PMTH212 ASSIGNMENT 5

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(1) $f(x,y) = e^{xy^2}$

$$
\begin{aligned}
f_x &= \frac{\partial}{\partial x}(e^{xy^2}) = y^2 e^{xy^2} \\
f_{xy} &= \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(y^2 e^{xy^2}) = 2ye^{xy^2} + 2xy^3 e^{xy^2} = 2ye^{xy^2}(1+xy^2) \\
f_{xyx} &= \frac{\partial}{\partial x}(f_{xy}) = \frac{\partial}{\partial x}(2ye^{xy^2}(1+xy^2)) = 2y^3 e^{xy^2} + 2y^3 e^{xy^2} + 2xy^5 e^{xy^2} \\
&= 4y^3 e^{xy^2} + 2xy^5 e^{xy^2} = 2y^3 e^{xy^2}(2+xy^2) \\
f_{xx} &= \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(y^2 e^{xy^2}) = y^4 e^{xy^2} \\
f_{xxy} &= \frac{\partial}{\partial y}(f_{xx}) = \frac{\partial}{\partial y}(y^4 e^{xy^2}) = 4y^3 e^{xy^2} + 2xy^5 e^{xy^2} = 2y^3 e^{xy^2}(2+xy^2) \\
f_y &= \frac{\partial}{\partial y}(e^{xy^2}) = 2xye^{xy^2} \\
f_{yx} &= \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2xye^{xy^2}) = 2ye^{xy^2} + 2xy^3 e^{xy^2} = 2ye^{xy^2}(1+xy^2) = f_{xy} \\
f_{yxx} &= \frac{\partial}{\partial x}(f_{yx}) = \frac{\partial}{\partial x}(f_{xy}) = 2y^3 e^{xy^2}(2+xy^2)
\end{aligned}
$$

Hence, $f_{xyx} = f_{xxy} = f_{yxx} = 2y^3 e^{xy^2}(2+xy^2)$.

(2)

$$
\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= 3\left(1 + \frac{v}{u}\right) + (-2)(2u) = 3 + \frac{3v}{u} - 4u \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= 3\ln u + (-2)(-\ln v - 1) = 3\ln u + 2\ln v + 2
\end{aligned}
$$

(3) From the Cauchy-Riemann equations, we know that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ such that

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial y}
$$

$$r = \frac{x}{\cos\theta}, \qquad \frac{\partial r}{\partial x} = \frac{1}{\cos\theta}$$

$$\theta = \sin^{-1}\left(\frac{y}{r}\right), \qquad \frac{\partial\theta}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}} = \frac{1}{r\sqrt{1-\sin^2\theta}} = \frac{1}{r\cos\theta}$$

Hence,

$$\frac{\partial u}{\partial r}\frac{1}{\cos\theta} = \frac{\partial v}{\partial\theta}\frac{1}{r\cos\theta}$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$$

We also know that $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ such that

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} = -\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} = -\frac{\partial v}{\partial x}$$

Hence,

$$\frac{\partial u}{\partial\theta}\frac{1}{r\cos\theta} = -\frac{\partial v}{\partial r}\frac{1}{\cos\theta}$$
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}$$

(4) $z = f(x,y) = \ln\left[(x^2+y^2)^{1/2}\right]$ at $P(-1,0,0)$

$$f(-1,0) = \ln\left[(1+0)^{1/2}\right] = \ln 1 = 0$$
$$f(x,0) = \ln\left[(x^2+0)^{1/2}\right] = \ln x$$
$$f_x(x,0) = \frac{1}{x}$$
$$f_x(-1,0) = \frac{1}{x}\Big|_{(-1,0)} = -1$$
$$f(-1,y) = \ln\left[(1+y^2)^{1/2}\right]$$
$$f_y(-1,y) = \frac{y}{1+y^2}$$
$$f_y(-1,0) = \frac{y}{1+y^2}\Big|_{(-1,0)} = 0$$

The equation of the tangent plane to the given surface is therefore

$$z = 0 - 1(x+1) + 0(y-0) \;\Rightarrow\; x + z + 1 = 0$$

Since $\vec{n} = \langle -1, 0, -1 \rangle$, the equation of the normal line at $(-1, 0, 0)$ is

$$\begin{aligned} &x + 1 = -t \;\Rightarrow x = -t - 1 \\ &y - 0 = 0 \;\Rightarrow y = 0 \\ &z - 0 = -t \;\Rightarrow z = -t \\ &-x - 1 = -z \;\Rightarrow\; x - z + 1 = 0 \end{aligned}$$

(5) $f(x, y) = 3x^2y - xy$ at $P(2, -3)$

$$\begin{aligned} \bigtriangledown f(2, -3) &= \langle 6xy - y, 3x^2 - x \rangle = \langle -36 + 3, 12 - 2 \rangle \\ &= \langle -33, 10 \rangle \text{ is perpendicular to the level curve at } (2, -3) \\ || \bigtriangledown f(2, -3)|| &= \sqrt{(-33)^2 + 10^2} = \sqrt{1189} \end{aligned}$$

$$\mathbf{u} = \frac{\bigtriangledown f(2, -3)}{|| \bigtriangledown f(2, -3)||} = \left\langle -\frac{33}{\sqrt{1189}}, \frac{10}{\sqrt{1189}} \right\rangle \text{ is the normalised unit vector}$$