Chapter 4

Geometric Interpretation

Our definitions are abstract because this level of abstractness has several distinct advantages.

(i) It allows us to isolate the essence of the matter.

(ii) It broadens the applicability of the theory we develop.

(iii) It makes proofs of general results simpler, more elegant and more transparent, even if this might not be apparent upon first encounter. For the abstractness forces us to use only general concepts rather than special tricks tailored to specific examples.

Nevertheless, this abstractness can be daunting, if one is unaccustomed to abstract methods.

It is therefore important to have a few standard examples, or common applications, both as a guide and as a warning: These examples offer concrete illustrations of the ideas investigated, show some of the difficulties which can arise and display what can "go wrong".

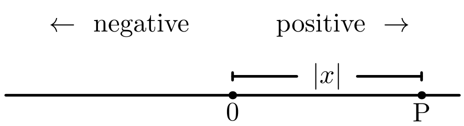
In the case of linear algebra, the oldest applications are to geometry and to physical situations, as the term *vector space* attests. More recent applications include number theory, statistics, economics and computer science.

In order to master the concepts in this course, it is important to bear these examples in mind, together with the caveat that examples often exhibit special features not shared by other examples.

Descartes introduced what we now call *Cartesian co-ordinates* to study geometry. (These are typically introduced in secondary school mathematics.)

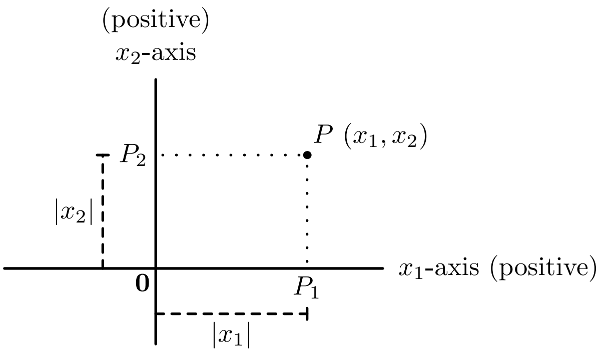
To study the geometry of the line, we draw a line, ℓ, and choose a fixed point $\mathbf{0}$ on it. We choose one side of the line (one direction) from $\mathbf{0}$ and call it *positive.* The other side (opposite direction) is called *negative.* Finally, the point P on the line ℓ is assigned the real number, x, as its *co-ordinate* if the distance from $\mathbf{0}$ to P is $|x|$, with x positive or negative according to whether P is on the positive or negative side of $\mathbf{0}$.

Assigning each point P of the line ℓ its co-ordinate x defines a bijection between ℓ and $\mathbb{R}$.



For plane geometry and spatial geometry, we need (precisely) two and three co-ordinates respectively. So we take two (resp. three) mutually perpendicular, concurrent lines in the plane (resp. in space). These are the *x_1- and x_2–axes* (resp. *x_1-, x_2- and x_3-axes*). Their point of intersection is called the *origin*, which we denote by $\mathbf{0}$. We choose *positive* and *negative* directions for each co-ordinate axis.

To each point, P, in the plane (resp. in space) we assign an *ordered pair* (resp. *ordered triple*) of real numbers, (x_1, x_2) (resp. (x_1, x_2, x_3)), called the *co-ordinates* of P. The i-th co-ordinate, x_i is obtained by taking the through P perpendicular to the i-th co-ordinate axis and finding P_i, the point of intersection with the i-th co-ordinate axis. Then the distance of P_i form $\mathbf{0}$ is $|x_i|$, with x_i positive or negative according to whether $P_i$is on the positive or negative side of the i-th co-ordinate axis. Clearly $\mathbf{0}$ has co-ordinates $(0,0)$ (resp. $(0,0,0)$.



Assigning each point P in the plane its co-ordinate pair (x_1, x_2) defines a bijection between the plane and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and assigning each point P in space its co-ordinate triple (x_1, x_2, x_3) defines a bijection between space and $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

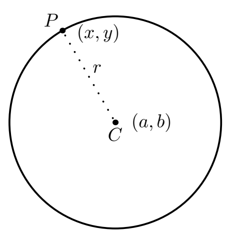
Observation 4.1. We often write (x, y) instead of (x_1, x_2) and (x, y, z) in place of (x_1, x_2, x_3) when dealing with plane geometry or spatial geometry.

The introduction of co-ordinates means that relations between points can be translated into relations between their co-ordinates.

For example if the points in space P and Q have co-ordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) respectively then the distance, d, between P and Q is given by

$$d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = \sqrt{\sum_{j=1}^{3} (x_j - y_j)^2}.$$

If C is the point in the plane with co-ordinates (a, b), then the circle of radius r with centre C comprises all points P whose co-ordinates (x, y) satisfy the equation $(x - a)^2 + (y - b)^2 = r^2$.



Conversely, this also means that we can give equations in two variables a geometric interpretation. Thus, given any three real numbers a, b, c with either a or b non-zero, the set of all points P in the plane whose co-ordinates (x, y) satisfy the equation

$$ax + by = c$$

comprise *a* line ℓ in the plane. In fact every line in the plane is obtained in this manner, with different lines corresponding to *essentially different* equations, where we consider two such equations to be *essentially the same* if one can be obtained form the other by multiplying through by a non-zero constant.

It is for this reasons that equations of the above form are called *linear equations*.

Continuing in this vein, we take fixed real numbers a, b, c, d, e and f, and consider the *system of linear equations*

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

where we assume that either $a \neq 0$ or $b \neq 0$ and that $c \neq 0$ or $d \neq 0$.

If we take a solution to be the co-ordinates of a point P in the plane, then the set of all solutions has a geometric interpretation.

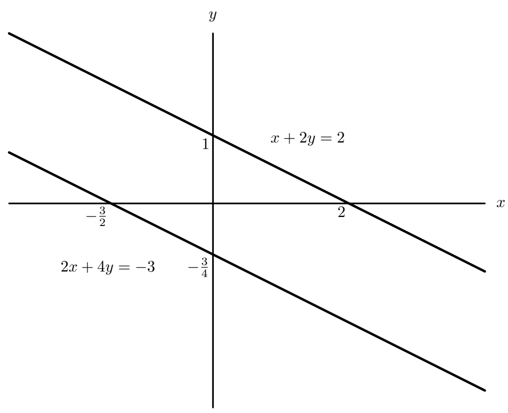
If the solutions to the first equation comprise the co-ordinates of the points of the line ℓ_1 and the solutions to the second equation are the co-ordinates of the point of the ℓ_2, then there are several possibilities.

(i) ℓ_1 and ℓ_2 represent the same line. In this case there are infinitely many solutions to the system of equations.

(ii) ℓ_1 and ℓ_2 represent parallel but distinct lines. In this case there is no solution of the system.

(iii) ℓ_1 and ℓ_2 are not parallel. In this case they have a unique point of intersection, P, whose co-ordinates are the unique solution to our system of linear equations.

We note that cases (i) and (ii) correspond to the relation $ad - bc = 0$.



The situation is slightly more involved, but still similar, when we move to spatial geometry. If we take real numbers a, b, c, and j with at least one of a, b, c non-zero, then points whose co-ordinates, (x, y, z), comprise all solutions of the linear equation

$$ax + by + cz = j$$

form a *plane*. Thus a given line is represented by a system of two linear equations with the property that the planes they represent intersect in the given line.

Another spatial phenomenon, is that a new possibility arises for two lines in space, say ℓ_1 and ℓ_2, in addition to the three above, namely, ℓ_1 and$_2$ could be skew. They are not parallel, yet they do not meet. Since each line in space is determined by two equations, two lines require, in general, four equations. Thus the system of equations we obtain comprises four linear equations in three unknowns – an *over-determined system* – and it is possible that any three have a common solution without the four having any solution, as the next example shows

$$\begin{array}{ccccccc} \mathrm{x} & & & & & = & 0 \\ & & \mathrm{y} & & & = & 0 \\ & & & & \mathrm{z} & = & 0 \\ \mathrm{x} & + & \mathrm{y} & + & \mathrm{z} & = & 1 \end{array}$$

The reader is invited to try to represent what we do geometrically, bearing in mind that our geometric representation is both illuminating and misleading. It is simple to avoid many pitfalls by remembering that the geometric representation depends intimately on properties of the real numbers and that in other situations those features which depend upon properties specific to $\mathbb{R}$ are not available.

4.1 Exercises

The purpose of these exercises is to provide practice in moving between equational, parametric and vectorial representations of lines, planes, etc. The questions are formulated in their general form.The reader who has difficulty with such generality should first attempt a numerical example, by taking, say, $a = 3, b = 4$ and $c = 5$.

Exercise 4.1. Choose a co-ordinate system for the plane, with origin $\mathbf{0}$. Let P have co-ordinates (x, y) and A have co-ordinates (a, b). Prove that if θ is the acute angle between the line through $\mathbf{0}$ and A and the line through $\mathbf{0}$ and P, then

$$\cos\theta = \frac{ax + by}{\sqrt{a^2 + b^2}\sqrt{x^2 + y^2}}.$$

Exercise 4.2. Choose a co-ordinate system for the plane. Let ℓ be the line comprising all points P in the plane whose co-ordinates (x, y) satisfy the equation $ax + by = c$. [To ensure that we do have a line, we require that either $a \neq 0$ or $b \neq 0$, or, equivalently, that $a^2 + b^2 \neq 0$.]

Find the co-ordinates of the point P_ℓ on ℓ which is closest to the origin, $\mathbf{0}$.

Exercise 4.3. A *parametric representation* of the line ℓ is a function

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R},\ t \longmapsto (x(t), y(t))$$

whose image is ℓ. Find a parametric representation of the line ℓ in Exercise 4.2.

Exercise 4.4. Having chosen a co-ordinate system for the plane, we assign to each point P a vector, called its *co-ordinate vector*: If P has co-ordinates (x, y), then its co-ordinate vector is $\begin{bmatrix} x \\ y \end{bmatrix}$. This allows us to represent geometric objects and relations using vectorial equations and to interpret vector equations geometrically. For example, the vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} + \lambda \begin{bmatrix} r \\ s \end{bmatrix} \qquad (\lambda \in \mathbb{R}),$$

with r, s, u, v fixed and x, y variable, represents a line in the plane.

Find a vectorial representation of the line ℓ in Exercise 4.2.