Sample Solutions for Tutorial 12

Question 1.

(a)
$$\begin{aligned}\begin{vmatrix}\lambda a & \lambda b\\ c & d\end{vmatrix} &:= (\lambda a)d - (\lambda b)c\\ &= \lambda(ad-bc)\\ &=: \lambda\begin{vmatrix}a & b\\ c & d\end{vmatrix}\end{aligned}$$

The second equality follows similarly.

(b)
$$\begin{aligned}\begin{vmatrix}a+e & b+f\\ c & d\end{vmatrix} &:= (a+e)d - (b+f)c\\ &= ad+ed-bc-fc\\ &= (ad-bc)+(ed-fc)\\ &=: \begin{vmatrix}a & b\\ c & d\end{vmatrix} + \begin{vmatrix}e & b\\ c & d\end{vmatrix}\end{aligned}$$

(c)
$$\begin{aligned}\begin{vmatrix}a & \lambda a\\ c & \lambda c\end{vmatrix} &:= a(\lambda c) - (\lambda c)c\\ &= \lambda(ac-ac)\\ &= 0\end{aligned}$$

The second equality follows similarly.

(d)
$$\begin{aligned}\begin{vmatrix}a & c\\ b & d\end{vmatrix} &:= ad-cb\\ &= ad-bc\\ &=: \begin{vmatrix}a & b\\ c & d\end{vmatrix}\end{aligned}$$

Question 2.

(i)
$$\begin{aligned}\begin{vmatrix}1 & 2 & 3\\ 1 & 4 & 5\\ 1 & 1 & 5\end{vmatrix} &= \begin{vmatrix}1 & 2 & 3\\ 0 & 2 & 2\\ 0 & -1 & 2\end{vmatrix} && \begin{matrix}R_2 - R_1\\ R_3 - R_1\end{matrix}\\ &= \begin{vmatrix}1 & 0 & 7\\ 0 & 0 & 6\\ 0 & 1 & -2\end{vmatrix} && \begin{matrix}R_1 + 2R_3\\ R_2 + 2R_3\end{matrix}\\ &= 6\begin{vmatrix}1 & 0 & 7\\ 0 & 0 & 1\\ 0 & -1 & -2\end{vmatrix} && \text{factoring out 6 from } R_2\\ &= 6\begin{vmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{vmatrix} && \begin{matrix}R_1 - 7R_2\\ R_2 + 2R_2\end{matrix}\\ &= 6\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix} && \text{expanding by } R_1\\ &= 6\end{aligned}$$

$$A_{(1)(1)} = \begin{vmatrix} 4 & 5 \\ 1 & 5 \end{vmatrix} = 15 \qquad A_{(1)(2)} = \begin{vmatrix} 1 & 5 \\ 1 & 5 \end{vmatrix} = 0 \qquad A_(1)(3) = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$$

(ii) $$A_{(2)(1)} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7 \qquad A_{(2)(2)} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2 \qquad A_{(2)(3)} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{(3)(1)} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{(3)(2)} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2 \qquad A_{(3)(3)} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$$

Thus

$$\begin{aligned} \operatorname{adj}(A) &= \begin{bmatrix} A_{(1)(1)} & -A_{(2)(1)} & A_{(3)(1)} \\ -A_{(1)(2)} & +A_{(2)(2)} & -A_{(3)(2)} \\ A_{(1)(3)} & -A_{(2)(3)} & A_{(3)(3)} \end{bmatrix} \\ &= \begin{bmatrix} 15 & -7 & -2 \\ 0 & 2 & -2 \\ -3 & 1 & 2 \end{bmatrix} \end{aligned}$$

(iii) $$\begin{aligned} A\operatorname{adj}(A) &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 15 & -7 & -2 \\ 0 & 2 & -2 \\ -3 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 15-9 & -7+4+3 & -2-4+6 \\ 15-15 & -7+8+5 & -2-8+10 \\ 15-15 & -7+2-5 & -2-3-10 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\ &= \det(A)\,\underline{\mathbf{1}}_3 && \text{as } \det(A) = 6 \end{aligned}$$

$$\begin{aligned} \operatorname{adj}(A)\,A &= \begin{bmatrix} 15 & -7 & -2 \\ 0 & 2 & -2 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 15-7-2 & 30-28-2 & 45-35-10 \\ 2-2 & 8-2 & 10-10 \\ -3+1+2 & -6+4+2 & -9+5+10 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \\ &= \det(A)\,\underline{\mathbf{1}}_3 && \text{as } \det(A) = 6 \end{aligned}$$

Question 3. We stick to the Gauß-Jordan Algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 4 & 5 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{c} \rightsquigarrow \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \rightsquigarrow \\ \frac{1}{2} \times R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 - 2R_2 \\ \rightsquigarrow \\ R_3 + R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \rightsquigarrow \\ \frac{1}{3}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{array}\right]$$

$$\begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \rightsquigarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -\frac{7}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & -0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{array}\right]$$

From this we see that the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 1 & 5 \end{bmatrix}$ is the matrix $\dfrac{1}{6}\begin{bmatrix} 15 & -7 & -2 \\ 0 & 2 & -2 \\ -3 & 1 & 2 \end{bmatrix}$.

Question 4.

The line segments OP, OQ, OR “are” the vectors $\mathbf{a} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}, \mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $\mathbf{c} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$ respectively.

The volume of the parallelepiped scanned by them is the absolute value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

Since $(\mathbf{b} \times \mathbf{c}) = (vz - wy)\mathbf{i} - (uy - wx)\mathbf{j} + (uy - vx)\mathbf{k}$,

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= r(vz - wy) - s(uz - wx) + (uy - vx) \\ &= r\begin{vmatrix} v & w \\ y & z \end{vmatrix} - s\begin{vmatrix} u & w \\ x & z \end{vmatrix} + t\begin{vmatrix} u & v \\ x & y \end{vmatrix} \\ &=: \begin{vmatrix} r & s & t \\ u & v & w \\ x & y & z \end{vmatrix} \end{aligned}$$