AMTH250 Assignment 5 Solutions

September 16, 2011

Question 1

Here is a script to generate the equations in matrix form:

```
k = sqrt(2)/2;

a = zeros(13,13);
b = zeros(13,1);

a(1,2) = 1;
a(1,6) = -1;

a(2,3) = 1;
b(2) = 10;

a(3,1) = k;
a(3,4) = -1;
a(3,5) = -k;

a(4,1) = k;
a(4,3) = 1;
a(4,5) = k;

a(5,4) = 1;
a(5,8) = -1;

a(6,7) = 1;

a(7,5) = k;
a(7,6) = 1;
a(7,9) = -k;
a(7,10) = -1;

a(8,5) = k;
a(8,7) = 1;
a(8,9) = k;
b(8) = 15;
```

```
a(9,10) = 1;
a(9,13) = -1;

a(10,11) = 1;
b(10) = 20;

a(11,8) = 1;
a(11,9) = k;
a(11,12) = -k;

a(12,9) = k;
a(12,11) = 1;
a(12,12) = k;

a(13,13) = 1;
a(13,12) = k;

% Exact solution

x0 =[-20*sqrt(2) 20 10 -30 10*sqrt(2) 20 0 -30 5*sqrt(2) 25 20 -25*sqrt(2) 25]';
```

(a)

Solving in Octave:

```
octave:> x = a\b
x =
  -28.28427
   20.00000
   10.00000
  -30.00000
   14.14214
   20.00000
    0.00000
  -30.00000
    7.07107
   25.00000
   20.00000
  -35.35534
   25.00000
```

The accuracy can be estimated using the condition number of the matrix:

```
octave:> rerr =cond(a)*eps
rerr =  2.3120e-15
```

(b)

The actual error is

```
octave:> err = norm(x-x0)/norm(x0)
err =  1.3820e-16
```

(c)

```
octave:> eig(a)
ans =
  -1.27279 + 0.00000i
  -0.49903 + 1.07138i
  -0.49903 - 1.07138i
  -0.52561 + 0.56675i
  -0.52561 - 0.56675i
  -0.12363 + 0.93904i
  -0.12363 - 0.93904i
   1.14770 + 0.96283i
   1.14770 - 0.96283i
   0.87329 + 0.70838i
   0.87329 - 0.70838i
   0.72297 + 0.00000i
   0.51150 + 0.00000i
```

The matrix `a` has three real eigenvalues The real eigenvalues and their corresponding eigenvectors are:

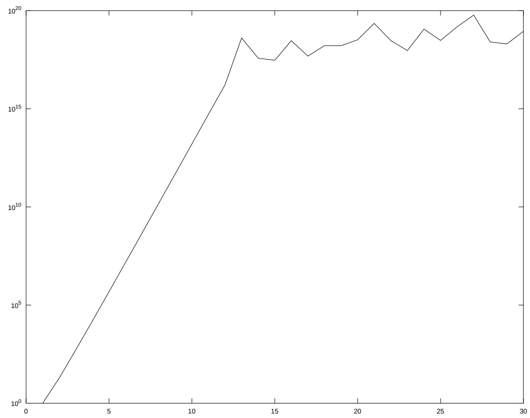
```
octave:> [V L] = eig(a);
octave:> real([L(1,1) L(12,12) L(13,13)])
ans =
  -1.27279   0.72297   0.51150

octave:> real([V(:,1) V(:,12) V(:,13)])
ans =
   0.2707445  -0.2173918  -0.1703240
   0.1057564   0.3980937   0.3917440
  -0.1346055   0.2878090   0.2003763
  -0.2821253  -0.1321592  -0.0946017
   0.4274406  -0.3247553  -0.1814827
   0.4503570   0.5552610   0.4788643
  -0.5732094   0.4014359   0.2449382
   0.2619164   0.1026285  -0.0017736
  -0.0882485  -0.1380302  -0.1661952
   0.0854298   0.1330010   0.3427691
  -0.1087340   0.0961555   0.1753257
   0.0864365  -0.0912040  -0.2955285
  -0.0268920   0.2327925   0.4277776
```

Question 2

(a)

```
octave:> for n = 1:30
>   chilb(n) = cond(hilb(n));
> end
octave:> semilogy(chilb)
```



(b)

This shows that the condition number grows exponentially with n up to about $n = 12$. After that it appears to to be fairly constant, but what has happened is that rounding errors have taken over and the results are unreliable.

Note that the departure from linearity in the graph occurs at a condition number of about $10^{16} \approx 1/\varepsilon_{\text{mach}}$.

(c)

An estimate for relative error in the solution of a system of linear equations is

$$E_{\text{rel}} \approx \operatorname{cond} \mathbf{H}_n \times \varepsilon_{\text{mach}}$$

Therefore to obtain a relative error less than 10^{-4} we require the condition number to be less than

```
octave:20> 1e-4/eps
ans =  4.5036e+11

octave:> chilb(8)
ans =  1.5258e+10
octave:> chilb(9)
ans =  4.9315e+11
```

So $n \leq 8$.

(d)

Here is a script to compute the maximum and minimum eigenvalues:

```
maxeig = zeros(30,1);
mineig = zeros(30,1);
for n = 1:30
  eighilb = abs(eig(hilb(n)));
  maxeig(n) = max(eighilb);
  mineig(n) = min(eighilb);
end
```

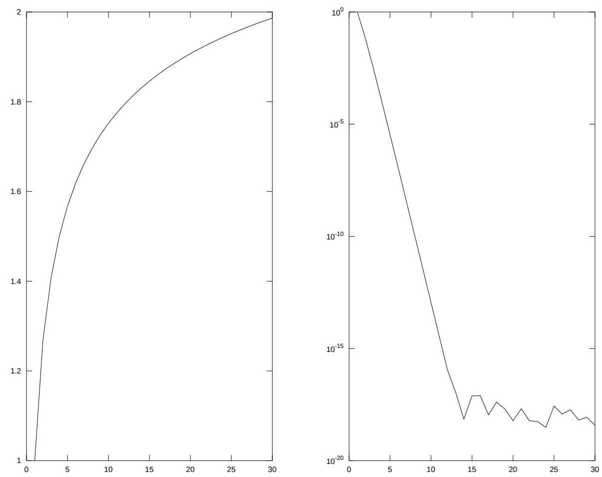


Figure 1: Maximum and minimum eigenvalues of $\mathbf{H}_n$ (n.b. minimum eigenvalues use semilog scale)

The maximum eigenvalue grows slowly with n increasing from 1 at $n = 1$ to just less than 2 at $n = 30$.

The minimum eigenvale decreases exponentially with n up to about $n = 12$. After that it appears to to be fairly constant, but again what has happened is that rounding errors have taken over and the results are unreliable.

The departure from linearity in the graph occurs at a minimum eigenvalue of about $10^{-16} \approx \varepsilon_{\text{mach}}$.

(e)

Experimentation shows that the computed eigenvalues are all positive up to $n = 12$. For $n \geq 13$ the computed eigenvalues contain some negative values.

(f)

```
octave:> [chilb(1:12) maxeig(1:12) mineig(1:12) maxeig(1:12)./mineig(1:12)]
ans =
   1.0000e+00   1.0000e+00   1.0000e+00   1.0000e+00
   1.9281e+01   1.2676e+00   6.5741e-02   1.9281e+01
   5.2406e+02   1.4083e+00   2.6873e-03   5.2406e+02
   1.5514e+04   1.5002e+00   9.6702e-05   1.5514e+04
   4.7661e+05   1.5671e+00   3.2879e-06   4.7661e+05
   1.4951e+07   1.6189e+00   1.0828e-07   1.4951e+07
   4.7537e+08   1.6609e+00   3.4939e-09   4.7537e+08
   1.5258e+10   1.6959e+00   1.1115e-10   1.5258e+10
   4.9315e+11   1.7259e+00   3.4997e-12   4.9315e+11
   1.6025e+13   1.7519e+00   1.0931e-13   1.6027e+13
   5.2153e+14   1.7749e+00   3.4480e-15   5.1476e+14
   1.6620e+16   1.7954e+00   1.0942e-16   1.6408e+16
```

$$\mathrm{cond}(\mathbf{H}_n) = \frac{\lambda_{\max}(\mathbf{H}_n)}{\lambda_{\min}(\mathbf{H}_n)}$$

(g)

Negative eigenvalues occur for $n \geq 13$. The values of the condition number and minimum eigenvalue exhibit a change after $n = 12$. That these changes occurs when the condition number reaches $1/\varepsilon_{\mathrm{mach}}$ and the minimum eigenvalue reaches $\varepsilon_{\mathrm{mach}}$ indicate that rounding error could well be the cause. Therefore it is reasonable to conclude that the results for the condition number and minimum eigenvalue are only reliable up to $n = 12$. There is no indication that the computation of the maximum eigenvalue is unreliable for any of the values of n we have looked at.

Question 3

The formulation as a linear programming problem is:

Variables: A, B, C, D, E — number of units of each product manufactured.

Objective: Maximize the profit:

$$\begin{aligned}
c =& 10A + 9B + 10C + 9.5D + 5E \\
&- \frac{9}{60}(15A + 8B + 8C + 12D + 9E) \\
&- \frac{9}{60}(8A + 10B + 12C + 4D + 4E) \\
&- \frac{12}{60}(6A + 9B + 10C + 12D + 0E) \\
=& 5.35A + 4.5B + 5C + 4.7D + 3.05E
\end{aligned}$$

Constraints:

$$\begin{aligned}
15A + 8B + 8C + 12D + 9E &\leq 4800 \\
8A + 10B + 12C + 4D + 4E &\leq 4800 \\
6A + 9B + 10C + 12D + 0E &\leq 4800
\end{aligned}$$

Lower Bounds:

$$A \geq 0, \quad B \geq 0, \quad C \geq 20, \quad D \geq 30, \quad E \geq 0$$

It is a little unclear whether this a linear programming problem or integer programming problem; i.e. whether fractions of a product can be produced and sold. Thefore we will solve both the LP and IP problems.

Here is the Octave script to solve both problems

```
obj = [5.35 4.5 5 4.7 3.05]';
vtype1 = "CCCCC";
vtype2 = "IIIII";
ptype = -1;
cnstr=[15 8 8 12 9; 8 10 12 4 4; 6 9 10 12 0];
lb = [0 0 20 30 0]';
ub = [];
rhs = [4800 4800 4800]';
ctype = "UUU";

[lx, lopt] = glpk(obj, cnstr, rhs, lb, ub, ctype, vtype1, ptype);
[ix, iopt] = glpk(obj, cnstr, rhs, lb, ub, ctype, vtype2, ptype);
```

The solutions are:

```
octave:> lx, lopt
lx =
     0.00000
     0.00000
   334.88372
   120.93023
    74.41860
lopt =  2469.8

octave:> ix, iopt
ix =
     1
     6
   330
   120
    73
iopt =  2469
```