AMTH250 Assignment 2

Due: 15th August

Reproduce the following *[2 marks for each question]*:

Question 1

The reduced cubic equation $y^3 + 3py + 2q = 0$ has one real and two complex solutions when $D = q^2 + p^3 > 0$. These are given by Cardano's formula as

$$y_1 = u + v, \quad y_2 = -\frac{u+v}{2} + \frac{i}{2}\sqrt{3}(u-v), \quad y_3 = -\frac{u+v}{2} - \frac{i}{2}\sqrt{3}(u-v)$$

where

$$u = \sqrt[3]{-q + \sqrt{q^2+p^3}}, \qquad v = \sqrt[3]{-q - \sqrt{q^2+p^3}}$$

Question 2

An $n \times n$ matrix $\mathbf{A}$ is *non-singular* if it satisfies any one of the following equivalent conditions:

1. $\mathbf{A}$ has an inverse, i.e. there is a matrix $\mathbf{A}^{-1}$ such that
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$
2. $\det \mathbf{A} \neq 0$.
3. $\operatorname{rank} \mathbf{A} = n$. (The rank of a matrix is the maximum number of linearly independent rows or columns of the matrix.)
4. For every non-zero vector $\mathbf{z}$, $\mathbf{A}\mathbf{z} \neq 0$.

Question 3

The *gamma function* $\Gamma(x)$ is defined by

$$\begin{aligned}\Gamma(x) &\equiv \lim_{n\to\infty} \prod_{\nu=0}^{n-1} \frac{n!\, n^{x-1}}{x+\nu} \\ &= \lim_{n\to\infty} \frac{n!\, n^{x-1}}{x(x+1)\cdots(x+n-1)} \\ &= \int_0^\infty e^{-t} t^{x-1}\, dt\end{aligned}$$

The integral definition is only valid for $x > 0$.

Question 4

Given an n-vector $\mathbf{a}$, we can annihilate **all** of its entries below the kth position, provided that $a_k \neq 0$, by the following transformation:

$$\mathbf{M}_k\mathbf{a} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & -m_{k+1} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -m_n & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $m_i = a_i/a_k$, $i = k+1, \dots, n$.

Question 5

The absolute and relative error are defined by

$$\text{Absolute error} = \text{Approximate value} - \text{True value} \tag{1}$$

$$\text{Relative error} = \frac{\text{Absolute error}}{\text{True value}} \tag{2}$$

A useful way to think of relative error is via the expression

$$\text{Approximate value} = \text{True value} \times (1 + \text{Relative error}) \tag{3}$$

Question 6

Differentiating the differential equation

$$\frac{dy}{dt} = f(t, y)$$

gives

$$\frac{d^2y}{dt^2} = \frac{d}{dt}f(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f$$

and we have the Taylor series approximation

$$\begin{aligned} y(t+h) &\approx y(t) + y'(t)h + \frac{1}{2}y''(t)h^2 \\ &= y(t) + hf(t,y) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)f(t,y)\right) \end{aligned}$$

Question 7

No Pivoting		Partial Pivoting		Complete Pivoting	
Error	Residual	Error	Residual	Error	Residual
8.964	2.07×10^{-14}	0.156	2.53×10^{-16}	0.164	2.53×10^{-16}
1.426	2.77×10^{-15}	0.113	2.04×10^{-16}	0.175	2.93×10^{-16}
0.883	3.60×10^{-16}	0.080	2.97×10^{-16}	0.036	3.48×10^{-16}

Table 1: Errors and residuals for 3 random 100×100 matrices.

Question 8

Payoff ($)

Player 1			Player 2	
1	2	3	1	3
4	5	6	2	5
			3	6

Doing Assignment 2

Make sure you have successfully completed Assignment 1 before you attempt Assignment 2.

1 Working with LaTeX

The process of creating a LaTeX document typically runs through a cycle:

1. Write some stuff.
2. Run LaTeX.
3. Correct errors.
4. View the `.pdf` file.
5. Make sure it looks right.
6. Write some more, etc.

It is important that you correct LaTeX errors as soon as they appear. Section 1.2 below has some hints on understanding LaTeX error messages.

1.1 LaTeX Source Files

Although LaTeX doesn't care about the formatting of input files you will find it easier to revise and edit your own work if you follow some simple rules when creating input files:

1. Don't make lines too long and most importantly don't make lines so long that they wrap around to the next line. This will require a change of habit for those used to working with Microsoft Word, but you need to remember that the appearance of LaTeX documents is determined by LaTeX commands and not how the source file is formatted.
2. Start new sentences on a new line.
3. Start displayed formulas on a new line. Don't be afraid of using line breaks in the middle of equations to improve readability.
4. Use indentation to make the structure of things like `enumerate`, `tabular` and `align` environments easier to comprehend. There are plenty of examples in the study guide.

1.2 A Point on LaTeX Style

Sometimes it can be difficult to decide when to switch to and from mathematics mode. When in doubt you should follow the logical structure of what you are working with. Here is an instructive example from the assignment:

In point 3 of Question 2 you might be tempted to try something like:

```
\item rank $A = n$
```

This is not good style since since "rank A" is a single mathematical expression. In any case this wouldn't work if the expression occurred in the middle rather than at the start of a formula. It is better to start with something like

```
\item $\text{rank} A = n$
```

even if you have to work harder on getting the spacing right.

1.3 Error Messages

The error messages produced by LaTeX can appear somewhat cryptic and difficult to understand.

1. Perhaps the most common error is misspelling LaTeX commands. For example, if in the previous sentence I had typed `\LateX{}` instead of `\LaTeX{}`, then LaTeX would respond with

   ```
   ! Undefined control sequence.
   l.65 The error messages produced by \LateX
                                                {} can appear somewhat
   ?
   ```

   The `l.65 ...` refers to line 65 of the input file which is a useful hint as to where the error occurred. If you type `h` Enter (in the bottom Console bar in Texworks or at the command line in Linux) LaTeX will respond with

   ```
   The control sequence at the end of the top line
   of your error message was never \def'ed. If you have
   misspelled it (e.g., `\hobx'), type `I' and the correct
   spelling (e.g., `I\hbox'). Otherwise just continue,
   and I'll forget about whatever was undefined.

   ?
   ```

   This pins down the error to a mistake in typing `\LaTeX`. The best thing to do is to type `x` Enter to exit LaTeX and go back to the source file and correct the error.

2. Another common error is forgetting to close off the maths environments, `$` and `$$`, or close off braces `{`. If you type something like

   ```
   $ x^2 + y^2 = z^2 .....
   ```

   and forget to close off the `$` sign, then you will see

   ```
   ! Missing $ inserted.
   <inserted text>
                   $
   l.110
   ```

   Knowing the line number and that a `$` is missing is enough to allow you to track down the error. If you ask for help, `h` Enter, LaTeX will respond with

   ```
   ? h
   I've inserted a begin-math/end-math symbol since I think
   you left one out. Proceed, with fingers crossed.
   ```

   Again, the best thing to do is to type `x` Enter to exit LaTeX and go back to the source file and correct the error. **Do not** rely on LaTeX to correct the error for you since this will not fix your source file and in any case LaTeX can only guess where the missing `$` should go.

3. LaTeX often gives messages beginning like

   ```
   Overfull \hbox (36.65945pt too wide) in paragraph at lines 160--160
   ```

   This is not a LaTeX error – it just means that a line is too long and LaTeX can't find a way to insert a line break. A point is 1/72 inch or about 1/3 mm so this line is a bit more than 1 cm too long.

   There are two common causes: (i) `\verb` commands or `verbatim` environments as in this example, or (ii) LaTeX can't work out to break things up so that the text can be justified while maintaining reasonable spacing and what it knows about hyphenation. In most day-to-day applications you can live with small `Overfull \hbox`'s, but if you need to fix the second cause a simple rewording of the offending sentence will usually do the trick.

2 Hints for Assignment 2

2.1 Outline

Your LaTeX input file should look something like:

```
\documentclass[11pt,a4paper]{article}

\usepackage{amsmath}

\title{AMTH250 \\ Assignment 2}
\author{...}

\begin{document}

\maketitle

\subsubsection*{Question 1}

The reduced cubic equation $y^3 + 3py + 2q = 0$ has ...

    .
    .
    .

\end{document}
```

2.2 General Remarks

In marking the assignment I will also take into account whether you have chosen the correct way to do things. Many constructions can be done more than one way, for example the numbering in Question 2, but there is always a simple direct way to do things.

I will also take into account the readability of your `.tex` file (see §1.1 above). Lack of indentation and wrapped lines are the two things that contribute most to lack of readability.

2.3 General Comments on the Questions

(a) Most of these questions have been taken from lecture notes I have written for other units. The aim has been to give examples of the type of mathematics as is written in practice.

(b) All you need to do the assignment has been covered in the study guide and I have attempted to ensure everything can be done in a straight-forward way.

(c) I have not included things that are deliberately tricky, but be aware that there are some subtleties.

(d) Be careful about the distinction between `\textsl` and `\textit`.

(e) Watch your indentation and spacing.

(f) I have consistently used `align` rather than `eqnarray`.

2.4 Comments on Individual Questions

Question 1

The only likely errors are getting the spacing, `\quad` and `\qquad`, between formulas wrong or having the "where" between the displayed formulas indented.

Question 2

(a) Getting the spacing in "rank A = n" correct is not easy, but there are at least two ways to do it (see the remarks in §1.2 on this point).

(b) The end of the last line could conceivably been written (ignoring the bold font) as either `$z, Az \neq 0$` or `$z$, $Az \neq 0$`. Which is correct? (The same applies to last line of Question 4.)

Question 3

I have used some maths spacing, i.e. `\thinspace` etc., in the formulas.

Question 5

Be careful about spacing and alignment.

Question 7

This uses a `table` environment. Be careful with the numbers in scientific notation.

Question 8

Here we have tables within a table. The two sub-tables form one row of the main table. Here is an outline of the construction:

```
\begin{center}
  \begin{tabular}{cc}      % main table
    ...
    ...
    \begin{tabular}{ccc}   % sub-table 1
      ....
      ....
    \end{tabular} &
    \begin{tabular}{cc}    % sub-table 2
      ....
      ....
    \end{tabular} \\
  \end{tabular}
\end{center}
```