AMTH250
Assignment 8

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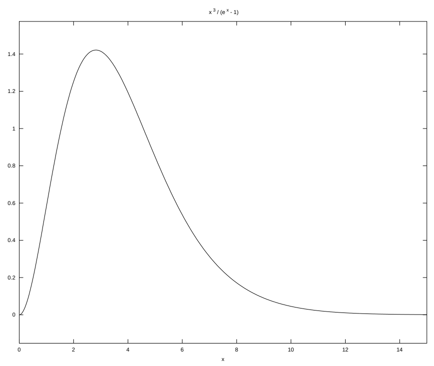
Question 1

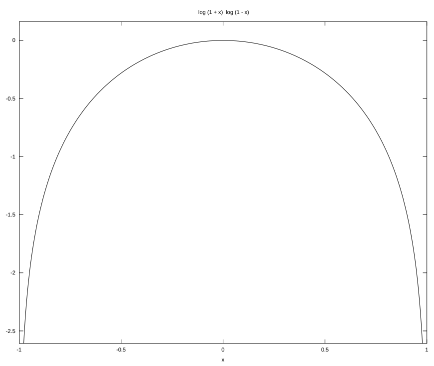
(a)

$$I_1 = \int_0^\infty \frac{x^3}{e^x - 1} \, dx = 6.4939$$



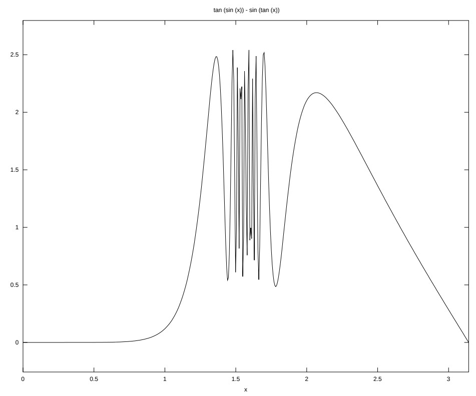
(b)

$$I_2 = \int_{-1}^1 \ln(1+x)\ln(1-x) \, dx = -1.1016$$



(c)

$$I_3 = \int_0^{\pi} \tan(\sin x) - \sin(\tan x)\ dx = 2.6643$$

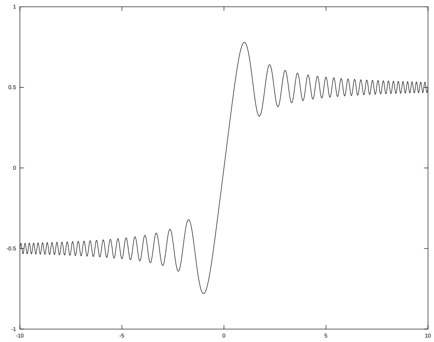


These results are reliable since I_1, I_2 and I_3 were successfully evaluated with an absolute error tolerance of $10^{-13}, 10^{-13}$ and 10^{-12} respectively.

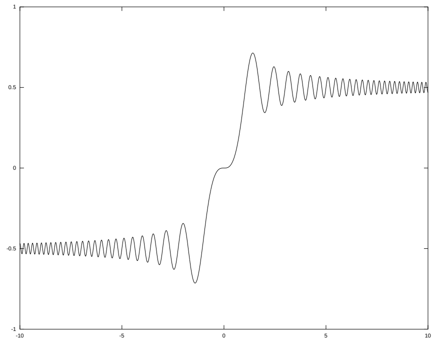
```
f1=@(x) x.^3./(e.^x-1);
[intf1,ierr,nf,eerr]=quad(f1,0,Inf,[1e-13 0])
intf1 =  6.4939
ierr1 = 0
nf1 =  345
eerr1 = 7.8500e-014

f2=@(x) log(1+x).*log(1-x);
[intf2,ierr2,nf2,eerr2]=quad(f2,-1,1,[1e-13 0])
intf2 = -1.1016
ierr2 = 0
nf2 =  735
eerr2 = 6.2172e-015

f3=@(x) tan(sin(x))-sin(tan(x));
[intf3,ierr3,nf3,eerr3]=quad(f3,0,pi,[1e-12 0])
intf3 =  2.6643
ierr3 = 0
nf3 =  735
eerr3 = 5.9641e-013
```

Question 2

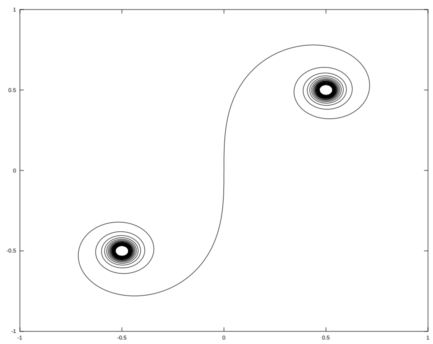
$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)\ dt$$



$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\ dt$$



Cornu or Euler spiral



Question 3

(a) We first identify the range of random numbers for each dimension.

$$x \in [0,1],\ y \in [0,2],\ z \in [0,4]$$

Then we enclose the ellipsoid in a 4×4 cube. We select $n = 100000$ points uniformly distributed within the cube and use the distance formula

$$d = \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2}$$

to check whether the point is contained within the ellipsoid. If $d \leq 1$, then the given point is inside the ellipsoid. We then estimate the volume of the ellipsoid by the formula

$$V_e \approx V_c \times \frac{k}{n} = \frac{64k}{n}$$

where V_c is the volume of the cube, k is the number of points in the ellipsoid and n is the number of points in the cube. Thus, by Monte Carlo approximation

$$V_e \approx 33.567$$

This is consistent with the exact result

$$V_e = \frac{4}{3}\pi abc = \frac{32}{3}\pi = 33.510$$

where $a = 1, b = 2, c = 4$.

(b) Since Ω is contained within the square $[0,1] \times [0,1]$, we generate x and y as uniform $[0,1]$ random numbers. We then generate n random numbers and use those lying in Ω to form the sum of the function. Equating the volume of Ω and the average value of f in Ω, our Monte Carlo integration is given by

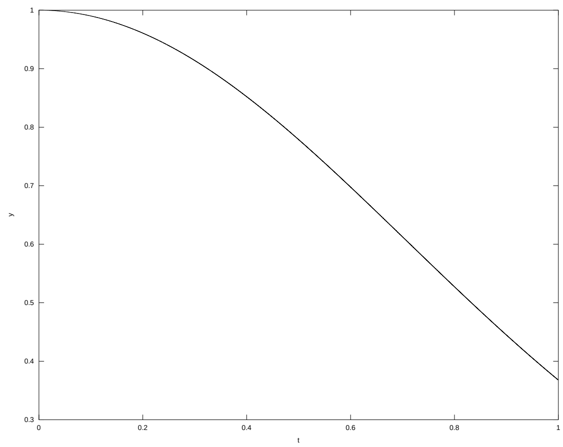
$$I = V_\Omega \times \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\sqrt{x_i^2+y_j^2}} \approx 0.83028$$

where $V_\Omega = \dfrac{\pi}{2}$. This is consistent with the exact result obtained from Wolfram.

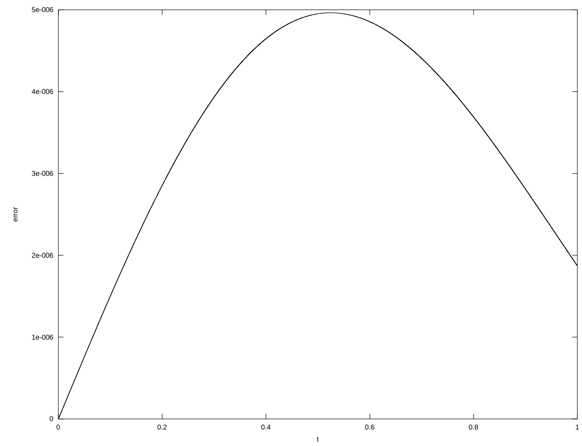
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} e^{-\sqrt{x^2+y^2}}\ dx\ dy = 0.830138$$

Question 4

(a) With initial conditions $t_0 = 0$ and $y(0) = 1$, we apply Euler's method under various step sizes, $h = \dfrac{1}{2^k}$ for $1 \le k \le 16$, to solve our initial value problem. The numerical solution is displayed in the graph below.



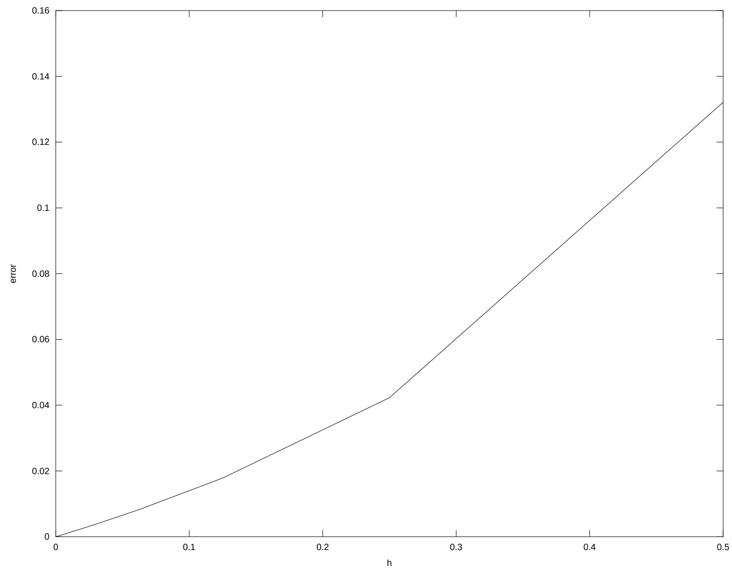
We also show the error curve below. We can see that the errors are insignificant enough, of the order of 10^{-6}, for the computed and exact solutions to be barely distinguishable on a graph. Note that the shape of the curve here is trivial.



At $t = 1$, the exact value of $y(t)$ is $e^{-1} = 0.3678794411171442$. We compare this with the computed values from the various step sizes in the table below.

k	$y(1)$	Error
1	0.50000	0.13212
2	0.41016	0.04228
3	0.38571	0.01784
4	0.37613	0.00825
5	0.37185	0.00398
6	0.36983	0.00195
7	0.36885	0.00097
8	0.36836	0.00048
9	0.36812	0.00024
10	0.36800	0.00012
11	0.36794	0.00006
12	0.36791	0.00003
13	0.36789	0.00001
14	0.36789	0.00001
15	0.36788	0.00000
16	0.36788	0.00000

In fact, at $k = 16$, the error is only 1.871×10^{-6}, which is consistent with our earlier findings.



The error curve as a function of h shows that the accuracy of our results improves as we decrease the step size. In other words, the bigger the step size, the larger the error.

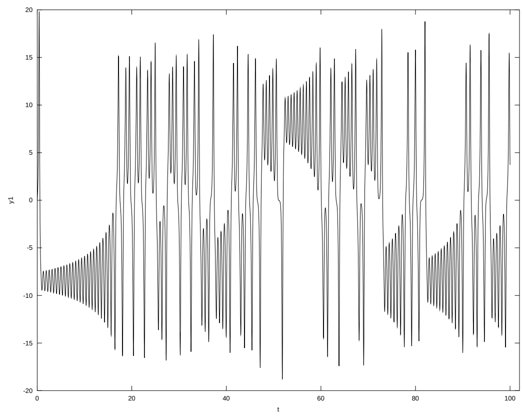
(b) This problem is unstable for $h = 1$. We also know by experiment that Euler's method gives 'accidental' stable solutions for some larger step sizes, with y decaying to 0 after just a few steps. This is true for $h = 1/2$ and $1/4$, converging after only 4 and 10 steps respectively. For this reason we ignore results for $k < 2$. Nonetheless, the various step sizes $h = 2^{-k}$ for integers $k \geq 3$ all give stable solutions. Thus in general, $h = x^{-k}$ where x is an *exponent of* 2, provide stable results. In other words,

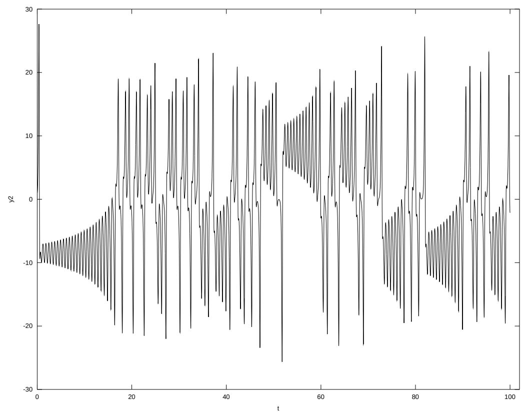
$$h = \frac{1}{2^k}, \frac{1}{4^k}, \frac{1}{8^k}, \frac{1}{16^k}, \ldots$$

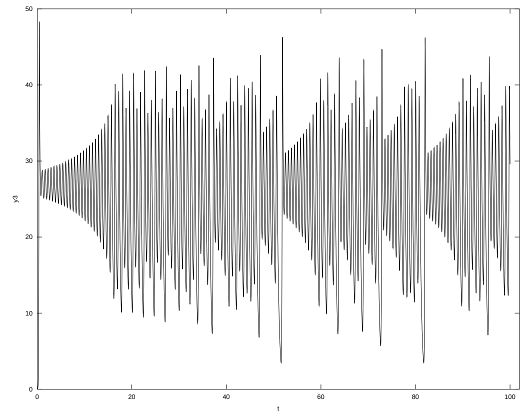
are all valid ranges of step-sizes for which $y \to 0$ as $t \to \infty$. By trial-and-error we found that for all other h, the solution diverges and thus gave unstable result.

Question 5

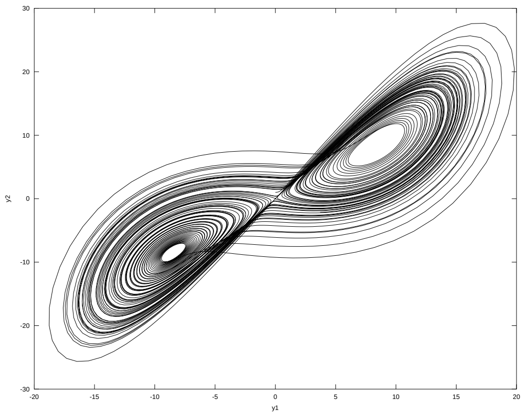
(a) **lsode** solution to the Lorenz system of differential equations

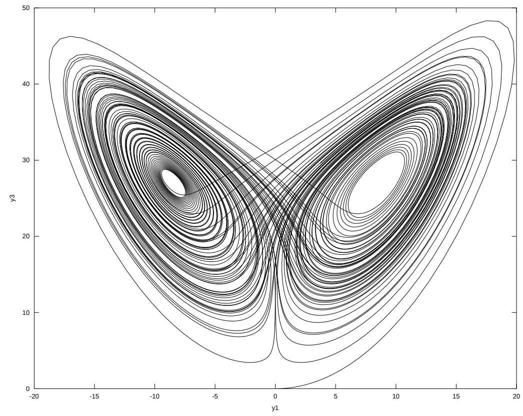


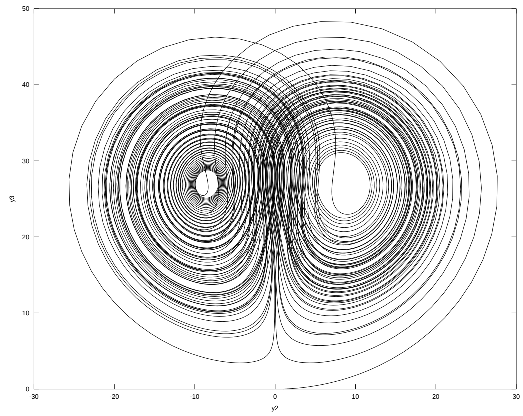




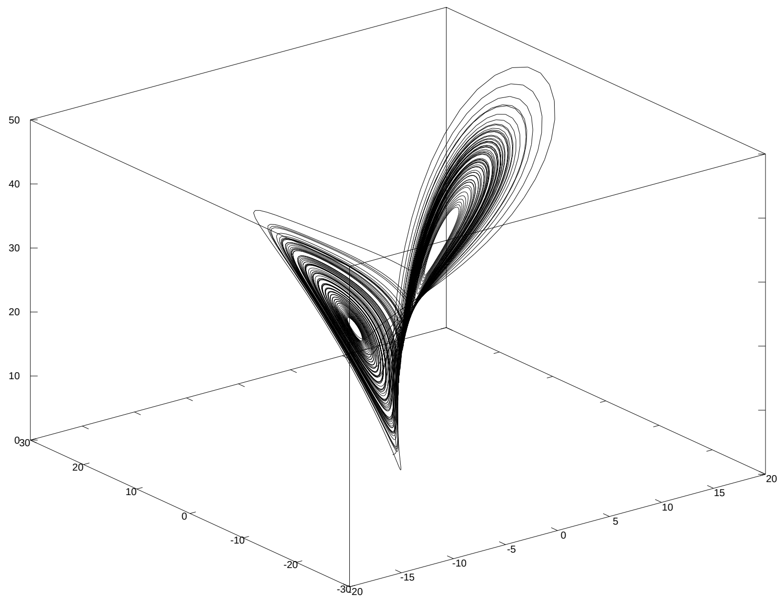
(b) Phase-plane plots







(c) 3D plot



We then experimented by changing the initial conditions by a tiny amount. Using a random number generator, we changed our initial values from $[y_1(0)\ y_2(0)\ y_3(0)] = [0\ 1\ 0]$ to

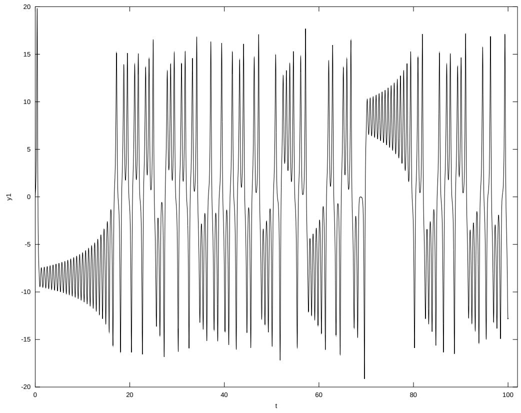
$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} -2.7813\mathrm{e}^{-11} \\ 1 + 2.6788\mathrm{e}^{-12} \\ -3.2696\mathrm{e}^{-11} \end{bmatrix}$$

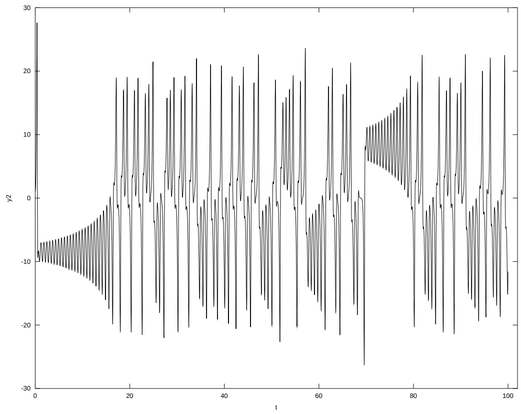
The following table compares the final values under both scenarios.

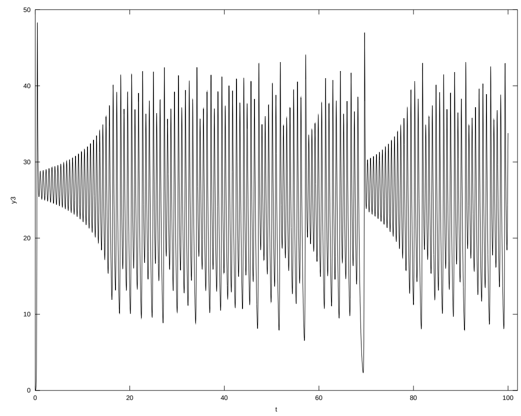
Final value	Original	Changed
$y_1(100)$	3.7149	-12.570
$y_2(100)$	-2.1299	-11.581
$y_3(100)$	29.570	33.796

It follows that numerical solutions to differential equations are very sensitive to even the tiniest of changes to initial conditions. We repeat parts (a)-(c) to check for any graphical differences.

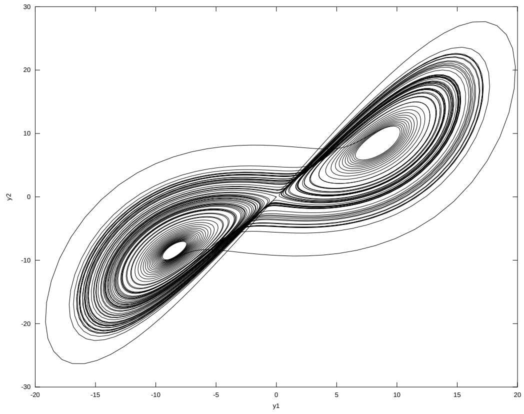
(i) **lsode** solution under changed initial values

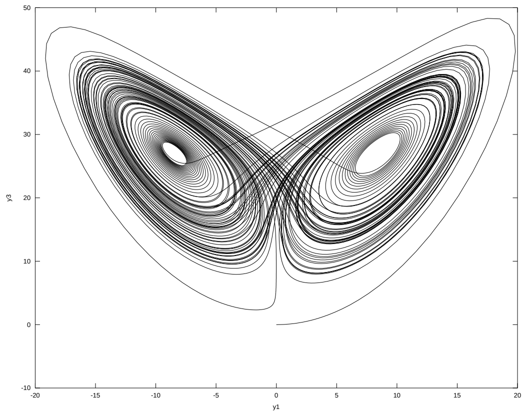


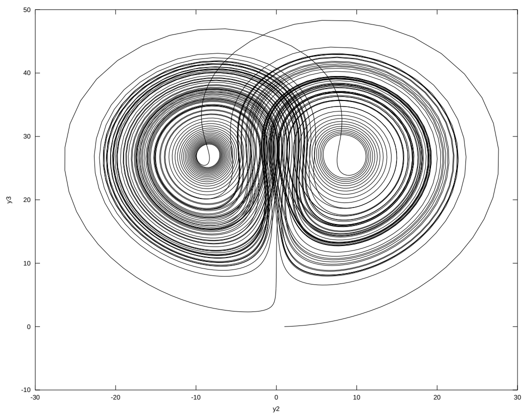




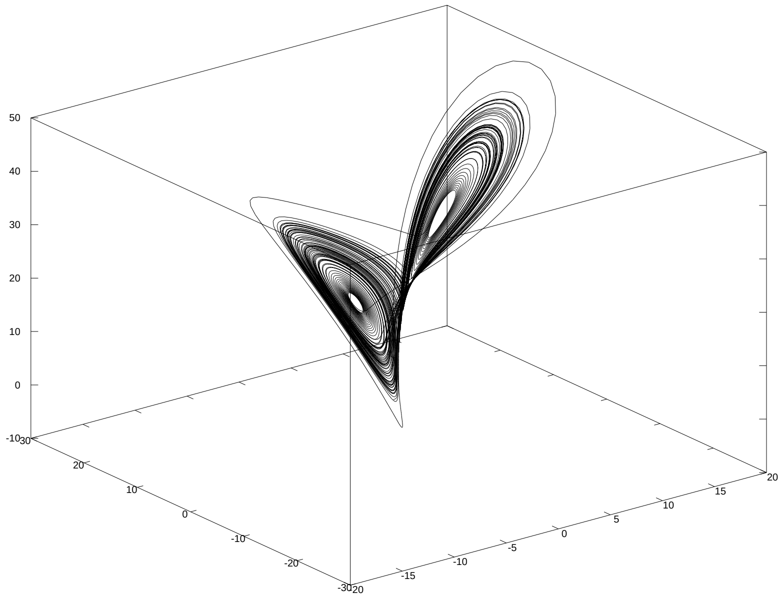
(ii) Phase-plane plots under changed initial values







(iii) 3D plot under changed initial values



We observe no significant change to the overall shapes and patterns of the phase-plane and 3D plots after changing initial values, except perhaps for minor displacements. However, there are some notable differences in the $\mathbf{y}$ over t plots, particularly for y_1 and y_2 after $t = 50$.

Appendix

1. (a)
   ```
   ezplot(f1, [0,15])
   print('f1.eps','-deps')
   ```
   (b)
   ```
   ezplot(f2, [-1,1])
   print('f2.eps','-deps')
   ```
   (c)
   ```
   ezplot(f3, [0,pi])
   print('f3.eps','-deps')
   ```

2.
   ```
   x=-10:0.01:10;
   c=@(x) cos(pi*x.^2/2);
   s=@(x) sin(pi*x.^2/2);

   for i=1:length(x)
   C(i)=quad(c,0,x(i));
   S(i)=quad(s,0,x(i));
   end

   plot(x,C)
   print('fresc.eps','-deps')
   plot(x,S)
   print('fress.eps','-deps')
   plot(S,C)
   print('cornu.eps','-deps')
   ```

3. (a)
   ```
   function v = monte4d(n)
   k=0;
   for i=1:n
   x=rand(1,1);
   y=2*rand(1,1);
   z=4*rand(1,1);
   if (x^2+y.^2/4+z.^2/16<=1)
   k=k+1;
   end
   end
   v=64*k/n;
   endfunction

   monte4d(100000)
   ```

(b)
```
function ii=monte2d(n)
k=0;
sumf=0;
while (k<n)
x=rand(1,1);
y=rand(1,1);
if (x.^2+y.^2<=1)
k=k+1;
sumf=sumf+exp(-sqrt(x.^2+y.^2));
end
end
ii=(pi/2)*(sumf/n);
endfunction

monte2d(100000)
```

4.
```
f=@(t,y) -2*t*y;
N=16;
h=zeros(1,N);
y1=zeros(1,N);
for k=1:N
hk=2^-k;
h(k)=hk;
[t y]=euler(f,0,1,hk,2^k);
y1(k)=y(end);
end

plot(t,y)
xlabel('t')
ylabel('y')
print('euler.eps','-deps')

err=y-exp(-t.^2);
plot(t,err)
xlabel('t')
ylabel('error')
print('error.eps','-deps')

err1=exp(-1)-y1
plot(h,err1)
xlabel('t')
ylabel('error')
print('err1.eps','-deps')
```

5.
```
global sigma b r
sigma=10;
b=8/3;
r=28;

function dy = lv(y,t)
global sigma b r
dy=zeros(3,1);
dy(1)=sigma*(y(2)-y(1));
dy(2)=r*y(1)-y(2)-y(1)*y(3);
dy(3)=y(1)*y(2)-b*y(3);
endfunction

t=(0:0.01:100)';
y=lsode(@lv,[0 1 0],t);
```

(a)
```
plot(t,y(:,1))
axis([0 102 -20 20])
xlabel('t')
ylabel('y1')
print('y1.eps','-deps')

plot(t,y(:,2))
axis([0 102 -30 30])
xlabel('t')
ylabel('y2')
print('y2.eps','-deps')

plot(t,y(:,3))
axis([0 102 0 50])
xlabel('t')
ylabel('y3')
print('y3.eps','-deps')
```

(b)
```
plot(y(:,1),y(:,2))
xlabel('y1')
ylabel('y2')
print('pp1q.eps','-deps')

plot(y(:,1),y(:,3))
xlabel('y1')
ylabel('y3')
print('pp2.eps','-deps')
```

```
plot(y(:,2),y(:,3))
xlabel('y2')
ylabel('y3')
print('pp3.eps','-deps')
```

(c)
```
plot3(y(:,1),y(:,2),y(:,3))
print('threed.eps','-deps')

c=1e-10*randn(1,1);
d=1e-10*randn(1,1);
g=1e-10*randn(1,1);

t=(0:0.01:100)';
y1=lsode(@lv,[c 1+d g],t);
```

i.
```
plot(t,y1(:,1))
axis([0 102 -20 20])
xlabel('t')
ylabel('y1')
print('y1s.eps','-deps')

plot(t,y1(:,2))
axis([0 102 -30 30])
xlabel('t')
ylabel('y2')
print('y2s.eps','-deps')

plot(t,y1(:,3))
axis([0 102 0 50])
xlabel('t')
ylabel('y3')
print('y3s.eps','-deps')
```

ii.
```
plot(y1(:,1),y1(:,2))
xlabel('y1')
ylabel('y2')
print('pp1s.eps','-deps')

plot(y1(:,1),y1(:,3))
xlabel('y1')
ylabel('y3')
print('pp2s.eps','-deps')
```

```
plot(y1(:,2),y1(:,3))
xlabel('y2')
ylabel('y3')
print('pp3s.eps','-deps')
```

iii.
```
plot3(y1(:,1),y1(:,2),y1(:,3))
print('threeds.eps','-deps')
```