Sample Solutions for Tutorial 3

Question 1.

(a) This f is not a function, because while every subscriber is assigned a telephone number, this does not need to be unique, since some subscribers have more than one telephone account.

(b) This f is a function, because the operation of multiplication assigns to each ordered pair of real numbers, (u, v), a uniquely determined real number, their product uv, and f is obtained by restricting to the case $u = x$.

(c) This f fails to be a function for two reasons.

(i) Since the square of no real number is negative, f does not assign anything to -1.
(ii) Since $(-y)^2 = y^2$, f assigns both -1 and 1 to 1.

(d) For each $x \in X = \mathbb{R}_0^+ := \{r \in \mathbb{R} \mid r \geq 0\}$, the equation $y^2 = x$ has the solutions $y = \pm\sqrt{x}$[1] When $x = 0$, the solutions are ± 0,which coincide. Otherwise precisely one is positive (this is conventional denoted $\sqrt{x}$). In other words, if $x \in \mathbb{R}_0^+$ then the equation $x = y^2$ has precisely one solution $y \in \mathbb{R}_0^+$, which shows that f is a function.

(b) It is immediate that f assigns to each real number x, a real number. The only obstruction to f's being a function is that it might assign two or more different real numbers to some real number x. It is immediate from the definition of f that this could only occur if $x \leq 0$ and $x \geq 0$, in which case both x and $-x$ are assigned to x. But $x \leq 0$ and $x \geq 0$ if and only if $x = 0$ and $0 = -0$, so that only one real number is assigned to 0. Hence the only obstruction vanishes, and f is indeed a function.

Question 2. Since each of the functions have the same set as domain and co-domain, any pair of them can be composed, so their is a total of nine compositions.

(i) $f \circ f \colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(f(x)) = f(x^2) = x^4$
(ii) $g \circ f \colon \mathbb{R} \to \mathbb{R},\ x \mapsto g(f(x)) = g(x^2) = 2x^2$
(iii) $h \circ f \colon \mathbb{R} \to \mathbb{R},\ x \mapsto h(f(x)) = h(x^2) = x^2 + 2$
(iv) $f \circ g \colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(g(x)) = f(2x) = (2x)^2 = 4x^2$
(v) $g \circ g \colon \mathbb{R} \to \mathbb{R},\ x \mapsto g(g(x)) = g(2x) = 2(2x) = 4x$
(vi) $h \circ g \colon \mathbb{R} \to \mathbb{R},\ x \mapsto h(g(x)) = h(2x) = 2x + 2$
(vi) $f \circ h \colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(h(x)) = f(x+2) = (x+2)^2 = x^2 + 4x + 4$
(vii) $g \circ h \colon \mathbb{R} \to \mathbb{R},\ x \mapsto g(h(x)) = g(x+2) = 2(x+2) = 2x + 4$
(ix) $h \circ g \colon \mathbb{R} \to \mathbb{R},\ x \mapsto h(h(x)) = h(x+2) = (x+2) + 2 = x + 4$

Question 3.

In order for f to be a function, the formula must produce a single real number as output for each input. This occurs if and only if we can extract the square root of $x^2 - 3x + 2$. In order to be able to extract a square root, the number in question cannot be negative. Thus we must have $x^2 - 3x + 2 \geq 0$. Since $x^2 - 3x + 2 = (x-1)(x-2)$, we must have either $x - 1, x - 2 \leq 0$ or $x - 1, x - 2 \geq 0$. The former is equivalent to $x \leq 1$ and the latter to $x \geq 2$.

[1] A proof is provided later in this course.

Hence, f is a function if and only if

$$X \subseteq]-\infty, 1] \cup [2, \infty[= \{x \in \mathbb{R} \mid x \leq 1\} \cup \{x \in \mathbb{R} \mid x \geq 2\} = \{x \in \mathbb{R} \mid x \leq 1 \text{ or } x \geq 2\}$$

Since $\tan\theta := \frac{\sin\theta}{\cos\theta}$, g is a function whenever this expression determines a unique real number. This is the case whenever $\cos\theta \neq 0$, that is, θ is not an odd multiple of $\frac{\pi}{2}$. Putting $\theta := 3x + 2$, we see that we must have $x \neq \frac{k\pi - 4}{6}$, with $k \in \mathbb{Z}$.

Hence, g is a function if and only if

$$X \subseteq \mathbb{R} \setminus \{\frac{k\pi - 4}{6} \mid k \in \mathbb{Z}\} == \{x \in \mathbb{R} \mid x \neq \frac{k\pi - 4}{6} \ (k \in \mathbb{Z})\}$$

Whether is injective depends on X.

If $X \subseteq [2, \infty[$, then f is injective. To see this, note that $\sqrt{x^2 - 3x + 2} = \sqrt{u^2 - 3u + 2}$ only if $(x^2 - 3x + 2) - (u^2 - 3u + 2) = (x - u)(x + u - 3) = 0$. Since $x, u \geq 2$, $x + u - 3 \neq 0$, so $x = u$.

If $X =]-\infty, 1] \cup [2, \infty[$, then f is not injective because $f(0) = \sqrt{2} = f(3)$

Similarly, if $X \subseteq]\frac{1\pi - 4}{6}, \frac{\pi - 2}{3}$, then g is injective

But if when, for example $X = X \subseteq \mathbb{R} \setminus \{\frac{k\pi - 4}{6} \mid k \in \mathbb{Z}\}$ g is not injective as $g(\frac{13\pi}{6}) = g(\frac{\pi}{6})$.

Question 4. Take functions $f: X \to Y$ and $g: Y \to Z$ and consider $g \circ f: X \to Z$.

(a) Suppose that f and g are both surjective. Take $z \in Z$.
Since g is surjective, there is a $y \in Y$ with $g(y) = z$.
Since f is surjective, there is an $x \in X$ with $f(x) = y$.
Then $(g \circ f)(x) := g(f(x)) = g(y) = z$, showing that $g \circ f$ is surjective.

(b) Suppose that $g \circ ff$ is surjective. Take $z \in Z$.
Since $g \circ f$ is surjective, there is an $x \in X$ with $(g \circ f)(x) = z$.
Put $y := f(x)$. Then $g(y) = g(f(x)) =: (g \circ f)(x) = z$, showing that g is surjective.

(c) Take $X = Z = \{0\} Y = \mathbb{R}$, $f: X \to Y,\ x \mapsto x$ and $g: Y \to Z,\ y \mapsto 0$.

Then $g \circ f: X \to Z$ is the identity function on $\{0\}$, and hence surjective. However, $\mathrm{im}(f) = \{0\} \neq \mathbb{R}$, so that f is not surjective.