Sample Solutions for Tutorial 4

Question 1.

(a) Put $V := \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \leq 0 \text{ for all } x \in \mathbb{R}\} \subseteq \mathcal{F}(\mathbb{R})$.
Then V is not a vector subspace of $\mathcal{F}(\mathbb{R})$ because while

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto -1$$

is clearly an element of V and $-1 \in \mathbb{R}$,

$$(-1).f : \mathbb{R} \longrightarrow \mathbb{R}, \quad \longmapsto (-1).1 = -1$$

is clearly not in V.

(b) Put $V := \{f : \mathbb{R} \to \mathbb{R} \mid f(7) = 0\} \subseteq \mathcal{F}(\mathbb{R})$.
Take $f, g \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha.f + \beta.g)\,(7) := \alpha f(7) + \beta g(7) = \alpha.0 + \beta.0 = 0,$$

showing that $\alpha.f + \beta.g \in V$, making V a vector subspace of $\mathcal{F}(\mathbb{R})$.

(c) Put $V := \{f : \mathbb{R} \to \mathbb{R} \mid f(1) = 2\} \subseteq \mathcal{F}(\mathbb{R})$.
Then V is not a vector subspace of $\mathcal{F}(\mathbb{R})$ because if $f \in V$, then $(f + f)(1) := f(1) + f(1) = 2 + 2 = 4 \neq 2$, so that $f + f \notin V$.

(d) Put $V := \{f : \mathbb{R} \to \mathbb{R} \mid \text{there are } a, b \in \mathbb{R} \text{ with } f(x) = a + b\sin x \text{ for all } x \in \mathbb{R}\} \subseteq \mathcal{F}(\mathbb{R})$.
Take $f, g \in V, \alpha, \beta \in \mathbb{R}$. Then there are $a, b, c, d \in \mathbb{R}$ such that

$$f(x) = a + b\sin x \qquad g(x) = c + d\sin x$$

for all $x \in \mathbb{R}$. Thus, for $x \in \mathbb{R}$

$$\begin{aligned}(\alpha.f + \beta.g)(x) &:= \alpha f(x) + \beta g(x)\\ &= \alpha.(a + b\sin x) + \beta.(c + d\sin x)\\ &= (\alpha a + \beta c) + (\alpha b + \beta d)\sin x.\end{aligned}$$

So, putting $A := \alpha a + \beta c$ and $B := \alpha b + \beta d$, we have $A, B \in \mathbb{R}$ and for all $x \in \mathbb{R}$,

$$(\alpha.f + \beta.g)(x) = A + B\sin x,$$

showing that $\alpha.f + \beta.g \in V$, which makes V a vector subspace of $\mathcal{F}(V)$.

(e) $\mathcal{D}^{(n)}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is } n \text{ times differentiable }\}$.
Let $D^n f$ denote the n^{th} derivative of f. We know from calculus that if $f, g \in \mathcal{D}^n(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$D^n(\alpha.f + \beta.g) = \alpha.D^n f + \beta.D^n g,$$

Thus $\alpha.f + \beta.g$ is n times differentiable, making V a vector subspace of $\mathcal{F}(\mathbb{R})$.

(f) $\mathcal{C}^{(n)}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid \text{ is } n \text{ times continuously differentiable }\}$.
Plainly, $\mathcal{C}^{(n)}(\mathbb{R}) \subset \mathcal{D}^{(n)}(\mathbb{R})$. It is thus sufficient to show that $\mathcal{C}^{(n)}(\mathbb{R})$ is a vector subspace of $\mathcal{D}^{(n)}(\mathbb{R})$. But this is a simple consequence of calculus, where we learnt that a linear combination of continuous functions is itself continuous.

Question 2.

$\mathcal{P}_n$ in a vector subspace of $\mathbb{R}[t]$ since the degree of the sum of two polynomial cannot exceed the maximum of their individual degrees, and multiplying a polynomial by a constant cannot increase its degree.

More formally, take $p, q \in \mathcal{P}_n$ and $\alpha, \beta \in \mathbb{R}$. Then there are $a_j, b_j \in \mathbb{R}$ $(j = 1, \dots, n)$ with

$$p = a_0 + a_1 t + \cdots + a_n t^n \qquad \text{and} \qquad q = b_0 + b_1 t + \cdots + b_n t^n.$$

Then

$$\begin{aligned}\alpha p + \beta q &:= \alpha(a_0 + a_1 t + \cdots + a_n t^n) + \beta(b_0 + b_1 t + \cdots + b_n t^n)\\ &= (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)t + \cdots + (\alpha a_n + \beta b_n)t^n\\ &= c_0 + c_1 t + \cdots + c_n t^n,\end{aligned}$$

where $c_j := \alpha a_j + \beta b_j \ (j = 1, \ldots, n)$.

Question 3.

(a) $\mathbf{M}(2;\mathbb{Z})$ is not a vector subspace of $\mathbf{M}(2;\mathbb{R})$, for $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \in \mathbf{M}(2;\mathbb{Z})$ and $\frac{1}{2} \in \mathbb{R}$, but $\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & 0\end{bmatrix} \notin \mathbf{M}(2;\mathbb{Z})$.

(b) Put $V := \left\{ \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \mathbf{M}(2;\mathbb{R}) \mid a+b+c+d = 0\right\}$.

Taking $\underline{\mathbf{A}}, \underline{\mathbf{B}} \in V$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\underline{\mathbf{A}} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad \text{and} \qquad \underline{\mathbf{B}} = \begin{bmatrix}e & f\\ g & h\end{bmatrix},$$

for some $a, b, c, d, e, f, g, h \in \mathbb{R}$ with $a+b+c+d = e+f+g+h = 0$. So then

$$\begin{aligned}\alpha\underline{\mathbf{A}} + \beta\underline{\mathbf{B}} &= \alpha\begin{bmatrix}a & b\\ c & d\end{bmatrix} + \beta\begin{bmatrix}e & f\\ g & h\end{bmatrix}\\ &= \begin{bmatrix}\alpha a + \beta e & \alpha b + \beta f\\ \alpha c + \beta g & \alpha d + \beta h\end{bmatrix}.\end{aligned}$$

Since

$\alpha a + \beta e + \alpha b + \beta f + \alpha c + \beta g + \alpha d + \beta g = \alpha(a+b+c+d) + \beta(e+f+g+h) = 0, \alpha\underline{\mathbf{A}} + \beta\underline{\mathbf{B}} \in V$, showing that V is a vector subspace of $\mathbf{M}(2;\mathbb{R})$.

(c) Put $V := \{\underline{\mathbf{A}}\mathbf{M}(2;\mathbb{R}) \mid \det(\underline{\mathbf{A}}) = 0\}$.

To see that V is not a vector subspace of $\mathbf{M}(2;\mathbb{R})$, take $\underline{\mathbf{A}} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \in V$ and $\underline{\mathbf{B}} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \in V$. Then

$$\underline{\mathbf{A}} + \underline{\mathbf{B}} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \notin V,$$

since $\det(\underline{\mathbf{A}} + \underline{\mathbf{B}}) = 1$.T

Comment. The tutorial questions have relied on your knowledge of earlier mathematics courses for such things as the properties of continuous and differentiable functions. Thus, you should by now be seeing some of the results you know from calculus in a new light.

But we have also relied on your prior knowledge of matrix algebra and determinants, at least for "small" matrices. In this case, no *understanding* has been presumed, just the ability to carry out the required calculations. During this course, you will be shown the significance of matrix multiplication from a theoretical point of view, you will see how the determinant fits in in general: it is uniquely determined by its characteristic properties!