MATH101 ASSIGNMENT 4

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(1) (a)

$$\lim_{x\to\infty} \frac{x^3+2x-1}{2x^3+x^2} = \lim_{x\to\infty} \frac{1+\frac{2}{x^2}-\frac{1}{x^3}}{2+\frac{1}{x}} = \frac{1+0-0}{2+0} = \frac{1}{2}$$

(b)

$$\lim_{x\to 1} \frac{\sqrt{2x+1}-\sqrt{x+1}}{x} = \frac{\sqrt{2\cdot 1+1}-\sqrt{1+1}}{1} = \sqrt{3}-\sqrt{2}$$

(c)

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} \\ &= 1\cdot \frac{1}{\cos 0} = \frac{1}{1} = 1 \end{aligned}$$

(d)

$$\begin{aligned} \lim_{x\to 0} \frac{\cos^2 x-\cos x}{x} &= \lim_{x\to 0} \frac{\cos x(\cos x-1)}{x} \\ &= \lim_{x\to 0} \frac{-\cos x(1-\cos x)}{x} \\ &= -\lim_{x\to 0} \cos x \cdot \lim_{x\to 0} \frac{1-\cos x}{x} \\ &= -1\cdot 0 = 0 \end{aligned}$$

(2) Let $\varepsilon > 0$ be given. Then we must find a number, $\delta > 0$, such that $|\sqrt{x}-1| < \varepsilon$ whenever $|x-1| < \delta$.

We know the domain of $\sqrt{x}$ is $\{x \in \mathbb{R} \mid x \geq 0\}$.

By definition of absolute values, we also know

$$\begin{aligned} |x-1| &= x-1 < \delta \implies x < 1+\delta \\ |x-1| &= -(x-1) < \delta \implies 1-x < \delta \implies x > 1-\delta \end{aligned}$$

Moreover

$$\begin{aligned} |\sqrt{x}-1| &= \sqrt{x}-1 \quad \text{if } \ x>1 \\ |\sqrt{x}-1| &= 1-\sqrt{x} \quad \text{if } \ 0<x<1 \end{aligned}$$

Algebraic steps yield

$$\sqrt{x}-1<\varepsilon \quad \text{whenever } \ 1<x<1+\delta \qquad \text{(i)}$$
$$1-\sqrt{x}<\varepsilon \quad \text{whenever } \ 1-\delta<x<1 \qquad \text{(ii)}$$

However we observe that

$$\sqrt{x}-1<\varepsilon \quad \text{if } \ x<(\varepsilon+1)^2 \qquad \text{(iii)}$$
$$1-\sqrt{x}<\varepsilon \quad \text{if } \ x>(1-\varepsilon)^2 \qquad \text{(iv)}$$

Equating (i) and (iii) gives

$$\begin{aligned} 1+\delta &= (\varepsilon+1)^2 \\ &= \varepsilon^2+2\varepsilon+1 \\ \delta &= \varepsilon^2+2\varepsilon \end{aligned}$$

Equating (ii) and (iv) gives

$$\begin{aligned} 1-\delta &= (1-\varepsilon)^2 \\ &= 1-2\varepsilon+\varepsilon^2 \\ \delta &= 2\varepsilon-\varepsilon^2 \end{aligned}$$

So given $\varepsilon>0$ there is a $\delta>0$ such that $|\sqrt{x}-1|<\varepsilon$ whenever $|x-1|<\delta$,

$$\begin{aligned} \delta &= \varepsilon^2+2\varepsilon \quad \text{whenever } \ x>1 \\ \delta &= 2\varepsilon-\varepsilon^2 \quad \text{whenever } \ 0<x<1 \end{aligned}$$

thereby completing the proof of

$$\lim_{x\to 1}\sqrt{x}=1$$

(3) (a) For continuity of f at $x = 1$ we require

$$\lim_{x\to 1} f(x) = f(1) = 2\cdot 1 + k = 2 + k$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{1+x^2} = \frac{1}{1+1^2} = \frac{1}{2}$$

$$2 + k = \frac{1}{2}$$

$$k = -\frac{3}{2}$$

(b) To show f is continuous on $(0, \infty)$ we must consider the limits of the piecewise function at the following points.

$\mathbf{x = 0}$:

$$\lim_{x\to 0} f(x) \;=\; \lim_{x\to 0} \frac{1}{1+x^2} \;=\; \frac{1}{1+0^2} \;=\; 1 \;=\; f(0)$$

$\mathbf{x = 1}$:

$$\lim_{x\to 1}\; 2x - \frac{3}{2} \;=\; 2\cdot 1 - \frac{3}{2} \;=\; \frac{1}{2} \;=\; f(1)$$

$\mathbf{x = +\infty}$:

$$\lim_{x\to +\infty}\; 2x - \frac{3}{2} \;= +\infty \qquad \text{(limit does not exist)}$$

The equality of the one-sided limits also show there is no discontinuity at $x = 1$.

$$\begin{aligned} \lim_{x\to 1^-} f(x) \;&=\; \lim_{x\to 1^-} \frac{1}{1+x^2} \;=\; \frac{1}{2} \\ \lim_{x\to 1^+} f(x) \;&=\; \lim_{x\to 1^+}\; 2x - \frac{3}{2} \;=\; \frac{1}{2} \end{aligned}$$

We now consider the intervals $(0, 1)$ and $[1, \infty)$.

$\mathbf{0 < x < 1}$:

$$\lim_{x\to a} f(x) \;=\; \lim_{x\to a} \frac{1}{1+x^2} \;=\; \frac{1}{1+a^2} \;=\; \frac{1}{1+a^2} \;=\; f(a)$$

$\mathbf{x \geq 1}$:

$$\lim_{x\to a} f(x) \;=\; \lim_{x\to a}\; 2x - \frac{3}{2} \;=\; 2a - \frac{3}{2} \;=\; f(a)$$

Thus we have shown that

$$\lim_{x\to a} f(x) = f(a)$$

for $\{x \in \mathbb{R} \mid x > 0\}$ and that there is no discontinuity at $x = 1$. Hence f is continuous on $(0, \infty)$.

(4) (i) $f : (0,1) \to \mathbb{R}, \quad x \mapsto x^2$

(a) *Monotone*
Let $a, b \in (0,1)$ and $a < b$. Then $a^2 < b^2$ **iff** $a + b > 0$. Since a, b are both positive, $a + b > 0$. Thus $f(a) < f(b)$ whenever $a < b$, and f is monotonically increasing.

(b) *Injective*
Given f is continuous and *strictly* monotone, it is necessarily injective. Moreover, $f(a) = f(b)$ **iff** $a = b$. We also note that there is no $x \in (0,1)$ such that $f(x) \leq 0$ or $f(x) \geq 1$, thus f is not surjective.

(c) $\inf(f) = 0$ and $\sup(f) = 1$
It is immediate from the definition of f that the infimum is 0 and the supremum is 1. It also follows that f attains neither of its bounds.

(ii) $g : \mathbb{R}\backslash\{-1\} \to \mathbb{R}, \quad x \mapsto \frac{1}{1+x}$

(a) *Non-monotone*
Let $a, b, c, d \in \mathbb{R}\backslash\{-1\}$ and $a < b < -1 < c < d$. Then by definition,

$$\begin{array}{ccccccc} f(a) & > & f(b) & \text{whenever} & a < b & \Rightarrow & \text{decreasing} \\ f(c) & > & f(d) & \text{whenever} & c < d & \Rightarrow & \text{decreasing} \end{array}$$

However,

$$\begin{array}{ccccccc} f(a) & < & f(d) & \text{whenever} & a < d & \Rightarrow & \text{increasing} \\ f(b) & < & f(c) & \text{whenever} & b < c & \Rightarrow & \text{increasing} \end{array}$$

Thus g is not monotonic since it is not entirely non-decreasing or non-increasing over its domain.

(b) *Injective*
Even though not monotonic, g is injective since $f(a) = f(b)$ **iff** $a = b$ (this is because g is not continuous). We also note that there is no $x \in \mathbb{R}\backslash\{-1\}$ such that $f(x) = 0$, therefore f is not surjective.

(c) *No infimum or supremum*
The one-sided limits at $x = -1$ show that g is unbounded above and below since both limits do not exist.

$$\begin{aligned}\lim_{x\to -1^+} g(x) &= \lim_{x\to -1^+} \frac{1}{1+x} = +\infty \\ \lim_{x\to -1^-} g(x) &= \lim_{x\to -1^-} \frac{1}{1+x} = -\infty\end{aligned}$$

(iii) $h : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{x^2+2}$

(a) *Non-monotone*
Let $a, b, c, d \in \mathbb{R}$ and $a < b < 0 < c < d$. Then by definition,

$$\begin{array}{ccccccc} h(a) & < & h(b) & \text{whenever} & a < b & \Rightarrow & \text{increasing} \\ h(c) & > & h(d) & \text{whenever} & c < d & \Rightarrow & \text{decreasing} \end{array}$$

Thus h is not monotonic since it is increasing in $\mathbb{R}_0^-$ and decreasing $\mathbb{R}_0^+$

(b) *Neither*
Knowing that h is continuous in $\mathbb{R}$ and non-monotone, we can say that h is not injective. Furthermore, there exists $a \neq b$ such that $h(a) = h(b)$, namely when $a = -b$ (as h is symmetric at $x = 0$). We also note that there is no $x \in \mathbb{R}$ such that $h(x) \leq 0$, thus h is not surjective.

(c) $\inf(h) = 0$ and $\sup(h) = \max(h) = \frac{1}{2}$
Since $x^2 \geq 0$, then $x^2 + 2 > 0$ and $\frac{1}{x^2+2} > 0$. It follows that h is bounded below by infimum 0, which it never attains. To find the supremum, we solve for x so that the denominator is smallest. Clearly, this is when $x = 0$ and thus $h(0) = \frac{1}{0^2+2} = \frac{1}{2}$. Hence h is bounded above by supremum $\frac{1}{2}$, which it attains.

(iv) $t : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}, \quad x \mapsto \tan(x)$

(a) *Monotone*
Let $a, b \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $a < b < 0 < c < d$. Then by definition,

$$\begin{array}{ccccccc} t(a) & < & t(b) & \text{whenever} & a < b & \Rightarrow & \text{increasing} \\ t(c) & < & t(d) & \text{whenever} & c < d & \Rightarrow & \text{increasing} \\ t(b) & < & t(c) & \text{whenever} & b < c & \Rightarrow & \text{increasing} \end{array}$$

Thus $t(u) < t(v)$ whenever $u < v$, and t is monotonically increasing.

(b) *Bijective*
Since $t(u) = t(v)$ **iff** $u = v$, t is injective. We also observe that $t \to +\infty$ as $x \to \frac{\pi}{2}$ while $t \to -\infty$ as $x \to -\frac{\pi}{2}$. Consequently, for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ there exists $t(x) \in \mathbb{R}$. Hence t is also surjective.

(c) *No infimum or supremum*
Since t decreases without bound as x tends to $-\frac{\pi}{2}$, there is no infimum. Similarly, since t increases without bound as x tends to $\frac{\pi}{2}$, there is no supremum. The one-sided limits confirm that t is unbounded above and below since the limits do not exist.

$$\begin{aligned}\lim_{x \to (-\frac{\pi}{2})^+} \tan(x) &= -\infty \\ \lim_{x \to (\frac{\pi}{2})^-} \tan(x) &= \infty\end{aligned}$$