Chapter 1

Notation, Logic and Proof

1.1 The Greek Alphabet

The Greek alphabet is frequently used in the mathematical sciences. We list its characters and their names for the benefit of those who are not yet familiar with them.

alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ
epsilon	ϵ, ε	E
zeta	ζ	Z
eta	η	H
theta	θ, ϑ	Θ
iota	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	N
xi	ξ	Ξ
omicron	o	O
pi	π, ϖ	Π
rho	ρ, ϱ	P
sigma	σ, ς	Σ
tau	τ	T
upsilon	υ	Υ
phi	ϕ, φ	Φ
chi	χ	X
psi	ψ	Ψ
omega	ω	Ω

1.2 Logic and Proof

Mathematics concerns itself with *statements*, or, more particularly, relationships between statements. Bertand Russell characterised mathematics as being the study of which statements follow logically from which statements. This places *proof* at the centre of mathematics.

We therefore begin with the logic we use. Since a complete, rigorous account would distract us too much from our principal aims, we only provide a brief summary here. A comprehensive account can be found in I.M.Copi's *Symbolic Logic*.

There are several different ways of providing a completely rigorous formulation. We do not provide any of these here, as it beyond the scope of this course.

Definition 1. A *proposition* or *statement* is any sentence about which it can be sensibly said that it is true.

A proposition is *compound* when it contains another proposition. Otherwise it is *simple*.

Example 2. We illustrate the above.

(i) Wash your hands before eating.

This is not a proposition (statement). It is an imperative.

(ii) What time does the next bus to town leave?

This is not a proposition (statement). It is a question.

(iii) Columbus discovered America.

This is a simple proposition.

(iv) All swans are white and some ducks are white.

This is a compound proposition, the *conjunction* of the two simple propositions
All swans are white.
and
Some ducks are white.

(v) The moon is not made of green cheese.

This is a compound proposition, the *negation* of the simple proposition
The moon is made of green cheese.

(vi) This cheese is Australian or it is imported.

This is a compound proposition, the *disjunction* of the two simple propositions
This cheese is Australian.
and
It (this cheese) is imported.

(vii) If woollen clothes are washed in hot water, then they shrink.

This is a compound proposition, the *conditonal* of the two simple propositions, the *antecedent*,
Woollen clothes are washed in hot water.
and the *consequent*,
They (woollen clothes) shrink.

(viii) The integer, n, is even if and only if there is a remainder of 0 on division by 2.

This is a compound proposition, the *biconditonal* of the two simple propositions
The integer, n, is even .
and
There is a remainder of 0 on division by 2..

Notation 3. If P and Q are propositions, we often write

1. $\neg P$ for the negation of P: not P,
2. $P \vee Q$ for the disjunction of P and Q: P or Q,
3. $P \wedge Q$ for the conjunction of P and Q: P and Q,
4. $P \Rightarrow Q$ for the conditional with antecedent P and consequent Q: If P, then Q
5. $P \Leftrightarrow Q$ for the biconditional of P and Q: P if and only if Q

Observation 4. For propositions P and Q, the following are equivalent.

1. "If P, then Q."
2. "P only if Q."
3. "Q whenever P."
4. "P is sufficient for Q."
5. "Q is necessary for P."

Observation 5. For propositions P and Q, the following are equivalent.

1. "P if and only if Q."
2. "If P, then Q, and, if Q, then P."

Using our symbolic notation, $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Observation 6. The proposition P *implies* the proposition Q if and only if the proposition
If P, then Q.
is true.

Definition 7. The propositions P and Q are *logically equivalent* if and only if each implies the other.

Observation 8. The propositions P and Q are logically equivalent if and only if the proposition
P if and only if Q
is true.

Observation 9. The negation of the conditional

If P, then Q.

is the conjunction

P and not Q.

Using our symbolic notation, $\neg(P \Rightarrow Q)$ is $P \wedge (\neg Q)$.

The conditional

If P, then Q.

is equivalent to

If not Q, then not P.

but not to

If Q, then P.

or to

If not P, then not Q.

Definition 10. The *converse* of the conditional

If P, then Q.

is the conditional

If Q, then P.

Observation 11. The reader should take care to avoid some mistakes commonly made when commencing logic. Neither

"If Q, then P."

nor

"If not P, then not Q."

follows from

"If Q, then P."

Propositions can refer to, or make statements about collections of objects. Such cases can be dealt with by means of *predicate* logic, which is the study of propositions of the form $P(x)$, where P is some predicate, and x is an object constrained to some fixed collection. Example 2 (iv) illustrates this. If we constrain x to the collection of all swans, y to the collection of all ducks and let P be the predicate is white, then

(a) All swans are white.

is of the form

For all x, $P(x)$,

where $P(x)$ is x is white. We denoted this symbolically by

$$(\forall x)P(x).$$

$\forall$ is the *universal quantifier.*

(b) Some ducks are white.

is of the form

There is at least one y with $P(y)$.

We denote this symbolically by

$$(\exists y)P(y).$$

$\exists$ is the *existential quantifier.*

Observation 12. We have the following important relations between negation and quantifiers

(i) The negation of, for example,

All swans are white.

is

There is at least one swan which is not white.

In our symbolic notation, $\neg\Big((\forall x)P(x)\Big)$ is $(\exists x)(\neg P(x))$,

(ii) The negation of, for example,

Some ducks are white.

is

All ducks are not white.,

or, less clumsily,

No duck is white.

In our symbolic notation $\neg\Big((\exists x)P(x)\Big)$ is $(\forall x)(\neg P(x))$,

There are several essentially synonymous expressions for mathematical statements, and some very general conventions have emerged.

Convention 13. An *axiom* is a proposition which is taken to be true, without further proof.
A *theorem* is a mathematical proposition of sufficient significance to be highlighted and one which has been proved to be true.
A *lemma* is a preliminary theorem, whose prime application is to assist in the proof of a result considered more significant.
A *corollary* is a theorem, which is an immediate or easy consequence of another theorem.

It should be emphasised that these terms are a matter of convenience and that there is no formal or logical difference between them.

Definition 14. A *proof* is a sequence of propositions, each of which is an axiom, a hypothesis, or follows from previous propositions by the application of one of the rules of inference.

While this states clearer what a proof is, it does not indicate how to actually prove anything. This is because there is no universal method or procedure for producing proofs. It requires effort and experience (practice!) to know how to present a proof. To know what may be assumed, what needs proof, how much detail is called for, requires judgment.
There are, however, several useful strategies. We illustrate three of them.

Example 15 (Direct Proof). A proof is *direct* if the conclusion is deduced from the hypotheses. We provide an example.

Lemma. *If n is a counting number, then $n^2 - n$ is an even number.*
We provide two proofs. The first is by "bull-at-a-gate" computation. The second is not.

First Proof. Let n be a counting number.
Then either $n = 2k$ or $n = 2k - 1$, for some counting number, k, depending on whether n is even or odd.

If $n = 2k$, then $n^2 = 4k^2$ and so

$$\begin{aligned} n^2 - n &= 4k^2 - 2k \\ &= 2(2k^2 - k), \end{aligned}$$

which is an even number.

If $n = 2k - 1$, then $n^2 = 4k^2 - 4k + 1$ and so

$$\begin{aligned} n^2 - n &= 4k^2 - 4k + 1 - (2k - 1) \\ &= 4k^2 - 6k + 2 \\ &= 2(2k^2 - 3k + 1) \end{aligned}$$

which is an even number. □

Second Proof. Let n be a counting number.

Then $n^2 - n = n(n-1)$ is the product of two consecutive integers.

Since one of any two consecutive integers must be even, their product is even. □

Example 16 (Indirect Proof). A proof is *indirect* if it proceeds by deducing a contradiction from assuming that the conclusion is false.

Lemma. *There are infinitely many prime numbers.*

The proof relies on knowing that a prime number is a number, other than ± 1, which has no divisor k, with $1 < k < p$.

Proof. Assume that there are only finitely many prime numbers, say $p_1, \ldots, p_n$.

Consider $N = (p_1 p_2 \cdots p_n) + 1$.

Suppose that for some j, p_j divides N.

Then p_j divides $N - p_1 p_2 \cdots p_n = 1$, which is impossible.

Hence none of $p_1, \ldots, p_n$ divides N.

Clearly, $N > p_j$ for $j = 1, 2, \ldots, n$.

If N is prime, it cannot be one of the p_js as it is larger than all of them.

If N is not prime, it must have a prime factor, p.

By the above, none of the p_js divide N, so p cannot be one of the p_js.

This contradicts the assumption that the list $p_1, \ldots, p_n$ contains all primes. □

Example 17 (Proof by Mathematical Induction). *Mathematical induction* is a method which is frequently used in the following special, but common, situation.

We have a family of propositions, P_n, one for each counting number, n, and we wish to prove that each and every one of the propositions, P_n, is true. The *Principle of Mathematical Induction* provides a strategy.

Principle of Mathematical Induction. *Suppose given propositions $P_1, P_2, \ldots,$ one for each counting number n.*

Suppose that P_1 is true and that P_{k+1} is true whenever P_k is true.

Then P_n is true for every n.

This is applied in two stages.

The first comprises verifying that P_1 is true. This is called *anchoring the induction*[1] .

The second comprises assuming that P_k is true for an arbitrary counting number, k, and then deducing that P_{k+1} must the also be true. This is called the *inductive step*. The assumption that P_k is true is the *inductive hypothesis*.

Lemma. *For every counting number, n,*

$$1 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. Take $n = 1$. Then

$$1 + \cdots + n = 1 \qquad \text{as there is only one summand, } 1.$$

and

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1,$$

showing that P_1 is true.

We make the inductive hypothesis that for some counting number, k, P_k is true, that is,

$$1 + \cdots + k = \frac{k(k+1)}{2}$$

Then

$$\begin{aligned}
1 + \cdots + (k+1) &= (1 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by the inductive hypothesis} \\
&= \frac{(k+1)}{2}(k+2) && \text{taking out the factor } \frac{k+1}{2} \\
&= \frac{(k+1)\big((k+1)+1\big)}{2},
\end{aligned}$$

showing that P_{k+1} is true whenever P_k is true.

Hence, by the Principle of Mathematical Induction, if n is any counting number,

$$1 + \cdots + n = \frac{n(n+1)}{2}.$$

□

The above example provides the opportunity and motivation to introduce Σ *notation*.

Notice that in $1 + \cdots + n$ the dots hint that the missing terms are of the form j for $j = 2, \ldots, n-1$. Then these are added together.

[1]There are variations, such as beginning with a natural number other than 1.

Definition 18. Let n be a counting number. For each $j \in \{1, \ldots, n\}$, let u_j be an expression of a form which can be added. Then

$$\sum_{j=1}^{n} u_j := u_1 + \cdots + u_n.$$

In the last lemma, $u_j = j$ and so the lemma states that for every counting number, n,

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

1.3 Logical Notation

We summarise the logical notation we use, whenever convenient.

$P \Longrightarrow Q$	for	"if P, then Q", or "Q whenever P", or "P only if Q";
$P \Longleftrightarrow Q$	for	"P if and only if Q", that is to say P and Q are logically equivalent;
$P :\Longleftrightarrow Q$	for	"P is defined to be equivalent to Q";
$\neg P$	for	"Not P"
$P \vee Q$	for	"P or Q"
$P \wedge Q$	for	"P and Q
$\forall$	for	"For every ...";
$\exists$	for	"There is at least one ...";
$\exists!$	for	"There is a unique ...", or "There is one and only one ...".
$A := B$	for	"A is defined to be (equal to/the same as) B."

1.4 Exercises

1.4.1. (a) Write down the negation and the converse of the proposition

If a positive integer is divisible by 4 and by 6, then it is divisible by 24.

(b) Write down the negation of the propositions

Every prime number is odd.

Some clever people do dumb things.

1.4.2. Let a be a real number. Prove that, if, for every real number, b,

$$(a+b)^2 = a^2 + b^2,$$

then $a = 0$.

1.4.3. Use induction to prove that for all counting numbers, n,

$$\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + \ldots + n^3 = \Big(\frac{1}{2}n(n+1)\Big)^2.$$

1.4.4. Let a be a positive real number. Prove that, for every counting number, n,

$$(1+a)^n \geq 1 + na.$$

1.4.5. Prove that if n is a counting number, then $3^{2n} - 1$ is divisible by 8.

1.4.6. Given any subset Y of the set X, we write $Y' = X \setminus Y$. Use Venn diagrams to demonstrate that, for all subsets A and B of X,

$$A \setminus B = A \cap B'.$$