UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101

005776

PAPER TITLE: Algebra & Differential Calculus

PAPER NUMBER: First and Only

DATE: **Monday 16 June 2008** **TIME:** **9:30 AM TO 12:30 PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: FIVE (5)

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE:

1	6 LEAF A4 BOOKS	0	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	0	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: **NIL**

POCKET CALCULATORS PERMITTED: **YES** **(SILENT TYPE)**

TEXTBOOKS OR NOTES PERMITTED: FIVE (5) A4 SHEETS (10 PAGES IF WRITTEN ON BOTH SIDES) OF HANDWRITTEN NOTES. NO PHOTOCOPIES, NO PRINTED NOTES PERMITTED. THE NOTES MUST BE HANDED IN WITH THE EXAMINATION SCRIPT.

INSTRUCTIONS FOR CANDIDATES:

- Candidates **MAY** make notes on this examination question paper during the fifteen minutes reading time
- Questions are of equal value
- SECTION A: - Answer ALL questions
- SECTION B: - answer only TWO (2) of the FOUR (4) questions provided
- Candidates may retain this examination question paper

SECTION A

You should attempt all questions in this section.

Question 1. *[10 marks]*

(a) A function f with domain $X = \{0, 1, 2, 3\}$ and codomain $Y = \{1, 2, 3, 4\}$ is given by the following table:

x	0	1	2	3
f(x)	1	1	2	2

Is f injective? Is f surjective? Justify your answer.

(b) Let a be a positive real number. Prove that for any positive integer n

$$(1 + a)^n \geq 1 + na.$$

(c) Determine supremum and infimum (if they exist) of the following sets

$$(i) \quad \{0.5, \pi, \sqrt{2}, 2.14\} \qquad (ii) \quad \{x \in \mathbb{R} : 3 - x^2 \geq 0\}.$$

Question 2. *[10 marks]*

For each of the following functions,

(a) state if the function is monotone.

(b) Decide if it is injective and/or surjective on the given domain.

(c) Find the supremum and infimum (if they exist); in each case state whether or not the function attains its bounds.

Justify your answers.

$$\text{(i)} \quad f(x) = x^2 : (0, 1) \to \mathbb{R} \qquad \text{(ii)} \quad f(x) = \frac{1}{1 + x^4} : \mathbb{R} \to (0, 1]$$

Question 3 is on page 3

Question 3. *[10 marks]*

(a) Determine which of the following sequences of real numbers $(u_n)_{n\in\mathbb{N}}$ is monotone and discuss the behaviour of u_n as $n \to \infty$.

(i) $u_n = (-1)^n \dfrac{n^2+3}{n^3+1}$ (ii) $u_n = \dfrac{1}{3^n}$.

(b) Determine which of the following series converge and which diverge, justifying your answer.

$$(i)\quad \sum_{n=1}^{\infty} \frac{n^2+1}{n} \qquad (ii)\quad \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Question 4. *[10 marks]*

(a) Find all points at which each of the following functions is well-defined and differentiable. Compute its derivative, where it exists. Where the function fails to be differentiable, explain why.

(i) $f(x) = \dfrac{1}{\cos^2 x}$ (ii) $g(x) = |x|$.

(b) Differentiate the functions

(i) $f(x) = \dfrac{x+1}{x^2-1}$, (ii) $g(x) = x^2 \cos x$, (iii) $h(x) = \cos(x^2)$.

Question 5. *[10 marks]*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^3 - 3x^2 - 12x + 6$.

(a) Determine the intervals on which f is (i) increasing or decreasing (ii) concave up or concave down.

(b) Find all the relative maxima and relative minima of f, the absolute maximum and the absolute minimum on $[-3, 5]$.

(c) Sketch the graph of f on the interval $[-3, 5]$ (Choose an appropriate scale).

Question 6 is on page 4

Question 6 *[10 marks]*

(a) Solve the system of linear equations below by transformation to the reduced row-echelon form.

$$\begin{aligned} x &+ 2y + 3z = 6 \\ x &- 2y + 3z = -2 \\ 2x &- y + z = 2. \end{aligned}$$

(b) For

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix}.$$

state whether the following products and (or) sums are defined: calculate those which are defined:

(i) AB (ii) $A - C$ (iii) BC.

(c) Evaluate the determinant

$$\begin{vmatrix} 2 & 13 & 22 & 9 \\ 0 & 1 & 37 & 12 \\ 0 & 0 & 4 & 31 \\ 0 & 0 & 0 & 2 \end{vmatrix}.$$

Question 7 *[10 marks]*

(a) For $z = 2 + 2\,\mathrm{i}$, write each of the following complex numbers in the form $x + \mathrm{i}\,y$

(i) $\bar{z}$, (ii) $|z|$, (iii) z^2, (iv) $\dfrac{1}{z}$.

(b) Find all complex numbers z that satisfy

$$z^5 = -1.$$

Question 8 is on page 5

Question 8. *[10 marks]*

(a) Find the inverse matrix of

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 5 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

Check your answer.

(b) Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}.$$

SECTION B

You should attempt only TWO questions in this section.

Question 9. *[10 marks]*

(a) Let $f : X \to Y$ be a function that is continuous at x_0 and $g : Y \to Z$ be a function that is continuous at $y_0 = f(x_0)$. Prove that $g \circ f$ is continuous at x_0.

(b) Give an example of a function defined on $\mathbb{R}$ that is **not** continuous at $c = 0$. Justify your answer.

Question 10. *[10 marks]*

(a) Prove that a function f that is differentiable at some point c must be continuous at c.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that between any two zeros of f, its derivative f' must also have a zero.

Question 11. *[10 marks]*

A rectangle has been fitted in a semicircle of radius r what are the lenghts of the side if it is known that the rectangle has maximal possible area?

Question 12. *[10 marks]*

Find a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ of degree 3 such that $p(0) = 0$, $p(1) = -1$, $p(-1) = 1$, $p(2) = -8$.