MATH101

Intensive School

Tutorial 1

Question 1.

State which of the following sequences, $(u_n)_{n\in N}$ are monotone. Are these increasing or decreasing? Describe the behaviour as $n \to \infty$.

(a) $u_n = \dfrac{n}{n^2+1}$

(b) $u_n = n^2$

(c) $u_n = n^n$

(d) $u_n = \dfrac{n^2+1}{n}$

(e) $u_n = 2n + (-1)^n$

(f) $u_n = \dfrac{n^2-1}{n^3-1}$

(g) $u_n = \dfrac{n^3+2n+1}{1-10n^2-n^3}$

(h) $u_n = 2^{-n}$

(i) $u_n = \dfrac{n!}{n^n}$

Question 2.

In relativistic quantum mechanics, the solution of the Dirac equation for the electron in the hydrogen atom gives energy levels

$$E_n = \frac{mc^2}{\sqrt{1 + \dfrac{\frac{\gamma^2}{c^2}}{n + \sqrt{\kappa^2 - \frac{\gamma^2}{c^2}}}}}$$

Here, m is the electron mass, γ a constant related to the electric charge, κ a constant related to the orbital quantum number and n the principal quantum number.

Is $(E_n)_{n\in\mathbb{N}}$ a monotonic sequence? Does it converge?

Question 3.

Prove formally that

(a) $2^n \to \infty$ as $n \to \infty$

(b) $\frac{1}{2^n} \to 0$ as $n \to \infty$

Question 4.

A ball is dropped from a height of 20 metres. On each bounce, the ball returns to a height $\frac{4}{5}^{\text{th}}$ of that of the previous bounce.

What is the height of the third bounce?

Find an expression for h_n, the height of the n^{th} bounce.

Find an expression for d_n, the distance travelled by the ball up to the n^{th} bounce.

What is the total distance travelled by the ball from the time it is fist dropped until it comes to rest?

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Tutorial 2

Question 1.

Discuss the whether each of the following series converges.

(a) $\dfrac{1}{3} + \dfrac{2}{5} + \dfrac{3}{7} + \cdots + \dfrac{n}{2n+1} + \cdots = \displaystyle\sum_{n=1}^{\infty} \dfrac{n}{2n+1}$

(b) $1 + \dfrac{1}{1} + \dfrac{1}{2\cdot 1} + \dfrac{1}{3\cdot 2\cdot 1} + \cdots + \dfrac{1}{n!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \dfrac{2^n + 1}{3^n + 1}$

(d) $\displaystyle\sum_{n=0}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$

(e) $\displaystyle\sum_{k=0}^{\infty} \cos(k\pi)$

(f) $\displaystyle\sum_{n=0}^{\infty} \dfrac{n^2}{2^n}$

(g) $\displaystyle\sum_{n=0}^{\infty} \dfrac{n!}{n^n}$

(h) $\displaystyle\sum_{k=1}^{\infty} \dfrac{k^{10}}{3^k}$

(i) $\displaystyle\sum_{k=1}^{\infty} \dfrac{\sin^2 k}{k^2}$

(j) $\displaystyle\sum_{n=0}^{\infty} \dfrac{(n!)^2}{(2n)!}$

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Tutorial 3

Question 1.

Use the definition of the derivative to prove the following.

(a) $\dfrac{d}{dx}\left(x^2\right) = 2x$

(b) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-1}{x^2} \qquad (x \neq 0)$

Question 2.

State at which points the following functions are differentiable, giving the derivative at these points. If a function fails to be differentiable at a point, explain why this is the case.

(a) $f\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{1}{x}$

(b) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 1 + x^2 & \text{for } x < 0 \\ (1+x)^2 & \text{for } x \geq 0 \end{cases}$

Question 3.

Find the derivative of each of the following functions.

(a) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^6$

(b) $f\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto 3x^{-4}$

(c) $f\colon \mathbb{R} \setminus \{-1\} \longrightarrow \mathbb{R}, \quad x \longmapsto x^2(1+x)^{-1}$

(d) $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto x^{\frac{3}{4}}$

(e) $f\colon \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{x^2+1}{x^3-1}$

(f) $f\colon \mathbb{R}_0^+ \setminus \{1\} \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{\sqrt{x}+1}{1-\sqrt{x}}$

Question 4. (**Falling into a Black Hole**)

We consider what happen to a scientist as (s)hr jumps, feet first towards a singularity in the universe called a "black hole".

Write ℓ for the height of our scientist at his/her proper time τ, τ_i for the time measured by the scientist when his/her feet hit the singularity, A for a representative cross-sectional area of the scientist's body and ℓ_0, resp. A_0 for the initial values of ℓ, resp. A.

According to Einstein's General Theory of Relativity,

$$\ell = \ell_0 \left(\frac{\tau_i}{\tau_i - \tau} \right)^{\frac{1}{3}}$$

$$A = A_0 \left(\frac{\tau_i - \tau}{\tau} \right)^{\frac{4}{3}}$$

(a) Find the time it take for the scientist to be stretched to twice his/her original height.

(b) Find the rate of increase in the scientist's height.
Does this mean that at some time the scientist's head will be moving away from his/her feet so quickly that (s)e will be unable to wiggle the toes, because the neural impulses from the brain won't reach the muscles in the feet?

(c) Determine the rate of decrease in the scientist's cross-sectional area.

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Tutorial 4

Question 1.

Determine whether the following functions are differentiable at the specified points. If so, determine the derivative at that point.

(a) $f\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} \sqrt{x} & \text{for } x \leq 1 \\ \frac{1}{2}(x+1) & \text{for } x > 1 \end{cases} \qquad \text{at } x = 1$

(b) $g\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{otherwise} \end{cases} \qquad \text{at } x = 0$

(c) $h\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad \longmapsto \begin{cases} x^4 + x^2 & \text{when } x \leq 1 \\ 4x^2 - 2x & \text{when } x \geq 1 \end{cases}$

Question 2.

Find the first three derivatives of $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad u \longmapsto \sqrt{u^2+1}$.

Question 3.

Determine the maximal subset, D, of $\mathbb{R}$ on which the following expressions define a function $D \to \mathbb{R}$. Determine where these functions are differentiable and where they fail to be differentiable. Find the derivatives where possible.

(a) $\dfrac{1}{1+\tan x}$

(b) $x^2 \cos x$

Question 4.

In the de Sitter cosmological solution of the equations of General Relativity, the expansion factor, a, for the universe is given by

$$a(t) = a_0 e^{Ht},$$

where H is Hubble's constant.

The volume of a cube of side length 1 today ($t = 0$) will have expanded to

$$V(t) = \big(a(t)\big)^3$$

at time t in the future.

Find the rate at which the volume is changing.

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Tutorial 5

Question 1.

For each of the following equations, suppose that y is a function of x satisfying the given equation. In each case find $\frac{dy}{dx}$ and explain where the function is not differentiable.

(a) $xy + x^2y^2 - 3x + 1 = 0$

(b) $y^2 \sin x + \cos y = 1$

Question 2.

For each of the following functions, f, verify that it satisfies the hypotheses of the Mean Value Theorem on the specified interval $[a, b]$, and find all $c \in]a, b[$ with $f'(c)(b - a) = f(b) - f(a)$

(a) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad u \longmapsto 1 + u^2 \qquad$ on $[0, 1]$

(b) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad y \longmapsto y^3 + y - 4 \qquad$ on $[-1, 2]$

Question 3.

Take $f\colon [0, \pi] \setminus \{\frac{\pi}{2}\} \longrightarrow \mathbb{R}, \qquad \theta \longmapsto \tan\theta$.
Show that, even though $f(0) = f(\pi)$, there is no $c \in]0, \pi[$ with $f'(c) = 0$.
Explain why this does not contradict Rolle's Theorem.

Question 4.

Let n be a counting number and consider the function $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto x^{\frac{1}{n}} - 1$.
Use the Mean Value Theorem to prove that for all $x \geq 1$,

$$x^{\frac{1}{n}} \;\geq\; 1 + \frac{1}{n}(x - 1)$$

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Tutorial 6

Question 1.

Determine the intervals on which the following functions are

(i) increasing, respectively decreasing,

(ii) concave up, respectively concave down.

(a) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x$

(b) $g\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad u \longmapsto e^{-\frac{u^2}{2}}$

(c) $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto 9t^3 - 9t^2 + 24t$

Question 2.

Find all critical points and determine all relative maxima and minima of the following functions.

(a) $k\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto xe^{x^2}$

(b) $\psi\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad y \longmapsto \frac{1}{y^2+1}$

(c) $\alpha\colon [-2, 2] \longrightarrow \mathbb{R}, \quad u \longmapsto (4 - u^2)^{\frac{3}{4}}$

(d) $m\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x^2$

Question 3.

Sketch the graphs of the functions in Question 1 and Question 2.

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Tutorial 7

Question 1.

Find all relative extrema of the following functions.

(a) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^3 + \frac{5}{2}x^2 - 2x + 1$

(b) $g\colon \mathbb{R} \longmapsto \mathbb{R}, \quad x \longmapsto (x^2+1)e^{-x^2}$

(c) $h\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad v \longmapsto \dfrac{(\ln v)^2}{v^2}$

(d) $k\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longrightarrow \dfrac{x}{x^2+1}$

(e) $\ell\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad v \longrightarrow \sqrt{x^4+x^2+1}$

(f) $\xi\colon [-1,2] \longrightarrow \mathbb{R}, \quad z \longmapsto \sqrt{2-z-z^2}$

Question 2.

Find the absolute extrema (if there are any) of the following functions.

(a) $\varrho\colon [-1,1] \longrightarrow, \quad x \longmapsto \sqrt{x^4-1}$

(b) $m\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{\ln x}{x}$

(c) $\tau\colon [1,16] \longrightarrow \mathbb{R}, \quad w \longmapsto \sqrt{w} + \dfrac{1}{\sqrt{w}}$

(d) $q\colon [0,1] \longrightarrow \mathbb{R}, \quad t \longmapsto \dfrac{t^2-1}{t^2+1}$

Question 3.

Sketch the graphs of the functions in Questions 1 and 2.

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Tutorial 8

Question 1.

The intensity of light from a point source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two point sources of light of strengths S and $8S$ are separated by a distance of 90cm. Where on the line segment joining the two sources is the intensity of light a minimum?

Question 2.

A piece of wire of length L units is used to make two figures, a square and a circle. How should the wire be cut to make the total area enclosed by the two figures is

(a) minimised,

(b) maximised.

Question 3.

Determine, where possible, the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^2}$

(c) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^2}$

(d) $\displaystyle\lim_{x\to \infty} \left(1 + \frac{1}{x^2}\right)^x$

(e) $\displaystyle\lim_{x\to \infty} \left(x - \ln(1 + e^x)\right)$

(f) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} (\cos x)^{\tan x}$