Tut. 5

Let $f(x,y) = \sqrt{x^2+y^2-1}$ Find f_{xy} and f_{yx} and verify their equality

$$f_x(x,y) = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{x^2+y^2-1}} = \frac{x}{\sqrt{x^2+y^2-1}}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2-1}}$$

$$f_{xy}(x,y) = -\frac{1}{\cancel{2}} \frac{x \times \cancel{2}y}{(x^2+y^2-1)^{3/2}} = -\frac{xy}{(x^2+y^2-1)^{3/2}}$$

$$f_{yx} = -\frac{xy}{(x^2+y^2-1)^{3/2}} = f_{xy}$$

$$f(x,y) = 4x^3y + 3x^2y$$

$$f_x = 12x^2y + 6xy \quad , \quad f_{xy} = 12x^2 + 6x$$

$$f_y = 4x^3 + 3x^2 \quad , \quad f_{yx} = 12x^2 + 6x$$

$$f_{xy} = f_{yx}$$

Let $T = x^2 y - xy^3 + 2 \quad : \mathbb{R}^2 \to \mathbb{R}$

$x = r\cos\theta,\ y = r\sin\theta \quad : \mathbb{R}^2 \to \mathbb{R}^2$

Find $\dfrac{\partial T}{\partial r}, \dfrac{\partial T}{\partial \theta}$

$$\left\langle \frac{\partial T}{\partial r}, \frac{\partial T}{\partial \theta} \right\rangle = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r} =$$

$$= (2xy - y^3)\cos\theta + (x^2 - 3xy^2)\sin\theta =$$

$$(2r^2\sin\theta\cos\theta - r^3\sin^3\theta)\cos\theta + (r^2\cos^2\theta - 3r^3\cos\theta\sin^2\theta)\sin\theta =$$

$$= r^2\sin\theta\cos\theta\,(2\cos\theta - r\sin^2\theta + \cos\theta - 3r\sin^2\theta) =$$

$$= r^2\sin\theta\cos\theta\,(3\cos\theta - 4r\sin^2\theta)$$

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta} =$$

$$= (2xy - y^3)(-r\sin\theta) + (x^2 - 3xy^2)\,r\cos\theta =$$

$$-r\sin\theta(2r^2\sin\theta\cos\theta - r^3\sin^3\theta) + r\cos\theta(r^2\cos^2\theta - 3r^2\cos\theta\sin^2\theta)$$

$$= -2r^3\sin^2\theta\cos^2\theta + r^4\sin^4\theta + r^3\cos^3\theta - 3r^4\cos^2\theta\sin^2\theta$$

Find an equation for the tangent plane and the normal line to the given surface at P

$z = 4x^3y^2 + 2y \qquad P(1, -2, 12)$

$z = f(P) + f_x(x - x_0) + f_y(y - y_0)$

$f(1, -2) = 12$

$f_x = \frac{\partial z}{\partial x} = 12x^2y^2 \qquad f_y = \frac{\partial z}{\partial y} = 8x^3y + 2$

$f_x(1, -2) = 12 \cdot 1 \cdot 4 = 48 \qquad f_y(1, -2, 12) = 8 \cdot 1 \cdot (-2) + 2 = -14$

$z = 12 + 48(x - 1) - 14(y + 2)$

$48x - 14y - z = 64$

$\vec{n} = \langle 48, -14, -1 \rangle$

normal line: $x = 1 + 48t$

$y = -2 - 14t$

$z = 12 - t$

$z = e^{3y} \sin 3x \quad , \quad P\left(\frac{\pi}{6}, 0, 1\right)$

$\frac{\partial z}{\partial x} = 3e^{3y} \cos 3x = 3e^{0} \cos \frac{\pi}{2} = 0$ at P

$\frac{\partial z}{\partial y} = 3e^{3y} \sin 3x = 3e^{0} \sin \frac{\pi}{2} = 3$ at P

$z = 1 + 0 \times (x - x_0) + 3(y - y_0) = 1 + 3y$

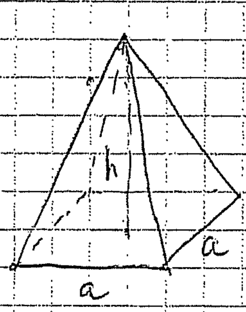
$3y - z = -1$

$\vec{n} = \langle 0, 3, -1 \rangle$

normal line: $x = \frac{\pi}{6}$

$y = 3t$

$z = 1 - t$



Given is a prism.
measure a with error 1%
h with error 2%. ~~to~~ Use differentials to estimate the max error in the calculated volume.

$V = \frac{1}{3}a^2 h. \quad V(a + 0{,}01a, h + 0{,}02h) \approx$

$V(a, h) + f_a(a, h)\Delta a + f_h(a, h)\Delta h =$

$\frac{1}{3}a^2 h + \frac{2}{3}ah \times 0{,}01a + \frac{1}{3}a^2 \times 0{,}02h =$

$\frac{1}{3}a^2 h(1 + 0{,}02 + 0{,}02) = \frac{1}{3}a^2 h(1 + 0{,}04)$

The approximate maximal error is 4%.

Find the directional derivative of f in the direction of $\vec{a}$

$f(x,y) = e^{x}\cos y \quad , \quad P(0, \frac{\pi}{4}) \; , \; \vec{a} = 5\vec{i} - 2\vec{j}$

$\|\vec{a}\| = \sqrt{5^2 + 2^2} = \sqrt{29}$

$= \frac{\vec{a}}{\|\vec{a}\|} = \frac{5}{\sqrt{29}}\vec{i} - \frac{2}{\sqrt{29}}\vec{j}$

$f_x = e^{x}\cos y = e^{0}\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ at P

$f_y = -e^{x}\sin y = -\frac{\sqrt{2}}{2}$ at P

$D_{\vec{a}} f(0, \frac{\pi}{4}) = f_x u_1 + f_y u_2 =$

$= f_x \frac{5}{\sqrt{29}} - f_y \frac{2}{\sqrt{29}} =$

$= \frac{5\sqrt{2}}{2\sqrt{29}} + \frac{2\sqrt{2}}{2\sqrt{29}} = \frac{7\sqrt{2}}{2\sqrt{29}} = \frac{7}{\sqrt{58}}$

Find a unit vector in the direction in which $f(x,y)$ increases (decreases) most rapidly at P and find the rate of change of f at P in that direction.

$f(x,y) = 4x^3y^2$, $P(-1,1)$

$\nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 12x^2y^2, 8x^3y \rangle = \langle 12, -8 \rangle$ at P

f increases most rapidly in the direction $\langle 12, -8 \rangle$

unit vector $\langle \frac{12}{\sqrt{12^2+8^2}}, -\frac{8}{\sqrt{12^2+8^2}} \rangle = \langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \rangle$

the rate of change: $\|\nabla f(-1,1)\| = 4\sqrt{13}$

f decreases most rapidly in $\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$

the rate of change: $-\|\nabla f(-1,1)\| = -4\sqrt{13}$