PMTH212 ASSIGNMENT 6

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(1) $f(x, y, z) = (x - 3y + 4z)^{1/2}$ at $P(0, -3, 0)$

$$\bigtriangledown f(x, y, z) = \left\langle \frac{1}{2\sqrt{x - 3y + 4z}},\ -\frac{3}{2\sqrt{x - 3y + 4z}},\ \frac{2}{\sqrt{x - 3y + 4z}} \right\rangle$$
$$= \left\langle \frac{1}{6},\ -\frac{1}{2},\ \frac{2}{3} \right\rangle \text{ at } P$$

f increases most rapidly in the direction $\left\langle \frac{1}{6},\ -\frac{1}{2},\ \frac{2}{3} \right\rangle$

$$||f(0, -3, 0)|| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{13}{18}} = \frac{1}{3}\sqrt{\frac{13}{2}}$$

$$\mathbf{u} = \frac{\bigtriangledown f(0, -3, 0)}{||f(0, -3, 0)||} = \left\langle \frac{1}{2}\sqrt{\frac{2}{13}},\ -\frac{3}{2}\sqrt{\frac{2}{13}},\ 2\sqrt{\frac{2}{13}} \right\rangle$$

The rate of increase of f in that direction is $||f(0, -3, 0)|| = \frac{1}{3}\sqrt{\frac{13}{2}}$

(2) $z = f(x^2 + y^2)$. Let $z = f(u)$, $u = x^2 + y^2$ such that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}(2x)$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}(2y)$$
$$\Rightarrow\ y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 2xy\frac{\partial z}{\partial u} - 2xy\frac{\partial z}{\partial u} = 0$$

(3) Denote $W(x, y, z) = F(x, y, z(x, y)) = 0$ such that

$$\frac{\partial W}{\partial x} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \text{ (chain rule)}$$
$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \ \Rightarrow\ \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$
$$\frac{\partial W}{\partial y} = \frac{\partial F}{\partial y}\frac{dy}{dy} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \text{ (chain rule)}$$
$$= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \ \Rightarrow\ \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

(4) $F(u,v)=\int_u^v f(t)\ dt$

$$\int_u^v f(t)\ dt = \Big[F(t)\Big]_u^v = F(v) - F(u) = F(u,v)$$

$$\frac{\partial F}{\partial u} = F_u(u,v) = -F'(u) = -f(u)$$

$$\frac{\partial F}{\partial v} = F_v(u,v) = F'(v) = f(v)$$