Chapter 5 solutions to Stochastic Calculus for Finance II by Steven E. Shreve

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1. (i) Note that

$$f'(x) = f(x); \quad f''(x) = f(x).$$

$$\begin{aligned} df\big(X(t)\big) &= f'\big(X(t)\big)d\big(X(t)\big) + \frac{1}{2}f''\big(X(t)\big)d\big(X(t)\big)d\big(X(t)\big) \\ &= f\big(X(t)\big)\cdot\big(\sigma(t)dW(t) + (\alpha(t) - R(t) - \frac{1}{2}\sigma^2(t))dt\big) + \frac{1}{2}f\big(X(t)\big)\sigma^2(t)dt \\ &= D(t)S(t)\cdot\big(\sigma(t)dW(t) + (\alpha(t) - R(t))dt\big) \end{aligned}$$

(ii)

$$dD(t) = -R(t)D(t)dt; \quad dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t).$$

By Ito's product rule,

$$\begin{aligned} d(D(t)S(t)) &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\ &= -S(t)R(t)D(t)dt + D(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) \\ &= D(t)S(t)\Bigg(\big(\alpha(t) - R(t)\big)dt + \sigma(t)dW(t)\Bigg), \end{aligned}$$

which agrees with the computation done in part (i).

2. The usual risk neutral pricing formula has the form

$$D(t)V(t) = \widetilde{\mathbf{E}}[D(T)V(T)\big|\mathcal{F}(t)].$$

Applying Lemma 5.2.2 to the right hand side of the equation, we have

$$D(t)V(t) = \widetilde{\mathbf{E}}[D(T)V(T)\big|\mathcal{F}(t)] = \frac{1}{Z(t)}\mathbf{E}[D(T)V(T)Z(T)\big|\mathcal{F}(t)].$$

3. (i) Let $h(s) := (s-K)^+$. Then the expected payoff can be written as

$$c(0,x) = \widetilde{\mathbf{E}}[e^{-rT}h\big(x\exp\{\sigma\widetilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\}\big)].$$

Differentiating with respect to x, we have

$$\begin{aligned} c_x(0,x) &= \widetilde{\mathbf{E}}[e^{-rT}\mathbf{1}_{(K,\infty)}\big(x\exp\{\sigma\widetilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\}\big)\cdot\exp\{\sigma\widetilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\}] \\ &= \widetilde{\mathbf{E}}[\mathbf{1}_{(K,\infty)}\big(x\exp\{\sigma\widetilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\}\big)\cdot\exp\{\sigma\widetilde{W}(T) - \frac{1}{2}\sigma^2T\}] \end{aligned}$$

(ii) Let $\Theta(t) = -\sigma$ and let

$$Z(t) = \exp\{-\int_0^t \Theta(u)d\widetilde{W}(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\} = \exp\{\sigma\widetilde{W}(t) - \frac{1}{2}\sigma^2 t\}.$$

Then by Girsanov's Theorem,

$$\widehat{\mathbf{P}}(A) := \int_A Z(\omega)d\widetilde{\mathbf{P}}(\omega) \text{ for all } A \in \mathcal{F}$$

is a probability measure and the process

$$\widehat{W}(t) := \widetilde{W}(t) - \sigma t$$

is a Brownian motion. Under this measure, $c_x(0,x)$ can be written as

$$c_x(0,x) = \widetilde{\mathbf{E}}[\mathbf{1}_{(K,\infty)}\big(S(T)\big)Z] = \widehat{\mathbf{E}}[\mathbf{1}_{(K,\infty)}\big(S(T)\big)] = \widehat{\mathbf{P}}[S(T) > K]$$

(iii) Rewriting $S(T)$ in terms of $\widehat{W}(T)$, we have

$$S(T) = x\exp\{\sigma\widehat{W}(T) + (r + \frac{1}{2}\sigma^2)T\}.$$

Therefore,

$$\begin{aligned} S(T) > K &\iff x\exp\{\sigma\widehat{W}(T) + (r + \frac{1}{2}\sigma^2)T\} > K \\ &\iff \sigma\widehat{W}(T) + (r + \frac{1}{2}\sigma^2)T > -\log\frac{x}{K} \\ &\iff -\frac{\widehat{W}(T)}{\sqrt{T}} < d_+(T,x), \end{aligned}$$

where $d_+(T,x) := \frac{1}{\sigma\sqrt{T}}[\log\frac{x}{K} + (r + \frac{1}{2}\sigma^2)T]$. This implies that

$$\widehat{\mathbf{P}}[S(T) > K] = \widehat{\mathbf{P}}[-\frac{\widehat{W}(T)}{\sqrt{T}} < d_+(T,x)].$$

Since $\widehat{W}$ is a Brownian motion, $-\frac{\widehat{W}(T)}{\sqrt{T}}$ is a standard normal distribution. Therefore,

$$\widehat{\mathbf{P}}[-\frac{\widehat{W}(T)}{\sqrt{T}} < d_+(T,x)] = N(d_+(T,x)).$$

4. (i) By Ito's formula,

$$\begin{aligned} d\log(S(t)) &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}dS(t)dS(t) \\ &= \frac{1}{S(t)}(r(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)) - \frac{1}{2S^2(t)}\sigma^2(t)S^(t)dt \\ &= \sigma(t)d\widetilde{W}(t) + (r(t) - \frac{1}{2}\sigma^2(t))dt \end{aligned}$$

This implies that

$$\log\big(\frac{S(T)}{S(0)}\big) = \int_0^T d\log(S(t)) = \int_0^T \sigma(t)d\widetilde{W}(t) + \int_0^T (r(t) - \frac{1}{2}\sigma^2(t))dt$$

In other word, if we let

$$X = \int_0^T \sigma(t)d\widetilde{W}(t) + \int_0^T (r(t) - \frac{1}{2}\sigma^2(t))dt,$$

then $S(T) = S(0)e^X$. Since $r(t)$ and $\sigma(t)$ are nonrandom, the second term in the definition of X, $\int_0^T (r(t) - \frac{1}{2}\sigma^2(t))dt$ is a constant.

Let $Y(t) := \int_0^t \sigma(s)d\widetilde{W}(s)$. Fix a number u and let

$$f(t,x) = \exp\{ux - \frac{1}{2}u^2 \int_0^t \sigma^2(s)ds\}.$$

We now compute the differential of $f(t, Y(t))$.

$$\begin{aligned} df(t,Y(t)) &= -\frac{1}{2}u^2\sigma^2(t)f(t,Y(t))dt + uf(t,Y(t))dY(t) + \frac{1}{2}u^2 f(t,Y(t))dY(t)dY(t) \\ &= uf(t,Y(t))\sigma(t)d\widetilde{W}(t) \end{aligned}$$

This implies that

$$f(t,Y(t)) = f(0,Y(0)) + u \cdot \int_0^t f(s,Y(s))\sigma(s)d\widetilde{W}(s).$$

Since the second term on the left is an Ito integral, its expectation, under the risk neutral measure, vanishes. Taking risk neutral expectation of the above equation, we have

$$\widetilde{\mathbf{E}}[\exp\{uY(T) - \frac{1}{2}u^2(\int_0^T \sigma^2(s)ds)] = \widetilde{\mathbf{E}}[f(T,Y(T))] = \widetilde{\mathbf{E}}[f(0,Y(0))] = 1$$

This shows that the random variable $Y(T)$ has moment generating function

$$\varphi(u) = \frac{1}{2}u^2(\int_0^T \sigma^2(s)ds),$$

which is the moment generating function of a normally distributed random variable with mean 0 and variance $\int_0^T \sigma^2(s)ds$. Thus, $Y(T)$ is normally distributed with mean 0 and variance $\int_0^T \sigma^2(s)ds$. And X is also normally distributed with mean $\int_0^T (r(s) - \frac{1}{2}\sigma^2(s))ds$ and variance $\int_0^T \sigma^2(s)ds$.

(ii) To simplify notations, we let

$$\bar{r} = \frac{1}{T}\int_0^T r(s)ds; \quad ||\sigma||_2 = \sqrt{\frac{1}{T}\int_0^T \sigma^2(s)ds}$$

$$\begin{aligned} c(0,S(0)) &= \widetilde{\mathbf{E}}[e^{-\bar{r}T}(S(0)e^X - K)^+] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\bar{r}T}\big(S(0)\exp\{-||\sigma||_2\sqrt{T}y + (\bar{r} - \frac{1}{2}||\sigma||_2^2)T\} - K\big)^+ e^{-\frac{1}{2}y^2}dy \end{aligned}$$

The term in the integrand

$$S(0)\exp\{-||\sigma||_2\sqrt{T}y + (\bar{r} - \frac{1}{2}||\sigma||_2^2)T\} - K$$

is positive if and only if

$$y < d_- := \frac{1}{||\sigma||_2\sqrt{T}}[\log\frac{S(0)}{K} + (\bar{r} - \frac{1}{2}||\sigma||_2^2)T]$$

Thus, the integral can be written as

$$\begin{aligned}
c(0,S(0)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-\bar{r}T}\big(S(0)\exp\{-||\sigma||_2\sqrt{T}y+(\bar{r}-\frac{1}{2}||\sigma||_2^2)T\}-K\big)e^{-\frac{1}{2}y^2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} \big(S(0)\exp\{-||\sigma||_2\sqrt{T}y-\frac{1}{2}||\sigma||_2^2T\}-Ke^{-\bar{r}T}\big)e^{-\frac{1}{2}y^2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} S(0)\exp\{-\frac{(y+||\sigma||_2\sqrt{T})^2}{2}\}dy-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} Ke^{-\bar{r}T}e^{-\frac{1}{2}y^2}dy \\
&= S(0)N(d_+)-Ke^{-\bar{r}T}N(d_-),
\end{aligned}$$

where we let

$$d_+ = d_- + ||\sigma||_2\sqrt{T}.$$

This completes the proof.

5. (i) We have

$$dZ(t) = -\Theta(t)Z(t)dW(t).$$

$$\begin{aligned}
d\big(\frac{1}{Z(t)}\big) &= -\frac{1}{Z^2(t)}dZ(t)+\frac{1}{2}\frac{2}{Z^3(t)}dZ(t)dZ(t) \\
&= -\frac{1}{Z^2(t)}(-\Theta(t)Z(t)dW(t))+\frac{1}{Z^3(t)}\Theta^2(t)Z^2(t)dt \\
&= \frac{1}{Z(t)}\big(\Theta(t)dW(t)+\Theta^2(t)dt\big)
\end{aligned}$$

(ii) If $\widetilde{M}(t)$ is a martingale under $\widetilde{\mathbf{P}}$, then

$$\widetilde{\mathbf{E}}[\widetilde{M}(t)\big|\mathcal{F}(s)] = \widetilde{M}(s).$$

Note that, by Lemma 5.2.2, the left hand side of this equation can be written as

$$\widetilde{\mathbf{E}}[\widetilde{M}(t)\big|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbf{E}[\widetilde{M}(t)Z(t)\big|\mathcal{F}(s)].$$

These two equations imply

$$\mathbf{E}[\widetilde{M}(t)Z(t)\big|\mathcal{F}(s)] = \widetilde{M}(s)Z(s).$$

In other words, $M(t) := \widetilde{M}(t)Z(t)$ is a martingale under $\mathbf{P}$.

(iii) The differential of $\widetilde{M}(t) = M(t)\cdot\frac{1}{Z(t)}$ is:

$$\begin{aligned}
d\big(\widetilde{M}(t)\big) &= M(t)d\big(\frac{1}{Z(t)}\big)+\frac{1}{Z(t)}dM(t)+dM(t)d\big(\frac{1}{Z(t)}\big) \\
&= M(t)\frac{1}{Z(t)}\big(\Theta(t)dW(t)+\Theta^2(t)dt\big)+\frac{1}{Z(t)}\Gamma(t)dW(t)+\Gamma(t)dW(t)\frac{1}{Z(t)}\big(\Theta(t)dW(t)+\Theta^2(t)dt\big) \\
&= \frac{1}{Z(t)}\big(M(t)\Theta(t)+\Gamma(t)\big)dW(t)+\frac{1}{Z(t)}\big(M(t)\Theta^2(t)+\Gamma(t)\Theta(t)\big)dt
\end{aligned}$$

(iv) Writing in terms of $d\widetilde{W}(t) = dW(t) + \Theta(t)dt$,

$$
\begin{aligned}
d\big(\widetilde{M}(t)\big) &= \frac{1}{Z(t)}\big(M(t)\Theta(t) + \Gamma(t)\big)(d\widetilde{W}(t) - \Theta(t)dt) + \frac{1}{Z(t)}\big(M(t)\Theta^2(t) + \Gamma(t)\Theta(t)\big)dt \\
&= \frac{1}{Z(t)}\big(M(t)\Theta(t) + \Gamma(t)\big)d\widetilde{W}(t)
\end{aligned}
$$

Integrating, we have

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \big(\widetilde{M}(s)\Theta(s) + \frac{\Gamma(s)}{Z(s)}\big)d\widetilde{W}(s)$$

6. Let us check that $Z(t)$ is a martingale. With $X(t) = -\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t ||\Theta(u)||^2 du$ and $f(x) = e^x$ so that $f'(x) = e^x$ and $f''(x) = e^x$, we have

$$
\begin{aligned}
dZ(t) &= df(X(t)) \\
&= Z(t)(-\Theta(t)\cdot dW(t) - \frac{1}{2}||\Theta(t)||^2 dt) + \frac{1}{2}e^{X(t)}||\Theta(t)||^2 dt \\
&= -Z(t)\Theta(t)\cdot dW(t).
\end{aligned}
$$

Integrating,

$$Z(t) = Z(0) - \int_0^t Z(u)\Theta(u)\cdot dW(u).$$

Since the second term on the left is a sum of Ito integrals, which are martingales, $Z(t)$ is a martingale. In particular, if we set $Z := Z(T)$

$$\mathbf{E}Z = \mathbf{E}Z(T) = Z(0) = 1.$$

$\widetilde{W}(0) = 0$; $\widetilde{W}(t)$ is continuous. For $i = 1, 2$,

$$d\widetilde{W}_i(t)d\widetilde{W}_i(t) = (dW_i(t) + \Theta_i(t)dt)(dW_i(t) + \Theta_i(t)dt) = dt.$$

$$d\widetilde{W}_1(t)d\widetilde{W}_2(t) = (dW_1(t) + \Theta_1(t)dt)(dW_2(t) + \Theta_2(t)dt) = 0,$$

since W_1 and W_2 are independent.

We claim that for $i = 1, 2$, $\widetilde{W}_i(t)Z(t)$'s are martingales under $\mathbf{P}$:

$$
\begin{aligned}
d\big(\widetilde{W}_i(t)Z(t)\big) &= \widetilde{W}_i(t)dZ(t) + Z(t)d\widetilde{W}_i(t) + d\widetilde{W}_i(t)dZ(t) \\
&= -\widetilde{W}_i(t)Z(t)\Theta(t)\cdot dW(t) + Z(t)dW_i(t) + Z(t)\Theta_i(t)dt - Z(t)\Theta_i(t)dt \\
&= -\widetilde{W}_i(t)Z(t)\Theta(t)\cdot dW(t) + Z(t)dW_i(t).
\end{aligned}
$$

This shows that

$$\widetilde{W}_i(t)Z(t) = \widetilde{W}_i(0)Z(0) + \int_0^t -\widetilde{W}_i(u)Z(u)\Theta(u)\cdot dW(u) + \int_0^t Z(t)dW_i(t).$$

Since the last two terms are Ito integrals, $\widetilde{W}_i(t)Z(t)$'s are martingales under $\mathbf{P}$.

We claim that for $i = 1, 2$, $\widetilde{W}_i(t)$'s are martingales under the measure

$$\widetilde{\mathbf{P}}(A) = \int_A Z(\omega)d\mathbf{P}(\omega) \text{ for all } A \in \mathcal{F}.$$

Indeed, using the martingale property of $\widetilde{W}_i(t)Z(t)$ under $\mathbf{P}$, we have

$$\widetilde{\mathbf{E}}[\widetilde{W}_i(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbf{E}[\widetilde{W}_i(t)Z(t)|\mathcal{F}(s)] = \frac{1}{Z(s)}\widetilde{W}_i(s)Z(s) = \widetilde{W}_i(s)$$

By the two-dimensional Levy's Theorem, $\widetilde{W}(t)$ is a two-dimensional Brownian motion. This completes the proof.

7. (i) At time 0, we invest $X_2(0)$ into the money market account and leave the money there until time T. We then follow the portfolio value process X_1 at all times. Doing so, this new portfolio value process X_2 can be described as

$$X_2(t) = \frac{X_2(0)}{D(t)} + X_1(t).$$

$$\mathbf{P}[X_2(T) \geq \frac{X_2(0)}{D(t)}] = \mathbf{P}[X_1(T) \geq 0] = 1;$$

$$\mathbf{P}[X_2(T) > \frac{X_2(0)}{D(t)}] = \mathbf{P}[X_1(T) > 0] > 0$$

(ii) We set up X_1 as follows. We borrow $X_2(0)$ from the money market at time 0 until time T and invest according to X_2 at all times $0 \leq t \leq T$. In other words,

$$X_1(t) = -\frac{X_2(0)}{D(t)} + X_2(t).$$

Then

$$\mathbf{P}[X_1(T) \geq 0] = \mathbf{P}[X_2(T) \geq \frac{X_2(0)}{D(t)}] = 1;$$

$$\mathbf{P}[X_1(T) > 0] = \mathbf{P}[X_2(T) > \frac{X_2(0)}{D(t)}] > 0.$$

8. (i) Note that the discounted payoff can be written as

$$D(t)V(t) = \widetilde{\mathbf{E}}[D(T)V(T)|\mathcal{F}(t)].$$

Therefore, it is a martingale under the risk neutral measure. Indeed, for $0 \leq s < t \leq T$,

$$\widetilde{\mathbf{E}}[D(t)V(t)|\mathcal{F}(s)] = \widetilde{\mathbf{E}}[\widetilde{\mathbf{E}}[D(T)V(T)|\mathcal{F}(t)]|\mathcal{F}(s)] = \widetilde{\mathbf{E}}[D(T)V(T)|\mathcal{F}(s)] = D(s)V(s).$$

By the Martingale Representation Theorem, there is an adapted process $\widetilde{\Gamma}(t)$ such that

$$d\big(D(t)V(t)\big) = \widetilde{\Gamma}(t)d\widetilde{W}(t).$$

Expanding the left hand side using Ito's product rule, while noting that $dD(t) = -R(t)D(t)dt$, we have

$$d\big(D(t)V(t)\big) = D(t)dV(t) + V(t)dD(t) + dD(t)dV(t) = D(t)\big(dV(t) - R(t)V(t)dt\big).$$

Putting these two equations together, we have

$$dV(t) = R(t)V(t)dt + \frac{\widetilde{\Gamma}(t)}{D(t)}d\widetilde{W}(t).$$

(ii) Suppose $D(t)V(t)$ is not almost surely positive. We define

$$A := \{D(t)V(t) \leq 0\},$$

seen as an element in the σ-algebra $\mathcal{F}(t)$. By the hypothesis,

$$\widetilde{\mathbf{P}}(A) > 0$$

Then

$$0 \geq \int_A D(t)V(t)d\widetilde{\mathbf{P}} = \int_A \widetilde{\mathbf{E}}[D(T)V(T)|\mathcal{F}(t)]d\widetilde{\mathbf{P}} = \int_A D(T)V(T)d\widetilde{\mathbf{P}}$$

Since $D(T)V(T)$ is almost surely positive, the latter integral is strictly positive. A contradiction. In summary, we have proved that $D(t)V(t)$, therefore $V(t)$, is almost surely positive.

(iii) Since the denominator $D(t)V(t)$ is almost surely positive, we may let

$$\sigma(t) := \frac{\widetilde{\Gamma}(t)}{D(t)V(t)}.$$

Then

$$dV(t) = R(t)V(t)dt + \sigma(t)V(t)d\widetilde{W}(t).$$

9.

$$\begin{aligned}
c_K(0,T,x,K) &= \frac{\partial}{\partial K}\big(e^{-rT}\int_K^\infty y\widetilde{p}(0,T,x,y)dy - e^{-rT}K\int_K^\infty \widetilde{p}(0,T,x,y)dy\big) \\
&= e^{-rT}\cdot(-K\widetilde{p}(0,T,x,K)) - e^{-rT}\int_K^\infty \widetilde{p}(0,T,x,y)dy - e^{-rT}K\cdot(-\widetilde{p}(0,T,x,K)) \\
&= -e^{-rT}\int_K^\infty \widetilde{p}(0,T,x,y)dy = -e^{-rT}\widetilde{\mathbf{P}}[S(T) > K].
\end{aligned}$$

Differentiating $c_K(0,T,x,K)$ with respect to K, we have

$$\begin{aligned}
c_{KK}(0,T,x,K) &= \frac{\partial}{\partial K}\big(-e^{-rT}\int_K^\infty \widetilde{p}(0,T,x,y)dy\big) \\
&= e^{-rT}\cdot\widetilde{p}(0,T,x,K).
\end{aligned}$$

10. (i) At time t_0, the owner of the chooser option may choose either the call or the put, which puts the value of the chooser option to be

$$\begin{aligned}
\max\{C(t_0), P(t_0)\} &= \max\{C(t_0), C(t_0) - F(t_0)\} \\
&= C(t_0) + \max\{0, -F(t_0)\} \\
&= C(t_0) + \big(e^{-r(T-t_0)}K - S(t_0)\big)^+.
\end{aligned}$$

(ii) Applying the risk neutral pricing formula, the time zero value of the chooser option is

$$\begin{aligned}
&\widetilde{\mathbf{E}}[e^{-rt_0}C(t_0) + \big(e^{-r(T-t_0)}K - S(t_0)\big)^+] \\
&= \widetilde{\mathbf{E}}[e^{-rt_0}\widetilde{\mathbf{E}}[e^{-r(T-t_0)}(S(T)-K)^+|\mathcal{F}(t)]] + \widetilde{\mathbf{E}}[e^{-rt_0}\big(e^{-r(T-t_0)}K - S(t_0)\big)^+] \\
&= \widetilde{\mathbf{E}}[e^{-rT}(S(T)-K)^+] + \widetilde{\mathbf{E}}[e^{-rt_0}\big(e^{-r(T-t_0)}K - S(t_0)\big)^+]
\end{aligned}$$

These two terms are the time zero prices of the call expiring at time T with strike K and the put expiring at time t_0 with strike $e^{-r(T-t_0)}K$.

11. Define the adapted process

$$\Theta(t) := \frac{\alpha(t) - R(t)}{\sigma(t)}.$$

Define

$$Z(t) = \exp\{-\int_0^t \Theta(u)dW(u) - \int_0^t \Theta^2(u)du\}; \quad \widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du.$$

Then by Girsanov's Theorem, under the measure $\widetilde{\mathbf{P}}(A) = \int_A Z(\omega)d\mathbf{P}(\omega)$, the process $\widetilde{W}(t)$ is a Brownian motion.

Rewriting the differential $dS(t)$ and $dX(t)$ in terms of $d\widetilde{W}(t)$, we have

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

(5.2.22) says that

$$d\big(D(t)S(t)\big) = \sigma(t)D(t)S(t)d\widetilde{W}(t).$$

Let $\Delta(t)$ be a portfolio process and $X(t)$ the associated portfolio value process. By Ito's product rule,

$$\begin{aligned}
d\big(D(t)X(t)\big) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= D(t)\big(\Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt - C(t)dt\big) + X(t)(-R(t)D(t)dt) + 0 \\
&= \Delta(t)D(t)\big(dS(t) - R(t)S(t)dt\big) - D(t)C(t)dt \\
&= \Delta(t)D(t)\sigma(t)S(t)d\widetilde{W}(t) - D(t)C(t)dt.
\end{aligned}$$

If we apply the Martingale Representation Theorem to the martingale process

$$\widetilde{M}(t) = \widetilde{\mathbf{E}}[\int_0^T D(u)C(u)du\big|\mathcal{F}(t)],$$

we shall get an adapted process $\widetilde{\Gamma}(t)$ such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u).$$

Let $X(0) = \widetilde{M}(0)$ and $\Delta(t) = \frac{\widetilde{\Gamma}(t)}{D(t)\sigma(t)S(t)}$. Then

$$d\big(D(t)X(t)\big) = \widetilde{\Gamma}(t)d\widetilde{W}(t) - D(t)C(t)dt.$$

Integrating both sides, we have

$$D(T)X(T) = \widetilde{M}(0) + \int_0^T \widetilde{\Gamma}(u)d\widetilde{W}(u) - \int_0^T D(u)C(u)du = 0,$$

where the last equality holds almost surely.

12. (i) It is clear that $\widetilde{B}_i(0) = 0$; $\widetilde{B}_i(t)$ continuous in t and $d\widetilde{B}_i(t)d\widetilde{B}_i(t) = dB_i(t)dB_i(t) = dt$.

$$d\widetilde{B}_i(t) = \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}dW_j(u) + \sum_{j=1}^d \frac{\sigma_{ij}(u)\Theta(u)}{\sigma_i(u)}du = \sum_{j=1}^d \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}d\widetilde{W}_j(u).$$

Therefore, $\widetilde{B}_i(t)$ is a martingale under the risk neutral measure. We may now apply Levy's Theorem to conclude that $\widetilde{B}_i(t)$ is a Brownian motion.

(ii)

$$
\begin{aligned}
dS_i(t) &= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dB_i(t) \\
&= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)\big(d\widetilde{B}_i(t) - \sum_{j=1}^{d}\frac{\sigma_{ij}(t)\Theta_j(t)}{\sigma_i(t)}dt\big) \\
&= \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)\big(d\widetilde{B}_i(t) - \frac{\alpha_i(t)-R(t)}{\sigma_i(t)}dt\big) \\
&= R(t)S_i(t)dt + \sigma_i(t)S_i(t)d\widetilde{B}_i(t)
\end{aligned}
$$

(iii) Since $dB_i(t)dt = 0$,

$$
d\widetilde{B}_i(t)d\widetilde{B}_k(t) = (dB_i(t)+\gamma_i(t)dt)(dB_k(t)+\gamma_k(t)dt) = dB_i(t)dB_k(t) = \rho_{ik}(t).
$$

(iv) Applying Ito's product rule and the fact that $dB_i(u)dB_k(u) = \rho_{ik}(t)dt$, we have

$$
\begin{aligned}
B_i(t)B_k(t) &= B_i(0)B_k(0) + \int_0^t d\big(B_i(u)B_k(u)\big) \\
&= \int_0^t \big(B_i(u)dB_k(u) + B_k(u)dB_i(u) + \rho_{ik}(u)du\big).
\end{aligned}
$$

Taking expectation, we have

$$
\mathbf{E}[B_i(t)B_k(t)] = \mathbf{E}[\int_0^t \rho_{ik}(u)du] = \int_0^t \rho_{ik}(u)du,
$$

where we used the hypothesis that $\rho_{ik}(t)$ is non random in the second equality. We can repeat the above argument to show that $\widetilde{\mathbf{E}}[\widetilde{B}_i(t)\widetilde{B}_k(t)] = \int_0^t \rho_{ik}(u)du$, taking risk neutral expectation in the last step.

(v) The first equality below follows from the argument given in part (iv).

$$
\mathbf{E}[B_1(t)B_2(t)] = \mathbf{E}[\int_0^t \rho_{12}(u)du] = \mathbf{E}[\int_0^t \operatorname{sign}\big(W_1(u)\big)du]
$$

Since $-W_1(u)$ is also a Brownian motion under $\mathbf{P}$,

$$
\mathbf{E}[\int_0^t \operatorname{sign}\big(W_1(u)\big)du] = \mathbf{E}[\int_0^t \operatorname{sign}\big(-W_1(u)\big)du] = \mathbf{E}[-\int_0^t \operatorname{sign}\big(W_1(u)\big)du],
$$

which implies that $\mathbf{E}[B_1(t)B_2(t)] = \mathbf{E}[\int_0^t \operatorname{sign}\big(W_1(u)\big)du] = 0$.

Let us compute $\widetilde{\mathbf{E}}[\widetilde{B}_1(t)\widetilde{B}_2(t)]$. Note that $\widetilde{W}_1(t) = W_1(t) + t$ is a Brownian motion under the risk neutral measure. By the argument in part (iv), we have

$$
\widetilde{\mathbf{E}}[\widetilde{B}_1(t)\widetilde{B}_2(t)] = \widetilde{\mathbf{E}}[\int_0^t \rho_{12}(u)du] = \widetilde{\mathbf{E}}[\int_0^t \operatorname{sign}\big(W_1(u)\big)du] = \widetilde{\mathbf{E}}[\int_0^t \operatorname{sign}\big(\widetilde{W}_1(u)-u\big)du].
$$

For $u > 0$,

$$
\widetilde{\mathbf{E}}[\operatorname{sign}\big(\widetilde{W}_1(u)-u\big)] = \widetilde{\mathbf{P}}[\widetilde{W}_1(u)-u \geq 0]\cdot 1 + \widetilde{\mathbf{P}}[\widetilde{W}_1(u)-u < 0]\cdot(-1) = 1 - 2N(u) < 0.
$$

By Fubini's theorem,

$$\widetilde{\mathbf{E}}[\int_0^t \operatorname{sign}\big(\widetilde{W}_1(u)-u\big)du] = \int_0^t \widetilde{\mathbf{E}}[\operatorname{sign}\big(\widetilde{W}_1(u)-u\big)]du < 0.$$

To summarize, we have shown that

$$\mathbf{E}[B_1(t)B_2(t)] = 0; \quad \widetilde{\mathbf{E}}[\widetilde{B}_1(t)\widetilde{B}_2(t)] < 0.$$

13. (i) Since $\widetilde{W}_1(t) = W_1(t)$, it is clear that

$$\widetilde{\mathbf{E}}W_1(t) = \widetilde{\mathbf{E}}\widetilde{W}_1(t) = 0.$$

$$\widetilde{\mathbf{E}}W_2(t) = \widetilde{\mathbf{E}}[\widetilde{W}_2(t) - \int_0^t \widetilde{W}_1(u)du] = 0 - \int_0^t \widetilde{\mathbf{E}}[\widetilde{W}_1(u)]du = -\int_0^t 0du = 0.$$

In the second equality, we applied Fubini's theorem. Indeed, $\int_0^t \widetilde{\mathbf{E}}[|\widetilde{W}_1(u)|]du = \int_0^t \sqrt{\frac{2u}{\pi}}du$ is finite. Therefore, $\widetilde{\mathbf{E}}[\int_0^t \widetilde{W}_1(u)du] = \int_0^t \widetilde{\mathbf{E}}[\widetilde{W}_1(u)]du$.

(ii) By Ito's product rule,

$$W_1(T)W_2(T) = \int_0^T \big(W_1(t)dW_2(t) + W_2(t)dW_1(t)\big) = \int_0^T \big(\widetilde{W}_1(t)d\widetilde{W}_2(t) - \widetilde{W}_1^2(t)dt + W_2(t)d\widetilde{W}_1(t)\big).$$

Upon taking risk neutral expectation, the terms involving $d\widetilde{W}_i(t)$ vanishes, and we have

$$\widetilde{\mathbf{E}}[W_1(T)W_2(T)] = -\widetilde{\mathbf{E}}[\int_0^T \widetilde{W}_1^2(t)dt] = -\int_0^T \widetilde{\mathbf{E}}[\widetilde{W}_1^2(t)]dt = -\int_0^T tdt = -\frac{1}{2}T^2$$

14. (i) Let us write the differential of the discounted portfolio value process in terms of $d\widetilde{W}(t)$ and dt.

$$\begin{aligned} d\big(e^{-rt}X(t)\big) &= -re^{-rt}X(t)dt + e^{-rt}dX(t) \\ &= -re^{-rt}X(t)dt + e^{-rt}\big(\Delta(t)dS(t) - a\Delta(t)dt + r(X(t)-\Delta(t)S(t))dt\big) \\ &= e^{-rt}\Delta(t)\big(dS(t) + (-a - rS(t))dt\big) \\ &= e^{-rt}\Delta(t)\sigma S(t)d\widetilde{W}(t). \end{aligned}$$

Since there is no dt term in the differential, $e^{-rt}X(t)$ is a martingale under $\widetilde{\mathbf{P}}$.

(ii) Applying Ito's formula to compute the differential of $Y(t)$, we have

$$\begin{aligned} dY(t) &= Y(t)\big(\sigma d\widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)dt\big) + \frac{1}{2}Y(t)\big(\sigma d\widetilde{W}(t) + (r-\frac{1}{2}\sigma^2)dt\big)\cdot\big(\sigma d\widetilde{W}(t) + (r-\frac{1}{2}\sigma^2)dt\big) \\ &= Y(t)\big(\sigma d\widetilde{W}(t) + rdt\big). \end{aligned}$$

Let us compute the differential of $e^{-rt}Y(t)$.

$$\begin{aligned} d\big(e^{-rt}Y(t)\big) &= e^{-rt}Y(t)\big(\sigma d\widetilde{W}(t) + rdt\big) - re^{-rt}Y(t)dt \\ &= e^{-rt}Y(t)\sigma d\widetilde{W}(t) \end{aligned}$$

Therefore, $e^{-rt}Y(t)$ is a martingale under $\widetilde{\mathbf{P}}$.

Let $S(t) = S(0)Y(t) + Y(t)\int_0^t \int_0^t \frac{a}{Y(s)}ds$ and compute its differential:

$$\begin{aligned} dS(t) &= S(0)dY(t) + Y(t)\frac{a}{Y(t)}dt + \Big(\int_0^t \frac{a}{Y(s)}ds\Big)dY(t) \\ &= \Big(S(0) + \int_0^t \frac{a}{Y(s)}ds\Big)Y(t)(\sigma d\widetilde{W}(t) + rdt) + adt \\ &= S(t)(\sigma d\widetilde{W}(t) + rdt) + adt. \end{aligned}$$

(iii) Using the fact that $e^{-rt}Y(t)$ is a martingale,

$$\begin{aligned} \widetilde{\mathbf{E}}[S(T)\big|\mathcal{F}(t)] &= S(0)\widetilde{\mathbf{E}}[Y(T)\big|\mathcal{F}(t)] + \widetilde{\mathbf{E}}[Y(T)\big|\mathcal{F}(t)]\int_0^t \frac{a}{Y(s)}ds + a\int_t^T \widetilde{\mathbf{E}}[\frac{Y(T)}{Y(s)}\big|\mathcal{F}(t)]ds \\ &= S(0)e^{r(T-t)}Y(t) + e^{r(T-t)}Y(t)\int_0^t \frac{a}{Y(s)}ds + a\int_t^T e^{r(T-s)}ds \\ &= e^{r(T-t)}S(t) + \frac{a(e^{r(T-t)}-1)}{r}. \end{aligned}$$

(iv) The differential of the process $\widetilde{\mathbf{E}}[S(T)\big|\mathcal{F}(t)]$ is

$$\begin{aligned} & d\Big(e^{r(T-t)}S(t) + \frac{a(e^{r(T-t)}-1)}{r}\Big) \\ &= e^{r(T-t)}\big(rS(t)dt + \sigma S(t)d\widetilde{W}(t) + adt\big) - re^{r(T-t)}S(t)dt - ae^{r(T-t)}dt \\ &= e^{r(T-t)}\sigma S(t)d\widetilde{W}(t). \end{aligned}$$

From this, we see that $\widetilde{\mathbf{E}}[S(T)\big|\mathcal{F}(t)]$ is a martingale under $\widetilde{\mathbf{P}}$.

(v) Let us rewrite the time t value of the forward contract

$$\widetilde{\mathbf{E}}[e^{-r(T-t)}(S(T)-K)\big|\mathcal{F}(t)] = S(t) + \frac{a}{r}(1 - e^{-r(T-t)}) - Ke^{-r(T-t)}.$$

If $\widetilde{\mathbf{E}}[e^{-r(T-t)}(S(T)-K)\big|\mathcal{F}(t)] = 0$, then

$$K = e^{r(T-t)}\Big(S(t) + \frac{a}{r}(1 - e^{-r(T-t)})\Big) = \widetilde{\mathbf{E}}[S(T)\big|\mathcal{F}(t)].$$

(vi) Reusing the computation done in part (i) and putting $\Delta(t) = 1$, the differential of the discounted portfolio value process is:

$$d\big(e^{-rt}X(t)\big) = e^{-rt}(dS(t) + (-a - rS(t))dt) = d\big(e^{-rt}S(t)\big) - ae^{-rt}dt.$$

Integrating, we have

$$\begin{aligned} e^{-rT}X(T) - X(0) &= \int_0^T d\big(e^{-rt}S(t)\big) - ae^{-rt}dt \\ &= e^{-rT}S(T) - S(0) + \frac{a}{r}(e^{-rT}-1). \end{aligned}$$

Since $X(0) = 0$, we may simplify the above equation to get

$$X(T) = S(T) - \Big(e^{rT}S(0) + \frac{a}{r}(e^{rT}-1)\Big) = S(T) - \mathrm{For}_S(0,T)$$