MATH101 ASSIGNMENT 8

MARK VILLAR

(1) (a) For $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x; \quad [0, 2\pi]$

(i)
$$f'(x) = \cos x$$
$$\begin{cases} < 0 & \text{for} \quad (\frac{\pi}{2}, \frac{3\pi}{2}) \\ = 0 & \text{for} \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \\ > 0 & \text{for} \quad (0, \frac{\pi}{2}) \quad \text{and} \quad (\frac{3\pi}{2}, 2\pi) \end{cases}$$

Thus f is monotonically decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$ while it is monotonically increasing on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$. It has critical points at $x = \frac{\pi}{2}, \frac{3\pi}{2}$. As f is an oscillating function, this pattern is repeated over $\mathbb{R}$ with periodicity 2π.

(ii)
$$f''(x) = -\sin x$$
$$\begin{cases} < 0 & \text{for} \quad (0, \pi) \\ = 0 & \text{for} \quad x = 0, \pi, 2\pi \\ > 0 & \text{for} \quad (\pi, 2\pi) \end{cases}$$

Thus f is concave down on $(0, \pi)$ while it is concave up on $(\pi, 2\pi)$. It has points of inflection at $x = 0, \pi, 2\pi$.

(b) For $g : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^x + e^{-x}$

(i)
$$\begin{aligned} g'(x) &= e^x + (-1)e^{-x} \\ &= e^x - e^{-x} \end{aligned}$$
$$\begin{cases} < 0 & \text{for} \quad x < 0 \\ = 0 & \text{for} \quad x = 0 \\ > 0 & \text{for} \quad x > 0 \end{cases}$$

Thus f is monotonically decreasing when $x < 0$ while it is monotonically increasing when $x > 0$. It has a critical point at $x = 0$.

(ii)
$$\begin{aligned} g''(x) &= e^x - (-1)e^{-x} \\ &= e^x + e^{-x} \\ &> 0 \quad \text{for} \quad x \in \mathbb{R}. \end{aligned}$$

Thus g is concave up for all x and has a turning point at $x = 0$.

(c) For $h : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto 9t^3 - 9t^2 + 3$

(i)
$$h'(t) = 27t^2 - 18t$$
$$\begin{cases} < 0 \quad \text{for} \quad (0, \frac{2}{3}) \\ = 0 \quad \text{for} \quad t = 0, \frac{2}{3} \\ > 0 \quad \text{for} \quad (-\infty, 0) \quad \text{and} \quad (\frac{2}{3}, \infty) \end{cases}$$

Thus f is monotonically decreasing on $(0, \frac{2}{3})$ while it is monotonically increasing when $x < 0$ and $x > \frac{2}{3}$. It has critical points at $t = 0, \frac{2}{3}$.

(ii)
$$h''(t) = 54t - 18$$
$$\begin{cases} < 0 \quad \text{for} \quad (-\infty, \frac{1}{3}) \\ = 0 \quad \text{for} \quad t = \frac{1}{3} \\ > 0 \quad \text{for} \quad (\frac{1}{3}, \infty) \end{cases}$$

Thus h is concave down when $t < \frac{1}{3}$ while it is concave up when $t > \frac{1}{3}$. It has a point of inflection at $t = \frac{1}{3}$.

(d) For $k : \mathbb{R} \longrightarrow \mathbb{R}, \quad u \longmapsto \ln(u^2 + 9)$

(i)
$$k'(u) = (\tfrac{1}{u^2+9})(2u) = \tfrac{2u}{u^2+9}$$
$$\begin{cases} < 0 \quad \text{for} \quad u < 0 \\ = 0 \quad \text{for} \quad u = 0 \\ > 0 \quad \text{for} \quad u > 0 \end{cases}$$

Thus f is monotonically decreasing when $u < 0$ while it is monotonically increasing when $u > 0$. It has a critical point at $u = 0$.

(ii)
$$k''(u) = \tfrac{(u^2+9)(2)-(2u)(2u)}{(u^2+9)^2} = \tfrac{18-2u^2}{(u^2+9)^2}$$
$$\begin{cases} < 0 \quad \text{for} \quad (-\infty, -3) \quad \text{and} \quad (3, \infty) \\ = 0 \quad \text{for} \quad u = -3, 3 \\ > 0 \quad \text{for} \quad (-3, 3) \end{cases}$$

Thus k is concave down when $u < -3$ and $u > 3$ while it is concave up when $-3 < u < 3$. It has points of inflection at $u = -3, 3$.

(2) For $f : (-1, \infty) \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(1 + x) - x$

(a)
$$f'(x) = \tfrac{1}{1+x} - 1 = -\tfrac{x}{1+x}$$
$$\begin{cases} < 0 \quad \text{for} \quad (0, \infty) \\ = 0 \quad \text{for} \quad x = 0 \\ > 0 \quad \text{for} \quad (-1, 0) \end{cases}$$

Thus f is monotonically decreasing on $\mathbb{R}_0^+$ since $f'(x) < 0$ for all $x > 0$. As f is decreasing,

$$f(x) \leq f(0) = 0$$

Hence, for all $x \geq 0$:

$$\ln(1+x) - x \leq 0$$
$$\ln(1+x) \leq x$$

(b) Let $g : (-1, \infty) \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(1+x) - x + \frac{x^2}{2}$

$$g'(x) = -\tfrac{x}{1+x} + x = \tfrac{x^2}{1+x}$$

$$\begin{cases} = 0 & \text{for} \quad x = 0 \\ > 0 & \text{for} \quad (-1,0) \quad \text{and} \quad (0,\infty) \end{cases}$$

Thus g is monotonically increasing on $\mathbb{R}_0^+$ since $g'(x) > 0$ for all $x > 0$. As g is increasing,

$$g(x) \geq g(0) = 0$$

Hence, for all $x \geq 0$:

$$\ln(1+x) - x + \frac{x^2}{2} \geq 0$$
$$\ln(1+x) \geq x - \frac{x^2}{2}$$

Let $h : (-1, \infty) \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}$

$$g'(x) = \tfrac{x^2}{1+x} - x^2 = -\tfrac{x^3}{1+x}$$

$$\begin{cases} < 0 & \text{for} \quad (0,\infty) \\ = 0 & \text{for} \quad x = 0 \\ > 0 & \text{for} \quad (-1,0) \end{cases}$$

Thus h is monotonically decreasing on $\mathbb{R}_0^+$ since $h'(x) < 0$ for all $x > 0$. As h is decreasing,

$$h(x) \leq h(0) = 0$$

Hence, for all $x \geq 0$:

$$\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} \leq 0$$
$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

Thus we have shown that

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

(3) (a) For $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto 3x^5 - 20x^3 + 45x + 1$

The domain of f is $\mathbb{R}$ which contains no boundary points. Moreover, since f is a polynomial function it is differentiable everywhere. Hence,

$$f'(x) = 15x^4 - 60x^2 + 45 = 15(x^2-3)(x-1)(x+1) = 0 \quad \text{if and only if} \quad x \in \{\pm 1, \pm\sqrt{3}\}$$

$$f''(x) = 60x^3 - 120x$$

Then $f''(-\sqrt{3}) = -60\sqrt{3}, \quad f''(-1) = 60, \quad f''(1) = -60, \quad f''(\sqrt{3}) = 60\sqrt{3}$

Thus f has relative minima at $x = -1, \sqrt{3}$ and relative maxima at $x = -\sqrt{3}, 1$.

$f(-\sqrt{3}) = -12\sqrt{3} + 1 \approx -19.78, \quad f(-1) = -27, \quad f(1) = 29,$
$f(\sqrt{3}) = 12\sqrt{3} + 1 \approx 21.78$

(b) For $g : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto t^2 e^{2t}$

The domain of g is $\mathbb{R}$ which contains no boundary points. Additionally, g is differentiable everywhere. Hence,

$$g'(t) = 2te^{2t} + 2t^2e^{2t} = 2te^{2t}(1+t) = 0 \quad \text{if and only if} \quad t \in \{-1, 0\}$$

$$g''(t) = 4te^{2t} + 2e^{2t} + 4t^2e^{2t} + 4te^{2t} = 4t^2e^{2t} + 8te^{2t} + 2e^{2t}$$

Then $g''(-1) = -\frac{2}{e^2}, \quad g''(0) = 2$

Thus g has relative maximum at $x = -1$ and relative minimum at $x = 0$.

$g(-1) = \frac{1}{e^2} \approx 0.135, \quad g(0) = 0$

(c) For $h : (-3, \infty) \longrightarrow \mathbb{R}, \quad x \longmapsto \sqrt{x^3 + 3x^2 + 3}$

We observe that h is differentiable over its domain $(-3, \infty)$. Hence,

$$h'(x) = \left(\tfrac{1}{2}\right)(x^3 + 3x^2 + 3)^{-\frac{1}{2}}(3x^2 + 6x) = \tfrac{3x^2+6x}{2\sqrt{x^3+3x^2+3}} = \tfrac{3x(x+2)}{2\sqrt{x^3+3x^2+3}} = 0$$

$$\text{if and only if} \quad x \in \{-2, 0\}$$

(The only critical points are given by $3x^2(x+2)=0$ since $x^3+3x^2+3>0$)

$h''(x) = \frac{1}{2}\left[\left(-\frac{1}{2}\right)(3x^2+6x)(x^3+3x^2+3)^{-\frac{3}{2}}(3x^2+6x)+(x^3+3x^2+3)^{-\frac{1}{2}}(6x+6)\right]$

$= \frac{6x+6}{2\sqrt{x^3+3x^2+3}} - \frac{(3x^2+6x)^2}{4(x^3+3x^2+3)^{\frac{3}{2}}}$

Then $h''(-2) = -\frac{3\sqrt{7}}{7}$, $\quad h''(0)=\sqrt{3}$

Thus h has relative maximum at $x=-2$ and relative minimum at $x=0$.

$h(-2)=\sqrt{7}$, $\quad h(0)=\sqrt{3}$

(4) For $n \in \mathbb{N}$ and $x \geq 0$, let $P(n)$ be the proposition:

$$\ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... - \frac{x^{2n}}{2n}$$

Equivalently,

$$\ln(1+x) \geq \sum_{n=1}^{\infty}\left[\frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n}\right]$$

$\mathbf{n=1}$:

$$\frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} = \frac{x^{2-1}}{2-1} - \frac{x^2}{2} = x - \frac{x^2}{2}$$

We showed in Question 2(b) that $\ln(1+x) \geq x - \frac{x^2}{2}$. Therefore $P(1)$ is true.
$\mathbf{n \geq 2}$: We make the inductive hypothesis that $P(n)$ is true for $n=k$.

$$\ln(1+x) \geq \sum_{k=1}^{\infty}\left[\frac{x^{2k-1}}{2k-1} - \frac{x^{2k}}{2k}\right]$$

We now prove $P(k+1)$:

$$\text{Let } \; j(x) = \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - ... + \frac{x^{2k}}{2k} - \frac{x^{2k+1}}{2k+1}$$

Then j is differentiable over its domain $(-1,\infty)$ and

$$\begin{aligned} j'(x) &= \frac{1}{1+x} - 1 + x - x^2 + x^3 - ... + x^{2k-1} - x^{2k} \\ &= -\frac{x}{1+x} + \sum_{k=1}^{\infty}\left[x^{2k-1} - x^{2k}\right] = -\frac{x}{1+x} + \frac{1}{x^2+x} = \frac{1-x}{x} \end{aligned}$$

It follows that

$$j'(x) = \begin{cases} < 0 & \text{for} \quad x > 1 \\ = 0 & \text{for} \quad x = 1 \\ > 0 & \text{for} \quad 0 < x < 1 \end{cases}$$

Thus j is monotonically increasing for $0 < x < 1$ and as such,

$$j(x) \geq j(0) = 0$$

Hence, for $0 < x < 1$:

$$\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - ... + \frac{x^{2k}}{2k} - \frac{x^{2k+1}}{2k+1} \geq 0$$

$$\ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... - \frac{x^{2n}}{2n} + \frac{x^{2k+1}}{2k+1}$$

Therefore $P(k+1)$ is true whenever $P(k)$ is true. So by the principle of mathematical induction, for all $n \in \mathbb{N}$:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... - \frac{x^{2n}}{2n} \leq \ln(1+x)$$

We then let $S(n)$ be the statement:

$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{x^{2n-1}}{2n-1}$$

Equivalently,

$$\ln(1+x) \leq x - \sum_{n=1}^{\infty} \left[\frac{x^{2n}}{2n} - \frac{x^{2n+1}}{2n+1} \right]$$

$\mathbf{n = 1}$:

$$x - \frac{x^{2n}}{2n} + \frac{x^{2n+1}}{2n+1} = x - \frac{x^2}{2} + \frac{x^3}{3}$$

We showed in Question 2(b) that $\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$. Therefore $S(1)$ is true.
$\mathbf{n \geq 2}$: We make the inductive hypothesis that $S(n)$ is true for $n = k$.

$$\ln(1+x) \leq x - \sum_{k=1}^{\infty} \left[\frac{x^{2k}}{2k} - \frac{x^{2k+1}}{2k+1} \right]$$

We now prove $S(k+1)$:

$$\text{Let} \ \ l(x) = \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - ... - \frac{x^{2k-1}}{2k-1} + \frac{x^{2k}}{2k}$$

Then l is differentiable over its domain $(-1, \infty)$ and

$$l'(x) = \frac{1}{1+x} - 1 + x - x^2 + x^3 - ... + x^{2k-2} - x^{2k-1}$$
$$= -\frac{x}{1+x} + \sum_{k=1}^{\infty} \left[x^{2k-2} - x^{2k-1} \right] = -\frac{x}{1+x} + \frac{1}{x^3+x^2} = \frac{1-x^3}{x^3+x^2}$$

It follows that

$$l'(x) = \begin{cases} < 0 & \text{for} \quad x > 1 \\ = 0 & \text{for} \quad x = 1 \\ > 0 & \text{for} \quad 0 < x < 1 \end{cases}$$

Thus l is monotonically decreasing for $x > 1$ and as such,

$$l(x) \leq l(0) = 0$$

Hence, for $x > 1$:

$$\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - ... - \frac{x^{2k-1}}{2k-1} + \frac{x^{2k}}{2k} \leq 0$$

$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{x^{2k-1}}{2k-1} - \frac{x^{2k}}{2k}$$

Therefore $S(k+1)$ is true whenever $S(k)$ is true. So by the principle of mathematical induction, for all $n \in \mathbb{N}$:

$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{x^{2n-1}}{2n-1}$$

Finally, we have shown that

$$x - \frac{x^2}{2} + \frac{x^3}{3} - ... - \frac{x^{2n}}{2n} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} - ... + \frac{x^{2n-1}}{2n-1}$$