Tut. 10

Calculate

~~cate~~ $\int_C \vec{F} \cdot d\vec{r}$ along the curve C.

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_a^b f(x(t),y(t))\,x'(t) + g(x(t),y(t))\,y'(t)\,dt$$

$$\vec{F}(x,y) = x^2\,\vec{i} + xy\,\vec{j}$$

$$C: \ \vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} \qquad (0 \le t \le \pi)$$

$$x = 2\cos t, \quad y = 2\sin t$$

$$\int_C x^2\,dx + xy\,dy = \int_0^{\pi} 4\cos^2 t\,(2\cos t)' + 4\sin t\cos t\,(2\sin t)'\,dt =$$

$$= \int_0^{\pi} -8\cos^2 t\sin t + 8\sin t\cos^2 t\,dt = 0$$

$$\vec{F}(x,y) = x^2 y\,\vec{i} + 4\,\vec{j}$$

$$C: \ \vec{r}(t) = e^t\,\vec{i} + e^{-t}\,\vec{j} \qquad (0 \le t \le 1)$$

$$\int_C x^2 y\,dx + 4\,dy = \int_0^1 e^{2t}e^{-t}e^{t} - 4e^{-t}\,dt =$$

$$= \int_0^1 e^{2t} - 4e^{-t}\,dt = \frac{1}{2}e^{2t}\Big|_0^1 + 4e^{-t}\Big|_0^1 =$$

$$= \frac{1}{2}e^2 - \frac{1}{2} + 4e - 4 = \frac{1}{2}e^2 + 4e - \frac{9}{2}$$

Find the work done by the force field F on a particle that moves along the curve C,

(1) $\vec{F}(x,y,z) = xy\,\vec{i} + yz\,\vec{j} + xz\,\vec{k}$

$C.\ \vec{r}(t) = t\,\vec{i} + t^2\vec{j} + t^3\vec{k} \qquad 0 \le t \le 1$

$x = t,\ y = t^2,\ z = t^3$

$$A = \int_C xy\,dx + yz\,dy + xz\,dz =$$

$$= \int_0^1 t\cdot t^2 (t)' + t^2\cdot t^3 (t^2)' + t\cdot t^3 (t^3)'\ dt =$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6\ dt = \frac{t^4}{4} + \frac{5t^7}{7}\Big|_0^1 =$$

$$= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

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(2) $\vec{F}(x,y,z) = (x+y)\,\vec{i} + xy\,\vec{j} - z^2\,\vec{k}$

along the line segments from $(0,0,0)$ to $(1,3,$ and from $(1,3,1)$ to $(2,-1,4)$

$C_1:\ x = t,\ y = 3t,\ z = t \qquad 0 \le t \le 1$

$C_2\quad \langle 1,-4,3\rangle$

$x = 1+t,\ y = 3-4t,\ z = 1+3t \qquad 0 \le t \le$

$$\int_C \ldots = \int_{C_1} \ldots + \int_{C_2} \ldots = \int_0^1 (t+3t)(t)' + t\times 3t\,(3t)' - t^2 (t)' -$$

$$+\int_0^1 \left((1+t)+(3-4t)\right)(1+t)' + (1+t)(3-4t)(3-4t)' + (1+3t)^2(1+3t)'\,dt$$

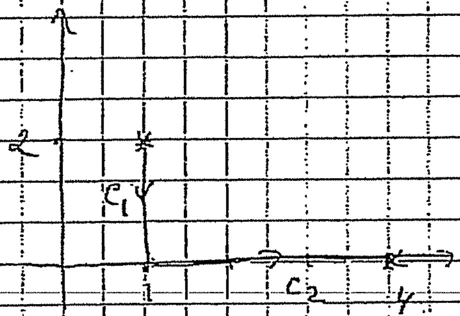
$$= \int_0^1 4t + 9t^2 - t^2 + 4 - 3t - 12 + 4t + 16t^2 + 3 + 18t + 27t^2\,dt =$$

$$= \int_0^1 \cancel{52t^2}\ 51t^2 + 23t - 5\,dt = \frac{51t^3}{3} + \frac{23t^2}{2} - 5t\Big|_0^1 =$$

$$= \frac{51}{3} + \frac{23}{2} - 5 = \frac{102+69-30}{6} = \frac{141}{6}$$

Show that the integral is independent of the path and evaluate the integral

(1) $\displaystyle\int_{(1,2)}^{(4,0)} 3y\,dx + 3x\,dy$



We must check that

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}, \quad f = 3y,\ g = 3x$$

$$\frac{\partial(3x)}{\partial x} = 3 = \frac{\partial(3y)}{\partial y} \Rightarrow$$ integral is independent of the path

1) way to evaluate the integral

find the potential φ

$$\varphi_x = 3y,\quad \varphi_y = 3x \quad (*)$$

$$\varphi = 3xy + C(y) \Rightarrow \varphi_y = 3x + C'(y) \overset{(*)}{=} 3x \Rightarrow C'(y) = 0$$

$$\Rightarrow C(y) = \text{const.} = C$$

$$\int_{(1,2)}^{(4,0)} 3y\,dx + 3x\,dy = \varphi(4,0) - \varphi(1,2) =$$

$$= 3\times 4\times 0 + \cancel{C} - 3\times 1\times 2 - \cancel{C} = -6$$

2) another way to calculate the integral: along the path $C_1 \cup C_2$ - see the sketch

C_1 : $x = 1$, $y = 2 - t$ $\quad 0 \leq t \leq 2$

C_2 : $x = t$, $y = 0$ $\quad 0 \leq t \leq 4$

$$\int_C \ldots = \int_{C_1} \ldots + \int_{C_2} \ldots = \int_0^2 3(1-t)\times 0 + 3(1-t)' \, dt +$$

$$+ \int_1^4 3\times 0\times 1 + 3t\times 0 \, dt = -6$$

Green's Theorem

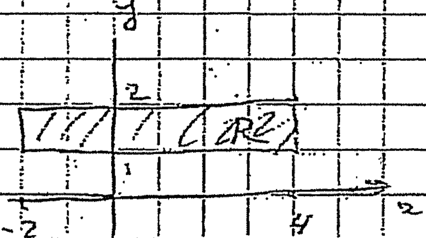
$$\oint f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Evaluate the line integral using the Green's Th

Curve is oriented counterclockwise

(1) $\displaystyle\int_C 3xy\,dx + 2xy\,dy$

C: $-2 \le x \le 4$, $1 \le y \le 2$



$$= \iint_R \frac{\partial(2xy)}{\partial x} - \frac{\partial(3xy)}{\partial y}\,dA =$$

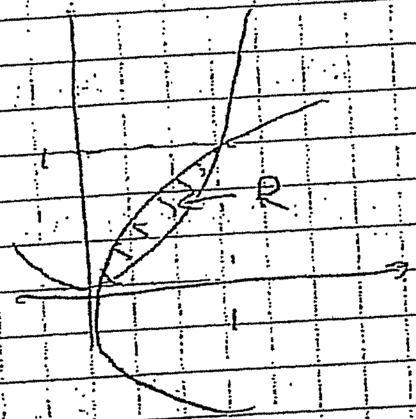
$$= \iint_R 2y - 3x\,dA = \int_{-2}^{4}\int_{1}^{2} 2y - 3x\,dy\,dx = \int_{-2}^{4} y^2 - 3xy\Big|_1^2\,dx$$

$$= \int_{-2}^{4} 4 - 6x - 1 + 3x\,dx = \int_{-2}^{4} 3x - 3\,dx =$$

$$= \oint \frac{3x^2}{2} - 3x\Big|_{-2}^{4} = 24 - 12 - 6 - 6 = 0$$

(2) $\displaystyle\int_C x^2 y\, dx + (y + xy^2)\, dy$

$C: \ y = x^2, \ x = y^2$



$$= \iint_R y^2 - x^2 \, dA =$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} y^2 - x^2 \, dy\, dx = \int_0^1 \left(\frac{y^3}{3} - x^2 y \Big|_{x^2}^{\sqrt{x}} \right) dx =$$

$$= \int_0^1 \frac{x^{3/2}}{3} - x^2\sqrt{x} - \frac{x^6}{3} + x^4 \, dx =$$

$$= \frac{2}{5} \cdot \frac{x^{5/2}}{3} - \frac{2x^{7/2}}{7} - \frac{x^7}{3 \cdot 7} + \frac{x^5}{5} \Big|_0^1 =$$

$$= \frac{2}{15} - \frac{2}{7} - \frac{1}{21} + \frac{1}{5} = \frac{14 - 30 - 5 + 21}{3 \cdot 5 \cdot 7} = 0$$