MATH102 ASSIGNMENT 11

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(1)

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\begin{aligned}
&2y'' + y' + y = 0 \;\Rightarrow\; 2m^2 + m + 1 = 0 \\
&\Rightarrow m = \frac{-1 \pm \sqrt{1-8}}{4} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4} i \\
&\Rightarrow y = e^{-\frac{1}{4}x}\left(C_1 \cos \frac{\sqrt{7}}{4} x + C_2 \sin \frac{\sqrt{7}}{4} x\right)
\end{aligned}
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(2)

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\begin{aligned}
&y'' + 10y' + 25y = 0 \;\Rightarrow\; m^2 + 10m + 25 = 0 \\
&\Rightarrow (m+5)^2 = 0 \;\Rightarrow\; m = -5 \\
&\Rightarrow y = C_1 e^{-5x} + C_2 x e^{-5x}
\end{aligned}
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(3)

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\begin{aligned}
&y'' + 2y' + 5y = 0 \;\Rightarrow\; m^2 + 2m + 5 = 0 \\
&\Rightarrow m = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i \\
&\Rightarrow y = e^{-x}\big(C_1 \cos 2x + C_2 \sin 2x\big)
\end{aligned}
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\begin{aligned}
y(0) &= 0 = e^0\big(C_1 \cos 0 + C_2 \sin 0\big) \;\Rightarrow C_1 = 0 \\
y'(x) &= C_1\big(-e^{-x}\cos 2x - 2e^{-x}\sin 2x\big) + C_2\big(-e^{-x}\sin 2x + 2e^{-x}\cos 2x\big) \\
&= -e^{-x}\big[C_1\big(\cos 2x + 2\sin 2x\big) + C_2\big(\sin 2x - 2\cos 2x\big)\big] \\
y'(0) &= 1 = -1\big[C_1(\cos 0 + 2\sin 0) + C_2(\sin 0 - 2\cos 0)\big] \\
&= 1 = -C_1 + 2C_2 \;\Rightarrow\; C_2 = \frac{1}{2} \quad (\text{since } C_1 = 0) \\
\Rightarrow y &= \frac{1}{2} e^{-x} \sin 2x
\end{aligned}
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(4)

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\begin{aligned}
&y'' + 7y' - 8y = e^x \\
&\text{Let } y = Axe^x \\
&\Rightarrow y' = Ae^x + Axe^x, \\
&\Rightarrow y'' = 2Ae^x + Axe^x
\end{aligned}
$$

Then,

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\begin{aligned}
&(2Ae^x + Axe^x) + 7(Ae^x + Axe^x) - 8(Axe^x) = 9Ae^x = e^x \\
&\Rightarrow A = \frac{1}{9} \\
&\Rightarrow y_p = \frac{1}{9}xe^x
\end{aligned}
$$

We then find the independent solutions to the corresponding homogeneous equation in order to write down the general solution $y = y_p + C_1y_1 + C_2y_2$.

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\begin{aligned}
&y'' + 7y' - 8y = 0 \;\Rightarrow\; m^2 + 7m - 8 = 0 \\
&\Rightarrow (m+8)(m-1) = 0 \;\Rightarrow\; m = -8, 1 \\
&\Rightarrow y = \frac{1}{9}xe^x + C_1\,e^{-8x} + C_2\,e^x
\end{aligned}
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(5)

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\begin{aligned}
&y'' - y' - 2y = \cos x - 5\sin x \\
&\text{Let } y = A_1\cos x + A_2\sin x \\
&\Rightarrow y' = -A_1\sin x + A_2\cos x, \\
&\Rightarrow y'' = -A_1\cos x - A_2\sin x
\end{aligned}
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Then,

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\begin{aligned}
&(A_1\cos x - A_2\sin x) - (-A_1\sin x + A_2\cos x) - 2(A_1\cos x + A_2\sin x) = \\
&(-A_1 - A_2 - 2A_1)\cos x + (-A_2 + A_1 - 2A_2)\sin x = \\
&(-3A_1 - A_2)\cos x + (A_1 - 3A_2)\sin x = \cos x - 5\sin x \\
&\Rightarrow -3A_1 - A_2 = 1,\; A_1 - 3A_2 = -5 \\
&\Rightarrow A_1 = 3A_2 - 5,\; -3(3A_2 - 5) - A_2 = 1 \\
&\Rightarrow -9A_2 + 15 - A_2 = 1,\; -10A_2 = -14 \\
&\Rightarrow A_2 = \frac{7}{5},\; A_1 = 3\left(\frac{7}{5}\right) - 5 = -\frac{4}{5} \\
&\Rightarrow y_p = -\frac{4}{5}\cos x + \frac{7}{5}\sin x
\end{aligned}
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The corresponding homogeneous equation give the following independent solutions and thus the general solution.

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\begin{aligned}
&y'' - y' - 2y = 0 \;\Rightarrow m^2 - m - 2 = 0\\
&(m-2)(m+1) = 0 \;\Rightarrow m = -1, 2\\
&y = -\frac{4}{5}\cos x + \frac{7}{5}\sin x + C_1 e^{-x} + C_2 e^{2x}
\end{aligned}
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(6) (a)

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\begin{aligned}
&4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 101y = 0 \;\Rightarrow 4m^2 + 4m + 101 = 0\\
&\Rightarrow m = \frac{-4 \pm \sqrt{16 - 1616}}{8} = \frac{-4 \pm 40i}{8} = -\frac{1}{2} \pm 5i\\
&\Rightarrow y = e^{-\frac{1}{2}t}\big(C_1 \cos 5x + C_2 \sin 5t\big)
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= 10 = e^0\big(C_1 \cos 0 + C_2 \sin 0\big) \;\Rightarrow C_1 = 10\\
y'(t) &= C_1\left(-\frac{1}{2}e^{-\frac{1}{2}t}\cos 5x - 5e^{-\frac{1}{2}t}\sin 5t\right) +\\
&\qquad C_2\left(-\frac{1}{2}e^{-\frac{1}{2}t}\sin 5t + 5e^{-\frac{1}{2}t}\cos 5t\right)\\
&= -e^{-\frac{1}{2}t}\left[C_1\left(\frac{1}{2}\cos 5t + 5\sin 5t\right) + C_2\left(\frac{1}{2}\sin 5t - 5\cos 5t\right)\right]\\
y''(0) &= 0 = -1\left[10\left(\frac{1}{2}\cos 0 + 5\sin 0\right) + C_2\left(\frac{1}{2}\sin 0 - 5\cos 0\right)\right]\\
&= 0 = -1\big(5 - 5C_2\big) = 5C_2 - 5 \;\Rightarrow\; C_2 = 1\\
&\Rightarrow y = e^{-\frac{1}{2}t}\big(10\cos 5t + \sin 5t\big)
\end{aligned}
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(b)

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\begin{aligned}
&y = \sqrt{101}\,e^{-\frac{1}{2}t}\left(\frac{10}{\sqrt{101}}\cos 5t + \frac{1}{\sqrt{101}}\sin 5t\right)\\
&\Rightarrow \sin\omega = \frac{10}{\sqrt{101}},\;\; \cos\omega = \frac{1}{\sqrt{101}}\\
&\Rightarrow \tan\omega = \frac{1}{10},\;\; \omega \approx 0.09967 \text{ radians or } 5.7^\circ\\
&\Rightarrow y = \sqrt{101}\,e^{-\frac{1}{2}t}\cos\big(5t - \omega\big)
\end{aligned}
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(c) The period of the oscillation is $\frac{2\pi}{5} \approx 1.26$ seconds.

(d) The frequency of the oscillation is $\frac{5}{2\pi} \approx 0.80$ oscillations per second.