Exercise Solutions (W12): MATH102–Statistics

24/10/06

1. Now
$$f(x;\alpha,\,\beta)=\frac{(x/\beta)^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\,\beta}$$
and
$$E(X)=\int_0^\infty xf(x;\alpha,\,\beta)dx=\int_0^\infty x\frac{(x/\beta)^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\,\beta}dx$$
$$=\int_0^\infty\frac{(x/\beta)^{\alpha}e^{-x/\beta}}{\Gamma(\alpha)}dx=\int_0^\infty\alpha\beta\frac{(x/\beta)^{\alpha+1-1}e^{-x/\beta}}{\Gamma(\alpha+1)\,\beta}dx=\alpha\beta$$

2. Now
$$V(X)=E(X^2)-[E(X)]^2$$
$$E(X^2)=\int_0^\infty x^2f(x;\alpha,\,\beta)dx=\int_0^\infty x^2\frac{(x/\beta)^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\,\beta}$$
$$=\int_0^\infty\frac{x^{\alpha+2-1}}{\beta^{\alpha-1}}\frac{e^{-x/\beta}}{\Gamma(\alpha)\beta}dx=\int_0^\infty\alpha(\alpha+1)\beta^2\frac{(x/\beta)^{\alpha+2-1}e^{-x/\beta}}{\Gamma(\alpha+2)\beta}dx$$
$$=\alpha(\alpha+1)\beta^2\int_0^\infty\frac{(x/\beta)^{\alpha+2-1}e^{-x/\beta}}{\Gamma(\alpha+2)\beta}dx=\alpha(\alpha+1)\beta^2$$
Hence $V(X)=\alpha^2\beta^2+\alpha\beta^2-\alpha^2\beta^2=\alpha\beta^2$
as required.

3. By definition
$$\mu_x=\alpha\beta,\;\;\sigma_x^2=\alpha\beta^2$$
So
$$\frac{\sigma_x^2}{\mu_x}=\frac{\alpha\beta^2}{\alpha\beta}=\beta$$
to give
$$\beta=\frac{\sigma_x^2}{\mu_x}$$

Also

$$\frac{\mu_x^2}{\sigma_x^2} = \frac{\alpha^2\beta^2}{\alpha\beta^2} = \alpha$$

giving

$$\alpha = \frac{\mu_x^2}{\sigma_x^2}$$

An alternative derivation given in the Notes uses

$$\beta = \frac{\sigma_x^2}{\mu_x}$$

with

$$\mu_x = \alpha\beta$$

by definition.

This gives

$$\alpha = \frac{\mu_x}{\beta}$$

but this is not really an answer to the question, as β needs to be calculated first.

4. By definition $\mu_y = \alpha\beta$ and $\nu = \alpha$, so $\beta = \mu/\alpha = \mu/\nu$.

   Thus

$$g(y; \alpha, \beta) = \frac{(y/\beta)^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha)\ \beta}$$

   giving

$$\begin{aligned} g(y; \mu, \nu) &= \frac{(y/[\mu/\nu])^{\nu-1} e^{-y/(\mu/\nu)}}{\Gamma(\nu)\ \mu/\nu} \\ &= \frac{(\nu y/\mu)^{\nu} (\nu y/\mu)^{-1} e^{-\nu y/\mu}}{\Gamma(\nu)\ \mu}\nu \\ &= \frac{1}{\Gamma(\nu)} \left(\frac{\nu y}{\mu}\right)^{\nu} \frac{e^{-\nu y/\mu}}{y} \end{aligned}$$

5. Now

$$\sigma_y^2 = \alpha\beta^2$$

and

$$\mu = \alpha\beta$$

giving

$$\beta = \frac{\mu}{\alpha} = \frac{\mu}{\nu}$$

and so

$$\sigma_y^2 = \nu\beta^2 = \nu\left(\frac{\mu}{\nu}\right)^2 = \frac{\mu^2}{\nu}$$

Thus

$$\sigma_y^2 = \frac{\mu^2}{\nu}$$