Arrows and Exact Sequences

S'il est vrai que la mathématique est
la reine des sciences, qui est la reine
de la mathématique? La suite exacte!

The above is a plaisanterie by Henri Cartan (1952 in an Oberwolfach meeting). It was not meant seriously; most mathematicians agree that number theory is the queen of mathematics. But it shows that, at that time, many people working in topology, algebra and related fields were greatly impressed by the impact and power of that simple quite new idea; it was able to facilitate arguments and intuition at the same time.

Did exact sequences "exist" before the period they were invented? Yes, of course; but they were expressed in a clumsy complicated way, not easy to handle. It seems to me that one of the crucial new elements was the use of arrows as notation for maps.

Some mathematical concepts have a curious history. They are discovered, then overlooked by the author himself. Soon later they conquer not only the specific field, but large parts of all of mathematics. Eventually people think that no reasonable work could have been done without them – which of course is not true. All this seems to apply to the exact sequence.

1. Preliminaries.

Let us first recall that a sequence

$$\ldots\longrightarrow A_n\longrightarrow A_{n-1}\longrightarrow\ldots\longrightarrow A_0\longrightarrow\ldots$$

of homomorphisms of Abelian groups A_i is called exact if at each stage the kernel of the outgoing homomorphism is equal to the image of the incoming homomophism. The range of indices $i \in \mathbb{Z}$ can be infinite or finite. If before or after an A_i there is no arrow then nothing is said about the kernel, or the image respectively. The A_i can be provided with operators from a ring Λ (Λ-modules). Although we always speak of groups, the A_i can also be other types of structures compatible with the homomorphisms called "maps" in the sequel.

A "short" exact seqence

$$0\longrightarrow A_3\longrightarrow A_2\longrightarrow A_1\longrightarrow 0$$

simply means that $A_3\longrightarrow A_2$ is injective and thus A_3 can be considered as a subgroup of A_2; further that $A_2/A_3 = A_1$ where $=$ means isomorphism. Thus the exactness implies a sequence of isomorphisms connecting the A_i with the successive kernel-images. If such isomorphisms are given and if the respective maps are specified, that collection of isomorphisms is equivalent to an exact sequence, A classical example of an exact sequence of Abelian groups, easy to describe, is the *homotopy group sequence of a fiber space* with total space E, base space B and fiber F (see [E1], [E2], for example)

$$...\longrightarrow\pi_i(F)\longrightarrow\pi_i(E)\longrightarrow\pi_i B)\longrightarrow\pi_{i-1}(F)\longrightarrow...$$

with $i > 2$ (we omit discussion of the changes necessary in the lowest dimensions).

The elements of $\pi_i(X)$, where X is a topological space, are homotopy classes of continuous maps from the i-dimensional sphere S^i to X. The map going from F to E is induced by the inclusion of the fiber, from E to B by the projection of the fiber space onto the basis. The essential step is the map from dimension i to $i-1$, based on the covering homotopy property (also homotopy lifting property) for fiber spaces: One can lift a map from S^i into B, representing en element of $\pi_i(B)$, to a map of the ball V^i into E and then restrict the latter to a map of the boundary sphere S^{i-1} of V^i into F.

2. Arrows.

Using an arrow

$$A\longrightarrow B$$

for a map with domain A and range B seems to be, since the middle of the last century the most natural thing. Because of its accuracy and vigor it provides strong intuition. In the "old" times, arrows were not used for maps (only sometimes to indicate where a certain element should go; or for limits; or for logical implications).

Although maps were used extensively (not just objects with some relation between them) no arrows appear in the whole work of Heinz Hopf nor in the papers of the author and other topologists and algebraists until about 1950. But then all for a sudden the new fashion began and "everybody" started using arrows, exact sequences, and even diagrams of arrows and sequences. This was probably first done systematically in the book by Eilenberg and Steenrod "Foundations of Algebraic Topology" (1952) and then in "Homological Algebra" (1956) by Cartan and Eilenberg. No wonder the use of arrows became a standard tool in Category Theory. This had not been the case in the very beginning of that field when categories and functors were just meant to explain "naturality", i.e. to explain what constructions are natural in the sense that they are compatible with maps between the objects. But later, when in connection with algebraic homology concepts Category Theory became a true mathematical field the use of arrows became indispensable.

2. First appearence.

For the first time a sequence of maps with the exactness property (the word exact is not yet used) appears in 1941 in a 2-page announcement without proofs by Witold Hurewicz [H]. It concerns cohomology groups (which cohomology theory?), and the maps are not all well-defined. The context in which Hurewicz mentioned the sequence did not seem to meet great interest. This may be one of the reasons why the sequence was overlooked. Another reason may have been the lack of communication between Europe and America due to World War II.

It seems that Hurewicz too overlooked the importance of his own new idea, new not in the sense of facts, but in the sense of formulating them.

Indeed, Hurewicz and Steenrod considered in 1941 in a note in the National Academy of Sciences USA [H-S] the exact relative homotopy sequence (discovered earlier by J.H.C.Whitehead), i.e. the relations between the absolute homotopy groups of a space X, a subspace Y, and the relative homotopy groups of X relative to Y. They did it in the old fashion without any arrows. Surprisingly enough, the exact homotopy group sequence for fiber spaces as described above was only implicit in that note: The covering homotopy property was formulated, and from it the fact that the homotopy group of E relative to F is just the homotopy group of B (see Section 1). But the exact sequence itself was not formulated neither with nor without arrows! It had been established by the author in his thesis in 1940 (and independently by Ehresmann and Feldbau). I had called them "Hurewicz Formulae" because they were generalizations of what Hurewics had proved in 1935/36 for Lie group fibrations. I applied them to the computation of many homotopy groups of spheres and of Lie groups.

It should be mentioned that in 1930 already Lefschetz had found something which could be called the relative exact homology sequence for a space and a subspace; combined with what later was called excision it implies Alexander duality.

In 1945 Eilenberg and Steenrod [E-S] announced what would later, in the book mentioned above, become a fundamental treatment of homology theory of spaces. In that preliminary announcement the exact homology sequence for a space and a subspace appears under the name of "natural system of groups and homomorphisms".

3. Exactness and chain complexes.

In 1947 finally the name exact sequence was invented by Kelley and Pitcher [K-P]. They showed that many known results of algebraic topology could be formulated in that very intuitive form. They emphasized the purely algebraic aspect of exact sequences, and showed that the homology and cohomology exact sequences of spaces can be obtained through the algebraic concept of chain complex and through limiting procedures; thus they were led to examine the behaviour of exact sequences under direct or inverse limits.

Immediately thereafter the world of topology adopted the new way of presenting theorems, definitions, axioms. The axiomatic treatment in the Eilenberg-Steenrod 1952 book relies heavily on diagrams of exact sequences. In the introduction the authors say: "The diagrams incorporate a large amount of information. Their use provides extensive savings in space and mental effort".

They noticed that, apart from exactness, an important property of a diagram is to be commutative; i.e., passing from one group to another one by two different arrow-paths should yield the same homomorphism.

As a simple but already very important example let me mention the case of two (long) exact sequences, written horizontally, and related by vertical homomorphisms. Horizontal exactness and vertical maps which are compatible, i.e. such that all squares are commutative, may contain in many cases deep information and difficult arguments, and are very suggestive. Just to examine whether commutativity is present may suggest new concepts and theorems. A remarkable fact: such commutative diagrams have inspired an artist (Bernar Venet) to beautiful monumental paintings.

I close with a famous theorem, the "five lemma". It relates two exact sequences (of length five) in the above way and formulates relations between the vertical maps; these relations are very useful in many applications. Without the arrow language and the concepts of exactness and commutativity its formulation would be very complicated – not to speak of the proof!

I have recently heard young, and even not so young, mathematicians ask

"Was there mathematical life before arrows and exact sequences?"

References

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