Sample Solutions for Tutorial 5

Question 1.

(a) Take $u_n = \dfrac{n}{2n+1}$. Then

$$\begin{aligned} u_m - u_n &= \frac{m}{2m+1} - \frac{n}{2n+1} \\ &= \frac{m(2n+1) - n(2m+1)}{(2m+1)(2n+1)} \\ &= \frac{m-n}{(2m+1)(2n+1)} \end{aligned}$$

Since $(2m+1)(2n+1) > 0$, $u_m - u_n > 0$ if and only if $m > n$, showing that $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing.

(b) Take $u_n = \dfrac{2^n}{n^2}$. Then $u_n > 0$ for all $n \in \mathbb{N}^*$, and

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^2}\frac{n^2}{2^n} = \frac{2n^2}{(n+1)^2}$$

Thus $u_{n+1} > u_n$ if and only if $(n+1)^2 < 2n^2$, that is, $n^2 - 2n - 1 > 0$, or $(n-1)^2 - 2 > 0$. This occurs if and only if $n \geq 2$. Thus $(u_n)_{n\in\mathbb{N}}$ is not monotonic. Explicitly,

$$u_1 = 2, \quad u_2 = 1, \quad u_3 = \frac{8}{9}, \quad u_4 = 1, \quad u_5 = \frac{32}{25},$$

so that $u_1 > u_2 > u_3 < u_4 < u_5$.

(c) Take $u_n = a + (n-1)d$ for $a, d \in \mathbb{R}$. [This sequence is an *arithmetic progression* because each term (except the first) is the arithmetic mean of its predecessor and it successor.] Then

$$u_{n+1} - u_n = (a + nd) - (a + (n-1)d) = d.$$

Thus $(u_n)_{n\in\mathbb{N}}$ is monotonic. It is increasing if $d > 0$ and decreasing if $d < 0$.

(d) Take $u_n = ar^{n-1}|$ for $a, r \in \mathbb{R}$ [This sequence is an *geometric progression* because each term (except the first) is the geometric mean of its predecessor and it successor.]

If $a = 0$ or $r = 0$, then $u_n = 0$ for all $n \in \mathbb{N}$.

Otherwise

$$\frac{u_{n+1}}{u_n} = \frac{ar^{n+1}}{ar^n} = r$$

Consequently,

(i) the sequence $(u_n)_{n\in\mathbb{N}}$ alternates if $r < 0$;
(ii) $u_{n+1} > u_n$ if $0 < r < 1$ and $u_n < 0$, that is, $a < 0$ and $0 < r < 1$;
(iii) $u_{n+1} < u_n$ if $0 < r < 1$ and $u_n > 0$, that is, $a > 0$ and $0 < r < 1$;
(iv) $u_{n+1} < u_n$ if $r > 1$ and $u_n < 0$, that is, $a < 0$ and $r > 1$;
(v) $u_{n+1} > u_n$ if $r > 1$ and $u_n > 0$, that is, $a > 0$ and $r > 1$;

Summarising, if $a, r \neq 0$, the sequence $(ar^n)_{n\in\mathbb{N}}$ is

(α) monotonically increasing when either $a < 0$ and $0 < r < 1$, or $a > 0$ and $r > 1$;
(β) monotonically decreasing when either $a > 0$ and $0 < r < 1$, or $a < 0$ and $r > 1$;
(γ) constant if $r = 1$;
(δ) alternating if < 0.

Question 2.

(a) Let $u_n = a + (n-1)d$.

(i) $S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-2)d) + (a+(n-1)d)$
Writing the terms in reverse order, we obtain
$S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + a + d + a$
Adding corresponding terms, we obtain
$2S = (2a+(n-1)d) + (2a+(n-1)d) + \cdots + (2a+(n-1)d) = n(2a+(n-1)d)$, so that

$$S_n = na + \frac{n(n-1)}{2}d$$

(ii) Alternatively, $S_1 = a = 1a + 0d$, anchoring a proof by induction.
Now suppose that $S_n = na + \frac{n(n-1)}{2}d$.
Then

$$\begin{aligned}
S_{n+1} &= S_n + u_{n+1} \\
&= (na + \frac{n(n-1)}{2}d) + (a + nd) \\
&= (n+1)a + \frac{n^2 - n + 2n}{2}d \\
&= (n+1)a + \frac{(n+1)n}{2}d,
\end{aligned}$$

completing the proof by induction.

(b) Let $u_n = ar^n$ with $a, r \in \mathbb{R} \setminus \{0\}$. $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$
Multiplying by r, we obtain
$rS_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$
Hence, $S - rS = a - ar^n$, or $(1-r)S_n = a(1-r^n)$, or $S_n = a\dfrac{1-r^n}{1-r}$ if $r \neq 1$.

Since $n(n-1) \to \infty$ as $\to \infty$, the arithmetic series converges if and only if $a = d = 0$.

As far as the geometric series is concerned, $S_n = a\dfrac{1-r^n}{1-r} = \dfrac{a}{1-r} - \dfrac{ar^n}{1-r}$.

Now $r^n \to 0$ if $|r| < 1$ and $|r^n| \to \infty$ if $|r| > 1$. If $r = 1$, $S_n = na$ is unbounded, whence the series diverges. Finally, if $r = -1$, $Sn = a$ if n is odd, and 0 if n is even, so the series does not converge.

Hence the series $\sum ar^n$ converges if and only if $|r| < 1$.

Question 3. $u_n = \dfrac{1}{n^2}$ $n \geq 1$. Clearly, $u_n > 0$ for every n, so that $(S_n)_{n\in\mathbb{N}^*}$ is monotonically increasing. Moreover,

$$\begin{aligned}
S_n &= 1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \tfrac{1}{6^2} + \tfrac{1}{7^2} + \tfrac{1}{2^2} \cdots + \tfrac{1}{n^2} \\
&= 1 + \tfrac{1}{2^2} + \tfrac{1}{3^2}) + (\tfrac{1}{4^2} + \tfrac{1}{5^2} + \tfrac{1}{6^2} + \tfrac{1}{7^2}) + \cdots + \tfrac{1}{n^2} \\
&\leq 1 + (\tfrac{1}{2^2} + \tfrac{1}{2^2}) + (\tfrac{1}{4^2} + \tfrac{1}{4^2} + \tfrac{1}{4^2} + \tfrac{1}{4^2}) + \cdots + \tfrac{1}{n^2} \\
&= 1 + \tfrac{2}{2^2} + \tfrac{4}{4^2} + \cdots + \tfrac{1}{n^2} \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{n^2} \\
&= 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + \tfrac{1}{n^2}
\end{aligned}$$

We formulate this more precisely.

Since $(n+j)^2 > n^2$ for $j > 0$, $0 < \frac{1}{(n+j)^2} < \frac{1}{n^2}$.

Taking $n = 2^m$ we obtain

$$\tfrac{1}{(2^m)^2} + \tfrac{1}{(2^m+1)^2} + \cdots + \tfrac{1}{(2^m+2^m-1)^2} \leq \tfrac{1}{(2^m)^2} + \cdots \tfrac{1}{(2^m)^2} = 2^m \tfrac{1}{(2^m)^2} = \tfrac{1}{2^m} = (\tfrac{1}{2})^m$$

or, equivalently,

$$\sum_{j=0}^{2^m-1} \frac{1}{(2^m+j)^2} \leq \sum_{j=0}^{2^m-1} \frac{1}{(2^m)^2} = 2^m \frac{1}{(2^m)^2} = \frac{1}{2^m}$$

Thus,

$$\begin{aligned}
S_{2^{m+1}-1} &= \sum_{r=1}^{2^{m+1}-1} \tfrac{1}{r^2} \\
&= \sum_{k=1}^{m} \sum_{j=0}^{2^k-1} \frac{1}{(2^k+j)^2} \\
&\leq \sum_{k=1}^{m} \frac{1}{2^k}
\end{aligned}$$

By the Comparison test, since $\sum \frac{1}{2^k}$, so does $\sum \frac{1}{n^2}$.