PMTH212 ASSIGNMENT 4

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(1) (a) $f(x,y) = (x-y)^{\frac{1}{2}} = \sqrt{x-y} \;\Rightarrow\; x - y > 0 \equiv y < x$

(b) $f(x,y) = \cos\left(\dfrac{xy}{1+x^2+y^2}\right) \;\Rightarrow\; 1 + x^2 + y^2 \neq 0$

Since $1 + x^2 + y^2 \geq 1$ for all x, y then $\mathbb{R}^2$ is the domain of f. We also conclude that f is continuous over its entire domain since

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

(2) (a) We simplify the function and observe continuity over its entire domain $\mathbb{R}^2$. Hence the limit is simply the value of the function at $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 16y^4}{x^2+4y^2} = \lim_{(x,y)\to(0,0)} x^2 - 4y^2 = 0 - 0 = 0$$

(b) Let $z = x^2 + y^2$ and $z \to 0$ if $(x,y) \to (0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{z\to 0} \frac{\sin z}{z} = 1$$

(3) (a) $C_m : y = mx$ or $x = t,\ y = mt$. Let $(x,y) \to (0,0)$ along C_m. Since $m \neq 0$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6+y^2} = \lim_{t\to 0} \frac{t^3(mt)}{2t^6+(mt)^2} = \lim_{t\to 0} \frac{mt^2}{2t^4+m^2} = 0$$

$C_k : y = kx^2$ or $x = t,\ y = kt^2$. Let $(x,y) \to (0,0)$ along C_k. Since $k \neq 0$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6+y^2} = \lim_{t\to 0} \frac{t^3(kt^2)}{2t^6+(kt^2)^2} = \lim_{t\to 0} \frac{kt}{2t^2+k^2} = 0$$

(b) $C_r : y = rx^3$ or $x = t,\ y = rt^3$. Let $(x,y) \to (0,0)$ along C_r. Since $r \neq 0$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6+y^2} = \lim_{t\to 0} \frac{t^3(rt^3)}{2t^6+(rt^3)^2} = \lim_{t\to 0} \frac{r}{2+r^2}$$

By choosing different values for r we obtain different limits along C_r, implying that the limit does not exist. Moreover, since the limits along C_m and C_k are different to the limit along C_r, we conclude that the function does not have a limit as $(x,y) \to (0,0)$.

(4)
$$f(x,y,z) = z\ln(x^2 y\cos z) = z(2\ln x + \ln y + \ln\cos z)$$
$$f_x = \frac{2z}{x}, \quad f_y = \frac{z}{y}, \quad f_z = \ln\cos z - z\tan z$$

Note:
$$w = \ln u, \;\; u = \cos z$$
$$\frac{dw}{dz} = \frac{dw}{du}\cdot\frac{du}{dz} = \frac{1}{u}\cdot(-\sin z)$$
$$= -\frac{\sin z}{\cos z} = -\tan z$$

(5) (a) $f(x,y) = \ln(x^2+y^2)$
$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}\ , \qquad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$$
$$\frac{\partial^2 f}{\partial x^2} = -\frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$
$$\frac{\partial^2 f}{\partial y^2} = -\frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)+2(x^2-y^2)}{(x^2+y^2)^2} = 0$$

(b) $u(x,y) = e^x\cos y,\ v(x,y) = e^x\sin y$
$$\frac{\partial u}{\partial x} = e^x\cos y\ , \qquad \frac{\partial v}{\partial y} = e^x\cos y$$
$$\frac{\partial u}{\partial y} = -e^x\sin y\ , \qquad \frac{\partial v}{\partial x} = e^x\sin y$$

Hence, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$

(6) $f(x,y) = (x^2+y^2)^{2/3}$
$$f(x,0) = (x^2+0)^{2/3} = x^{4/3}$$
$$f_x(x,0) = \frac{4}{3}x^{1/3}$$
$$f_x(0,0) = \frac{4}{3}(0) = 0$$