ASSIGNMENT 1

Question 1.

Find all real 2×2 matrices, $\underline{\mathbf{A}}$, such that $\underline{\mathbf{A}}^2 = \underline{\mathbf{1}}_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Question 2.

Let $\underline{\mathbf{1}}_r$ is the $r \times r$ identity matrix.
Let $\underline{\mathbf{N}}_r := [x_{ij}]_{r \times r}$ be the real $r \times r$ matrix given by

$$x_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Put $\underline{\mathbf{A}} = a\underline{\mathbf{1}}_r + \underline{\mathbf{N}}_r$.

Find $\underline{\mathbf{A}}^m$ for $m \in \mathbb{N}$.

Question 3.

Find $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^n$ for $n \in \mathbb{N} \setminus \{0\}$.

Question 4.

Let X be the set $\{a, b, c, d\}$, with all elements distinct.
Define binary operations $+$ and $\cdot$, on X by

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

and

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

Show that X is a field with respect to these operations.
[This field is usually denoted by $\mathbb{F}_4$, or $\mathbb{F}_{2^2}$.]

ASSIGNMENT 2

Question 1.

Let V and W be vector spaces over $\mathbb{F}$ and $T: V \longrightarrow W$ a linear transformation.
Prove that $\ker(T) := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$ is a vector subspace of V.

Question 2.

Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ as real vector spaces with respect to component-wise operations.

Prove that the function $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation if and only if there are real numbers a, b, c, d, e, f such that for all $(x, y) \in \mathbb{R}^2$

$$\varphi(x, y) = (ax + by, cx + dy, ex + fy).$$

Question 3.

Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through an angle of θ radians about the origin. Prove that φ is an isomorphism.

Question 4.

Let $\mathbb{R}[t]$ denote the set of all polynomials in the indeterminate t with real coefficients. Show that $\mathbb{R}[t]$ is a real vector space with respect to the usual operations on polynomials.

We regard each polynomial $p(t) \in \mathbb{R}[t]$ as defining a function

$$p : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto p(x).$$

Prove that

$$\varphi : \mathbb{R}[t] \longrightarrow \mathbb{R}[t], \quad p(t) \longmapsto \int_0^t p(x)dx$$

defines an injective linear transformation. Find a left inverse for φ.

ASSIGNMENT 3

Question 1.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for the vector space V over the field $\mathbb{F}$.
Put $\mathbf{f}_1 := -\mathbf{e}_1$, $\mathbf{f}_2 := \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{f}_3 := \mathbf{e}_1 + \mathbf{e}_3$.
Prove that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is also a basis for V.

Question 2.

Let $\mathcal{P}_2$ be the set of all real polynomials of degree no greater than 2. It is a real vector space with respect to the usual addition of polynomials and multiplication of a polynomial by a constant.

Show that both $\mathcal{B} := \{1, t, t^2\}$ and $\mathcal{B}' := \{t, t^2 + t, t^2 + t + 1\}$ are bases for $\mathcal{P}_2$.

If we regard the polynomial p as defining the function $\mathbb{R} \to \mathbb{R}$, $x \mapsto p(x)$, then p is differentiable. Then, as we know from calculus,

$$D : \mathcal{P}_2 \longrightarrow \mathcal{P}_2, \quad p \longmapsto p' = \frac{dp}{dx}$$

defines a linear transformation.

Find the matrix of D with respect to the bases

(i) $\mathcal{B}$ in both the domain and co-domain,
(ii) $\mathcal{B}$ in the domain and $\mathcal{B}'$ in the co-domain,
(iii) $\mathcal{B}'$ in the domain and $\mathcal{B}$ in the co-domain,
(iv) $\mathcal{B}'$ in both the domain and co-domain.

Question 3.

Let V be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which solve the differential equation

$$\frac{d^2y}{dx^2} = y.$$

Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for V, where

$$\mathbf{e}_1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^x$$

$$\mathbf{e}_2 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \cosh x$$

Find the matrix representation with respect to this basis of the linear transformation

$$D : V \longrightarrow V, \quad y \longmapsto \frac{dy}{dx}.$$

ASSIGNMENT 4

Question 1.

Find the determinant of the matrix

$$\begin{bmatrix}1\\3\\9\end{bmatrix}\begin{bmatrix}1 & 7 & 49\end{bmatrix}$$

Question 2.

Show that the $n \times n$ matrix $\underline{\mathbf{A}}$ is invertible if and only if its determinant is non-zero.

Question 3.

Recall that $\mathbb{F}_{(p)} := \left\{ \begin{bmatrix}x_1\\ \vdots \\ x_p\end{bmatrix} \middle| x_1, \ldots, x_p \in \mathbb{F} \right\}$, and that the $m \times n$ matrix $\underline{\mathbf{A}}$ can be identified with the linear transformation

$$T_{\underline{\mathbf{A}}} : \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(m)}, \quad \underline{\mathbf{x}} \longmapsto \underline{\mathbf{A}}\,\underline{\mathbf{x}}.$$

In each case below, find a basis for the image of $\underline{\mathbf{A}}$ as well as a basis for the kernel of $\underline{\mathbf{A}}$.

(a) $\underline{\mathbf{A}} = \begin{bmatrix}2 & 1 & 3\\ 0 & 2 & 5\\ 1 & 1 & 1\end{bmatrix}$

(b) $\underline{\mathbf{A}} = \begin{bmatrix}1 & 6 & 3 & 5\\ 2 & 11 & 3 & 7\\ 3 & 16 & 5 & 9\end{bmatrix}$

Question 4.

Find all $\lambda \in \mathbb{R}$ such that there is a non-zero $\mathbf{v} \in \mathbb{R}_{(2)}$ such that $\underline{\mathbf{A}}\,\mathbf{v} = \lambda\mathbf{v}$, where

(a) $\underline{\mathbf{A}} = \begin{bmatrix}4 & -3\\ 1 & 0\end{bmatrix}$,

(b) $\underline{\mathbf{A}} = \begin{bmatrix}4 & -4\\ 1 & 0\end{bmatrix}$,

(c) $\underline{\mathbf{A}} = \begin{bmatrix}4 & -5\\ 1 & 0\end{bmatrix}$,

(d) $\underline{\mathbf{A}} = \begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix}$.

ASSIGNMENT 5

Question 1.

Let $\underline{\mathbf{A}} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real 2×2 matrix. Show that

(a) $\underline{\mathbf{A}}$ is diagonalisable whenever $(a-d)^2 + 4bc > 0$.
(b) $\underline{\mathbf{A}}$ cannot be diagonalised (over $\mathbb{R}$) if $(a-d)^2 + 4bc < 0$.

Discuss what occurs when $(a-d)^2 + 4bc = 0$.

Question 2.

Recall that the matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are said to be similar if there is an invertible matrix $\underline{\mathbf{C}}$ such that $\underline{\mathbf{B}} = \underline{\mathbf{C}}\,\underline{\mathbf{A}}\,\underline{\mathbf{C}}^{-1}$.

Show that

(i) the matrices $\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ are similar, but
(ii) $\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$.

Question 3.

Take $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$. Prove or disprove each of the following statements.

(a) λ is an eigenvalue of $\underline{\mathbf{A}}$ if and only if it is an eigenvalue of $\underline{\mathbf{A}}^t$.
(b) $\mathbf{v}$ is an eigenvector of $\underline{\mathbf{A}}$ if and only if it is an eigenvector of $\underline{\mathbf{A}}^t$.
(c) If $\mathbf{v}$ is an eigenvector of $\underline{\mathbf{A}}$ for the eigenvalue λ and if p is any polynomial over $\mathbb{F}$, then $\mathbf{v}$ is an eigenvalue of $p(\underline{\mathbf{A}})$ for the eigenvalue $p(\lambda)$.
(d) $\underline{\mathbf{A}}$ is invertible if and only if 0 is not an eigenvalue of $\underline{\mathbf{A}}$.

Question 4.

Find a matrix $\underline{\mathbf{B}}$ which diagonalises $\underline{\mathbf{A}} := \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$.

Determine both $\underline{\mathbf{B}}\,\underline{\mathbf{A}}\,\underline{\mathbf{B}}^{-1}$ and $\underline{\mathbf{B}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{B}}$.

Do the same for $\underline{\mathbf{A}} := \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

ASSIGNMENT 6

Question 1.

We consider $\mathcal{P}_2$, the vector space of all real polynomials of degree at most 2.
Show that

$$\langle\!\langle f, g \rangle\!\rangle := f(-1)g(-1) + 2f(0)g(0) + f(1)g(1)$$

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx$$

both define inner products on $\mathcal{P}_2$.

Use the Gram-Schmidt Procedure with respect to each of these to construct orthonormal bases for $\mathcal{P}_2$ from the basis $\{1 - t, 1 + t^2, 1 - t^2\}$

Question 2.

Find an orthogonal matrix $\underline{\mathbf{A}}$ and an upper triangular matrix $\underline{\mathbf{B}}$ such that

$$\underline{\mathbf{A}}\,\underline{\mathbf{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$$

Question 3.

Given the symmetric real matrix $\underline{\mathbf{A}} := \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, find an orthogonal matrix $\underline{\mathbf{B}}$ such that $\underline{\mathbf{C}} := \underline{\mathbf{B}}\,\underline{\mathbf{A}}\,\underline{\mathbf{B}}^t$ is a diagonal matrix and find $\underline{\mathbf{C}}$.