MATH101

Intensive School

Sample Solutions to Tutorial 1

Question 1. (a)

$$\begin{aligned}
u_{n+1} - u_n &= \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} \\
&= \frac{(n+1)(n^2+1) - n((n+1)^2+1)}{((n+1)^2+1)(n^2+1)} \\
&= \frac{-(2n^2+2n-1)}{((n+1)^2+1)(n^2+1)} \\
&= -\frac{2(n+\frac{1}{2})^2 - \frac{3}{2}}{((n+1)^2+1)(n^2+1)}
\end{aligned}$$

Thus $0 \leq u_{n+1} < u_n$ for all $n \geq 1$.
On the other hand, $u_0 = 0 < \frac{1}{2} = u_1$, whence the sequence $(u_n)_{n\in\mathbb{N}}$ is not monotonic.

$$0 \leq \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we have $\lim_{n\to\infty} u_n = 0$

(b) $u_{n+1} - u_n = (n+1)^2 - n^2 = 2n+1 > 0$
Thus $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing.
Moreover, since $n^2 > n$ for $n > 1$ and $n \to \infty$ as $n \to \infty$, $u_n \to \infty$ as $n \to \infty$.

(c) $\dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)^{n+1}}{n^n} = (n+1)(1+\frac{1}{n})^n > 1$
Since $u_n > 0$ for every $n \in \mathbb{N}^*$, $(u_n)_{n=1}^{\infty}$ is monotonically increasing.
Moreover, since $n^n > n$ for $n > 1$ and $n \to \infty$ as $n \to \infty$, $u_n \to \infty$ as $n \to \infty$.

(d) Since $\dfrac{n+1}{(n+1)^2+1} < \dfrac{n}{n^2+1}$ for $n \geq 1$, we see that $u_{n+1} > u_n$ for all $n \geq 1$. Since there is no u_0, $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing.
Since $u_n = \dfrac{n^2+1}{n} = n + \frac{1}{n} > n$, $u_n \to \infty$ as $n \to \infty$.

(e) $u_{n+1} - u_n = 2(n+1) - (-1)^{n+1} - 2^n + (-1)^n = 2(1+(-1)^n) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$
Thus $u_{n+1} \geq u_n$ for all n. Thus $(u_n)_{n\in\mathbb{N}}$ is monotonically non-decreasing.
Since $u_n \geq 2n-1 \geq n$, $u_n \to \infty$ as $n \to \infty$.

(f) $u_n = \dfrac{n^2-1}{n^3-1} = \dfrac{n+1}{n^2+n+1}$ for $n > 1$. Thus

$$\begin{aligned}
u_{n+1} - u_n &= \frac{n+2}{(n+1)^2+(n+1)1} - \frac{n+1}{n^2+n+1} \\
&= \frac{(n+2)(n^2+n+1) - (n+1)(n^2+3n+3)}{((n^2+3n+3)(n^2+n+1)} \\
&= \frac{-(n^2+3n+1)}{(n^2+3n+3)(n^2+n+1)} \\
&< 0
\end{aligned}$$

Hence, $(u_n)_{n=2}^{\infty}$ is monotonically decreasing.

Since $u_n = \dfrac{n+1}{n^2+n+1} < \dfrac{2n}{n^2+n+1} < \dfrac{2}{n}$, we see that $\lim\limits_{n\to\infty} u_n = 0$.

(g) $\quad u_{n+1} - u_n = \dfrac{(n+1)^3+2(n+1)+1}{1-10(n+1)^2-(n+1)^3} - \dfrac{n^3+2n+1}{1-10n^2-n^3}$

Since we are only interested in whether this is positive or negative, it is enough to work with ν, the numerator obtain when $u_{n+1} - u_n$ is written as a rational function of n.

Since $(n+1)^3 = n^3+3n^2+3n+1$ and $(n+1)^2 = n^2+2n+1$, we obtain, successively,

$$\begin{aligned}
(n+1)^3+2(n+1)+1 &= n^3+3n^2+5n+4\\
1-10(n+1)^2-(n+1)^3 &= -(n^3+13n^2+23n+10)\\
((n+1)^3+2(n+1)+1)(1-10n^2-n^3) &= -(n^6+13n^5+35n^4+53n^3+37n^2-5n-4)\\
(n^3+2n+1)(-(n^3+13n^2+23n+10) &= -(n^6+13n^5+25n^4+37n^3+59n^2+43n+11)\\
\nu &= -(10n^4+16n^3-22n^2+38n+7)\\
&= -(10n^2(n^2-1)+12n^2(n-1)+4n^3+38n+7)\\
&< 0 \qquad \text{for all } n \in \mathbb{N}
\end{aligned}$$

Thus $(u_n)_{n\in\mathbb{N}}$ is monotonically decreasing. Moreover, since $\lim\limits_{n\to\infty} \dfrac{1}{n^k} = 0$ for $k > 0$,

$$\begin{aligned}
u_n &= \frac{n^3+2n+1}{1-10n^2-n^3}\\
&= -\frac{1+\frac{2}{n^2}+\frac{1}{n^3}}{1+10\frac{1}{n}-\frac{1}{n^3}}\\
&\to -1 \quad \text{as } n\to\infty
\end{aligned}$$

(h) $\quad 0 < u_{n+1} := 2^{-(n+1)} = \frac{1}{2}2^{-n} = \frac{1}{2}u_n$

Hence, $(u_n)_{n\in\mathbb{N}}$ is monotonically decreasing.

Moreover, since $2^n > n$ for all $n \in \mathbb{N}$, $0 < u_n < \dfrac{1}{n}$, whence $\lim\limits_{n\to\infty} u_n = 0$.

(i) $\quad \dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)!n^n}{(n+1)^{n+1}n!} = \left(\dfrac{n}{n+1}\right)^n < 1$

Since $u_n > 0$ for $n \in \mathbb{N}^*$, $(u_n)_{n\in\mathbb{N}^*}$ is monotonically decreasing.

Moreover, since $\frac{j}{n}$ <1 for $1 < j < n$,

$$\frac{n!}{n^n} = \frac{n.(n-1)\dots 2.1}{n.n\dots n} < \frac{1}{n},$$

whence $0 < u_n < \dfrac{1}{n}$ and so $\lim\limits_{n\to\infty} u_n = 0$.

Question 2.

Since $(n)_{n\in\mathbb{N}}$ is monotonically increasing, so is $\left(n + \sqrt{\kappa^2 - \frac{\gamma^2}{c^2}}\right)_{n\in\mathbb{N}}$.

Since this is a sequence of positive terms, each of

$$\left(\frac{\frac{\gamma^2}{c^2}}{n+\sqrt{\kappa^2-\frac{\gamma^2}{c^2}}}\right)_{n\in\mathbb{N}}$$

$$\left(1+\frac{\frac{\gamma^2}{c^2}}{n+\sqrt{\kappa^2-\frac{\gamma^2}{c^2}}}\right)_{n\in N}$$

$$\left(\sqrt{1+\frac{\frac{\gamma^2}{c^2}}{n+\sqrt{\kappa^2-\frac{\gamma^2}{c^2}}}}\right)_{n\in\mathbb{N}}$$

is monotonically decreasing. Thus $(E_n)_{n\in\mathbb{N}}$ is monotonically increasing.

Since $n \to \infty$ as $n \to \infty$, we have $\dfrac{\frac{\gamma^2}{c^2}}{n+\sqrt{\kappa^2-\frac{\gamma^2}{c^2}}} \to 0$ and $E_n \to mc^2$ as $n \to \infty$.

Question 3.

Recall that $2^n > n$ for every $n \in \mathbb{N}$,[1] or, equivalently, $0 < 2^{-n} < \dfrac{1}{n}$ for all $n \in \mathbb{N}$.

(a) Take $K \in \mathbb{R}$.
Since $\mathbb{N}$, the set of all natural number, is not bounded above, there is an $N \in \mathbb{N}$ with $N > K$.
If $n \geq N$, then $2^n > n \geq N > K$.
Thus, given $K \in \mathbb{R}$, there is an $N \in \mathbb{N}$ with $2^n > K$ whenever $n \geq N$, proving that

$$2^n \to \infty \quad \text{as} \quad n \to \infty.$$

(b) Take $\varepsilon > 0$. Then $\frac{1}{\varepsilon} > 0$.
Since $\mathbb{N}$ is not bounded above, there is an $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$, or, equivalently, $0 < \dfrac{1}{N} < \dfrac{!}{\varepsilon}$.
If $n \geq N$, then $0 < 2^{-n} < \dfrac{1}{n} \leq \dfrac{1}{N} < \varepsilon$.
Thus, given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ with $|2^{-n} - 0| < \varepsilon$ whenever $n \geq N$, proving that

$$\lim_{x\to\infty} 2^{-n} = 0.$$

Question 4.

Let h_n be the height of the ball after the n^{th} bounce, and d_n the distance travelled by the ball upto the n^{th} bounce.
Then $h_0 = 20$ and $h_{n+1} = \frac{4}{5}h_n$ for every $n \in \mathbb{N}$. Thus $h_n = 20\left(\frac{4}{5}\right)^n$.
In particular, $h_3 = 20\frac{64}{125} = \frac{256}{25}$, so that the height of the third bounce is 10·24 metres.
Now $d_0 = h_0 = 20$ and for $n \in \mathbb{N}$

$$d_{n+1} = d_n + h_n + h_{n+1} = d_n + \frac{9}{5}h_n = d_n + 36\left(\frac{4}{5}\right)^n$$

Thus,

$$d_{n+1} = 36\left(1 + \frac{4}{5} + \ldots + \left(\frac{4}{5}\right)^n\right) = 36\frac{1-(\frac{4}{5})^{n+1}}{1-\frac{4}{5}} = 180\left(1 - (\frac{4}{5})^{n+1}\right)$$

Since $\lim\limits_{n\to\infty} d_n = 180$, the ball travels 180 metres before coming to rest.

[1] We repeat a proof for the benefit of those who have forgotten,
Since $2^0 = 1 > 0$, and $2^1 = 2 > 1$, the proposition is true for $n = 0, 1$.
Suppose that for some $n \in \mathbb{N}^*$, $2^n > n$. Then $2^{n+1} = 2.2^n > 2n \geq n + 1$.
By the Principle of Mathematical Induction, $2^n > n$ for every $n \in \mathbb{N}$.