UNIVERSITY OF NEW ENGLAND

UNIT NAME: PMTH 213

PAPER TITLE: Linear Algebra

PAPER NUMBER: First and Only

DATE: **Monday 17 November 2008** **TIME:** **9:30 AM TO 12:30 PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: THREE (3)

NUMBER OF QUESTIONS ON PAPER: FIVE (5)

NUMBER OF QUESTIONS TO BE ANSWERED: FIVE (5)

STATIONERY PER CANDIDATE:

0	6 LEAF A4 BOOKS	**0**	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	**0**	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: NIL

POCKET CALCULATORS PERMITTED: YES **(SILENT TYPE)**

TEXTBOOKS OR NOTES PERMITTED: **FIVE (5) A4 sheets of handwritten double sided notes (10 pages). No photocopies. No printed pages.**

INSTRUCTIONS FOR CANDIDATES:

- Candidates may make notes on this examination question paper during the fifteen minutes reading time
- Answer all FIVE (5) questions
- Questions are not of equal value
- Candidates may retain this examination question paper

Question 1 *[20 marks]*

(a) Consider the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2,\ (x, y, z) \longmapsto (x + y + z, x - z).$$

Find $ker(T)$ and $im(T)$.

(b) Find all linear transformations

$$S : \mathbb{R}^3 \longrightarrow \mathbb{R}^2,\ (x, y, z) \longmapsto (u, v)$$

which map the plane determined by the equation $z = x$ in $\mathbb{R}^3$ onto the line $v = u$ in $\mathbb{R}^2$.

Question 2 *[20 marks]*

(a) *A subset U of a vector space V over the field $\mathbb{F}$ is a vector subspace of V if and only if*

$$\lambda u + \mu v \in U \ \ \forall u, v \in U,\ \forall \lambda, \mu \in \mathbb{F}.$$

Let

$$U = \{A \mid A \text{ is a real } n \times n \text{ matrix satisfying } A^t = A\},$$

where A^t denotes the transpose of A. Use the above statement/definition to show that U is a vector subspace of the vector space of all real $n \times n$ matrices over the field $\mathbb{R}$.

(b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for the vector space V over the field of real numbers, and $T : V \to V$ a linear transformation.

(i) Show that if

$$\mathbf{f}_1 = \mathbf{e}_1,\ \mathbf{f}_2 = \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3,$$

then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is also a basis for V.

(ii) Find the matrix, B, of T with respect to $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, given that its matrix with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the identity matrix.

Question 3 *[20 marks]*

Given the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, find

(a) its eigenvalues,

(b) bases for its eigenspaces,

(c) an orthogonal matrix, P, which diagonalises A,

(d) $P^{-1}AP$,

(e) A^n for an arbitrary positive integer n.

Question 4 *[20 marks]*

(a) Let $C([0,1])$ be the real vector space of all continuous functions from $[0,1]$ to $\mathbb{R}$. Show that

$$\langle p, q \rangle := \int_0^1 2p(t)q(t)dt$$

defines an inner product in $C([0,1])$.

(b) *Let $A = [a_{ij}]_{n\times n}$ be a symmetric real $n \times n$ matrix. The real quadratic form*

$$Q(x_1, ..., x_n) := \sum_{i,j=1}^{n} a_{ij}x_ix_j = (x_1, ..., x_n)A(x_1, ..., x_n)^t$$

is positive definite if and only if all the eigenvalues of A are positive.

Use this statement or otherwise to determine whether

$$Q_0(x, y) = x^2 + 2y^2 + xy$$

is positive definite.

Question 5 *[20 marks]*

(a) Find all 2×2 real matrices such that $A^2 = I$, where I stands for the 2×2 identity matrix.

(b) Prove that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism if and only if $\{T(\mathbf{e}_i) \mid j = 1, ..., n\}$ is a basis for $\mathbb{R}^n$ whenever $\{\mathbf{e}_j \mid j = 1, ..., n\}$ is a basis for $\mathbb{R}^n$.