Chapter 5 solutions to Stochastic Calculus for Finance I by Steven E. Shreve

Chung Ching Lau

1. (i)

$$\begin{aligned}
\mathbf{E}[\alpha^{\tau_2}] &= \mathbf{E}[\alpha^{(\tau_2-\tau_1)}\alpha^{\tau_1}] \\
&= \mathbf{E}[\alpha^{(\tau_2-\tau_1)}]\cdot\mathbf{E}[\alpha^{\tau_1}] \quad \text{(independence of } (\tau_2-\tau_1) \text{ and } \tau_1) \\
&= \mathbf{E}[\alpha^{\tau_1}]^2 \quad ((\tau_2-\tau_1) \text{ and } \tau_1 \text{ have the same distribution.)}
\end{aligned}$$

(ii) The argument is similar:

$$\begin{aligned}
\mathbf{E}[\alpha^{\tau_m}] &= \mathbf{E}[(\Pi_{i=1}^{m-1}\alpha^{(\tau_{i+1}-\tau_i)})\cdot\alpha^{\tau_1}] \\
&= \Pi_{i=1}^{m-1}(\mathbf{E}[\alpha^{(\tau_{i+1}-\tau_i)}])\cdot\mathbf{E}[\alpha^{\tau_1}] \quad \text{(independence of } (\tau_{i+1}-\tau_i) \text{ and } \tau_i \text{ for } 1\le i\le m-1.) \\
&= \mathbf{E}[\alpha^{\tau_1}]^m \quad ((\tau_{i+1}-\tau_i) \text{ and } \tau_1 \text{ have the same distribution.)}
\end{aligned}$$

(iii) Note that we have not used symmetry of the random walk. Therefore the equality still holds for asymmetric random walk.

2. (i) For $\sigma > 0$

$$\begin{aligned}
f'(\sigma) &= pe^{\sigma} - qe^{-\sigma} \\
&> pe^{\sigma} - pe^{-\sigma} && (p > q) \\
&> 0 && (e^{\sigma} > e^{-\sigma} \text{ for } \sigma > 0.)
\end{aligned}$$

Therefore f is strictly increasing for $\sigma > 0$. Thus, $f(\sigma) > f(0) = 1$ for $\sigma > 0$.

(ii)

$$\begin{aligned}
\mathbf{E}_n[S_{n+1}] &= \mathbf{E}_n[e^{\sigma M_n}e^{\sigma X_{n+1}}(\frac{1}{f(\sigma)^n})(\frac{1}{f(\sigma)})] \\
&= S_n\cdot\frac{1}{f(\sigma)}\cdot\mathbf{E}_n[e^{\sigma X_{n+1}}] \\
&= S_n\cdot\frac{1}{f(\sigma)}\cdot(pe^{\sigma}+qe^{-\sigma}) \\
&= S_n.
\end{aligned}$$

(iii)

$$\begin{aligned}
1 &= S_0 \\
&= \mathbf{E}[S_{\tau_1\wedge n}] && \text{(A martingale stopped at a stopping time is a martingale.)} \\
&= \mathbf{E}[e^{\sigma M_{\tau_1\wedge n}}(\frac{1}{f(\sigma)^{\tau_1\wedge n}})].
\end{aligned}$$

Since $f(\sigma) > 1$,

$$\lim_{n\to\infty}\frac{1}{f(\sigma)^{\tau_1\wedge n}} = \begin{cases}\frac{1}{f(\sigma)^{\tau_1}} & \tau_1 = \infty \\ 0 & \tau_1 < \infty.\end{cases} \tag{1}$$

Since the process $M_{\tau_1 \wedge n}$ is stopped once it reaches 1,

$$M_{\tau_1 \wedge n} \leq 1.$$

Therefore,

$$e^{\sigma M_{\tau_1 \wedge n}} \leq e^{\sigma}. \tag{2}$$

Using (1) and (2) and the fact that $M_{\tau_1} = 1$ if $\tau_1 < \infty$,

$$\lim_{n\to\infty} e^{\sigma M_{\tau_1 \wedge n}}(\frac{1}{f(\sigma)^{\tau_1 \wedge n}}) = \mathbf{1}_{\tau_1<\infty} e^{\sigma M_{\tau_1}}(\frac{1}{f(\sigma)^{\tau_1}}) = \mathbf{1}_{\tau_1<\infty} e^{\sigma}(\frac{1}{f(\sigma)^{\tau_1}}).$$

Using (2) again,

$$e^{\sigma M_{\tau_1 \wedge n}}(\frac{1}{f(\sigma)^{\tau_1 \wedge n}}) \leq e^{\sigma}.$$

Then dominated convergence theorem implies

$$1 = \lim_{n\to\infty} \mathbf{E}[e^{\sigma M_{\tau_1 \wedge n}}(\frac{1}{f(\sigma)^{\tau_1 \wedge n}})] = \mathbf{E}[\lim_{n\to\infty} e^{\sigma M_{\tau_1 \wedge n}}(\frac{1}{f(\sigma)^{\tau_1 \wedge n}})] = \mathbf{E}[\mathbf{1}_{\tau_1<\infty} e^{\sigma}(\frac{1}{f(\sigma)^{\tau_1}})].$$

In other words,

$$e^{-\sigma} = \mathbf{E}[\mathbf{1}_{\tau_1<\infty}\frac{1}{f(\sigma)^{\tau_1}}]. \tag{3}$$

$$\begin{aligned}
1 &= \lim_{\sigma\to 0^+} e^{-\sigma} \\
&= \lim_{\sigma\to 0^+} \mathbf{E}[\mathbf{1}_{\tau_1<\infty}\frac{1}{f(\sigma)^{\tau_1}}] && \text{(by (3))} \\
&= \mathbf{E}[\lim_{\sigma\to 0^+} \mathbf{1}_{\tau_1<\infty}\frac{1}{f(\sigma)^{\tau_1}}] && \text{(by dominated convergence theorem)} \\
&= \mathbf{E}[\mathbf{1}_{\tau_1<\infty}].
\end{aligned}$$

(iv) We let $\alpha = \frac{1}{f(\sigma)}$ and solve for $e^{-\sigma}$ in terms of α. From

$$\begin{aligned}
\alpha &= \frac{1}{f(\sigma)} \\
&= \frac{1}{pe^{\sigma} + qe^{-\sigma}}
\end{aligned}$$

We have the quadratic equation in $e^{-\sigma}$

$$q\alpha e^{-2\sigma} - e^{-\sigma} + p\alpha = 0.$$

Solving it, we have

$$e^{-\sigma} = \frac{1 \pm \sqrt{1 - 4pq\alpha^2}}{2q\alpha}. \tag{4}$$

Letting $\sigma \to 0$ and therefore $\alpha \to 1$ on both sides of the above equation, we have

$$\lim_{\sigma \to 0} e^{-\sigma} = 1.$$

$$\lim_{\alpha \to 1} \frac{1 + \sqrt{1 - 4pq\alpha^2}}{2q\alpha} = \frac{1-q}{q}.$$

$$\lim_{\alpha \to 1} \frac{1 - \sqrt{1 - 4pq\alpha^2}}{2q\alpha} = 1.$$

For the equality in (4) to hold as $\sigma \to 0$,

$$e^{-\sigma} = \frac{1 - \sqrt{1 - 4pq\alpha^2}}{2q\alpha}.$$

We then apply (3) to get

$$\begin{aligned} \frac{1 - \sqrt{1 - 4pq\alpha^2}}{2q\alpha} &= \mathbf{E}[\mathbf{1}_{\tau_1 < \infty} \alpha^{\tau_1}] && \text{(by (3))} \\ &= \mathbf{E}[\alpha^{\tau_1}] && (\alpha^{\tau_1} = 0 \text{ when } \tau = \infty) \end{aligned}$$

(v) Differentiating both sides of $\mathbf{E}[\alpha^{\tau_1}] = \frac{1 - \sqrt{1 - 4pq\alpha^2}}{2q\alpha}$ with respect to α, we have

$$\begin{aligned} \mathbf{E}[\tau_1 \alpha^{\tau_1 - 1}] &= \frac{d}{d\alpha} \mathbf{E}[\alpha^{\tau_1}] \\ &= \frac{d}{d\alpha} \frac{1 - \sqrt{1 - 4pq\alpha^2}}{2q\alpha} \\ &= \frac{2q\alpha(4pq\alpha(1 - 4pq\alpha^2)^{-\frac{1}{2}}) - (1 - (1 - 4pq\alpha^2)^{\frac{1}{2}})2q}{(2q\alpha)^2} \\ &= \frac{(1 - 4pq\alpha^2)^{-\frac{1}{2}} - 1}{2q\alpha^2} \end{aligned}$$

Letting $\alpha \to 1^-$, we get

$$\mathbf{E}[\tau_1] = \frac{1}{1 - 2q}.$$

3. (i) We solve for $1 = f(\sigma) = pe^{\sigma} + qe^{-\sigma}$. It is equivalent to solving the following quadratic equation in e^{σ}:

$$pe^{2\sigma} - e^{\sigma} + q = 0.$$

The solutions are:

$$e^{\sigma} = \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm (1-2p)}{2p} = \begin{cases} \frac{q}{p} & \text{or} \\ 1 \end{cases}$$

Therefore the solutions to $1 = f(\sigma)$ are $\sigma = \ln \frac{q}{p}$ and $\sigma = 0$. We compute the derivative of f for $\sigma > \ln \frac{q}{p} > 0$,

$$\begin{aligned} f'(\sigma) &= pe^{\sigma} - qe^{-\sigma} \\ &> p(\frac{q}{p}) - qe^{-\sigma} && (e^{\sigma} < \frac{q}{p}) \\ &> 0 && (\sigma > 0 \Rightarrow e^{-\sigma} < 1). \end{aligned}$$

Therefore $f(\ln \frac{q}{p}) = 1$ and $f(\sigma) > 1$ for $\sigma > \ln \frac{q}{p}$.

(ii) Repeating part ii and the earlier part of iii of exercise 2 while letting $\sigma > \ln \frac{q}{p} > 0$, we still have

$$e^{-\sigma} = \mathbf{E}[\mathbf{1}_{\tau_1 < \infty} \frac{1}{f(\sigma)^{\tau_1}}].$$

Letting $\sigma \to \ln \frac{q}{p}$ in the above equation and applying dominated convergence theorem, we have

$$\frac{p}{q} = \mathbf{E}[\mathbf{1}_{\tau_1 < \infty}] < 1.$$

(iii) We let $\alpha = \frac{1}{f(\sigma)}$ and solve for σ in terms of α.

$$e^{-\sigma} = \frac{1 \pm \sqrt{1-4pq\alpha^2}}{2q\alpha}.$$

Letting $\sigma \to \ln \frac{q}{p}$ and therefore $\alpha \to 1$ in the above equation, we have

$$\lim_{\alpha \to 1} \frac{1 + \sqrt{1-4pq\alpha^2}}{2q\alpha} = \frac{1 + \sqrt{1-4pq}}{2q} = \frac{1-q}{q}$$

$$\lim_{\alpha \to 1} \frac{1 - \sqrt{1-4pq\alpha^2}}{2q\alpha} = \frac{1 - \sqrt{1-4pq}}{2q} = 1$$

$$\lim_{\sigma \to \ln \frac{q}{p}} e^{-\sigma} = \frac{1-q}{q}.$$

For equality to hold as $\sigma \to \ln \frac{q}{p}$,

$$e^{-\sigma} = \frac{1 + \sqrt{1-4pq\alpha^2}}{2q\alpha}.$$

And

$$\mathbf{E}[\alpha^{\tau_1}] = \frac{1 + \sqrt{1-4pq\alpha^2}}{2q\alpha} \tag{5}$$

(iv) Since $\mathbf{P}[\tau_1 = \infty] > 0$, $\mathbf{E}[\tau_1] = \infty$.

4. (i) Taking the coefficient of the α^{2k} term in

$$\sum_{k=0}^{\infty} \alpha^{2k} \mathbf{P}[\tau_2 = 2k] = \mathbf{E}[\alpha^{\tau_2}] = \sum_{k=0}^{\infty} (\frac{\alpha}{2})^{2k} \frac{(2k)!}{(k+1)!k!},$$

we have

$$\mathbf{P}[\tau_2 = k] = \frac{1}{2^{2k}} \frac{(2k)!}{(k+1)!k!}.$$

(ii) Let $k \geq 1$.

$$\begin{aligned} \mathbf{P}[\tau_2 \leq 2k] &= \mathbf{P}[M_{2k} = 2] + 2\mathbf{P}[M_{2k} \geq 4] \qquad \text{(Reflection principle)} \\ &= 1 - (\mathbf{P}[M_{2k} = 0] + \mathbf{P}[M_{2k} = 2]). \end{aligned}$$

Plugging in $k-1$ into the above equality, we have

$$\mathbf{P}[\tau_2 \leq 2k-2] = 1 - (\mathbf{P}[M_{2k-2} = 0] + \mathbf{P}[M_{2k-2} = 2])$$

$$\begin{aligned} \mathbf{P}[\tau_2 = 2k] &= \mathbf{P}[\tau_2 \leq 2k] - \mathbf{P}[\tau_2 \leq 2k-2] \\ &= 1 - (\mathbf{P}[M_{2k} = 0] + \mathbf{P}[M_{2k} = 2]) - \big(1 - (\mathbf{P}[M_{2k-2} = 0] + \mathbf{P}[M_{2k-2} = 2])\big) \\ &= (\mathbf{P}[M_{2k-2} = 0] + \mathbf{P}[M_{2k-2} = 2]) - (\mathbf{P}[M_{2k} = 0] + \mathbf{P}[M_{2k} = 2]) \\ &= \frac{1}{2^{2k-2}} (\frac{(2k-2)!}{(k-1)!(k-1)!} + \frac{(2k-2)!}{k!(k-2)!}) - \frac{1}{2^{2k}} (\frac{(2k)!}{k!k!} + \frac{(2k)!}{(k+1)!(k-1)!}) \\ &= \frac{1}{2^{2k}} \frac{(2k-2)!}{(k+1)!k!} \big(4(k+1)k^2 + 4(k+1)k(k-1) - 2k(2k-1)(k+1) - 2k(2k-1)k\big) \\ &= \frac{1}{2^{2k}} \frac{(2k-2)!}{(k+1)!k!} \big(2(k+1)2k(2k-1) - 2k(2k-1)(2k+1)\big) \\ &= \frac{1}{2^{2k}} \frac{(2k)!}{(k+1)!k!}. \end{aligned}$$

5. (i) Consider the set of paths that reaches m at some time $t = k, 1 \leq k \leq n$ and ends up at $b \leq m$ at $t = n$. We reflect these paths after the first passage time τ_m, i.e.

$$M_i' := \begin{cases} M_i & \text{if } i \leq \tau_m \\ 2m - M_i & \text{else if } i > \tau_m. \end{cases}$$

Then $M_n' = 2m - b > m$. Conversely, we can recover the set of aforementioned paths by reflecting the paths that ends at strictly above m over the portion after the first passage time τ_m. Therefore,

$$\mathbf{P}[M_n^* \geq m, M_n = b] = \mathbf{P}[M_n = 2m - b]$$

To reach $2m - b$ at time n,

$$\#\text{heads} - \#\text{tails} = (2m - b).$$

Since the length of the path is n,

$$\#\text{heads} + \#\text{tails} = n$$

Solving this system of linear equations, we have

$$\#\text{heads} = \frac{n + 2m - b}{2}; \quad \#\text{tails} = \frac{n - (2m - b)}{2}$$

Therefore, the number of distinct paths that ends at $2m - b$ equals $\frac{n!}{(\frac{n+2m-b}{2})!(\frac{n-(2m-b)}{2})!}$, and

$$\mathbf{P}[M_n = 2m - b] = (\frac{1}{2})^n \frac{n!}{(\frac{n+2m-b}{2})!(\frac{n-(2m-b)}{2})!}$$

(ii) Since the number of distinct paths that ends at $2m - b$ equals $\frac{n!}{(\frac{n+2m-b}{2})!(\frac{n-(2m-b)}{2})!}$ and probability of getting each of these paths is $p^{\#\text{heads}} q^{\#\text{tails}}$,

$$\mathbf{P}[M_n^* \geq m, M_n = b = \mathbf{P}[M_n = 2m - b] = (p^{\frac{n+2m-b}{2}} q^{\frac{n-(2m-b)}{2}}) \frac{n!}{(\frac{n+2m-b}{2})!(\frac{n-(2m-b)}{2})!}.$$

6. We first compute the price of an American put that expires at $t = 1$.

$$v_0(4) = \max\{(4-4)^+, \frac{4}{5}(\frac{1}{2}v_1(8) + \frac{1}{2}v_1(2))\} = \frac{2}{5}((4-8)^+ + (4-2)^+) = \frac{4}{5}$$

For expiry time $= 3$, we first compute the value of the put at $t = 3$:

$$v_3(32) = (4-32)^+ = 0; \quad v_3(8) = (4-8)^+ = 0; \quad v_3(2) = (4-2)^+ = 2; \quad v_3(\frac{1}{2}) = (4 - \frac{1}{2})^+ = \frac{7}{2}$$

We then compute the value of the put at $t = 2$:

$$v_2(16) = \max\{(4-16)^+, \frac{4}{5}(\frac{1}{2}v_3(32) + \frac{1}{2}v_3(8))\} = 0;$$

$$v_2(4) = \max\{(4-4)^+, \frac{4}{5}(\frac{1}{2}v_3(8) + \frac{1}{2}v_3(2))\} = \frac{4}{5};$$

$$v_2(1) = \max\{(4-1)^+, \frac{4}{5}(\frac{1}{2}v_3(2) + \frac{1}{2}v_3(\frac{1}{2}))\} = 3$$

We then compute the value of the put at $t = 1$:

$$v_1(8) = \max\{(4-8)^+, \frac{4}{5}(\frac{1}{2}v_2(16) + \frac{1}{2}v_2(4))\} = \frac{8}{25};$$

$$v_1(2) = \max\{(4-2)^+, \frac{4}{5}(\frac{1}{2}v_2(4) + \frac{1}{2}v_2(1))\} = 2$$

Finally, the time zero value of the put is:

$$v_0(4) = \max\{(4-4)^+, \frac{4}{5}(\frac{1}{2}v_1(8) + \frac{1}{2}v_1(2))\} = \frac{116}{125} = 0.928.$$

For expiry time $=5$, we first compute the value of the put at $t=5$:

$$v_5(128) = v_5(32) = v_5(8) = 0; \quad v_5(2) = 2; \quad v_5(\frac{1}{2}) = \frac{7}{2}; \quad v_5(\frac{1}{8}) = \frac{31}{8}.$$

For $t=4$,

$$v_4(64) = v_4(16) = 0; \quad v_4(4) = \frac{4}{5}; \quad v_4(1) = 3; \quad v_4(\frac{1}{4}) = \frac{15}{4}.$$

For $t=3$,

$$v_3(32) = 0; \quad v_3(8) = \frac{8}{25}; \quad v_3(2) = 2; \quad v_3(\frac{1}{2}) = \frac{7}{2}.$$

For $t=2$,

$$v_2(16) = \frac{16}{125}; \quad v_2(4) = \frac{116}{125}; \quad v_2(1) = 3.$$

For $t=1$,

$$v_1(8) = \frac{264}{625}; \quad v_1(2) = 2.$$

For $t=0$,

$$v_0(4) = \frac{3028}{3125} = 0.96896.$$

7. (i) We have the consumption function expressed in terms of the price function:

$$c(s) = v(s) - \frac{2}{5}(v(2s) + v(\frac{s}{2})). \tag{6}$$

and the price function is already computed explicity for $s = 2^j$.

$$v(2^j) = \begin{cases} \frac{4}{2^j} & j \geq 1 \\ 4 - 2^j & j \leq 1. \end{cases} \tag{7}$$

Plugging in (7) into (6), we have for $j \geq 2$,

$$\begin{aligned} c(2^j) &= \frac{4}{2^j} - \frac{2}{5}(\frac{4}{2^{j+1}} + \frac{4}{2^{j-1}}) \\ &= \frac{4}{2^j} - \frac{2}{5}(\frac{2}{2^j} + \frac{4 \cdot 2}{2^j}) \\ &= 0. \end{aligned}$$

For $j=1$,

$$\begin{aligned} c(2) &= 2 - \frac{2}{5}(1+3) \\ &= \frac{2}{5}. \end{aligned}$$

For $j \leq 0$,

$$\begin{aligned} c(2^j) &= (4 - 2^j) - \frac{2}{5}((4 - 2^{j+1}) + (4 - 2^{j-1})) \\ &= \frac{4}{5}. \end{aligned}$$

Summarizing,

$$c(2^j) = \begin{cases} 0 & j \geq 2 \\ \frac{2}{5} & j = 1 \\ \frac{4}{5} & j \leq 0. \end{cases}$$

(ii) Plugging in (7) into

$$\delta(s) = \frac{v(2s) - v(\frac{s}{2})}{2s - \frac{s}{2}}, \tag{8}$$

we have for $j \geq 2$,

$$\begin{aligned} \delta(2^j) &= \frac{\frac{4}{2^{j+1}} - \frac{4}{2^{j-1}}}{\frac{3 \cdot 2^j}{2}} \\ &= \frac{-4}{2^{2j}}. \end{aligned}$$

For $j = 1$,

$$\delta(2) = \frac{-2}{3}.$$

For $j \leq 0$,

$$\begin{aligned} \delta(2^j) &= \frac{4 - 2^{j+1} - (4 - 2^{j-1})}{\frac{3 \cdot 2^j}{2}} \\ &= -1. \end{aligned}$$

Summarizing,

$$\delta(2^j) = \begin{cases} \frac{-4}{2^{2j}} & j \geq 2; \\ \frac{-2}{3} & j = 1; \\ -1 & j \leq 0. \end{cases}$$

(iii) Assume that the value of the hedging portfolio equals to the value of the perpetual put at time n, i.e. $X_n = v(2^j)$. We want to check that the value of the hedging portfolio must equal to the value of the put at time $n+1$ regardless of the outcome of the coin toss. The value of the hedging portfolio at time $n+1$:

$$X_{n+1} = \delta_n(S_n) S_{n+1} + (1+r)(X_n - c(S_n) - \delta_n(S_n) S_n)$$

Let us first work out the case when $j \geq 2$. If $\omega_{n+1} = H$,

$$\begin{aligned} X_{n+1}(H) &= \frac{-4}{2^{2j}} \cdot 2^{j+1} + \frac{5}{4}\left(\frac{4}{2^j} - 0 - \frac{-4}{2^{2j}} \cdot 2^j\right) \\ &= \frac{-8}{2^j} + \frac{5}{2^j} + \frac{5}{2^j} \\ &= \frac{4}{2^{j+1}} = v(2^{j+1}). \end{aligned}$$

Similarly, if $\omega_{n+1} = T$,

$$\begin{aligned}
X_{n+1}(T) &= \frac{-4}{2^{2j}} \cdot 2^{j-1} + \frac{5}{4}\left(\frac{4}{2^j} - 0 - \frac{-4}{2^{2j}} \cdot 2^j\right) \\
&= \frac{-2}{2^j} + \frac{5}{2^j} + \frac{5}{2^j} \\
&= \frac{4}{2^{j-1}} = v(2^{j-1}).
\end{aligned}$$

Now for the case when $j = 1$. If $\omega_{n+1} = H$,

$$\begin{aligned}
X_{n+1}(H) &= \frac{-2}{3} \cdot 4 + \frac{5}{4}\left(2 - \frac{2}{5} - \frac{-2}{3} \cdot 2\right) \\
&= \frac{-8}{3} + \frac{5}{4}\left(\frac{30 - 6 + 20}{15}\right) \\
&= 1 = v(4).
\end{aligned}$$

If $\omega_{n+1} = T$,

$$\begin{aligned}
X_{n+1}(T) &= \frac{-2}{3} \cdot 1 + \frac{5}{4}\left(2 - \frac{2}{5} - \frac{-2}{3} \cdot 2\right) \\
&= \frac{-2}{3} + \frac{11}{3} \\
&= 3 = v(1).
\end{aligned}$$

Finally for the case when $j \leq 0$. If $\omega_{n+1} = H$,

$$\begin{aligned}
X_{n+1}(H) &= -1 \cdot 2^{j+1} + \frac{5}{4}\left((4 - 2^j) - \frac{4}{5} - (-1) \cdot 2^j\right) \\
&= -2^{j+1} + 4 \\
&= v(2^{j+1}).
\end{aligned}$$

If $\omega_{n+1} = T$,

$$\begin{aligned}
X_{n+1}(T) &= -1 \cdot 2^{j-1} + \frac{5}{4}\left((4 - 2^j) - \frac{4}{5} - (-1) \cdot 2^j\right) \\
&= -2^{j-1} + 4 \\
&= v(2^{j-1}).
\end{aligned}$$

8. (i) Clearly,

$$v(S_n) = S_n > (S_n - K)^+ = g(S_n).$$

Since $\frac{1}{(1+r)^n} S_n$ is a martingale, $\frac{1}{(1+r)^n} v(S_n) = \frac{1}{(1+r)^n} S_n$ is a martingale.

(ii) If the purchaser execute the call at time n, her payoff is $S_n - K$. The risk neutral expectation of the payoff at time zero is

$$\begin{aligned}
\tilde{\mathbf{E}}\left[\frac{1}{(1+r)^n}(S_n - K)\right] &= \tilde{\mathbf{E}}\left[\frac{1}{(1+r)^n} S_n\right] - \frac{1}{(1+r)^n} K \\
&= S_0 - \frac{1}{(1+r)^n} K \qquad \left(\frac{1}{(1+r)^n} S_n \text{ is a martingale.}\right)
\end{aligned}$$

(iii) Let us check that $v(s) = s$ satisfies the Bellman equation.

$$\begin{aligned}\max\{g(s), \frac{1}{1+r}(\tilde{p}v(us) + \tilde{q}v(ds))\} &= \max\{s-K, \frac{1}{1+r}(\frac{(1+r-d)us + (u-1-r)ds}{u-d})\} \\ &= \max\{s-K, \frac{1}{1+r}(\frac{(1+r)(us-ds)}{u-d})\} \\ &= s = v(s).\end{aligned}$$

Now let us prove that $v(s) = s$ satisfies the boundary conditions:

$$\lim_{s\to 0} v(s) = 0 \tag{9}$$

and

$$\lim_{s\to\infty} \frac{v(s)}{s} = 1. \tag{10}$$

For (9),

$$\lim_{s\to 0} v(s) = \lim_{s\to 0} s = 0.$$

For (10),

$$\lim_{s\to\infty} \frac{v(s)}{s} = \lim_{s\to\infty} \frac{s}{s} = \lim_{s\to\infty} 1 = 1$$

(iv) Let τ^* be an optimal stopping time for the perpetual American call. Then

$$\begin{aligned}\tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{S_{\tau^*}-K}{(1+r)^{\tau^*}}] &\geq \tilde{\mathbf{E}}[\frac{S_n-K}{(1+r)^n}] \qquad \text{(consider the constant stopping time } \tau = n) \\ &= S_0 - \frac{K}{(1+r)^n}\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have

$$\tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{S_{\tau^*}-K}{(1+r)^{\tau^*}}] \geq S_0. \tag{11}$$

In particular, $\mathbf{P}[\mathbf{1}_{\tau^*<\infty}] > 0$. Therefore,

$$\tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{K}{(1+r)^{\tau^*}}] > 0,$$

which implies

$$\tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{S_{\tau^*}-K}{(1+r)^{\tau^*}}] < \tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{S_{\tau^*}}{(1+r)^{\tau^*}}]. \tag{12}$$

We consider the latter term.

$$\begin{aligned}\tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{S_{\tau^*}}{(1+r)^{\tau^*}}] &\leq \liminf_{n\to\infty} \tilde{\mathbf{E}}[\mathbf{1}_{\tau^*<\infty}\frac{S_{\tau^*\wedge n}}{(1+r)^{\tau^*\wedge n}}] \qquad \text{(by Fatou's lemma)} \\ &\leq \lim_{n\to\infty} \tilde{\mathbf{E}}[\frac{S_{\tau^*\wedge n}}{(1+r)^{\tau^*\wedge n}}] \\ &= S_0 \qquad (\frac{S_{\tau^*\wedge n}}{(1+r)^{\tau^*\wedge n}} \text{ is a martingale.})\end{aligned}$$

This contradicts with (11) and (12).

9. (i) Plugging in $v(s) = s^p$ into the equation

$$v(s) = \frac{2}{5}v(2s) + \frac{2}{5}v(\frac{s}{2}), \tag{13}$$

we have

$$\begin{aligned} s^p &= \frac{2}{5}(2s)^p + \frac{2}{5}(\frac{s}{2})^p \\ \Leftrightarrow s^p &= \frac{2}{5}s^p \cdot (2^p + \frac{1}{2^p}) \\ \Leftrightarrow 0 &= 2 \cdot 2^{2p} - 5 \cdot 2^p + 2. \end{aligned}$$

Solving this quadratic equation in 2^p, we have

$$\begin{aligned} 2^p &= \frac{5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} \\ &= \frac{5 \pm 3}{4} \\ &= 2 \quad \text{or} \quad \frac{1}{2}. \end{aligned}$$

Therefore, $p = 1$ or $p = -1$ and $v_1(s) = s$ and $v_2(s) = \frac{1}{s}$ are solutions to (13).

(ii) The general solution to (13) is given by

$$v(s) = As + \frac{B}{s}.$$

Note that $v(s)$ must satisfy the boundary condition $\lim_{s\to\infty} v(s) = 0$. If $A \neq 0$,

$$\lim_{s\to\infty}(As + \frac{B}{s}) = \operatorname{sgn}(A)\infty.$$

This forces $A = 0$.

(iii) We want to find the range of B such that $f_B(s) = \frac{B}{s} - (4 - s) = 0$ admits solutions in s such that $s > 0$.

$$\begin{aligned} \frac{B}{s} - (4 - s) &= 0 \\ \Leftrightarrow s^2 - 4s + B &= 0 \end{aligned}$$

For the above quadratic equation to admit real solution, the discriminant

$$\begin{aligned} \Delta = (-4)^2 - 4B &\geq 0 \\ B &\leq 4. \end{aligned}$$

In that case, both roots $s = \frac{4 \pm \sqrt{16 - 4B}}{2}$ are positive.

(iv) Take $B = 4$ and $s_B = 2$ be the unique root of the equation $f_B(s) = 0$ maximizes $v_B(s)$.

(v) $v_B'(s_B) = -\frac{B}{s_B^2} = -\frac{4}{2^2} = -1$.