Chapter 2 solutions to Stochastic Calculus for Finance I by Steven E. Shreve

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1. (i)

$$\begin{aligned}
&\sum_{\omega\in\Omega}\mathbf{P}(\omega) &&= 1\\
\iff &\sum_{\omega\in A}\mathbf{P}(\omega)+\sum_{\omega\in A^c}\mathbf{P}(\omega) &&= 1\\
\iff &\mathbf{P}(A)+\mathbf{P}(A^c) &&= 1.
\end{aligned}$$

(ii)

$$\mathbf{P}(\bigcup_{n=1}^N A_n) = \sum_{\omega\in\bigcup_{n=1}^N A_n}\mathbf{P}(\omega) \le \sum_{n=1}^N\sum_{\omega\in A_n}\mathbf{P}(\omega) = \sum_{n=1}^N \mathbf{P}(A_n).$$

The inequality above holds because any term $\mathbf{P}(\omega)$ on the left must appear at least once on the right, since

$$\omega\in\bigcup_{n=1}^N A_n \iff \omega\in A_n \text{ for some } n.$$

The inequality above is an equality if and only if any term on the left side appears exactly once on the right hand side, i.e. $A_1,\cdots,A_n$ are disjoint.

2. (i)

$$\tilde{\mathbf{P}}(S_3=32)=\tilde{\mathbf{P}}(HHH)=\frac{1}{2^3}.$$

$$\tilde{\mathbf{P}}(S_3=8)=\tilde{\mathbf{P}}(HHT)+\tilde{\mathbf{P}}(HTH)+\tilde{\mathbf{P}}(THH)=3\cdot\frac{1}{2^3}.$$

$$\tilde{\mathbf{P}}(S_3=2)=\tilde{\mathbf{P}}(HTT)+\tilde{\mathbf{P}}(THT)+\tilde{\mathbf{P}}(TTH)=3\cdot\frac{1}{2^3}.$$

$$\tilde{\mathbf{P}}(S_3=0.5)=\tilde{\mathbf{P}}(TTT)=\frac{1}{2^3}.$$

(ii)

$$\tilde{\mathbf{E}}[S_1]=\frac{1}{2}\cdot 8+\frac{1}{2}\cdot 2=5;$$

$$\tilde{\mathbf{E}}[S_2]=\frac{1}{4}\cdot 16+\frac{1}{2}\cdot 4+\frac{1}{4}\cdot 1=\frac{25}{4};$$

$$\tilde{\mathbf{E}}[S_3]=\frac{1}{8}\cdot 32+\frac{3}{8}\cdot 8+\frac{3}{8}\cdot 2+\frac{1}{8}\cdot 0.5=\frac{125}{16}.$$

The growth rate is .25.

(iii)

$$\mathbf{P}(S_3=32)=\mathbf{P}(HHH)=\frac{8}{27}.$$

$$\mathbf{P}(S_3=8)=\mathbf{P}(HHT)+\mathbf{P}(HTH)+\mathbf{P}(THH)=3\cdot\frac{4}{27}.$$

$$\mathbf{P}(S_3=2)=\mathbf{P}(HTT)+\mathbf{P}(THT)+\mathbf{P}(TTH)=3\cdot\frac{2}{27}.$$

$$\mathbf{E}[S_1] = \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2 = 6;$$
$$\mathbf{E}[S_2] = \frac{4}{9}\cdot 16 + \frac{4}{9}\cdot 4 + \frac{1}{9}\cdot 1 = 9;$$
$$\tilde{\mathbf{E}}[S_3] = \frac{8}{27}\cdot 32 + \frac{12}{27}\cdot 8 + \frac{6}{27}\cdot 2 + \frac{1}{27}\cdot 0.5 = \frac{729}{54} = 13.5.$$

The growth rate is .5.

3.

$$\begin{aligned}\mathbf{E}_n[\varphi(M_{n+1})] &\geq \varphi(\mathbf{E}_n[M_{n+1}]) && \text{(Jensen's inequality)}\\ &= \varphi(M_n) && (M_n \text{ is a martingale})\end{aligned}$$

4. (i)

$$\begin{aligned}\mathbf{E}_n[M_{n+1}] &= \mathbf{E}_n[M_n + X_{n+1}]\\ &= M_n + \mathbf{E}_n[X_{n+1}] && (M_n \text{ is known at time } n)\\ &= M_n + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1)\\ &= M_n\end{aligned}$$

(ii)

$$\begin{aligned}\mathbf{E}_n[S_{n+1}] &= \mathbf{E}_n[e^{\sigma M_{n+1}}(\frac{2}{e^{\sigma}+e^{-\sigma}})^{n+1}]\\ &= \mathbf{E}_n[S_n e^{\sigma X_{n+1}}(\frac{2}{e^{\sigma}+e^{-\sigma}})]\\ &= S_n(\frac{2}{e^{\sigma}+e^{-\sigma}})\mathbf{E}_n[e^{\sigma X_{n+1}}] && \text{(Taking out what is known)}\\ &= S_n(\frac{2}{e^{\sigma}+e^{-\sigma}})(\frac{1}{2}e^{\sigma}+\frac{1}{2}e^{-\sigma})\\ &= S_n\end{aligned}$$

5. (i) We prove the equation

$$I_n = \frac{1}{2}M_n^2 - \frac{n}{2}. \tag{1}$$

by induction on n. For $n = 1$, $I_1 = M_0(M_1 - M_0) = 0$ and

$$\begin{aligned}&\frac{1}{2}M_1^2 - \frac{1}{2}\\ &= \frac{1}{2}(1^2) - \frac{1}{2} = 0 \quad \text{or} \quad \frac{1}{2}((-1)^2) - \frac{1}{2} = 0\end{aligned}$$

Assume (1) holds for n. Then

$$\begin{aligned}
I_{n+1} &= \sum_{j=0}^{n} M_j(M_{j+1} - M_j) \\
&= I_n + M_n(M_{n+1} - M_n) \\
&= \frac{1}{2}M_n^2 - \frac{n}{2} + M_n(M_{n+1} - M_n) && \text{(By induction)} \\
&= \frac{1}{2}(M_{n+1} - X_{n+1})^2 + (M_{n+1} - X_{n+1})X_{n+1} - \frac{n}{2} && (M_n = M_{n+1} - X_{n+1}) \\
&= \frac{1}{2}(M_{n+1}^2 - 2M_{n+1}X_{n+1} + X_{n+1}^2) + \frac{1}{2}(2M_{n+1}X_{n+1} - 2X_{n+1}^2) - \frac{n}{2} \\
&= \frac{1}{2}M_{n+1}^2 - \frac{1}{2}X_{n+1}^2 - \frac{n}{2} \\
&= \frac{1}{2}M_{n+1}^2 - \frac{n+1}{2} && (X_{n+1} = \pm 1)
\end{aligned}$$

(ii) Let us express I_{n+1} in terms of I_n.

$$\begin{aligned}
I_{n+1} &= \frac{1}{2}M_{n+1}^2 - \frac{n+1}{2} \\
&= \frac{1}{2}(M_n + X_{n+1})^2 - \frac{n+1}{2} \\
&= \frac{1}{2}(M_n^2 + 2M_nX_{n+1} + X_{n+1}^2) - \frac{n+1}{2} \\
&= I_n + \frac{1}{2}M_nX_{n+1} - \frac{n}{2}
\end{aligned}$$

Note that $M_n = \pm\sqrt{2I_n + n}$. Then

$$\mathbf{E}_n[f(I_{n+1})] = \frac{1}{2}f(I_n + \frac{1}{2}\sqrt{2I_n + n} - \frac{n}{2}) + \frac{1}{2}f(I_n - \frac{1}{2}\sqrt{2I_n + n} - \frac{n}{2}).$$

So we take g be the function on I_n and n on the right hand side.

6.

$$\begin{aligned}
\mathbf{E}_n[I_{n+1}] &= \mathbf{E}_n[\sum_{j=0}^{n-1} \Delta_j(M_{j+1} - M_j) + \Delta_n(M_{n+1} - M_n)] \\
&= I_n + \mathbf{E}_n[\Delta_n(M_{n+1} - M_n)] \\
&= I_n + \Delta_n\mathbf{E}_n[(M_{n+1} - M_n)] && (\Delta_n \text{ is known at time } n) \\
&= I_n + \Delta_n(\mathbf{E}_n[M_{n+1}] - M_n) \\
&= I_n + \Delta_n(M_n - M_n) && (M_n \text{ is a martingale}) \\
&= I_n
\end{aligned}$$

7. Consider a two period model with stochastic process X_0, X_1 and X_2. Let ω_1 and ω_2 be independent and

$$\mathbf{P}(\omega_1 = H) = \mathbf{P}(\omega_2 = H) = \frac{1}{2}.$$

Let

$$X_0 = X_1(H) = X_1(T) = X_2(TH) = X_2(TT) = 1; \quad X_2(HH) = 2; \quad X_2(HT) = 0.$$

Then it is easy to check that

$$\mathbf{E}[X_1] = \mathbf{E}_1[X_2] = 1.$$

So they form a martingale. On the other hand, the distribution of X_2 given X_1 depends on the first coin toss. Therefore X_0, X_1, X_2 is not a Markov process.

Let us show this via the book's definition of a Markov process. Let $f(x) = x^2$. Then

$$\mathbf{E}_1[f(X_2)](H) = 2; \quad \mathbf{E}_1[f(X_2)](T) = 1.$$

But $X_1(H) = X_1(T) = 1$, so $\mathbf{E}_1[f(X_2)]$ cannot be expressed as $g(X_1)$.

8. (i) We prove by backward induction on n. Suppose $M_k = M'_k$ for $k \geq n+1$.

$$M_n = \tilde{\mathbf{E}}_n[M_{n+1}] = \tilde{\mathbf{E}}_n[M'_{n+1}] = M'_n$$

The first and third equalities hold because of the martingale property of M_n and M'_n respectively. The second equality follows by induction.

(ii)

$$\begin{aligned}
&\tilde{\mathbf{E}}_n[\frac{V_{n+1}}{(1+r)^{n+1}}](\bar{\omega_1}, \cdots, \bar{\omega_n}) \\
&= \frac{1}{(1+r)^{n+1}} \cdot (\frac{1+r-d}{u-d} V_{n+1}(\bar{\omega_1}, \cdots, \bar{\omega_n}, H) + \frac{u-1-r}{u-d} V_{n+1}(\bar{\omega_1}, \cdots, \bar{\omega_n}, T)) \\
&= \frac{1}{(1+r)^{n+1}} \cdot (\frac{1+r-d}{u-d} uV_n(\bar{\omega_1}, \cdots, \bar{\omega_n}) + \frac{u-1-r}{u-d} dV_n(\bar{\omega_1}, \cdots, \bar{\omega_n})) \\
&= \frac{1}{(1+r)^n} V_n(\bar{\omega_1}, \cdots, \bar{\omega_n})
\end{aligned}$$

(iii)

$$\tilde{\mathbf{E}}_n[\frac{V'_{n+1}}{(1+r)^{n+1}}] = \tilde{\mathbf{E}}_n[\tilde{\mathbf{E}}_{n+1}[\frac{V_N}{(1+r)^N}]] = \frac{1}{(1+r)^n}\tilde{\mathbf{E}}_n[\frac{V_N}{(1+r)^{N-n}}] = \frac{V'_n}{(1+r)^n},$$

where the second equality holds by iterated conditioning.

(iv) Clearly, $\frac{V_N}{(1+r)^N} = \frac{V'_N}{(1+r)^N}$ holds. Since $\frac{V_n}{(1+r)^n}$ and $\frac{V'_n}{(1+r)^n}$ are martingales, part one implies $\frac{V_n}{(1+r)^n} = \frac{V'_n}{(1+r)^n}$.

9. (i) Recall that the risk neutral probabilities are given by

$$\tilde{p} = \frac{1+r-d}{u-d}; \quad \tilde{q} = \frac{u-1-r}{u-d}.$$

Let us compute the up and down factors at time zero and one:

$$u_0 = 2; \quad d_0 = \frac{1}{2}$$

$$u_1(H) = \frac{3}{2}; \quad d_1(H) = 1$$
$$u_1(T) = 4; \quad d_1(T) = 1.$$

Therefore,

$$\tilde{p}_0 = \frac{1+\frac{1}{4}-\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2}; \quad \tilde{q}_0 = \frac{1}{2}$$

$$\tilde{p}_1(H) = \frac{1+\frac{1}{4}-1}{\frac{1}{2}} = \frac{1}{2}; \quad \tilde{q}_1(H) = \frac{1}{2}$$

$$\tilde{p}_1(T) = \frac{1+\frac{1}{2}-1}{3} = \frac{1}{6}; \quad \tilde{q}_1(T) = \frac{5}{6}.$$

It follows that

$$\tilde{\mathbf{P}}(HH) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}; \quad \tilde{\mathbf{P}}(HT) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}; \quad \tilde{\mathbf{P}}(TH) = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}; \quad \tilde{\mathbf{P}}(TT) = \frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}.$$

(ii)

$$V_2(HH) = (12-7)^+ = 5; \quad V_2(HT) = (8-7)^+ = 1; \quad V_2(TH) = (8-7)^+ = 1 \quad V_2(TT) = (2-7)^+ = 0.$$

$$V_1(H) = \mathbf{E}_1[\frac{V_2}{1+r_1}](H) = \tilde{p}_1(H)\frac{V_2(HH)}{1+r_1(H)} + \tilde{q}_1(H)\frac{V_2(HT)}{1+r_1(H)} = \frac{1}{2}\cdot\frac{5}{1+\frac{1}{4}} + \frac{1}{2}\cdot\frac{1}{1+\frac{1}{4}} = \frac{12}{5}$$

$$V_1(T) = \mathbf{E}_1[\frac{V_2}{1+r_1}](T) = \tilde{p}_1(T)\frac{V_2(TH)}{1+r_1(T)} + \tilde{q}_1(T)\frac{V_2(TT)}{1+r_1(T)} = \frac{1}{6}\cdot\frac{1}{1+\frac{1}{2}} + 0 = \frac{1}{9}$$

$$V_0 = \mathbf{E}[\frac{V_1}{1+r_0}] = \tilde{p}_0\frac{V_1(H)}{1+r_0} + \tilde{q}_0\frac{V_1(T)}{1+r_0} = \frac{1}{2}\cdot\frac{\frac{12}{5}}{1+\frac{1}{4}} + \frac{1}{2}\cdot\frac{\frac{1}{9}}{1+\frac{1}{4}} = \frac{226}{225}$$

(iii) The value of the hedging portfolio is:

$$X_1 = \Delta_0 S_1 + (1+r_0)(V_0 - \Delta_0 \cdot S_0).$$

Since X_1 has to equal V_1 no matter the result of the first coin toss, we assume $\omega_1 = H$ and solve for

$$\begin{aligned}\frac{12}{5} &= V_1(H)\\ &= \Delta_0 S_1(H) + (1+r_0)(V_0 - \Delta_0 \cdot S_0)\\ &= \Delta_0 \cdot 8 + (1+\frac{1}{4})(\frac{226}{225} - \Delta_0 \cdot 4)\\ &= 3\Delta_0 + \frac{113}{90}.\end{aligned}$$

This holds if and only if $\Delta_0 = \frac{103}{270}$. Let us check that $X_1(T) = V_1(T)$ with $\Delta_0 = \frac{103}{270}$.

$$\begin{aligned}X_1(T) &= \Delta_0 S_1(T) + (1+r_0)(V_0 - \Delta_0 \cdot S_0)\\ &= \frac{103}{270}\cdot 2 + \frac{5}{4}(\frac{226}{225} - \frac{103}{270}\cdot 4) = \frac{1}{9} = V_1(T)\end{aligned}$$

(iv) We assume $\omega_2 = H$ and solve $X_2(HH) = V_2(HH)$ for $\Delta_1(H)$.

$$\begin{aligned}
5 &= V_2(HH) \\
&= \Delta_1(H)S_2(HH) + (1 + r_1(H))(V_1(H) - \Delta_1(H) \cdot S_1(H)) \\
&= \Delta_1(H) \cdot 12 + \frac{5}{4}(\frac{12}{5} - \Delta_1(H) \cdot 8) \\
&= 2\Delta_1(H) + 3.
\end{aligned}$$

This holds if and only if $\Delta_1(H) = 1$. Let us check that $X_2(HT) = V_2(HT)$ with $\Delta_1(H) = 1$.

$$\begin{aligned}
X_2(HT) &= \Delta_1(H)S_2(HT) + (1 + r_1(H))(V_1(H) - \Delta_1(H) \cdot S_1(H)) \\
&= 1 \cdot 8 + \frac{5}{4}(\frac{12}{5} - 1 \cdot 8) = 1 = V_2(HT)
\end{aligned}$$

10. (i)

$$\begin{aligned}
&\tilde{\mathbf{E}}_n[\frac{X_{n+1}}{(1+r)^{n+1}}] \\
&= \tilde{p}(\Delta_n Y_{n+1}(H)S_n + (1+r)(X_n - \Delta_n S_n)) + \tilde{q}(\Delta_n Y_{n+1}(T)S_n + (1+r)(X_n - \Delta_n S_n)) \\
&= \Delta_n S_n(u\tilde{p} + d\tilde{q}) + (\tilde{p} + \tilde{q})(1+r)(X_n - \Delta_n S_n) \\
&= \Delta_n S_n(1+r) + (1+r)(X_n - \Delta_n S_n) = X_n.
\end{aligned}$$

(ii) Let V_N be a derivative security paying off at time N. Then there is a replicating portfolio process $\Delta_0, \cdots, \Delta_{N-1}$ that generates a wealth process $X_1, \cdots, X_N$ with initial wealth X_0, that satisfies $X_N = V_N$. We set the price of the derivative at time n $V_n := X_n$. Since we have shown that the discounted price process $X_0, \cdots, \frac{X_n}{1(1+r)^n}, \cdots$ is a martingale,

$$\frac{V_n}{(1+r)^n} = \tilde{\mathbf{E}}_n[\frac{V_N}{(1+r)^N}].$$

Equivalently, the risk neutral pricing formula holds:

$$V_n = \tilde{\mathbf{E}}_n[\frac{V_N}{(1+r)^{N-n}}].$$

(iii) Let us show that the discounted stock price is not a martingale.

$$\begin{aligned}
\tilde{\mathbf{E}}_n[\frac{S_{n+1}}{(1+r)^{n+1}}] &= \tilde{\mathbf{E}}_n[\frac{(1 - A_{n+1})Y_{n+1}S_n}{(1+r)^{n+1}}] \\
&= \frac{S_n}{(1+r)^{n+1}}(\tilde{\mathbf{E}}_n[Y_{n+1}] - \tilde{\mathbf{E}}_n[A_{n+1}Y_{n+1}]) \\
&= \frac{S_n}{(1+r)^{n+1}}(1 + r - \tilde{\mathbf{E}}_[A_{n+1}Y_{n+1}]) \\
< \frac{S_n}{(1+r)^n},
\end{aligned}$$

where the last strict inequality holds because $A_{n+1}Y_{n+1}$ are strictly positive for any outcome of the $n+1$ coin toss.

If $A_{n+1} = a \in (0,1)$ for any n and any coin tossing results, then

$$
\begin{aligned}
&\tilde{\mathbf{E}}_n[\frac{S_{n+1}}{(1-a)^{n+1}(1+r)^{n+1}}] \\
&= \tilde{\mathbf{E}}_n[\frac{(1-A_{n+1})Y_{n+1}S_n}{(1-a)^{n+1}(1+r)^{n+1}}] \\
&= \tilde{\mathbf{E}}_n[\frac{(1-a)Y_{n+1}S_n}{(1-a)^{n+1}(1+r)^{n+1}}] && (A_{n+1} = a) \\
&= \frac{(1-a)S_n}{(1-a)^{n+1}(1+r)^{n+1}}\tilde{\mathbf{E}}_n[Y_{n+1}] \\
&= \frac{(1-a)S_n}{(1-a)^{n+1}(1+r)^{n+1}}(1+r) = \frac{S_n}{(1-a)^n(1+r)^n}
\end{aligned}
$$

11. (i) Let us examine the term on the right hand side of the equation.

$$
F_N + P_N = (K - S_N)^+ + (S_N - K) = \begin{cases} (K - S_N) + (S_N - K) = 0 & \text{if } K \geq S_N \\ S_N - K & \text{if } K \leq S_N. \end{cases}
$$

This is just $(S_N - K)^+ = C_N$.

(ii)

$$
C_n = \tilde{\mathbf{E}}_n[\frac{C_N}{(1+r)^{N-n}}] = \tilde{\mathbf{E}}_n[\frac{F_N + P_N}{(1+r)^{N-n}}] = F_n + P_n,
$$

where we used the result in part one for the second equality.

(iii)

$$
F_0 = \tilde{\mathbf{E}}[\frac{F_N}{(1+r)^N}] = \tilde{\mathbf{E}}[\frac{S_N}{(1+r)^N}] - \tilde{\mathbf{E}}[\frac{K}{(1+r)^N}] = S_0 - \frac{K}{(1+r)^N},
$$

where we used the fact that the discounted stock price is a martingale in the last equality.

(iv) The portfolio consists of one share of stock and $F_0 - S_0$ amount of cash at $t = 0$. At time N, the share of stock is now worth S_n and the cash position grows to $(1+r)^N(F_0 - S_0)$, i.e. the value of the portfolio at $t = N$ is:

$$
S_N + (1+r)^N(F_0 - S_0) = S_N + (1+r)^N(S_0 - \frac{K}{(1+r)^N} - S_0) = S_N - K.
$$

Note that we used the result in part three in the first equality.

(v) By part one, $C_0 = F_0 + P_0$, where the strike price for the call and the put, and the forward price of the forward contract are K. But K is chosen such that $F_0 = 0$, so the above equality becomes $C_0 = P_0$.

(vi) No. Take $n = N$. If $S_N > K$, $C_N = (S_N - K)^+ > 0$, but $P_N = (K - S_N)^+ = 0$.

12. At time m, the owner of the chooser option can receive a put or a call, where both options have strike price K. By put call parity, owning the call is equivalent to owning the put, plus a forward contract with forward price K. So at time m, the owner will receive a put with strike price K and he may choose whether or not to own a forward contract with forward price K. The value of the forward contract with forward price K at time m is

$$(S_m - \frac{K}{(1+r)}^{N-m})^+.$$

Therefore the time zero value of the chooser option is:

$$P_0 + \frac{1}{(1+r)^m}\tilde{\mathbf{E}}[(S_m - \frac{K}{(1+r)}^{N-m})^+]$$

The latter term is just the price of call expiring at time m with strike price $\frac{K}{(1+r)^{N-m}}$.

13. (i) Let us express S_{n+1} and Y_{n+1} in terms of S_n and Y_n. Define $Z := \frac{S_{n+1}}{S_n}$ which depends only on the $(n+1)$-st coin toss. Then

$$S_{n+1} = ZS_n$$

and

$$Y_{n+1} = Y_n + S_{n+1} = Y_n + ZS_n.$$

Let $f_1(s, y)$ be a function with two variables.

$$\mathbf{E}_n[f_1(S_{n+1}, Y_{n+1})] = \mathbf{E}_n[f_1(ZS_n, Y_n + ZS_n)]$$

Let $g(s, y) := \mathbf{E}_n[f_1(Zs, y + Zs)]$. Then by the independence lemma,

$$g(S_n, Y_n) = \mathbf{E}_n[f_1(ZS_n, Y_n + ZS_n)] = \mathbf{E}_n[f_1(S_{n+1}, Y_{n+1})]$$

(ii) Let us express V_N in terms of S_N and Y_N.

$$V_N = f(\frac{1}{N+1}\sum_{n=0}^{N} S_n) = f(\frac{Y_N}{N+1}).$$

If we let $v_N(s, y) := f(\frac{y}{N+1})$, then $V_N = v_N(S_N, Y_N)$.

Let us compute $v_n(s, y)$ in terms of the function v_{n+1}, again letting $Z = \frac{S_{n+1}}{S_n}$.

$$V_n = \frac{1}{(1+r)}\tilde{\mathbf{E}}_n[V_{n+1}] = \frac{1}{(1+r)}\tilde{\mathbf{E}}_n[v_{n+1}(S_{n+1}, Y_{n+1})] = \frac{1}{(1+r)}\tilde{\mathbf{E}}_n[v_{n+1}(ZS_n, Y_n + ZS_n)]$$

If we let $v_n(s, y) = \frac{1}{(1+r)}\tilde{\mathbf{E}}_n[v_{n+1}(Zs, y + Zs)]$, then $V_n = v_n(s, y)$.

14. (i) Let $f_1(s,y)$ be any function in two variables. Fix $0 \leq n \leq N$. Let $Z := \frac{S_{n+1}}{S_n}$ which depends only on the $(n+1)$-st coin toss. Then

$$S_{n+1} = ZS_n$$

and

$$Y_{n+1} = \begin{cases} 0 & (\text{if } 0 \leq n \leq M-1) \\ Y_n + ZS_n & (\text{if } M \leq n \leq N). \end{cases}$$

$$\tilde{\mathbf{E}}_n[f_1(S_{n+1}, Y_{n+1})] = \begin{cases} \tilde{\mathbf{E}}_n[f_1(ZS_n, 0)] & (\text{if } 0 \leq n \leq M-1) \\ \tilde{\mathbf{E}}_n[f_1(ZS_n, Y_n + ZS_n)] & (\text{if } M \leq n \leq N). \end{cases}$$

We now let

$$g(s,y) := \begin{cases} \tilde{\mathbf{E}}_n[f_1(Zs, 0)] & (\text{if } 0 \leq n \leq M-1) \\ \tilde{\mathbf{E}}_n[f_1(Zs, y + Zs)] & (\text{if } M \leq n \leq N). \end{cases}$$

Then $\tilde{\mathbf{E}}_n[f_1(S_{n+1}, Y_{n+1})] = g(S_n, Y_n)$.

(ii) Let us express V_N in terms of S_N and Y_N.

$$V_N = f\left(\frac{1}{N-M} \sum_{n=M+1}^{N} S_n\right) = f\left(\frac{Y_N}{N-M}\right)$$

We let $v_N(s,y) = f(\frac{y}{N-M})$. Then $V_N = v_N(S_N, Y_N)$.

Let us compute $v_n(s,y)$ in terms of the function v_{n+1}, again letting $Z = \frac{S_{n+1}}{S_n}$. Note that

$$\begin{aligned} V_n &= \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[V_{n+1}] \\ &= \begin{cases} \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[v_{n+1}(S_{n+1}, Y_{n+1})] & (M \leq n \leq N) \\ \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[v_{n+1}(S_{n+1})] & (0 \leq n \leq M-1) \end{cases} \\ &= \begin{cases} \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[v_{n+1}(ZS_n, Y_n + ZS_n)] & (M \leq n \leq N) \\ \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[v_{n+1}(ZS_n)] & (0 \leq n \leq M-1) \end{cases}. \end{aligned}$$

We let

$$\begin{aligned} v_n(s,y) &= \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[v_{n+1}(Zs, y + Zs)] && (M+1 \leq n \leq N) \\ v_M(s) &= \frac{1}{(1+r)} \tilde{\mathbf{E}}_M[v_{M+1}(Zs, Zs)] \\ v_n(s) &= \frac{1}{(1+r)} \tilde{\mathbf{E}}_n[v_{n+1}(Zs)] && (0 \leq n \leq M-1) \end{aligned}$$

Then

$$V_n = \begin{cases} v_n(S_n) & (0 \leq n \leq M) \\ v_n(S_n, Y_n) & (M+1 \leq n \leq N). \end{cases}$$