AMTH140 ASSIGNMENT 1

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(1) (a) $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 4\}$
$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$
$A \cap B = \{3\}$
$$A \times B = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4), (5, 2), (5, 3), (5, 4)\\ (7, 2), (7, 3), (7, 4), (9, 2), (9, 3), (9, 4)\}$$
(b) $\mathcal{P}(B) = \big\{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\big\}$
(c) Let $B_1 = \{2\}, B_2 = \{3\}, B_3 = \{4\}$. Then $\{B_1, B_2, B_3\}$ is a partition of B that has 3 elements. In other words, $\{\{2\}, \{3\}, \{4\}\}$ is the required partition.

(2) From the set identities in the Lecture Notes (p.18) we show for sets A, B and C,

$$\begin{aligned}
(A - B) - C &= (A \cap B') - C && \text{(S10)}\\
&= (A \cap B') \cap C' && \text{(S10)}\\
&= A \cap (B' \cap C') && \text{(S2)}\\
&= A \cap (C' \cap B') && \text{(S1)}\\
&= (A \cap C') \cap B' && \text{(S2)}\\
&= (A - C) - B && \text{(S10)}
\end{aligned}$$

(3) Let $S(n)$ be the statement that $2^{3n} - 1$ is divisible by 7 for $\{n \in \mathbb{Z} \mid n \geq 1\}$.
$\mathbf{n = 1}:$ $2^{3(1)} - 1 = 2^8 - 1 = 7$. Hence $S(1)$ is true.
$\mathbf{n \geq 1}$: We make the inductive hypothesis that $S(k)$ is true, that is, $2^{3k} - 1 = 7m$ for $m \in \mathbb{Z}$. Then,

$$\begin{aligned}
2^{3(k+1)} - 1 &= 2^{3k} \cdot 2^3 - 1\\
&= (7m + 1) \cdot 2^3 - 1\\
&\quad \text{(by the inductive hypothesis)}\\
&= 56m + 8 - 1\\
&= 56m + 7\\
&= 7(8m + 1)
\end{aligned}$$

where $8m + 1$ is an integer. Hence $S(k + 1)$ is true whenever $S(k)$ is true. Thus by the principle of mathematical induction, 7 divides $2^{3n} - 1$ for each integer $n \geq 1$.

(4)

$$\begin{aligned}
f(x) &= 3x^3 + 2x^2 - 5x - 4 \\
&= x\big[x(3x+2) - 5\big] - 4 \\
f(1) &= 1\big[1(3 \cdot 1 + 2) - 5\big] - 4 \\
&= \big[3 + 2 - 5\big] - 4 \\
&= 0 - 4 = -4 \\
f(-2) &= -2\big[-2(3 \cdot (-2) + 2) - 5\big] - 4 \\
&= -2\big[-2(-4) - 5\big] - 4 \\
&= -2\big[8 - 5\big] - 4 \\
&= -2(3) - 4 \\
&= -6 - 4 = -10
\end{aligned}$$

(5) $f(x) = x^3 + 2x^2 - x + 2$. By the generalised triangle inequality,

$$\begin{aligned}
|x^3 + 2x^2 - x + 2| &\leq |x^3| + |2x^2| + |x| + |2| \\
&= x^3 + 2x^2 + x + 2 \quad (\text{if } x \geq 0) \\
&\leq x^3 + 2x^2 \times x + x \times x^2 + 2 \times x^3 \quad (\text{if } x \geq 1) \\
&\leq x^3 + 2x^3 + x^3 + 2x^3 \\
&\leq 6|x^3|
\end{aligned}$$

Since $|f(x)| \leq C|x^3|$ if $x \geq M$, where $C = 6$ and $M = 1$, then $f(x) = O(x^3)$.

(6)

$$f(n) = \frac{n^2 + 2\log_2 n}{n+1}$$

$$\begin{aligned}
\left|\frac{n^2 + 2\log n}{n+1}\right| &= \frac{n^2 + 2\log n}{n+1} \quad (\text{if } n \geq 1) \\
&\leq \frac{n^2 + 2n}{n+1} \quad (\text{since } \log n \leq n) \\
&\leq \frac{n^2 + 2n}{n} = n + 2 \leq n + 2n = 3n \\
|f(n)| &\leq 3|n| \text{ if } n \geq 1
\end{aligned}$$

$$\begin{aligned}
\left|\frac{n^2 + 2\log n}{n+1}\right| &\geq \frac{n^2}{n+1} \quad (\text{since } 2\log n > 0 \text{ if } n \geq 2) \\
&\geq \frac{n^2}{n+n} = \frac{n^2}{2n} = \frac{n}{2}
\end{aligned}$$

$$|f(n)| \geq \frac{1}{2}|n| \text{ if } n \geq 1$$

Therefore, $f(n) = \Theta(n)$ since

$$\frac{1}{2}|n| \leq |f(n)| \leq 3|n|$$

where $D = \frac{1}{2}$, $C = 3$ and $M = 1$.