MATH101 ASSIGNMENT 3

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(1) (a) $\bar{z} = 7$ and $|z| = 7$

(b) $\bar{z} = \sqrt{3} - i$ and $|z| = \sqrt{3+1} = 2$

(c) $z = \frac{2+i}{2-i} \times \frac{2+i}{2+i} = \frac{4+4i+i^2}{4+1} = \frac{3+4i}{5} = \frac{3}{5} + \frac{4}{5}i$

$\bar{z} = \frac{3}{5} - \frac{4}{5}i$ and $|z| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

(d) $z = (i^2)^2 = (-1)^2 = 1$

$\bar{z} = 1$ and $|z| = 1$

(e) $z = 2^3 - 3(2)^2(3i) + 3(2)(3i)^2 - (3i)^3 = 8 - 36i - 54 + 27i$

$z = -46 + 9i$ and $\bar{z} = -46 - 9i$

$|z| = \sqrt{2116 + 81} = 13\sqrt{13}$

(f) $z = \frac{2^3+3(2)^2(3i)+3(2)(3i)^2+(3i)^3}{5^3-3(5)^2(2i)+3(5)(2i)^2-(2i)^3} = \frac{8+36i-54-27i}{125-150i-60+8i} = \frac{-46+9i}{65-142i}$

$z = \frac{-46+9i}{65-142i} \times \frac{65+142i}{65+142i} = \frac{-2990-6532i+585i+1278i^2}{4225+20164} = \frac{-4268-5947i}{24389}$

$z = -\frac{4268}{24389} - \frac{5947}{24389}i$ and $\bar{z} = -\frac{4268}{24389} + \frac{5947}{24389}i$

$|z| = \sqrt{\left(\frac{4268}{24389}\right)^2 + \left(\frac{5947}{24389}\right)^2} = \sqrt{\frac{2197}{24389}}$ or

$|z| = \sqrt{\frac{46^2+9^2}{65^2+142^2}} = \sqrt{\frac{2116+81}{4225+20164}} = \sqrt{\frac{2197}{24389}}$

(g) $z = \frac{(3+i)-(1-i)}{(1-i)(3+i)} = \frac{2+2i}{3+i-3i-i^2} = \frac{2+2i}{4-2i}$

$z = \frac{2+2i}{4-2i} \times \frac{4+2i}{4+2i} = \frac{8+4i+8i+4i^2}{16+4} = \frac{4+12i}{20} = \frac{1}{5} + \frac{3}{5}i$

$\bar{z} = \frac{1}{5} - \frac{3}{5}i$ and $|z| = \sqrt{\frac{1}{25} + \frac{9}{25}} = \frac{\sqrt{10}}{5}$

(2) In order for f to be a well-defined function, we must have $x \geq 0$ (since we cannot take the square root of a negative number in $\mathbb{R}$). Hence, the maximum subset is $X_f \subseteq [0, \infty) := \text{dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\} := \mathbb{R}_0^+$

For X_g we must have $x^2 - 1 \geq 0$ or equivalently, $|x| \geq 1$. This implies $X_g \subseteq (-\infty, -1] \cup [1, \infty) := \text{dom}(g) = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 1\}$.

For X_h we must have $x \geq 1$. This is because g and h differ only by their codomains, with codom(h) restricted to $\mathbb{R}_0^+$. So by deduction, we find $X_h \subseteq [1, \infty) := \text{dom}(h) = \{x \in \mathbb{R} \mid x \geq 1\}$.

As square roots take only non-negative values in $\mathbb{R}$, the range (or image) of the functions f, g and h are all $\{x \in \mathbb{R} \mid x \geq 0\} := \mathbb{R}_0^+$.

$f \circ g : \text{im}(g) = \text{dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\}$
$f \circ h : \text{im}(h) = \text{dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\}$
$g \circ f : \text{im}(f) \neq \text{dom}(g) \Leftrightarrow \{x \in \mathbb{R} \mid x \geq 0\} \neq \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 1\}$
$h \circ f : \text{im}(f) \neq \text{dom}(h) \Leftrightarrow \{x \in \mathbb{R} \mid x \geq 0\} \neq \{x \in \mathbb{R} \mid x \geq 1\}$

Of the four compositions above, only $f \circ g$ and $f \circ h$ are defined.

$f \circ g : X_g \to \mathbb{R}, \quad x \mapsto f(g(x)) = f(\sqrt{x^2 - 1}) = \sqrt{\left(\sqrt{x^2 - 1}\right)}$ where

$$\text{dom}(f \circ g) = \text{dom}(g) := \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 1\}$$
$$\text{codom}(f \circ g) = \text{codom}(f) := \mathbb{R}$$
$$\text{im}(f \circ g) = \text{im}(g) := \{x \in \mathbb{R} \mid x \geq 0\}$$

$f \circ h : X_h \to \mathbb{R}, \quad x \mapsto f(h(x)) = f(\sqrt{x^2 - 1}) = \sqrt{\left(\sqrt{x^2 - 1}\right)}$ where

$$\text{dom}(f \circ h) = \text{dom}(h) := \{x \in \mathbb{R} \mid x \geq 1\}$$
$$\text{codom}(f \circ h) = \text{codom}(f) := \mathbb{R}$$
$$\text{im}(f \circ h) = \text{im}(h) := \{x \in \mathbb{R} \mid x \geq 0\}$$

(3) (a) We must have $x \neq -1$ for f to be a function (since we cannot divide by 0). Thus, the maximum subset is $X := \{x \in \mathbb{R} \mid x \neq -1\}$ and the function $f : X \to \mathbb{R}, \quad x \mapsto \frac{1}{1+x}$ has $\text{im}(f) = \{x \in \mathbb{R} \mid x \neq 0\}$.

Clearly, f is not surjective since codom$(f) \neq \text{im}(f)$, as there is no $x \in \mathbb{R}$ such that $f(x) = 0$. We give the following proof below.

Let y be in the codomain $\mathbb{R}$. We must find an x in the domain X such that $f(x) = \frac{1}{1+x} = y$. Solving for x, we find $x = \frac{1-y}{y}$ where $y \neq 0$. Thus, there is no $x \in \mathbb{R}$ which satisfies $f(x) = 0$.

However, the function is injective because $f(x) = f(x')$ if and only if $x = x'$. We show this by direct proof. Assume $f(x) = f(x')$. That is, $\frac{1}{1+x} = \frac{1}{1+x'}$. Hence, $1 + x = 1 + x'$ and $x = x'$. Hence, f is injective.

Geometrically, this means every horizontal line intersects the graph of f in at most one point.

(b) If we write $\sqrt{x^4 - x^2}$ in factorised form, $\sqrt{x^2(x^2-1)}$, then we must have $|x| \geq 1$ or $x = 0$ for f to be a function. Thus, $X := \{x \in \mathbb{R} \mid |x| \geq 1 \text{ or } x = 0\}$ and $f : X \to \mathbb{R}, \quad x \mapsto \sqrt{x^4 - x^2}$ has $\operatorname{im}(f) = \{x \in \mathbb{R} \mid x \geq 0\}$.

Again, f is not surjective since $\operatorname{codom}(f) \neq \operatorname{im}(f)$, as there is no $x \in \mathbb{R}$ such that $f(x) < 0$. The function is also not injective and we prove this by contradiction.

Assume $f(x) = f(x')$ such that $\sqrt{x^4 - x^2} = \sqrt{x'^4 - x'^2}$. Hence, $x^4 = x'^4$ and $x^2 = x'^2$. Observe that one solution to either equation is $x = 2$ and $x' = -2$. Hence, we have the following counterexample to f being injective. Suppose $x = 2$ and $x' = 2$, then

$$f(x) = f(2) = \sqrt{2^4 - 2^2} = \sqrt{16 - 4} = \sqrt{12}$$
$$f(x') = f(-2) = \sqrt{(-2)^4 - (-2)^2} = \sqrt{16 - 4} = \sqrt{12}$$

but $x \neq x'$. Thus f is neither injective nor surjective.

(c) For f to be a well-defined function, the maximum subset X is the set of all real numbers $\mathbb{R}$. So the function $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3 - 13$ has $\operatorname{im}(f) = \mathbb{R}$.

This function is surjective since $\operatorname{codom}(f) = \operatorname{im}(f)$. In other words, its range coincides with its codomain. Geometrically, this means that every horizontal line intersects the graph of f in one or more points. In this particular case, each horizontal line intersects f exactly once.

It is also injective because $f(x) = f(x')$ if and only if $x = x'$. Again, we assume $f(x) = f(x')$ such that $x^3 = x'^3$. By taking cube roots, we find $x = x'$. Hence, f is injective.

Moreover, f is bijective.

(4) Let $\varepsilon > 0$ be given. Then we must find a number, $\delta > 0$, such that $|x| < \delta$ guarantees $\left|\frac{1}{1+x^2} - 1\right| < \varepsilon$. Algebraic steps yield

$$\left|\frac{1}{1+x^2} - 1\right| = \left|\frac{1-(1+x^2)}{1+x^2}\right| = \left|\frac{-x^2}{1+x^2}\right| = \frac{|-x^2|}{|1+x^2|}$$

The term $1 + x^2$ is always positive (since $x^2 > 0$). So $1 + x^2 = |1 + x^2|$. We also know that $|-x^2| = |x^2| = |x||-x| = |x||x|$. Therefore,

$$\left|\frac{1}{1+x^2} - 1\right| = \frac{|x||x|}{1+x^2} = \frac{|x|}{1+x^2}\ |x|$$

Now we must estimate the largest value that the term $\frac{|x|}{1+x^2}$ can have for x in an interval centred at 0. We choose arbitrarily $-\frac{1}{2} < x < \frac{1}{2}$ which gives us $|x| < \frac{1}{2}$. Moreover $x^2 < \frac{1}{4}$, which implies $1 + x^2 < \frac{5}{4}$ and thus $\frac{1}{1+x^2} < \frac{4}{5}$.

Combining these estimates together yields

$$\frac{|x|}{1+x^2} = |x|\ \frac{1}{1+x^2} < \frac{1}{2} \cdot \frac{4}{5} = \frac{2}{5}$$

and hence,

$$\left|\frac{1}{1+x^2} - 1\right| = \frac{|x|}{1+x^2}\ |x| < \frac{2}{5}\ |x|$$

For this expression to be smaller than ε, we need $|x| < \frac{5}{2}\ \varepsilon$. Thus, given any $\varepsilon > 0$ we have found $\delta = \min\left[\frac{1}{2}, \frac{5}{2}\ \varepsilon\right]$ guarantees $\left|\frac{1}{1+x^2} - 1\right| < \varepsilon$ whenever $|x| < \delta$.