K$ fixed, there is N such that for all $n \geq N$, $x \leq K - \frac{1}{2n}$. Therefore, $\lim_{n\to\infty} f_n'(x) = 0$. Similarly, for any $x > K$ fixed, there is N such that for all $n \geq N$, $x \geq K + \frac{1}{2n}$. Therefore, $\lim_{n\to\infty} f_n'(x) = 1$.

On the other hand,

$$\lim_{n\to\infty} f_n'(K) = \lim_{n\to\infty} n(K-K) + \frac{1}{2} = \frac{1}{2}.$$

Putting these together, we have shown that

$$f_n'(x) = \begin{cases} 0 & \text{if } x < K \\ \frac{1}{2} & \text{if } x = K \\ 1 & \text{if } x > K. \end{cases}$$

(v) Say the path of the Brownian motion $W(t)$ stays strictly below K on $[0,T]$, i.e.

$$M := \max_{0\leq t\leq T}\{W(t)\} < K.$$

Then there is N such that for all $n \geq N$, $K - \frac{1}{2n} \geq M$. Thus, for $n \geq N$, $\mathbf{1}_{(K-\frac{1}{2n}, K+\frac{1}{2n})}(W(t)) = 0$ for any $t \in [0,T]$. This implies $L_K(T) = 0$ for this path.

(vi) Since the integrand $n \cdot \mathbf{1}_{(K-\frac{1}{2n}, K+\frac{1}{2n})}(W(t))$ is always greater than or equal to zero for all t. The limit of their integrals $L_K(T)$ must be greater than or equal to zero as well.

From part (i),

$$\mathbf{E}[L_K(T)] = \mathbf{E}[(W(T)-K)^+] - \mathbf{E}[\int_0^T \mathbf{1}_{(K,\infty)}(W(t))dW(t)] > 0.$$

These two facts imply that $\mathbf{P}(L_K(T) > 0) > 0$.

21. (i) When the price of the asset rises to K, we may not be able to buy quickly enough; likewise, when the asset price falls to K, we may not sell quickly enough.

(ii) Being an Ito integral, X is a martingale. In particular, $\mathbf{E}[X(T)] = X(0) = 0$. But $\mathbf{E}[(S(T)-K)^+] > 0$. Therefore, $X(T) = (S(T)-K)^+$ cannot hold.