Chapter 3

Vector Spaces

Linear algebra is the theory of *vector spaces*, and this chapter begins their formal study. The previous chapter looked in detail at examples of a vector space: solutions to homogeneous systems of linear equations with real coefficients. The exercises looked at other examples. The definitions below express the essential features of our examples without reference to their special features.

In our original example, there were actually two distinct sets: the solutions to the homogeneous systems of equations on the one hand, and the real numbers on the other. We had operations defined on each of these sets and a way of combining them.

The other examples in the last chapter were similar.

These considerations lead us to regard a vector space as a set with additional structure.

The additional structure is *algebraic* in nature. It allows us to compute and solve numerous problems explicitly with relative ease. This ease of computation and broad range of application comes at the price of requiring a relatively large number of axioms to describe the structure.

The structure is a mixed structure, for a vector space is actually a set-with-structure upon which another set-with-structure acts. The latter set is a *field* and we begin with the axioms for a field.

3.1 Fields

Definition 3.1. A *field* comprises a set, $\mathbb{F}$, together with two binary operations

$$\begin{aligned} \alpha &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \quad (x, y) \longmapsto \alpha(x, y) \\ \mu &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \quad (x, y) \longmapsto \mu(x, y) \end{aligned}$$

together with distinguished elements $0_{\mathbb{F}} \neq 1_{\mathbb{F}}$ satisfying the following axioms.

It is customary to call α *addition*, μ *multiplication* and to write $x +_{\mathbb{F}} y$ and $x \times_{\mathbb{F}} y$ for $\alpha(x, y)$ and $\mu(x, y)$ respectively. Then, given any $x, y, z \in \mathbb{F}$,

A1 $\quad x +_{\mathbb{F}} (y +_{\mathbb{F}} z) = (x +_{\mathbb{F}} y) +_{\mathbb{F}} z$

A2 $\quad x +_{\mathbb{F}} 0_{\mathbb{F}} = x = 0_{\mathbb{F}} +_{\mathbb{F}} x$

A3 $\quad$ There is an element $-x \in \mathbb{F}$ with $x +_{\mathbb{F}} (-x) = 0_{\mathbb{F}} = (-x) +_{\mathbb{F}} x$

A4 $\quad y +_{\mathbb{F}} x = x +_{\mathbb{F}} y$

M1 $\quad x \times_{\mathbb{F}} (y \times_{\mathbb{F}} z) = (x \times_{\mathbb{F}} y) \times_{\mathbb{F}} z$

M2 $\quad x \times_{\mathbb{F}} 1_{\mathbb{F}} = x = 1_{\mathbb{F}} \times_{\mathbb{F}} x$

M3 $\quad$ If $x \neq 0_{\mathbb{F}}$, then there is an $x^{-1} \in \mathbb{F}$ with $x \times_{\mathbb{F}} (x^{-1}) = 1_{\mathbb{F}} = (x^{-1}) \times_{\mathbb{F}} x$

M4 $\quad yx = xy$

D $\quad x \times_{\mathbb{F}} (y +_{\mathbb{F}} z) = (x \times_{\mathbb{F}} y) +_{\mathbb{F}} (x \times_{\mathbb{F}} z)$ and $(x +_{\mathbb{F}} y) \times_{\mathbb{F}} z = (x \times_{\mathbb{F}} z) +_{\mathbb{F}} (y \times_{\mathbb{F}} \times_{\mathbb{F}} z)$

Observation 3.2. Axioms A1 and M1 assert the *associativity* of addition and multiplication respectively.

Axioms A2 and M2 assert the *existence of a neutral element* for addition and multiplication respectively.

Axioms A3 and M3 assert the *existence of inverses* for addition and multiplication respectively.

Axioms A4 and M4 assert the *commutativity* of addition and multiplication respectively.

Finally, Axiom D asserts that *multiplication distributes over addition.*

Observation 3.3. Axioms A1, A2 and A3 assert that $\mathbb{F}$ is a *group* with respect to addition.

Axioms M1, M2 and M3 assert that $\mathbb{F} \setminus \{0_{\mathbb{F}}\}$ is a group with respect to multiplication.

Axioms A4 and M4 assert that these two group structures commutative (or *abelian*).

Finally, Axiom D describes how these two group structures interact.

Observation 3.4. Axioms A1, A2, A3, A4, M1, M2 and D assert that $\mathbb{F}$ is a *(unital) ring* with respect to addition and multiplication.

Axiom M4 asserts that this ring structure is *commutative*.

Example 3.5. The rational numbers, $\mathbb{Q}$, the real numbers, $\mathbb{R}$, and the complex numbers, $\mathbb{C}$, all form fields with respect to their usual addition and multiplication.

Example 3.6. The set of integers, $\mathbb{Z}$, forms a commutative ring, but not a field, with respect to their usual addition and multiplication. $\mathbb{Z}$ fails to be a field because there is no integer, x, with $3x = 1$..

Example 3.7. The set of all natural numbers, $\mathbb{N}$, does not form a group with respect to its addition, since, for example, there is no $x \in \mathbb{N}$ with $1 + x = 0$.

Example 3.8. The set of all 2×2 matrices with integer coefficients, $\mathbf{M}(2; \mathbb{Z})$, forms a ring with respect to the usual addition and matrix multiplication. This ring is plainly not commutative, because

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 3.9. Let $\mathbb{F} := \{x \in \mathbb{R} \mid x > 0\}$ and define

$$\begin{aligned} +_{\mathbb{F}} &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \quad (x, y) \longmapsto xy \\ \times_{\mathbb{F}} &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \quad (x, y) \longmapsto x^{\ln y}. \end{aligned}$$

These operations render $\mathbb{F}$ a field. The proof — and the identification of $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ — is left to the reader as an exercise.

Example 3.10. Let $\mathbb{F}$ be the set of all matrices of the form $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ with $x, y \in \mathbb{R}$ Define $+_{\mathbb{F}}$, $\times_{\mathbb{F}}$ to be the usual addition and multiplication of matrices.

The verification, that $\mathbb{F}$ is a field, is left as an exercise.

Example 3.11. Let $\mathbb{R}[t]$ denote the set of all polynomials in the indeterminate t, with real coefficients, so that

$$\mathbb{R}[t] := \{a_0 + a_1 t + \cdots + a_n t^n \mid a_j \in \mathbb{R}\ (j = 1, \ldots, n) \ \text{ and } \ a_n \neq 0 \text{ if } n \neq 0\,\}$$

Define $\sim$ on $\mathbb{R}[t]$ by

$$p(t) \sim q(t) \ \text{ if and only if } \ t^2 + 1 \ \text{ divides } \ p(t) - q(t).$$

Direct verification shows that $\sim$ is an equivalence relation on $\mathbb{R}[t]$. Let $\mathbb{F}$ denote the set of all $\sim$-equivalence classes, and denote the equivalence class of $p(t)$ by $[p(t)]$. $p(t)$ is called a *representative* of $[p(t)]$.

Every equivalence class can be represented by a polynomial of the form $a + bt$, with $a, b \in \mathbb{R}$, so that

$$\mathbb{F} = \{\,[a + bt] \mid a, b \in \mathbb{R}\,\}\,.$$

Then $\mathbb{F}$ is a field with respect to addition and multiplication defined by

$$\begin{aligned} +_{\mathbb{F}} : \mathbb{F} \times \mathbb{F} &\longrightarrow \mathbb{F}, \quad ([a + bt]\,, [c + dt]) \longmapsto [(a + c) + (b + d)t] \\ \times_{\mathbb{F}} : \mathbb{F} \times \mathbb{F} &\longrightarrow \mathbb{F}, \quad ([a + bt]\,, [c + dt]) \longmapsto [(ac - bd) + (ad + bc)t] \end{aligned}$$

The verifications are left to the reader as an exercise.

3.2 Vector Spaces

The motivating example for the notion of a vector space is provided by the notion of a vector from physics. Vectors, such as forces, can be added: if two forces act upon a given object, there is a net resultant vector. In addition, vectors can be scaled, that is multiplied by a *scalar*.

With the preliminary definitions above, we can define what we mean by a vector space.

Definition 3.12. A *vector space over the field* $\mathbb{F}$ — or an $\mathbb{F}$*-vector space* — is a set, V, together with a binary operation,

$$\boxplus : V \times V \longrightarrow V, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \mathbf{u} \boxplus \mathbf{v},$$

and an operation of the field $\mathbb{F}$ on V

$$\boxdot : \mathbb{F} \times V \longrightarrow V, \quad (\lambda, \mathbf{v}) \longmapsto \lambda \boxdot \mathbf{v}$$

satisfying the axioms listed below. Explicitly, we write $\mathbf{x}, \mathbf{y}$ and so on for the elements of V, $\mathbf{x} \boxplus \mathbf{y}$ for the binary operation on them and $\lambda \boxdot \mathbf{x}$ for the action of $\lambda \in \mathbb{F}$ on $\mathbf{x} \in V$.

Given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\lambda, \mu \in \mathbb{F}$,

VS1 $\quad \mathbf{x} \boxplus (\mathbf{y} \boxplus \mathbf{z}) = (\mathbf{x} \boxplus \mathbf{y}) \boxplus \mathbf{z}$

VS2 $\quad \mathbf{x} \boxplus \mathbf{0}_V = \mathbf{x} = \mathbf{0}_V \boxplus \mathbf{x}$

VS3 $\quad$ There is a $-\mathbf{x}$ such that $\mathbf{x} \boxplus (-\mathbf{x}) = \mathbf{0}_V = (-\mathbf{x}) \boxplus \mathbf{x}$

VS4 $\quad \mathbf{y} \boxplus \mathbf{x} = \mathbf{x} \boxplus \mathbf{y}$

VS5 $\quad 1_{\mathbb{F}} \boxdot \mathbf{x} = \mathbf{x}$

VS6 $\lambda \boxdot (\mathbf{x} \boxplus \mathbf{y}) = (\lambda \boxdot \mathbf{x}) \boxplus (\lambda \boxdot \mathbf{y})$

VS7 $(\lambda +_{\mathbb{F}} \mu) \boxdot \mathbf{x} = (\lambda \boxdot \mathbf{x}) \boxplus (\mu \boxdot \mathbf{x})$

VS8 $(\lambda \times_{\mathbb{F}} \mu) \boxdot \mathbf{x} = \lambda \boxdot (\mu \boxdot \mathbf{x})$

If V is a vector space over $\mathbb{F}$, then the elements of V are called *vectors* and the elements of $\mathbb{F}$ are called *scalars*

Thus a vector space V over the field $\mathbb{F}$ comprises an abelian group, V (Axioms VS1 to VS4) together with an *action* of the field $\mathbb{F}$ on V (Axioms VS5 to VS8).

Example 3.13. Let $\mathbb{F}$ be a field and put $\mathbb{F}^n := \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{F}\}$. Define addition and scalar multiplication by

$$\begin{aligned}(x_1, \dots, x_n) \boxplus (y_1, \dots, y_n) &:= (x_1 +_{\mathbb{F}} y_1, \dots, x_n +_{\mathbb{F}} y_n)\\ \lambda \boxdot (x_1, \dots, x_n) &:= (\lambda \times_{\mathbb{F}} x_1, \dots, \lambda \times_{\mathbb{F}} x_n)\end{aligned}$$

Then $\mathbb{F}^n$ is a vector space over $\mathbb{F}$. When we refer to $\mathbb{F}^n$ as a vector space over $\mathbb{F}$, we shall always mean the vector space structure just defined.

A familiar case is $\mathbb{R}^2 := \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$, the set of all ordered pairs of real numbers. [Here $\mathbb{F} := \mathbb{R}$ and $n = 2$.]

Notice in particular that when $n = 1$, the addition of vectors, $\boxplus$, coincides with the addition, $+_{\mathbb{F}}$, in the field $\mathbb{F}$ and multiplication of a vector by a scalar, $\boxdot$, is just the multiplication, $\times_{\mathbb{F}}$, in the field. Thus a field is always a vector space over itself and any property of an arbitrary vector space over $\mathbb{F}$ is also a property of $\mathbb{F}$ itself. Thus we may think of the notion of a vector space as being a generalisation of the notion of a field.

Example 3.14. Though the reader has already met matrices elsewhere, we revise the formal definiton and show that the set of all matrices of a fixed size forms a vector space.

Definition 3.15. An $m \times n$ matrix over $\mathbb{F}$ is an array of mn elements of $\mathbb{F}$ arranged into m rows and n columns. We write $[a_{ij}]$ to denote the $m \times n$ matrix over $\mathbb{F}$ with a_{ij} the ij-th *entry* or *coefficient*. Here, the first subscript indicates the rows and the second indicates the columns. Thus,

$$\underline{\mathbf{A}} = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

We write $\mathbf{M}(m \times n; \mathbb{F})$ for the set of all $m \times n$ matrices over $\mathbb{F}$, abbreviating this to $\mathbf{M}(n; \mathbb{F})$ when $m = n$.

Given $\underline{\mathbf{A}} = [a_{ij}]_{m \times n}$ and $\underline{\mathbf{B}} = [b_{ij}]_{m \times n}$ as well as $\lambda \in \mathbb{F}$ we define

$$\begin{aligned}[a_{ij}]_{m \times n} \boxplus [b_{ij}]_{m \times n} &:= [a_{ij} +_{\mathbb{F}} b_{ij}]_{m \times n}\\ \lambda \boxdot [a_{ij}]_{m \times n} &:= [\lambda a_{ij}]_{m \times n}.\end{aligned}$$

This renders $\mathbf{M}(m \times n; \mathbb{F})$ an $\mathbb{F}$ vector space.

Example 3.16. Let $\mathbb{F}$ be any field and X a non-empty set. Take

$$V = \mathcal{F}(X) := \{f : X \longrightarrow \mathbb{F} \mid f \text{ is a function } \}$$

so that V is the set of all $\mathbb{F}$-valued functions defined on X. Define

$$\begin{aligned}&\boxplus : V \times V \longrightarrow V, (f, g) \longmapsto f \boxplus g \qquad \text{where } (f \boxplus g)(x) := f(x) +_{\mathbb{F}} g(x) \text{ for all } x \in X\\ &\boxtimes : \mathbb{F} \times V \longrightarrow V, (\lambda, f) \longmapsto \lambda \boxdot f \qquad \text{where } (\lambda \boxdot f)(x) := \lambda \times_{\mathbb{F}} f(x) \text{ for all } x \in X\end{aligned}$$

It is easy to verify that this defines an $\mathbb{F}$-vector space structure on V.

Example 3.17. Take $\mathbb{F} = \mathbb{R}$ and $V = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid xf'(x) - f(x) = 0 \text{ for all } x \in \mathbb{R}\}$. V is a vector space over $\mathbb{R}$ with respect to the operations defined in Example 3.16.

Notice that there is no need to solve the differential equation in order to see that the set of all solutions forms a vector space.

Notational Convention. Except when there is danger of confusion, or for emphasis, we shall avail ourselves of *systematic ambiguity*:

1. We write 0 and 1 for $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ respectively.
2. We write $\lambda + \mu$ and $\lambda\mu$ for $\lambda +_{\mathbb{F}} \mu$ and $\lambda \times_{\mathbb{F}} \mu$ respectively.
3. We write $\mathbf{x} + \mathbf{y}$ for $\mathbf{x} \boxplus \mathbf{y}$ and $\lambda\mathbf{x}$ for $\lambda \boxdot \mathbf{x}$.
4. We write $\mathbf{0}$ for $\mathbf{0}_V$.

There is little danger of confusion, for it is usually clear from the context whether two vectors or two scalars are being added, and whether two scalars are being multiplies, or a vector by a scalar.

Theorem 3.18. *Take a vector space, V, over the field $\mathbb{F}$, $\lambda \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. Then*

(a) $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$ *if only if* $\mathbf{y} = \mathbf{z}$ *(In particular,* $\mathbf{x} + \mathbf{y} = \mathbf{0}_V$ *if and only if* $\mathbf{y} = -\mathbf{x}$*.)*

(b) $-\mathbf{0}_V = \mathbf{0}_V$

(c) $\lambda\mathbf{0}_V = \mathbf{0}_V$

(d) $0\mathbf{x} = \mathbf{0}_V$

(e) $\lambda\mathbf{x} = \mathbf{0}_V$ *if and only if either* $\lambda = 0$ *or* $\mathbf{x} = \mathbf{0}_V$

(f) $(-1)\mathbf{x} = -\mathbf{x}$

Proof. Take $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\lambda \in \mathbb{F}$.

(a) Suppose $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$.

$$\begin{aligned} -\mathbf{x} + (\mathbf{x} + \mathbf{y}) &= -\mathbf{x} + (\mathbf{x} + \mathbf{z}) && \text{by VS3, whence} \\ (-\mathbf{x} + \mathbf{x}) + \mathbf{y} &= (-\mathbf{x} + \mathbf{x}) + \mathbf{z} && \text{by VS1, whence} \\ \mathbf{0}_V + \mathbf{y} &= \mathbf{0}_V + \mathbf{z} && \text{by VS3} \end{aligned}$$

Hence, by VS2, $\mathbf{y} = \mathbf{z}$

(b)

$$\begin{aligned} -\mathbf{0}_V &= -\mathbf{0}_V + \mathbf{0}_V && \text{by VS2} \\ &= \mathbf{0}_V && \text{by VS3} \end{aligned}$$

(c)

$$\begin{aligned} \lambda\mathbf{x} + \lambda\mathbf{0}_V &= \lambda(\mathbf{x} + \mathbf{0}_V) && \text{by VS6} \\ &= \lambda\mathbf{x} && \text{by VS2} \\ &= \lambda\mathbf{x} + \mathbf{0}_V && \text{by VS2.} \end{aligned}$$

Hence, by (a), $\lambda\mathbf{0}_V = \mathbf{0}_V$.

(d)

$$\begin{aligned}
\mathbf{x} + 0\mathbf{x} &= 1\mathbf{x} + 0\mathbf{x} && \text{by VS5}\\
&= (1+0)\mathbf{x} && \text{by VS7}\\
&= 1\mathbf{x} && \text{by properties of fields}\\
&= \mathbf{x} && \text{by VS5}\\
&= \mathbf{x} + \mathbf{0}_V && \text{by VS2}
\end{aligned}$$

Hence, by (a), $0\mathbf{x} = \mathbf{0}_V$.

(e) Suppose that $\lambda\mathbf{x} = \mathbf{0}_V$ and $\lambda \neq 0$.

$$\begin{aligned}
\mathbf{x} &= 1\mathbf{x} && \text{by VS5}\\
&= (\frac{1}{\lambda}\lambda)\mathbf{x} && \text{by properties of fields}\\
&= \frac{1}{\lambda}(\lambda\mathbf{x}) && \text{by VS8}\\
&= \frac{1}{\lambda}\mathbf{0}_V && \text{by hypothesis}\\
&= \mathbf{0}_V && \text{by (c).}
\end{aligned}$$

(f)

$$\begin{aligned}
\mathbf{x} + (-1)\mathbf{x} &= 1\mathbf{x} + (-1)\mathbf{x} && \text{by VS5}\\
&= (1+(-1))\mathbf{x} && \text{by VS7}\\
&= 0\mathbf{x} && \text{by properties of fields}\\
&= \mathbf{0}_V && \text{by (d)}\\
&= \mathbf{x} + (-\mathbf{x}) && \text{by VS3}
\end{aligned}$$

Thus, by (a), $(-1)\mathbf{x} = -\mathbf{x}$. □

Corollary 3.19. *Let F be a field and take $x \in F$. Then $-(-x) = x$.*

Proof. Since $(-x) + x = 0$ and $(-x) + (-(-x)) = 0$, the conclusion follows immediately from the first part of Theorem 3.18. □

Observation 3.20. A set, V, may be a vector space over the same field in more than one way, even when the vector addition is the same in both cases. As an example, let V be a vector space over $\mathbb{C}$, with operations $\boxplus$ and $\boxtimes$. We define a new vector space structure on V by defining

$$\begin{aligned}
\overline{\boxplus} &: V \times V \longrightarrow V, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \mathbf{u} \boxplus \mathbf{v}\\
\overline{\boxtimes} &: \mathbb{C} \times V \longrightarrow V, \quad (\alpha, \mathbf{v}) \longmapsto \overline{\alpha} \boxtimes \mathbf{v}.
\end{aligned}$$

In other words, we "twist" multiplication by a scalar: instead of multiplying vetors by a given complex number, we multiply them by its complex conjugate.

To see that this is a genuinely different vector space, observe that for any vector, $\mathbf{v} \in V$,

$$\begin{aligned}
i \boxtimes \mathbf{v} = i \overline{\boxtimes} \mathbf{v} \quad &\text{if and only if} \quad i \boxtimes \mathbf{v} = -i \boxtimes \mathbf{v}\\
&\text{if and only if} \quad 2i \boxtimes \mathbf{v} = \mathbf{0}_V\\
&\text{if and only if} \quad \mathbf{v} = \mathbf{0}_V && \text{by Theorem 3.18(e)}
\end{aligned}$$

3.3 Exercises

Exercise 3.1. Show that $\mathbb{F} = \{a, b\}$, with $a \neq b$ is a field with respect to the operations $+$ and $.$ defined by:

$+$	a	b
a	a	b
b	b	a

and

$.$	a	b
a	a	a
b	a	b

Here we have defined the two binary operations by means of their *Cayley tables*: The value of the operation at (x, y) is the entry in the row labelled by x and column labelled by y.

[This field is often denoted by $\mathbb{F}_2$.]

Exercise 3.2. Show that $\mathbb{F} := \{a, b, c, d\}$, with all elements distinct, is a field with respect to the operations $+$ and $.$ defined by:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

and

$.$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

[This field is usually denoted by $\mathbb{F}_4$, or $\mathbb{F}_{2^2}$.]

Exercise 3.3. Show that the usual addition and multiplication render

(a) $\mathbb{C}$ a vector space over $\mathbb{R}$;

(b) $\mathbb{R}$ a vector space over $\mathbb{Q}$;

(c) $\mathbb{C}$ a vector space over $\mathbb{Q}$.

Exercise 3.4. Decide whether the following are vector spaces.

(a) Take $\mathbb{F} := \mathbb{C}$ and $V := \mathbb{C}$. Define $\boxplus$ to be the usual addition of complex numbers, and $\boxdot$ by

$$\alpha \boxdot z := \alpha^2 z \qquad (\alpha, z \in \mathbb{C}).$$

(b) Let $\mathbb{F}$ be any field and $V := \mathbb{F}^2$. Define $\boxplus$ to be the usual (component-wise) addition of ordered pairs, and $\boxdot$ by

$$\alpha \boxdot (\beta, \gamma) := (\alpha, \beta) \qquad (\alpha, \beta, \gamma \in \mathbb{F}).$$

(c) Take $\mathbb{F} := \mathbb{F}_{2^2}$ and $V := \mathbb{F}^2$. Define $\boxplus$ to be the usual addition of complex numbers, and $\boxdot$ by

$$\alpha \boxdot \mathbf{v} := \begin{cases} (\alpha\beta, \alpha\gamma) & \text{if } \gamma \neq 0 \\ (\alpha^2\beta, 0) & \text{if } \gamma = 0 \end{cases}.$$

(d) Take $\mathbb{F} := \mathbb{C}$ and $V := \mathbb{C}$. Define $\boxplus$ to be the usual addition of complex numbers, and $\boxdot$ by

$$\alpha \boxdot z := \Re(\alpha) z \qquad (\alpha, z \in \mathbb{C}),$$

where $\Re(\alpha)$ denotes the real part of the complex number α.

(e) Take $\mathbb{F} := \mathbb{R}$ and $V := \mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$. Define $\boxplus$ and $\boxdot$ by

$$\begin{aligned} x \boxplus y &:= xy \qquad && (x, y \in \mathbb{R}^+) \\ \alpha \boxdot x &:= x^\alpha && (\alpha \in \mathbb{R}, x \in \mathbb{R}^+) \end{aligned}$$

Exercise 3.5. Let $\mathbb{F} := \{x \in \mathbb{R} \mid x > 0\}$ and define

$$\begin{aligned} +_{\mathbb{F}} &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \quad (x, y) \longmapsto xy \\ \times_{\mathbb{F}} &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \quad (x, y) \longmapsto x^{\ln y}. \end{aligned}$$

Prove that these operations render $\mathbb{F}$ a field.

Exercise 3.6. Put $\mathbb{F} := \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$. Define $+_{\mathbb{F}}$, $\times_{\mathbb{F}}$ to be the usual addition and multiplication of matrices.

Prove that $\mathbb{F}$ is a field.

Exercise 3.7. Let $\mathbb{R}[t]$ denote the set of all polynomials in the indeterminate t, with real coefficients, so that

$$\mathbb{R}[t] := \{a_0 + a_1 t + \cdots + a_n t^n \mid a_j \in \mathbb{R}\ (j = 1, \ldots, n) \text{ and } a_n \neq 0 \text{ if } n \neq 0\}$$

(a)Define a relation, $\sim$, on $\mathbb{R}[t]$ by

$$p(t) \sim q(t) \text{ if and only if } t^2 + 1 \text{ divides } p(t) - q(t).$$

Prove that $\sim$ is an equivalence relation on $\mathbb{R}[t]$.

(b) Let $\mathbb{F}$ denote the set of all $\sim$-equivalence classes, and denote the equivalence class of $p(t)$ by $[p(t)]$.

Prove that every equivalence class can be represented by a polynomial of the form $a + bt$, with $a, b \in \mathbb{R}$. Thus

$$\mathbb{F} = \{\, [a + bt] \mid a, b \in \mathbb{R} \,\}.$$

(c) Define

$$\begin{aligned} +_{\mathbb{F}} &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, \qquad && ([a + bt], [c + dt]) \longmapsto [(a + c) + (b + d)t] \\ \times_{\mathbb{F}} &: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, && ([a + bt], [c + dt]) \longmapsto [(ac - bd) + (ad + bc)t] \end{aligned}$$

Prove that these definitions render $\mathbb{F}$ a field.

Exercise 3.8. Let $\mathbb{F}$ be any field and X a non-empty set and V the set all $\mathbb{F}$-valued functions defined on X, so that $V := \{f : X \longrightarrow \mathbb{F}\}$.

Define

$$\begin{aligned} &\boxplus : V \times V \longrightarrow V, \quad (f, g) \longmapsto f \boxplus g \qquad \text{where, for all } x \in X, \quad (f \boxplus g)(x) := f(x) +_{\mathbb{F}} g(x) \\ &\boxtimes : \mathbb{F} \times V \longrightarrow V, \quad (\lambda, f) \longmapsto \lambda \boxdot f \qquad \text{where, for all } x \in X, \quad (\lambda \boxdot f)(x) := \lambda \times_{\mathbb{F}} f(x). \end{aligned}$$

Prove that these definitions renders V a vector space over $\mathbb{F}$.