Tutorial 7

The second derivative test:

$f(x,y): \mathbb{R}^2 \supset X \to \mathbb{R}$ -

(x_0, y_0) is a critical point

$$D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$$

a) $D > 0$, $f_{xx}(x_0,y_0) > 0 \Rightarrow$ relative min
b) $D > 0$, $f_{xx}(x_0,y_0) < 0 \Rightarrow$ relative max
c) $D(x_0,y_0) < 0 \Rightarrow$ saddle point
d) $D = 0 \Rightarrow$ no conclusion.

Locate all relative max, min, saddle points.

① $f(x,y) = x^2 + xy + y^2 - 3x$

$f_x(x,y) = 2x + y - 3 \qquad = 0$

$f_y(x,y) = x + 2y \qquad = 0$

$x = -2y$

$-4y + y - 3 = -3y - 3 = 0$

$y = -1,\ x = 2$

$(2,-1)$ - critical point

$f_{xx} = 2$

$f_{xy} = 1 \qquad \Rightarrow \quad D = 2\cdot 2 - 1^2 = 3 > 0$

$f_{yy} = 2$

$f_{xx} = 2 > 0$

$\Rightarrow (2,-1)$ - relat. min

(2) $f(x,y) = xe^y$

$f_x = e^y \neq 0 \quad \Rightarrow$ no critical points
$f_y = x\cancel{y}e^y$

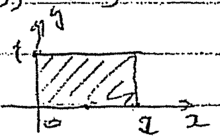
(3) $f(x,y) = x^2 + y - e^y$

$f_x = 2x \quad = 0 \qquad x = 0$
$f_y = 1 - e^y \quad = 0 \qquad y = 0$

$(0,0)$ - critical point

$f_{xx} = 2$
$f_{xy} = 0 \qquad D(0,0) = 2\times(-1) - 0^2 = -2 < 0$
$f_{yy} = -e^y = -1$ ~~for~~ at $(0,0)$ $\qquad \Downarrow$
$(0,0)$ is a saddle point.

(4). Find the absolute max and min over the R - rectangle region with verticals $(0,0), (0,1), (2,1), (2,0)$
$f(x,y) = xe^y - x^2 - e^y$



$f_x = e^y - 2x \quad = 0$
$f_y = xe^y - e^y \quad = 0$
$e^y(x-1) = 0 \Rightarrow x = 1, \quad e^y = 2 \Rightarrow y = \ln 2$
$(1, \ln 2)$ is not in the rectangle.

The sides of the rectangle:

1) $x = 0 \qquad 0 \leq y \leq 1$
$f(0,y) = -e^y$ - monotone, no critical points
$f(0,0) = -1, \quad f(0,1) = -e$

2) ~~f~~ $x = 2, \quad 0 \leq y \leq 1$
$f(2,y) = 2e^y - 4 - e^y = e^y - 4$ - monotone
$f(2,0) = -3, \quad f(2,1) = e - 4$

3). $y = 0 \qquad 0 \leq x \leq 2$
$f(x,0) = x - x^2 - 1$
$f'(x,0) = -2x + 1 = 0 \quad$ for $\quad x = \frac{1}{2}$
$f(\frac{1}{2}, 0) = \frac{1}{2} - \frac{1}{4} - 1 = -\frac{3}{4}$

4) $y=1 \qquad 0 \le x \le 2$

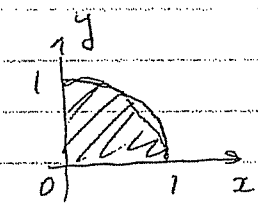
$f(x,1) = ex - x^2 - e = -x^2 + ex - e$

$f'(x,1) = -2x + e = 0 \quad$ for $\quad x = \frac{e}{2}$

$\underline{f\left(\frac{e}{2}, 1\right)} = -\frac{e^2}{4} + \frac{e^2}{2} - e = \frac{e^2}{4} - e \underline{\approx -0.88}$

Absolute Maximum is at $\left(\frac{1}{2}, 0\right)$: $f\left(\frac{1}{2}, 0\right) = -\frac{3}{4}$
absolute min at $(2,0)$: $f(2,0) = -3$

5 $\quad f(x,y) = xy^2 \qquad x \ge 0,\ y \ge 0,\ x^2+y^2 \le 1$



$f_x = y^2 = 0 \qquad y = 0$
$f_y = 2xy = 0$

$(x,0) \cdot 0 \le x \le 1$ - critical points

1) $\underline{f(x,0) = 0}$ for $\forall \cdot x \in [0,1]$

2) $x = 0 \quad 0 \le y \le 1 \Rightarrow \underline{f(0,y) = 0 \cdot y^2 = 0}$

3) $x^2+y^2 = 1 \Rightarrow y^2 = 1 - x^2$

$f(x,y) = xy^2 = x(1-x^2) = x - x^3$

$f' = 1 - 3x^2 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$, but $x \in [0,1] \Rightarrow x = \frac{1}{\sqrt{3}}$

$y = \sqrt{1-x^2} = \frac{\sqrt{2}}{\sqrt{3}}$

$\underline{f\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) = \frac{1}{\sqrt{3}} \cdot \frac{2}{3} = \frac{2}{3\sqrt{3}}}$ - absolute max

$\{(x,y) \in \mathbb{R}^2 ; \{x = 0, y \in [0,1]\} \cup \{y = 0, x \in [0,1]\}\}$ - abs. min.

Method of Lagrange multipliers

$f(x,y) \in C^1(R)$

$g(x,y) = 0 \quad , \nabla g \ne 0$

If $f(x,y)$ has a constrained relative extr. at (x_0, y_0)
then $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

Use Lagrange multipliers to find the max and min values of $f(x,y)$ subject to the given constraint.

① $f(x,y) = x^2 - y^2$, $x^2+y^2=25$

$g(x,y) = x^2+y^2-25=0$

$\nabla f = \langle 2x, -2y \rangle$, $\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2x = \lambda 2x \\ -2y = \lambda 2y \\ x^2+y^2=25 \end{cases} \qquad \begin{cases} 2x(1-\lambda)=0 \\ 2y(1+\lambda)=0 \\ x^2+y^2=25 \end{cases}$$

if $\lambda = 1 \Rightarrow y=0$, $x=\pm 5$

if $\lambda = -1 \Rightarrow x=0$, $y=\pm 5$

$f(0,\pm 5) = 0-25=-25$ - min at $(0,5)$, $(0,-5)$

$f(\pm 5, 0) = 25$ - max at $(5,0)$, $(-5,0)$

② Find the point on the plane $4x+3y+z=2$ that is closest to $(1,-1,1) = P_0$.

$d^2(P,P_0) = (x-1)^2+(y+1)^2+(z-1)^2 = f(x,y,z)$

$g(x,y,z) = 4x+3y+z-2=0$

$\nabla f = \langle 2(x-1), 2(y+1), 2(z-1) \rangle$

$\nabla g = \langle 4, 3, 1 \rangle$

$$\begin{cases} 2(x-1) = 4\lambda & x = 2\lambda+1 \\ 2(y+1) = 3\lambda & y = \frac{3}{2}\lambda - 1 \\ 2(z-1) = \lambda & z = \frac{\lambda}{2}+1 \\ 4x+3y+z-2=0 & 8\lambda+4+\frac{9}{2}\lambda-3+\frac{\lambda}{2}+1-2=0 \Rightarrow 13\lambda-1=0 \end{cases}$$

$\lambda = \frac{1}{13}$ $\quad x = \frac{2}{13}+1 = \frac{15}{13}$, $\quad y = \frac{3}{26}-1 = -\frac{23}{26}$, $z = \frac{1}{26}+1 = \frac{27}{26}$

$\left(\frac{15}{13}, -\frac{23}{26}, \frac{27}{26}\right)$ is the closest point to P_0

Find a vector in 3-space whose length is 5 and whose components have the largest possible sum.

$v = (x, y, z)$

$|v| = 5 \Rightarrow x^2 + y^2 + z^2 = 25$

$f(x,y,z) = x + y + z, \quad g(x,y,z) = x^2 + y^2 + z^2 - 25$

$\nabla f = \langle 1, 1, 1 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ 1 = 2\lambda z \\ x^2 + y^2 + z^2 = 25 \end{cases} \qquad x = \frac{1}{2\lambda}, \quad y = \frac{1}{2\lambda}, \quad z = \frac{1}{2\lambda}$$

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{3}{4\lambda^2} = 25$$

$$\lambda = \pm \frac{\sqrt{3}}{10}$$

$\lambda = \frac{\sqrt{3}}{10} \Rightarrow x = y = z = \frac{5}{\sqrt{3}}$

$\lambda = -\frac{\sqrt{3}}{10} \Rightarrow x = y = z = -\frac{5}{\sqrt{3}}$

$f\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right) = \frac{15}{\sqrt{3}}$ — max.

$f\left(-\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right) = -\frac{15}{\sqrt{3}}$

Hint for the last problem in ass. 7.:

1) $(a^x)' = (e^{x \ln a})' = \ln a \times e^{x \ln a} = a^x \ln a$

2) Use $\frac{d}{dx} \int_0^1 t^x \, dt = \int_0^1 \frac{\partial}{\partial x} t^x \, dt$