PURE MATHEMATICS 212

Multivariable Calculus

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PMTH212, Multivariable Calculus

Assignment Summary – 2012

For External Students

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This assignment summary may be detached for your notice board.

PURE MATHEMATICS 212

Multivariable Calculus

Summary

Please refer to the online resources at http://moodle.une.edu.au and

http://turing.une.edu.au/ pmth212 for detailed information.

Lecture Notes and Reference Book

Lecture notes will be provided.

The reference book is *Calculus Late Transcendentals Combined, Ninth Edition* by **H. Anton, I. Bivens, S. Davis**, Wiley, ISBN-13 978-0470183496. You may also use older editions.

Assessment

The 11 assignments contribute 30% of the final assessment. You must submit reasonable attempts on at least 8 assignments.

The final examination is compulsory. It contributes 70% of the final assessment. You must achieve at least 40% in the examination.

Examination

The examination will be of three hours duration. You are allowed to bring with you three A4 sized pieces of paper of hand-written notes. Both sides can be used. No printed and photocopied materials are allowed.

Unit Coordinator

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ASSIGNMENT 1

1. Find the initial point of $\mathbf{v}$=$\mathbf{i}$+2$\mathbf{j}$−3$\mathbf{k}$ if the terminal point is $(-2, 1, 4)$.

2. Find the distance from the point $(4, 5, 3)$ to the

(a) xz-plane (b) y-axis.

3. Find the area of the parallelogram spanned by the vectors $\mathbf{v} = \langle 1, 1, -3\rangle$, and $\mathbf{w} = \langle -1, 2, -1\rangle$.

4. Find a vector $\mathbf{n}$ which is perpendicular to the plane determined by the points

$A(0, -2, 1)$, $B(1, -1, -2)$, and $C(-1, 1, 0)$.

5. Find parametric equations for the line in $\mathbb{R}^2$ through (1,1) and parallel to the line $x = -5 + t,\ y = 1 - 2t$;

6. Determine whether the planes are perpendicular.

(a) $x - 3y - 2z - 2 = 0,\ 2x + z = 1$

(b) $3x - 2y + z = 1,\ 4x + 5y + 2z = 4$.

7. Find equations of the planes.

(a) Through $(-1, 4, -3)$ and perpendicular to the line

$$x - 2 = t, \quad y + 3 = 2t, \quad z = -t$$

(b) Through $(-1, 2, -5)$ and perpendicular to the planes $2x - y + z = 1$ and $x + y - 2z = 3$.

ASSIGNMENT 2

1. Name and sketch the surface: $z = 4x^2 + y^2 + 8x - 2y + 4z$

2. Find a parametric equation of the curve of intersection of the paraboloid $z = 4 - x^2 - y^2$ and the parabolic cylinder $z = 3y^2$ (Hint. Use the parametrization $y = \sin t$).
Compute the orthogonal projection of this curve to the xy-plane.

3. The curves below are given by their vector equations. Describe them in Cartesian coordinates. What are their geometric names?

(a) $\mathbf{r}{=}(3\sin 2t)\,\mathbf{i} + (3\cos 2t)\,\mathbf{j}$. (b) $\mathbf{r} = -2\,\mathbf{i} + t\,\mathbf{j} + (t^2 - 1)\,\mathbf{k}$.

4. For which values of the parameter t is the curve

$$\mathbf{r} = t\cos(t^2)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k}$$

smooth?

5. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be differentiable vector-valued functions of t. Prove that

$$\frac{d}{dt}[\mathbf{u} \cdot [\mathbf{v} \times \mathbf{w})] = \frac{d\mathbf{u}}{dt} \cdot [\mathbf{v} \times \mathbf{w}] + \mathbf{u} \cdot [\frac{d\mathbf{v}}{dt} \times \mathbf{w}] + \mathbf{u} \cdot [\mathbf{v} \times \frac{d\mathbf{w}}{dt}]$$

6. (a) Evaluate $\int [(t\sin t)\,\mathbf{i} + \log t\,\mathbf{j}]\,dt$;

(b) Find the arc length of the curve $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k};\ 0 \le t \le 2\pi$.

ASSIGNMENT 3

1. Find the unit tangent vector **T** and the principal unit normal vector **N** for the given value of t: $x = \cosh t,\ y = \sinh t,\ z = t;\ t = \ln 2$.

2. Find the curvature at the indicated point: $x = e^t, y = e^{-t}, z = t;\ t = 0$.

3. Show that for a plane curve described by $y = f(x)$ the curvature $\kappa(x)$ is

$$\kappa(x) = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$

[Hint: Let x be the parameter so that $\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j}$.]

4. Let $f(x, y) = x + (xy)^{1/3}$. Find
(a) $f(t, t^2)$, (b) $f(x, x^2)$, (c) $f(2y^2, 4y)$.

5. Find $g(u(x, y), v(x, y))$ if $g(x, y) = y \sin(x^2 y),\ u(x, y) = x^2 y^3,\ v(x, y) = \pi xy$.

6. Let $f(x, y, z) = zxy + x$. Find
(a) $f(x + y, x - y, x^2)$, (b) $f(xy, y/x, xz)$.

7. Determine the level surfaces passing through the point $(1, 1, 1)$.
(a) $f(x, y, z) = 3x - y + 2z$ (b) $f(x, y, z) = z - x^2 - y^2$.

ASSIGNMENT 4

1. Find the natural domains of the functions below. Determine where the function f is continuous. Justify your asnwer using the properties of continuous functions.

(a) $f(x,y) = (x-y)^{1/2}$.

(b) $f(x,y) = \cos(\dfrac{xy}{1+x^2+y^2})$.

2. Find the limit, if it exists.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+8y^3}{x+2y}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\ln(1+x^2+y^2)}{x^2+y^2}$.

3. (a) Show that the value of $\dfrac{x^3y}{2x^6+y^2}$ approaches 0 as $(x,y)\to(0,0)$ along any straight line $y=mx$, or along any parabola $y=kx^2$.

(b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6+y^2}$ does not exist.
[Hint: Let $(x,y)\to(0,0)$ along the curve $y=x^3$, and then compare the result with the results in (a).]

4. Find $f_x,\ f_y$ and f_z: $f(x,y,z) = z\ln(x^2y\cos z)$.

5. (a) Show that $f(x,y) = \ln(x^2+y^2)$ satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

This is called the **Laplace's Equation**.

(b) Show that $u(x,y) = e^x\cos y$ and $v(x,y) = e^x\sin y$ satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are called the **Cauchy-Riemann Equations**.

6. Let $f(x,y) = (x^2+y^2)^{2/3}$. Show that $f_x(0,0)=0$. [Hint. Use the definition $f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$.]

ASSIGNMENT 5

1 Let $f(x,y) = e^{xy^2}$. Find f_{xyx}, f_{xxy} and f_{yxx} and verify their equality.

2. Let $z = 3x - 2y$, where $x = v \ln u$, $y = u^2 - v \ln v$. Find $\dfrac{\partial z}{\partial v}$ and $\dfrac{\partial z}{\partial u}$ by the chain rule.

3. Show that if $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations (See Assignment 4, Question 5), and if $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

4. Find equations for the tangent plane and normal line to the surface given by $z = \mathrm{e}^{x^2+y^2}$ at the point $P(0,0,1)$.

5. Find a unit vector $\mathbf{u}$ that is perpendicular at $P(2,-3)$ to the level curve of $f(x,y) = 3x^2y - xy$ through P.

6. Find a unit vector in the direction in which $f(x,y,z) = (x - 3y + 4z)^{1/2}$ increases most rapidly at $P(0,-3,0)$, and find the rate of increase of f in that direction.

7. Let $G(u,v) = \int_u^v f(t)\,dt$. Show that $\dfrac{\partial G}{\partial u} = -f(u)$, $\dfrac{\partial G}{\partial v} = f(v)$;

ASSIGNMENT 6

1. Write down the second order Taylor polynomial of $f(x,y) = \mathrm{e}^{\sin x + y}$ at $P(0,0)$.

2. Locate all relative maxima, relative minima and saddle points: $f(x,y) = x^2 + xy - 2y - 2x + 1$.

3. Find the absolute maximum and minimum of $f(x,y) = xy - 2x$ on the the triangle with vertices (0,0), (0,4) and (4,0).

4. Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 - y$ subject to the constraint: $x^2 + y^2 = 25$.

5. Find the point on the line $x + 2y = 5$ that is closest to the origin.

ASSIGNMENT 7

1. Evaluate the iterated integral: $\int_0^1 \int_0^x e^{x^2}\, dy\, dx$.

2. Evaluate the double integral.

   (a) $\iint_R (x+y)\, dA$; R is the region enclosed between the curves $y = x^2$ and $y = x^{1/2}$.

   (b) $\iint_R x \cos y\, dA$; R is the triangular region bounded by y=x, y=0 and x=π.

3. Evaluate the integral by first reversing the order of integration.

   (a) $\int_0^2 \int_{y/2}^1 \cos(x^2)\, dx\, dy$,

   (b) $\int_1^3 \int_0^{lnx} x\, dy\, dx$.

4. (a) Use polar coordinates to evaluate $\iint_R (9 - x^2 - y^2)^{1/2}\, dA$, where R is the region in the first quadrant within the circle $x^2 + y^2 = 9$.

   (b) Evaluate the iterated integral by converting to polar coordinates.

$$\int_{-2}^2 \int_{-(4-y^2)^{1/2}}^{(4-y^2)^{1/2}} e^{-(x^2+y^2)}\, dx\, dy.$$

5. Find the surface area of the portion of the plane $2x + 2y + z = 8$ in the first octant that is cut off by the three coordinate planes.

ASSIGNMENT 8

1. Evaluate the triple integral $\iiint_R \cos(z/y)\, dV$, where R is defined by

$$\pi/6 \leq y \leq \pi/2, \quad y \leq x \leq \pi/2, \quad 0 \leq z \leq xy.$$

2. Express the integral as an equivalent integral in which the z-integration is performed first, the y-integration second and the x-integration last:

$$\int_0^3 \int_0^{(9-z^2)^{1/2}} \int_0^{(9-y^2-z^2)^{1/2}} f(x,y,z)\, dx\, dy\, dz.$$

3. Let G be the rectangular box defined by $a \leq x \leq b, c \leq y \leq d, k \leq z \leq l$. Show that

$$\iiint_G f(x)g(y)h(z)\, dV = [\int_a^b f(x)\, dx][\int_c^d g(y)\, dy][\int_k^l h(z)\, dz].$$

4. (a) Use cylindrical coordinates to find the volume of the solid bounded above and below by the sphere $x^2 + y^2 + z^2 = 9$ and laterally by the cylinder $x^2 + y^2 = 4$.

(b) Use spherical coordinates to find the volume of the solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$.

ASSIGNMENT 9

1. Let **F**=(3x+2y)**i**+(2x-y)**j**. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is

    (a) the line segment from (0,0) to (1,1);

    (b) the curve $y = x^2$ from (0,0) to (1,1)

2. Find the work done by a force **F**(x,y,z)=xy**i**+yz**j**+xz**k** acting on a particle that moves along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \leq t \leq 1$.

3. Show that $\displaystyle\int_{(0,1)}^{(\pi,-1)} y \sin x \, dx - \cos x \, dy$ is independent of the path, and evaluate the integral by

    (a) using the fundamental theorem of line integrals;

    (b) integrating along the line segment from (0,1) to $(\pi, -1)$.

4. Show that the integral $\displaystyle\int_{(0,0)}^{(1,\pi/2)} e^x \sin y \, dx + e^x \cos y \, dy$ is independent of the path, and find its value by any method.

5. Use Green's Theorem to calculate $\displaystyle\int_C (e^x + y^2) dx + (e^y + x^2) \, dy$, where C is the boundary of the region between $y = x^2$ and $y = x$ oriented counterclockwise.

ASSIGNMENT 10

1. Evaluate the surface integral $\iint_\sigma xyz\,dS$ where σ is the portion of the plane x+y+z=1 lying in the first octant.

2. Evaluate the surface integral $\iint_\sigma (x^2+y^2)\,dS$, where σ is the portion of the sphere $x^2+y^2+z^2=4$ above the plane $z=1$.

3. Evaluate $\displaystyle\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS$ where

$\mathbf{F}(x,y,z)=(x+y)\mathbf{i}+(y+z)\mathbf{j}+(z+x)\mathbf{k}$ and σ is the portion of the plane $x+y+z=1$ in the first octant, oriented by unit normals with positive components.

4. Use the Divergence Theorem to evaluate $\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ is the outer unit normal to σ.

(a) $\mathbf{F}(x,y,z)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$, σ is the sphere $x^2+y^2+z^2=9$.

(b) $\mathbf{F}(x,y,z)=z^3\mathbf{i}-x^3\mathbf{j}+y^3\mathbf{k}$, σ is the sphere $x^2+y^2+z^2=a^2$.

ASSIGNMENT 11

1. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$\mathbf{F}(x, y, z) = -3y^2\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$, C is the triangle in the plane $z = \frac{1}{2}y$ with vertices $(2, 0, 0)$, $(0, 2, 1)$ and $(0, 0, 0)$, with a counterclockwise orientation looking down the positive z-axis.

2. Show that if the components of $\mathbf{F}$ and their first- and second-order partial derivatives are continuous, then $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

3. Find the net volume of the fluid that crosses the surface σ in the direction of the orientation in one unit of time where the flow field is given by $\mathbf{F}(x, y, z) = x^2\mathbf{i} + yx\mathbf{j} + zx\mathbf{k}$, σ is the portion of the plane 6x+3y+2z=6 in the first octant oriented by upward unit normals.