Sample Examination

Question 1: [10 *marks*]

Find all injective (that is, 1–1) linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ which map the line with equation $v = 0$ onto the line with equations $x = y = z$.

Question 2: [8 *marks*]

Determine whether the real quadratic form

$$Q(x, y, z) = 2x^2 + 4xy + y^2 + 2yz + 4z^2 + 4zx$$

is positive definite.

Question 3: [10 *marks*]

Find all real 2×2 matrices that are both symmetric and orthogonal.

Question 4: [10 *marks*]

Show that the skew-symmetric $n \times n$ matrices form a real vector subspace of the real vector space of all $n \times n$ matrices with real coefficients and finds its dimension.

Question 5: [12 *marks*]

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for the vector space V and $T : V \longrightarrow V$ a linear transformation.

(a) Show that if $\mathbf{f}_1 := -\mathbf{e}_1,\ \mathbf{f}_2 := \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{f}_3 := \mathbf{e}_1 + \mathbf{e}_3$, then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is also a basis for V.

(b) Find the matrix B of T with respect to $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ given that its matrix with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Question 6: [10 *marks*]

Prove that the linear transformation $T : V \to W$ is injective (that is 1–1) if and only if $T(v_1) \dots T(v_n)$ are linearly independent in W whenever $v_1 \dots v_n$ are linearly independent in V.

Question 7: [15 *marks*]

Given the symmetric matrix $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$, find

(a) its eigenvalues,

(b) bases for its eigenspaces,

(c) an orthogonal matrix P which diagonalises A, and

(d) $P^{-1}AP$.

Question 8: [25 *marks*]

Let $\mathbb{R}[t]$ be the real vector space of all real polynomials, so that $\mathbb{R}[t] = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \dots, a_n \in \mathbb{R},\ n \in \mathbb{N}\}$.

(a) Prove that

$$\langle\langle f, g \rangle\rangle := \int_0^1 f(x)g(x)dx \qquad (f, g \in \mathbb{R}[t])$$

defines an inner product on $\mathbb{R}[t]$.

(b) Apply the Gram-Schmidt procedure with respect to this inner product to find an orthonormal basis for the vector subspace of $\mathbb{R}[t]$ generated by $\{\, t, t^2 \,\}$.

Sample Solutions

1

1.1 Solution A

Take any non-zero vector, $\mathbf{e}_1$, on the line in $\mathbb{R}^2$ with equation $v = 0$. Let $\mathbf{e}_2$ be any vector in $\mathbb{R}^2$ which is not linearly dependent on $\mathbf{e}_1$. Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$.

Let $\mathbf{f}_1$ be any non-zero vector in $\mathbb{R}^3$ on the line given by $x = y = z$, and let $\mathbf{f}_2$ be any vector in $\mathbb{R}^3$ not linearly dependent on $\mathbf{f}_1$.

Then the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ uniquely determined by requiring that $T(\mathbf{e}_\jmath) = \mathbf{f}_\jmath$, $(\jmath = 1, 2)$ is injective and maps the line with equation $v = 0$ onto the line with equations $x = y = z$. Moreover every such injective linear transformation is of this form.

1.2 Solution B

Every linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by an equation of the form $T(u, v) = (au + bv, cu + dv, eu + fv)$ with $a, b, c, d, e, f \in \mathbb{R}$.

Then $T(u, 0) = (au, cu, eu)$ which lies on the line given by $x = y = z$ for all $u \in \mathbb{R}$ if and only if $a = c = e$.

Moreover it is a map **onto** the line if and only if $a \neq 0$, so that $T(u, v) = (au + bv, au + dv, au + fv)$. This is injective if and only if $T(u, v) = (0, 0, 0) \implies u = v = 0$. But $T(u, v) = (0, 0, 0)$ if and only if $au + bv = au + dv = au + fv$, which occurs only if $(b - d)v = (d - f)v = (f - b)v = 0$. These equations only have the solution $v = 0$ if and only if at least one of $b - d, d - f$ and $f - b$ is non-zero, that is, $|b - d| + |d - f| + |f - b| \neq 0$. Conversely, if $|b - d| + |d - f| + |f - b| \neq 0$, then T is injective.

Thus T must be given by

$$T(u, v) = (au + bv, au + dv, au + fv) \quad \text{with } a \neq 0 \text{ and } |b - d| + |d - f| + |f - b| \neq 0.$$

1.3 Solution C

The standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is

$$\underline{\mathbf{A}} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

The line in $\mathbb{R}^2$ given by $v = 0$ is represented by $\begin{bmatrix} u \\ 0 \end{bmatrix}$ $(u \in \mathbb{R})$ in the standard basis.

Similarly, the line in $\mathbb{R}^3$ given by $x = y = z$ is represented in the standard basis by $\begin{bmatrix} x \\ x \\ x \end{bmatrix}$ $(x \in \mathbb{R})$.

Thus we have

$$\begin{bmatrix} x \\ x \\ x \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} au \\ cu \\ eu \end{bmatrix} \quad \text{for all } u \in \mathbb{R},$$

which is the case if and only if $a = c = e$.

Moreover T is injective if and only if $\operatorname{rk}(\underline{\mathbf{A}}) = 2$ in which case the line given by $v = 0$ is automatically mapped **onto** the line given $x = y = z$. But $\operatorname{rk}(\underline{\mathbf{A}}) = 2$ if and only if at least two of b, d, f are distinct. Thus T is as required if and only if

$$\underline{\mathbf{A}} = \begin{bmatrix} a & b \\ a & d \\ a & f \end{bmatrix} \quad \text{with } a \neq 0 \text{ and } |b - d| + |d - f| + |f - b| > 0.$$

2

2.1 Solution A

$$\begin{aligned} Q(x,y,z) &= 2x^2+4xy+y^2+2yz+4z^2+4zx \\ &= 2(x+y+z)^2-y^2-2yz+2z^2 \\ &= 2(x+y+z)^2-(y+z)^2+3z^2, \end{aligned}$$

which is not positive definite, since, for example, $Q(-1,1,0)=-1<0$.

2.2 Solution B

The standard matrix of Q is

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix},$$

whose

eigenvalues are the solutions of the equation

$$\det\left(\begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & 1-\lambda & 1 \\ 2 & 1 & 4-\lambda \end{bmatrix}\right)=0,$$

that is to say, the solutions of

$$\lambda^3-7\lambda^2-5\lambda+6=0.$$

Since the constant term of $(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)$ is $-\alpha\beta\gamma$, it follows that the product of the eigenvalues of the standard matrix for Q is $-6<0$, so that an odd number of the eigenvalues must be negative and an even number positive. Hence Q cannot be positive definite, since at least one eigenvalue must be negative.

3 Solution

A is a symmetric 2×2 real matrix if and only if $A=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ with $a,b,c\in\mathbb{R}$. Further, A is orthogonal if and only if $A^tA=\underline{\mathbf{1}}$.

Thus A is an orthogonal and symmetric 2×2 real matrix if and only if

$$a^2+b^2=1 \tag{1}$$

$$(a+d)b=0 \tag{2}$$

$$b^2+d^2=1 \tag{3}$$

Suppose that $b\neq 0$. Then, by (2), $d=-a$, and from (1), $a=\cos\theta, b=\sin\theta$ for some $\theta\in[0,2\pi[$.

If, on the other hand $b=0$, then it follows from (1) and (3), $a^2=d^2=1$.

Thus A is a real symmetric orthogonal 2×2 matrix if and only if

$$A=\begin{cases} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} & \text{for some } \theta\in[0,2\pi[\text{, or} \\[2ex] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{or} \\[2ex] \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \end{cases}$$

4 Solution

Given any $n \times n$ real matrices A, B and real numbers λ, μ,

$$\begin{aligned}
(\lambda A + \mu B)^t &= (\lambda A)^t + (\mu B)^t \\
&= \lambda A^t + \mu B^t \\
&= \lambda(-A) + \mu(-B) && \text{if } A, B \text{ are skew-symmetric} \\
&= -(\lambda A + \mu B),
\end{aligned}$$

showing that the skew-symmetric matrices form a vector subspace of the space of all real $n \times n$ matrices.

Given $1 \leq k < \ell \leq n$, put $E_{k\ell} = [x_{\imath\jmath}]_{n\times n}$ where

$$x_{\imath\jmath} = \begin{cases} 1 & \text{if } \imath = k, \jmath = \ell \\ -1 & \text{if } \imath = \ell, \jmath = k \\ 0 & \text{otherwise} \end{cases}.$$

Now $A = [a_{\imath\jmath}]_{n\times n}$ is skew-symmetric if and only if $a_{\jmath\imath} = -a_{\imath\jmath}$ for all $\imath, \jmath$, which is the case if and only if

$$A = \sum_{1 \leq \imath < \jmath \leq n} a_{\imath\jmath} E_{\imath\jmath}.$$

Plainly $\{ E_{\imath\jmath} \mid 1 \leq \imath < \jmath \leq n \}$ is linearly independent, and hence forms a basis for the skew-symmetric matrices. Thus their dimension is $\frac{n(n-1)}{2}$.

5 Solution

(a) Since

$$\begin{aligned}
\mathbf{e}_1 &= -\mathbf{f}_1 \\
\mathbf{e}_2 &= \mathbf{f}_2 - \mathbf{e}_1 = \mathbf{f}_1 + \mathbf{f}_2 \\
\mathbf{e}_3 &= \mathbf{f}_3 - \mathbf{e}_1 = \mathbf{f}_1 + \mathbf{f}_3,
\end{aligned}$$

$V = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle \subseteq \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \rangle \subseteq V$. Thus, since $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are three vectors generating the three dimensional vector space V, they must be linearly independent and so form a basis for V.

(b) From the data, the change of basis matrix is

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and from (a), it is its own inverse. Hence

$$\begin{aligned}
B &= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 11 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

6 Solution

$\underline{\Rightarrow}$: Suppose that T is injective and that $v_1, \dots, v_n$ are linearly independent in V. Then

$$
\begin{aligned}
\lambda_1 T(v_1) + \cdots + \lambda_n T(v_n) = \mathbf{0}_W &\Leftrightarrow T(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \mathbf{0}_W && \text{as } T \text{ is linear} \\
&\Leftrightarrow \lambda_1 v_1 + \cdots + \lambda_n v_n = \mathbf{0}_V && \text{as } T \text{ is injective} \\
&\Leftrightarrow \lambda_1 = \cdots = \lambda_n = 0 && \text{as } v_1, \dots, v_n \text{ are linearly independent,}
\end{aligned}
$$

showing that $T(v_1), \dots, T(v_n)$ are linearly independent.
$\underline{\Leftarrow}$: Suppose that $T(v_1), \dots T(v_n)$ are linearly independent in W whenever $v_1, \dots, v_n$ are linearly independent in V, and take $v \in V, v \neq \mathbf{0}_V$. Then v is linearly independent in V. Thus, by hypothesis $T(v)$ is linearly independent in W, so that $T(v) \neq \mathbf{0}_W$. Thus $\ker(T) = \{\mathbf{0}_V\}$, showing that T is injective.

7 Solution

(a) To find the eigenvalues of A, apply elementary row operations to $A - \lambda \underline{\mathbf{1}}_2$:

$$
\begin{bmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix}
\begin{matrix} \mathrm{R}_1 & \rightsquigarrow & \mathrm{R}_1 \\ \mathrm{R}_2 & \rightsquigarrow & (3-\lambda)\mathrm{R}_2 + 2\mathrm{R}_3 \end{matrix}
\begin{bmatrix} 3-\lambda & -2 \\ 0 & 5 - 6\lambda + \lambda^2 \end{bmatrix}
=
\begin{bmatrix} 3-\lambda & -2 \\ 0 & (\lambda-1)(\lambda-5) \end{bmatrix},
$$

so that the eigenvalues of A are 1 and 5.

(b) $\lambda = 1$ $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector for $\lambda = 1$ if and only if $2x - 2y = 0$, i.e. $y = x$. Hence $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ is a basis for the eigenspace for $\lambda = 1$.

$\lambda = 5$ $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector for $\lambda = 5$ if and only if $-2x - 2y = 0$, i.e. $y = -x$. Hence $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ is a basis for the eigenspace for $\lambda = 5$.

(c) Put $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then

$$
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}
\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
=
\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}
= 2 \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}
$$

is diagonal and $\langle\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rangle\rangle = 0$. Since $\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \| = \| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \| = \sqrt{2}$, put $P := \frac{1}{\sqrt{2}} M$. Then P is orthogonal and diagonalises A.

(d) By the above,

$$
P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.
$$

8 Solution

(a) Since each polynomial defines a continuous (and hence integrable) function $[0, 1] \longrightarrow \mathbb{R}$, putting $\langle\langle f, g \rangle\rangle := \int_0^1 f(x) g(x) dx$ defines a function $\langle\langle\ ,\ \rangle\rangle : \mathbb{R}[t] \times \mathbb{R}[t] \longrightarrow \mathbb{R}$.

- $\langle\langle f, f \rangle\rangle = \int_0^1 f(x)^2 dx \geq 0$ from the properties of the integral.
- Now $\langle\langle f, f \rangle\rangle = 0 \Leftrightarrow \int_0^1 f(x)^2 dx = 0 \Leftrightarrow f(x) \equiv 0$ as f^2 is continuous and non-negative.
- $\langle\langle g, f \rangle\rangle = \int_0^1 g(x) f(x) dx = \int_0^1 f(x) g(x) dx = \langle\langle f, g \rangle\rangle$ as $g(x) f(x) = f(x) g(x)$ $(f, g \in \mathbb{R}[t], x \in [0, 1])$.

– Take $f, g, h \in \mathbb{R}[t]$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned}
\langle\!\langle \lambda f + \mu g, h \rangle\!\rangle &:= \int_0^1 \big(\lambda f(x) + \mu g(x)\big) h(x) dx \\
&= \lambda \int_0^1 f(x)h(x)dx + \mu \int_0^1 g(x)h(x)dx \quad \text{by properties of the integral} \\
&=: \lambda \langle\!\langle f, g \rangle\!\rangle + \mu \langle\!\langle f, g \rangle\!\rangle.
\end{aligned}$$

Thus $\langle\!\langle\ ,\ \rangle\!\rangle$ is a real inner product.

(b) Put $\mathbf{v}_\imath := t^\imath$ $(\imath = 1, 2)$. Then $\langle\!\langle \mathbf{v}_\imath, v_\jmath \rangle\!\rangle = \int_0^1 x^{\imath+\jmath} dx = \frac{1}{\imath+\jmath+1}$ $(\imath, \jmath = 1, 2)$.

Thus $\langle\!\langle \mathbf{v}_1, \mathbf{v}_1 \rangle\!\rangle = \frac{1}{3}$. So put $\mathbf{e}_1 := \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \sqrt{3}t$.

Now put $\mathbf{f}_2 := \mathbf{v}_2 - \langle\!\langle \mathbf{v}_2, \mathbf{e}_1 \rangle\!\rangle \mathbf{e}_1 = v_2 - 3\langle\!\langle \mathbf{v}_2, \mathbf{v}_1 \rangle\!\rangle \mathbf{v}_1 = \mathbf{v}_2 - \frac{3}{4}\mathbf{v}_1$.

Then $\langle\!\langle \mathbf{f}_2, \mathbf{f}_{2\rangle\!\rangle} = \langle\!\langle \mathbf{v}_1, \mathbf{v}_1 \rangle\!\rangle - \frac{3}{2}\langle\!\langle \mathbf{v}_2, \mathbf{v}_1 \rangle\!\rangle + \frac{9}{16}\langle\!\langle \mathbf{v}_1, \mathbf{v}_1 \rangle\!\rangle = \frac{1}{5} - \frac{3}{8} + \frac{3}{16} = \frac{1}{80}$.

Put $\mathbf{e}_2 := \frac{\mathbf{f}_2}{\|\mathbf{f}_2\|} = 4\sqrt{5}\mathbf{f}_2 = \sqrt{5}(4t^2 - 3t)$.

Thus the orthonormal basis for the subspace $\langle t, t^2 \rangle$ of $\mathbb{R}[t]$ provided by the Gram-Schmidt procedure is

$$\{\sqrt{3}t, \sqrt{5}(4t^2 - 3t)\}.$$