Chapter 3

Real Valued Functions

Few people study or pursue mathematics out of interest in mathematics itself. Most do so because mathematics is needed for their main interest, most commonly for *modelling* phenomena, processes or complex systems in the natural sciences, in engineering, in economics and elsewhere.

Mathematical modelling is used where numerically quantifiable observations and data are available. One is typically interested in the relationship between different observables, and is seeking both an explanation for their behaviour and the ability to predict the behaviour.

When controlled, repeatable experiments are not available, we rely on repeated observations and rely on statistical analysis to make inferences. When controlled, repeatable experiments are available, we typically use real valued functions, that is functions of the form $f : X \to \mathbb{R}$. In the simplest cases, we use real valued functions of a real variable, that is to say, when $X \subseteq \mathbb{R}$. The bulk of this course is devoted to their study: univariate calculus.

The fact that the co-domain of the functions we study is $\mathbb{R}$ has significant consequences. The set of all functions from a fixed set, X, to $\mathbb{R}$, which we denote by $\mathcal{F}(X)$ admits significant structure independent of their common domain. We study these explicitly, because this structure simplifies subsequent analysis, applications and computations.

Since the reader has been subjected to mainly abstract generalities until now, we discuss some specific important functions before turning to the general structure.

3.1 Constant Functions

Choose a fixed real number r. For any non-empty set X we have the *constant function*

$$f_r \colon X \longrightarrow \mathbb{R}, \quad x \longmapsto r.$$

Different choices of r correspond to different choices of constant function $X \to \mathbb{R}$.

So, identifying the real number r with the function $X \to \mathbb{R}$ which only takes the value r allows us to regard $\mathbb{R}$ as a subset of $\mathcal{F}(X)$ whenever X is not empty.

3.2 The Case of X a Finite Set

3.2.1 $X = \{*\}$

When X is a singleton — a set with precisely one element — a function $f\colon X \to \mathbb{R}$ is the same thing as choosing a single real number: the function

$$f\colon \{*\} \longrightarrow \mathbb{R}, \quad * \longmapsto f(*)$$

may be identified with the real number $f(*)$. Hence $\mathcal{F}(X)$ can be identified with $\mathbb{R}$ in this case.

3.2.2 $X = \{1, 2\}$

A function $f\colon \{1, 2\} \to \mathbb{R}$ is completely determined by the pair of real numbers $(f(1), f(2))$. So, if we write x_1 for $f(1)$ and x_2 for $f(2)$, then we see that $\mathcal{F}(X)$ may be identified with

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\},$$

the set of all ordered pairs of real numbers, $\mathbb{R} \times \mathbb{R}$, which is also denoted $\mathbb{R}^2$.

3.2.3 $X = \{1, \dots, n\}$

A function $f\colon \{1, \dots, n\} \to \mathbb{R}$ is completely determined by the n-tuple of real numbers $(f(1), \dots, f(n))$. So, if we write x_j for $f(j)$ with $j \in \{1, \dots, n\}$, then we see that $\mathcal{F}(X)$ may be identified with

$$\{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\},$$

the set of all ordered pairs of real numbers, $\mathbb{R} \times \cdots \times \mathbb{R}$, which is also denoted $\mathbb{R}^n$ or

$$\prod_{j=1}^{n} \mathbb{R}.$$

We often write $(x_j)_{j=1}^{n}$ for the n-tuple $(x_1, \dots, x_n)$.

3.3 The Case of $X = \mathbb{N}$

A function $f\colon \mathbb{N} \to \mathbb{R}$ is the same thing as choosing for each natural number, n. a corresponding real number, $f(n)$. Thus f is completely determined by the *sequence* of real numbers $f(0), f(1), \dots, f(n), \dots$, which we denote by

$$(x_n)_{n \in \mathbb{N}},$$

where $x_n := f(n)$.

3.4 The Case of $X = \mathbb{R}$

3.4.1 The Identity Function

The most important function $\mathbb{R} \to \mathbb{R}$ is the *identity function*

$$id_{\mathbb{R}}\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x$$

3.4.2 The Absolute Value Function

The *absolute value* or *modulus* may be thought of geometrically by representing each real number as a point on a line, in which case the absolute value of a real number is the distance of the point representing it from the point representing 0. Formally

$$|\ |: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

It is left to the reader to verify that $x \mapsto |x|$ does indeed define a function $\mathbb{R} \to \mathbb{R}$.

We summarise the properties of this function.

Theorem 87. *Take $x, y \in \mathbb{R}$.*

(i) $|x| \geq 0$ *and* $|x| = 0$ *if and only if* $x = 0$. *Moreover,* $|x| = \max\{-x, x\}$.

(ii) $|xy| = |x|\,|y|$

(iii) $|x^2| = x^2$

(iv) $-|x| \leq x \leq |x|$

(v) $|x + y| \leq |x| + |y|$

(vi) $\big||x| - |y|\big| \leq |x - y|$

Proof. Take $x, y \in \mathbb{R}$.

(i) This is immediate from the definition.

(ii) $xy < 0$ if and only if either $x < 0$ and $y > 0$, or $x > 0$ and $y < 0$. Then either $|x| = -x$ and $|y| = y$, or $|x| = x$ and $|y| = -y$. In both cases $|x|\,|y| = -xy = |xy|$.

$xy = 0$ if and only if $x = 0$ or $y = 0$, that is, $|x| = 0$ or $|y| = 0$. In both cases $|x|\,|y| = 0 = |xy|$.

$xy > 0$ if and only if either $x < 0$ and $y < 0$, or $x > 0$ and $y > 0$. Then either $|x| = -x$ and $|y| = -y$, or $|x| = x$ and $|y| = y$. In both cases $|x|\,|y| = xy = |xy|$.

(iii) This follows from the fact that x^2 is never negative.

(iv) Note that since $|x| \geq 0$, we always have $-|x| \leq |x|$. The result follows from the fact that either $x = -|x|$ or $x = |x|$. (This argument also shows that $-|x| \leq -x \leq |x|$.)

(v) Since $|x|, |y|, |x + y| \geq 0$, it is enough to show that

$$|x + y|^2 \leq \big(|x| + |y|\big)^2$$

But

$$\begin{aligned} |x + y|^2 &= (x + y)^2 && \text{by (iii)} \\ &= x^2 + 2xy + y^2 \\ &\leq x^2 + 2|xy + |y^2 && \text{by (iv)} \\ &= |x|^2 + 2|x|\,|y| + |y|^2 && \text{by (ii) and (iii)} \\ &= \big(|x| + |y|\big)^2 \end{aligned}$$

(vi) By (v), $|x| = |x - y + | \leq |x - y| + |y|$. Thus

$$|x| - |y| \leq |x - y|$$

Similarly, $|y| = |y - x + x| \leq |y - x| + |x|$, whence

$$-|x| - |y| \leq |y - x| = |x - y|.$$

Hence

$$\big||x| - |y|\big| = \max\{-\big(|x| - |y|\big), \big(|x| - |y|\big)\} \leq |x - y|.$$

□

3.4.3 Polynomial Functions

Definition 88. A *polynomial with real coefficients in the indeterminate* t, or *real polynomial* is an expression of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n = \sum_{j=1}^{n} a_j t^j$$

where each a_j is a real number and $a_n \neq 0$ if $n \neq 0$.

If $a_n \neq 0$, then n is the *degree* of p or $p(t)$, and we write

$$\deg p = n.$$

We write $\mathbb{R}[t]$ for the set of all real polynomials in the indeterminate t.

Observation 89. It is important to remember that the t in a polynomial is *not* a real number. It is an indeterminate. The "addition" and the "multiplication" in the notation are *not* (arithmetic) addition and multiplication, they are *formal operations.*

This is often difficult to grasp at first and, at this stage, may seem to be mere sophistry. However the point is an important one.

Lemma 90. *Each polynomial,* $p = \sum_{j=1}^{n} a_j t^j \in \mathbb{R}[t]$*, defines a function*

$$f_p \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sum_{j=1}^{n} a_j x^j,$$

where now the addition and multiplication are the usual arithmetic operations and we adopt the convention that for every real number, x, $x^0 = 1$.

Proof. The fact that f_p is a function follow from the definition and properties of the usual arithmetic operations. □

Observation 91. We regard the function f_p as obtained by *evaluating* the polynomial p.

Definition 92. The function $f : \mathbb{R} \to \mathbb{R}$ is called a *polynomial function* is there is a polynomial $p \in \mathbb{R}[t]$ with $f = f_p$.

Example 93. The functions

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto 2x^2 + 3x \ -1$$
$$id_{\mathbb{R}}\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x$$

are polynomial functions, with $f = f_p$ and $g = f_q$ where $p(t) = -1 + 3t + 2t^2$ and $q(t) = t$.

Example 94. Even though $\cos^2 t + \sin^2 t$ is not a polynomial, the function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \cos^2 x + \sin^2 x$$

is a polynomial function. For by Pythagoras' Theorem $\cos^2 x + \sin^2 x = 1$ for every real number x. Hence $f = f_1$, where 1 is the polynomial a_0 of degree zero with $a_0 = 1$. [This example helps explain why we distinguish between polynomials and polynomial functions.]

3.4.4 Trigonometric Functions

The two *trigonometric functions* $\mathbb{R} \to \mathbb{R}$ are

$$\sin\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x$$
$$\cos\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \cos x$$

should be familiar from high school. They are*not* polynomial functions. This fact is not immediately obvious, but we shall provide a simple proof using differentiation.

We summarise the principal properties of the trigonometric functions.

Theorem 95. *Take $A, B \in \mathbb{R}$.*

(i) $\operatorname{im}(\cos) = \operatorname{im}(\sin) = [-1, 1] := \{y \in \mathbb{R} \mid -1 \leq y \leq 1\}$.

(ii) $\cos 0 = 1, \cos \frac{\pi}{2} = 0, \sin 0 = 0, \sin \frac{\pi}{2} = 1$

(iii) $\cos(-A) = \cos A, \sin(-A) = -\sin A$.

(iv) $\cos(A + 2\pi) = \cos A, \sin(A + 2\pi) = \sin A$.

(v) $\cos(A + B) = \cos A \cos B - \sin A \sin B, \sin(A + B) = \sin A \cos B + \cos A \sin B$

Proof. The proofs are left to the reader as revision. □

3.4.5 Exponential Functions

For each positive real number, a, there is a *exponential function*

$$a^{-}\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto a^x.$$

It is not difficult to define a^x for rational numbers x. However, to define it for all real numbers x is beyond our means at the moment. However, the theory developed in MATH102 provides an elegant proof that the above is indeed a function. We introduce it here because it is one of the most important functions in mathematics and its applications, despite our reluctance to do so before we can rigorously establish its properties, which we summarise in our next theorem.

Theorem 96. *Let a be a positive real number. Take $x, y \in \mathbb{R}$. Then*

(i) $\operatorname{im}(a-) =]0, \infty[:= r \in \mathbb{R} \mid r > 0 =: \mathbb{R}^+$.

(ii) $a^0 = 1$ *and* $a^1 = a$.

(iii) $a^{x+y} = a^x a^y$. *In particular,* $a^{-x} = \dfrac{1}{a^x}$

(iv) $a^{xy} = \big(a^x\big)^y$. *In particular,* $a^{\frac{1}{x}} = \sqrt[x]{a}$ *whenever* $x \neq 0$.

Proof. Deferred to MATH102. □

Observation 97. Amazingly, the functions we have listed, together with functions we can construct from them, suffice for nearly all applications and modelling.

3.5 The Algebra $\mathcal{F}(X)$

We now turn investigating the operations on $\mathcal{F}(X)$ induced by the operations on $\mathbb{R}$.

Definition 98. Given a set X, we define two (binary) operations on the set of real-valued functions defined on X, $\mathcal{F}(X)$, called *addition* and *multiplication (of functions).*

$$\alpha : \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow \mathcal{F}(X), \quad (f, g) \longmapsto f \boxplus g,$$

where $f \boxplus g$ is defined by

$$(f \boxplus g)(x) := f(x) + g(x)$$

and

$$\mu : \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow \mathcal{F}(X), \quad (f, g) \longmapsto f \boxtimes g,$$

where $f \boxtimes g$ is defined by

$$(f \boxtimes g)(x) := f(x)g(x).$$

Lemma 99. *α and μ are functions.*

Proof. We are required to show that If $f, g : X \to \mathbb{R}$ are functions, so are $f \boxplus g$ and $f \boxtimes g$, and that these are each uniquely determined by f and g.

The uniqueness follows immediately from the explicit formulæ in the definitions of $f \boxplus g$ and $f \boxtimes g$.

It only remains to show that each is, indeed, a function. To see this, note that each is defined on X, which we know to be a set, and take values in $\mathbb{R}$, which is also a set.

Now each $x \in X$ uniquely determines real numbers $f(x)$ and $g(x)$, since f and g are functions. But these determine $f(x) + g(x)$ and $f(x)g(x)$ uniquely by the properties of addition and multiplication of real numbers. □

Lemma 100. *For each $r \in \mathbb{R}$, let f_r be the constant function determined by r. Then, for all $r, s \in \mathbb{R}$,* *(i)* $f_r = f_s$ *if and only if* $r = s$*;*

(ii) $f_r \boxplus f_s = f_{r+s}$*;*

(iii) $f_r \boxtimes f_s = f_{rs}$*.*

Proof. We prove (ii) and leave the rest to the reader.

By definition, $\mathrm{dom}(f_r \boxplus f_s) = \mathrm{dom}(f_{r+s})$ and $\mathrm{codom}(f_r \boxplus f_s) = \mathrm{codom}(f_{r+s})$.

Now take $x \in X$. Then

$$\begin{aligned}(f_r \boxplus f_s)(x) &:= f_r(x) + f_s(x)\\ &:= r + s\\ &=: f_{r+s}(x).\end{aligned}$$

□

Observation 101. We saw in 3.1 that for a non-empty set X, we may identify $\mathbb{R}$ with the subset of $\mathcal{F}(X)$ comprising the constant functions. Lemma 100 shows that this identification is compatible with t the arithmetic properties of $\mathbb{R}$. Thus, we may regard the addition and multiplication on $\mathcal{F}(X)$ as extending those on $\mathbb{R}$.

We show that many of the properties of $\mathbb{R}$ extend to $\mathcal{F}(X)$.

Theorem 102. *Take* $f, g, h \in \mathcal{F}(X)$*. Then*

(A1) $(f \boxplus g) \boxplus h = f \boxplus (g \boxplus h)$

(A2) $f_0 \boxplus f = f = f \boxplus f_0$

(A3) *There is a* $(-f) \in \mathcal{F}(X)$ *such that* $(-f) + f = f_0 = f + (-f)$

(A4) $f \boxplus g = g \boxplus f$

(M1) $(f \boxtimes g) \boxtimes h = f \boxtimes (g \boxtimes h)$

(M2) $f_1 \boxtimes f = f = f \boxtimes f_1$

(M4) $f \boxtimes g = g \boxtimes f$

(D) $(f \boxplus g) \boxtimes h = (f \boxtimes h) \boxplus (g \boxtimes h)$ *and* $f \boxtimes (g \boxplus h) = (f \boxtimes g) \boxplus (f \boxtimes h)$

Proof. We establish (A1) and leave the rest as an exercise.

$\mathrm{dom}\big((f \boxplus g) \boxplus h\big) = \mathrm{dom}\big(f \boxplus (g \boxplus h)\big)$ and $\mathrm{codom}\big((f \boxplus g) \boxplus h\big) = \mathrm{codom}\big(f \boxplus (g \boxplus h)\big)$ by definition.

Take $x \in X$. Then

$$\begin{aligned}\big((f \boxplus g) \boxplus h\big)(x) &:= (f \boxplus g)(x) + h(x)\\ &:= (f(x) + g(x)) + h(x)\\ &:= f(x) + (g(x) + h(x)) \quad \text{by the associativity of addition of real numbers}\\ &=: f(x) + (g{+}h)(x)\\ &=: \big(f \boxplus (g \boxplus h)\big)(x)\end{aligned}$$

□

Observation 103. Notice that these properties hold independently of $X \neq \emptyset$.

Observation 104. Notice that all but one of the field axioms hold for $\mathcal{F}(X)$. The one exception is the existence of multiplicative inverses — labelled (M3) above. But there is a weaker form of this, namely,

(M3′) If $f(x) \neq 0$ for all $x \in X$, then is a $\frac{1}{f} \in \mathcal{F}(X)$ such that $(\frac{1}{f}) \boxtimes f = f_1 = f \boxtimes (\frac{1}{f})$.

3.5.1 A Partial Order on $\mathcal{F}(X)$

We have seen that the algebraic structure of $\mathbb{R}$ induces an algebraic structure on $\mathcal{F}(X)$, which has almost identical properties, the only weakening being that not every non-zero function $f \colon X \to \mathbb{R}$ has a multiplicative inverse.

In addition to its algebraic structure, $\mathbb{R}$ also has an order structure: $\mathbb{R}$ is a (totally) ordered set. This ordering induces a partial ordering on $\mathcal{F}(X)$, as we next show.

Definition 105. Let X be a set. Give $f, g \in \mathcal{F}(X)$

$$f \preceq g \quad \text{if and only if} \quad f(x) \leq g(x) \text{ for every } x \in X.$$

Lemma 106. *$\preceq$ is a partial order on $\mathcal{F}(X)$, but not, in general, a total order.*

Proof. To show that $\preceq$ is a partial ordering on $\mathcal{F}(X)$ follows by direct calculation. We carry this out for transitivity, leaving reflexivity and symmetry to the reader.

Take $f, g, h \in \mathcal{F}(X)$ with $f \preceq g$ and $g \preceq h$. Choose $x \in X$. Then

$$\begin{aligned} f(x) &\leq g(x) && \text{as } f \preceq g \\ g(x) &\leq h(x) && \text{as } g \preceq h \end{aligned}$$

Since $f(x), g(x), h(x) \in \mathbb{R}$ and $\leq$ is an order on $\mathbb{R}$, it follows that $f(x) \leq h(x)$

Since $x \in X$ is arbitrary, it follows that $f(x) \leq h(x)$ for every $x \in X$, that is $f \preceq g$.

To see that, in general, $\preceq$ is not a total ordering, we need to find a set X and $f, g \in \mathcal{F}(X)$ with $f \not\preceq g$ and $g \not\preceq f$.

We take $X := [-1, 1] := \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ and $f, g \in \mathcal{F}(X)$ given by

$$\begin{aligned} f \colon X &\longrightarrow \mathbb{R}, \quad x \longmapsto x \\ g \colon X &\longrightarrow \mathbb{R}, \quad x \longmapsto x^2 \end{aligned}$$

Since $f(\frac{1}{2}) = \frac{1}{2} > \frac{1}{4} = g(\frac{1}{2})$,we see that $f \not\preceq g$.

Since $g(-1) = 1 > -1 = f(-1)$, we see that $g \not\preceq f$. □

Observation 107. In order to simplify notation, it is usual to write $\leq$ instead of $\preceq$ for the above partial order on $\mathcal{F}(X)$, relying on the good sense f the reader to recognise when functions, rather than real numbers are being compared.

Observation 108. In this course, we shall be studying $\mathcal{F}(X)$ and some of its subsets, in the case that X is a non-empty subset of $\mathbb{R}$.

3.6 The Algebra $\mathcal{F}(\mathbb{R})$

In the special case where $X = R$, the domain and co-domain of each function coincide, and so we have a third (binary) operation on $\mathcal{F}(X) = \mathcal{F}(\mathbb{R})$, namely the composition of functions,

$$\circ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R}), \quad (f, g) \longmapsto g \circ f$$

where

$$f \circ g : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto g\big(f(x)\big).$$

We next study of this operation and its behaviour with respect to the other operations.

Theorem 109. *Take $f, g, h \in \mathcal{F}(\mathbb{R})$.*

(a) $(f \circ g) \circ h) = f \circ (g \circ h)$.

(b) $f \circ id_{\mathbb{R}} = f = id_{\mathbb{R}|} \circ f$.

(c) $(f \boxplus g) \circ h = (f \circ h) \boxplus (g \circ h)$.

(d) $(f \boxtimes g) \circ h = (f \circ h) \boxtimes (g \circ h)$.

(e) In general, $f \circ (g \boxplus h) \neq (f \circ g) \boxplus (f \circ h)$.

(f) In general, $f \circ (g \boxtimes h) \neq (f \circ h) \boxtimes (f \circ h)$.

Proof. Since all functions share a common domain which is also their common co-domain, it is sufficient to consider the values the functions take on the same element of the domain.
(a): This follows directly from Lemma 74.
(b): This follows directly from Lemma 75.
(c): For $x \in \mathbb{R}$,

$$\begin{aligned}
\big((f \boxplus g) \circ h\big)(x) &:= (f \boxplus g)\big(h(x)\big) \\
&:= \Big(f\big(h(x)\big)\Big).\Big(g\big(h(x)\big)\Big) \\
&=: (f \circ h)(x).(g \circ h)(x) \\
&=: \big((f \circ h) \boxtimes (g \circ h)\big)(x),
\end{aligned}$$

showing that the functions in question do, indeed, agree.
(d): Exercise.
(e): Consider $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad y \longmapsto y^2$ and $g = h = id_{\mathbb{R}}$. Then, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\big(f \circ (g \boxplus h)\big)(x) &= \big(g(x) + h(x)\big)^2 \\
&= (x + x)^2 \\
&= 4x^2,
\end{aligned}$$

whereas

$$\begin{aligned}
\big((f \circ g) \boxplus (f \circ h)\big)(x) &= (f \circ g)(x) + (f \circ h)(x) \\
&= \big(g(x)\big)^2 + \big(h(x)\big)^2 \\
&= 2x^2.
\end{aligned}$$

Choose $x := 1$. Then $\big((f \boxplus g) \circ h\big)(1) = 4 \neq 2 = \big((f \circ g) \boxplus (f \circ h)\big)(1)$.

(f): Exercise. □

Convention 110. To avoid a proliferation of notation, it is customary to write $+$ and $\times$ instead of $\boxplus$ and $\boxtimes$ trusting the good sense of the reader to recognise whether, for example, the addition of functions or the addition of real numbers is intended.

Finally we consider the relationship between composition of functions and the partial ordering on $\mathcal{F}(\mathbb{R})$ induced by the ordering of $\mathbb{R}$.

Lemma 111. *The composition of monotonic functions* $\mathbb{R} \to \mathbb{R}$ *is a monotonic function.*

Specifically, if $f, g\colon \mathbb{R} \to \mathbb{R}$ *are both non-decreasing or both non-increasing, then* $f \circ g$ *is non-decreasing, and if one of* f, g *is non-decreasing while the other is non-increasing, then* $f \circ g$ *is non-increasing.*

Proof. Exercise. □

We shall repeatedly exploit the algebraic structure of $\mathcal{F}(\mathbb{R})$ (and $\mathcal{F}(X)$ for $X \subseteq \mathbb{R}$) to clarify concepts as well as to simplify proofs and calculations.