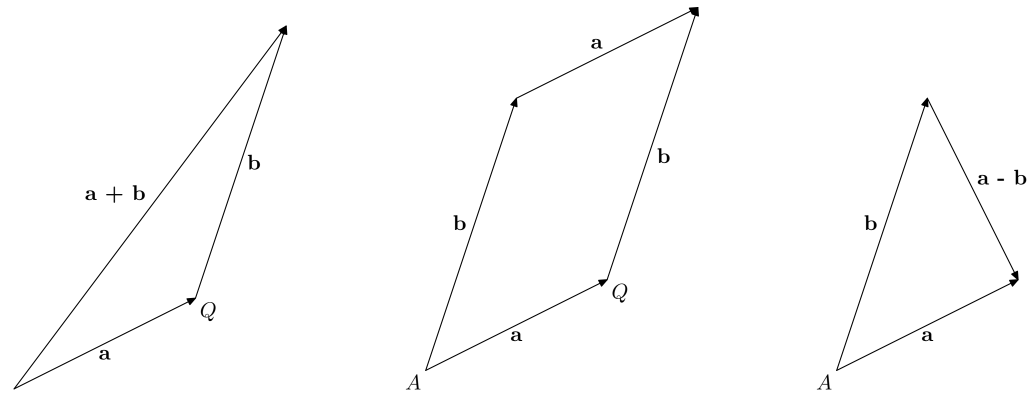
Sample Solutions for Tutorial 10

Question 1.

In the following diagrams, let α be the angle at Q and β the angle at A.



Using the cosine rule in the first diagram, we get

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\alpha \qquad (\bullet)$$

From the third diagram, we get

$$\|\mathbf{a}\text{-}\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\beta \qquad (\circ)$$

From the middle diagram we see that $\alpha + \beta - \pi$, whence $\cos\beta = -\cos\alpha$.
Using this, and adding $(\bullet)$ to $(\circ)$, we see that

$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}\text{-}\mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$$

Question 2.

The vector, $\mathbf{b}$, from $(1,3,5)$ to $(6,9,17)$ is $(5,6,12) = 5\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}$. Hence a vectorial representation of the line in question is

$$\mathbf{r}(t) = (1,3,5) + t(5,6,12) = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + t(5\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}).$$

If $\mathbf{r}(t) = (x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we see that

$$x = 1 + 5t$$
$$y = 3 + 6t$$
$$z = 5 + 12t$$

It follows form the first two that $6x - 6 = 5y - 15$, or $6x - 5y = -9$.
From the last two we see that $2y - 6 = z - 5$, or $2y - z = 1$.
Hence the line consists of

$$\{(x,y,z) \in \mathbb{R}^2 \mid 6x - 5y + 9 = 2y - z - 1 = 0\}$$

The vectors $\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $6\mathbf{i} + 9\mathbf{j} + 17\mathbf{k}$ span a parallelogram whose area,

$$
\begin{aligned}
\|(\mathbf{i}+3\mathbf{j}+5\mathbf{k})\times(6\mathbf{i}+9\mathbf{j}+17\mathbf{k})\| & \\
&= \|(3\times 17-5\times 9)\mathbf{i}+(5\times 6-1\times 17)\mathbf{j}+(1\times 9-3\times 6)\mathbf{k}\| \\
&= \|6\mathbf{i}+13\mathbf{j}-9\mathbf{k}\| \\
&= \sqrt{36+169+81} \\
&= \sqrt{286},
\end{aligned}
$$

is twice the area of the triangle spanned by $\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and $6\mathbf{i}+9\mathbf{j}+17\mathbf{k}$.
Since $\mathbf{b}$ may be regarded as the base of this triangle and the perpendicular distance of the origin from our line as its height, the perpendicular distance is the area of the parallelogram divided by the length of b $=\sqrt{25+36+144}=\sqrt{205}$, making the perpendicular distance of the origin from our line $\sqrt{\dfrac{286}{205}}$ units.

Question 3.

(a)

$$
\begin{array}{lrcrcrl}
 & 3x & + & 4y & = & 10 & \text{(i)} \\
 & 7x & - & 5y & = & 9 & \text{(ii)} \\
 & 5x & + & 6y & = & 16 & \text{(iii)} \\
2\times\text{(iii)}-\text{(ii)}-\text{(i)}: & & & 13y & = & 13 & \text{(iv)} \\
\text{substitute in (iii):} & 5x & & & = & 10 & \text{(v)}
\end{array}
$$

Thus $x=2$ and $y=1$.

(b)

$$
\begin{array}{lrcrcrcrl}
 & 2x & + & 4y & + & 4z & = & 7 & \text{(i)} \\
 & 3x & - & 7y & - & 2z & = & 15 & \text{(ii)} \\
 & 5x & - & 3y & + & 2z & = & 20 & \text{(iii)} \\
\text{(ii)}-\text{(ii)}-\text{(i)}: & 0x & + & 0y & + & 0z & = & -2 & \text{(iv)}
\end{array}
$$

This last equation shows the system is inconsistent, and so it cannot have any solutions.