MATH102 ASSIGNMENT 10

MARK VILLAR

(1)

$$\begin{aligned}
\frac{dy}{dx} = \frac{x^2 y}{1+x^3} \ &\Rightarrow \ \int \frac{1}{y}\ dy = \int \frac{x^2}{1+x^3}\ dx \\
&\Rightarrow \ \log|y| = \frac{1}{3}\log|1+x^3| + c \\
&\Rightarrow |y| = e^{\log|1+x^3|^{\frac{1}{3}}} e^c \\
&\quad \text{Since } e^x > 0 \text{ for all } x \\
&\Rightarrow y = \sqrt[3]{|1+x^3|}\ e^c \\
&\qquad = C\sqrt[3]{|1+x^3|} \ \ (\text{where } C = e^c)
\end{aligned}$$

The zero function $y = 0$ (for all values of x) also satisfies the differential equation trivially so the solutions of the DE are

$$y = C\sqrt[3]{|1+x^3|} \text{ and } y = 0$$

(2)

$$\begin{aligned}
y^3\ \frac{dy}{dx} = \sin 3x \ &\Rightarrow \ \int y^3\ dy = \int \sin 3x\ dx \\
&\Rightarrow \frac{1}{4}y^4 = -\frac{1}{3}\cos 3x + c \\
&\Rightarrow y^4 = 4c - \frac{4}{3}\cos 3x \\
&\Rightarrow y = \pm\sqrt[4]{C - \frac{4}{3}\cos 3x} \ \ (\text{where } C = 4c)
\end{aligned}$$

Since $y = 1$ when $x = 0$

$$\begin{aligned}
&\Rightarrow 1 = \pm\sqrt[4]{C - \frac{4}{3}\cdot 1} \\
&\Rightarrow 1^4 = C - \frac{4}{3} \\
&\Rightarrow C = \frac{7}{3}
\end{aligned}$$

The solution to the initial value problem is therefore

$$\begin{aligned} y &= \pm\sqrt[4]{\frac{7}{3} - \frac{4}{3}\cos 3x} \\ &= \pm\sqrt[4]{\frac{1}{3}\big(7 - 4\cos 3x\big)} \end{aligned}$$

(3)

$$x\frac{dy}{dx} - 2y = x + 1 \ \Rightarrow\ y' - \frac{2}{x}y = 1 + \frac{1}{x}$$

$$\begin{aligned} p(x) = -\frac{2}{x},\ \rho(x) = e^{\int -\frac{2}{x}\ dx} &= e^{-2\log x} \\ = e^{\log x^{-2}} = x^{-2} &= \frac{1}{x^2} \\ (\text{for} \ \ x > 0)& \end{aligned}$$

Multiplying both sides by the integrating factor $\rho(x)$

$$\begin{aligned} & x^{-2}y' - 2x^{-3}y = x^{-2} + x^{-3} \\ & \Rightarrow \big(x^{-2}y\big)' = x^{-2} + x^{-3} \\ & \Rightarrow\ x^{-2}y = \int x^{-2} + x^{-3}\ dx \\ & \qquad\qquad = -\frac{1}{x} - \frac{1}{2x^2} + C \\ & \Rightarrow y = Cx^2 - x - \frac{1}{2} \end{aligned}$$

(4)

$$\frac{dy}{dx} + y - x = 0 \ \Rightarrow\ y' + y = x$$

$$p(x) = 1,\ \rho(x) = e^{\int 1\ dx} = e^x$$

Multiplying both sides by the integrating factor $\rho(x)$

$$\begin{aligned} & e^x y' + e^x y = xe^x \\ & \Rightarrow \big(e^x y\big)' = xe^x \\ & \Rightarrow e^x y = \int xe^x\ dx \\ & \qquad\quad = xe^x - e^x + C \\ & \Rightarrow y = Ce^{-x} + x - 1 \end{aligned}$$

Since $y = 1$ when $x = 0$

$$\begin{aligned} &\Rightarrow 1 = Ce^0 + 0 - 1 \\ &\Rightarrow C = 2 \end{aligned}$$

The solution to the initial value problem is therefore

$$y = 2^{-x} + x - 1$$

(5) If 85% of a fossil's carbon-14 has decayed, then there is 15% remaining. We also know that $k = -\frac{\ln 2}{T} = -\frac{\ln 2}{5750}$ and hence

$$\begin{aligned} y = y_0 e^{-\frac{\ln 2}{5750}t} &\Rightarrow \frac{y}{y_0} = 0.15 = e^{-\frac{\ln 2}{5750}t} \\ \Rightarrow \ln 0.15 &= -\frac{\ln 2}{5750}t \\ \Rightarrow t &= -\frac{\ln 0.15}{\ln 2} \times 5750 \\ &\approx 15738 \text{ years} \end{aligned}$$

(6)

$$\frac{dP}{dt} = 0.0001P\ (10000 - P),\ (P > 0)$$

(a)

$$\int \frac{dP}{P\ (10000 - P)} = \int 0.0001\ dt = 0.0001t + c$$

Using partial fractions,

$$\frac{1}{P\ (10000 - P)} = \frac{a}{P} + \frac{b}{10000 - P}$$

Substituting $P = 0$ shows that $a = \frac{1}{10000}$ while $P = 10000$ shows $b = \frac{1}{10000}$.

$$\begin{aligned} \Rightarrow \int \frac{dP}{P\ (10000 - P)} &= \frac{1}{10000}\left(\int \frac{dP}{P} + \int \frac{dP}{10000 - P}\right) \\ &= \frac{1}{10000}\big(\ln P - \ln(10000 - P)\big) \\ &= \frac{1}{10000}\ln\left(\frac{P}{10000 - P}\right) \end{aligned}$$

$$\Rightarrow \frac{1}{10000} \ln\left(\frac{P}{10000-P}\right) = 0.0001t + c \quad \text{for} \quad 0 < P < 10000$$
$$\Rightarrow \ln\left(\frac{P}{10000-P}\right) = t + 10000c = t + C, \text{ where } C = 10000c$$
$$\Rightarrow \frac{P}{10000-P} = e^t e^C = Ae^t, \text{ where } A = e^C$$

Since $P = 1000$ when $t = 0$

$$\frac{P}{10000-P} = \frac{1000}{10000-1000} = Ae^0 \;\Rightarrow\; A = \frac{1}{9}$$
$$= \frac{1}{9}e^t$$

$$\Rightarrow 9P = (10000-P)e^t = 10000e^t - Pe^t$$
$$\Rightarrow P(9+e^t) = 10000e^t$$
$$\Rightarrow P = \frac{10000e^t}{9+e^t} = \frac{10000}{9e^{-t}+1}$$

(b) $P(5) \;=\; \dfrac{10000}{9e^{-5}+1} \;\approx\; 9428$ fish

(c) $7500 \;=\; \dfrac{10000}{9e^{-t}+1} \;\Rightarrow\; 9e^{-}t + 1 = \dfrac{4}{3}$

$\Rightarrow\; e^{-t} = \dfrac{1}{27} \;\Rightarrow\; -t = -\ln 27 \;\approx\; 3.3$ years