MATH101 ASSIGNMENT 11

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(1) To show that the system is inconsistent if $\gamma \neq 2\alpha - 3\beta$, we can write the equivalent statement: the system is consistent if and only if $\gamma = 2\alpha - 3\beta$. We now use Gauss-Jordan elimination to solve for the three unknowns x, y and z.
Assume $\gamma = 2\alpha - 3\beta$. The augmented matrix then becomes

$$\begin{array}{rrr|c} 2 & -1 & 3 & \alpha \\ 3 & 1 & -5 & \beta \\ -5 & -5 & 21 & 2\alpha - 3\beta \end{array}$$

Next we convert the matrix to its reduced row echelon form.

$$\begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array} \begin{array}{rrr|c} 5 & 0 & -2 & \alpha + \beta \\ 3 & 1 & -5 & \beta \\ -5 & -5 & 21 & 2\alpha - 3\beta \end{array}$$

$$\begin{array}{c} \\ R_2 + R_1 \\ \\ \end{array} \begin{array}{rrr|c} 5 & 0 & -2 & \alpha + \beta \\ 8 & 1 & -7 & \alpha + 2\beta \\ -5 & -5 & 21 & 2\alpha - 3\beta \end{array}$$

$$\begin{array}{c} \\ \\ R_3 + R_1 \end{array} \begin{array}{rrr|c} 5 & 0 & -2 & \alpha + \beta \\ 8 & 1 & -7 & \alpha + 2\beta \\ 0 & -5 & 19 & 3\alpha - 2\beta \end{array}$$

$$\begin{array}{c} \frac{1}{5}R_1 \\ \\ \\ \end{array} \begin{array}{rrr|c} 1 & 0 & -\frac{2}{5} & \frac{\alpha+\beta}{5} \\ 8 & 1 & -7 & \alpha + 2\beta \\ 0 & -5 & 19 & 3\alpha - 2\beta \end{array}$$

$$\begin{array}{c} \\ R_2 - 8R_1 \\ \\ \end{array} \begin{array}{rrr|c} 1 & 0 & -\frac{2}{5} & \frac{\alpha+\beta}{5} \\ 0 & 1 & -\frac{19}{5} & \frac{2\beta-3\alpha}{5} \\ 0 & -5 & 19 & 3\alpha - 2\beta \end{array}$$

$$\begin{array}{c} \\ \\ -\frac{1}{5}R_3 \end{array} \begin{array}{rrr|c} 1 & 0 & -\frac{2}{5} & \frac{\alpha+\beta}{5} \\ 0 & 1 & -\frac{19}{5} & \frac{2\beta-3\alpha}{5} \\ 0 & 1 & -\frac{19}{5} & \frac{2\beta-3\alpha}{5} \end{array}$$

$$\begin{array}{c} \\ \\ R_3 - R_2 \end{array} \begin{array}{rrr|c} 1 & 0 & -\frac{2}{5} & \frac{\alpha+\beta}{5} \\ 0 & 1 & -\frac{19}{5} & \frac{2\beta-3\alpha}{5} \\ 0 & 0 & 0 & 0 \end{array}$$

The last row implies that the equations are not independent and that the planes are intersecting in a single straight line. From the first two rows we can write

$$x - \frac{2}{5}z = \frac{\alpha+\beta}{5} \Rightarrow x = \frac{2z+\alpha+\beta}{5}$$
$$y - \frac{19}{5}z = \frac{2\beta-3\alpha}{5} \Rightarrow y = \frac{19z+2\beta-3\alpha}{5}$$

Thus we have shown that the system requires $\gamma = 2\alpha - 3\beta$ to solve for two of the unknowns, x and y, in terms of the third, z. Consequently, the system is consistent if and only if $\gamma = 2\alpha - 3\beta$.

(2) Let x, y and z be the number of shares the investor owns in BHP-Billiton, AMP and WMC respectively. Given the changes in both daily share prices and the value of total shares, the following system summarises the data available.

$$-x - 1.5y + 0.5z = -350$$
$$1.5x - 0.5y + z = 600$$

Thus we have a system of two linear equations in three unknowns. To solve this 2×3 system we use Gauss-Jordan elimination.

$$\left.\begin{array}{ccc|c} -1 & -\frac{3}{2} & \frac{1}{2} & -350 \\ \frac{3}{2} & -\frac{1}{2} & 1 & 600 \end{array}\right.$$

$$-R_1 \qquad \left.\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 350 \\ \frac{3}{2} & -\frac{1}{2} & 1 & 600 \end{array}\right.$$

$$R_2 - \tfrac{3}{2}R_1 \qquad \left.\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 350 \\ 0 & -\frac{11}{4} & \frac{7}{4} & 75 \end{array}\right.$$

$$-\tfrac{4}{11}R_2 \qquad \left.\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & 350 \\ 0 & 1 & -\frac{7}{11} & -\frac{300}{11} \end{array}\right.$$

$$R_1 - \tfrac{3}{2}R_2 \qquad \left.\begin{array}{ccc|c} 1 & 0 & \frac{5}{11} & \frac{4300}{11} \\ 0 & 1 & -\frac{7}{11} & -\frac{300}{11} \end{array}\right.$$

This is the last row elimination possible and consequently we can solve for only two of the unknowns, x and y, in terms of the third, z.

$$x + \frac{5}{11}z = \frac{4300}{11} \Rightarrow x = \frac{4300-5z}{11}$$
$$y - \frac{7}{11}z = -\frac{300}{11} \Rightarrow y = \frac{7z-300}{11}$$

Hence the stockbroker does not have enough information to calculate the number of shares the investor owns in each company.

But if the investor tells the stockbroker that $z = 200$ then,

$$x = \frac{4300 - 5(200)}{11} = 300$$
$$y = \frac{7(200) - 300}{11} = 100$$

(3) Given the matrices A, B and C, only $A - B$ is undefined as a 3×2 matrix cannot be subtracted from a 3×3 matrix. The rest are calculated below.

$$AB = \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 & -10 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 50+9-1 & -50-6+0 \\ 10+3-1 & -10-2+0 \\ 30+0+1 & -30+0+0 \end{bmatrix} = \begin{bmatrix} 58 & -56 \\ 12 & -12 \\ 31 & -30 \end{bmatrix}$$

$$AC = \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 0 \\ 4 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 10+0-4 & 15-3-2 & -5+0+3 \\ 2+0-4 & 3-1-2 & -1+0+3 \\ 6+0+4 & 9+0+2 & -3+0-3 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 10 & -2 \\ -2 & 0 & 2 \\ 10 & 11 & -6 \end{bmatrix}$$

$$CB = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 0 \\ 4 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 10 & -10 \\ 3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 20+9-1 & -20-6+0 \\ 0-3+0 & 0+2+0 \\ 40+6-3 & -40-4+0 \end{bmatrix} = \begin{bmatrix} 28 & -26 \\ -3 & 2 \\ 43 & -44 \end{bmatrix}$$

$$A + 3C = \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 9 & -3 \\ 0 & -3 & 0 \\ 12 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -4 \\ 1 & -2 & -1 \\ 15 & 6 & -8 \end{bmatrix}$$

$$AC - CA = \begin{bmatrix} 6 & 10 & -2 \\ -2 & 0 & 2 \\ 10 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1 & 0 \\ 4 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 10 & -2 \\ -2 & 0 & 2 \\ 10 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 10+3-3 & 6+3+0 & -2-3-1 \\ 0-1+0 & 0-1+0 & 0+1+0 \\ 20+2-9 & 12+2+0 & -4-2-3 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 10 & -2 \\ -2 & 0 & 2 \\ 10 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 10 & 9 & -6 \\ -1 & -1 & 1 \\ 13 & 14 & -9 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 4 \\ -1 & 1 & 1 \\ -3 & -3 & 3 \end{bmatrix}$$

(4) Since A is a 2×2 matrix and $\det A \neq 0$,

$$A^{-1} = \frac{1}{2 \times 2 - 1 \times 2} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix}$$

To find B^{-1} we consider the following augmented matrix and perform the row reductions below. Note that we are simply solving the system given by $BB^{-1} = I$.

$$\left.\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right.$$

$$\begin{array}{l} R_1 - 2R_3 \\ \\ \\ \end{array} \quad \begin{array}{ccr|ccr} 1 & 0 & -1 & 1 & 0 & -2 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{l} \\ R_2 - R_1 \\ \\ \end{array} \quad \begin{array}{ccr|rcr} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{l} \\ \\ R_3 - R_2 \end{array} \quad \begin{array}{ccr|rrr} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 & -1 & -1 \end{array}$$

$$\begin{array}{l} R_1 - R_3 \\ \\ \\ \end{array} \quad \begin{array}{ccr|rrr} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 & -1 & -1 \end{array}$$

$$\begin{array}{l} \\ \\ -R_3 \end{array} \quad \begin{array}{ccc|rcr} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 3 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}$$

$$\begin{array}{l} \\ R_2 - 3R_3 \\ \\ \end{array} \quad \begin{array}{ccc|rrr} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}$$

Thus,

$$B^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

We perform the same procedure on following system to find C^{-1}.

$$\left.\begin{array}{crc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right.$$

$$\begin{array}{l} \\ -R_2 \\ \\ \end{array} \quad \begin{array}{rcr|crc} 3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{rrrr|rrr} R_1 - 2R_3 & 1 & -1 & -2 & 1 & 0 & -2 \\ & -1 & 1 & -2 & 0 & -1 & 0 \\ & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{rrrr|rrr} & 1 & -1 & -2 & 1 & 0 & -2 \\ & -1 & 1 & -2 & 0 & -1 & 0 \\ R_3 - R_1 & 0 & 2 & 3 & -1 & 0 & 3 \end{array}$$

$$\begin{array}{rrrr|rrr} R_1 + \frac{1}{2}R_3 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ & -1 & 1 & -2 & 0 & -1 & 0 \\ & 0 & 2 & 3 & -1 & 0 & 3 \end{array}$$

$$\begin{array}{rrrr|rrr} & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ R_2 + R_1 & 0 & 1 & -\frac{5}{2} & \frac{1}{2} & -1 & -\frac{1}{2} \\ & 0 & 2 & 3 & -1 & 0 & 3 \end{array}$$

$$\begin{array}{rrrr|rrr} & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ & 0 & 1 & -\frac{5}{2} & \frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{8}(R_3 - 2R_2) & 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}$$

$$\begin{array}{rrrr|rrr} R_1 - \frac{1}{2}R_3 & 1 & 0 & 0 & \frac{3}{8} & \frac{1}{8} & -\frac{1}{4} \\ R_2 + \frac{5}{2}R_3 & 0 & 1 & 0 & -\frac{1}{8} & -\frac{3}{8} & \frac{3}{4} \\ & 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}$$

Thus,

$$C^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{8} & -\frac{3}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & 1 & -2 \\ -1 & -3 & 6 \\ -2 & 2 & 4 \end{bmatrix}$$

(5) We calculate the determinant by an expansion on the first row where

$$\det = c_{11} \times 3 + c_{12} \times 0 + c_{13} \times 2$$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 1(1 \times 4 - 1 \times 1) = 3$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1(-1 \times 1 - 1 \times 0) = -1$$

$$\Rightarrow \det = 3 \times 3 + 0 - 1 \times 2 = 7$$

Expanding by the third row we find

$$\det = c_{31} \times 0 + c_{32} \times 1 + c_{33} \times 4$$

$$c_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1(3 \times 1 + 2 \times 1) = -5$$

$$c_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 1(3 \times 1 + 0 \times 1) = 3$$

$$\Rightarrow \det = 0 - 5 \times 1 + 3 \times 4 = 7$$

Thus we have verified that the two expansions yield the same determinant.

(6) We construct the following system to find an interpolating polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2$ for the given data points.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 16 \end{bmatrix}$$

We then solve for the unknown coefficients a_0, a_1 and a_2 using Cramer's Rule. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 12 \\ 15 \\ 16 \end{bmatrix}$$

Then,

$$A_0 = \begin{bmatrix} 12 & 1 & 1 \\ 15 & 2 & 4 \\ 16 & 3 & 9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 12 & 1 \\ 1 & 15 & 4 \\ 1 & 16 & 9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 12 \\ 1 & 2 & 15 \\ 1 & 3 & 16 \end{bmatrix}$$

The determinants of each 3×3 matrix are given below.

$$|A| = 2 \ (\neq 0), \ |A_0| = 14, \ |A_1| = 12, \ |A_2| = -2$$

Hence,

$$a_0 = \frac{|A_0|}{|A|} = \frac{14}{2} = 7$$

$$a_1 = \frac{|A_1|}{|A|} = \frac{12}{2} = 6$$

$$a_2 = \frac{|A_2|}{|A|} = -\frac{2}{2} = -1$$

The interpolating polynomial is therefore

$$p(x) = 7 + 6x - x^2$$