THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101/101A

PAPER TITLE: Algebra & Differential Calculus

PAPER NUMBER: First and Only

DATE: **TIME:**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: SEVEN (7)

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE: $\boxed{1}$ X 6 LEAF A4 BOOKS $\boxed{1}$ X 12 LEAF A4 BOOKS

$\boxed{0}$ X ROUGH WORK BOOKS

GRAPH PAPER **NIL** (NUMBER OF SHEETS)

POCKET CALCULATORS PERMITTED YES (SILENT TYPE)

OTHER AIDS REQUIRED: NIL

INSTRUCTIONS FOR CANDIDATES:

- Questions are of equal value
- Candidates may make notes on this paper during the 15 minutes reading time.
- Candidates may retain their copy of this examination paper
- Answer ALL questions in SECTION A and answer only TWO questions in SECTION B.

TEXTBOOKS OR NOTES PERMITTED: FIVE (5) A4 SHEETS (10 PAGES IF WRITTEN ON BOTH SIDES OF HANDWRITTEN NOTES. NO PHOTOCOPIES, NO PRINTED NOTES PERMITTED.

SECTION A

You should attempt all questions in this section.

Question 1

(a) Decide whether each of the following subsets of $\mathbb{R}$ has an infimum and/or a supremum. If so, decide whether the infimum (respectively supremum) is an element of the set. Justify your answer.

(i) $\{x \in \mathbb{N}, | \, x^3 \geq \sqrt{5}\,\}$ (ii) $\{x \in \mathbb{R}, | \, \sqrt{16 - x^4} > 0\}$

[4 *marks*]

(b) Prove that if n is a non-zero natural number, then

$$5^n > n^2.$$

[4 *marks*]

(c) Prove, by contradiction, that for every real number, a,

$$a^2 - 3a > -3.$$

[2 *marks*]

Question 2

(a) For $z = 2 + 3i$, write each of the following complex numbers in the form $x + iy$.

(i) $4i - z$ (ii) $\dfrac{z - 2i}{z + 1 + i}$ (iii) $\overline{3z + 2}$

(iv) $\left|\dfrac{z + 2 - i}{1 + z + i}\right|$ (v) $\dfrac{1}{\overline{z}^2}$

[6 *marks*]

(b) Find all complex numbers, z, which satisfy

$$z^4 = i.$$

[4 *marks*]

[Question 3 is on page 3.]

Question 3

(a) For each of the following formulæ, find the maximal subset, X, of $\mathbb{R}$ on which

$$f : X \longrightarrow \mathbb{R}, \quad x \longmapsto f(x)$$

defines a function (X is then the "natural domain" of f) and determine $\operatorname{im}(f)$, the range of f.

(i) $f(x) = \dfrac{x}{\sqrt{1-x^2}}$ (ii) $f(x) = \dfrac{x}{x^2+1}$

[4 *marks*]

(b) Take X as in Part (a) (ii). Decide whether

$$f : X \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x}{x^2+1}$$

is injective, surjective or bijective. Find, if possible, $\inf f$ and $\sup f$. Justify your answer.

[4 *marks*]

(c) Find a real number, k, which renders continuous the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} \dfrac{\sin x^2}{x} & x < 0 \\ x^2 + k & x \geq 0 \end{cases}$$

[2 *marks*]

Question 4

(a) Determine which of the following sequences of real numbers, $(u_n)_{n \in \mathbb{N}}$, is monotone, and discuss the behaviour of u_n as $n \to \infty$.

(i) $u_n := r^{n+1} \quad (r \in \mathbb{R})$ (ii) $u_n := \dfrac{n+1}{n^2+1}$

[4 *marks*]

(b) Determine which of the following series converge, and which diverge, justifying your answer.

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{n!}$ (ii) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{\sqrt{n^2+1}}$ (iii) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^4+1}$

[6 *marks*]

[Question 5 is on page 4.]

Question 5

(a) Find all points at which each of the following functions is differentiable, as well as its derivative. Where the function fails to be differentiable, explain why.

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sqrt{1 + \cos x}$$
$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{-x^2}$$

[5 *marks*]

(b) Determine $\dfrac{dy}{dx}$ for a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto y$ satisfying

$$xe^{-(x^2+y^2)} + x^2 + y^2 = 7.$$

[5 *marks*]

Question 6

Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^4 - 4x^3 + 4x^2 + 1.$$

(a) Determine the intervals on which f is (i) increasing or decreasing (ii) concave up or concave down. [4 *marks*]

(b) Find all the relative maxima and relative minima of f, and the absolute maximum and absolute minimum on $[-1, 3]$. [3 *marks*]

(c) Sketch the graph of f on the interval $[-1, 3]$. [3 *marks*]

Question 7

(a) Find all real numbers x, y, z, such that

$$\begin{array}{ccccccc} 4x & + & 3y & + & z & = & 19 \\ 2x & - & y & + & 2z & = & 6 \\ x & + & y & + & z & = & 6 \end{array}$$

[4 *marks*]

[Question 7(b) is on page 5.]

(b) Given the matrices

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad C := \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

calculate, where possible, the following, justifying your answer.

(i) AB (ii) $2A + C$ (iii) BA

[4 *marks*]

(c) Evaluate the determinant

$$\begin{vmatrix} 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}.$$

[2 *marks*]

Question 8 Let P, Q, R, S be the four points in $\mathbb{R}^3$ with co-ordinates $(-1, 0, 0)$, $(1, 0, 0)$, $(1, 1, 1)$ and $(2, 1, 2)$ respectively.

(a) Write $\vec{PQ}$ and $\vec{PR}$ in terms of the standard unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. [2 *marks*]

(b) Find the orthogonal projection of $\vec{PQ}$ onto $\vec{PR}$. [3 *marks*]

(c) Find the area of the triangle with vertices P, Q and R. [3 *marks*]

(d) Find the volume of the parallelepiped with sides given by the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$. [2 *marks*]

[SECTION B is on page 6.]

SECTION B

You should attempt only TWO questions in this section.

Question 9

(a) Prove from first principles that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^2$$

is continuous. [4 *marks*]

(b) Take functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$. Suppose that for some $K > 0$ and all all $u, v \in \mathbb{R}$,

$$|f(u) - f(v)| \leq K|g(u) - g(v)|$$

Prove that if g is continuous at $a \in \mathbb{R}$, so is f. [6 *marks*]

Question 10

(a) Show that

$$f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(x+1)$$

satisfies the hypotheses of the Mean Value Theorem on every interval $[0, c]$, with $c > 0$. Use the Mean Value Theorem to show that for all $x > 0$,

$$\frac{x}{1+x} < \ln(1+x).$$

[Recall that $\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$.] [6 *marks*]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Show that between any two zeroes of f, its derivative f' must also have a zero. [4 *marks*]

Question 11

A piece of wire of length L units is used to construct the outline of two plane figures, one an equilateral triangle, the other a square. How should the wire be cut in order to make the total area enclosed (i) a maximum, (ii) a minimum. [10 *marks*]

[Question 12 is on page 7.]

Question 12

(a) Prove that for any vectors, $\mathbf{u}$ and $\mathbf{v}$, in $\mathbb{R}^3$,

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v})$$

or, equivalently,

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

[4 *marks*]

(b) Let A, B be distinct points in $\mathbb{R}^3$. Show that the distance, d, of the point, $P \in \mathbb{R}^3$ from the line through A and B is given by

$$d = \frac{|\vec{PA} \times \vec{PB}|}{|\vec{AB}|}.$$

[6 *marks*]