Chapter 4 solutions to Stochastic Calculus for Finance I by Steven E. Shreve

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1. (i) Let's determine the value of the American put at $t = 3$ by executing it if it is in the money.

$$v_3(32) = v_3(8) = 0; \quad v_3(2) = 2; \quad v_3(\frac{1}{2}) = \frac{7}{2}.$$

For $t \leq 2$, we compute the value of the american put by

$$v_i(s) = \max\{(4-s)^+, \frac{2}{5}(v_{i+1}(2s) + v_{i+1}(\frac{s}{2})\}.$$

We compute the value of the put at $t = 2$ using its value at $t = 3$:

$$v_2(16) = 0; \quad v_2(4) = \frac{4}{5}; \quad v_2(1) = \max\{3, \frac{2}{5}(2 + \frac{7}{2}) = \frac{11}{5}\} = 3,$$

where the put should be executed in the last state. We compute the value of the put at $t = 1$ using its value at $t = 2$:

$$v_1(8) = \frac{8}{25}; \quad v_1(2) = \max\{2, \frac{2}{5}(\frac{4}{5} + 3) = \frac{38}{25}\} = 2,$$

where the put should be executed in the latter state. Lastly, for $t = 0$,

$$v_0(4) = \frac{2}{5}(\frac{8}{25} + 2) = \frac{116}{125}.$$

The time-zero price of the put is $\frac{116}{125}$.

(ii) Time three price of the call:

$$v_3(32) = 28; \quad v_3(8) = 4; \quad v_3(2) = v_3(\frac{1}{2}) = 0$$

Time two price of the call:

$$v_2(16) = \max\{(16-4)^+, \frac{2}{5}(28+4)\} = \frac{64}{5};$$

$$v_2(4) = \frac{8}{5}; \quad v_2(\frac{1}{2}) = 0.$$

Time one price of the call:

$$v_1(8) = \max\{4, \frac{2}{5}(\frac{64}{5} + \frac{8}{5})\} = \frac{144}{25};$$

$$v_1(2) = \frac{16}{25}.$$

Time zero price of the call:

$$v_0(4) = \frac{2}{5}(\frac{144}{25} + \frac{16}{25}) = \frac{64}{25}.$$

(iii) Time three price of the straddle:

$$v_3(32) = 28; \quad v_3(8) = 4; \quad v_3(2) = 2; \quad v_3(\frac{1}{2}) = \frac{7}{2}.$$

Time two price of the straddle:

$$v_2(16) = \max\{12, \frac{2}{5}(28+4)\} = \frac{64}{5};$$

$$v_2(4) = \frac{2}{5}(4+2) = \frac{12}{5};$$

$$v_2(1) = \max\{3, \frac{2}{5}(2+\frac{7}{2})\} = 3.$$

Time one price of the straddle:

$$v_1(8) = \max\{4, \frac{2}{5}(\frac{64}{5}+\frac{12}{5})\} = \frac{152}{25};$$

$$v_1(2) = \max\{2, \frac{2}{5}(\frac{12}{5}+3)\} = \frac{54}{25}.$$

Time one price of the straddle:

$$v_0(4) = \frac{2}{5}(\frac{152}{25}+\frac{54}{25}) = \frac{412}{125}.$$

(iv) The time zero price of the straddle is less than the sum of the price of the put and the call:

$$V_0^S = \frac{412}{125} < V_0^P + V_0^C = \frac{116}{125} + \frac{64}{25} = \frac{436}{125}.$$

The owner of the put and the call can execute these options at different times. This causes the discrepancy.

2. The time zero value of the hedging portfolio $X_0 = -1.36$. Let Δ_0 be the number of stock held in the portfolio. Since we are hedging over a *long* position of the put,

$$\Delta_0 = \frac{(-V_1(H)) - (-V_1(T))}{S_1(H) - S_1(T)} = \frac{-\frac{2}{5}+3}{8-2} = \frac{13}{30}$$

(note the minus signs), where V_1 and S_1 are the values of the put and the stock at time 1 respectively. Recall that the optimal exercise time is $\tau(HH) = \infty$; $\tau(HT) = 2$; $\tau(TH) = 1$; $\tau(TT) = 1$. So we do not exercise the put at $t = 0$.

Let us check the value of the our portfolio plus our long position in the put adds up to zero at $t = 1$. The time one value of the hedging portfolio is given by the formula

$$X_1 = \Delta_0 S_1 + (1+r) \cdot (X_0 - \Delta_0 S_0).$$

If $\omega_1 = H$,

$$X_1(H) = \frac{13}{30}\cdot 8 + \frac{5}{4}\cdot(-\frac{34}{25} - \frac{13}{30}\cdot 4) = \frac{52}{15} - \frac{58}{15} = -\frac{2}{5} = -V_1(H).$$

We do not exercise the option in this case and will continue to hedge. We will hold

$$\Delta_1(H) == \frac{(-V_2(HH)) - (-V_2(HT))}{S_2(HH) - S_2(HT)} = \frac{1}{12}$$

unit of stock in our portfolio at $t = 1$.

If $\omega_1 = T$,

$$X_1(T) = \frac{13}{30}\cdot 2 + \frac{5}{4}\cdot(-\frac{34}{25} - \frac{13}{30}\cdot 4) = \frac{13}{15} - \frac{58}{15} = -3 = -V_1(T).$$

We exercise the option in this case to pay off our loan.

If $\omega_1 = H$ and $\omega_2 = H$,

$$\begin{aligned}
X_2(HH) &= \Delta_1(H)S_2(HH) + (1+r)(X_1(H) - \Delta_1(H)S_1(H)) \\
&= \frac{1}{12}\cdot 16 + \frac{5}{4}\cdot(-\frac{2}{5} - \frac{1}{12}\cdot 8) \\
&= \frac{4}{3} - \frac{4}{3} \\
&= 0 = -V_2(HH).
\end{aligned}$$

We let the put expire in this case, and there is no loan to be paid off.

If $\omega_1 = H$ and $\omega_2 = T$,

$$\begin{aligned}
X_2(HH) &= \Delta_1(H)S_2(HT) + (1+r)(X_1(H) - \Delta_1(H)S_1(H)) \\
&= \frac{1}{12}\cdot 4 + \frac{5}{4}\cdot(-\frac{2}{5} - \frac{1}{12}\cdot 8) \\
&= \frac{1}{3} - \frac{4}{3} \\
&= -1 = -V_2(HT).
\end{aligned}$$

We exercise the put, getting 1 dollar and pay off our loan with it.

3. Let us first compute the time three price of this American derivative security, which is just its intrinsic value at time three.

$$V_3(HHH) = V_3(HHT) = V_3(HTH) = V_3(HTT) = V_3(THH) = 0;$$

$$V_3(THT) = (4 - \frac{1}{4}(4+2+4+2))^+ = 1;$$

$$V_3(TTH) = (4 - \frac{1}{4}(4+2+1+2))^+ = \frac{7}{4};$$

$$V_3(TTT) = (4 - \frac{1}{4}(4+2+1+\frac{1}{2}))^+ = \frac{17}{8}.$$

Let us compute the intrinsic value of this American derivative security at time two:

$$G_2(HH) = G_2(HT) = 0; \quad G_2(TH) = \left(4 - \frac{1}{3}(4+2+4)\right)^+ = \frac{2}{3}; \quad G_2(TT) = \left(4 - \frac{1}{3}(4+2+1)\right)^+ = \frac{5}{3}.$$

Then we compute the time two price of this American derivative security.

$$V_2(HH) = \max\{G_2(HH), \frac{1}{1+r}(\tilde{p}V_3(HHH) + \tilde{q}V_3(HHT))\} = 0;$$

$$V_2(HT) = \max\{G_2(HT), \frac{1}{1+r}(\tilde{p}V_3(HTH) + \tilde{q}V_3(HTT))\} = 0;$$

$$V_2(TH) = \max\{G_2(TH), \frac{1}{1+r}(\tilde{p}V_3(THH) + \tilde{q}V_3(THT))\} = \max\{\frac{2}{3}, \frac{4}{5}(\frac{1}{2}0 + \frac{1}{2}1)\} = \frac{2}{3};$$

$$V_2(TT) = \max\{G_2(TT), \frac{1}{1+r}(\tilde{p}V_3(TTH) + \tilde{q}V_3(TTT))\} = \max\{\frac{5}{3}, \frac{4}{5}(\frac{1}{2}\frac{7}{4} + \frac{1}{2}\frac{17}{8})\} = \frac{5}{3};$$

Intrinsic value at time one:

$$G_1(H) = 0; G_1(T) = (4 - \frac{1}{2}(4+2))^+ = 1.$$

Time one price of this American derivative security:

$$V_1(H) = \max\{G_1(H), \frac{1}{1+r}(\tilde{p}V_2(HH) + \tilde{q}V_2(HT))\} = 0;$$

$$V_1(T) = \max\{G_1(T), \frac{1}{1+r}(\tilde{p}V_2(TH) + \tilde{q}V_2(TT))\} = \max\{1, \frac{4}{5}(\frac{1}{2}\frac{2}{3} + \frac{1}{2}\frac{5}{3})\} = 1.$$

Time zero price of this American derivative security:

$$V_0 = \frac{1}{1+r}(\tilde{p}V_1(H) + \tilde{q}V_1(T)) = \frac{4}{5}(\frac{1}{2}0 + \frac{1}{2}1) = \frac{2}{5}.$$

Optimal stopping time:

$$\tau(HHH) = \tau(HHT) = \tau(HTH) = \tau(HTT) = \infty;$$

$$\tau(THH) = \tau(THT) = \tau(TTH) = \tau(TTT) = 1,$$

since $G_1(T) > \frac{1}{1+r}(\tilde{p}V_2(TH) + \tilde{q}V_2(TT))$.

4. No, it is not necessary to charge more for the put since our hedging portfolio can always cover the short position in the put.

5. Let us first list the 15 stopping times that exercises when the option is out of money.

$$\tau(HH) = 2; \quad \tau(HT) = 2; \quad \tau(TH) = 1; \quad \tau(TT) = 1$$

$$\tau(HH) = 2; \quad \tau(HT) = 2; \quad \tau(TH) = 2; \quad \tau(TT) = 2$$

$$\tau(HH) = 2; \quad \tau(HT) = 2; \quad \tau(TH) = 2; \quad \tau(TT) = \infty$$

$$\tau(HH) = 2; \quad \tau(HT) = 2; \quad \tau(TH) = \infty; \quad \tau(TT) = 2$$

$$\tau(HH) = 2; \quad \tau(HT) = 2; \quad \tau(TH) = \infty; \quad \tau(TT) = \infty.$$

$$\tau(HH) = 2; \quad \tau(HT) = \infty; \quad \tau(TH) = 1; \quad \tau(TT) = 1$$

$$\tau(HH) = 2; \quad \tau(HT) = \infty; \quad \tau(TH) = 2; \quad \tau(TT) = 2$$

$$\tau(HH) = 2; \quad \tau(HT) = \infty; \quad \tau(TH) = 2; \quad \tau(TT) = \infty$$

$$\tau(HH) = 2; \quad \tau(HT) = \infty; \quad \tau(TH) = \infty; \quad \tau(TT) = 2$$

$$\tau(HH) = 2; \quad \tau(HT) = \infty; \quad \tau(TH) = \infty; \quad \tau(TT) = \infty$$

In the above 10 stopping times, the option is executed when out of money at the state HH.

$$\tau(HH) = 1; \quad \tau(HT) = 1; \quad \tau(TH) = 1; \quad \tau(TT) = 1$$

$$\tau(HH) = 1; \quad \tau(HT) = 1; \quad \tau(TH) = 2; \quad \tau(TT) = 2$$

$$\tau(HH) = 1; \quad \tau(HT) = 1; \quad \tau(TH) = 2; \quad \tau(TT) = \infty$$

$$\tau(HH) = 1; \quad \tau(HT) = 1; \quad \tau(TH) = \infty; \quad \tau(TT) = 2$$

$$\tau(HH) = 1; \quad \tau(HT) = 1; \quad \tau(TH) = \infty; \quad \tau(TT) = \infty$$

In the above 5 stopping times, the option is executed when out of money at the state H.

Let us now list the 11 stopping times that never exercises when the option is out of money and compute their time zero risk neutral expected value.

$$\tau(HH) = 0; \quad \tau(HT) = 0; \quad \tau(TH) = 0; \quad \tau(TT) = 0.$$

$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}} \cdot \frac{G_\tau}{(1+r)^\tau}] = 1.$$

$$\tau(HH) = \infty; \quad \tau(HT) = 2; \quad \tau(TH) = 1; \quad \tau(TT) = 1.$$

$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}} \cdot \frac{G_\tau}{(1+r)^\tau}] = \frac{1}{4}\frac{1}{(1+r)^2}G_2(HT) + \frac{1}{2}\frac{1}{1+r}G_1(T) = \frac{4}{25} + \frac{6}{5} = \frac{34}{25}.$$

$$\tau(HH) = \infty; \quad \tau(HT) = 2; \quad \tau(TH) = 2; \quad \tau(TT) = 2.$$

$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}} \cdot \frac{G_\tau}{(1+r)^\tau}] = \frac{1}{4}\frac{1}{(1+r)^2}(G_2(HT) + G_2(TH) + G_2(TT)) = \frac{4}{25}(1 + 1 + \frac{7}{2}) = \frac{22}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=2;\quad \tau(TH)=2;\quad \tau(TT)=\infty.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{4}\frac{1}{(1+r)^2}(G_2(HT)+G_2(TH))=\frac{4}{25}(1+1)=\frac{8}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=2;\quad \tau(TH)=\infty;\quad \tau(TT)=2.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{4}\frac{1}{(1+r)^2}(G_2(HT)+G_2(TT))=\frac{4}{25}(1+\frac{7}{2})=\frac{18}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=2;\quad \tau(TH)=\infty;\quad \tau(TT)=\infty.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{4}\frac{1}{(1+r)^2}G_2(HT)=\frac{4}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=\infty;\quad \tau(TH)=1;\quad \tau(TT)=1.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{2}\frac{1}{(1+r)}G_1(T)=\frac{6}{5}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=\infty;\quad \tau(TH)=2;\quad \tau(TT)=2.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{4}\frac{1}{(1+r)^2}(G_2(TH)+G_2(TT))=\frac{4}{25}(1+\frac{7}{2})=\frac{18}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=\infty;\quad \tau(TH)=2;\quad \tau(TT)=\infty.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{4}\frac{1}{(1+r)^2}G_2(TH)=\frac{4}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=\infty;\quad \tau(TH)=\infty;\quad \tau(TT)=2.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=\frac{1}{4}\frac{1}{(1+r)^2}G_2(TT)=\frac{4}{25}\frac{7}{2}=\frac{14}{25}.$$

$$\tau(HH)=\infty;\quad \tau(HT)=\infty;\quad \tau(TH)=\infty;\quad \tau(TT)=\infty.$$
$$\tilde{\mathbf{E}}[\mathbf{I}_{\{\tau<\infty\}}\cdot\frac{G_\tau}{(1+r)^\tau}]=0.$$

These computation verifies that the stopping time

$$\tau(HH)=\infty;\quad \tau(HT)=2;\quad \tau(TH)=\infty;\quad \tau(TT)=2$$

is optimal.

6. (i) The discounted payoff $\frac{1}{(1+r)^n}(K - S_n)$ is a supermartingale. Indeed,

$$\tilde{\mathbf{E}}_n[\frac{1}{(1+r)^{n+1}}(K - S_{n+1})] = \frac{K}{(1+r)^{n+1}} - S_n < \frac{1}{(1+r)^n}(K - S_n).$$

By optional sampling - part I (Theorem 4.3.2), for any stopping time $\tau \leq N$, $\frac{1}{(1+r)^{\tau\wedge n}}(K - S_{\tau\wedge n})$ is also a supermartingale. Thus,

$$K - S_0 = \frac{1}{(1+r)^{\tau\wedge 0}}(K - S_{\tau\wedge 0}) \geq \tilde{\mathbf{E}}[\frac{1}{(1+r)^{\tau\wedge N}}(K - S_{\tau\wedge N})] = \tilde{\mathbf{E}}[\frac{1}{(1+r)^{\tau}}(K - S_{\tau})].$$

Therefore, the optimal stopping time is $\tau = 0$; the time zero value of the option is $K - S_0$.

(ii) Suppose we hold a long position of the aforementioned derivative security and the European call option and a short position of the American put. It suffices to show that under all scenarios, we can close our position with a nonnegative payoff.

Suppose the owner of the American put executes it at some time $t \leq N$. We can then execute the option in part one that cancels out the payment to the owner of the put, and we are left with an European call, which has nonnegative value.

Suppose the owner of the American put never executes it. Then at time $t = N$, we execute the option in part one, and execute the European call if it is in the money. From these actions, we receive $(K - S_n) + (S_n - K)^+ \geq 0$.

(iii) Consider a forward contract that pays $F_N = (S_N - K)$ at $t = N$. Let $C_N = (S_N - K)^+$ and $P_N = (K - S_N)^+$ be the payoff of an European call and European put with strike price K and expiry date $t = N$. Then $C_N = P_N + F_N$. Therefore, the time zero value of the European call must equal to sum of the time zero value of the forward contract and the European put.

$$V_0^{EC} = V_0^{forward} + V_0^{EP}. \tag{1}$$

Let us compute the time zero value of the forward contract:

$$V_0^{forward} = \tilde{\mathbf{E}}[\frac{S_N - K}{(1+r)^N}] = S_0 - \frac{K}{(1+r)^N}. \tag{2}$$

On the other hand, the value of an European put has to be less than the value of an American put with the same strike price and expiry date.

$$V_0^{EP} \leq V_0^{AP} \tag{3}$$

Combining (1), (2) and (3), we have

$$V_0^{EC} \leq S_0 - \frac{K}{(1+r)^N} + V_0^{AP}.$$

7. Note that the discounted payoff $\frac{1}{(1+r)^n}(S_n - K)$ is a submartingale. Indeed,

$$\tilde{\mathbf{E}}_n[\frac{1}{(1+r)^{n+1}}(S_{n+1} - K)] = S_n - \frac{K}{(1+r)^{n+1}} > S_n - \frac{K}{(1+r)^n}.$$

By optional sampling - part II (Theorem 4.3.3),

$$\tilde{\mathbf{E}}[\frac{1}{(1+r)^{\tau\wedge N}}(S_{\tau\wedge N} - K)] \leq \tilde{\mathbf{E}}[\frac{1}{(1+r)^N}(S_N - K)] = S_0 - \frac{K}{(1+r)^N}.$$

Thus, the optimal stopping time is $\tau = N$; the time zero value of the option is $S_0 - \frac{K}{(1+r)^N}$.