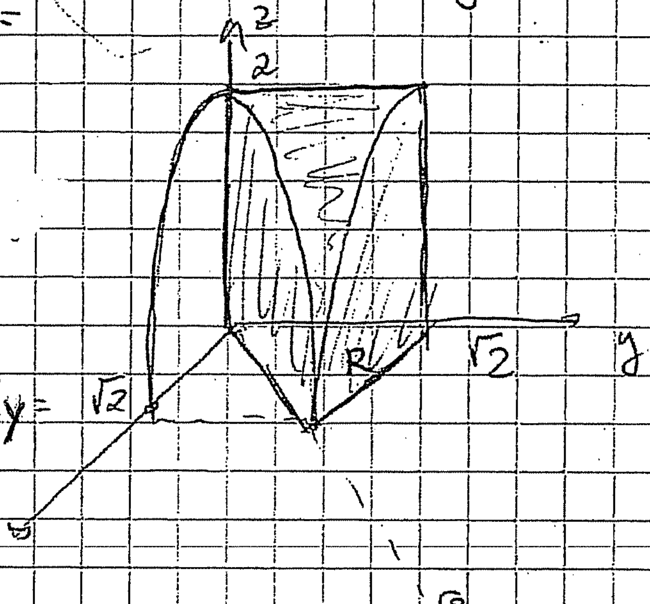
Test 9.

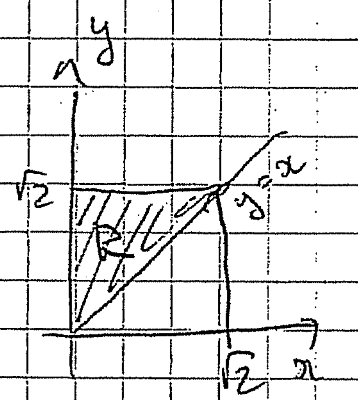
Evaluate the triple integral

$\iiint_G xyz\,dV$ where G is in the first octant and bounded by

$z = 2 - x^2$, $z = 0$, $y = x$, $x = 0$

$$\int_0^{\sqrt{2}}\int_0^{y}\int_0^{2-x^2} xyz\,dz\,dx\,dy =$$





type II region

$$\int_0^{\sqrt{2}}\int_0^{y} xy\frac{z^2}{2}\Big|_0^{2-x^2} dx\,dy =$$

$$\frac{1}{2}\int_0^{\sqrt{2}}\int_0^{y}(4xy - 4x^3y + x^5y)\,dx\,dy = \frac{1}{2}\int_0^{\sqrt{2}}\left(2x^2y - x^4y + \frac{x^6}{6}y\Big|_0^{y}\right)dy =$$

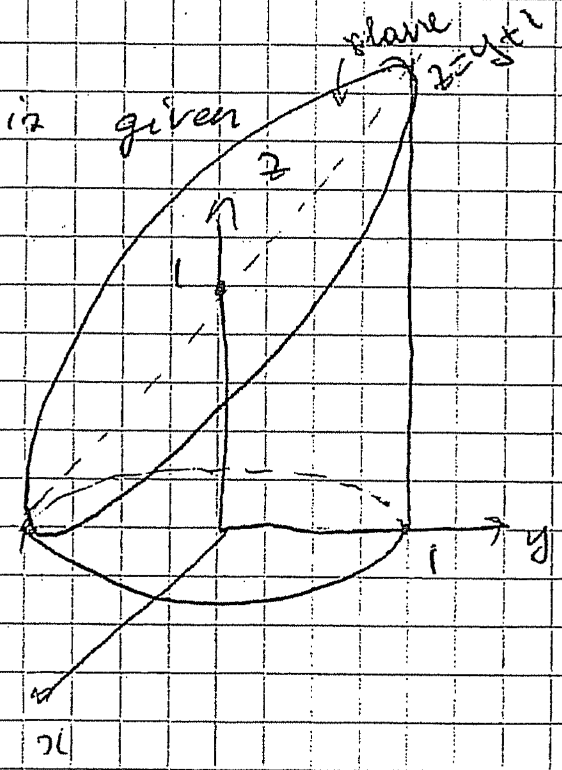
$$\frac{1}{2}\int_0^{\sqrt{2}} 2y^3 - y^5 + \frac{y^7}{6}\,dy = \frac{1}{2}\left(\frac{y^4}{2} - \frac{y^6}{6} + \frac{y^8}{8\cdot 6}\right)\Big|_0^{\sqrt{2}} =$$

$$= \frac{1}{2}\left(\frac{4}{2} - \frac{8}{6} + \frac{16}{8\cdot 6}\right) = \frac{1}{2}$$

Sketch the solid whose volume is given

by the integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{y+1} dz\,dy\,dx$$



$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ - cylinder $x^2+y^2 \le 1$

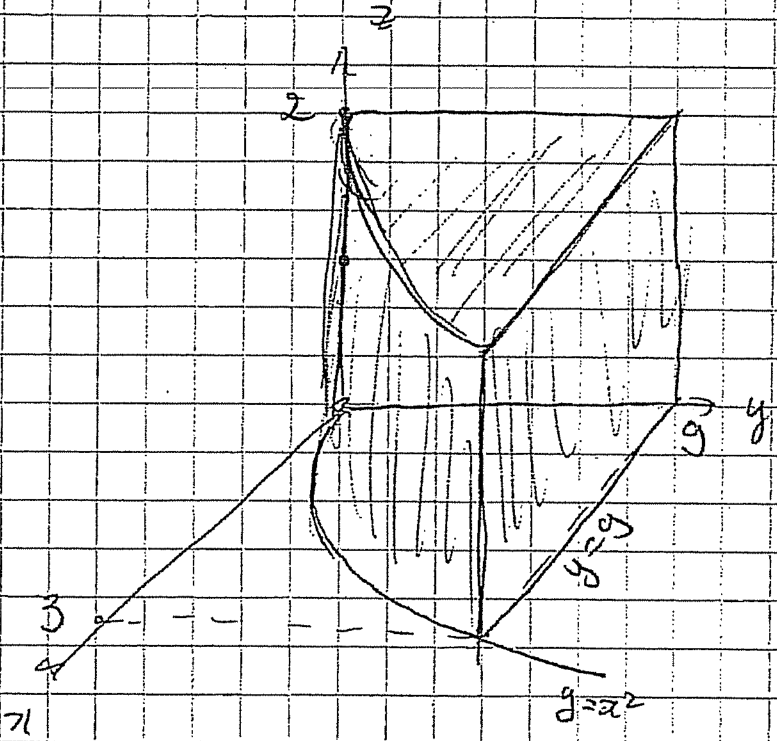
$0 \le z \le y+1$ - between two planes

$$\int_{0}^{3}\int_{x^2}^{9}\int_{0}^{2} dz\,dy\,dx$$

$0 \le x \le 3$

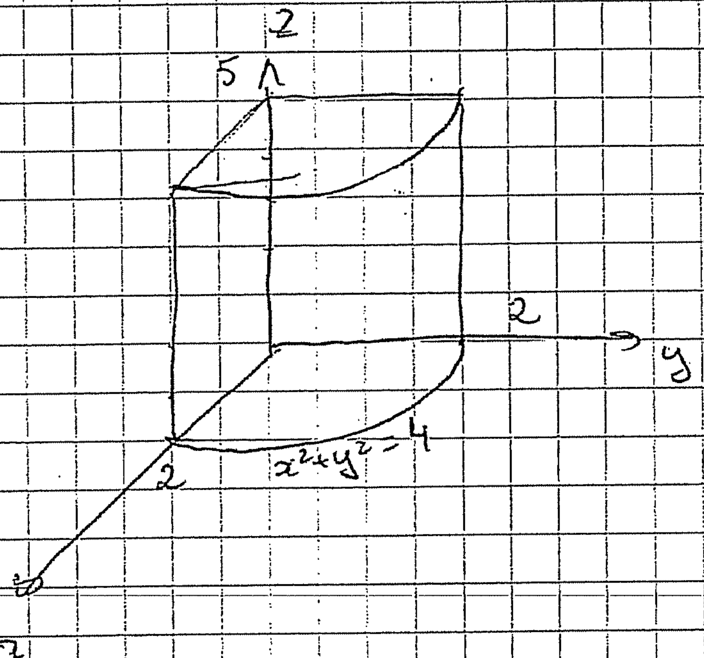
$x^2 \le y \le 9$

$0 \le z \le 2$



...s each integral as an equivalent integral in which the z-integration is performed first, the y-integral second, and the x-int. last.

$$\int_0^5 \int_0^2 \int_0^{\sqrt{4-y^2}} f(x,y,z)\, dx\, dy\, dz$$



$0 \le z \le 5$

$0 \le y \le 2$

$x \le \sqrt{4-y^2} \Rightarrow x^2+y^2=4$

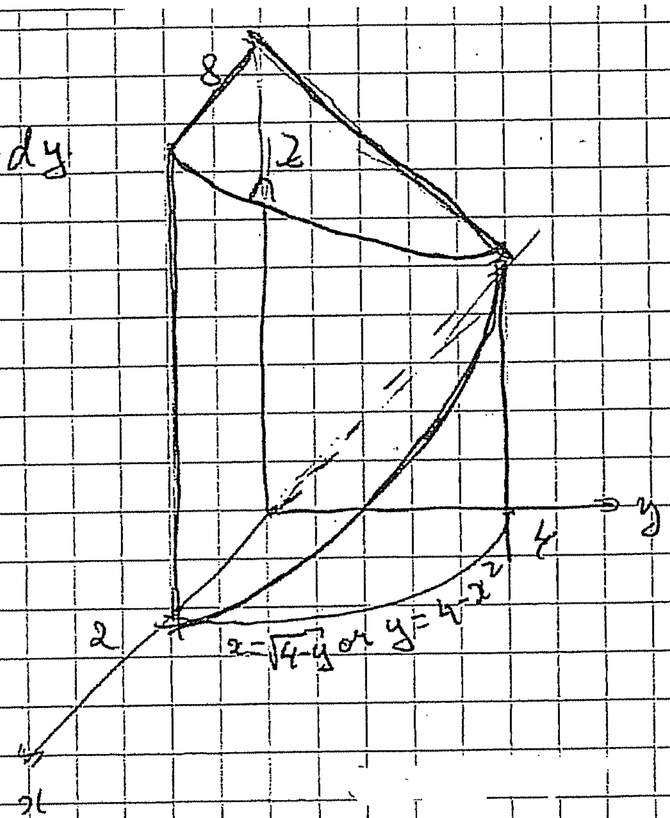
$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^5 f(x,y,z)\, dz\, dy\, dx$$

$0 \le z \le 5$

$0 \le y \le \sqrt{4-x^2}$

$0 \le x \le 2$

$$\int_0^4 \int_y^{8-y} \int_0^{\sqrt{4-y}} f(x,y,z)\, dx\, dz\, dy$$



$0 \le y \le 4$

$y \le z \le 8-y$ - planes

$0 \le x \le \sqrt{4-y}$

$$\int_0^2 \int_0^{4-x^2} \int_y^{8-y} f(x,y,z)\, dz\, dy\, dx$$

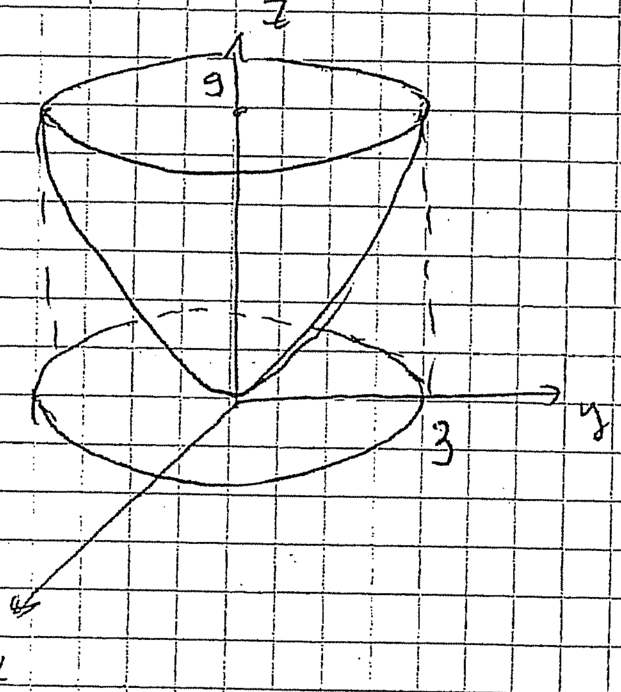
$y \le z \le 8-y$

$0 \le y \le 4-x^2$

$0 \le x \le 2$

~~Use~~

cylindrical coordinates to find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ plane $z = 9$



$$\iiint_G 1 \, dV =$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\iiint_G r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^3\int_{r^2}^9 r \, dz\, dr\, d\theta =$$

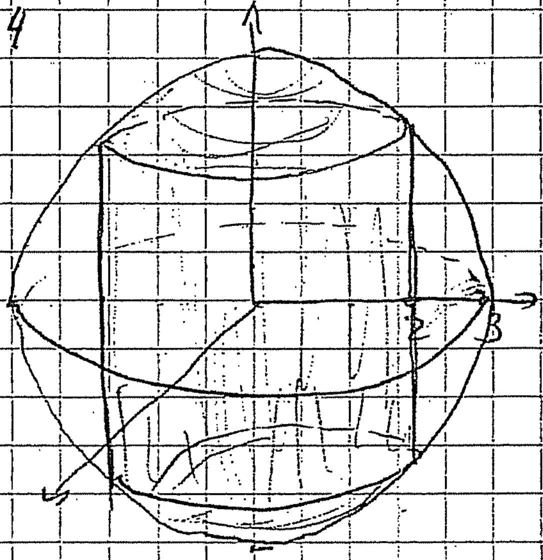
$$\int_0^{2\pi}\int_0^3 rz \Big|_{r^2}^9 dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^3 (9r - r^3)\, dr\, d\theta = \int_0^{2\pi} \frac{9r^2}{2} - \frac{r^4}{4}\Big|_0^3 \, d\varphi =$$

$$= \int_0^{2\pi} \frac{81}{2} - \frac{81}{4} \, d\theta = \frac{81\pi}{2}$$

between the sphere $x^2+y^2+z^2=9$

and the cylinder $x^2+y^2=4$



$$\iiint dx\,dy\,dz =$$

$$2\int_0^{2\pi}\int_0^2\int_0^{\sqrt{9-x^2-y^2}} r\,dz\,dr\,d\theta =$$

$$2\int_0^{2\pi}\int_0^2 rz\Big|_0^{\sqrt{9-\cancel{x^2}-\cancel{y^2}-r^2}} dr\,d\theta =$$

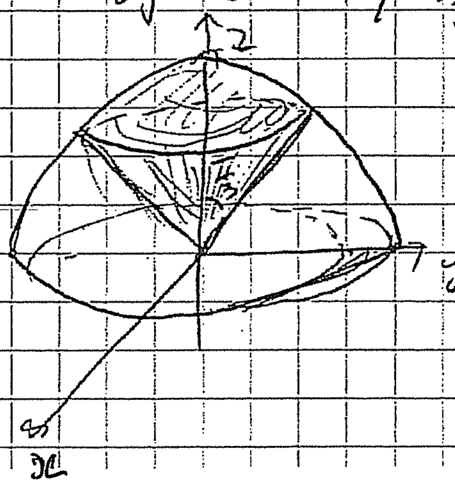
$$2\int_0^{2\pi}\int_0^2 r\sqrt{9-r^2}\,dr\,d\theta = 2\int_0^{2\pi} -\frac{1}{2}\times\frac{2}{3}\left(9-r^2\right)^{3/2}\Big|_0^2\,d\theta =$$

$$= -\frac{2}{3}\int_0^{2\pi}\left(5\sqrt{5}-27\right)d\theta = \frac{4\pi}{3}\left(27-5\sqrt{5}\right)$$

Spherical coordinates :

Find the volume of the solid bounded

above by the sphere $\rho=4$ and below by the cone $\varphi=\frac{\pi}{3}$



$$\iiint dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

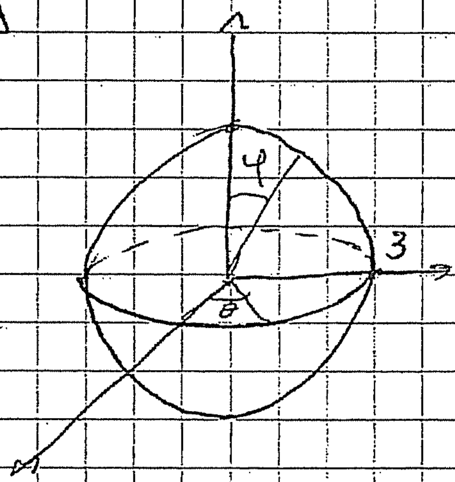
$$\int_0^{2\pi}\int_0^{\pi/3}\frac{64}{3}\sin\varphi\,d\varphi\,d\theta = \frac{64}{3}\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi/3}\,d\theta$$

$$\int_0^{2\pi} -\frac{1}{2}+1\,d\theta = \frac{64}{3}\pi$$

Compute

the integral

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\; dz\,dx\,dy$$



$$\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho^3 \sin\varphi \; d\rho\, d\varphi\, d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{\rho^4}{4}\sin\varphi \Big|_0^3 \; d\varphi\, d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{81}{4}\sin\varphi \; d\varphi\, d\theta =$$

$$\frac{81}{4}\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi}\; d\varphi\, d\theta = \frac{81}{4}\int_0^{2\pi} 2\, d\theta = 81\pi$$