Sample Solutions for Tutorial 11

Question 1.

(a)

$$
\begin{array}{rcccccccl}
 & 2x & + & 3y & + & 4z & = & 14 & \text{(i)} \\
 & 3x & + & 6y & - & z & = & 8 & \text{(ii)} \\
2\times\text{(ii)} - 3\times\text{(i)} : & & & 3y & - & 14z & = & -26 & \text{(iii)} \\
\text{(i)} - \text{(ii)} : & 2x & & & + & 18z & = & 40 & \text{(iv)}
\end{array}
$$

Hence (x, y, z) is a solution if and only if $x = 20 - 9z$ and $y = \frac{2}{3}(-13 + 7z)$, or, equivalently, the set of all solutions is

$$\{(20 - 27\lambda, -26 + 14\lambda, \lambda) \mid \lambda \in \mathbb{R}\}$$

(b)

$$
\begin{array}{rcccccccl}
 & & & 3y & + & z & = & 5 & \text{(i)} \\
 & 3x & + & 7y & - & 5z & = & 25 & \text{(ii)} \\
 & 2x & - & y & + & 3z & = & -1 & \text{(iii)} \\
\text{(ii)} + 5\times\text{(i)} : & 3x & + & 22y & & & = & 50 & \text{(iv)} \\
\text{(iii)} - 3\times\text{(i)} : & 2x & - & 10y & & & = & -16 & \text{(v)} \\
2\times\text{(iv)} - 3\times\text{(v)} : & & & 74y & & & = & 148 & \text{(vi)} \\
\text{Substitute (vi) in (v)} : & 2x & & & & & = & 4 & \text{(vii)} \\
\text{Substitute (vi) in (i)} : & & & & & z & = & -1 & \text{(viii)}
\end{array}
$$

Hence $(2, 2, -1)$ is the unique solution of the given system of equation.

Question 2.

$$
\begin{aligned}
r &= 2x + 3y \\
&= 2(3v + 4w) + 3(3u + 7v - 5w) \\
&= 9u + (2 \times 3 + 3 \times 7)v + (2 \times 4 + 3 \times (-5))w \\
&= 9u + 27v - 11w
\end{aligned}
$$

and

$$
\begin{aligned}
s &= 3x + 6y \\
&= 3(3v + 4w) + 6(3u + 7v - 5w) \\
&= 18u + (9 + 42)v + (12 - 30)w \\
&= 18u + 51v - 18w
\end{aligned}
$$

Or, suggestively,

$$
\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 3 & 7 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 9 & 27 & -11 \\ 18 & 51 & -18 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}
$$

Question 3.

$$
\begin{aligned}
r + u &= (2x + 3y + 4z) + (3y + z) \\
&= 2x + 6y + 5z \\
s + v &= (3x + 6y - z) + (3x + 7y - 5z) \\
&= 6x + 13y - 6z \\
t + w &= (2x + y + 6z) + (2x - y + 3z) \\
&= 4x + 9z
\end{aligned}
$$

Or, suggestively

$$\begin{aligned}
\begin{bmatrix} r+u \\ s+v \\ t+w \end{bmatrix} &= \begin{bmatrix} r \\ s \\ t \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
&= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & -1 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 & 3 & 1 \\ 3 & 7 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} 2 & 6 & 5 \\ 6 & 13 & -6 \\ 4 & 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}$$

Question 4.

$$\begin{array}{lcccccl}
 & 5x & - & 4y & = & \lambda x & \\
\text{and} & 3x & - & 2y & = & \lambda y & \\
\text{if and only if} & (5-\lambda)x & - & 4y & = & 0 & \text{(i)} \\
\text{and} & 3x & - & (2+\lambda)y & = & 0 & \text{(ii)} \\
3 \times \text{(i)} + (\lambda - 5) \times \text{(ii)} : & & & (-12 - (\lambda-5)(\lambda+2))y & = & 0 & \text{(iii)} \\
4 \times \text{(ii)} - (\lambda + 2) \times \text{(i)} : & (12 + (\lambda+2)(\lambda-5)x & & & = & 0 & \text{(iv)}
\end{array}$$

Since $12 + (\lambda-5)(\lambda+2) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$, (iii) and/or (iv) have a non-zero solution if and only if $\lambda = 1, 2$.

$\lambda = \mathbf{1}$: It follows from (ii) that $3x - (\lambda+2)y = 0$, or, equivalently, $3x = 3y$. Hence the solution set of the system of equations

$$\begin{array}{lccccc}
 & 5x & - & 4y & = & x \\
\text{and} & 3x & - & 2y & = & y
\end{array}$$

is

$$\{(t,t) \mid t \in \mathbb{R}\}$$

$\lambda = \mathbf{2}$: It follows from (ii) that $3x - (\lambda+2)y = 0$, or, equivalently, $3x = 4y$. Hence the solution set of the system of equations

$$\begin{array}{lccccc}
 & 5x & - & 4y & = & 2x \\
\text{and} & 3x & - & 2y & = & 2y
\end{array}$$

is

$$\{(4t,3t) \mid t \in \mathbb{R}\}$$