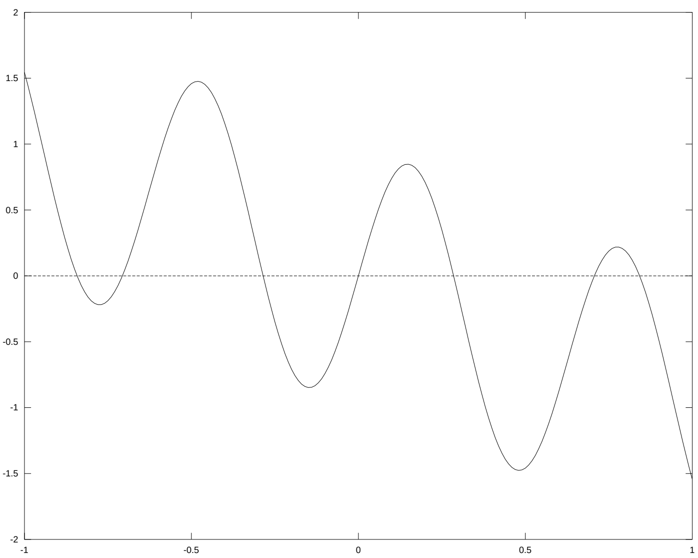
AMTH250
Assignment 7

Mark Villar

October 24, 2011

Question 1

$$f(x) = \sin(10x) - x,\ \ x \in [-1, 1]$$



The zeros of the function are $\pm 0.28523, \pm 0.70682, \pm 0.84232$ and 0.

Question 2

We plot $y = x$ and $y = \tan(x)$ and deduce that the curves will intersect for the millionth time in the interval $\left[1000000\pi, \frac{2000001}{2}\pi\right]$. Applying the fzero command to the function $f(x) = x - \tan(x)$, we obtain

$$x_{1000000} \approx 3141594.22438580$$

Question 3

(a) (i) $g_1(x) = (x^2 + 2)/3$
$g_1'(x) = 2x/3 \;\Rightarrow\; g_1'(2) = 4/3$ (divergent)

(ii) $g_2 = \sqrt{3x - 2}$
$$g_2'(x) = \frac{3}{2\sqrt{3x-2}} \;\Rightarrow\; g_2'(2) = 3/4 \quad \text{(linearly convergent)}$$

(iii) $g_3 = 3 - 2/x$
$g_3'(x) = 2/x^2 \;\Rightarrow\; g_3'(2) = 1/2$ (linearly convergent)

(iv) $g_4 = (x^2 - 2)/(2x - 3)$
$$g_4'(x) = \frac{2(x^2 - 3x + 2)}{(3 - 2x)^2} \;\Rightarrow\; g_4'(2) = 0 \quad \text{(quadratically convergent)}$$

(b) (i) This fixed point iteration fails to converge at $x = 2$.
For $x(0) < 2$, convergence does occur but at the other root, $x = 1$.

```
x1 =
Columns 1 through 7:
1.9000   1.8700   1.8323   1.7858   1.7297   1.6639   1.5895

Columns 8 through 14:
1.5089   1.4256   1.3441   1.2688   1.2033   1.1493   1.1070

Columns 15 through 21:
1.0751   1.0520   1.0356   1.0241   1.0163   1.0109   1.0073

Columns 22 through 28:
1.0049   1.0033   1.0022   1.0015   1.0010   1.0007   1.0004

Columns 29 through 35:
1.0003   1.0002   1.0001   1.0001   1.0001   1.0000   1.0000
```

For $x(0) > 2$, we can see that the iteration diverges.

```
x1 =
Columns 1 through 6:
2.1000   2.1367   2.1884   2.2631   2.3739   2.5451

Columns 7 through 12:
2.8258   3.3285   4.3595   7.0019   17.009   97.098

Columns 13 through 16:
3.1434e+003   3.2936e+006   3.6159e+012   4.3581e+024

Columns 17 through 20:
6.3311e+048   1.3361e+097   5.9506e+193   Inf
```

(ii) We confirm below that convergence occurs at $x = 2$ for this fixed point form.

```
x2 =
Columns 1 through 7:
1.9000   1.9235   1.9418   1.9559   1.9666   1.9748   1.9810

Columns 9 through 15:
1.9857   1.9893   1.9919   1.9939   1.9954   1.9966   1.9974

Columns 16 through 22:
1.9981   1.9986   1.9989   1.9992   1.9994   1.9995   1.9997

Columns 22 through 28:
1.9997   1.9998   1.9999   1.9999   1.9999   1.9999   2.0000

Columns 45 through 51:
2.0000   2.0000   2.0000   2.0000   2.0000   2.0000   2.0000
```

The ratio of errors at each step confirms that the convergence rate is approximately $3/4$.

```
ratio2 =
Columns 1 through 6:
0.76462   0.76107   0.75837   0.75631   0.75475   0.75358

Columns 7 through 12:
0.75269   0.75202   0.75152   0.75114   0.75085   0.75064

Columns 13 through 18:
0.75048   0.75036   0.75027   0.75020   0.75015   0.75011

Columns 19 through 24:
0.75009   0.75006   0.75005   0.75004   0.75003   0.75002

Columns 25 through 30:
0.75002   0.75001   0.75001   0.75001   0.75000   0.75000

Columns 45 through 50:
0.75000   0.75000   0.75000   0.75000   0.75000   0.75000
```

(iii) Convergence also occurs at $x = 2$ under this fixed point form, with an approximate rate of convergence of $1/2$.

```
x3 =
Columns 1 through 7:
1.9000   1.9474   1.9730   1.9863   1.9931   1.9965   1.9983

Columns 8 through 14:
1.9991   1.9996   1.9998   1.9999   1.9999   2.0000   2.0000

Columns 46 through 51:
2.0000   2.0000   2.0000   2.0000   2.0000   2.0000   2.0000

ratio3 =
Columns 1 through 6:
0.52632   0.51351   0.50685   0.50345   0.50173   0.50087

Columns 7 through 12:
0.50043   0.50022   0.50011   0.50005   0.50003   0.50001

Columns 13 through 18:
0.50001   0.50000   0.50000   0.50000   0.50000   0.50000

Columns 44 through 49:
0.49123   0.50000   0.50000   0.57143   0.50000   0.50000
```

(iv) The rapid convergence under this fixed point form confirms our claim of (at least) quadratic convergence at $x = 2$.

```
x4 =
1.9000   2.0125   2.0002   2.0000   2.0000   2.0000

ratio4 =
0.12500   0.01220   0.00015   0.00000   0.00000
```

Question 4

(a) We define the reciprocal of some number $y > 0$ as a zero of the function

$$f(x) = x - \frac{1}{y} = 0$$

Rewriting this equation gives

$$\begin{aligned} f(x) &= xy - 1 = 0 \\ &= y - \frac{1}{x} = 0 \end{aligned}$$

Applying Newton's method,

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{y - \frac{1}{x}}{\frac{1}{x^2}} \\ &= x_k + (x_k - yx_k^2) \\ &= x_k(2 - yx_k) \end{aligned}$$

(b) Please see Appendix for Octave output.

```
function x = reciprocal(x0, n)
x = zeros(1, n+1);
x(1) = x0;
for k = 1:n
x(k+1) = x(k)*(2-2*x(k));
end
endfunction
```

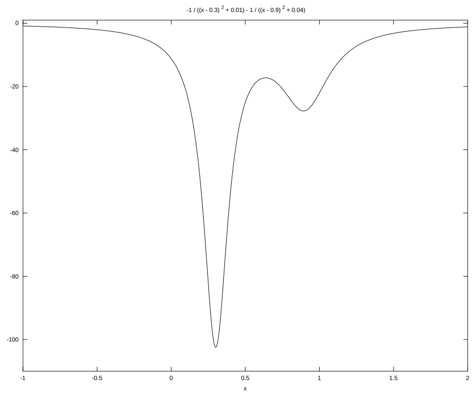
(i) $x(0) > 1$ or $x(0) < 0$ diverges

(ii) $x(0) = 0.99999999999999999444991910$ converges to 0 at $x(1)$

(iii) $0 < x(0) < 0.99999999999999999444991910$ converges to $1/2$

Question 5

(a) We find the global minimum of $-f(x)$ to determine the maximum of $f(x)$. The graph below shows that the interval $\left(0, \frac{1}{2}\right)$ contains the minimum of $-f(x)$.



(i) We use $\left(0, \frac{1}{2}\right)$ as the initial interval.

```
[a,b] = goldsec(f, 0, 0.5, 1e-8)
a =  0.300375617070495
b =  0.300375626326580
```

$$-f(\mathrm{a}) = 102.501408560372$$
$$-f(\mathrm{b}) = 102.501408560372$$

(ii) We use $\left(0, \frac{1}{5}, \frac{1}{2}\right)$ as the initial bracket for the minimum.

```
p = parab(f, 0, 0.2, 0.5, 10)
Columns 1 through 3:
0.273575619912928   0.335215298966089   0.297714018039367

Columns 4 through 6:
0.300794515485728   0.300508503190500   0.300375573369255

Columns 7 through 9:
0.300375603984782   0.300375621620731   0.300375621641481

Column 10:
0.300375621631106
```

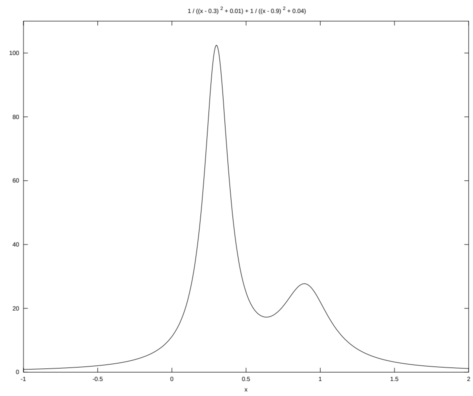
$$-f(\mathrm{p}) = 102.501408560372$$

(iii) We use $\left(0, \frac{1}{2}\right)$ as the initial interval.

```
m = fminbnd(f, 0, 0.5)
m =  0.300375621982956
```

$$-f(\mathrm{m}) = 102.501408560372$$

The graph of $f(x)$ below confirms that its global maximum is approximately 102.5 attained at $x \approx 0.30$.



(b) To determine our best estimate, we compare the different function values under each method and check for the largest $-f(x^*)$.

$$\begin{aligned}
-f(x_{\mathrm{a}}) - (-f(x_{\mathrm{b}})) = 0 &\Rightarrow -f(x_{\mathrm{a}}) = -f(x_{\mathrm{b}}) \\
-f(x_{\mathrm{p}}) - (-f(x_{\mathrm{a}})) \approx 2.1316 \times 10^{-13} &\Rightarrow -f(x_{\mathrm{p}}) > -f(x_{\mathrm{a}}) \\
-f(x_{\mathrm{m}}) - (-f(x_{\mathrm{a}})) \approx 2.1316 \times 10^{-13} &\Rightarrow -f(x_{\mathrm{m}}) > -f(x_{\mathrm{a}}) \\
-f(x_{\mathrm{m}}) - (-f(x_{\mathrm{p}})) = 0 &\Rightarrow -f(x_{\mathrm{m}}) = -f(x_{\mathrm{p}})
\end{aligned}$$

We conclude that parab and fminbnd provide a better estimate than goldsec. Moreover, there is negligible difference (if any at all) between the former two methods. We estimate the accuracy of each method by evaluating $f''(x^*)$.

$$\begin{aligned}
f''(x) &= \frac{8(x-0.9)^2}{\left((x-0.9)^2+0.04\right)^3} + \frac{8(x-0.3)^2}{\left((x-0.3)^2+0.01\right)^3} \\
&\quad - \frac{2}{\left((x-0.9)^2+0.04\right)^2} - \frac{2}{\left((x-0.3)^2+0.01\right)^2} \\
f''(x_{\mathrm{p}}) &= -19965.73926483396 \\
f''(x_{\mathrm{m}}) &= -19965.73926159880
\end{aligned}$$

Analysis of rounding errors show that parab has greater accuracy than fminbnd since $\Delta x_{\mathrm{p}} < \Delta x_{\mathrm{m}}$.

$$\Delta x_{\mathrm{p}} = \sqrt{\frac{2\varepsilon_{\mathrm{mach}}}{|f''(x_{\mathrm{p}})|}} = 1.49139407334926 \times 10^{-10}$$

$$\Delta x_{\mathrm{m}} = \sqrt{\frac{2\varepsilon_{\mathrm{mach}}}{|f''(x_{\mathrm{m}})|}} = 1.49139407347009 \times 10^{-10}$$

Thus, successive parabolic interpolation gives the best estimate for finding the maximum of $f(x)$.

Question 6

We express the relationship between the angle α and distance x by the following implicit function.

$$f(x, \alpha) = 0.0122583125x^2 - x \sin\alpha \cos\alpha + 13\cos^2\alpha = 0$$

To find the maximum distance x at which the target can be reached, we define $x = g(\alpha)$ where g takes a value of α and returns x as the larger root of $f(x, \alpha)$. We then determine the range of α for which $f(x, \alpha)$ has real roots to know where $g(\alpha)$ is defined. Using an iterative loop to compute the roots of each polynomial generated and disregarding any complex roots, we find the maximum distance $x \approx 24.56$ metres.

Appendix

1. ```
function y=wave(x)
y=sin(10*x)-x;
endfunction

x=linspace(-1,1,201);
plot(x,wave(x))
hold on
plot([-1 1],[0 0])
print('wave.eps','-deps')

fzero(@wave,[-1,-0.8])
fzero(@wave,[-0.8,-0.5])
fzero(@wave,[-0.5,-0.1])
fzero(@wave,[-0.1,0.1])
fzero(@wave,[0.1,0.5])
fzero(@wave,[0.5,0.8])
fzero(@wave,[0.8,1])
```

2. ```
function y=tangent(x)
y=x-tan(x);
endfunction

fzero(@tangent,[1000000*pi,1000000.5*pi])
```

3. ```
function x=iterate(g,x0,n)
x=zeros(1,n+1);
x(1)=x0;
for i=1:n
x(i+1)=g(x(i));
end
endfunction

function y=gone(x)
y=(x.^2+2)/3;
endfunction

x1=iterate(@gone,1.9,100)
x1=iterate(@gone,2.1,100)

function y=gtwo(x)
y=sqrt(3*x-2);
endfunction
```
```

```
x2=iterate(@gtwo,1.9,50)
err2=abs(x2-2);
ratio2=err2(2:51)./err2(1:50)

function y=gthree(x)
y=3-2./x;
endfunction

x3=iterate(@gthree,1.9,50)
err3=abs(x3-2);
ratio3=err3(2:50)./err3(1:49)

function y=gfour(x)
y=(x.^2-2)/(2*x-3);
endfunction

x4=iterate(@gfour,1.9,5)
err4=abs(x4-2);
ratio4=err4(2:6)./err4(1:5)
```

4. (a) `reciprocal(0.9,20)`

```
Columns 1 through 6:
0.90000   0.18000   0.29520   0.41611   0.48593   0.49960

Columns 7 through 13:
0.50000   0.50000   0.50000   0.50000   0.50000   0.50000
```

(b) (i) `reciprocal(1.01,20)`

```
Columns 1 through 4:
1.0100e+000   -2.0200e-002   -4.1216e-002   -8.5830e-002

Columns 14 through 17:
-1.4177e+070   -4.0196e+140   -3.2315e+281   -Inf

reciprocal(-0.99,20)
Columns 1 through 4:
-9.9000e-001   -3.9402e+000   -3.8931e+001   -3.1091e+003

Columns 8 through 11:
-2.5041e+060   -1.2541e+121   -3.1456e+242    -Inf
```

(ii) `reciprocal(0.99999999999999999944499191,10)`

```
1   0   0   0   0   0   0   0   0   0   0
```

(iii) `reciprocal(0.99999999999999994449919099999,60)`

```
Columns 1 through 4:
1.0000e+000  2.2204e-016  4.4409e-016  8.8818e-016

Columns 51 through 54:
1.1060e-001  1.9673e-001  3.1606e-001  4.3233e-001

Columns 55 through 59:
4.9084e-001  4.9983e-001  5.0000e-001  5.0000e-001
```

5. (a)
```
f=@(x) -1./((x-0.3).^2+0.01)-1./((x-0.9).^2+0.04);
ezplot(f,[-1,2])
axis([-1 2 -110 1])
print('goldsec.eps','-deps')
```

   (b)
```
d2f=@(x) (8.*(x-.9).^2)./((x-.9).^2+.04).^3
+ (8.*(x-.3).^2)./((x-.3).^2+.01).^3
- 2./((x-.9).^2+.04).^2 - 2./((x-.3).^2+.01).^2

d2fp=d2f(p(10));
rp=sqrt(2*eps/abs(d2fp))
rm=sqrt(2*eps/abs(d2f(m)))
```

6.
```
a=linspace(0,pi/2,100)';
c=9.80665/800;
p=[c*ones(length(a),1) -sin(a).*cos(a) 13*cos(a).^2];
pr=zeros(length(a),2);
maxpr=zeros(length(a),1);

for i=1:length(a)
pr(i,1:2)=roots(p(i,:));
maxpr(i)=max(pr(i,1:2));
xmax=max(real(maxrp))
end

xmax
```