Chapter 3 solutions to Stochastic Calculus for Finance II by Steven E. Shreve

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1. Let $A \in \sigma(W(u_2) - W(u_1))$ and $B \in \mathcal{F}(t)$. By the "information accumulates" property, $B \in \mathcal{F}(u_1)$. We know that $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(u_1)$. It then follows from the definition of independence, that $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. This proves that $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

2.
$$\begin{aligned}
&\mathbb{E}[W^2(t) - t|\mathcal{F}(s)] \\
&= \mathbb{E}[(W(t) - W(s))^2 + 2W(s)W(t) - W^2(s) - t|\mathcal{F}(s)] \\
&= \mathbb{E}[(W(t) - W(s))^2|\mathcal{F}(s)] + 2W(s)\mathbb{E}[W(t)|\mathcal{F}(s)] - W^2(s) - t && \text{(Taking out what is known.)} \\
&= \mathbb{E}[(W(t) - W(s))^2] + 2W(s)\mathbb{E}[W(t)|\mathcal{F}(s)] - W^2(s) - t && (W(t) - W(s) \text{ independent of } \mathcal{F}(s)) \\
&= t - s + 2W(s)\mathbb{E}[W(t)|\mathcal{F}(s)] - W^2(s) - t && \text{(By definition of Brownion motion.)} \\
&= t - s + 2W^2(s) - W^2(s) - t && \text{(Brownian motion is a martingale.)} \\
&= W^2(s) - s.
\end{aligned}$$

3. Let us compute the third derivative:
$$\phi^{(3)}(u) = \mathbb{E}[(X-\mu)^3 e^{u(X-\mu)}] = (3\sigma^4 u + \sigma^6 u^3) e^{\frac{1}{2}\sigma^2 u^2}.$$
For the fourth derivative:
$$\phi^{(4)}(u) = \mathbb{E}[(X-\mu)^4 e^{u(X-\mu)}] = (3\sigma^4 + 6\sigma^6 u^2 + \sigma^8 u^4) e^{\frac{1}{2}\sigma^2 u^2}.$$
In particular,
$$\phi^{(4)}(0) = \mathbb{E}[(X-\mu)^4] = 3\sigma^4.$$

4. (i) Note that W is uniform continuous over the interval $[0, T]$, i.e. for any $\epsilon > 0$, there is $\delta > 0$ such that
$$|x - y| < \delta \Rightarrow |W(x) - W(y)| < \epsilon.$$
In particular, for any partition $\Pi = \{t_0, \cdots, t_n\}$ such that the $||\Pi|| < \delta$, $\max_{1 \le k \le n-1} |W(t_{k+1}) - W(t_k)| < \epsilon$. In other words,
$$\max_{1 \le k \le n-1} |W(t_{k+1}) - W(t_k)| \to 0 \quad \text{as} \quad ||\Pi|| \to 0.$$
On the other hand, we have the following inequality
$$\frac{\sum_{k=1}^{n-1}(W(t_{k+1}) - W(t_k))^2}{\max_{1 \le k \le n-1} |W(t_{k+1}) - W(t_k)|} \le \sum_{k=1}^{n-1} |W(t_{k+1}) - W(t_k)|.$$
As $||\Pi|| \to 0$,
$$\sum_{k=1}^{n-1}(W(t_{k+1}) - W(t_k))^2 \to T \quad \text{almost surely.}$$

Therefore
$$\sum_{k=1}^{n-1} |W(t_{k+1}) - W(t_k)| \to \infty \quad \text{almost surely.}$$

(ii)
$$\sum_{k=1}^{n-1} |W(t_{k+1}) - W(t_k)|^3 \le \max_{1\le k\le n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{k=1}^{n-1} (W(t_{k+1}) - W(t_k))^2.$$

As $||\Pi|| \to 0$,
$$\max_{1\le k\le n-1} |W(t_{k+1}) - W(t_k)| \to 0$$

and
$$\sum_{k=1}^{n-1} (W(t_{k+1}) - W(t_k))^2 \to T \quad \text{almost surely.}$$

Therefore, as $||\Pi|| \to 0$,
$$\sum_{k=1}^{n-1} (W(t_{k+1}) - W(t_k))^3 \to 0 \quad \text{almost surely.}$$

5. Define the random variable
$$X := \frac{W(T)}{\sqrt{T}}.$$

We know that X has a standard normal distribution. Then writing the stock price $S(T)$ in terms of X, we have

$$\begin{aligned}
& S(T) - K \ge 0 \\
\iff & S(0)e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}X} - K \ge 0 \\
\iff & (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}X \ge -\log\frac{S(0)}{K} \\
\iff & X \ge -\frac{1}{\sigma\sqrt{T}}\big(\log\frac{S(0)}{K} + (r - \frac{1}{2}\sigma^2)T\big) = -d_-(T, S(0)).
\end{aligned}$$

We now compute the expectation of the discounted price of the call option.

$$\begin{aligned}
& \mathbb{E}[e^{-rT}\big(S(T) - K\big)^+] \\
&= \frac{1}{\sqrt{2\pi}} \cdot e^{-rT} \int_{-\infty}^{\infty} (S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K)^+ \cdot e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot e^{-rT} \int_{-d_-(T,S(0))}^{\infty} (S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K) \cdot e^{-\frac{x^2}{2}}\,dx \\
&= -Ke^{-rT}N(d_-(T,S(0))) + \frac{1}{\sqrt{2\pi}} \cdot e^{-rT} \int_{-d_-(T,S(0))}^{\infty} S(0)e^{rT - \frac{1}{2}(x-\sigma\sqrt{T})^2}\,dx \\
&= -Ke^{-rT}N(d_-(T,S(0))) + S(0) \cdot \frac{1}{\sqrt{2\pi}} \int_{-d_-(T,S(0))-\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}y^2}\,dy \\
&= -Ke^{-rT}N(d_-(T,S(0))) + S(0)N(d_+(T,S(0))).
\end{aligned}$$

For the second equality above, we used the result $S(T) - K \ge 0 \iff X \ge -d_-(T, S(0))$ that we just proved. For the fourth equality, we applied a change of variable $y = x - \sigma\sqrt{T}$.

6. (i) Since $W(t) - W(s)$ is independent of $\mathcal{F}(s)$,

$$X(t) - X(s) = W(t) - W(s) + \mu(t-s) \quad \text{is independent of} \quad \mathcal{F}(s).$$

(Adding the constant $(t-s)$ to $W(t) - W(s)$ does not change the sigma algebra generated by it.) Let

$$g(x) = \mathbb{E}[f\big((X(t) - X(s)) + x\big)].$$

Then by the independence lemma,

$$\mathbb{E}[f\big(X(t)\big)|\mathcal{F}(s)] = \mathbb{E}[f\big((X(t) - X(s)) + X(s)\big)|\mathcal{F}(s)] = g(X(s)).$$

Let us compute $g(x)$. First we define the random variable $Z := W(t) - W(s)$. It is normally distributed with mean 0 and variance $t-s$. Therefore, $f\big((X(t) - X(s)) + x\big) = f(Z + \mu(t-s) + x)$.

$$\begin{aligned}\mathbb{E}[f(Z + \mu(t-s) + x)] &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(z + \mu(t-s) + x) e^{-\frac{z^2}{2(t-s)}} dz \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{\left(y - x - \mu(t-s)\right)^2}{2(t-s)}} dy,\end{aligned}$$

where we used the change of variable $y = z + \mu(t-s) + x$ in the second equality.

(ii) Since $W(t) - W(s)$ is independent of $\mathcal{F}(s)$,

$$\frac{S(t)}{S(s)} = \exp\{\sigma\big(W(t) - W(s)\big) + \nu(t-s)\} \quad \text{is independent of} \quad \mathcal{F}(s).$$

Let

$$g(x) = \mathbb{E}[f\big(\frac{S(t)}{S(s)} \cdot x\big)].$$

Then by the independence lemma,

$$\mathbb{E}[f\big(S(t)\big)|\mathcal{F}(s)] = \mathbb{E}[f\big(\frac{S(t)}{S(s)} \cdot S(s)\big)|\mathcal{F}(s)] = g(S(s)).$$

Let us compute $g(x)$. Again we let $Z := W(t) - W(s)$. Then we can write

$$f\big(\frac{S(t)}{S(s)} \cdot x\big) = f(x \exp\{\sigma Z + \nu(t-s)\}).$$

Taking expectation over Z, we have

$$\mathbb{E}[f\big(\frac{S(t)}{S(s)} \cdot x\big)] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(x \exp\{\sigma z + \nu(t-s)\}) e^{-\frac{z^2}{2(t-s)}} dz \tag{1}$$

Letting $\tau = t - s$ and $y = x\exp\{\sigma z + \nu\tau\}$. Then $z = \frac{1}{\sigma}\big(\log\frac{y}{x} - \nu\tau\big)$ and

$$dz = \frac{1}{\sigma y} dy.$$

Plugging these new variables into (1), we have

$$\mathbb{E}[f\big(\frac{S(t)}{S(s)} \cdot x\big)] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(\log\frac{y}{x} - \nu\tau)^2}{2\sigma^2\tau}} \cdot \frac{1}{\sigma y} dy.$$

7. (i) Note that $Z(t)$ can be written as

$$\exp\{\sigma X(t) - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)t\} = \exp\{\sigma\big(\mu t + W(t)\big) - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)t\} = \exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t\},$$

which is the geometric Brownian motion, a martingale.

(ii) Using the facts that $Z(t)$ is a martingale, and that a stopped martingale is a martingale,

$$1 = Z(0) = \mathbb{E}[Z(t\wedge\tau_m)] = \mathbb{E}[\exp\{\sigma X(t\wedge\tau_m) - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)(t\wedge\tau_m)\}]$$

(iii) Let us analyze the behaviour of $Z(t\wedge\tau_m)$ as $t\to\infty$.

$$0 \le \limsup_{t\to\infty} \exp\{\sigma X(t\wedge\tau_m)\} \le \exp\{\sigma m\}$$

and

$$\lim_{t\to\infty} \exp\{\sigma X(t\wedge\tau_m)\} = \exp\{\sigma m\} \quad \text{if} \quad \tau_m < \infty.$$

$$\lim_{t\to\infty} \exp\{-\big(\sigma\mu + \frac{1}{2}\sigma^2\big)(t\wedge\tau_m)\} = \mathbf{1}_{\{\tau_m<\infty\}} \exp\{-\big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}$$

These equations imply that

$$\lim_{t\to\infty} Z(t\wedge\tau_m) = \mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}$$

Since $Z(t\wedge\tau_m) \le \exp\{\sigma m\}$ for any t, we may apply dominated convergence theorem:

$$\mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}] = \mathbb{E}[\lim_{t\to\infty} Z(t\wedge\tau_m)] = \lim_{t\to\infty} \mathbb{E}[Z(t\wedge\tau_m)] = 1.$$

Since $\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\} \le \exp\{\sigma m\}$ for any $\sigma > 0$, we may apply the dominated convergence theorem again to get

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}}] &= \mathbb{E}[\lim_{\sigma\to 0^+} \mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}] \\ &= \lim_{\sigma\to 0^+} \mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}] = 1.\end{aligned}$$

This implies that $\mathbf{1}_{\{\tau_m=\infty\}}$ has measure zero. Therefore,

$$1 = \mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}] = \mathbb{E}[\exp\{\sigma m - \big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}]$$

and

$$\mathbb{E}[\exp\{-\big(\sigma\mu + \frac{1}{2}\sigma^2\big)\tau_m\}] = e^{-\sigma m}. \tag{2}$$

If we let $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$ and solve for σ, we get

$$\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}$$

by solving the quadratic equation. The negative root is ignored here because we assume that $\sigma > 0$. Plugging this into (2), we have

$$\mathbb{E}[\exp\{-\alpha\tau_m\}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}}. \tag{3}$$

(iv) Differentiating (3) with respect to α, we have

$$\mathbb{E}[-\tau_m \cdot \exp\{-\alpha\tau_m\}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \cdot \frac{-m}{\sqrt{2\alpha+\mu^2}},$$

and

$$\mathbb{E}[\tau_m \cdot \exp\{-\alpha\tau_m\}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \cdot \frac{m}{\sqrt{2\alpha+\mu^2}},$$

For $0 < \alpha_1 < \alpha_2$,

$$\tau_m \cdot \exp\{-\alpha_1\tau_m\} \geq \tau_m \cdot \exp\{-\alpha_2\tau_m\}.$$

Therefore, we can apply the monotone convergence theorem to get

$$\mathbb{E}[\tau_m] = \mathbb{E}[\lim_{\alpha\to 0^+} \tau_m \cdot \exp\{-\alpha\tau_m\}] = \lim_{\alpha\to 0^+} e^{m\mu - m\sqrt{2\alpha+\mu^2}} \cdot \frac{m}{\sqrt{2\alpha+\mu^2}} = \frac{m}{\mu}.$$

(v) If $\sigma > -2\mu$,

$$\sigma\mu + \frac{1}{2}\sigma^2 > 0.$$

This implies that

$$\lim_{t\to\infty} \exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)(t\wedge\tau_m)\} = \mathbf{1}_{\{\tau_m<\infty\}} \exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}$$

and that $Z(t\wedge\tau_m) \leq \exp\{\sigma m\}$ still holds. We may then repeat the dominated convergence theorem argument in part three to show that

$$\mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] = \mathbb{E}[\lim_{t\to\infty} Z(t\wedge\tau_m)] = \lim_{t\to\infty} \mathbb{E}[Z(t\wedge\tau_m)] = 1.$$

Since $\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\} \leq \exp\{\sigma m\}$ for any $\sigma > -2\mu$, we may apply the dominated convergence theorem again to get

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{-2\mu m\}] &= \mathbb{E}[\lim_{\sigma\to -2\mu^+} \mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] \\ &= \lim_{\sigma\to -2\mu^+} \mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] = 1,\end{aligned}$$

i.e.

$$\mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}}] = \exp\{2\mu m\} < 1.$$

If $\{\tau_m = \infty\}$, then $\exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\} = 0$. Therefore,

$$1 = \mathbb{E}[\mathbf{1}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] = \mathbb{E}[\exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}].$$

We put $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$ and solve for σ in this quadratic equation to get

$$\sigma = -\mu + \sqrt{\mu^2 + 2\alpha}.$$

Note that $\sigma \neq -\mu - \sqrt{\mu^2 + 2\alpha}$ because the latter term is negative. Plugging in, we have

$$\exp\{m\mu - m\sqrt{2\alpha + \mu^2}\} = \mathbb{E}[\exp\{-\alpha\tau_m\}].$$

8. (i)

$$\begin{aligned}
\varphi_n(u) &= \mathbb{E}[\exp\{u \cdot \frac{1}{\sqrt{n}} M_{nt,n}\}] \\
&= \mathbb{E}[\exp\{u \cdot \frac{1}{\sqrt{n}} \sum_{i=k}^{nt} X_{k,n}\}] \\
&= \prod_{i=k}^{nt} \mathbb{E}[\exp\{u \cdot \frac{1}{\sqrt{n}} X_{k,n}\}] \\
&= [e^{\frac{u}{\sqrt{n}}} \tilde{p}_n + e^{\frac{-u}{\sqrt{n}}} \tilde{q}_n]^{nt} \\
&= \big[e^{\frac{u}{\sqrt{n}}} \cdot \frac{\frac{r}{n} + 1 - e^{-\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}} - e^{-\frac{\sigma}{\sqrt{n}}}} - e^{\frac{-u}{\sqrt{n}}} \cdot \frac{\frac{r}{n} + 1 - e^{\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}} - e^{-\frac{\sigma}{\sqrt{n}}}}\big]^{nt}
\end{aligned}$$

where we used independence of $X_{k,n}$ in the third equality.

(ii) Plugging in $x = \frac{1}{\sqrt{n}}$ into the result in part one, we have

$$\begin{aligned}
\varphi_{\frac{1}{x^2}}(u) &= \big[e^{ux} \cdot \frac{rx^2 + 1 - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} - e^{-ux} \cdot \frac{rx^2 + 1 - e^{\sigma x}}{e^{\sigma x} - e^{-\sigma x}}\big]^{\frac{t}{x^2}} \\
&= \big[\frac{(rx^2 + 1)\sinh ux + \sinh{(\sigma - u)x}}{\sinh \sigma x}\big]^{\frac{t}{x^2}}
\end{aligned}$$

Taking log, we have

$$\begin{aligned}
\log \varphi_{\frac{1}{x^2}}(u) &= \frac{t}{x^2} \log\big(\frac{(rx^2 + 1)\sinh ux + \sinh{(\sigma - u)x}}{\sinh \sigma x}\big) \\
&= \frac{t}{x^2} \log\big(\frac{(rx^2 + 1)\sinh ux + \sinh \sigma x \cosh ux - \cosh \sigma x \sinh ux}{\sinh \sigma x}\big) \\
&= \frac{t}{x^2} \log\big(\cosh ux + \frac{(rx^2 + 1 - \cosh \sigma x)\sinh ux}{\sinh \sigma x}\big).
\end{aligned}$$

(iii) The taylor series expansion of the first term is:

$$\cosh ux = 1 + \frac{1}{2}u^2x^2 + \mathcal{O}(x^4).$$

For the second term,

$$\begin{aligned}
\frac{(rx^2 + 1 - \cosh \sigma x)\sinh ux}{\sinh \sigma x} &= \frac{\big(rx^2 + 1 - (1 + \frac{1}{2}\sigma^2x^2 + \mathcal{O}(x^4))\big)(ux + \mathcal{O}(x^3))}{\sigma x + \mathcal{O}(x^3)} \\
&= \frac{rux^3 - \frac{1}{2}ux^3\sigma^2 + \mathcal{O}(x^5)}{\sigma x + \mathcal{O}(x^3)} \\
&= \frac{rux^2}{\sigma} - \frac{1}{2}ux^2\sigma + \mathcal{O}(x^4).
\end{aligned}$$

Adding this two equations, we have

$$\cosh ux + \frac{(rx^2 + 1 - \cosh \sigma x)\sinh ux}{\sinh \sigma x} = 1 + \frac{1}{2}u^2x^2 + \frac{rux^2}{\sigma} - \frac{1}{2}ux^2\sigma + \mathcal{O}(x^4).$$

(iv) We apply the Taylor series expansion $\log(1+x) = x + \mathcal{O}(x^2)$ to expand

$$\begin{aligned}\log \varphi_{\frac{1}{x^2}}(u) &= \frac{t}{x^2}\log\big(\cosh ux + \frac{(rx^2+1-\cosh\sigma x)\sinh ux}{\sinh\sigma x}\big)\\ &= \frac{t}{x^2}(\frac{1}{2}u^2x^2 + \frac{rux^2}{\sigma} - \frac{1}{2}ux^2\sigma + \mathcal{O}(x^4))\\ &= t(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2}) + \mathcal{O}(x^2)),\end{aligned}$$

Taking limit $x \to 0$, we have

$$\lim_{x\to 0}\log\varphi_{\frac{1}{x^2}}(u) = t(\frac{u^2}{2} + \frac{ru}{\sigma} - \frac{u\sigma}{2})$$

Therefore, the limiting distribution of the moment generating function of $\frac{\sigma}{\sqrt{n}}M_{nt,n}$ is

$$\lim_{x\to 0}\varphi_{\frac{1}{x^2}}(\sigma u) = \exp\{t(\frac{\sigma^2u^2}{2} + ru - \frac{u\sigma^2}{2})\}. \tag{4}$$

On the other hand, let X be a normally distributed random variable with mean μ and variance σ_x^2. Let us compute its moment generating function:

$$\begin{aligned}\mathbb{E}[e^{uX}] &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{ux - \frac{(x-\mu)^2}{2\sigma_x^2}\}dx\\ &= \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{-\frac{x^2-2(\mu+u\sigma_x^2)x+\mu^2}{2\sigma_x^2}\}dx\\ &= \exp\{\frac{(\mu+u\sigma_x^2)^2-\mu^2}{2\sigma_x^2}\}\cdot\frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{-\frac{\big(x-(\mu+u\sigma_x^2)\big)^2}{2\sigma_x^2}\}dx\\ &= \exp\{\frac{2\mu u\sigma_x^2+u^2\sigma_x^4}{2\sigma_x^2}\}\cdot 1 = \exp\{\mu u + \frac{1}{2}u^2\sigma_x^2\}\end{aligned}$$

If $\mu = (r-\frac{1}{2}\sigma^2)t$ and $\sigma_x^2 = \sigma^2 t$, the moment generating function of X is:

$$\exp\{(r-\frac{1}{2}\sigma^2)tu + \frac{1}{2}u^2\sigma^2 t\},$$

coinciding with (4).

9. (i) Write $\alpha_k'(m) := \frac{d\alpha_k(m)}{dm}$. Then

$$\alpha_k'(m) = -m\frac{1}{\sqrt{2\pi}}\int_0^{\infty}t^{-\frac{k+2}{2}}\exp\{-\alpha t - \frac{m^2}{2t}\}dt = -m\alpha_{k+2}(m)$$

Differentiating

$$g(\alpha,m) = ma_3(m)$$

with respect to m, we have

$$g_m(\alpha,m) = a_3(m) + ma_3'(m) = a_3(m) - m^2a_5(m).$$

and

$$g_{mm}(\alpha,m) = a_3'(m) - 2ma_5(m) - m^2a_5'(m) = -3ma_5(m) + m^3a_7(m).$$

(ii) We let $u(t) := \exp\{-\alpha t - \frac{m^2}{2t}\}$ and $v(t) = \frac{-2}{3}t^{-\frac{3}{2}}$. Then $u'(t) = (-\alpha + \frac{m^2}{2t^2})\exp\{-\alpha t - \frac{m^2}{2t}\}$ and $v'(t) = t^{-\frac{5}{2}}$. We then use integration by parts to compute $\alpha_5(m)$.

$$\begin{aligned}\alpha_5(m) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty u(t)v'(t)dt \\ &= \frac{1}{\sqrt{2\pi}}\big(u(t)v(t)\big)|_{t=0}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_0^\infty u'(t)v(t)dt \\ &= 0 - \frac{1}{\sqrt{2\pi}}\int_0^\infty(-\alpha + \frac{m^2}{2t^2})\exp\{-\alpha t - \frac{m^2}{2t}\}\frac{-2}{3}t^{-\frac{3}{2}}dt \\ &= -\frac{2\alpha}{3}\alpha_3(m) + \frac{m^2}{3}\alpha_7(m).\end{aligned}$$

(iii) By part two,

$$m^3 a_7(m) = 3ma_5(m) + 2\alpha m a_3(m).$$

Plugging this into the equation on $g_{mm}(m)$, we have

$$g_{mm}(\alpha, m) = -3ma_5(m) + 3ma_5(m) + 2\alpha m a_3(m) = 2\alpha m a_3(m) = 2\alpha g(\alpha, m).$$

(iv) The roots to the quadratic equation

$$\lambda^2 - 2\alpha = 0$$

are $\lambda_1 = \sqrt{2\alpha}$ and $\lambda_2 = -\sqrt{2\alpha}$. Therefore, the general solution to our second order ODE

$$g_{mm}(\alpha, m) - 2\alpha g(\alpha, m) = 0$$

is

$$g(\alpha, m) = A_1 e^{\sqrt{2\alpha}m} + A_2 e^{-\sqrt{2\alpha}m}$$

(v) We can split $g(\alpha, m)$ into two terms:

$$g(\alpha, m) = \frac{m}{\sqrt{2\pi}}\int_0^m t^{-\frac{3}{2}}\exp\{-\alpha t - \frac{m^2}{2t}\}dt + \frac{m}{\sqrt{2\pi}}\int_m^\infty t^{-\frac{3}{2}}\exp\{-\alpha t - \frac{m^2}{2t}\}dt \tag{5}$$

For the first term, note that

$$e^{-\alpha t} \le 1 \quad \text{for } t > 0.$$

and

$$1 \le \sqrt{\frac{m}{t}} \quad \text{for } 0 < t < m.$$

Therefore, we can bound the first term as follows:

$$\frac{m}{\sqrt{2\pi}}\int_0^m t^{-\frac{3}{2}}\exp\{-\alpha t - \frac{m^2}{2t}\}dt \le \frac{m}{\sqrt{2\pi}}\int_0^m \sqrt{\frac{m}{t}}t^{-\frac{3}{2}}\exp\{-\frac{m^2}{2t}\}dt. \tag{6}$$

For the second term, note that

$$t^{-\frac{3}{2}} \le m^{-\frac{3}{2}} \quad \text{for } t > m$$

and

$$\exp\{-\frac{m^2}{2t}\} \le 1 \quad \text{for } t > m.$$

Therefore, we can bound the second term as follows:

$$\frac{m}{\sqrt{2\pi}}\int_m^\infty t^{-\frac{3}{2}}\exp\{-\alpha t - \frac{m^2}{2t}\}dt \le \frac{1}{\sqrt{2\pi m}}\int_m^\infty \exp\{-\alpha t\}dt \tag{7}$$

Plugging the two inequalities (6) and (7) into (5), we get

$$g(\alpha, m) \le \frac{m}{\sqrt{2\pi}} \int_0^m \sqrt{\frac{m}{t}} t^{-\frac{3}{2}} \exp\{-\frac{m^2}{2t}\} dt + \frac{1}{\sqrt{2\pi m}} \int_m^\infty \exp\{-\alpha t\} dt.$$

Applying a change of variable $s = \frac{t}{m}$ to the first term on the right hand side, we have

$$\frac{m}{\sqrt{2\pi}} \int_0^m \sqrt{\frac{m}{t}} t^{-\frac{3}{2}} \exp\{-\frac{m^2}{2t}\} dt = \frac{1}{\sqrt{2\pi}} \int_0^1 s^{-\frac{5}{2}} \exp\{-\frac{m}{2s}\} ds,$$

which goes to zero as $m \to \infty$. It is obvious that $\lim_{m\to\infty} \frac{1}{\sqrt{2\pi m}} \int_m^\infty \exp\{-\alpha t\} dt = 0$. Therefore, $\lim_{m\to\infty} g(\alpha, m) = 0$.

In part four, we showed that

$$g(\alpha, m) = A_1 e^{\sqrt{2\alpha} m} + A_2 e^{-\sqrt{2\alpha} m}.$$

Assume $A_1 \neq 0$. Letting $m \to \infty$, the right hand side converges to $\pm\infty$, contradicting to the result we just proved. This implies $A_1 = 0$ and

$$g(\alpha, m) = A_2 e^{-\sqrt{2\alpha} m}. \tag{8}$$

(vi) We first apply the change of variables $s = \frac{t}{m^2}$to the integral form of $g(\alpha, m)$:

$$\begin{aligned} g(\alpha, m) &= \frac{m}{\sqrt{2\pi}} \int_0^\infty t^{-\frac{3}{2}} \exp\{-\alpha t - \frac{m^2}{2t}\} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty s^{-\frac{3}{2}} \exp\{-\alpha m^2 s - \frac{1}{2s}\} ds. \end{aligned}$$

Then apply the change of variables $y = \frac{1}{\sqrt{s}}$, we have

$$g(\alpha, m) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \exp\{-\frac{\alpha m^2}{y^2} - \frac{y^2}{2}\} dy.$$

Note that we have used the facts $dy = -\frac{1}{2} s^{-\frac{3}{2}} ds$; $y \to \infty$ as $s \to 0$ and $y \to 0$ as $s \to \infty$.

For $0 < m_1 < m_2$, we have the following inequality on integrands:

$$\exp\{-\frac{\alpha m_2^2}{y^2} - \frac{y^2}{2}\} \le \exp\{-\frac{\alpha m_1^2}{y^2} - \frac{y^2}{2}\}.$$

Also note that

$$\lim_{m\to 0^+} \exp\{-\frac{\alpha m^2}{y^2} - \frac{y^2}{2}\} = \exp\{-\frac{y^2}{2}\}.$$

We may then apply monotone convergence theorem to prove that

$$\lim_{m\to 0^+} g(\alpha, m) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \lim_{m\to 0^+} \exp\{-\frac{\alpha m^2}{y^2} - \frac{y^2}{2}\} dy = \frac{2}{\sqrt{2\pi}} \int_0^\infty \exp\{-\frac{y^2}{2}\} dy = 1.$$

On the other hand, taking limit $m \to 0$ in (8), we have

$$1 = \lim_{m\to 0^+} g(\alpha, m) = \lim_{m\to 0^+} A_2 e^{-\sqrt{2\alpha} m} = A_2,$$

implying that $A_2 = 1$ and

$$g(\alpha, m) = e^{-\sqrt{2\alpha} m}.$$