Sample Solutions for Tutorial 9

Question 1.

(i) Since $\dfrac{1}{\sin x} - \dfrac{1}{x} = \dfrac{x - \sin x}{x \sin x}$ for $0 < |x| < \pi$,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0}\frac{x - \sin x}{x\sin x} \\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} && \text{by the Bernoulli-de l'Hôpital Rule} \\
&= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} && \text{by the Bernoulli-de l'Hôpital Rule} \\
&= 0
\end{aligned}$$

(ii) Given $u > 0$, $u = e^{\ln u}$. Since both the exponential and natural logarithm functions are continuous.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = a \quad \text{if and only if} \quad \lim_{x\to\infty}\frac{\ln x}{x} = \ln a$$

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln x}{x} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} && \text{by the Bernoulli-de l'Hôpital Rule} \\
&= \lim_{x\to\infty}\frac{1}{x} \\
&= 0
\end{aligned}$$

Thus, $\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$.

(iii) Since $\dfrac{x + \sin x}{x} = 1 + \dfrac{\sin x}{x}$ and $\lim_{x\to\infty}\dfrac{\sin x}{x} = 0$,

$$\lim_{x\to\infty}\frac{x+\sin x}{x} = 1$$

iv Since $\dfrac{d}{dx}(x + \sin x) = 1 + \cos x$ and $\dfrac{d}{dx}x = 1$.

$$\lim_{x\to\infty}\frac{\frac{d}{dx}(x+\sin x)}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{1+\cos x}{1} = 1 + \lim_{x\to\infty}\cos x$$

and there is no such limit, as $\cos x$ is bounded, but does not converge as $x \to \infty$.

Question 2.

Let the internal radius of the cylindrical tank be r metres. Then its volume is $V = 10\pi r^2$ cubic metres. Let the error in the measure of the radius be δr and the error in the volume be δV. Then, for small errors,

$$\delta V \approx \frac{dV}{dr}\delta r = 20\pi r\delta r,$$

so that the relative error is

$$\frac{\delta V}{V} \approx \frac{20\pi r\delta r}{10\pi r^2} = \frac{2\delta r}{r}$$

Thus, to ensure that $\dfrac{\delta V}{V} < 1\% = 0.01$, we must have $\dfrac{2\delta r}{r} < 0.01$ so that we must measure the internal radius within 0.5% of the correct value, that is, to the nearest 5 cm.

Question 3.

Given $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{-x}\cos x$, its domain contains no boundary points and

$$\begin{aligned}
f'(x) &= -e^{-x}\cos x + e^{-x}(-\sin x) \\
&= -e^{-x}(\cos x + \sin x) \\
&= -e^{-x} 2 \sin\tfrac{\pi}{4}\cos(x - \tfrac{\pi}{4}) \\
&\begin{cases} < 0 & \text{if } \frac{(8m-1)\pi}{4} < x < \frac{(8m+3)\pi}{4} \\ = 0 & \text{if } x = \frac{(4n+3)\pi}{4} \\ > 0 & \text{if } \frac{(8m+3)\pi}{4} < x < \frac{(8m+7)\pi}{4} \end{cases} \qquad (m, n \in \mathbb{Z})
\end{aligned}$$

Thus f is differentiable everywhere.
Hence the extrema occur at the points where the derivative is 0.
It is monotonically decreasing on the intervals $[\frac{(8m-1)\pi}{4} < \frac{(8m+3)\pi}{4}]$ and monotonically increasing on the intervals $[\frac{(8m+3)\pi}{4} < \frac{(8m+7)\pi}{4}] \ \ (m \in \mathbb{Z})$

$$\begin{aligned}
f''(x) &= \frac{d}{dx}\bigl(-e^{-x}(\cos x + \sin x)\bigr) \\
&= e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x) \\
&= 2e^{-x}\sin x \\
&\begin{cases} > 0 & \text{if } 2m\pi < x < (2m+1)\pi \\ = 0 & \text{if } x = n\pi \\ < 0 & \text{if } (2m-1)\pi < x < 2m\pi \end{cases} \qquad (m, n \in \mathbb{Z})
\end{aligned}$$

Hence f is concave up on $[2m\pi, (2m+1)\pi]$ and concave down on $[(2m-1)\pi, 2m\pi]$ with a local maximum at $\frac{(8n-1)\pi}{4}$ and a local minimum at $\frac{(8n+3)\pi}{4} \ \ (m \in \mathbb{Z})$.

The graph of f on $[0, 4\pi]$ is