# MATH102 ASSIGNMENT 7

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(1)

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(a)

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + ... + \frac{x^{n+2}}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

(b)

$$\begin{aligned} e^{-x^2} &= 1 + \left(-x^2\right) + \frac{\left(-x^2\right)^2}{2!} + ... + \frac{\left(-x^2\right)^n}{n!} + ... \\ &= 1 - x^2 + \frac{x^4}{2!} - ... + (-1)^n \ \frac{x^{2n}}{n!} + ... = \sum_{n=0}^{\infty} (-1)^n \ \frac{x^{2n}}{n!} \end{aligned}$$

(c)

$$\begin{aligned} \int e^{-x^2} \ dx &= \int 1 - x^2 + \frac{x^4}{2!} - ... + (-1)^n \ \frac{x^{2n}}{n!} + ... \ dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} - ... + (-1)^n \ \frac{x^{2n+1}}{n!(2n+1)} + ... \\ &= \sum_{n=0}^{\infty} (-1)^n \ \frac{x^{2n+1}}{n!(2n+1)} \end{aligned}$$

(2)

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n = 1 + kx + \binom{k}{2} x^2 + \binom{k}{3} x^3 + ... + x^k$$

(a)

$$\begin{aligned} \left(1 - x^2\right)^k &= 1 + k(-x^2) + \binom{k}{2} \left(-x^2\right)^2 + \binom{k}{3} \left(-x^2\right)^3 + ... + \left(-x^2\right)^k \\ &= 1 - kx^2 + \binom{k}{2} x^4 - \binom{k}{3} x^6 + ... + (-1)^n \ x^{2k} \end{aligned}$$

Since $\dfrac{1}{\sqrt{1-x^2}}$ requires $k = -\dfrac{1}{2}$ (not a positive integer) the truncated result of the general Binomial Theorem is applicable.

$$
\begin{aligned}
\left(1-x^2\right)^{-\frac{1}{2}} = 1 - \left(-\frac{1}{2}\right) x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\, x^4 - \\
\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\, x^6 + ... + \\
(-1)^n \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)...\left(-\frac{1}{2}-n+1\right)}{n!}\, x^{2n} \\
= \quad 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + ...
\end{aligned}
$$

(b)

$$
\begin{aligned}
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}} \\
&= \int_0^x 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \frac{35}{128}t^8 + ...\; dt \\
&= \left[ t + \frac{1}{6}t^3 + \frac{3}{40}t^5 + \frac{5}{112}t^7 + \frac{35}{1152}t^9 + ... \right]_0^x \\
&= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + ...
\end{aligned}
$$

(3)

$$
\begin{aligned}
f(x) &= 1 + 3x + 5x^2 - 2x^3 & f(-1) &= 1 - 3 + 5 + 2 = 5 \\
f'(x) &= 3 + 10x - 6x^2 & f'(-1) &= 3 - 10 - 6 = -13 \\
f''(x) &= 10 - 12x & f''(-1) &= 10 + 12 = 22 \\
f'''(x) &= -12 & f'''(-1) &= -12
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= f(-1) + f'(-1)(x+1) + \frac{f''(-1)(x+1)}{2!} + \frac{f'''(-1)(x+1)}{3!} \\
&= 5 - 13(x+1) + \frac{22}{2}(x+1) - \frac{12}{6}(x+1) \\
&= 5 - 13(x+1) + 11(x+1) - 2(x+1) \\
&= 5 - 13x - 13 + 11x + 11 - 2x - 2 \\
&= 1 - 4x
\end{aligned}
$$

(4) (a)

$$\int_1^\infty \frac{dx}{x^2} = \lim_{R\to\infty} \left[-\frac{1}{x}\right]_1^R = \lim_{R\to\infty} \left(1 - \frac{1}{R}\right) = 1$$

(b)

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{r\to 0^+} \left[2\sqrt{x}\right]_r^1 = \lim_{r\to 0^+} 2\left(1 - \sqrt{r}\right) = 2$$

(c)

$$\begin{aligned}\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{\substack{r\to -1\\ R\to 1}} \left[\sin^{-1} x\right]_r^R = \lim_{\substack{r\to -1\\ R\to 1}} \left(\sin^{-1} R - \sin^{-1} r\right)\\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi\end{aligned}$$