UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101

PAPER TITLE: Algebra & Differential Calculus

PAPER NUMBER: First and Only

DATE: **Thursday 8 November 2007** **TIME:** **9:30 AM TO 12:30 PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: FIVE (5)

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE:

1	6 LEAF A4 BOOKS	0	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	0	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: **NIL**

POCKET CALCULATORS PERMITTED: **YES** **(SILENT TYPE)**

TEXTBOOKS OR NOTES PERMITTED: FIVE A4 sheets (10 pages if written on both sides) of HANDWRITTEN notes. No photocopies, no printed notes permitted.

INSTRUCTIONS FOR CANDIDATES:

- Candidates may make notes on this examination question paper during the fifteen minutes reading time
- Questions are of equal value
- **SECTION A:** - Answer all questions
- **SECTION B:** - Answer only TWO (2) of the FOUR (4) questions provided
- Candidates may retain this examination question paper

SECTION A

You should attempt all questions in this section.

Question 1 [10 marks]

(a) Assuming that a is a negative number and $b < a$ prove that $b^2 > a^2$;

(b) Prove by the mathematical induction that for any constant $a \neq 1$, $n = 1, 2, \cdots$,

$$1 + a + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

(c) Determine supremum and infimum (if exist) of the following sets

$$(i) \quad \{n \in \mathbb{N} : 65 > n^3 > 7\} \qquad (ii) \quad \{x \in \mathbb{R} : 81 - x^2 < 0\}.$$

Question 2 [10 marks]

(a) For $z = 3 - 2\,\mathrm{i}$, write each of the following complex numbers in the form $x + \mathrm{i}\,y$

(i) $\bar{z}$, (ii) $|z|$, (iii) $\bar{z}^2$, (iv) $\dfrac{1}{z}$, (v) $\dfrac{z^2 + \mathrm{i} - 1}{\bar{z}}$.

(b) Find all complex numbers z that satisfy

$$z^3 = \mathrm{i}\,.$$

Question 3 [10 marks]

(a) Find the natural domain X and the range Y of the functions defined by the following formulae

(i) $f(x) = 1 - x^2$, (ii) $f(x) = \dfrac{1}{1+x}$, (iii) $f(x) = \sqrt{x^3 - 1}$.

(b) Sketch the graph of the function $f : X \to \mathbb{R}$ from Part (a) (ii) $f(x) = \dfrac{1}{1+x}$. Decide whether this function is injective or surjective.

(c) Find a real number k that renders continuous the function

$$f : x \mapsto \begin{cases} x^2 + 3, & x < 1 \\ \sin(\pi x) + k, & x \geq 1. \end{cases}$$

Question 4 [10 marks]

(a) Determine which of the following sequences of real numbers $(u_n)_{n\in\mathbb{N}}$ is monotone and discuss the behaviour of u_n as $n \to \infty$.

(i) $u_n = 2 - \dfrac{1}{n+1}$ (ii) $u_n = (-2)^n$.

(b) Determine which of the following series converge and which diverge, justifying your answer.

$$(i)\ \sum_{n=1}^{\infty} \frac{n}{(-3)^n} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{n^2+1}{n} \qquad (iii)\ \sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}.$$

Question 5 [10 marks]

(a) Find all points at which each of the following functions is well-defined and differentiable, as well as its derivative. Where the function fails to be differentiable, explain why.

(i) $f(x) = \dfrac{1}{\sin^2 x}$ (ii) $g(x) = 2|x| + x^2$.

(b) Differentiate the functions

(i) $f(x) = \dfrac{2x+1}{x^4+1}$, (ii) $g(x) = e^{2x}\sin x$, (iii) $h(x) = \cos(e^x)$.

Question 6 [10 marks]

Consider the function $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 1$.

(a) Determine the intervals on which f is (i) increasing or decreasing (ii) concave up or concave down.

(b) Find all the relative maxima and relative minima of f and the absolute maximum a the absolute minimum on $[-3, 3]$.

(c) Sketch the graph of f on the interval $[-3, 3]$ (Choose an appropriate scale).

Question 7 is on page 4

Question 7 [10 marks]

(a) Find all real numbers x, y, z such that

$$\begin{array}{rcrcrcl} x & + & 3y & - & z & = & 4 \\ 2x & - & y & + & 2z & = & 6 \\ x & + & y & + & z & = & 6. \end{array}$$

(b) For

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -2 \\ -1 & 1 & -2 \end{pmatrix}.$$

state whether the following products and (or) sums are defined; calculate those which are defined:

(i) CA (ii) $A - B$ (iii) BC.

(c) Evaluate the determinant

$$\begin{vmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 10 & -99 & 897 & 3 \\ 3 & 1 & -1 & -3 \end{vmatrix}.$$

Question 8 [10 marks]

(a) Find the inverse matrix of

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Check your answer.

(b) Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

SECTION B

You should attempt only TWO questions in this section.

Question 9 [10 marks]

(a) Let $f(x) = \sqrt{x}$. Find $\delta > 0$ such that $|x - 1| < \delta$ guarantees $|f(x) - f(1)| < 0.01$.

(b) Let

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x \geq 1; \\ x, & \text{if } x < 1. \end{cases}$$

Prove that $f(x)$ is not continuous at $x = 1$.

Question 10 [10 marks]

(a) Prove that if the function $f(x)$ is continuous at $x = c$, then there is a constant $\delta > 0$, such that for all $x \in (c - \delta, c + \delta)$, $|f(x)| \leq 1 + |f(c)|$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that if there are constants $b > a$, such that $f(a) = f(b)$, then there is a constant $c \in (a, b)$, such that $f'(c) = 0$.

Question 11 [10 marks]

Find the largest possible area for a rectangle with base on the x-axis and upper vertices on the curve $y = 4 - x^2$.

Question 12 [10 marks] Find a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ of degree 3 such that $p(0) = 0$, $p'(0) = 1$, $p(-1) = -1$, $p(1) = 1$.