Sample Solutions for Tutorial 6

Question 1.

(a) We consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longmapsto (x, y)$

(i) Take $\mathbf{e}_1 := (1, 0), \mathbf{e}_2 := (0, 1), \mathbf{f}_1 := (1, 0), \mathbf{f}_2 := (0, 1)$. Then

$$T(\mathbf{e}_1) = (1, 0) = \mathbf{f}_1 = 1.\mathbf{f}_1 + 0.\mathbf{f}_2$$
$$T(\mathbf{e}_2) = (0, 1) = \mathbf{f}_2 = 0.\mathbf{f}_1 + 1.\mathbf{f}_2$$

Hence the matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) Take $\mathbf{e}_1 := (1, 0), \mathbf{e}_2 := (0, 1), \mathbf{f}_1 := (0, 1), \mathbf{f}_2 := (1, 0)$. Then

$$T(\mathbf{e}_1) = (1, 0) = \mathbf{f}_2 = 0.\mathbf{f}_1 + 1.\mathbf{f}_2$$
$$T(\mathbf{e}_2) = (0, 1) = \mathbf{f}_1 = 1.\mathbf{f}_1 + 2.\mathbf{f}_2$$

Hence the matrix of T is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iii) Take $\mathbf{e}_1 := (1, 2), \mathbf{e}_2 := (3, 4), \mathbf{f}_1 := (1, 0), \mathbf{f}_2 := (0, 1)$. Then

$$T(\mathbf{e}_1) = (1, 2) = 1(0, 1) + 2(0, 1) = 1.\mathbf{f}_1 + 2.\mathbf{f}_2$$
$$T(\mathbf{e}_2) = (3, 4) = 3(1, 0) + 4(0, 1) = 3.\mathbf{f}_1 + 4.\mathbf{f}_2$$

Hence the matrix of T is $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(iv) Take $\mathbf{e}_1 := (1, 0), \mathbf{e}_2 := (0, 1), \mathbf{f}_1 := (1, 2), \mathbf{f}_2 := (3, 4)$. Then

$$T(\mathbf{e}_1) = (1, 0) = a(1, 2) + c(3, 4) \;\Leftrightarrow\; a + 3c = 1 \;\text{ and }\; 2a + 4c = 0$$
$$T(\mathbf{e}_2) = (0, 1) = b(1, 2) + d(3, 4) \;\Leftrightarrow\; b + 3d = 0 \;\text{ and }\; 2b + 4d = 1$$

Now the solutions to these equations is easily seen to be $a = -2, c = 1$ for the first pair and $b = \frac{3}{2}, d = \frac{-1}{2}$, so that the matrix of T is

$$\begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

(v) Take $\mathbf{e}_1 := (3, 4), \mathbf{e}_2 := (1, 2), \mathbf{f}_1 := (1, 2), \mathbf{f}_2 := (3, 4)$. Then

$$T(\mathbf{e}_1) = (3, 4) = \mathbf{f}_2 = 0.\mathbf{f}_1 + 1.\mathbf{f}_2$$
$$T(\mathbf{e}_2) = (1, 2) = \mathbf{f}_1 = 1.\mathbf{f}_1 + 0.\mathbf{f}_2$$

Hence the matrix of T is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

[This should be reminiscent of Part (ii).]

(b) From calculus we know that $D : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ is given by

$$D : a + bt + ct^2 + dt^3 \longmapsto b + 2ct + 3dt^2$$

and that for all $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathcal{P}_3$

$$D(\alpha p + \beta q) = \alpha D(p) + \beta D(q).$$

(i) Take $\mathbf{e}_1 := 1, \mathbf{e}_2 := t, \mathbf{e}_3 := t^2, \mathbf{e}_4 := t^3, \mathbf{f}_1 := 1, \mathbf{f}_2 := t, \mathbf{f}_3 := t^2$. Then

$$\begin{aligned}
D(\mathbf{e}_1) &= 0 = 0.\mathbf{f}_1 + 0.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_2) &= 1 = 1.\mathbf{f}_1 + 0.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_3) &= 2t = 0.\mathbf{f}_1 + 2.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_4) &= 3t^2 = 0.\mathbf{f}_1 + 0.\mathbf{f}_2 + 3.\mathbf{f}_3
\end{aligned}$$

Hence the matrix of D is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(ii) Take $\mathbf{e}_1 := 1, \mathbf{e}_2 := t, \mathbf{e}_3 := t^2, \mathbf{e}_4 := t^3, \mathbf{f}_1 := 6, \mathbf{f}_2 := 6t, \mathbf{f}_3 := 3t^2$. Then

$$\begin{aligned}
D(\mathbf{e}_1) &= 0 = 0.\mathbf{f}_1 + 0.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_2) &= 1 = \frac{1}{6}.\mathbf{f}_1 + 0.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_3) &= 2t = 0.\mathbf{f}_1 + \frac{1}{3}.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_4) &= 3t^2 = 0.\mathbf{f}_1 + 0.\mathbf{f}_2 + 1.\mathbf{f}_3
\end{aligned}$$

Hence the matrix of D is

$$\begin{bmatrix} 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) Take $\mathbf{e}_1 := 1, \mathbf{e}_2 := 1+t, \mathbf{e}_3 := 1+t+t^2, \mathbf{e}_4 := 1+t+t^2+t^3, \mathbf{f}_1 := 1, \mathbf{f}_2 := 1+t, \mathbf{f}_3 := 1+t+t^2$. Then

$$\begin{aligned}
D(\mathbf{e}_1) &= 0 = 0.\mathbf{f}_1 + 0.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_2) &= 1 = 1.\mathbf{f}_1 + 0.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_3) &= 1 + 2t = -1 + (1+t) = (-1).\mathbf{f}_1 + 2.\mathbf{f}_2 + 0.\mathbf{f}_3 \\
D(\mathbf{e}_4) &= 1 + 2t + 3t^2 = -1 - 1(1+t) + 3(1+t+t^2) = (-1).\mathbf{f}_1 + (-1).\mathbf{f}_2 + 3.\mathbf{f}_3
\end{aligned}$$

Hence the matrix of D is

$$\begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Question 2.

Let $T : V \longmapsto W$ be a linear transformation. Then T is an isomorphism if and only if there is a linear transformation $S : W \longrightarrow V$ with $S \circ T = id_V$ and $T \circ S = id_W$. Choose a basis for V and one for W. Let $\underline{\mathbf{A}}$ be the matrix of T and $\underline{\mathbf{B}}$ that of S with respect to the bases chosen. Then the matrix of $S \circ T$ is $\underline{\mathbf{B}}\,\underline{\mathbf{A}}$ and that of $T \circ S$ is $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$. But the matrices of id_V and id_W are $\underline{\mathbf{1}}$ since we have a fixed basis for each vector space. Thus $\underline{\mathbf{A}}, \underline{\mathbf{B}} = \underline{\mathbf{1}}$ and $\underline{\mathbf{B}}\,\underline{\mathbf{A}} = \underline{\mathbf{1}}$, showing that $\underline{\mathbf{A}}$ is an invertible matrix.

Conversely, let the matrix $\underline{\mathbf{A}}$ of T with respect to the bases $\mathcal{B}$ for V and $\mathcal{B}\prime$ for W be invertible, with inverse $\underline{\mathbf{B}} = [b_{ij}]$. Then $\underline{\mathbf{B}}$ represent the linear transformation

$$S : W \longrightarrow V, \quad \sum_j y_j \mathbf{e}'_j \longmapsto \sum_j \sum_i b_{ij} y_j \mathbf{e}_i$$

Moreover, since $\underline{\mathbf{B}}\,\underline{\mathbf{A}} = \underline{\mathbf{1}}$ and $\underline{\mathbf{A}}\,\underline{\mathbf{B}} = \underline{\mathbf{1}}$, we have

$$(S \circ T)(\mathbf{e}_j) = \mathbf{e}_j \qquad \text{and} \qquad (T \circ S)(\mathbf{e}'_j) = \mathbf{e}'_j$$

for all i, j. But since any linear transformation is uniquely determined by its values on a basis, this means that $S \circ T = id_V$ and $T \circ S = id_W$, which proves that T is an isomorphism.