MATH101 ASSIGNMENT 10

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(1) (a) $\vec{PQ} = [1-(-1), 0-0, 0-0] = [2,0,0] = 2\mathbf{i}$

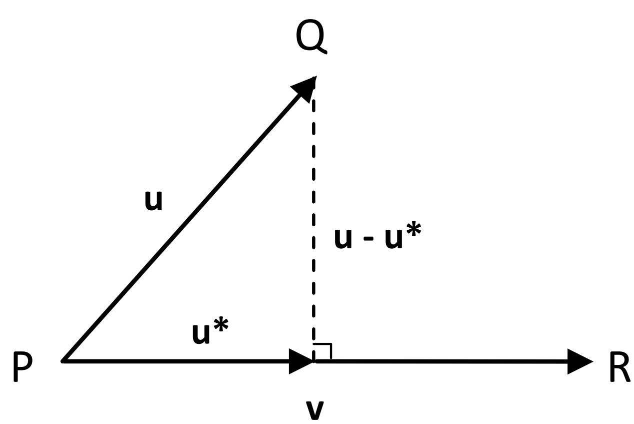
$\vec{PR} = [1-(-1), 1-0, 1-0] = [2,1,1] = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) Let $\mathbf{u} = \vec{PQ}$ and $\mathbf{v} = \vec{PR}$. Then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{u}^* = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\,\mathbf{v} = \frac{4+0+0}{\left(\sqrt{4+1+1}\right)^2}\,\mathbf{v} = \frac{2}{3}\,[2,1,1]$$
$$= \left[\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right] = \frac{4}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

Moreover, the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is given by

$$\mathbf{u} - \mathbf{u}^* = \left[2 - \frac{4}{3}, 0 - \frac{2}{3}, 0 - \frac{2}{3}\right]$$
$$= \left[\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right] = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

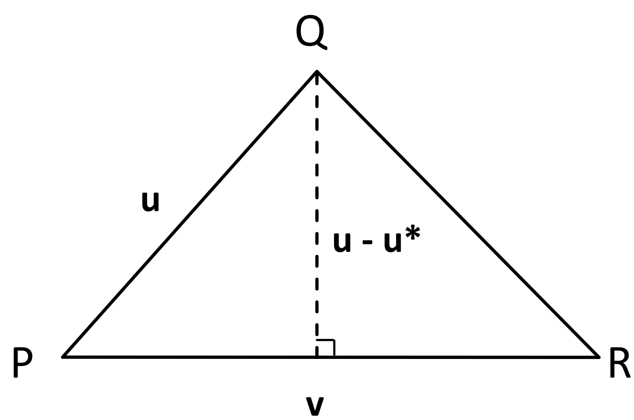


(c) The triangle with vertices P, Q and R has the following dimensions,

$$b = \|\vec{PR}\| = \|\mathbf{v}\| = \sqrt{6}$$
$$h = \|\mathbf{u} - \mathbf{u}^*\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \frac{2\sqrt{3}}{3}$$

where b is the base and h is the (perpendicular) height. Its area is therefore

$$A = \frac{1}{2}\, bh = \frac{1}{2}\left(\sqrt{6}\right)\left(\frac{2\sqrt{3}}{3}\right) = \sqrt{2}$$



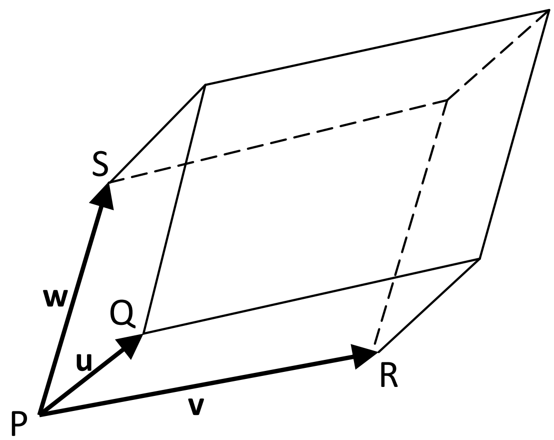
(d) Let $\mathbf{w} = \vec{PS} = [2 - (-1), 1 - 0, 2 - 0] = [3, 1, 2] = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The volume of the parallelepiped with sides given by the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ is given by the absolute value of the scalar triple product below.

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \quad \text{where} \quad \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding by the first row, where c_{ij} are the cofactors of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$:

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1(1 \times 2 - 1 \times 1) = 1$$

Hence $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = c_{11} \times 2 + c_{12} \times 0 + c_{13} \times 0 = 1 \times 2 = 2$ and $V = |2| = 2$



(2) If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ are three vectors in $\mathbb{R}^3$ then,

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

Furthermore the vector triple product is given by

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & -(v_1w_3 - v_3w_1) & v_1w_2 - v_2w_1 \end{vmatrix}$$

Thus the $\mathbf{i}$-component of $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is given by

$$\begin{aligned} &= u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ &= u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3 \\ &= v_1(u_2w_2 + u_3w_3) - w_1(u_2v_2 + u_3v_3) \\ &= v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v}) \\ &= (\mathbf{u} \cdot \mathbf{w})v_1 - (\mathbf{u} \cdot \mathbf{v})w_1 \end{aligned}$$

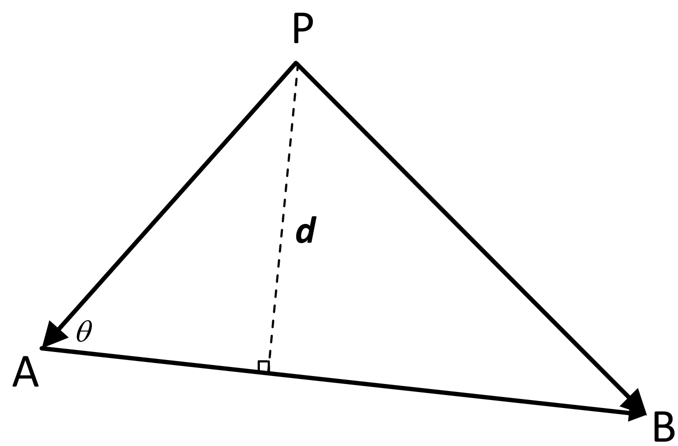
By symmetry, $(\mathbf{u} \cdot \mathbf{w})v_2 - (\mathbf{u} \cdot \mathbf{v})w_2$ and $(\mathbf{u} \cdot \mathbf{w})v_3 - (\mathbf{u} \cdot \mathbf{v})w_3$ are the $\mathbf{j}$- and $\mathbf{k}$-components of $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. Combining all three components therefore gives

$$\begin{aligned} &= (\mathbf{u} \cdot \mathbf{w})(v_1 + v_2 + v_3) - (\mathbf{u} \cdot \mathbf{v})(w_1 + w_2 + w_3) \\ &= (\mathbf{u} \cdot \mathbf{w})\ \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\ \mathbf{w} \end{aligned}$$

(3) I was unable to answer this successfully so my first attempt is shown instead, which assumes a slight variation to the phrasing of the question.

Let $\angle P\hat{A}B = \theta$. Then,

$$\|\vec{PA} \times \vec{AB}\| = \|\vec{PA}\|\|\vec{AB}\| \sin\theta$$



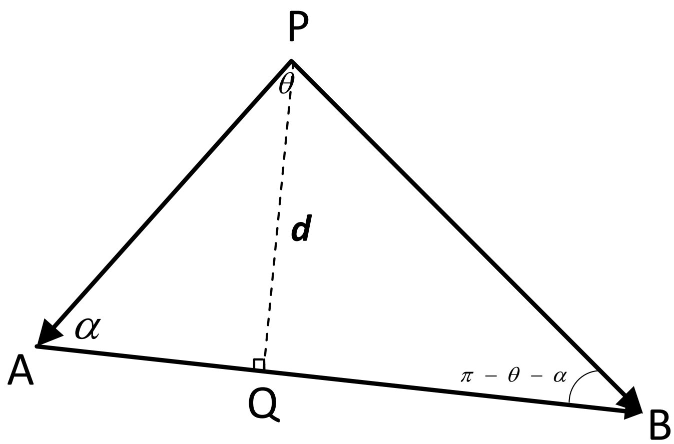
If d is the perpendicular distance from the point P to the line AB, then the right-angle $\triangle AQP$ (where Q is a point on $\vec{AB}$ and $\angle A\hat{Q}P = 90°$) implies

$$\sin\theta = \frac{d}{\|\vec{PA}\|}$$

It follows that $d = \|\vec{PA}\| \sin\theta$ and thus $\|\vec{PA} \times \vec{AB}\| = d\ \|\vec{AB}\|$. Rearranging gives

$$d = \frac{\|\vec{PA} \times \vec{AB}\|}{\|\vec{AB}\|}$$

However, if we were to answer the original question we produce the following diagram and calculations.



Let $\angle A\hat{P}B = \theta$. Then,

$$\|\vec{PA} \times \vec{PB}\| = \|\vec{PA}\|\|\vec{PB}\| \sin\theta \Rightarrow \|\vec{PA}\| = \frac{\|\vec{PA} \times \vec{PB}\|}{\|\vec{PB}\| \sin\theta}$$

If $\angle P\hat{A}B = \alpha$, it follows that

$$\sin\alpha = \frac{d}{\|\vec{PA}\|} \Rightarrow d = \|\vec{PA}\| \sin\alpha$$

$$= \frac{\|\vec{PA} \times \vec{PB}\| \sin\alpha}{\|\vec{PB}\| \sin\theta}$$

In order to show that

$$d = \frac{\|\vec{PA} \times \vec{PB}\|}{\|\vec{AB}\|}$$

we must therefore show that

$$\frac{1}{\|\vec{AB}\|} = \frac{\sin\alpha}{\|\vec{PB}\| \sin\theta}$$

which we fail to do due to some flaw in our reasoning.

(4) We solve the following systems of linear equations using Gauss-Jordan elimination.

(a)

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ -3 & -6 & 12 & -12 \end{array}\right]$$

$$\begin{array}{c} \\ -\frac{1}{3}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ 1 & 2 & -4 & 4 \end{array}\right]$$

Hence the second equation is simply a multiple of the first. It follows that the two planes coincide and that all points (x, y, z) on the plane satisfy the pair of equations.

(b)

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -18 \\ 0 & 2 & -1 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3 + 4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -18 \\ 0 & 2 & -1 & 8 \\ 0 & -3 & 13 & -81 \end{array}\right]$$

$$\begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 2 & -1 & 8 \\ 0 & -3 & 13 & -81 \end{array}\right]$$

$$\begin{array}{c} \\ \frac{1}{2}R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & -3 & 13 & -81 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3 + 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & \frac{23}{2} & -69 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -\frac{1}{23}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & -\frac{1}{2} & 3 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} & 3 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -2R_3 \end{array} \quad \left.\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -6 \end{array}\right.$$

Thus,

$$x=-10, \quad y=1, \quad z=-6$$

(c)

$$\left.\begin{array}{rrr|r} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right.$$

$$\begin{array}{l} R_1+\frac{1}{2}R_2 \\ \\ \\ \end{array} \quad \left.\begin{array}{rrr|r} 1 & -\frac{1}{2} & -3 & \frac{17}{2} \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right.$$

$$\begin{array}{l} \\ R_2-2R_1 \\ \\ \end{array} \quad \left.\begin{array}{rrr|r} 1 & -\frac{1}{2} & -3 & \frac{17}{2} \\ 0 & -2 & 8 & -16 \\ 5 & -8 & 7 & 1 \end{array}\right.$$

$$\begin{array}{l} \\ \\ R_3-5R_1 \end{array} \quad \left.\begin{array}{rrr|r} 1 & -\frac{1}{2} & -3 & \frac{17}{2} \\ 0 & -2 & 8 & -16 \\ 0 & -\frac{11}{2} & 22 & -\frac{83}{2} \end{array}\right.$$

$$\begin{array}{l} R_1-\frac{1}{4}R_2 \\ \\ \\ \end{array} \quad \left.\begin{array}{rrr|r} 1 & 0 & -5 & \frac{25}{2} \\ 0 & -2 & 8 & -16 \\ 0 & -\frac{11}{2} & 22 & -\frac{83}{2} \end{array}\right.$$

$$\begin{array}{l} \\ -\frac{1}{2}R_2 \\ \\ \end{array} \quad \left.\begin{array}{rrr|r} 1 & 0 & -5 & \frac{25}{2} \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{11}{2} & 22 & -\frac{83}{2} \end{array}\right.$$

$$\begin{array}{l} \\ \\ -\frac{2}{11}R_3 \end{array} \quad \left.\begin{array}{rrr|r} 1 & 0 & -5 & \frac{25}{2} \\ 0 & 1 & -4 & 8 \\ 0 & 1 & -4 & \frac{83}{11} \end{array}\right.$$

$$\begin{array}{l} \\ \\ R_3-R_2 \end{array} \quad \left.\begin{array}{rrr|r} 1 & 0 & -5 & \frac{25}{2} \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & -\frac{5}{11} \end{array}\right.$$

The system is therefore inconsistent and there is no unique solution.

(5) A condition on the coefficients of a_{ij} that guarantees a consistent system is

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

This reduces the general system of linear equations in three unknowns to the following augmented matrix.

$$\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}$$

Thus we have a unique solution in

$$x_1 = b_1, \quad x_2 = b_2, \quad x_3 = b_3$$

which implies that the system is consistent.