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Assignment 1

1) Direct proof: Let x be a positive real number less than 1. Symbolically, $\{x \in \mathbb{R} \mid 0 < x < 1\}$. Squaring the inequality gives $0 < x^2 < 1$, thus showing $x^2 < 1$ when x is real and between 0 and 1.

Alternatively, if we multiply the inequality by x, we find $0 < x^2 < x$, showing that the square of x is less than x itself. By transitivity, since $x < 1$, x^2 is also less than 1.

Indirect proof: Assume $x^2 \geq 1$.
Consider first $x^2 = 1$. Solving for x by taking square roots give $x = \pm 1$. Clearly, these values of x are not in the set defined above, $\{x \in \mathbb{R} \mid 0 < x < 1\}$.
Similarly for $x^2 > 1$, we find $x < -1$ and $x > 1$. Again, these inequalities lie outside the set, thus contradicting our assumption that x is a positive real number less than 1.

2) Proof by induction: Let $P(n)$ be the proposition $\sum_{k=1}^{n} (2k-1) = n^2$

$n = 1$: $\sum_{k=1}^{1} (2 \times 1 - 1) = 1 = 1^2$

Hence $P(1)$ is true.

$n \geq 1$: We make the inductive hypothesis that $P(n)$ is true, such that
$$\sum_{k=1}^{n} (2k-1) = n^2$$

Then,
$$\sum_{k=1}^{n+1} (2k-1) = \sum_{k=1}^{n} (2k-1) + \left(2(n+1) - 1\right)$$
$$= n^2 + 2n + 1 \text{ (by inductive hypothesis)}$$
$$= (n+1)^2$$

Mark Villar

Hence $P(n+1)$ is true whenever $P(n)$ is true.
By the principle of mathematical induction,
$\sum_{k=1}^{n} (2k-1) = n^2$ for every counting number n.

3) Let $P(n)$ be the proposition that if $n \in \mathbb{N}$, then $3^{4n} - 1$ is divisible by 80.

$n=1$: $3^4 - 1 = 81 - 1 = 80$

Hence $P(1)$ is true.

$n \geqslant 1$: We make the inductive hypothesis that $P(n)$ is true, that is, for $m \in \mathbb{N}$, $3^{4n} - 1 = 80m$.

$$
\begin{aligned}
\text{Then, } 3^{4(n+1)} - 1 &= 3^{4n+4} - 1 \\
&= 81 \cdot 3^{4n} - 1 \\
&= 81(80m+1) - 1 \quad \left(\text{by inductive hypothesis}\right) \\
&= 80(81m) + 80 \\
&= 80(81m+1)
\end{aligned}
$$

which is divisible by 80.
Hence $P(n+1)$ is true whenever $P(n)$ is true.
By the principle of mathematical induction, 80 divides $3^{4n} - 1$ for $n \in \mathbb{N}$.

4) Let P be $A \subseteq B$, Q be $A \cap B = A$ and R be $A \cup B = B$. Prove $P \Leftrightarrow Q \Leftrightarrow R$.

* <u>$P \Leftrightarrow Q$</u>

Assume $A \subseteq B$. Symbolically, $\forall x \, (x \in A \Rightarrow x \in B)$
Let $x \in A \cap B$. By definition of intersection, $x \in A$ and $x \in B$. If $x \in A$, then $A \cap B \subseteq A$ (since $A \subseteq B$). Similarly, if $x \in B$, $A \cap B \subseteq B$.
Now let $x \in A$. Then $x \in A$ and $x \in B$. That is, $x \in A \cap B$. Hence $A \subseteq A \cap B$.
Since $A \cap B \subseteq A$ and $A \subseteq A \cap B$, then $A \cap B = A$.
Next assume $A \cap B = A$. Since $A \cap B \subseteq B$, if $A \cap B = A$, then $A \subseteq B$.
Thus, $P \Leftrightarrow Q$.

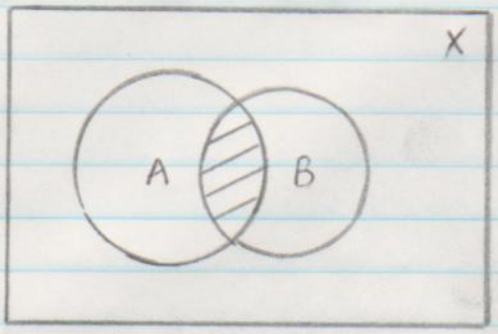
* <u>$P \Leftrightarrow R$</u>

Assume $A \subseteq B$. Let $x \in A \cup B$. By definition of union, $x \in A$ or $x \in B$. But if $x \in A$, then $x \in B$ (since $A \subseteq B$). So in either

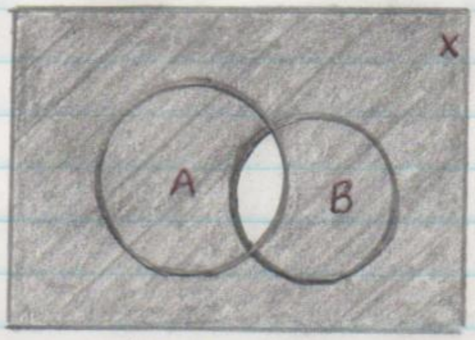
Mark Villar

case, $x \in B$ and $A \cup B \subseteq B$.
Now let $x \in B$. Then $x \in A$ or $x \in B$. That is, $x \in A \cup B$. Hence $B \subseteq A \cup B$.
Since $A \cup B \subseteq B$ and $B \subseteq A \cup B$, then $A \cup B = B$.
Next assume $A \cup B = B$. Since $A \subseteq A \cup B$, if $A \cup B = B$, then $A \subseteq B$.
Thus, $P \Leftrightarrow R$.
* Since $P \Leftrightarrow Q$ and $P \Leftrightarrow R$, then it follows that $Q \Leftrightarrow R$ and therefore $P \Leftrightarrow Q \Leftrightarrow R$

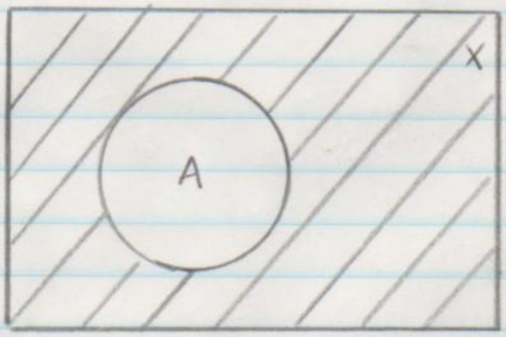
5) Let A and B be subsets on X. We illustrate successively that $(A \cap B)' = A' \cup B'$



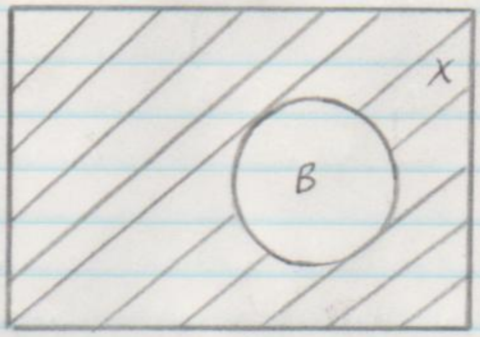
$A \cap B$



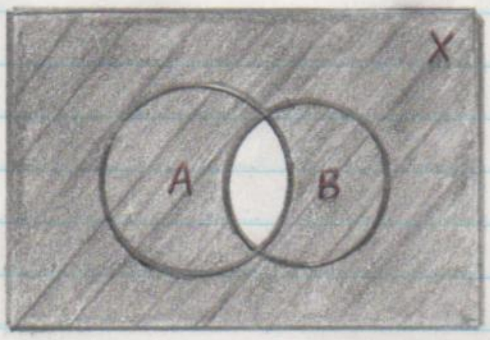
$(A \cap B)'$



A'



B'



$A' \cup B'$

Since the second and last diagrams agree, the sets they depict also agree.