Chapter 10

Rank and Nullity

Let $T\colon V \to W$ be a linear transformation of vector spaces over the field $\mathbb{F}$. We have seen that $\ker(T)$ is a vector subspace of V and that $\operatorname{im}(T)$ is a vector subspace of W. The dimensions of these two subspaces determined by T contain significant information about T.

Definition 10.1. The *rank* of T, $\operatorname{rk}(T)$, is the dimension of $\operatorname{im}(T)$ and the *nullity* of T, $\operatorname{n}(T)$, is the dimension of $\ker(T)$:

$$\begin{aligned}\operatorname{rk}(T) &:= \dim_{\mathbb{F}}(\operatorname{im}(T))\\ \operatorname{n}(T) &:= \dim_{\mathbb{F}}(\ker(T))\end{aligned}$$

When the domain of the linear transformation $T\colon V \to W$ is finitely generated, the rank and the nullity of determine each other.

Theorem 10.2. *Let $T\colon V \to W$ be a linear transformation with V finitely generated. Then $\operatorname{rk}(T) + n(T) = \dim V$.*

Proof. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_p\}$ be a basis for $\ker(T)$.

Extend this to a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_{p+q}\}$ of V.

We show that $\{T(\mathbf{e}_{p+1}), \dots, T(\mathbf{e}_{p+q})\}$ is a basis for $\operatorname{im}(T)$.

Take $\mathbf{w} \in \operatorname{im}(T)$.

Then $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$.

But $\mathbf{v} = \sum_{j=1}^{p+q} \lambda_j \mathbf{e}_j$, whence

$$\begin{aligned}\mathbf{w} = T(\mathbf{v}) &= T\left(\sum_{j=1}^{p+q} \lambda_j \mathbf{e}_j\right)\\ &= \sum_{j=1}^{p+q} \lambda_j T(\mathbf{e}_j)\\ &= \sum_{j=p+1}^{p+q} \lambda_j T(\mathbf{e}_j) && \text{since } T(\mathbf{e}_j) = \mathbf{0}_W \text{ for all } j \le p,\end{aligned}$$

showing that $\{T(\mathbf{e}_{p+1}), \dots, T(\mathbf{e}_{p+q})\}$ generates $\operatorname{im}(T)$.

Now suppose that $\lambda_{p+1}T(\mathbf{e}_{p+1}) + \cdots + \lambda_{p+q}T(\mathbf{e}_{p+q}) = \mathbf{0}_W$.

Then $\lambda_{p+1}\mathbf{e}_{p+1} + \cdots + \lambda_{p+q}\mathbf{e}_{p+q} \in \ker(T) = \langle \mathbf{e}_1, \ldots, \mathbf{e}_p \rangle$.
Thus $\lambda_{p+1}\mathbf{e}_{p+1} + \cdots + \lambda_{p+q}\mathbf{e}_{p+q} = \mu_1\mathbf{e}_1 + \cdots + \mu_p\mathbf{e}_p$.
Putting $\lambda_j := -\mu_j$ for $1 \le j \le p$, we see that

$$\sum_{j=1}^{p+q} \lambda_j \mathbf{e}_j = \mathbf{0}_V.$$

But $\{\mathbf{e}_1, \ldots, \mathbf{e}_{p+q}\}$ is a basis for V. Hence $\lambda_j = 0$ for all j. In particular, $\lambda_{p+i} = 0$ for $i = 1, \ldots, q$.
Thus $T(\mathbf{e}_{p+1}), \ldots, T(\mathbf{e}_{p+q})$ are linearly independent. □

Corollary 10.3. *Let $T : V \to W$ be a linear transformation. Then*

$$V \cong \ker(T) \oplus \operatorname{im}(T).$$

Proof. Exercise. □

Lemma 10.4. *Take a linear transformation $T\colon V \to W$.*

(i) T is injective if and only if $\mathrm{n}(T) = 0$.

(ii) If W is finitely generated, then T is surjective if and only if $\mathrm{rk}(T) = \dim_{\mathbb{F}} W$.

Proof. Let $T : V \to W$ be a linear transformation.
(i) $\mathrm{n}(T) = 0$ if and only if $\ker(T) = \{\mathbf{0}_V\}$ if and only if T is injective.

(ii) $\Rightarrow$: Immediate from the definition.

$\Leftarrow$: Suppose that $\mathrm{rk}(T) := \dim_{\mathbb{F}}(\operatorname{im}(T)) = \dim_{\mathbb{F}}(W) = n$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for $\operatorname{im}(T)$.
Then $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a set of n linearly independent vectors in the n dimensional vector space W.
By Theorem 8.6, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for W, showing that $\operatorname{im}(T) = W$. □

The hypothesis that W be finitely generated cannot be dispensed with in Lemma 10.4(ii), as our next example shows.

Example 10.5. Consider $T : \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$ given, heuristically, by

$$T(p) := \int_0^t p(x)dx,$$

so that

$$T(a_0 + a_1 t + \cdots + a_n t^4) = a_0 t + \tfrac{a_1}{2} t^2 + \cdots + \tfrac{a_n}{n+1} t^{n+1}$$

By definition, $\{1, t, t^2, \ldots\}$ is a basis for $\mathbb{R}[t]$.
Clearly, $\{t, t^2, \ldots\}$ is a basis for $\operatorname{im}(T)$.
Thus $\operatorname{im}(T) \neq \mathbb{R}[t]$, as $1 \notin \operatorname{im}(T)$; in other words, T is not surjective.
Since the function

$$\{1, t, t^2, \ldots\} \longrightarrow \{t, t^2, \ldots\}, \qquad t^j \longmapsto t^{j+1}$$

is a bijection between our basis for $\mathbb{R}[t]$ and a basis for $\operatorname{im}(T)$, we have $\dim_{\mathbb{F}}(W) = \mathrm{rk}\,(T)$.

We restrict attention to finitely generated vector spaces. Thus we may choose a basis for each vector space and so represent each linear transformation by a matrix. This leads us we to speak of rank and nullity of matrices.

10.1 Rank and Nullity for Matrices

We introduce two useful vector spaces isomorphic to $\mathbb{F}^n$.

Definition 10.6. We write $\mathbb{F}^{(n)}$ for $\mathbf{M}(1 \times n; \mathbb{F})$ and call this the *space of n-row vectors over* $\mathbb{F}$. We write $\mathbb{F}_{(n)}$ for $\mathbf{M}(n \times 1; \mathbb{F})$ and call this the *space of n-column vectors over* $\mathbb{F}$.

Observation 10.7. Both $\mathbb{F}^{(n)}$ and $\mathbb{F}_{(n)}$ are isomorphic to $\mathbb{F}^n$ as vector spaces over $\mathbb{F}$. In this case, we have obvious isomorphisms,

$$\mathbb{F}^n \longrightarrow \mathbb{F}^{(n)}, \quad (x_1, \dots, x_n) \longmapsto \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

$$\mathbb{F}^{(n)} \longrightarrow \mathbb{F}_{(n)}, \quad \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

It could be objected that the three vector spaces $\mathbb{F}^{(n)}, \mathbb{F}_{(n)}$ and $\mathbb{F}^n$ are not really different, just different ways of writing the same thing, with the only differences being whether parentheses and commas are used instead of brackets, or whether the coefficients are arranged in a row or in a column, purely notational differences.

But that is *precisely* the significance of the notion of isomorphism — two vector spaces are isomorphic if and only if the differences between them are purely notational.

While in the case of, say, $\mathbb{R}^2, \mathbb{R}^{(2)}$ and $\mathbb{R}_{(2)}$, there are obvious isomorphisms, that is not always the case. The real vector space $\{f\colon \mathbb{R} \to \mathbb{R} \mid \frac{d^2 f}{dx^2} + f = 0\}$ is also isomorphic with $\mathbb{R}^2$, but we need the theory of differential equations to construct an isomorphism.

Observation 10.8. Given the $m \times n$ matrix $\underline{\mathbf{A}} := [a_{ij}]_{m\times n}$ with coefficients in $\mathbb{F}$, we may regard it as comprising n vectors, $\mathbf{c}_1, \dots, \mathbf{c}_n$ from $\mathbb{F}_{(m)}$, or, equally, as comprising m vectors, $\mathbf{r}_1, \dots, \mathbf{r}_m$ from $\mathbb{F}^{(n)}$. Explicitly,

$$\underline{\mathbf{A}} = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$$

where, for $i = 1, \dots, m$ and $j = 1, \dots, n$,

$$\mathbf{c}_j := \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \text{and} \quad \mathbf{r}_i := \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}.$$

Definition 10.9. The *column space of the matrix* $\underline{\mathbf{A}}$ is the subspace of $\mathbb{F}_{(m)}$ generated by the columns of $\underline{\mathbf{A}}$ and the *row space of the matrix* $\underline{\mathbf{A}}$ is the subspace of $\mathbb{F}^{(n)}$ generated its rows.

Using our notation above, the column space of $\underline{\mathbf{A}}$ is $\langle \mathbf{c}_1, \dots, \mathbf{c}_n \rangle \leq \mathbb{F}_{(m)}$ and the row space of $\underline{\mathbf{A}}$ is $\langle \mathbf{r}_1, \dots, \mathbf{r}_m \rangle \leq \mathbb{F}^{(n)}$

The *row rank* of $\underline{\mathbf{A}} := [a_{ij}]_{m\times n}$, is $\operatorname{rowrk}(\underline{\mathbf{A}}) := \mathit{dim}_{\mathbb{F}}(\langle \mathbf{r}_1, \dots, \mathbf{r}_m \rangle)$ and its *column rank* is $\operatorname{colrk}(\underline{\mathbf{A}}) := \dim_{\mathbb{F}}(\langle \mathbf{c}_1, \dots, \mathbf{c}_n \rangle)$.

The *null space* of $\underline{\mathbf{A}}$ is $N(\underline{\mathbf{A}}) := \{\mathbf{x} \in \mathbb{F}_{(n)} \mid \underline{\mathbf{A}}\,\mathbf{x} = \underline{\mathbf{0}}\}$ and the *nullity* is $\mathrm{n}(\underline{\mathbf{A}}) := \dim_{\mathbb{F}}(N(\underline{\mathbf{A}}))$.

Observation 10.10. A linear transformation $\mathbb{F}_{(n)} \to \mathbb{F}_{(m)}$ is precisely multiplication (on the left) by an $m \times n$ matrix . So we may regard the $m \times n$ matrix, $\underline{\mathbf{A}}$ as the linear transformation

$$L_{\underline{\mathbf{A}}}\colon \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(m)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\mathbf{x}.$$

Then the null space of $\underline{\mathbf{A}}$ is $\ker(L_{\underline{\mathbf{A}}})$, the kernel of $L_{\underline{\mathbf{A}}}$, and the column space of $\underline{\mathbf{A}}$ is $\operatorname{im}(L_{\underline{\mathbf{A}}})$, the image of $L_{\underline{\mathbf{A}}}$.

We can apply the above to solving systems of linear equations. We represent the system of m equations in n unknowns

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{‡}$$

by the matrix equation

$$\underline{\mathbf{A}}\mathbf{x} = \mathbf{b},$$

where

$$\underline{\mathbf{A}} = [a_{ij}]_{m\times n}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Theorem 10.11. *The system of equations, (‡), has a solution if and only if* $\mathbf{b}$ *is in the column space of* $\underline{\mathbf{A}}$.

Proof. Immediate. □

The central fact about the row rank and the column rank of a matrix may be somewhat surprising.

Theorem 10.12. *The row rank and the column rank of a matrix agree.*

Proof. Let $\underline{\mathbf{A}} := [a_{ij}]_{m\times n}$ have row rank p and column rank q. Let $\mathbf{r}_i \in \mathbb{F}_{(n)}$ be the i-th row and $\mathbf{c}_j \in \mathbb{F}^{(m)}$, so that

$$\mathbf{c}_j := \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \text{and} \quad \mathbf{r}_i := \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \qquad (i = 1, \ldots, m, \quad j = 1, \ldots, n).$$

Since the column rank of $\underline{\mathbf{A}}$ is not altered if we permute the columns of $\underline{\mathbf{A}}$, and since the row rank is unaltered by permuting its rows, we may assume, without loss of generality, that $\{\mathbf{c}_1, \ldots, \mathbf{c}_q\}$ and $\{\mathbf{r}_1, \ldots, \mathbf{r}_p\}$ are linearly independent.

Suppose that $q < n$. Then $\mathbf{c}_n$ is a linear combination of $\{\mathbf{c}_1, \ldots, \mathbf{c}_{n-1}\}$, say

$$\mathbf{c}_n = \alpha_1\mathbf{c}_1 + \cdots + \alpha_{n-1}\mathbf{c}_{n-1}$$

so that

$$a_{in} = \sum_{j=1}^{n-1} \alpha_j a_{ij} \qquad (i = 1, \ldots, m).$$

Then, for each i,

$$\mathbf{r}_i \in W := \{[x_1 \cdots x_n] \in \mathbb{F}^{(n)} \mid x_n = \sum_{j=1}^{n-1} \alpha_j x_j\}$$

and W is, plainly, a vector subspace of $\mathbb{F}^{(n)}$.

Then the row space of $\underline{\mathbf{A}}$ is

$$V := \langle \mathbf{r}_1, \ldots, \mathbf{r}_m \rangle \leq W,$$

and so the row rank of $\underline{\mathbf{A}}$ is $\dim V$.

Let $\underline{\mathbf{A}}'$ be the $m \times (n-1)$ matrix obtain from $\underline{\mathbf{A}}$ by deleting its n^{th} column $\mathbf{c}_n$. Let $\mathbf{r}'_i \in \mathbb{F}^{(n-1)}$ be the i^{th} row of $\underline{\mathbf{A}}'$.

The function

$$T \colon V \longrightarrow \mathbb{F}^{(n-1)}, \quad [x_1 \cdots x_{n-1}\ x_n] \longmapsto [x_1 \cdots x_{n-1}]$$

is clearly a linear transformation.

Since $T(\mathbf{r}_i) = \mathbf{r}'_i$, $\operatorname{im}(T)$ is the row space of $\underline{\mathbf{A}}'$.

Thus, the rank of T is the row rank of $\underline{\mathbf{A}}'$.

Take $[x_1 \cdots x_n] \in \ker(T)$.

Then $T([x_1 \cdots x_n]) = [x_1 \cdots x_{n-1}] = [0 \cdots 0,]$ whence $x_j = 0$ for $1 \leq j < n$.

Since $x_n = \displaystyle\sum_{j=1}^{n-1} \alpha_j x_j$, we have $x_n = 0$ as well.

Hence $\ker(T) = \{[0 \cdots 0,]\}$, so that $\mathrm{n}(T) = 0$. It follows that

$$\begin{aligned}
\operatorname{rowrk}(\underline{\mathbf{A}}') &= \operatorname{rk}(L_{\underline{\mathbf{A}}}) \\
&= \dim(V) - \mathrm{n}(T) && \text{by Theorem 10.2} \\
&= \operatorname{rowrk}(\underline{\mathbf{A}}) - \mathrm{n}(T) \\
&= \operatorname{rowrk}(\underline{\mathbf{A}}) && \text{as } \mathrm{n}(T) = 0
\end{aligned}$$

Thus $\underline{\mathbf{A}}$ and $\underline{\mathbf{A}}'$ have the same row ranks and the same column ranks.

If $p < m$, we may eliminate the last row, $\mathbf{r}'_m$, from $\underline{\mathbf{A}}'$ to form $\underline{\mathbf{A}}''$, an $(m-1) \times (n-1)$ matrix, which, by a similar argument to the one just presented, has the same column and row rank as $\underline{\mathbf{A}}$.

We continue this process of eliminating rows and columns until we arrive at a matrix $\widehat{\underline{\mathbf{A}}}$ with the same column and row rank as $\underline{\mathbf{A}}$, but whose columns and rows are all linearly independent.

By hypothesis, $\widehat{\underline{\mathbf{A}}}$ is a $p \times q$ matrix, whose rows, $\widehat{\mathbf{r}}_i \quad (i = 1, \ldots p)$, and columns, $\widehat{\mathbf{c}}_j \quad (j = 1, \ldots q)$, are linearly independent.

Since $\widehat{\mathbf{r}}_i \in \mathbb{F}^{(q)}$, it follows from Theorem 8.4, that $p \leq q$.

Similarly, since $\widehat{\mathbf{c}}_j \in \mathbb{F}_{(p)}$, $q \leq p$.

Thus $p = q$. □

Theorem 10.12 allows us to speak unambiguously of the *rank* of a matrix.

Definition 10.13. The *rank* of the matrix $\underline{\mathbf{A}}$, $\operatorname{rk}(\underline{\mathbf{A}})$, is its row rank, or, equivalently, its column rank.

Theorem 10.14. *Let $T : V \to W$ be a linear transformation of finitely generated vector spaces. Let $\underline{\mathbf{A}}$ be any matrix representing T. Then $\operatorname{rk}(\underline{\mathbf{A}}) = \operatorname{rk}(T)$ and $n(\underline{\mathbf{A}}) = n(T)$.*

Proof. Exercise. □

Lemma 10.15. *The $m \times n$ matrix, $\underline{\mathbf{A}}$, is invertible only if $m = n$.*

Proof. $\underline{\mathbf{A}}$ as the linear transformation

$$L_{\underline{\mathbf{A}}} : \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(m)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\mathbf{x}.$$

$\underline{\mathbf{A}}$ is invertible if and only if $L_{\underline{\mathbf{A}}}$ is an isomorphism.

In this case $L_{\underline{\mathbf{A}}}$ is surjective and injective, whence $\mathrm{rk}(L_{\underline{\mathbf{A}}}) = \dim(\mathbb{F}_{(m)}) = m$ and $\mathrm{n}(L_{\underline{\mathbf{A}}}) = 0$, respectively.

By Theorem 10.2, $m + 0 = \dim(\mathbb{F}_{(n)}) = n$. □

Finally, we turn to the relation between the ranks of two matrices and the rank of their product.

Lemma 10.16. *Take* $\underline{\mathbf{A}} \in \mathbf{M}(p \times q; \mathbb{F})$ *and* $\underline{\mathbf{B}} \in \mathbf{M}(q \times r; \mathbb{F})$. *Then*

$$\mathrm{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) \;\leq\; \min\{\mathrm{rk}(\underline{\mathbf{A}}), \mathrm{rk}(\underline{\mathbf{B}})\}$$

Proof. Recall that the rank of a matrix is the number of linearly independent rows it contains, or, equivalently, the number of linearly independent columns.

We showed (cf. Section 9.5) that the rows of $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$ are linear combinations of the rows of $\underline{\mathbf{B}}$. Hence, there cannot be more linearly independent rows in $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$ than there are in $\underline{\mathbf{B}}$. So $\mathrm{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) \leq \mathrm{rk}(\underline{\mathbf{B}})$.

We also showed that the columns of $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$ are linear combinations of the columns of $\underline{\mathbf{A}}$. Thus, there cannot be more linearly independent columns in $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$ than there are in $\underline{\mathbf{A}}$. So $\mathrm{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) \leq \mathrm{rk}(\underline{\mathbf{A}})$.

Hence $\mathrm{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) \leq \min\{\mathrm{rk}(\underline{\mathbf{A}}), \mathrm{rk}(\underline{\mathbf{B}})\}$. □

Corollary 10.17. *If* $\underline{\mathbf{A}}$ *is invertible, then* $\mathrm{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \mathrm{rk}(\underline{\mathbf{B}})$

Proof. If $\underline{\mathbf{A}} \in \mathbf{M}(q; \mathbb{F})$ is invertible, then

$$L_{\underline{\mathbf{A}}} : \mathbb{F}_{(m)} \longrightarrow \mathbb{F}_{(m)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\mathbf{x}$$

is an isomorphism.

Let $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{F}_{(m)}$ be the columns of $\underline{\mathbf{B}}$.

Then $\underline{\mathbf{A}}\mathbf{c}_1 = L_{\underline{\mathbf{A}}}(\mathbf{c}_1), \ldots, \underline{\mathbf{A}}\mathbf{c}_n = L_{\underline{\mathbf{A}}}(\mathbf{c}_n)$ are the columns of $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$.

Since $L_{\underline{\mathbf{A}}}$ is an isomorphism, it follows by Lemma 8.8(iii) that $\{\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_r}\}$ is a basis for the column space of $\underline{\mathbf{B}}$ if and only if $\{\underline{\mathbf{A}}\mathbf{c}_{j_1}, \ldots, \underline{\mathbf{A}}\mathbf{c}_{j_r}\}$ is a basis for the column space of $\underline{\mathbf{A}}\,\underline{\mathbf{B}}$. □

10.2 Calculating the Column Space and the Null Space of a Matrix

Given their significance, it is important to be able to calculation of the column space and the null space of a matrix.

It may surprise the reader that the calculations required are just the familiar transformation to echelon form (cf. MATH101). This comprises applying the elementary row operations to the matrix in question. As indicated in Observation 9.31, each elementary row operation can be performed on the matrix $\underline{\mathbf{B}}$ by mulitiplying it on the left by a suitable matrix. The details follow.

10.2.1 Elementary Row Operations

The elementary row operations apply to matrices and we can regard each of these operations as a function $\mathbf{M}(m \times n; \mathbb{F}) \longrightarrow \mathbf{M}(m \times n; \mathbb{F})$. Recall that we write $\mathbf{M}(n; \mathbb{F})$ for $\mathbf{M}(n \times n; \mathbb{F})$.

The elementary row operations are

ER01 **M**ultiplication of the $\mathbf{i}^{\text{th}}$ row by $\lambda \in \mathbb{F} \quad (\lambda \neq 0)$.

ER02 **A**ddition of μ times the $\mathbf{j}^{\text{th}}$ row to the $\mathbf{i}^{\text{th}}$ row $\quad (\mathbf{i} \neq \mathbf{j})$.

ER03 **S**wapping the $\mathbf{i}^{\text{th}}$ row with the $\mathbf{j}^{\text{th}}$ row $\quad (\mathbf{i} \neq \mathbf{j})$.

Clearly, each the elementary row operations defines a function $\mathbf{M}(n;\mathbb{F}) \longrightarrow \mathbf{M}(n;\mathbb{F})$. Moreover, each of these functions is bijective, with obvious inverses, namely, multiplying the i^{th} row by $\frac{1}{\lambda}$, adding $-\mu$ times the j^{th} row to the i^{th} row, and, finally, swapping the i^{th} row with the j^{th} row.

We illustrate how the operations can be performed using matrix multiplication on the left, using concrete examples for $\mathbf{M}(2;\mathbb{F})$, before presenting the general form.

Example 10.18. Take $\underline{\mathbf{B}} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}(2;\mathbb{F})$

ER01 If we multiply the second row of $\underline{\mathbf{B}}$ by λ, we obtain

$$\begin{bmatrix} a & b \\ \lambda c & \lambda d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

ER02 If we add μ times the second row of $\underline{\mathbf{B}}$ to the first row of $\underline{\mathbf{B}}$ we obtain

$$\begin{bmatrix} a+\mu c & b+\mu d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

ER03 If we swap the first and second rows of $\underline{\mathbf{B}}$, we obtain

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The general form of the matrices performing the elementary row operations is now determined, and we list them next, illustrating each using examples from $\mathbf{M}(4;\mathbb{F})$.

ERO1 $\quad \mathbf{M}(m \times n;\mathbb{F}) \longrightarrow \mathbf{M}(m \times n;\mathbb{F}), \quad \underline{\mathbf{B}} \longmapsto \underline{\mathbf{M}}(i,\lambda)\underline{\mathbf{B}} \qquad$ where

$$\underline{\mathbf{M}}(i,\lambda) := [m_{k\ell}]_{n\times n}, \quad \text{with } \; m_{k\ell} = \begin{cases} \lambda & \text{if } k = \ell = i \\ 1 & \text{if } k = \ell \neq i \\ 0 & \text{if } k \neq \ell \end{cases}. \tag{\textbf{ero1}}$$

Example 10.19. $\underline{\mathbf{M}}(2,\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ERO1 $\quad \mathbf{M}(m \times n;\mathbb{F}) \longrightarrow \mathbf{M}(m \times n;\mathbb{F}), \quad \underline{\mathbf{B}} \longmapsto \underline{\mathbf{A}}(i,\mu j)\underline{\mathbf{B}} \qquad$ where

$$\underline{\mathbf{A}}(i,\mu j) := [a_{k\ell}]_{n\times n}, \quad \text{where } \; a_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ \mu & \text{if } k = i, \; \ell = j \\ 0 & \text{otherwise} \end{cases}. \tag{\textbf{ero2}}$$

Example 10.20. $\underline{\mathbf{A}}(1,\mu 3) = \begin{bmatrix} 1 & 0 & \mu & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ERO3 $\quad \mathbf{M}(m \times n; \mathbb{F}) \longrightarrow \mathbf{M}(m \times n; \mathbb{F}), \quad \underline{\mathbf{B}} \longmapsto \underline{\mathbf{S}}(i,j)\underline{\mathbf{B}}$ where

$$\underline{\mathbf{S}}(i,j) := [s_{k\ell}]_{n\times n}, \quad \text{where } s_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \neq i,j \\ 1 & \text{if } k = i,\ \ell = j \\ 1 & \text{if } k = j,\ \ell = i \\ 0 & \text{otherwise} \end{cases} \tag{ero3}$$

Example 10.21. $\underline{\mathbf{S}}(2,4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

The next theorem summarises these considerations.

Theorem 10.22. *Take $\lambda, \mu \in \mathbb{F}$, with $\lambda \neq 0$, $m, n \in \mathbb{N}$ and $i, j \in \mathbb{N}$ with $1 \leq i \leq m,\ 1 \leq j \leq n$ and $i \neq j$. Then we have isomorphisms.*

$$\begin{aligned} M_{(i,\lambda)} \colon \mathbf{M}(m \times n; \mathbb{F}) &\longrightarrow \mathbf{M}(m \times n; \mathbb{F}), & \underline{\mathbf{B}} &\longmapsto \underline{\mathbf{M}}(i,\lambda)\underline{\mathbf{B}} \\ A_{(\imath,\mu j)} \colon \mathbf{M}(m \times n; \mathbb{F}) &\longrightarrow \mathbf{M}(m \times n; \mathbb{F}), & \underline{\mathbf{B}} &\longmapsto \underline{\mathbf{A}}(i,\mu j)\underline{\mathbf{B}} \\ S_{(i,j)} \colon \mathbf{M}(m \times n; \mathbb{F}) &\longrightarrow \mathbf{M}(m \times n; \mathbb{F}), & \underline{\mathbf{B}} &\longmapsto \underline{\mathbf{S}}(i,j)\underline{\mathbf{B}} \end{aligned}$$

Proof. That these functions are linear transformations follows directly from the definition of matrix multiplication, and we have already seen that they are bijective. $\square$

Theorem 10.22, together with the discussion in Section 9.5, provides us with a procedure for determining the column space and the null space of a given matrix, which we state before illustrating with a concrete example.

Let the $\mathbf{c}_1, \ldots \mathbf{c}_n$ be the columns of the $m \times n$ matrix. $\underline{\mathbf{B}}$. Apply elementary row operations to reduce $\underline{\mathbf{B}}$ to a matrix $\underline{\mathbf{E}}$ in echelon form.

Since each step comprises multiplication on the left by a suitable matrix of the form $\underline{\mathbf{M}}(i,\lambda), \underline{\mathbf{A}}(i,\mu j)$ or $\underline{\mathbf{S}}(i,j)$, it comprise the application of an isomorphism of the form $M_{(i,\lambda)}, A_{(i,\mu j)}$ or $S_{(i,j)}$.

Since we obtain $\underline{\mathbf{E}}$ by composing isomorphisms, we have an isomorphism

$$T \colon \mathbf{M}(m \times n; \mathbb{F}) \longrightarrow \mathbf{M}(m \times n; \mathbb{F}), \qquad \underline{\mathbf{X}} \longmapsto \underline{\mathbf{C}}\,\underline{\mathbf{X}}$$

with $\underline{\mathbf{C}}$ a product of matrices of the form $\underline{\mathbf{M}}(i,\lambda), \underline{\mathbf{A}}(i,\mu j)$ or $\underline{\mathbf{S}}(i,j)$.

Thus, $\underline{\mathbf{C}}$ is invertible and $\underline{\mathbf{E}} = \underline{\mathbf{C}}\,\underline{\mathbf{B}}$.

By the proof of Corollary 10.17, $\{\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_r}\}$ is a basis for the column space of $\underline{\mathbf{B}}$ if and only if $\{\underline{\mathbf{C}}\mathbf{c}_{j_1}, \ldots, \underline{\mathbf{C}}\mathbf{c}_{j_r}\}$ is a basis for the column space of $\underline{\mathbf{C}}\,\underline{\mathbf{B}} = \underline{\mathbf{E}}$.

The columns of $\underline{\mathbf{E}}$ which contain the "pivot 1"s — that is, the first non-zero element in a row — form a basis for the column space of $\underline{\mathbf{E}}$.

Hence the corresponding columns of $\underline{\mathbf{B}}$ form a basis for the column space of $\underline{\mathbf{B}}$.

Example 10.23.

$$\underline{\mathbf{B}} = \begin{bmatrix} 1 & 2 & 2 & 5 \\ 3 & 6 & 1 & 10 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

We apply reduce $\underline{\mathbf{B}}$ to echelon form

$$\begin{aligned}
\begin{bmatrix}1 & 2 & 2 & 5\\ 3 & 6 & 1 & 10\\ 1 & 2 & -1 & 2\end{bmatrix} &\rightsquigarrow \begin{bmatrix}1 & 2 & 2 & 5\\ 0 & 0 & -5 & -5\\ 0 & 0 & -3 & -3\end{bmatrix} && \begin{matrix}R_1 \rightsquigarrow R_1\\ R_2 \rightsquigarrow \mathbb{R}_2 - 3R_1\\ R_3 \rightsquigarrow R_3 - R_1\end{matrix}\\
&\rightsquigarrow \begin{bmatrix}1 & 2 & 2 & 5\\ 0 & 0 & 1 & 1\\ 0 & 0 & -3 & -3\end{bmatrix} && \begin{matrix}R_1 \rightsquigarrow R_1\\ R_2 \rightsquigarrow \frac{1}{-5}\mathbb{R}_2\\ R_3 \rightsquigarrow R_3\end{matrix}\\
&\rightsquigarrow \begin{bmatrix}1 & 2 & 2 & 5\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} && \begin{matrix}R_1 \rightsquigarrow R_1\\ R_2 \rightsquigarrow \mathbb{R}_2\\ R_3 \rightsquigarrow R_3 + 3R_2\end{matrix}
\end{aligned}$$

Thus $\underline{\mathbf{E}} = \begin{bmatrix}1 & 2 & 2 & 5\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$.

Since the pivot elements are in the first and third columns of $\underline{\mathbf{E}}$, a basis for the column space of $\underline{\mathbf{B}}$ is given by the first and third columns of $\underline{\mathbf{B}}$.

In other words, $\begin{bmatrix}1\\3\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\\-1\end{bmatrix}$ comprise a basis for the column space of $\underline{\mathbf{B}}$.

We can also read off a basis for the null space of $\underline{\mathbf{B}}$ from its echelon form.

Example 10.24. Turning to $N(\underline{\mathbf{B}})$, the null space of $\underline{\mathbf{B}}$, recall that $\mathbf{x} \in \mathbb{F}_{(n)}$ is an element of $N(\underline{\mathbf{B}})$ if and only if $\underline{\mathbf{B}}\mathbf{x} = \mathbf{0} \in \mathbb{F}_{(m)}$.

Since $\underline{\mathbf{C}}$ is an invertible matrix, this is equivalent to $\underline{\mathbf{E}}\mathbf{x} = \underline{\mathbf{CB}}\mathbf{x} = \mathbf{0}$.

Putting $\mathbf{x} = \begin{bmatrix}w\\x\\y\\z\end{bmatrix}$, we obtain

$$\begin{aligned}
w &= -2x - 2y - 5z\\
y &= -z
\end{aligned}$$

or, equivalently

$$\mathbf{x} = \begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}-2x - 3z\\x\\-z\\z\end{bmatrix} = x\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + z\begin{bmatrix}-3\\0\\-1\\1\end{bmatrix}$$

In other words, $\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}-3\\0\\-1\\1\end{bmatrix}$ comprise a basis for the null space of $\underline{\mathbf{B}}$.

Observation 10.25. The above allows us to find a basis for the image and kernel of a linear transformation. We illustrate this using the matrix above.

Example 10.26. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be a basis for V and $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ a basis for W.

Let $T\colon V \to W$ be a linear transformation whose matrix with respect to these bases is

$$\underline{\mathbf{B}} = \begin{bmatrix} 1 & 2 & 2 & 5 \\ 3 & 6 & 1 & 10 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

It is immediate from the above that $\mathbf{f}_1 + 3\mathbf{f}_2 + \mathbf{f}_3$ and $2\mathbf{f}_1 + \mathbf{f}_2 - \mathbf{f}_3$ form a basis for $\mathrm{im}(T)$ and that $-2\mathbf{e}_1 + \mathbf{e}_2$ and $-3\mathbf{e}_1 - \mathbf{e}_3 + \mathbf{e}_4$ comprise a basis for $\ker(T)$.

10.3 Exercises

Exercise 10.1. Let $T : V \longrightarrow W$ be a linear transformation of finitely generated vector spaces. Let $\underline{\mathbf{A}}$ be any matrix representing T. Prove that $\mathrm{rk}(\underline{\mathbf{A}}) = \mathrm{rk}(T)$ and $\mathrm{n}(\underline{\mathbf{A}}) = \mathrm{n}(T)$.

Exercise 10.2. Let $T : V \longrightarrow W$ be a linear transformation. Prove that

(i) $V \cong \ker(T) \oplus \mathrm{im}(T)$.

(ii) $\mathrm{im}\, T \cong {}^{V}\!/_{\ker(T)}$.

Exercise 10.3. Prove that an $n \times n$ matrix has a left inverse if and only if it has a right inverse.

Exercise 10.4. Find a basis for the column space and a basis for the null space of each of the following real matrices.

(i) $\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ 3 & 6 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 6 & 5 \\ 1 & 0 & 8 & 7 \\ 6 & 8 & 1 & 3 \\ 7 & 14 & -11 & -8 \end{bmatrix}$