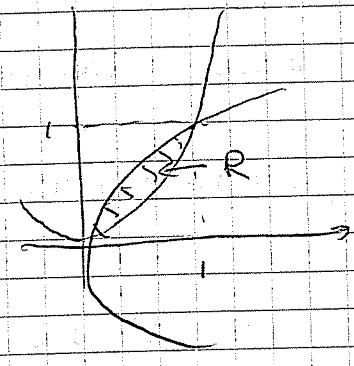
(2) $\int_C x^2y\,dx + (y+xy^2)\,dy$

C: $y=x^2$, $x=y^2$



$= \iint_R y^2 - x^2 \, dA =$

$= \int_0^1\int_{x^2}^{\sqrt{x}} y^2-x^2\,dy\,dx = \int_0^1 \left(\frac{y^3}{3} - x^2y\Big|_{x^2}^{\sqrt{x}}\right)dx =$

$= \int_0^1 \frac{x^{3/2}}{3} - x^2\sqrt{x} - \frac{x^6}{3} + x^4\,dx =$

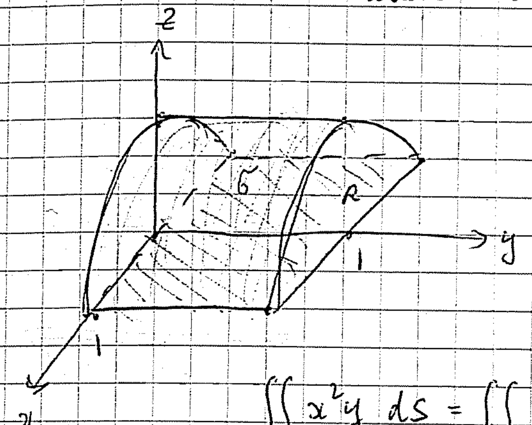
$= \frac{2}{5}\cdot\frac{x^{5/2}}{3} - \frac{2x^{7/2}}{7} - \frac{x^7}{3\cdot 7} + \frac{x^5}{5}\Big|_0^1 =$

$= \frac{2}{15} - \frac{2}{7} - \frac{1}{21} + \frac{1}{5} = \frac{14-30-5+21}{3\cdot 5\cdot 7} = 0$

Tut 11

Evaluate

(1) $\iint_\sigma x^2y\,dS$, where σ is the portion of the cylinder $x^2+\cancel{y^2}z^2=1$ between planes $y=0$, $y=1$ above the xy-plane



$z=\sqrt{1-\cancel{y^2}} = \sqrt{1-x^2}$

$z_x = \frac{1}{2}\frac{-2x}{\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$

$z_y = 0$

$\iint_\sigma x^2y\,dS = \iint_R x^2y\sqrt{z_x^2+z_y^2+1}\,ds =$

$= \iint_R x^2y\sqrt{\frac{x^2}{1-x^2}+0+1}\,ds = \iint_R x^2y\sqrt{\frac{x^2+1-x^2}{1-x^2}}\,ds$

$= \iint_R \frac{x^2y}{\sqrt{1-x^2}}\,ds = \int_{-1}^1\int_0^1 \frac{x^2y}{\sqrt{1-x^2}}\,dy\,dx = \int_{-1}^1\frac{x^2}{\sqrt{1-x^2}}\,dx\int_0^1 y\,dy$

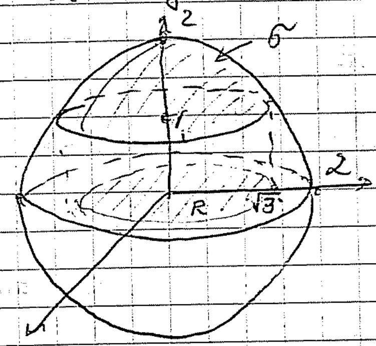
$x=\sin t$
$dx=\cos t\,dt$

$= \int_\pi^{2\pi}\frac{\sin^2 t}{\cancel{\cos t}}\times\cancel{\cos t}\,dt \times \left(\frac{y^2}{2}\right)_0^1 = \frac12\int \frac{1-\cos 2t}{2}\,dt =$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$= \frac14\left(t-\frac12\sin 2t\right)\Big|_\pi^{2\pi} = \frac{\pi}{4}$

② $\iint_{\sigma} (x^2+y^2)z \, dS$, where $\sigma: x^2+y^2+z^2=4$ above $z=1$



$z=1$ and $x^2+y^2+z^2=4 \Rightarrow$

$x^2+y^2=3$, $r=\sqrt{3}$

$$\iint_{\sigma} (x^2+y^2)z \, dS =$$

$$= \iint_R (x^2+y^2) z \sqrt{z_x^2+z_y^2+1} \, dA =$$

$z = \sqrt{4-x^2-y^2}$

$z_x = -\frac{x}{\sqrt{4-x^2-y^2}}$

$z_y = -\frac{y}{\sqrt{4-x^2-y^2}}$

$$= \iint_R (x^2+y^2) z \sqrt{\frac{\cancel{x^2+y^2}+4-\cancel{x^2-y^2}}{4-x^2-y^2}} \, dA =$$

polar coordinates

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^2 \, z \, \frac{2}{\sqrt{4-r^2}} \times r \, dr \, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^2 \cancel{\sqrt{4-r^2}} \times \frac{2r}{\cancel{\sqrt{4-r^2}}} \, dr \, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} 2r^3 \, dr \, d\theta =$$

$$= 2\int_0^{2\pi} \frac{r^4}{4}\Big|_0^{\sqrt{3}} d\theta = \frac{1}{2} \times 9\theta \Big|_0^{2\pi} = 9\pi$$

Calculate the indicated unit normal

① $\sigma: 2x^2+y^2+z^2=4$ at $(1,1,1)$ that points toward the xy-plane

$G(x,y,z) = 2x^2+y^2+z^2-4=0$

$\nabla G(x,y,z) = \langle 4x, 2y, 2z \rangle$

at $(1,1,1)$ $\langle 4,2,2 \rangle$ or $\langle -4,-2,-2 \rangle$ - toward the xy-plane

$\|\nabla G(x,y,z)\| = \sqrt{16x^2+4y^2+4z^2}$

$\|\nabla G(1,1,1)\| = \sqrt{16+4+4} = 2\sqrt{6}$

$\vec{n} = \langle -\frac{4}{2\sqrt{6}}, -\frac{2}{2\sqrt{6}}, -\frac{2}{2\sqrt{6}} \rangle$

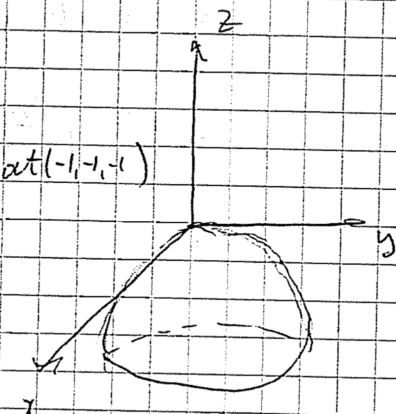
$= \langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \rangle$

② $\sigma: x^2+y^2=-z$, normal points away the xz-plane at $(-1,-1,-1)$

$G(x,y,z) = x^2+y^2+z=0$

$\nabla G(x,y,z) = \langle 2x, 2y, 1 \rangle$

$\|\nabla G(x,y,z)\| = \sqrt{4x^2+4y^2+1} = \sqrt{9} = 3$ at $(-1,-1,-1)$



at $(-1,-1,-1)$ $\nabla G = \langle -2,-2,1 \rangle$

or $\langle 2,2,-1 \rangle$ -

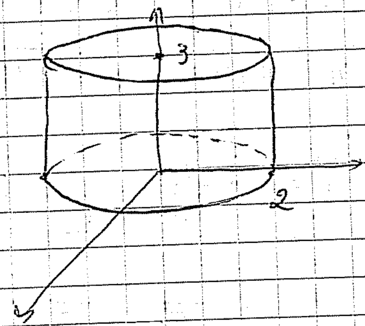
away the xz-plane

$\vec{n} = \langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle$

Use the divergence theorem to evaluate $\iint_\sigma \vec{F}\cdot\vec{n}\, dS$ where $\vec{n}$ is the outer unit normal to σ

① $\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

σ: cylindrical solid $x^2+y^2=4$, $z=0$, $z=3$



$$\iint_\sigma \vec{F}\cdot\vec{n}\, ds = \iiint_G \operatorname{div}\vec{F}\, dv =$$

$$= \iiint_G 3x^2+3y^2+3z^2\, dv$$

cylindrical coordinates:

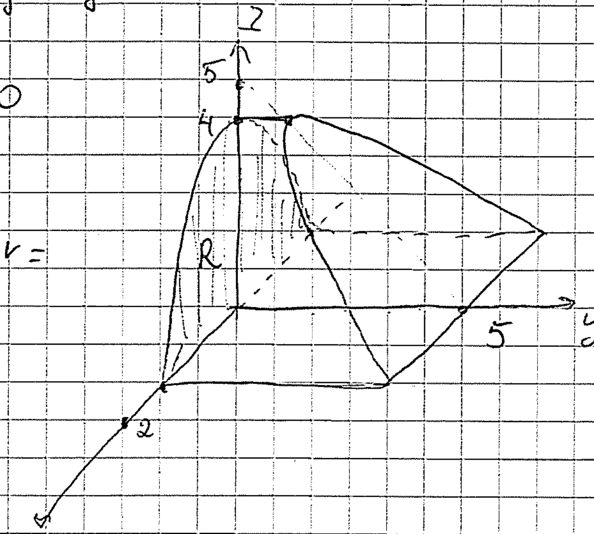
$$= 3\int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)\, r\, dz\, dr\, d\theta =$$

$$= 3\int_0^{2\pi} d\theta \int_0^2 \left(r^3 z + \frac{z^3}{3} r \Big|_0^3\right) dr = 3\times 2\pi \times \int_0^2 3r^3+9r\, dr =$$

$$= 6\pi \times \left(\frac{3r^4}{4} + \frac{9r^2}{2}\Big|_0^2\right) = 6\pi(12+18) = 180\pi$$

② $\vec{F}(x,y,z) = x^3\vec{i} + x^2 y\vec{j} + xy\vec{k}$

σ: $z = 4-x^2$
$y+z=5$, $z=0$, $y=0$



$$\iint_\sigma \vec{F}\cdot\vec{n}\, ds = \iiint_G \operatorname{div}\vec{F}\, dv =$$

$$= \iiint_G 3x^2+x^2+0\, dv =$$

$$= \iint_R \left(\int_0^{5-z} 4x^2\, dy\right) dA =$$

$$= \int_{-2}^{2}\int_0^{4-x^2}\int_0^{5-z} 4x^2\, dy\, dz\, dx = 4\int_{-2}^{2}\int_0^{4-x^2} x^2 y \Big|_0^{5-z} dz\, dx =$$

$$= 4\int_{-2}^{2}\int_0^{4-x^2} \left(5x^2 - x^2 z\right) dz\, dx = 4\int_{-2}^{2} 5x^2 z - x^2\frac{z^2}{2}\Big|_0^{4-x^2} dx =$$

$$= 4\int_{-2}^{2} 5x^2(4-x^2) - \frac{x^2}{2}(16-18x^2+x^4)\, dx =$$

$$= 4\int_{-2}^{2} 12x^2 - x^4 - \frac{x^6}{2}\, dx = 4\left[4x^3 - \frac{x^5}{5} - \frac{x^7}{14}\right]_{-2}^{2} =$$

$$= 8\left[4\cdot 8 - \frac{32}{5} - \frac{64}{7}\right] = \frac{4608}{35}$$