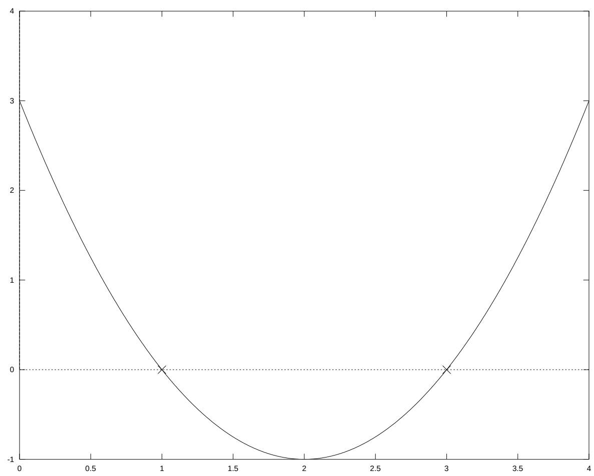
AMTH250 Assignment 4
Solutions

August 30, 2011

Question 1

```
octave:> x = 0:0.01:4;
octave:> y = (x-1).*(x-3);
octave:> plot(x,y)
octave:> hold on
octave:> plot([0 4],[0 0])
octave:> plot(1,0,'x')
octave:> plot(3,0,'x')
```



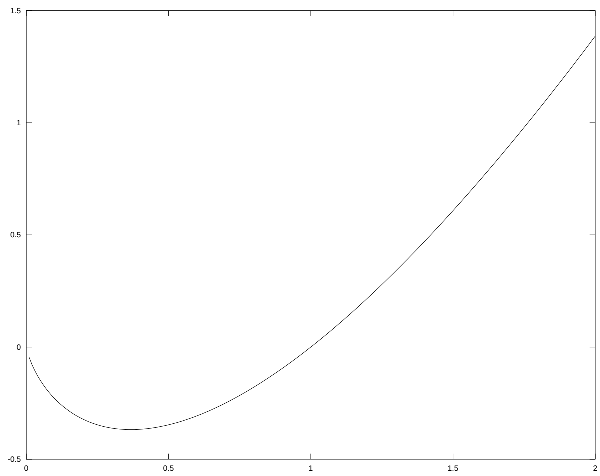
Question 2

Note that

$$\lim_{x \to 0} x \ln x = 0$$

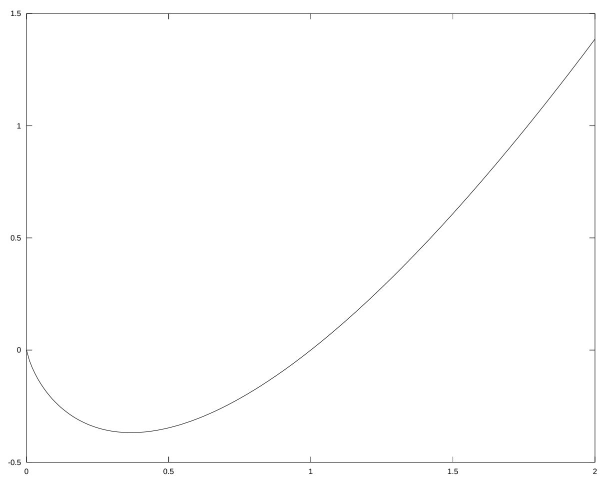
A simple attempt misses the limiting point at the origin because in Octave evaluates `0*log(0)` to a `NaN`.

```
octave:> x = 0:0.01:2;
octave:> y = x.*log(x);
octave:> plot(x,y)
```



Replacing `y(1)` by `0` does the trick:

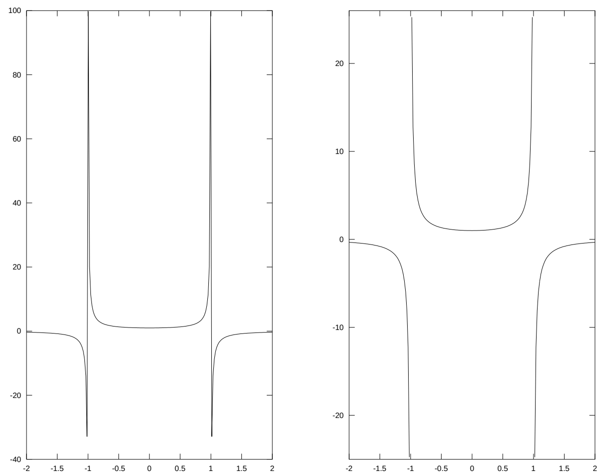
```
octave:> y(1) = 0;
octave:> plot(x,y)
```



Question 3

The appearance of the graph depends on whether or not the values $x = \pm 1$ at which the function has singularities are included among the abscissa. If $x = \pm 1$ are not included (see the first subplot below) then a large negative value on one side of the singularity will be joined to a large positive value on the other side, giving the appearance of of an asymptote. If $x = \pm 1$ are included (see the second subplot below) then the function will evaluate to `Inf` at $x = \pm 1$ in Octave and these points will not be plotted breaking the graph at the singularities.

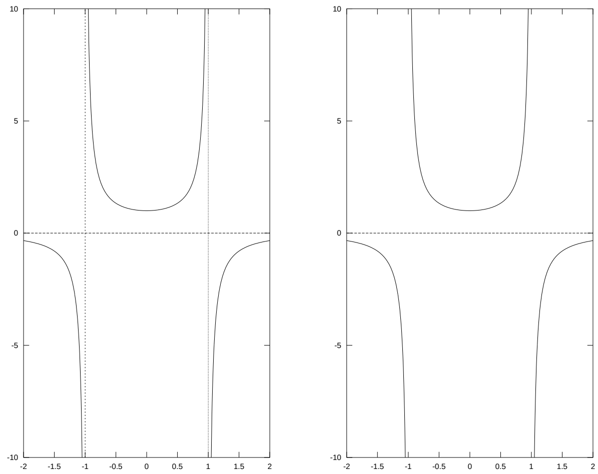
```
octave:> x = linspace(-2,2,200);
octave:> y = 1./(1 - x.^2);
octave:> subplot(1,2,1)
octave:> plot(x,y)
octave:> x = linspace(-2,2,201);
octave:> y = 1./(1 - x.^2);
octave:> subplot(1,2,2)
octave:> plot(x,y)
```



To do the final plot, we plot the curve without asymptotes and then add asymptotes as separate lines.

```
octave:> x = linspace(-2,2,1001);
octave:> y = 1./(1 - x.^2);
octave:> subplot(1,2,1)
octave:> plot(x,y)
octave:> hold on
octave:> axis([-2 2 -10 10])
octave:> plot([-2 2],[0 0])
octave:> plot([-1 -1],[-10 10])
octave:> plot([1 1],[-10 10])
```

```
octave:> hold off
octave:> subplot(1,2,2)
octave:> plot(x,y)
octave:> hold on
octave:> axis([-2 2 -10 10])
octave:> plot([-2 2],[0 0])
```



Question 4

Addition, multiplication and division can produce overflow.

```
octave:> x = realmax
x =  1.7977e+308
octave:> y = realmin
y =  2.2251e-308
octave:> x+x
ans = Inf
octave:> x*x
ans = Inf
octave:> x/y
ans = Inf
```

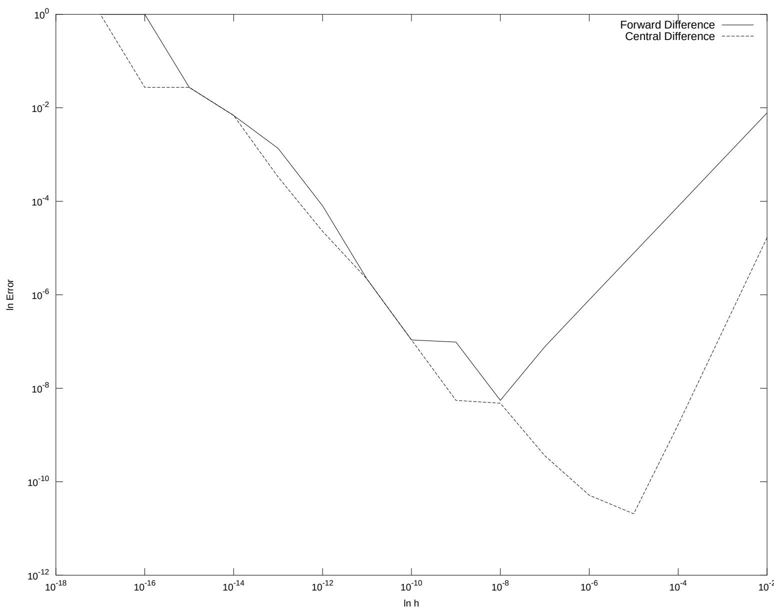
Subtraction of two positive numbers cannot overflow. If x and y are positive then $-y < x - y < x$.

Question 5

(a) and **(c)** Here are scripts for the forward difference and central difference approximations

```
% Forward difference approximation to f'(x) for f(x) = sin(x) at x=1.
n = -2:-1:-17;
h = 10 .^n;
exact =cos(1);
approx = (sin(1+h) - sin(1))./h;
err = abs(approx - exact) / exact;
loglog(h, err,'r')
legend('Forward Difference')
xlabel('ln h')
ylabel('ln Error')
```

```
% Central difference approximation to f'(x) for f(x) = sin(x) at x=1.
n = -2:-1:-17;
h = 10 .^n;
exact =cos(1);
approx = (sin(1+h) - sin(1-h))./(2*h);
err = abs(approx - exact) / exact;
loglog(h, err,'g')
legend('Central Difference')
xlabel('ln h')
ylabel('ln Error')
```



(b)

Method	Min Error	h
Forward Difference	$\approx 10^{-8}$	$\approx 10^{-8}$
Central Difference	$\approx 10^{-11}$	$\approx 10^{-5}$

(d)

Recall from the discussion in the notes that the error in these calculations has two components; (a) the truncation error due to the approximation itself, and (b) cancellation error occurring in the subtraction of two nearly equal numbers. As h decreases, the truncation error decreases but the cancellation error increases.

The two types of error are apparent in the graphs; the truncation error dominating in the right hand side of the graphs and the cancellation error dominating in the left hand side of the graphs. The cancellation error is roughly the same for the two methods, but the truncation decreases more rapidly for the central difference method.

The total error is the sum of the two types of error. The effect of the error of the central difference formula decreasing more rapidly with h is that minimum error is smaller and occurs at a larger value of h than for the forward difference formula.

Question 6

(a)

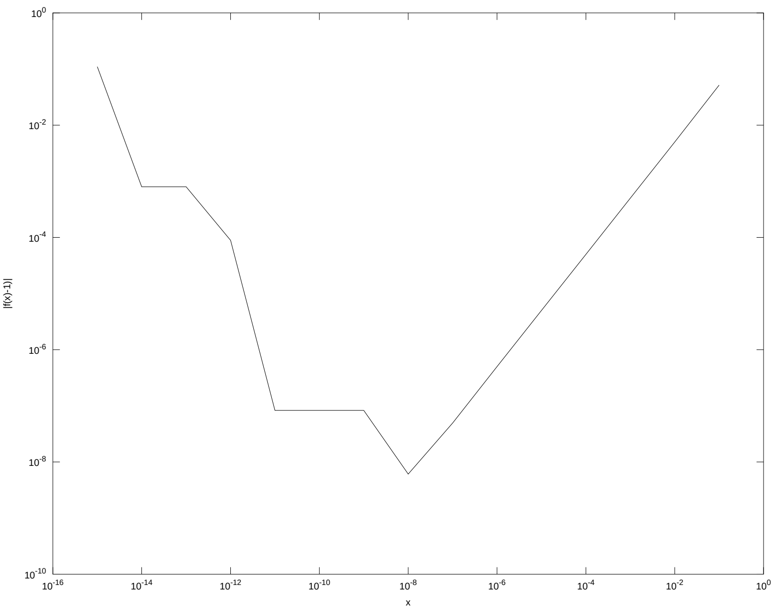
$$
\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{e^x - 1}{x} \\
&= \frac{\lim\limits_{x\to 0} \dfrac{d}{dx}(e^x - 1)}{\lim\limits_{x\to 0} \dfrac{d}{dx} x} \\
&= \frac{\lim\limits_{x\to 0} e^x}{\lim\limits_{x\to 0} 1} \\
&= 1
\end{aligned}
$$

(b)

Mathematically as $x \to 0$, $f(x) \to 1$. Computationally, as $x \to 0$ the numerator, $e^x - 1$ will be formed from the subtraction of two nearly equal terms and cancellation will occur.

This shows up in the log-log plot of $|f(x) - 1|$ against x as $x \to 0$:

```
octave:> k = 1:15;
octave:> x = 10.^(-k);
octave:> fx = (exp(x)-1)./x;
octave:> loglog(x, abs(fx-1))
```



This is exactly the same type of graph we saw for the finite difference approximations to a derivative. A $x \to 0$ the the truncation error, in this case the difference between $f(x)$ and its limiting value $f(1) = 1$, decreases. At a certain point cancellation error takes over and the difference $|f(x) - 1|$ begins to increase.

Aside: The formula for $f(x)$

$$f(x) = \frac{e^x - 1}{x}$$

is the forward difference approximation to $e^x - 1$ at $x = 0$. This explains why the graphs are so similar in shape. Also it is not accidental that the minimum error is about 10^{-8} and occurs at $x \approx 10^{-8}$. It can be shown that the minimum error in the forward difference approximation is expected to occur at $h \approx \sqrt{\varepsilon_{\text{mach}}}$. Similarly the minimum error in the central difference approximation is expected to occur at $h \approx \sqrt[3]{\varepsilon_{\text{mach}}}$.