Chapter 15

Matrix Representations of Inner Products

Just matrices enables us to represent linear transformations and perform computations on them, so we now turn to representing inner products using matrices, with a view to allowing us to carry out concrete computations.

It is actually simpler to discuss general sesqui-linear forms and bi-linear forms, restricting to the appropriate special cases as required.

Take finitely generated vector spaces U and V over the sub-field $\mathbb{F}$ of $\mathbb{C}$ let $\beta : U \times V \to \mathbb{F}$ be a sesqui-linear function. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ be a basis for U and $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ a basis for V.

Given $\mathbf{u} \in U$ and $\mathbf{v} \in V$, we have

$$\mathbf{u} = \sum_{i=1}^{m} x_i \mathbf{e}_i \qquad \text{and} \qquad \mathbf{v} = \sum_{j=1}^{n} y_j \mathbf{f}_j,$$

whence

$$\begin{aligned}
\beta(\mathbf{u}, \mathbf{v}) &= \beta(\sum_{i=1}^{m} x_i \mathbf{e}_i, \sum_{j=1}^{n} y_j \mathbf{f}_j) \\
&= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} x_i \overline{y_j} \beta(\mathbf{e}_i, \mathbf{f}_j) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} \overline{y_j},
\end{aligned}$$

where $a_{ij} := \beta(\mathbf{e}_i, \mathbf{f}_j) \qquad (1 \leq i \leq m,\ 1 \leq j \leq n)$

Let

$$\underline{\mathbf{x}} := \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad \text{and} \qquad \underline{\mathbf{y}} \begin{bmatrix} y_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

be the co-ordinate vectors of $\mathbf{u}$ and $\mathbf{v}$ with respect to the given bases, and put $\underline{\mathbf{A}} := [a_{ij}]_{m \times n}$ with $a_{ij} := \beta(\mathbf{e}_i, \mathbf{f}_j)$. Direct calculation shows that $\beta(\mathbf{u}, \mathbf{v}) = \underline{\mathbf{x}}^t \underline{\mathbf{A}}\, \overline{\underline{\mathbf{y}}}$.

Since our purposes require us to consider only the case $U = V$ and $\mathbf{f}_j = \underline{\mathbf{e}}_j$, we dispense with the greater generality for the rest of this chapter. But the importance and usefulness of the more general approach cannot be over-emphasised, for it marks the beginnings of tensor analysis, which has many applications in statistics, geometry, physics, chemistry and engineering.

Definition 15.1. Given a sesqui-linear form $\beta : V \times V \to \mathbb{F}$, the *matrix of* β *with respect to the basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *of* V is the matrix

$$\underline{\mathbf{A}} := \left[\beta(\mathbf{e}_i, \mathbf{e}_j)\right]_{n \times n}$$

Since an inner product is a sesqui-linear form, we can already deduce an important fact.

Theorem 15.2. *Let* $\langle\!\langle\ ,\ \rangle\!\rangle$ *be an inner product on* V*. Let* $\underline{\mathbf{A}}$ *be the matrix of* $\langle\!\langle\ ,\ \rangle\!\rangle$ *with respect to the basis* $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$*. The* $\mathcal{B}$ *is an orthogonal basis if and only if* $\underline{\mathbf{A}}$ *is a diagonal matrix, and* $\mathcal{B}$ *is an orthonormal basis if and only if* $\underline{\mathbf{A}} = \underline{\mathbf{1}}_n$

The above discussion forms the basis of our computational techniques. We summarise the above discussion in our next theorem.

Theorem 15.3. *Let* $\{\mathbf{e}_i, \ldots, \mathbf{e}_n\}$ *be a basis for the vector space* V*. Let* $\underline{\mathbf{x}}$ *is the co-ordinate vector of* $\mathbf{u} \in V$ *and* $\underline{\mathbf{y}}$ *that of* $\mathbf{v} \in V$*.*
If $\beta : V \times V \to \mathbb{F}$ *is a sesqui-linear form on* V*, then*

$$\beta(\mathbf{u}, \mathbf{v}) = \underline{\mathbf{x}}^t \underline{\mathbf{A}}\, \overline{\underline{\mathbf{y}}}$$

and if β *is bi-linear, then*

$$\beta(\mathbf{u}, \mathbf{v}) = \underline{\mathbf{x}}^t \underline{\mathbf{A}}\, \underline{\mathbf{y}}.$$

We investigate the relationship between endomorphisms (or changes of basis) on the one hand, and sesqui-linear forms on the other.

Lemma 15.4. *Let* $\beta : V \times V \to \mathbb{F}$ *be sesqui-linear and* $T : V \to V$ *an endomorphism. Then*

$$\gamma : V \times V \longrightarrow \mathbb{F}, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \beta(T(\mathbf{u}), T(\mathbf{v}))$$

is a sesqui-linear form, denoted $\gamma = \beta \circ (T \times T)$*.*

Proof. Take $\lambda, \mu \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Then

$$\begin{aligned}
\gamma(\lambda\mathbf{u} + \mu\mathbf{v}, \mathbf{w}) &= \beta(T(\lambda\mathbf{u} + \mu\mathbf{v}), T(\mathbf{w})) \\
&= \beta(\lambda T(\mathbf{u}) + \mu T(\mathbf{v}), T(\mathbf{w})) \\
&= \lambda\beta(T(\mathbf{u}), T(\mathbf{w})) + \mu\beta(T(\mathbf{v}), T(\mathbf{w})) \\
&= \lambda\gamma(\mathbf{u}, \mathbf{w}) + \mu\gamma(\mathbf{v}, \mathbf{w}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma(\mathbf{u}, \lambda{+}\mu\mathbf{w}) &= \beta(T(\mathbf{u}), T(\lambda{+}\mu\mathbf{w})) \\
&= \beta(\mathbf{u}, \lambda T(\mathbf{v}) + \mu T(\mathbf{w})) \\
&= \overline{\lambda}\beta(T(\mathbf{u}), T(\mathbf{v})) + \overline{\mu}\beta(T(\mathbf{u}), T(\mathbf{w})) \\
&= \overline{\lambda}\gamma(\mathbf{u}, \mathbf{v}) + \overline{\mu}\gamma(\mathbf{u}, \mathbf{w}),
\end{aligned}$$

which shows that γ is sesqui-linear. □

Corollary 15.5. *Let* $\beta : V \times V \to \mathbb{F}$ *be bi-linear and* $T : V \to V$ *an endomorphism. Then* $\beta \circ (T \times T)$ *is a bi-linear form.*

Theorem 15.6. *Let $\beta, \gamma : V \times V \to \mathbb{F}$ be sesqui-linear forms and $T : V \to V$ an endomorphism. Choose a basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V. If the matrices of β, γ and T with respect to $\mathcal{B}$ are $\underline{\mathbf{B}}, \underline{\mathbf{C}}$ and $\underline{\mathbf{A}}$ respectively, then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \overline{\underline{\mathbf{A}}}.$$

Corollary 15.7. *Let $\beta, \gamma : V \times V \to \mathbb{F}$ be bi-linear forms and $T : V \to V$ an endomorphism. Choose a basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V. If the matrices of β, γ and T with respect to $\mathcal{B}$ are $\underline{\mathbf{B}}, \underline{\mathbf{C}}$ and $\underline{\mathbf{A}}$ respectively, then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \underline{\mathbf{A}}.$$

Corollary 15.8. *Let $\beta : V \times V \to \mathbb{F}$ be a sesqui-linear form. Choose bases $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\mathbb{C} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ for V. If the matrices of β with respect to $\mathcal{B}$ is $\underline{\mathbf{B}}$ and that with respect to $\mathcal{C}$ is $\underline{\mathbf{C}}$ then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \overline{\underline{\mathbf{A}}},$$

where $\underline{\mathbf{A}}$ is the "change of basis matrix" from the basis $\mathcal{C}$ to $\mathcal{B}$.

Proof. Recall that if $\mathbf{v} \in V$ has $\underline{\mathbf{x}}$ as co-ordinate vector with respect to $\mathcal{B}$ and $\underline{\mathbf{y}}$ with respect of $\mathcal{C}$, then $\underline{\mathbf{x}} = \underline{\mathbf{A}}\, \underline{\mathbf{y}}$. □

Corollary 15.9. *Let $\beta : V \times V \to \mathbb{F}$ be a bi-linear form. Choose bases $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\mathcal{C} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ for V. If the matrices of β with respect to $\mathcal{B}$ is $\underline{\mathbf{B}}$ and that with respect to $\mathcal{C}$ is $\underline{\mathbf{C}}$ then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \underline{\mathbf{A}},$$

where $\underline{\mathbf{A}}$ is the "change of basis matrix" from the basis $\mathcal{C}$ to $\mathcal{B}$.

In order to represent inner products on finitely generated spaces by matrices, we have only exploited the fact that an inner product on V is a sesqui-linear form on V. The other requirements impose conditions on the matrix representing our form.

Because $\langle\!\langle \mathbf{u}, \rangle\!\rangle = \overline{\langle\!\langle , \mathbf{u} \rangle\!\rangle}$ for all $\mathbf{u}, \mathbf{v} \in V$, we must have

$$a_{ji} := \langle\!\langle \mathbf{e}_j, \mathbf{e}_i \rangle\!\rangle = \overline{\langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle} =: a_{ij}$$

for any basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$. Thus, if $\underline{\mathbf{A}}$ is the matrix of the inner product, we must have

$$\underline{\mathbf{A}}^t = \overline{\underline{\mathbf{A}}}$$

Definition 15.10. The complex matrix $\underline{\mathbf{A}}$ is called *Hermitian* if and only if

$$\underline{\mathbf{A}}^t = \overline{\underline{\mathbf{A}}}.$$

Of course, if the $\mathbb{F}$ is a subset of $\mathbb{R}$, then the sesqui-liniearity becomes bi-linearity and $\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle = \langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle$ for all $\mathbf{u}, \mathbf{v} \in V$, so that if $\underline{\mathbf{A}}$ is the matrix of the inner product, then

$$\underline{\mathbf{A}}^t = \underline{\mathbf{A}}$$

Definition 15.11. The real matrix $\underline{\mathbf{A}}$ is called *symmetric* if and only if

$$\underline{\mathbf{A}}^t = \underline{\mathbf{A}}.$$

We summarise our discussion in the next theorem.

Theorem 15.12. *Any matrix representing a complex inner product must be Hermitian, and any matrix representing a real inner product must be symmetric.*

So far, we have not exploited the positive definiteness of inner products. This also has consequences for the matrix representation of inner products. It is again more convenient to present the discussion at the level of sesqui-linear forms, rather than restricting only to inner products.

First observe that for any inner product space V, given $\lambda \in \mathbb{F}$ and $\mathbf{x} \in V$, it follows from (IP1) and (IP2) that

$$\langle\!\langle \lambda\mathbf{x}, \lambda\mathbf{x} \rangle\!\rangle = \lambda\overline{\lambda}\langle\!\langle \mathbf{x}, \mathbf{x} \rangle\!\rangle = |\lambda|^2 \langle\!\langle \mathbf{x}, \mathbf{x} \rangle\!\rangle$$

In particular, given a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, we have $\mathbf{x} = \sum x_j \mathbf{e}_j$, for suitable x_j $(j = 1, \dots, n)$, whence $\langle\!\langle \mathbf{x}, \mathbf{x} \rangle\!\rangle = \sum a_{ij} x_i \overline{x_j}$, with $a_{ij} := \langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle$.

Thus we require that for all $x_1, \dots, x_n$

$$\sum_{j=1}^{n} a_{ij} x_i \overline{x_j} \geq 0,$$

with equality if and only if each $x_j = 0$.

This last expression is a homogeneous quadratic polynomial in the co-ordinates of $\mathbf{x}$.

Given the importance of such functions, especially in the real case, we later devote a chapter to them.

15.1 Exercises

Exercise 15.1. Let V and W be finitely generated vector spaces over the subfield $\mathbb{F}$ of $\mathbb{C}$. Let $\gamma : W \times W \to \mathbb{F}$ be a bi-linear form on W and $T : V \to W$ a linear transformation.

Show that

$$\beta : V \times V \longrightarrow \mathbb{F}, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \gamma(T(\mathbf{u}), T(\mathbf{v}))$$

defines a bi-linear form on V. [We write $\beta = \gamma \circ (T \times T)$.]

Show that if, instead, γ is sesqui-linear, then so is β.

Choose bases $\mathcal{B} = \{\mathbf{e}_i, \dots, \mathbf{e}_n\}$ for V and $\mathcal{C} = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ for W. Let the matrix of T with respect to these bases be $\underline{\mathbf{A}}$. Let the matrix of γ with respect to $\mathcal{C}$ be $\underline{\mathbf{C}}$ and that of β with respect to $\mathcal{B}$ be $\underline{\mathbf{B}}$.

Show that if γ is bi-linear, then

$$\underline{\mathbf{B}} = \underline{\mathbf{A}}^t \underline{\mathbf{C}}\, \underline{\mathbf{A}},$$

and if γ is sesqui-linear, then

$$\underline{\mathbf{B}} = \underline{\mathbf{A}}^t \underline{\mathbf{C}}\, \overline{\underline{\mathbf{A}}}.$$

Exercise 15.2. Show that if $\underline{\mathbf{A}}$ is a complex Hermitian $n \times n$ matrix, and $\underline{\mathbf{B}}$ is any other complex $n \times n$ matrix, then $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \overline{\underline{\mathbf{B}}}$ is also Hermitian.

Show that if $\underline{\mathbf{A}}$ is a real symmetric $n \times n$ matrix, and $\underline{\mathbf{B}}$ is any other real $n \times n$ matrix, then $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \underline{\mathbf{B}}$ is also symmetric.