Chapter 5

Sequences and Series

5.1 Sequences

Definition 5.1. A *sequence* in X is a function $f\colon \mathbb{N} \to X$.

It is customary to write $(x_n)_{n\in\mathbb{N}}$ for the sequence $f\colon \mathbb{N} \to X,\ n \mapsto x_n$. In this notation, x_n is the n^{th} *term* of the sequence.

The sequence of real numbers, $(x_n)_{n\in\mathbb{N}}$, is *bounded* if and only if the set $\{x_n | n \in \mathbb{N}\}$ is a bounded set, that is, there is a $K \in \mathbb{R}$ with $|x_n| < K$ for all $n \in \mathbb{N}$.

The sequence of real numbers, $(x_n)_{n\in\mathbb{N}}$, is *monotone* if and only if the function $f\colon \mathbb{N} \to \mathbb{R}$ determined by the sequence is a monotone function.

When we regard a sequence of real numbers as a function $\mathbb{N} \to \mathbb{R}$, it is continuous everywhere, as the next lemma shows.

Lemma 5.2. *Let $f\colon \mathbb{N} \to \mathbb{R}$ be a function. Then f is continuous at every $n \in \mathbb{N}$.*

Proof. Given $n \in \mathbb{N}$ and $\varepsilon > 0$, put $\delta = 1$

If $x \in \mathbb{N}$ satisfies $|n - x| < 1$, then $x = n$ as the difference between two integers is always an integer.

Thus $f(x) = f(n)$, whence $|f(x) - f(n)|0 < \varepsilon$. □

Hence, it is the behaviour as $n \to \infty$ which is of interest in the case of sequences.

Definition 5.3. The sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ *converges to* $\ell \in \mathbb{R}$ if and only if given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ with $|x_n - \ell| < \varepsilon$ whenever $n \geq N$. We write

$$x_n \to \ell \ \text{ as } \ n \to \infty \qquad \text{or} \qquad \lim_{x\to\infty} x_n = \ell$$

$(x_n)_{n\in\mathbb{N}}$ *diverges to* ∞ (resp. to $-\infty$) if and only if given any $K \in \mathbb{R}$, there is an $N \in \mathbb{N}$ with $x_n > K$ (resp. $x_n < K$) whenever $n > N$. We write

$$x_n \to \infty \ \text{ as } \ n \to \infty.$$

In other words, a sequence converges to a real number if, given any tolerance about that number, all but finitely many terms of the sequence lie within the tolerance of the number.

Example 5.4. Every constant sequence converges.

Example 5.5. If the sequence $(u_n)_{n\in\mathbb{N}}$ is given by $u_n = n$, then $u_n \to \infty$ as $n \to \infty$.

In other words, $n \to \infty$ as $n \to \infty$.

Example 5.6. If the sequence $(u_n)_{n\in\mathbb{N}}$ is given by $u_n = \dfrac{1}{n+1}$, then $u_n \to 0$ as $n \to \infty$.

To see this, choose $\varepsilon > 0$. Then $\frac{1}{\varepsilon} > 0$.

Since the natural numbers are unbounded, there is an $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$.

Take $n \geq N$. Then

$$|u_n - 0| = |\frac{1}{n}| < \frac{1}{N} < \varepsilon.$$

Lemma 5.7. *Every convergent sequence is bounded.*

Proof. Let $x_n \to \ell$ as $n \to \infty$.

Since $1 > 0$, there is an $N \in \mathbb{N}$ with $|x_n - \ell| < 1$ whenever $n > N$.

Put $K := 1 + \max\{|x_0|, \dots |x_N|, |\ell - 1|, |\ell + 1|\}$

Plainly, $|x_n| < K$ for all $n \in \mathbb{N}$. □

The converse of Lemma 5.7 is false.

Example 5.8. The sequence $\big((-1)^n\big)_{n\in\mathbb{N}}$ is bounded but does not converge.

It is bounded because $|(-1)^n)| = 1 < 2$ for all $n \in \mathbb{N}$.

Suppose that it converged to $\ell \in \mathbb{R}$. Since $1 > 0$, there is an $N \in \mathbb{N}$ with $|x_n - \ell| < 1$ whenever $n \geq N$. Then

$$\begin{aligned}
2 &= |(-1)^{2N} - (-1)^{2N+1}| \\
&= |(-1)^{2N} - \ell + \ell - (-1)^{2N+1}| \\
&\leq |(-1)^{2N} - \ell| + |(-1)^{2N+1} - \ell| \\
&< 1 + 1 && \text{as since } 2N \geq N \\
&= 2, && \text{which is a contradiction.}
\end{aligned}$$

While the converse of Lemma 5.7 is false in general, it holds for monotone sequences, as we now show.

Lemma 5.9. *A monotonically increasing (resp. decreasing) sequence of real numbers is bounded below (resp. above) and converges if and only if it is bounded above (resp.below).*

Proof. If $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing, then $u_n \geq u_0$ for all $n \in \mathbb{N}$.

Let $(u_n)_{n\in\mathbb{N}}$ be bounded above. By the completeness property of the real numbers, $\{u_n \mid n \in \mathbb{N}\}$ has a supremum, ℓ.

Take $\varepsilon >$.

Since $\ell - \varepsilon < \ell$, it is not an upper bound for or sequence.

Hence there is an $N \in \mathbb{N}$ with $\ell - \varepsilon < u_N \leq \ell$.

Take $n \geq N$. Since $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing and ℓ is an upper bound, we have

$$\ell - \varepsilon < u_N \leq u_n \leq \ell,$$

whence $|u_n - \ell| < \varepsilon$ for all $n \geq N$.

The case of decreasing sequences is left to the reader. □

There is a stronger condition than boundedness, the *Cauchy condition*, which is equivalent to convergence in the case of sequences of real numbers.

Definition 5.10. The sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ satisfies the *Cauchy condition* or is a *Cauchy sequence* if and only if given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ with $|x_m - x_n| < \varepsilon$, whenever $m, n \geq N$.

Lemma 5.11. *Every convergent sequence satisfies the Cauchy criterion.*

Proof. Let $x_n \to \ell$ as $n \to \infty$ and take $\varepsilon > 0$.

Since $\frac{\varepsilon}{2} > 0$, there is an $N \in N$ with $|x_k - \ell| < \frac{\varepsilon}{2}$ whenever $k \geq N$. Thus, if $m, n \geq N$,

$$|x_m - x_n| = |x_m - \ell + \ell - x_n| \leq |x_m - \ell| + |\ell - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Lemma 5.12. *Every Cauchy sequence is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of real numbers.

Take $N \in \mathbb{N}$ with $|x_m - x_N| < 1$ for $m \geq N$, that is $x_N - 1 < x_m < x_N + 1$ if $m \geq N$.

Put $K := \max\{|x_0|, \ldots, |x_{N-1}|, |x_N - 1|, |x_N + 1|\}$.

Plainly, $|x_n| \leq K$ for all $n \in \mathbb{N}$. □

We construct two monotonic sequences associated with the bounded sequence, $(x_n)_{n\in\mathbb{N}}$.

For each $m \in \mathbb{N}$, put $i_m := \inf\{x_k | k \geq m\}$ and $s_m := \sup\{x_k | k \geq m\}$.

Take $m \in \mathbb{N}$.

Since $\inf A \leq \inf B$ and $\sup A \geq \sup B$ whenever $A \subseteq B$,

$$\inf(x_n)_{n\in\mathbb{N}} = i_0 \leq i_m \leq i_{m+1} \leq s_{m+1} \leq s_m \leq s_0 = \sup(x_n)_{n\in\mathbb{N}}$$

Thus, $(i_m)_{m\in\mathbb{N}}$ is a monotonically increasing sequence, bounded above (by $\sup(x_n)_{n\in\mathbb{N}}$) and $(s_m)_{m\in\mathbb{N}}$ is a monotonically decreasing sequence, bounded below (by $\inf(x_n)_{n\in\mathbb{N}}$).

By Lemma 5.9 both sequences converge.

Definition 5.13. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Then its *limes inferior* and its *limes superior* are defined by

$$\liminf x_n := \lim_{m\to\infty} \inf_{k\geq m} x_k \qquad \text{and} \qquad \limsup x_n := \lim_{m\to\infty} \sup_{k\geq m} x_k$$

From the considerations above, if $(x_n)_{n\in\mathbb{N}}$ is bounded,

$$\inf x_n \leq \liminf x_n \leq \limsup x_n \leq \sup x_n.$$

Example 5.14. If $x_n = (-1)^n$, then $\liminf x_n = -1$ and $\limsup x_n = 1$.

Example 5.15. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = \begin{cases} n & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$.

Then $\liminf x_n = 0$ but there is no $\limsup x_n$.

Theorem 5.16. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $\lim x_n = \ell$ if and only if $\liminf x_n = \limsup x_n = \ell$.*

Proof. Let the sequence $(x_n)_{n\in\mathbb{N}}$ converge to ℓ.

Take $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$, and so there is an $N \in \mathbb{N}$ with $|x_m - \ell| < \frac{\varepsilon}{2}$ for all $m \geq N$. Thus, if $m \geq N$,

$$\ell - \frac{\varepsilon}{2} \leq \inf_{k\geq m} x_k \leq x_m \leq \sup_{k\geq m} x_k \leq \ell + \frac{\varepsilon}{2}.$$

or, equivalently,

$$0 \leq \sup_{k\geq m} x_k - \inf_{k\geq m} x_k \leq \varepsilon.$$

Hence

$$0 \leq \limsup x_n - \liminf x_n = \lim_{m\to\infty}\big(\sup_{k\geq m} x_k - \inf_{k\geq m} x_k\big) \leq \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we must have $\limsup x_n - \liminf x_n = 0$.

For the converse, suppose that $\limsup x_n = \liminf x_n = \ell$.

Take $\varepsilon > 0$.

There is an $N \in \mathbb{N}$ with $|\sup_{k\geq m} x_k - \ell| < \varepsilon$ and $|\inf_{k\geq m} x_k - \ell| < \varepsilon$.

Then, for $m \geq N$

$$\ell - \varepsilon \leq \inf_{k\geq m} x_k \leq x_m \leq \sup_{k\geq m} x_k \leq \ell + \varepsilon,$$

or, equivalently, $|x_m - \ell| < \varepsilon$ whenever $m \geq N$, showing that $(x_n)_{n\in\mathbb{N}}$ converges (to ℓ). □

Lemma 5.17. *If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers, then $\liminf x_n = \limsup x_n$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of real numbers.

Take $\varepsilon > 0$.

Since $\frac{\varepsilon}{2} > 0$, there is an $N \in \mathbb{N}$ with $|x_j - x_m| < \frac{\varepsilon}{2}$ whenever $j \geq m \geq N$.

Thus, $x_m - \frac{\varepsilon}{2} < x_j < x_m + \frac{\varepsilon}{2}$ for all $j \geq m \geq N$, whence

$$x_m - \frac{\varepsilon}{2} \leq \inf_{k\geq m} x_k \leq x_j \leq \sup_{k\geq m} x_k \leq x_m + \frac{\varepsilon}{2}$$

Hence $0 \leq \sup_{k\geq m} x_k - \inf_{k\geq m} x_k \leq \varepsilon$ for all $m \geq N$, whence

$$0 \leq \limsup x_n - \liminf x_n := \lim_{m\to\infty}\big(\sup_{k\geq m} x_k - \inf_{k\geq m} x_k\big) \leq \varepsilon.$$

Since this holds for all $\varepsilon > 0$, the conclusion follows. □

Theorem 5.18. *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence.

By Lemma 5.12, $(x_n)_{n\in\mathbb{N}}$ is bounded.

By Lemma 5.17, $\liminf x_n = \limsup xn$.

By Theorem 5.16, $(x_n)_{n\in\mathbb{N}}$ converges.

The converse is Lemma 5.7. □

Sometimes it is possible to determine whether a sequence is convergent without recourse to the Cauchy condition.

Lemma 5.19 (Squeezing Theorem). *Let $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that there is an $N \in \mathbb{N}$ with $u_n \leq x_n \leq v_n$ whenever $n \geq N$.*

If both $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to ℓ, then $(x_n)_{n\in\mathbb{N}}$ also converges to ℓ.

Proof. Take $\varepsilon > 0$. There are $K, L \in \mathbb{N}$ such that $|u_n - \ell| < \varepsilon$ when $n > K$ and $|v_n - \ell| < \varepsilon$ whenever $n > L$.

Put $M := \max\{K, L, N\}$ and take $n > M$. From the above, if $k \geq n$

$$\ell - \varepsilon < u_k \leq x_k \leq v_k < \ell + \varepsilon,$$

so that

$$\ell - \varepsilon \leq \inf_{k\geq n} x_k \leq \sup_{k\geq n} x_k \leq \ell + \varepsilon,$$

whence $\liminf x_n = \limsup x_n = \ell$.

By Theorem 5.16. $(x_n)_{n\in\mathbb{N}}$ converges to ℓ. □

5.2 Series

Associated with each sequence of real numbers, $(x_n)_{n\in\mathbb{N}}$ is another sequence of real numbers, namely, its *sequence of partial sums.*

Definition 5.20. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Its *sequence of partial sums* is the sequence of real numbers $(S_n)_{n\in\mathbb{N}}$ defined by

$$S_n := \sum_{j=1}^{n} x_n := x_0 + \cdots + x_n$$

The *series* associated with $(x_n)_{n\in\mathbb{N}}$ is

$$\sum_{n=0}^{\infty} x_n := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{n} x_n$$

The series $\sum_{n=-0}^{\infty} x_n$ *converges to* ℓ if the sequence of partial sums, $(S_n)_{n\in\mathbb{N}}$, converges to ℓ and *diverges* if $S_N \to \infty$ or $S_n \to -\infty$ as $n \to \infty$.

Observation 5.21. The sequence $(x_n)_{n\in\mathbb{N}}$ and its sequence of partial sums, $(S_n)_{n\in\mathbb{N}}$ determine each other completely, for

$$x_n = \begin{cases} S_0 & \text{if } n = 0 \\ S_n - S_{n-1} & \text{if } n > 0. \end{cases}$$

Convergence of the series $\sum x_n$ imposes a strong restriction on the sequence $(x_n)_{n\in\mathbb{N}}$.

Lemma 5.22. *If the series* $\sum_{n\in\mathbb{N}} x_n$ *converges, then* $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose that $\sum x_n$ converges to $\ell \in \mathbb{R}$.

Then $(S_n)_{n\in\mathbb{N}}$, the sequence of partial sums must be a Cauchy sequence.

Take $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ with $|S_m - S_n| < \varepsilon$ whenever $m, n \geq N$.

In particular, putting $m := n + 1$, $|x_{n+1} - 0| = |S_{n+1} - S_n| < \varepsilon$ whenever $n > N$.

Thus $\lim_{n\to\infty} x_n = 0$. □

Example 5.23 (**Arithmetic Series**). Let $(x_n)_{\in\mathbb{N}}$ be an arithmetic progression, so that $x_n = a + nd$ for some $a, d \in \mathbb{R}$.

Let $n \to \infty$. Then

$$a + nd \longrightarrow \begin{cases} -\infty & \text{if } d < 0 \\ 0 & \text{if } d = 0 \\ \infty & \text{if } d > 0 \end{cases}$$

Hence the series $\sum(a + nd)$ can only converge if $a = d = 0$, which is of little interest.

Example 5.24 (**Geometric Series**). Let $(x_n)_{\in\mathbb{N}}$ be a geometric progression: $x_n = ar^n$ for some $a, r \in \mathbb{R}(a \neq 0)$. Then

$$S_n = a + ar + \cdots + ar^n$$

so that

$$rS_n = ar + \cdots + ar^n + ar^{n+1}$$

whence

$$S_n - rS_n = a_ar^{n+1}.$$

So

$$S_n = \begin{cases} (n+1)a & \text{if } r = 1 \\ a\dfrac{1 - r^{n+1}}{1 - r} & \text{otherwise} \end{cases}$$

Thus, as $n \to \infty$, S_n

(i) oscillates without bound if $r < -1$;

(ii) alternates between a and $-a$ if $r = -1$;

(iii) converges to $\dfrac{a}{1-r}$ if $-1 < r < 1$;

(iv) diverges to $\pm\infty$, depending as $a > 0$ or $a < 0$ if $r \geq 1$.

We summarise the above in the next lemma.

Lemma 5.25. *The geometric series $\sum ar^n$ converges if and only if $|r| < 1$, in which case it converges to $\dfrac{a}{1-r}$.*

The converse of Lemma 5.22 does not hold: not every sequence which converges to zero gives rise to a convergent series

Example 5.26 (**The Harmonic Series**). The sequence $(\frac{1}{n+1})_{n\in\mathbb{N}}$ clearly converges to 0 as n tends to ∞. But, the *harmonic series*, $\sum \frac{1}{n+1}$, does not converge, as we now show.

Since $\frac{1}{n+1} > 0$ for every $n \in \mathbb{N}$, the sequence of partial sums is monotonically strictly increasing.

Hence it converges if and only if it is bounded above.

Since $2^n + k \leq 2^{n+1}$ for $k = 1, \ldots 2^n$, we see that $\frac{1}{2^{n+1}} \leq \frac{1}{2^n+k}$ for $k = 1, \ldots, 2^n$.

Hence

$$\sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} \geq \sum_{j=1}^{2^n} \frac{1}{2^{n+1}} = \frac{1}{2},$$

and so

$$\sum_{j=0}^{2^{n+1}} \frac{1}{n+1} \geq 1 + \frac{n+1}{2} = \frac{n+3}{2},$$

which is, clearly, unbounded.

A convenient method for testing series for convergence is provided by comparing the series in question with one whose behaviour is known.

Theorem 5.27 (Comparison Test). *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences of non-negative real numbers such that for some $k > 0$, $u_n \leq kv_n$ for all $n \in \mathbb{N}$. Then*

(a) $\sum u_n$ converges whenever $\sum v_n$ converges.

(b) $\sum v_n$ diverges whenever $\sum u_n$ diverges.

Proof. Put $S_n := \sum_{j=1}^{n} u_n$ and $T_n := \sum_{j=1}^{n} kv_n$.

Since $u_n, v_n \geq 0$ for all $n \in \mathbb{N}$, both $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ are monotonically increasing sequence. Hence they converge if and only if they are bounded above.

Since $S_n \leq T_n$ for all $n \in \mathbb{N}$, $(S_n)_{n\in\mathbb{N}}$ is bounded above whenever $(T_n)_{n\in\mathbb{N}}$ is bounded above, and $(T_n)_{n\in\mathbb{N}}$ is unbounded whenever $(S_n)_{n\in\mathbb{N}}$ is unbounded. □

Our first application of the comparison test is to *absolute convergence*.

Definition 5.28. The series $\sum u_n$ *converges absolutely* if and only if the series $\sum |u_n|$ converges,

Theorem 5.29. *Every absolutely convergent series converges.*

Proof. Let $\sum a_n$ be an absolutely convergent series.

Since $-|a_n| \leq a_n \leq |a_n|$ for all $n \in \mathbb{N}$, we have

$$0 \leq a_n + |a_n| \leq 2|a_n| \qquad \text{for all } n \in \mathbb{N}.$$

Hence, since $\sum 2|a_n|$ converges, it follows from the comparison test that $\sum(a_n + |a_n|)$ also converges.

Since $\sum a_n$ is the difference between two convergent series, it also converges. □

We next apply the comparison test to investigate the convergence of $\sum \frac{1}{n^k}$, for $k \in \mathbb{R}$.

Example 5.30 (The Series $\sum \frac{1}{n^k}$). We investigate the dependence of the behaviour when k varies, writing S_n for the n^{th} partial sum.

(i) If $k < 0$, then $\frac{1}{n^k} = n^{|k|} \geq 1$, whence $S_n \geq n + 1$, so that $\sum \frac{1}{j^k}$ diverges.

(ii) If $k = 0$, then $\frac{1}{n^k} = 1$ for all $n > 0$, so that $S_n = n + 1$, which is unbounded.

(iii) If $0 < k \leq 1$, then $0 < n^k \leq n$ for all $n \geq 1$. Thus $0 < \frac{1}{n} \leq \frac{1}{n^k}$. Since the harmonic series diverges, the comparison test shows that $\sum \frac{1}{n^k}$ diverges if $0 < k \leq 1$.

(iv) Finally, take $k \geq 1$.

For $j > 0$, $(n+j)^k > n^k > 0$, or, equivalently $0 < \frac{1}{(n+j)^k} < \frac{1}{n^k}$ for $j > 0$.

In particular, taking $n = 2^m$, we have

$$\begin{aligned}
\frac{1}{(2^m)^k} + \frac{1}{(2^m+1)^k} + \cdots \frac{1}{(2^m+2^m-1)^k} &= \sum_{j=0}^{2^m-1} \frac{1}{(2^m+j)^k} \\
&\leq \sum_{j=0}^{2^m-1} \frac{2^m}{(2^m)^k} \\
&= \frac{1}{(2^m)^{k-1}} \\
&= \left(\frac{1}{2^{k-1}}\right)^m
\end{aligned}$$

Thus

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{(2^{m+1}-1)^k} \leq 1 + \frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}}\right)^2 + \cdots + \left(\frac{1}{2^{k-1}}\right)^m$$

This last series is a geometric series with common ratio $\frac{1}{2^{k-1}}$.
Since $k > 1, |\frac{1}{2^{k-1}}| < 1$, whence the geometric series converges.
Hence the comparison test shows that $\sum \frac{1}{n^k}$ converges if $k > 1$.

The next lemma summarises this.

Lemma 5.31. *The series $\sum \frac{1}{n^k}$ converges if $k > 1$ and diverges if $k \leq 1$.*

While the comparison test is powerful, it has the disadvantage that it requires finding a second series to test the convergence/divergence of a given series.

There are tests for convergence which depend only on the series in question.

Theorem 5.32 (Ratio Test). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = r.$$

If $r < 1$, then $\sum u_n$ converges absolutely.

If $r > 1$, then $\sum u_n$ is unbounded.

Proof. Put $\varepsilon := \frac{|1-r|}{2}$. Since $\left|\frac{u_{n+1}}{u_n}\right| \to r$ as $n \to \infty$, there is an $N \in \mathbb{N}$ with

$$\left|\frac{|u_{n+1}|}{|u_n|} - r\right| < \varepsilon,$$

or equivalently,

$$(r-\varepsilon)|u_n| < |u_{n+1}| < (r+\varepsilon)|u_n|$$

whenever $n \geq N$.

If $r < 1$, then $|1-r| = 1-r$. Put $s := r + \varepsilon = \frac{1+r}{2}$.
Then $|u_{n+1}| < |u_n|s$, for all $n \geq N$, whence $|u_{N+k}| < |u_N|s^k$ for all $k \in \mathbb{N}$. Thus,

$$\sum_{n=N}^{\infty} |u_n| \leq \sum_{n\in\mathbb{N}} |u_N|s^n = \frac{|u_N|}{1-s} \qquad \text{since } s < 1.$$

Thus $\sum u_n$ converges absolutely.

If $r > 1$, then $|1-r| = r-1$. Put $s := r - \varepsilon = \frac{1+r}{2}$.
Then, arguing as above, $|_n|s < |u_{n+1}|$ for all $n \geq N$, whence $|u_N|s^k < |u_{N+k}|$ for all $k \in \mathbb{N}$. Thus,

$$\sum_{n=N}^{\infty} |u_n| \geq \sum_{n\in\mathbb{N}} |u_N|s^n,$$

which diverges since $s > 1$, showing that $\sum u_n$ is unbounded. □

We present another intrinsic test for convergence.

Theorem 5.33 (**Root Test**). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

$$\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = r.$$

If $r < 1$, then $\sum u_n$ converges absolutely.
If $r > 1$, then $\sum u_n$ is unbounded.

Proof. Put $\varepsilon := \dfrac{|1-r|}{2}$, and choose $N \in \mathbb{N}$ such that for all $n \geq N$, $\left||_n|^{\frac{1}{n}} - r\right| < \varepsilon$, r, equivalently $(r-\varepsilon)^n < |u_n| < (r+\varepsilon)^n$

If $r < 1$, then $|1-r| = 1-r$. Put $s := r + \varepsilon = \frac{1+r}{2}$.
Since $|u_n| < s^n$ for all $n \geq N$, and $s < 1$,

$$\sum_{n=N}^{\infty} |u_n| \leq \sum_{n=N}^{\infty} s^n = \frac{1}{1-s},$$

showing that $\sum u_n$ converges absolutely.

If $r > 1$, then $|1-r| = r-1$. Put $s := r - \varepsilon = \frac{1+r}{2}$.
Since $|u_n| > s^n$ for all $n \geq N$, and $s < 1$,

$$\sum_{n=N}^{\infty} |u_n| \geq \sum_{n=N}^{\infty} s^n,$$

showing that $\sum u_n$ is not bounded. □

If $|u_n|^{\frac{1}{n}} \to 1$ as $n \to \infty$, then the root test is inconclusive.

Example 5.34. We use a fact, to be proved later, namely that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

It follows that for any k, $\lim |\frac{1}{n^k}|^{\frac{1}{n}} = 1$. But, by Lemma 5.31, $\sum \frac{1}{n^k}$ converges if $k > 1$ and diverges if $k \leq 1$.

There is another useful test for convergence.

Theorem 5.35 (Alternating Series Test). *Let $(u_n)_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of positive real numbers, whose limit is* 0.

Then the series $\displaystyle\sum_{n\in\mathbb{N}}(-1)^n u_n$ *converges.*

Proof. Put $S_n := \displaystyle\sum_{j=0}^{n}(-1)^j u_j$.

For $j \in \mathbb{N}$,

$$
\begin{aligned}
S_{2j+1} &= S_{2j} - u_{2j+1} < S_{2j}\\
S_{2j+2} &= S_{2j+1} + u_{2j+2} > S_{2j+1}\\
S_{2j+2} &= S_{2j} - (u_{2j+1} - u_{2j+2}) < S_{2j}\\
S_{2j+3} &= S_{2j+1} + (u_{2j+2} - u_2 j + 3) > S_{2j+1}
\end{aligned}
$$

It follows from these inequalities that $(S_{2n})_{n\in\mathbb{N}}$ is monotonically decreasing, that $(S_{2n+1})_{n\in\mathbb{N}}$ is monotonically increasing, and that

$$
\sup_{k>m} S_k = \begin{cases} S_m & \text{if } m \text{ is even}\\ S_{m+1} & \text{if } m \text{ is odd}\end{cases}
$$
$$
\inf_{k>m} S_k = \begin{cases} S_{m+1} & \text{if } m \text{ is even}\\ S_m & \text{if } m \text{ is odd}\end{cases}
$$

whence

$$
\sup_{k\geq m} S_m - \inf_{k\geq m} S_m = u_{m+1}
$$

and so

$$
\limsup S_n - \liminf S_n = \lim_{n\to\infty} u_{n+1} = 0,
$$

showing that $(S_n)_{n\in\mathbb{N}}$ converges. $\square$

We a now show that if $\left|\dfrac{u_{n+1}}{u_n}\right| \to 1$ as $n \to \infty$, then no conclusion can be drawn about the convergence or divergence of the series $\sum u_n$.

Example 5.36 (The Alternating Harmonic Series). Put $u_n := \frac{1}{n+1}$, so that $(u_n)_{n\in\mathbb{N}}$ is the harmonic sequence.

It was shown in Example 5.26, that the sequence $\sum \frac{1}{n}$ diverges.

It follows from the alternating series test that $\sum \frac{(-1)^n}{n}$ converges.

Yet both of the sequences satisfy $\left|\frac{u_{n+1}}{u_n}\right| \to 1$ as $n \to \infty$.

Observation 5.37. The alternating harmonic series is an example of a convergent series which is not absolutely convergent, which shows that the converse of Theorem 5.29 is false in general.

5.3 Power Series

Our investigation of sequences and series of real numbers examined them one at a time: we looked at sequences of the form $(u_n)_{n\in\mathbb{N}}$, with $u_n \in \mathbb{R}$ fixed, or, equivalently, we regarded $(u_n)_{n\in\mathbb{N}}$ as the function $f\colon \mathbb{N} \longrightarrow \mathbb{R},\ \ n \longmapsto u_n$.

We now turn to investigating several sequences and their associated series simultaneously, by varying the terms of the sequence simultaneously. We restrict attention to a particular form of variation suited to our programme: we consider sequences of the form $(c_n x^n)_{n\in\mathbb{N}}$, where we allow x to be any real number.

This links our previous study to our programme of finding the "curve of best fit", by simultaneously generalising both. For the sequences we have studied till now are of the form of our new sequences if we let $x = 1$, and the n^{th} partial sum of $(c_n x^n)_{n\in\mathbb{N}}$ is the polynomial $c_0 + c_1 x + \cdots + c_n x^n$.

Observation 5.38. If we put

$$X := \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} c_n x^n \text{ converges}\}$$

then $X \neq \emptyset$, as $0 \in X$. Moreover, the n^{th} partial sum of the sequence $(c_n x^n)_{n\in\mathbb{N}}$ is the polynomial $c_0 + c_1 x + \ldots + c_n x^n$, whence

$$f\colon X \longrightarrow \mathbb{R}, \quad x \longmapsto \sum_{n=0}^{\infty} c_n x^n$$

is a function which we may fruitfully regard as the limit of a sequence of polynomial functions.

Definition 5.39. A *power series* is a series of the form $\sum c_n x^n$. The c_n are the *coefficients* of the power series.

Since the behaviour of the power series $\sum c_n x^n$, is clearly determined by its sequence of coefficients, we first investigate $(c_n)_{n\in\mathbb{N}}$

We consider the associated sequence $\left(|c_n|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$.

Suppose that $\left(|c_n|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ is unbounded.
Take $x \neq 0$.
Then, for each $N \in \mathbb{N}$, there is a $k > N$ such that $|u_k|^{\frac{1}{k}} > |\frac{1}{x}|$.
Thus $|c_k x^k| > 1$.
But then the sequence $(c_n x^n)_{n\in\mathbb{N}}$ cannot converge to 0.
Consequently, $\sum c_n x^n$ cannot converge.
If $\left(|c_n|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ is bounded, put $C := \limsup(|c_n|^{\frac{1}{n}})$.
Suppose that $C = 0$. Take $x \neq 0$. Then there is an $N \in \mathbb{N}$ with

$$\sup\{|c_k|^{\frac{1}{k}} \mid k \geq N\} < \left|\frac{1}{2x}\right|,$$

or, equivalently,

$$|c_k x^k| < \frac{1}{2^k},$$

whenever $k \geq N$. Thus, by the comparison test, $\sum c_n x^n$ converges absolutely.

Hence, $X = \mathbb{R}$ on this case.

If $C \neq 0$, put $R := \dfrac{1}{C}$.

Take $x \in \mathbb{R}$ with $|x| < R$.

Choose S with $|x| < S < R$, so that $\limsup(|c_n|^{\frac{1}{n}}) = \dfrac{1}{R} < \dfrac{1}{S} < \dfrac{1}{|x|}$.

Then there is an $N \in \mathbb{N}$ with $\sup\limits_{k \geq n} |c_k|^{\frac{1}{k}} < \dfrac{1}{S}$ for all $n \geq N$.

Consequently, $|c_k|^k < \dfrac{1}{S^k}$ for all $k \geq N$.

Hence, if $k \geq N$,

$$|c_k x^k| < \left(\frac{|x|}{S}\right)^k$$

Since $|\dfrac{x}{S}| < 1$, the comparison test shows that $\sum\limits_{n \in \mathbb{N}} c_n x^n$ converges uniformly.

Take $x \in \mathbb{R}$ with $|x| > R$.

Choose S with $R < S < |x|$, so that $\dfrac{1}{|x|} < \dfrac{1}{S} < \limsup(|c_n|^{\frac{1}{n}}) = \dfrac{1}{R}$.

Then there is an $N \in \mathbb{N}$ with $\sup\limits_{k \geq n} |c_k|^{\frac{1}{k}} > \dfrac{1}{S}$ for all $n \geq N$.

Thus, for each $n \geq N$ there is a $k \geq n$ with $|c_k| > \dfrac{1}{S^k}$, for which

$$|c_k x^k| > \left|\frac{x}{S}\right|^k > 1.$$

Thus $c_n x^n$ cannot converge to 0, whence $\sum c_n x^n$ cannot converge.

We summarise the above in the next theorem.

Theorem 5.40. *Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of real number, for which* $\limsup\left(|u_n|^{\frac{1}{n}}\right) = C$.

(a) If $C = 0$, then $\sum c_n x^n$ converges absolutely for every real number x.

(b) If $C \neq 0$, put $R := \dfrac{1}{C}$. Then

  (i) $\sum c_n x^n$ converges absolutely whenever $|x| < R$.

  (ii) $\sum c_n x^n$ is unbounded whenever $|x| > R$.

Definition 5.41. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of real numbers, for which $\limsup\left(|u_n|^{\frac{1}{n}}\right) = C$.

The *radius of convergence* of the power series $\sum c_n x^n$ is infinite if $C = 0$ and $\dfrac{1}{C}$ if $C \neq 0$.

Definition 5.41 allows us to reformulate Theorem 5.40.

Theorem 5.42. *Let $\sum c_n x^x$ be a power series with radius of convergence R.*

Then

$$f\colon]-S, S[\longrightarrow \mathbb{R}, \quad x \longmapsto \sum_{n \in \mathbb{N}} c_n x^n$$

defines a function if and only if $S \leq R$.

[Here we allow $R = \infty$.]

Example 5.43. Consider the sequence $(c_n)_{n\in\mathbb{N}}$ given by $c_n = 1$ for all $n \in \mathbb{N}$.

Then $|c_n|^{\frac{1}{n}} = 1^{\frac{1}{n}} = 1$ for all n, whence $\limsup(|c_n|^{\frac{1}{n}}) = 1$.

The series $\sum c_n x^n$ is the geometric series $\sum x^n$, which converges if and only if $|x| < 1$, whence the radius of convergence is 1. Since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

if $|x| < 1$, we have the function,

$$f\colon]-1,1[\longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{1-x}$$

Example 5.44. Consider the sequence $(c_n)_{n\in\mathbb{N}}$ given by $c_n = (-1)^n$ for all $n \in \mathbb{N}$. Then $|c_n|^{\frac{1}{n}} = 1^{\frac{1}{n}} = 1$ for all n, whence $\limsup(|c_n|^{\frac{1}{n}}) = 1$.

The series $\sum c_n x^n$ is the geometric series $\sum (-x)^n$, which converges if and only if $|x| < 1$, whence the radius of convergence is 1. Since

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x},$$

if $|x| < 1$, we have the function,

$$f\colon]-1,1[\longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{1+x}$$

Example 5.45. Consider the sequence $(c_n)_{n\in\mathbb{N}}$ given by $c_{2n} = 1$ and $c_{2n+1} = 0$ for all $n \in \mathbb{N}$. Then $|c_{2n}|^{\frac{1}{2n}} = 1^{\frac{1}{2n}} = 1$ and $|c_{2n+1}|^{\frac{1}{2n+1}} = 0^{\frac{1}{2n+1}} = 0$ for all n, whence $\limsup(|c_n|^{\frac{1}{n}}) = 1$.

The series $\sum c_n x^n$ is the geometric series $\sum x^{2n}$, which converges if and only if $|x| < 1$, whence the radius of convergence is 1. Since

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2},$$

if $|x| < 1$, we have the function,

$$f\colon]-1,1[\longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{1-x^2}$$