Study Guide for Linear Algebra

(PMTH213   2011)

I. Bokor

What Is Linear Algebra?

Why is linear algebra called "linear algebra"?
What is it about?
Why study it?

These are natural questions deserving at least provisional answers.

A brief, if crude, initial answer to the first question is that linear algebra is the study of mappings of co-ordinate systems sending straight lines through the origin to straight lines through the origin. (By the end of the semester you should have a good understanding both of that answer and of why it is only a crude, provisional one.)

There are two principal aspects to linear algebra: theoretical and computational. A major part of mastering the subjecting consists of learning how these two aspects are related and how to move from one to the other.

Many computations will seem very similar, and therefore they will be confusing without a grasp of their theoretical context and significance. It will be very tempting to draw false conclusions.

On the other hand, while many statements are easier to express elegantly and to understand from a purely theoretical point of view, you will need to "get your hands dirty" to apply them to concrete problems.

There are often different formulations of the same concept or theorem, when it is not obvious that they are equivalent. It is common for one formulation to express general aspects with clarity, but be all but useless for practical calculation, and another formulation to be computationally convenient, but give no insight into the underlying structure and concepts. Both are indispensable. Without the conceptual formulation, there can be no understanding of the underlying structures, and so no theoretical development, or expansion of the range of applicability. On the other hand, concrete applications require computation.

We develop the theory first, starting with a handful of familiar examples, on which the development can be tested, and then show how this leads concrete calculations.

Where You Have Already Met Linear Algebra

You have alreadymet a number of aspects of linear algebra already in your study of mathematics, although your attention may not have been drawn to this fact at the time. Some of the occasions you have met linear algebra — or seen its applications — include

(1) In the algebra section of MATH101, matrices and determinants are studied, including eigenvalues and eigenvectors, algebraic operations on matrices and determinants.

(2) The calculus section of MATH101 studies an example of a real vector space (even though it is not called one) and show explicitly and in detail that it is, in fact, a vector space. Certain important subspaces are also explicitly studied. It is also shown in detail in MATH101 that differentiation is a linear transformation.

(3) MATH102 continues the study of these vector spaces, showing explicitly and in detail that integration (the definite integral at least) is a linear transformation.

(4) The differential equations section of MATH102 studies, at some length and in some detail, ordinary linear differential equations with constant coefficients. The techniques for solving them are, historically,amongst the first applications of linear algebra, and illustrate the power and importance of *characteristic equations*, *characteristic values* and *characteristic vectors*. (The last two terms are synonymous with eigenvalues and eigenvectors)

(5) You have met and used matrices in AMTH140, learning some of their algebraic properties and see them applied to computing such things as the number of different paths between any two vertices in a graph.

(6) In MATH140 you have studied at some length and in some detail, *linear difference equations with constant coefficients.* The techniques for solving them are identical to those used in the differential equations section of MATH102 — in fact the two correspond perfectly if "λ^n" and "n" in the case of difference equations are replaced, respectively by"$e^{\lambda x}$" "x^n" in the case of differential equations — and illustrate, once again, the power and importance of characteristic equations, characteristic values, and characteristic vectors.

(7) The first part of PMTH212 deals with two and three dimensional real vector spaces, including the notion of "inner product". Matrix products arise as the "Chain Rule" for differentiation, and determinants enter when integrating by substitution. The axioms governing vector spaces are given explicitly and the study of quadrics treats an application of the theory of bilinear forms.

Overview

An analysis of the features common to the examples listed above leads to the notions of *vector space* and *linear transformation* Linear algebra is their study.

A *vector space* is a mixed object. It consists of two components, *vectors* and *scalars.* Scalars behave like the rational numbers in that they can be added, subtracted and multiplied. Division by any non-zero scalar is also possible. We say that the scalars form a *field.* Vectors can be added and subtracted but not, in general, multiplied by each other. The interaction between vectors and scalars consists of "multiplying" a vector by a scalar. We say that the vectors form an *abelian group* on which the field of scalars *act.*

To go from one vector space to another, to *transform* vector spaces, or to compare them, we have *linear transformations,*. These are functions between vector spaces (with common scalars) respecting the vector space structure. Intuitively, two vector spaces with common scalars are essentially the same if the only difference between them is purely notational: what the elements are called, or how they are designated. This intuition is formulated mathematically by the notion of *isomorphism.*: an isomorphism of vector spaces is a linear transformation for which there is an inverse linear transformation.

We commence with a handful of basic examples of vector spaces. We show how other vector spaces can be constructed from these. In particular, we construct the *direct sum* of vector spaces with common scalars and determine when a subset of a vector space forms a *sub-space.* We also show that the set of all linear transformations between two vector spaces again forms a vector space.

It is natural to ask whether there are vector spaces essentially different from those we have constructed. This leads to one of the central tasks of linear algebra is to *classify* vector spaces with common scalars into *isomorphism classes.* It is an amazing fact that the question as to whether two vector spaces with common scalars are isomorphic can be decided by computing a single numerical invariant for each, namely the *dimension*, and comparing these: Two vector spaces with common scalars are isomorphic if and only if they have the same dimension. We prove this by showing that each vector space has a basis and that two vector spaces with common scalars are isomorphic if and only if any basis for one has the same number of elements as any basis for the other. This number is the dimension.

We shall see that every vector space can be written as the direct sum of non-trivial subspaces, none of which can be further decomposed in this manner, the number of direct summands being precisely the dimension of the vector space: Each indecomposable summand has dimension 1.

If we choose a basis for each of our vector spaces, and if each has finite dimension, then we can represent each linear transformation by a *matrix,* whose coefficients are scalars. We define a *multiplication* for matrices to represent the composition of linear transformations. It is this use of matrices which makes many things computable. For example, if the matrix has non-zero *determinant*, then it represents an isomorphism, and vice-versa.

We can also form a *direct sum of linear transformations*. But while every vector space is a direct sum of as many one-dimensional subspaces as its dimension, it is not true in general that every linear transformation of a vector space to itself can be expressed as the direct sum of that many linear transformations between such subspaces! Indeed, this is possible if and only if the vector space has a basis consisting of *eigenvectors*. Then the matrix (in the finite-dimensional case) of the linear transformation, with respect to such a basis, has all of its non-diagonal coefficients 0, and the diagonal entries are precisely the *eigenvalues*.

The above applies to any vector space with, at most, restriction to finite dimensionality.

Some vector spaces admit additional structure. One such structure is an *inner product*. This allows us to study the vector space in question geometrically. The inner product allows us to measure angles and speak of distances in the vector space concerned. This richer structure allows for a deeper study and finds wide application both within mathematics, statistics and in other sciences and technology.

One example which should be familiar is digital recording of sound and pictures.

Other important applications of vector spaces with inner product include quantum mechanics and relatitivity theory.

Literature

There is no single set textbook for this course in linear algebra. Instead, you have a complete, self-contained set of notes, and there are two recommended texts. Both cover the content of this course, albeit differently. I find both of them eminently suitable as textbooks, but for different reasons. They complement each other. You are strongly advised to obtain a copy of at least one of them. They are:

Klaus Jänich's *Linear Algebra*. This is published by Springer-Verlag in its "Undergraduate Texts in Mathematics" series.

It is the translation into English of one of the common textbooks for the first semester of first year at German universities. It is thoroughly modern in its approach. The author is a master expositor. He explains concepts with simplicity and great clarity, showing how linear algebra is used.

Charles W. Curtis' *Linear Algebra, An Introductory Approach*. This is another book in Springer's "Undergraduate Texts in Mathematics" series.

It is by an American author, and is written from a more traditional perspective than Jänich's book. It has different intentions, as the last few chapters show. It develops the theory further. The usefulness of linear algebra is illustrated in the last chapter, whose sections are devoted to applications in geometry, differential equations and number theory respectively.

There are many other books on the subject in most libraries. You should go to a library with a proper mathematics section and browse. Look for books with "Linear Algebra", "Matrix Theory" and/or "Vector Spaces" in the title. In addition, most books on modern algebra or abstract algebra will contain at least a chapter on vector spaces or linear algebra.

You will find that some books will suit you, others will not. If you pursue mathematics further, you may well discover that some books which appeal to you now become less attractive later – and vice versa!

I would like to draw your attention to several books, at least one of which should suit your current tastes.

(1) If you think you need a large number of worked examples, the Schaum's Outline Series' volume on linear algebra is certainly full of them and you may find the book worth looking at.

(2) One of the classics on the subject is Paul Halmos' *Finite Dimensional Vector Spaces.* The author treats the subject from an "invariant" perspective: conceptual proofs are preferred over computational ones and the theory is developed as far as possible without appealing to explicit bases.
(3) A more recent book by Paul Halmos is *Linear Algebra Problem Book*, which gives a lucid exposition accompanied by a pertinent question in each section, together with (separate) hints and complete solutions.
(4) Another book which I can highly recommend you to look at is M. Hirsh and S. Smale *Differential Equations, Dynamical Systems, and Linear Algebra.* As the title suggests, the principal application the authors have in mind is to the theory of differential equations. Interestingly, this not only shows the unity of mathematics, illustrating the interconnection between topics, but also reflects the historical development.

Problems and Assessment

Tutorials. While the tutorial problems are not compulsory, it is difficult to imagine that someone could master the material without enough exercises. I strongly suggest you attempt at least the ones listed.

Assignments. There are six assignments. You must submit your attempted solutions to **all** of the assignments in order to complete the unit.

Examination. The examination will be an "open book" examination. You will be allowed to take in with you five A4 sheets — ten pages if you write on both sides — of **hand-written** notes. **no** printed notes, **no** photocopied material, **no** scanned material will be allowed into the examination room.

Final Assessment. Your final mark will be determined by your performance in the assignments and in the examination. Of the 100 marks available, 70 are for the examination and 30 for the assignments.

The examination is by far more the more important. You must pass the examination to pass the course, so that at least 35 of your final mark must come from the examination to pass. You must receive a distinction on the examination to receive a distinction for the course, so that 53 of your marks must come from the examination for a distinction.

The assignment marks may help you improve your final assessment from a pass to a credit, or a distinction to a high distinction.

Comments and Questions

Feel free to contact me with any questions, problems, comments, complaints and/or suggestions. The best ways to reach me are

- by post, care of — Mathematics
Science & Technology
UNE
Armidale 2351
- by telephone, on — **(02) 6773 2213**
- by fax, on — **(02) 6773 3312**
- electronically, at — `imi@turing.une.edu.au`

Imi Bokor

Tutorial Questions

Tutorial 1

Question 1.

Find all real numbers x, y, z satisfying the following system of equations.

$$\begin{array}{rcrcrcl} x & + & 7y & + & 4z & = & 21 \\ 3x & - & 6y & + & 5z & = & 2 \\ 5x & + & y & - & 3z & = & 14 \end{array}$$

Question 2.

For each of the following matrices, $\underline{\mathbf{A}}$, below, find $\underline{\mathbf{A}}^n \quad (n \in \mathbb{N} \setminus \{0\})$.

(i) $\underline{\mathbf{A}} := \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$.

(ii) $\underline{\mathbf{A}} := \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$.

(iii) $\underline{\mathbf{A}} := \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$.

Question 3.

Prove that if $\underline{\mathbf{A}} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then, for all $n \in \mathbb{N}$,

$$\underline{\mathbf{A}}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Question 4.

Let $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ both satisfy the differential equation

$$(*) \qquad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Show that for all real numbers λ, μ, the function

$$h \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \lambda f(x) + \mu g(x)$$

also satisfies $(*)$.

Tutorial 2

Question 1.

Show that $\mathbb{R}$ is a vector space over $\mathbb{Q}$ with respect to the usual addition and multiplication of real numbers.

Question 2.

Take $X = \{a, b, c\}$, with all elements distinct.
Define binary operations, $+$ and $\cdot$, on X by

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

and

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

Prove that X is a field with respect to $+$ and $\cdot$. This field is usually written as$\mathbb{F}_3$.

Question 3.

Decide whether the following are vector spaces.

(a) Take $\mathbb{F} := \mathbb{C}$ and $V := \mathbb{C}$.
Let $\boxplus$ to be the usual addition of complex numbers, and define $\boxdot$ by

$$\alpha \boxdot z := \alpha^2 z \qquad (\alpha, z \in \mathbb{C}).$$

(b) Let $\mathbb{F}$ be any field and take $V := \mathbb{F}^2$.
Let $\boxplus$ be the usual (component-wise) addition of ordered pairs, and define $\boxdot$ by

$$\alpha \boxdot (\beta, \gamma) := (\alpha\beta, 0) \qquad (\alpha, \beta, \gamma \in \mathbb{F}).$$

(c) Take $\mathbb{F} := \mathbb{F}_2 = \{0, 1\}$ with operations $+$ and $\cdot$ defined by

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Let $V := (\mathbb{F}_2)^2 = \{(0,0), (0,1), (1,0), (1,1)\}$.
Define $\boxdot \colon \mathbb{F}_2 \times V \longrightarrow V$ by

$$\alpha \boxdot (\beta, \gamma) := \begin{cases} (\alpha\beta, \alpha\gamma) & \text{if } \gamma \neq 0 \\ (\alpha^2 & \end{cases}$$

Define $\boxplus \colon V \times V \longrightarrow V$ by

$\boxplus$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(0,0)$

(d) Take $\mathbb{F} := \mathbb{C}$ and $V := \mathbb{C}$.
Let $\boxplus$ be the usual addition of complex numbers, and define $\boxdot$ by

$$\alpha \boxdot z := \mathbb{R}e(\alpha)z \qquad (\alpha, z \in \mathbb{C}),$$

where $\mathbb{R}e(\alpha)$ denotes the real part of the complex number α.

(e) Take $\mathbb{F} := \mathbb{R}$ and $V := \mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$.
Define $\boxplus$ and $\boxdot$ by

$$x \boxplus y := xy \qquad (x, y \in \mathbb{R}^+)$$
$$\alpha \boxdot x := x^\alpha \qquad (\alpha \in \mathbb{R}, x \in \mathbb{R}^+)$$

Question 4.

Let V be a vector space over the field $\mathbb{F}$.
Take $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{F}$.

Prove each of the following statements.

(i) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
(ii) The equation $\mathbf{u} + \mathbf{x} = \mathbf{v}$ has a unique solution, $\mathbf{x}$.
(iii) $-(-\mathbf{u}) = \mathbf{u}$.
(iv) $0\mathbf{v} = \mathbf{0}_V$.
(v) $-(\alpha\mathbf{u}) = (-\alpha)\mathbf{u} = \alpha(-\mathbf{u})$.
(vi) $(-\alpha)(-\mathbf{u}) = \alpha\mathbf{u}$.
(vii) If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{v}$.

Tutorial 3

Unless otherwise specified, we regard $\mathbb{R}^n$ as a vector space over $\mathbb{R}$ with the vector space operations defined component-wise.

Question 1.

Determine $T \circ S$ and $S \circ T$ for

$$T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (u,v) \longmapsto (u+2v, 2u+5v)$$
$$S\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \longmapsto (x+y, x)$$

Question 2.

Find all linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ which maps the line with equation $u = v$ onto the line with equations $x = y = 0$.

Question 3.

Find, if possible, linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the conditions specified.

(i) $T(1,0,0) = (1,0,0),\ T(1,1,0) = (0,1,0),\ T(1,1,1) = (0,0,1)$

(ii) $T(1,2,3) = (1,0,0),\ T(3,1,2) = (0,0,1),\ T(2,3,1) = (0,1,0)$

(iii) $T(1,2,1) = (1,0,0),\ T(1,2,2) = (1,1,0),\ T(0,0,1) = (0,0,0)$

(iv) $T(1,0,0) = (1,2,3),\ T(0,2,2) = (6,1,0),\ T(1,0,1) = (2,0,1),\ T(5,2,5) = (14,5,9)$

Where there is no solution, explain why not.

Question 3.

Take a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T(1,0,0) = (1,0,0), \quad T(0,1,0) = (0,0,1) \quad \text{and} \quad T(0,0,1) = (1,0,1).$$

Find all solutions $(x,y,z) \in \mathbb{R}^3$, of

(i) $T(x,y,z) = (6,0,7)$

(ii) $T(x,y,z) = (6,1,7)$.

Tutorial 4

Question 1.

Given a set X, let $\mathcal{F})(X)$ be the set of all real valued functions defined on X. This forms a real vector space with respect to "point-wise" operations: Given $f, g \in \mathcal{F})(X)$ and $\lambda \in \mathbb{R}$, $f+g$ and $\lambda.f$ are defined by

$$\begin{aligned} f+g\colon X &\longrightarrow \mathbb{R}, \quad x \longmapsto f(x)+g(x) \\ \lambda.f\colon X &\longrightarrow \mathbb{R}, \quad x \longmapsto \lambda f(x) \end{aligned}$$

Decide which of the following subsets of $\mathcal{F})(\mathbb{R})$ are vector subspaces.

(a) $\{f \in \mathcal{F})(\mathbb{R}) \mid f(x) \leq 0 \text{ for all } x \in \mathbb{R}\}$
(b) $\{f \in \mathcal{F})(\mathbb{R}) \mid f(7) = 0\}$
(c) $\{f \in \mathcal{F})(\mathbb{R}) \mid f(1) = 2\}$
(d) $\{f \in \mathcal{F})(\mathbb{R}) \mid \text{there are } a, b \in \mathbb{R} \text{ with } f(x) = a + b\sin x \text{ for all } x \in \mathbb{R}\}$
(e) $\mathcal{D}^n(\mathbb{R}) := \{f \in \mathcal{F})(\mathbb{R}) \mid f \text{ is } n \text{ times differentiable}\}$ $(n \in \{1, 2, \ldots\})$
(f) $\mathcal{C}^n(\mathbb{R}) := \{f \in \mathcal{F})(\mathbb{R}) \mid f \text{ is } n \text{ times continuously differentiable}\}$ $(n \in \{1, 2, \ldots\})$

Question 2.

Let $\mathbb{R}[t]$ denote the set of all polynomials with real coefficients, so that

$$\mathbb{R}[t] := \{a_0 + a_1 t + \cdots + a_m t^m \mid m \in \mathbb{N} \text{ and } a_j \in \mathbb{R} \text{ for } 0 \leq j \leq m\}$$

This forms a real vector space with respect to the usual addition of polynomials and multiplication of a polynomial by a fixed real number.

Show that for each $n \in \mathbb{N}$,

$$\mathcal{P}_n := \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$$

forms a vector subspace of $\mathbb{R}[t]$.

Question 3.

$\mathbf{M}(2;\mathbb{R})$, the set of all 2×2 matrices with real coefficients, is a real vector space with respect to the usual operations on real matrices.

Determine which of the following subsets of $\mathbf{M}(2;\mathbb{R})$ form vector subspaces.

(a) $\mathbf{M}(2;\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$

(b) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}(2;\mathbb{R}) \mid \ a+b+c+d = 0 \right\}$

(c) $\{\underline{\mathbf{A}} \in \mathbf{M}(2;\mathbb{R}) \mid \det(\underline{\mathbf{A}}) = 0\}$

Tutorial 5

Question 1.

Determine which of $(2, 2, 2)$ and $(3, 1, 5) \in \mathbb{R}^3$ are linear combinations of $\mathbf{u} := (0, -1, 1)$ and $\mathbf{v} := (1, -1, 3)$.

Question 2.

Using the notation in Question 2 of Tutorial 4, decide which of the following sets of elements of $\mathcal{P}_2$ are linearly independent.

(a) $\{4t^2 - t + 2, 2t^2 + 6t + 3, -4t^2 + 10t + 2\}$
(b) $\{4t^2 - t + 2, 2t^2 + 6t + 3, 6t^2 + 5t + 5\}$
(c) $\{t^2 + t + 23, 5t^2 - t, +2\}$
(d) $\{3t^2 + 3t + 1, t^2 + 6t + 3, 5t^2 + t + 2, -t^2 + 2t + 7\}$

Question 3.

Let $\mathcal{F}(\mathbb{R})$ be the set of all real valued functions defined on $\mathbb{R}$. This is a real vector space with respect to point-wise defined addition of functions and multiplication of a function by a real constant. Decide whether $\{f, g, h\}$ is a linearly independent set of elements of $\mathcal{F}(\mathbb{R})$ when f, g and h are defined by

(a) $f(x) := \cos 2x, \quad g(x) := \sin x, \quad h(x) := 7 \qquad (x \in \mathbb{R})$
(b) $f(x) := \ln(x^2 + 1), \quad g(x) := \sin x, \quad h(x) := e^x \qquad (x \in \mathbb{R})$

Tutorial 6

Question 1.

This question investigates findng the matrix representation of a linear transformation $T\colon V \longrightarrow W$. To do so, we specify bases $\{\mathbf{e}_j\}$ for V and $\{\mathbf{f}_i\}$ for W.

(a) Take $V = W = \mathbb{R}^2$ and $T = id_{\mathbb{R}^2}$, so that $T(x, y) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$.

Find the matrix $\underline{\mathbf{A}}_T$ in each of the following cases.

(i) $\mathbf{e}_1 := (1, 0),\ \mathbf{e}_2 := (0, 1)$ and $\mathbf{f}_1 := (1, 0),\ \mathbf{f}_2 := (0, 1)$

(ii) $\mathbf{e}_1 := (1, 0),\ \mathbf{e}_2 := (0, 1)$ and $\mathbf{f}_1 := (0, 1),\ \mathbf{f}_2 := (1, 0)$

(iii) $\mathbf{e}_1 := (1, 2),\ \mathbf{e}_2 := (3, 4)$ and $\mathbf{f}_1 := (1, 0),\ \mathbf{f}_2 := (0, 1)$

(iv) $\mathbf{e}_1 := (1, 0),\ \mathbf{e}_2 := (0, 1)$ and $\mathbf{f}_1 := (1, 2),\ \mathbf{f}_2 := (3, 4)$

(v) $\mathbf{e}_1 := (3, 4),\ \mathbf{e}_2 := (1, 2)$ and $\mathbf{f}_1 := (1, 2),\ \mathbf{f}_2 := (3, 4)$

(b) Let $\mathcal{P}_n$ be the set of all real polynomials in the indeterminate t of degree at most n.
The polynomial $p \in \mathbb{R}[t]$ induces the function

$$f_p\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto p(x)$$

This allows us to define the derivative of p, $D(p)$, to be the polynomial q that $f_q(x) = f_p'(x)$ for all $x \in \mathbb{R}$.

Prove that the function

$$D\colon \mathcal{P}_3 \longrightarrow \mathcal{P}_2, \quad p \mapsto D(p)$$

is a linear transformation and find its matrix with respect to each of the following bases.

(i) $\mathbf{e}_1 := 1,\ \mathbf{e}_2 := t,\ \mathbf{e}_3 := t^2,\ \mathbf{e}_4 := t^3$ and $\mathbf{f}_1 := 1,\ \mathbf{f}_2 := t,\ \mathbf{f}_3 := t^2$

(ii) $\mathbf{e}_1 := 1,\ \mathbf{e}_2 := t,\ \mathbf{e}_3 := t^2,\ \mathbf{e}_4 := t^3$ and $\mathbf{f}_1 := 6,\ \mathbf{f}_2 := 6t,\ \mathbf{f}_3 := 3t^2$

(iii) $\mathbf{e}_1 := 1,\ \mathbf{e}_2 := 1{+}t,\ \mathbf{e}_3 := 1{+}t^2,\ \mathbf{e}_4 := 1{+}t{+}t^2{+}t^3$ and $\mathbf{f}_1 := 1,\ \mathbf{f}_2 := 1{+}t,\ \mathbf{f}_3 := 1{+}t{+}t^2$

Question 2.

Prove that a linear transformation between finitely generated vector spaces is an isomorphism if and only if every matrix representing it is invertible.

Tutorial 7

Question 1.

Consider the system of linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Prove that there is a solution $(x_1, \ldots, x_n) \in \mathbb{F}^n$ if and only if $(b_1, \ldots, b_m) \in \mathbb{F}^m$ is an element of the vector subspace of $\mathbb{F}^m$ generated by

$$\{(a_{11}, \ldots a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn})\}.$$

When is the solution unique?

Question 2.

Let $T \colon V \longrightarrow W$ be a linear transformation.

Prove that if the matrix of T with respect to some choice of bases is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then T is neither injective nor surjective.

Question 3.

Show that $\mathcal{B} := \{(1,2),(3,4)\}$ and $\mathcal{B}' := \{(2,1),(4,3)\}$ are bases for $\mathbb{R}^2$.

Suppose that the matrix with respect to $\mathcal{B}$ of the linear transformation $T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

What is the matrix of T with respect to $\mathcal{B}'$?

Tutorial 8

Question 1.

Evaluate the determinant of each of the following matrices.

(i) $\begin{bmatrix} 1 & 6 & 4 & 7 \\ 4 & 5 & 0 & 8 \\ 6 & 2 & 1 & 9 \\ 7 & 3 & 5 & 6 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 9 & 15 \end{bmatrix}$

(iii) $\begin{bmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{bmatrix}$

(iv) $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Question 2.

(a) Find the determinant and trace of each of the following matrices.

(i) $\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$

(b) Find the determinant of each of the following matrices.

(i) $\begin{bmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{bmatrix}$

(ii) $\begin{bmatrix} 4-\lambda & -4 \\ 1 & -\lambda \end{bmatrix}$

(iii) $\begin{bmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{bmatrix}$

(c) Compare the corresponding results in parts (a) and (b).

Question 3.

Decide which of the following sets of vectors form a basis for $\mathbb{R}^3$.

(i) $\{(1,2,3),(6,5,4),(31,20,9)\}$

(ii) $\{(1,2,3),(1,4,9),(1,8,27)\}$

(iii) $\{(1,2,3),(2,3,1),(3,1,2)\}$

Tutorial 9

Question 1.

Show that eigenvectors for different eigenvalues of the same endomorphism must be linearly independent.

Question 2.

Consider the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where

(a) $T(x, y, z) = (-2x + 4y + 4z, -2x + 4y + 2z, -x + y + 3z)$
(b) $T(x, y, z) = (-2x + 8y, -2x + 6y, -x + 2y + 2z)$
(c) $T(x, y, z) = (-3x + 9y + 2z, -3x + 7y + 2z, -x + 2y + 2z)$

For each of the above

(i) find the matrix of T with respect to the standard basis for $\mathbb{R}^3$;
(ii) find the eigenvalues of T;
(iii) find the eigenvectors of T for each eigenvalue;
(iv) find, if possible, a basis for $\mathbb{R}^3$ consisting of eigenvectors of T;
(v) find the matrix of T with respect to this new basis for $\mathbb{R}^3$.

Question 3.

Let $\mathcal{C}^\infty(\mathbb{R})$ be the set of all infinitely differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. It is a real vector space with respect to point-wise addition of functions and point-wise multiplication of functions by real numbers.

Show that each of the following mappings, $T\colon \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})$, is a linear transformation and find all eigenvalues of each T, as well as their corresponding eigenvectors.

(i) $T(f) := \frac{df}{dx}$
(ii) $T(f) := \frac{d^2f}{dx^2}$
(iii) $T(f) := \frac{d^2f}{dx^2} - 4\frac{df}{dx}$

Tutorial 10

Question 1.

Show that in each of the cases below, $\beta\colon V \times V \longrightarrow \mathbb{R}$ defines an inner product on the real vector space V.

(a)

$$V := \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$
$$\beta(f,g) := t\int_0^1 f(t)g(t)dt$$

(b)

$$V := \mathbb{R}_{(2)} = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \}$$

$$\beta(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}) := [u \quad v] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c)

$$V := \mathcal{P}_2,$$
$$\beta(p,q) := p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

(d)

$$V := \mathbf{M}(m \times n; \mathbb{R}),$$
$$\beta(\underline{\mathbf{A}}, \underline{\mathbf{B}}) := \mathrm{tr}(\underline{\mathbf{A}}^t \underline{\mathbf{B}})$$

Question 2.

Let $\langle\!\langle \ , \ \rangle\!\rangle$ be an inner product on the real vector space V.

Prove that if $\mathbf{u}, \mathbf{v} \neq \mathbf{0}_V$ and $\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle = 0$ then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

Question 3.

Let β be a bilinear form on the finitely generated real vector space V.

Let $T\colon V \longrightarrow V$ be a linear transformation.

Show that

$$\gamma\colon V \times V \longrightarrow \mathbb{R}, \quad (\mathbf{x}, \mathbf{y}) \longmapsto \beta\left(T(\mathbf{x}), T(\mathbf{y})\right)$$

is also a bilinear form on V.

Choose a fixed basis for V.

Show that if $\underline{\mathbf{A}}$ is the matrix of β, and $\underline{\mathbf{B}}$ that of T, then $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \underline{\mathbf{B}}$ is the matrix of γ.

Question 4. Classify each of the following real quadratic forms according to definiteness properties:

(a) $q(x,y) := x^2 + 4xy + 5y^2$

(b) $q(x,y,z) := 4x^2 - 4xy + 5y^2 - 2yz + 3z^2 - 4zx$

(c) $q(x,y,z) = 2x^2 + 3y^2 + 2z^2 + 6xy + 6yz + 4zx$

Question 5.

Let $\langle\!\langle\ ,\rangle\!\rangle$ be an inner product on the real vector space V.
Let $\mathbf{u} \in V$ be a fixed non-zero vector in V.
Let ℓ be the line determined by $\mathbf{u}$, so that $\ell = \{\lambda\mathbf{u} \mid \lambda \in \mathbb{R}\}$.
Show that if $T\colon V \to V$ is reflection in ℓ, then

$$T(\mathbf{x}) = \frac{2\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle}{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\mathbf{u} - \mathbf{x}.$$

Tutorial 11

Question 1.

Find the matrix of the inner product

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathcal{P}_2 \times \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad (p,q) \longmapsto p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

with respect to the basis $\{1, t, t^2\}$ of $\mathcal{P}_2$.

Do the same for

$$\langle\ \mid\ \rangle : \mathcal{P}_2 \times \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad (p,q) \longmapsto \int_{-1}^{1} p(x)q(x)dx.$$

Question 2.

The matrix $\underline{\mathbf{B}} = [b_{ij}]_{n\times n} \in \mathbf{M}(n;\mathbb{R})$ is *orthogonal* if and only if $\underline{\mathbf{B}}^t\underline{\mathbf{B}} = \underline{\mathbf{1}}_n$ and it is *upper triangular* if and only if $b_{ij} = 0$ whenever $i > j$.

Prove that if $\underline{\mathbf{A}}$ is an invertible real $n \times n$ matrix, then there are an orthogonal matrix $\underline{\mathbf{Q}}$ and an upper triangular matrix $\underline{\mathbf{R}}$ such that

$$\underline{\mathbf{A}} = \underline{\mathbf{Q}}\,\underline{\mathbf{R}}.$$

Find an orthogonal matrix, $\underline{\mathbf{Q}}$, and an upper triangular matrix, $\underline{\mathbf{R}}$, such that

$$\underline{\mathbf{Q}}\,\underline{\mathbf{R}} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}.$$

Question 3.

Take the real vector space $V := \{\varphi : [0, 2\pi] \longrightarrow \mathbb{R} \mid \varphi \text{ is continuous }\}$.

Show that

$$\langle\!\langle \varphi, \psi \rangle\!\rangle := \frac{1}{\pi}\int_0^{2\pi} \varphi(x)\psi(x)dx$$

is an inner product on V.

For $n \in \mathbb{N} \setminus \{0\}$, define

$$\varphi_n(x) := \cos(nx)$$
$$\psi_n(x) := \sin(nx)$$

Show that $\{\varphi_n, \psi_n \mid n = 1, 2, \ldots\}$ is a family of orthonormal elements of $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$.

Assignment Problems

ASSIGNMENT 1

Question 1.

Find all real 2×2 matrices, $\underline{\mathbf{A}}$, such that $\underline{\mathbf{A}}^2 = \underline{\mathbf{1}}_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Question 2.

Let $\underline{\mathbf{1}}_r$ is the $r \times r$ identity matrix.
Let $\underline{\mathbf{N}}_r := [x_{ij}]_{r \times r}$ be the real $r \times r$ matrix given by

$$x_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Put $\underline{\mathbf{A}} = a\underline{\mathbf{1}}_r + \underline{\mathbf{N}}_r$.

Find $\underline{\mathbf{A}}^m$ for $m \in \mathbb{N}$.

Question 3.

Find $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^n$ for $n \in \mathbb{N} \setminus \{0\}$.

Question 4.

Let X be the set $\{a, b, c, d\}$, with all elements distinct.
Define binary operations $+$ and $\cdot$, on X by

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

and

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

Show that X is a field with respect to these operations.
[This field is usually denoted by $\mathbb{F}_4$, or $\mathbb{F}_{2^2}$.]

ASSIGNMENT 2

Question 1.

Let V and W be vector spaces over $\mathbb{F}$ and $T\colon V \longrightarrow W$ a linear transformation.

Prove that $\ker(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$ is a vector subspace of V.

Question 2.

Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ as real vector spaces with respect to component-wise operations.

Prove that the function $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation if and only if there are real numbers a, b, c, d, e, f such that for all $(x, y) \in \mathbb{R}^2$

$$\varphi(x, y) = (ax + by, cx + dy, ex + fy).$$

Question 3.

Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through an angle of θ radians about the origin. Prove that φ is an isomorphism.

Question 4.

Let $\mathbb{R}[t]$ denote the set of all polynomials in the indeterminate t with real coefficients. Show that $\mathbb{R}[t]$ is a real vector space with respect to the usual operations on polynomials.

We regard each polynomial $p(t) \in \mathbb{R}[t]$ as defining a function

$$p : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto p(x).$$

Prove that

$$\varphi : \mathbb{R}[t] \longrightarrow \mathbb{R}[t], \quad p(t) \longmapsto \int_0^t p(x)dx$$

defines an injective linear transformation. Find a left inverse for φ.

ASSIGNMENT 3

Question 1.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for the vector space V over the field $\mathbb{F}$.
Put $\mathbf{f}_1 := -\mathbf{e}_1,\ \mathbf{f}_2 := \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{f}_3 := \mathbf{e}_1 + \mathbf{e}_3$.
Prove that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is also a basis for V.

Question 2.

Let $\mathcal{P}_2$ be the set of all real polynomials of degree no greater than 2. It is a real vector space with respect to the usual addition of polynomials and multiplication of a polynomial by a constant.

Show that both $\mathcal{B} := \{1, t, t^2\}$ and $\mathcal{B}' := \{t, t^2 + t, t^2 + t + 1\}$ are bases for $\mathcal{P}_2$.

If we regard the polynomial p as defining the function $\mathbb{R} \to \mathbb{R},\ x \mapsto p(x)$, then p is differentiable. Then, as we know from calculus,

$$D : \mathcal{P}_2 \longrightarrow \mathcal{P}_2, \quad p \longmapsto p' = \frac{dp}{dx}$$

defines a linear transformation.

Find the matrix of D with respect to the bases

(i) $\mathcal{B}$ in both the domain and co-domain,
(ii) $\mathcal{B}$ in the domain and $\mathcal{B}'$ in the co-domain,
(iii) $\mathcal{B}'$ in the domain and $\mathcal{B}$ in the co-domain,
(iv) $\mathcal{B}'$ in both the domain and co-domain.

Question 3.

Let V be the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ which solve the differential equation

$$\frac{d^2y}{dx^2} = y.$$

Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for V, where

$$\mathbf{e}_1 \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^x$$

$$\mathbf{e}_2 \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \cosh x$$

Find the matrix representation with respect to this basis of the linear transformation

$$D\colon V \longrightarrow V, \quad y \longmapsto \frac{dy}{dx}.$$

ASSIGNMENT 4

Question 1.

Find the determinant of the matrix

$$\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \begin{bmatrix} 1 & 7 & 49 \end{bmatrix}$$

Question 2.

Show that the $n \times n$ matrix $\underline{\mathbf{A}}$ is invertible if and only if its determinant is non-zero.

Question 3.

Recall that $\mathbb{F}_{(p)} := \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \middle| x_1, \ldots, x_p \in \mathbb{F} \right\}$, and that the $m \times n$ matrix $\underline{\mathbf{A}}$ can be identified with the linear transformation

$$T_{\underline{\mathbf{A}}} : \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(m)}, \quad \underline{\mathbf{x}} \longmapsto \underline{\mathbf{A}}\,\underline{\mathbf{x}}.$$

In each case below, find a basis for the image of $\underline{\mathbf{A}}$ as well as a basis for the kernel of $\underline{\mathbf{A}}$.

(a) $\underline{\mathbf{A}} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 5 \\ 1 & 1 & 1 \end{bmatrix}$

(b) $\underline{\mathbf{A}} = \begin{bmatrix} 1 & 6 & 3 & 5 \\ 2 & 11 & 3 & 7 \\ 3 & 16 & 5 & 9 \end{bmatrix}$

Question 4.

Find all $\lambda \in \mathbb{R}$ such that there is a non-zero $\mathbf{v} \in \mathbb{R}_{(2)}$ such that $\underline{\mathbf{A}}\,\mathbf{v} = \lambda \mathbf{v}$, where

(a) $\underline{\mathbf{A}} = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$,

(b) $\underline{\mathbf{A}} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$,

(c) $\underline{\mathbf{A}} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$,

(d) $\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

ASSIGNMENT 5

Question 1.

Let $\underline{\mathbf{A}} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real 2×2 matrix. Show that

(a) $\underline{\mathbf{A}}$ is diagonalisable whenever $(a-d)^2 + 4bc > 0$.
(b) $\underline{\mathbf{A}}$ cannot be diagonalised (over $\mathbb{R}$) if $(a-d)^2 + 4bc < 0$.

Discuss what occurs when $(a-d)^2 + 4bc = 0$.

Question 2.

Recall that the matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are said to be similar if there is an invertible matrix $\underline{\mathbf{C}}$ such that $\underline{\mathbf{B}} = \underline{\mathbf{C}}\,\underline{\mathbf{A}}\,\underline{\mathbf{C}}^{-1}$.

Show that

(i) the matrices $\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ are similar, but

(ii) $\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$.

Question 3.

Take $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$. Prove or disprove each of the following statements.

(a) λ is an eigenvalue of $\underline{\mathbf{A}}$ if and only if it is an eigenvalue of $\underline{\mathbf{A}}^t$.
(b) $\mathbf{v}$ is an eigenvector of $\underline{\mathbf{A}}$ if and only if it is an eigenvector of $\underline{\mathbf{A}}^t$.
(c) If $\mathbf{v}$ is an eigenvector of $\underline{\mathbf{A}}$ for the eigenvalue λ and if p is any polynomial over $\mathbb{F}$, then $\mathbf{v}$ is an eigenvalue of $p(\underline{\mathbf{A}})$ for the eigenvalue $p(\lambda)$.
(d) $\underline{\mathbf{A}}$ is invertible if and only if 0 is not an eigenvalue of $\underline{\mathbf{A}}$.

Question 4.

Find a matrix $\underline{\mathbf{B}}$ which diagonalises $\underline{\mathbf{A}} := \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$.

Determine both $\underline{\mathbf{B}}\,\underline{\mathbf{A}}\,\underline{\mathbf{B}}^{-1}$ and $\underline{\mathbf{B}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{B}}$.

Do the same for $\underline{\mathbf{A}} := \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$.

ASSIGNMENT 6

Question 1.

We consider $\mathcal{P}_2$, the vector space of all real polynomials of degree at most 2.

Show that

$$\langle\langle f, g\rangle\rangle := f(-1)g(-1) + 2f(0)g(0) + f(1)g(1)$$

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx$$

both define inner products on $\mathcal{P}_2$.

Use the Gram-Schmidt Procedure with respect to each of these to construct orthonormal bases for $\mathcal{P}_2$ from the basis $\{1 - t, 1 + t^2, 1 - t^2\}$

Question 2.

Find an orthogonal matrix $\underline{\mathbf{A}}$ and an upper triangular matrix $\underline{\mathbf{B}}$ such that

$$\underline{\mathbf{A}}\,\underline{\mathbf{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$$

Question 3.

Given the symmetric real matrix $\underline{\mathbf{A}} := \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, find an orthogonal matrix $\underline{\mathbf{B}}$ such that $\underline{\mathbf{C}} := \underline{\mathbf{B}}\,\underline{\mathbf{A}}\,\underline{\mathbf{B}}^t$ is a diagonal matrix and find $\underline{\mathbf{C}}$.