# MATH101 ASSIGNMENT 7

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(1) (a) Let $f : \mathbb{R}\backslash\{\left(\frac{(2n+1)\pi}{2}\right)^{\frac{1}{3}} \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \tan(x^3)$
By the chain rule and Lemma 7.7(i) of the lecture notes (p.94)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\tan(x^3) = \left(\frac{d}{dx}x^3\right)\sec^2(x^3) \\ &= 3x^2\sec^2(x^3) \end{aligned}$$

(b) Let $g : \mathbb{R}\backslash\{n\pi \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad t \longmapsto \frac{1-\cos t}{1+\sin t}$
By the quotient rule

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left(\frac{1-\cos t}{1+\sin t}\right) \\ &= \frac{(1+\sin t)(\sin t) - (1-\cos t)(\cos t)}{(1+\sin t)^2} \\ &= \frac{\sin t + \sin^2 t - \cos t + \cos^2 t}{(1+\sin t)^2} \\ &= \frac{1+\sin t - \cos t}{(1+\sin t)^2} \\ &= \frac{1}{1+\sin t} - \frac{\cos t}{(1+\sin t)^2} \end{aligned}$$

(c) Let $h : \mathbb{R} \longrightarrow \mathbb{R}, \quad s \longmapsto \ln(1+s^4)$
By the chain rule

$$\begin{aligned} h'(s) &= \frac{d}{ds}\left(1+s^4\right) \cdot \left(\frac{1}{1+s^4}\right) \\ &= \frac{4s^3}{1+s^4} \end{aligned}$$

(d) Let $j : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^x \sin x$
By the product rule

$$\begin{aligned} j'(x) &= e^x\left(\frac{d}{dx}\sin x\right) + \sin x\left(\frac{d}{dx}e^x\right) \\ &= e^x \cdot \cos x + \sin x \cdot e^x \\ &= e^x(\cos x + \sin x) \end{aligned}$$

(e) Let $k : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \frac{e^t}{e+t^2}$
By the quotient rule

$$\begin{aligned} k'(t) &= \frac{d}{dt}\left(\frac{e^t}{e+t^2}\right) \\ &= \frac{(e+t^2)(e^t) - (e^t)(2t)}{(e+t^2)^2} \\ &= \frac{e^t(e+t^2-2t)}{(e+t^2)^2} \\ &= \frac{e^t(e+t^2)}{(e+t^2)^2} - \frac{(e^t)(2t)}{(e+t^2)^2} \\ &= \frac{e^t}{e+t^2} - \frac{2e^t t}{(e+t^2)^2} \end{aligned}$$

(f) Let $m : \mathbb{R}\backslash\{n\pi \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(\sin^2 x)$
By the chain rule

$$\begin{aligned} m'(x) &= \frac{d}{dx}\sin^2 x \cdot \left(\frac{1}{\sin^2 x}\right) \\ &= \frac{2\sin x\cos x}{\sin^2 x} \\ &= \frac{2\cos x}{\sin x} \\ &= 2\cot x \end{aligned}$$

(g) Let $p : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{\cos x}$
By the chain rule

$$\begin{aligned} p'(x) &= \left(\frac{d}{dx}\cos x\right) e^{\cos x} \\ &= -\sin x \cdot e^{\cos x} \\ &= -e^{\cos x}\sin x \end{aligned}$$

(2) Using implicit differentiation, we find $\frac{dy}{dx}$ for the following equations.

(a)

$$\begin{aligned} \frac{d}{dx}\left(x^3y + x^2y^2 - 5x - 10\right) &= 3x^2 + x^3\,\frac{dy}{dx} + 2xy^2 + 2x^2y\,\frac{dy}{dx} - 5 = 0 \\ &= (x^3 + 2x^2y)\,\frac{dy}{dx} + 3x^2 + 2xy^2 - 5 = 0 \end{aligned}$$

If $x^3 + 2x^2y \neq 0$,

$$\frac{dy}{dx} = \frac{5 - 3x^2y - 2xy^2}{x^3 + 2x^2y}$$

(b)

$$\frac{d}{dx}\left(y^2\cos x + xe^y\right) = -y^2\sin x + 2y\cos x\ \frac{dy}{dx} + e^y + xe^y\ \frac{dy}{dx} = 0$$
$$= (2y\cos x + xe^y)\ \frac{dy}{dx} - y^2\sin x + e^y = 0$$

If $2y\cos x + xe^y \neq 0$,

$$\frac{dy}{dx} = \frac{y^2\sin x - e^y}{2y\cos x + xe^y}$$

(c)

$$\frac{d}{dx}\left(x^3 + \tan(x+y) - 2\right) = 3x^2 + \sec^2(x+y)\left(1 + \frac{dy}{dx}\right) = 0$$
$$= \sec^2(x+y)\ \frac{dy}{dx} + 3x^2 + \sec^2(x+y) = 0$$

If $\sec^2(x+y) \neq 0$,

$$\frac{dy}{dx} = \frac{-3x^2 - \sec^2(x+y)}{\sec^2(x+y)}$$
$$= -3x^2\cos^2(x+y) - 1$$

(d)

$$\frac{d}{dx}\left(xy - x^3\ln(x+y)\right) = y + x\ \frac{dy}{dx} - 3x^2\ln(x+y) - x^3\left(\frac{1}{x+y}\right)\left(1 + \frac{dy}{dx}\right) = 0$$
$$= \left(x - \frac{x^3}{x+y}\right)\frac{dy}{dx} + y - 3x^2\ln(x+y) - \frac{x^3}{x+y} = 0$$

$$\left(x^2 + xy - x^3\right)\frac{dy}{dx} = -xy - y^2 + 3x^3\ln(x+y) + 3x^2y\ln(x+y) + x^3$$

If $x^2 + xy - x^3 \neq 0$,

$$\frac{dy}{dx} = \frac{-xy - y^2 + 3x^3\ln(x+y) + 3x^2y\ln(x+y) + x^3}{x^2 + xy - x^3}$$
$$= \frac{x^3 - y^2 - xy + 3xy + 3x^2y\ln(x+y)}{x^2 + xy - x^3}$$
$$= \frac{x^3 - y^2 + 2xy + 3x^2y\ln(x+y)}{x^2 + xy - x^3}$$

(3) (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^3; \quad [0,1]$
Since $f$ is a polynomial function, it is differentiable everywhere and hence it satisfies the hypotheses of the Mean Value Theorem on [0,1]. Now

$$\frac{f(1)-f(0)}{1-0} = \frac{1^3-0^3}{1} = 1$$

As $f'(x) = 3x^2$,

$$f'(c) = \frac{f(1)-f(0)}{1-0} = 1 \text{ if and only if } 3c^2 = 1$$

Equivalently, $c^2 = \frac{1}{3}$ and $c = \pm\frac{1}{9}$. But only $c = \frac{1}{9}$ lies in $[0,1]$. Thus,

$$f^c = \frac{f(1)-f(0)}{1-0} \text{ for } c \in [0,1] \text{ if and only if } c = \frac{1}{9}$$

(b) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{x}; \quad [1,2]$
Since $f$ is a hyperbolic function, it is differentiable (and continuous) over $\mathbb{R}^+\backslash\{0\}$ and hence it satisfies the hypotheses of the Mean Value Theorem on [1,2]. Now

$$\frac{f(2)-f(1)}{2-1} = \frac{\frac{1}{2}-\frac{1}{1}}{1} = -\frac{1}{2}$$

As $f'(x) = -\frac{1}{x^2}$,

$$f'(c) = \frac{f(2)-f(1)}{2-1} = -\frac{1}{2} \text{ if and only if } -\frac{1}{c^2} = -\frac{1}{2}$$

Equivalently, $c^2 = 2$ and $c = \pm\sqrt{2}$. But only $c = \sqrt{2}$ lies in $[1,2]$. Thus,

$$f^c = \frac{f(2)-f(1)}{2-1} \text{ for } c \in [1,2] \text{ if and only if } c = \sqrt{2}$$

(c) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \ln x; \quad [1,e]$
Since $f$ is a logarithmic function, it is differentiable over $\mathbb{R}^+\backslash\{0\}$ and hence it satisfies the hypotheses of the Mean Value Theorem on [1,e]. Now

$$\frac{f(e)-f(1)}{e-1} = \frac{\ln e - \ln 1}{e-1} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

As $f'(x) = \frac{1}{x}$,

$$f'(c) = \frac{f(e)-f(1)}{e-1} = \frac{1}{e-1} \text{ if and only if } \frac{1}{c} = \frac{1}{e-1}$$

Equivalently, $c = e-1$. Thus,

$$f^c = \frac{f(e)-f(1)}{e-1} \text{ for } c \in [1,e] \text{ if and only if } c = e-1 \approx 1.71828$$

(4) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(1+x)$
Since $f(x) = \ln(1+x)$ is differentiable on $\mathbb{R}^+$, we apply the Mean Value Theorem for all $x \in (0, \infty)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \quad \text{for some} \quad c \in (0, \infty)$$

As $f'(x) = \frac{1}{1+x}$,

$$f'(c) = \frac{1}{1+c} = \frac{\ln(1+x) - \ln(1+0)}{x - 0} = \frac{\ln(1+x) - 0}{x} = \frac{\ln(1+x)}{x}$$

Thus,

$$\begin{aligned} x &= (1+c)\ln(1+x) \\ &> \ln(1+x) \quad \text{as} \quad (1+c) > 1 \end{aligned}$$

Now let $g : \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x}{1+x}$

Since $g(x) = \frac{x}{1+x}$ is differentiable on $\mathbb{R}^+$, we apply the Mean Value Theorem for all $x \in (0, \infty)$ such that

$$g'(c) = \frac{g(x) - g(0)}{x - 0} \quad \text{for some} \quad c \in (0, \infty)$$

As $g'(x) = \frac{1}{(1+x)^2}$,

$$g'(c) = \frac{1}{(1+c)^2} = \frac{\frac{x}{1+x} - \frac{0}{1+0}}{x - 0} = \frac{\frac{x}{1+x}}{x} = \frac{1}{1+x}$$

Thus,

$$\begin{aligned} 1 + x = (1+c)^2 \Rightarrow \frac{x}{1+x} &= \frac{x}{(1+c)^2} \\ &= \frac{(1+c)\ln(1+x)}{(1+c)^2} \\ &= \frac{1+c}{(1+c)^2}\ln(1+x) \\ &< \ln(1+x) \quad \text{as} \quad \frac{1+c}{(1+c)^2} < 1 \end{aligned}$$

Hence we have shown that

$$\frac{x}{1+x} < \ln(1+x) < x \quad \text{for all} \quad x > 0$$

(5) Let

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$$

Since $f$ and $g$ are both continuous on [a,b] and differentiable on (a,b), then $h$ is also continuous on $[a, b]$ and differentiable on $(a, b)$. Moreover,

$$\begin{aligned} h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\ &= f(a)g(b) - f(a)g(a) - g(a)f(b) + f(a)g(a) \\ &= f(a)g(b) - g(a)f(b) \\ h(b) &= f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)] \\ &= f(b)g(b) - g(a)f(b) - f(b)g(b) + f(a)g(b) \\ &= f(a)g(b) - g(a)f(b) \end{aligned}$$

Since $h(a) = h(b)$ it follows from Rolle's Theorem (p.86 of the lecture notes) that $h'(c) = 0$ for some $c \in (a, b)$. Consequently,

$$\begin{aligned} h'(c) &= f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0 \\ &\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \\ &\quad \text{if } g(b) \neq g(a) \text{ and } g'(c) \neq 0. \end{aligned}$$