Chapter 13

Inner Product Spaces

The discussion and the theory developed so far have applied to vector spaces over any field. The only restriction made has been to vector spaces which are finitely generated. Even then, many of our results did not actually depend on this hypothesis.

On the other hand, we have frequently appealed to geometry to provide motivation, illustration or graphical representation of concepts and theorems, which meant restricting attention to $\mathbb{R}^n$ ($n \in \mathbb{N}$). This is hardly surprising, since $\mathbb{R}^n$ is not only the most familiar vector space, but also the locus of analytic geometry since Descartes.

We now turn our attention to formulating such informal discussion and heuristic arguments more rigorously. Specifically, we investigate the additional structure a vector space must support in order for us to be able to "do geometry", that is, to speak of *distances* and *angles*. [Recall that we have already discussed what we mean by a "line", a "plane", and so on, in any vector space.]

Surprisingly, measuring angles also provides a way of measuring distance. However, the converse is not true. We do not enter a discussion here of why, for this and related questions are discussed in detail in courses on functional analysis, which may be fruitfully thought of as the study of infinite dimensional real and complex vector spaces, requiring the additional ingredient of concepts from topology.

Our approach is to first briefly discuss making sense of the "length" of a vector, show how this permits us to define a notion of distance and to define continuity of functions between vector spaces. It follows that all linear transformations are continuous.

We then introduce the additional structure required to make sense of the notion of an "angle" between vectors and show how this allows us to speak of length, hence distance and hence continuity.

Our intuition is based on our experience with Euclidean space, which is a real vector space. The discussion actually applies to vector spaces over any sub-field of the field of complex numbers, although not for finite fields, or fields constructed from finite fields[1]. Even more, the proofs of the central results are simplest when we work over the complex numbers, and the more familiar cases are easy applications.

13.1 Normed Vector Spaces

We begin by examining the notion of *length*, or *magnitude*, of a vector.

(i) It should be clear that length should be a non-negative real number, which is 0 for, and only

[1]The reasons for this are beyond the scope of this course.

for, the zero vector.

(ii) If we scale a vector, its length is multiplied by the magnitude of the scaling factor.

(iii) The length of the sum of two vectors cannot exceed the sum of the lengths of the two vectors.

We mention here, without further explanation, that it is essentially the second condition which forces us to restrict ourselves to vector spaces over sub-fields of $\mathbb{C}$. So, unless otherwise specified, $\mathbb{F}$ henceforth denotes a sub-field of $\mathbb{C}$. This means, in particular, that $\mathbb{F}$ contains $\mathbb{Q}$, the field of rational numbers.

We now express the properties above formally, turning them into a definition.

Definition 13.1. A *norm* on the vector space V over the sub-field $\mathbb{F}$ of $\mathbb{C}$ is a function $\|\ \| : V \to \mathbb{R}_0^+$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$

N1 $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}_V$

N2 $\|\lambda \mathbf{u}\| = |\lambda|\, \|\mathbf{u}\|$

N3 $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

A *normed vector space* is a vector space, V, over the field $\mathbb{F}$ ($\subseteq \mathbb{C}$), equipped with a norm, $\|\ \|$. It is denoted by $(V, \|\ \|)$, or simply by V when the norm is understood.

The vector, $\mathbf{v}$, in the normed vector space $(V, \|\ \|)$ is *normal* or *normalised*, or a *unit vector* if and only if $\|\mathbf{v}\| = 1$.

Example 13.2. The *absolute value* or *modulus* of a complex number defines a norm on any sub-field $\mathbb{F}$ of $\mathbb{C}$. The verification is left as an exercise.

Example 13.3. Let $\mathbb{F}$ be a sub-field of $\mathbb{C}$. Then

$$\|\ \|_{\mathbb{F}^n} : \mathbb{F}^n \longrightarrow \mathbb{R}_0^+, \quad (x_1, \dots, x_n) \longmapsto \big(\sum_{j=1}^{n} |x_j|^2\big)^{\frac{1}{2}}$$

defines a norm on $\mathbb{F}^n$, called the *Euclidean norm* on $\mathbb{F}$. The verification is left as an exercise.

Example 13.2 is just the case $n = 1$, and when $\mathbb{F} \subseteq \mathbb{R}$, we may replace $|x_j|^2$ by x_j^2.

Example 13.4. Recall that $G\colon \mathbb{R}^2 \to \mathbb{C},\ \ (x, y) \mapsto z := x + iy$ is an isomorphism of real vector spaces, and that

$$\|\ \|_{\mathbb{R}^2} = \sqrt{x^2 + y^2} = |x + iy| = \|G(x, y)\|_{\mathbb{C}^1}$$

Similarly,

$$G_n\colon \mathbb{R}^{2n} \longrightarrow \mathbb{C}^n, \quad (x_1, \dots, x_{2n}) \longmapsto (x_1 + ix_2, \dots, x_{2n-1} + ix_{2n}),$$

is an isomorphism of real vector spaces, with

$$\|(x_1, \dots, x_{2n})\|_{\mathbb{R}^{2n}} = \left(\sum_{k=1}^{2n} |x_k|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} = \|G_n(x_1, \dots, x_n)\|_{\mathbb{C}^n}.$$

Example 13.5. Let $V = \{f : [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$ denote the real vector space of all continuous real valued functions defined on the closed unit interval.

$$\|\ \|_1 : V \to \mathbb{R}_0^+, \quad f \mapsto \int_0^1 |f(t)|\, dt$$

defines a norm on V, called the $\mathcal{L}^1$ *norm* on V.

Observation 13.6. If $(V, \| \quad \|)$ is a non-trivial normed vector space then every $\mathbf{v} \in V$ in is a multiple of a unit vector. For if $\mathbf{v} \neq \mathbf{0}_V$, define $\mathbf{v}_u := \frac{\mathbf{v}}{\|\mathbf{v}\|}$. Then $\mathbf{v} = \|\mathbf{v}\|\mathbf{v}_u$. If, on the other hand, $\mathbf{v} = \mathbf{0}_V$, take any $\mathbf{w} \neq \mathbf{0}_V$ and define $\mathbf{u} := \frac{\mathbf{w}}{\|\mathbf{w}\|}$.

In either case, $\mathbf{v} = \mathbf{0}_V = 0\mathbf{v}_u = \|\mathbf{v}\|\mathbf{v}_u$.

The norm on a vector space can be used to measure of distance between any two elements of V. We first characterise what we mean by the *distance* between two points in a set.

(i) The distance between two points should be a non-negative real number, which is 0 if and only if the two points coincide.

(ii) The distance from one point to another is the same as the distance from the second to the first.

(iii) The distance between two points cannot exceed the sum of the distances of the first to any point plus the distance from that point to the second.

We mention here, without further explanation, that it these properties do not require any structure beyond being a set — in particular, there is no need to consider vector spaces. The study of sets equipped with a notion of distance between its points is the *theory of metric* spaces, a part of the study of topology.

We now express the properties above formally, turning them into a definition.

Definition 13.7. A *metric* (or *distance function*) on the set X is a function

$$d : X \times X \longrightarrow \mathbb{R}_0^+$$

such that for all $x, y, z \in X$

MS1 $d(x, y)$ if and only if $x = y$

MS2 $d(y, x) = d(x, y)$

MS3 $d(x, z) \leq d(x, y) + d(y, z)$.

A *metric space* comprises a set, X, equipped with a metric, d. We denote it by (X, d), writing only X when the metric is understood.

We now show that every normed vector space is a metric space in a natural way.

Definition 13.8. For the normed vector space, $(V, \| \quad \|)$, over the field $\mathbb{F}$, define

$$d_{\| \quad \|} : V \times V \longrightarrow \mathbb{R}_0^+, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \|\mathbf{u} - \mathbf{v}\|$$

Lemma 13.9. *Let $(V, \| \quad \|)$ be a normed vector space over the field $\mathbb{F}$. Then $(V, d_{\| \quad \|})$ is a metric space.*

Proof. Take $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Since $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in V$, $d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\| \geq 0$ for all $\mathbf{u}, \mathbf{v} \in V$, showing that $d_{\| \ \|}$ is well defined. Moreover,

$$\begin{aligned}
d(\mathbf{u},\mathbf{v}) = 0 \ \ &\text{if andf only if } \ \|\mathbf{u}-\mathbf{v}\| \\
&\text{if andf only if } \ \mathbf{u}-\mathbf{v} = \mathbf{0}_V && \text{by } \mathbf{N1} \\
&\text{if andf only if } \ \mathbf{u} = \mathbf{v}, && \text{verifying } \mathbf{MS1}. \\
d(\mathbf{v},\mathbf{u}) &:= \|\mathbf{v}-\mathbf{u}\| \\
&= |-1|\|\mathbf{u}-\mathbf{v}\| && \text{by } \mathbf{N2} \\
&= \|\mathbf{u}-\mathbf{v}\| \\
&=: d(\mathbf{u},\mathbf{v}), && \text{verifying } \mathbf{MS2}. \\
d(\mathbf{u},\mathbf{w}) &:= \|\mathbf{u}-\mathbf{w}\| \\
&= \|\mathbf{u}-\mathbf{v}+\mathbf{v}-\mathbf{w}\| \\
&\leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\| && \text{by } \mathbf{N3} \\
&=: d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}), && \text{verifying } \mathbf{MS3}.
\end{aligned}$$

□

Definition 13.10. If $(V, \| \ \|)$ is a normed vector space over the field $\mathbb{F}$, then $d_{\| \ \|}$ is the metric on V *induced by the norm* $\| \ \|$.

In particular, the Euclidean distance between points in $\mathbb{R}^n$ is the metric induced by the Euclidean norm. This allows us to reformulate the definition of continuity met in univariate and multivariate calculus in terms of metrics and so extend the notion of continuity to more general spaces.

Definition 13.11. Given metric spaces (X, d) and (Y, e), the function $f : X \to Y$ is *continuous* at $a \in X$ if and only if given any $\varepsilon > 0$ there is a $\delta > 0$ such that $e(f(x), f(a)) < \varepsilon$ whenever $d(x, a) < \delta$.

We do not pursue these ideas further here, but return to our main interest, the quest for the structure required to be able to sense of "angle" between two vectors.

13.2 Inner Products

Recall that if we take two points P and Q in the Cartesian plane, neither of which is the origin, O, with co-ordinates (x, y) and (u, v) respectively, then we can compute the cosine of the angle $\angle POQ$ directly from the co-ordinates. Suppose that the angle in question is θ. We express (x, y) in polar co-ordinates,

$$x = r\cos\alpha \quad \text{and} \quad y = r\sin\alpha$$

for uniquely determined $r > 0$ and $0 \leq \alpha < 2\pi$, so that, $r = \sqrt{x^2+y^2}$. Then, without loss of generality,

$$u = s\cos(\alpha+\theta) \quad \text{and} \quad v = s\sin(\alpha+\theta),$$

so that $s = \sqrt{u^2+v^2}$. Since $\theta = \alpha + \theta - \alpha$, it follows that

$$\begin{aligned}
\cos\theta &= \cos(\alpha+\theta)\cos(\alpha) + \sin(\alpha+\theta)\sin(\alpha) \\
&= \frac{u}{s}\frac{x}{r} + \frac{v}{s}\frac{y}{r} \\
&= \frac{ux+vy}{\sqrt{u^2+v^2}\sqrt{x^2+y^2}}.
\end{aligned}$$

Define

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad \big((u,v),(x,y)\big) \longmapsto ux + vy.$$

Then

$$\cos\theta = \frac{\langle\!\langle (u,v),(x,y)\rangle\!\rangle}{\sqrt{\langle\!\langle (u,v),(u,v)\rangle\!\rangle}\sqrt{\langle\!\langle (x,y),(x,y)\rangle\!\rangle}}. \tag{$*$}$$

and we can define the angle between (u,v) and (x,y) to be the unique angle $\theta \in [0,\pi]$ satisfying

$$\sqrt{\langle\!\langle (u,v),(u,v)\rangle\!\rangle}\sqrt{\langle\!\langle (x,y),(x,y)\rangle\!\rangle}\cos\theta = \langle\!\langle (u,v),(x,y)\rangle\!\rangle.$$

Since we can define the angle purely in terms of the function $\langle\!\langle\ ,\ \rangle\!\rangle$, its characteristic properties provide a basis for the definition of a general notion allowing the use of Equation $(*)$ to define an angle between two vectors in a real vector space. We first characterise $\langle\!\langle\ ,\ \rangle\!\rangle$.

Lemma 13.12. *Take $(x,y),(u,v),(r,s) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then*

(i) $\langle\!\langle (x,y),(x,y)\rangle\!\rangle \geq 0$ *with equality if and only if* $(x,y) = (0,0)$.

(ii) $\langle\!\langle (x,y),(u,v)\rangle\!\rangle = \langle\!\langle (u,v),(x,y)\rangle\!\rangle$

(iii) $\langle\!\langle \alpha(x,y),(u,v)\rangle\!\rangle = \alpha\langle\!\langle (x,y),(u,v)\rangle\!\rangle$

(iv) $\langle\!\langle (r,s)+(x,y),(u,v)\rangle\!\rangle = \langle\!\langle (r,s),(u,v)\rangle\!\rangle + \langle\!\langle (x,y),(u,v)\rangle\!\rangle$

Proof. The verifications are routine and left as an exercise. □

The function $\langle\!\langle\ ,\ \rangle\!\rangle$ just introduced leads naturally to

$$\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle} : \mathbb{R}^2 \longrightarrow \mathbb{R}^+_O, \quad (x,y) \longmapsto \sqrt{\langle\!\langle (x,y),(x,y)\rangle\!\rangle}.$$

Lemma 13.13. $\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle}$ *is a norm on* $\mathbb{R}^2$

Proof. It is routine to verify that $\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle}$ is well defined and that **N1** and **N2** hold. On the other hand, the verification of **N3** is not quite as trivial.

We leave these as an exercise, since we prove a more general version a little later. □

We extend the above discussion to complex vector spaces, using the results of Lemma 13.12 and Lemma 13.13 as a guide. One obvious generalisation, namely,

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \quad \big((u,v),(x,y)\big) \longmapsto ux + vy$$

will not do, for neither does Lemma 13.12 (i) hold, nor do we obtain a norm, since, by this definition, $\langle\!\langle (i,1),(i,1)\rangle\!\rangle = 0$.

If, on the other hand, we define

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \quad \big((u,v),(x,y)\big) \longmapsto u\overline{x} + v\overline{y},$$

then all our desired results hold, except for Lemma 13.12 (ii), which must be replaced by

(ii)′ $\langle\!\langle (x,y),(u,v)\rangle\!\rangle = \overline{\langle\!\langle (u,v),(x,y)\rangle\!\rangle}$

We take this as the model for our definition, and the characteristic properties serve as axioms.

Definition 13.14. Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. An *inner product* on the $\mathbb{F}$-vector space, V, is a function

$$\langle\!\langle\ ,\ \rangle\!\rangle : V \times V \longrightarrow \mathbb{F}$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{F}$

IP1 $\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle \geq 0$[2], with equality when, and only when, $\mathbf{u} = \mathbf{0}_V$;

IP2 $\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle = \overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}$;

IP3 $\langle\!\langle \lambda\mathbf{u}, \mathbf{v} \rangle\!\rangle = \lambda\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle$;

IP4 $\langle\!\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle\!\rangle = \langle\!\langle \mathbf{u}, \mathbf{w} \rangle\!\rangle + \langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle$.

Observation 13.15. When $\mathbb{F}$ is a subfield of $\mathbb{R}$, condition **IP2** reduces to Lemma 13.12 (ii).

Example 13.16. Take $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$ $(n \in \mathbb{N} \setminus \{0\})$ Then

$$\langle\!\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle\!\rangle := \sum_{j=1}^{n} w_j \overline{z_j}$$

defines an inner product, called the *Euclidean inner product*. It (and its restriction to $\mathbb{F}^n$ for a subfield, $\mathbb{F}$, of $\mathbb{C}$) is frequently also referred to as the *standard inner product* on $\mathbb{C}^n$ or $\mathbb{F}^n$.

Example 13.17. Take $\mathbb{F} = \mathbb{C}$ and $V := \{f : [0,1] \to \mathbb{C} \mid f \text{ is continuous}\}$ Then

$$\langle\!\langle f, g \rangle\!\rangle := \int_0^1 f(t)\overline{g(t)}dt$$

defines an inner product on V. This V is usually denoted $\mathcal{L}^2([0,1])$. It is a focus of study in functional analysis as well as in measure and integration theory. You will also meet it and related spaces in statistics, the theory of differential equations and theoretical physics.

The verification that the inner product axioms hold requires a little of the theory of complex variables, namely, the fact that we may write $f(t)$ as $x(t) + iy(t)$ (where $i^2 = -1$) and then

$$\int_0^1 f(t)dt := \int_0^1 x(t)dt + i\int_0^1 y(t)dt.$$

The crucial properties of inner product spaces which enables the definition of the angle between two vectors is the Cauchy-Schwarz Inequality, which we establish next.

Theorem 13.18 (Cauchy-Schwarz Inequality). *Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space V over the subfield $\mathbb{F}$ of $\mathbb{C}$. Then for all $\mathbf{u}, \mathbf{v} \in V$*

$$|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle| \leq \sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}.$$

Proof. Take $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned} 0 &\leq \langle\!\langle \alpha\mathbf{u} - \beta\mathbf{v}, \alpha\mathbf{u} - \beta\mathbf{v} \rangle\!\rangle \\ &= \alpha\overline{\alpha}\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle - \alpha\overline{\beta}\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle - \beta\overline{\alpha}\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle + \beta\overline{\beta}\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \end{aligned}$$

Put $\alpha := \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle$ and $\beta := \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle$. Then, since $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \in \mathbb{R}$ and $z\overline{z} = |z|^2$,

$$\begin{aligned} 0 &\leq \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle - \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle - \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \\ &= \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \left(\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle - |\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle|^2\right) \end{aligned}$$

Now $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}_V$, in which case $|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle| = 0 = \sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}$.
Otherwise, $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle > 0$, and so, $|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle|^2 \leq \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle$. □

[2]Here we use the convention that when we write $z \geq 0$ for $z \in \mathbb{C}$, we assert that z is, in fact, a real number.

It is this last result which allows us to define the *angle* between two vectors in an inner product space. We do this only for vector spaces when the scalars are real numbers in order to avoid questions about the meaning of "complex angles".

Definition 13.19. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space V over the subfield $\mathbb{F}$ of $\mathbb{R}$. Given $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}_V\}$, the *angle between* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $\angle\mathbf{uv}$, is the unique real number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}{\sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}}$$

We show that each inner product space is a normed vector space in a natural way.

Definition 13.20. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space, V, over the subfield, $\mathbb{F}$, of $\mathbb{C}$. Define

$$\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle} : V \longrightarrow \mathbb{F}, \quad \mathbf{v} \longmapsto \sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}$$

Theorem 13.21. *If* $\langle\!\langle\ ,\ \rangle\!\rangle$ *is an inner product on the vector space,* V*, over the field* $\mathbb{F}$*, then* $\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle}$ *is a norm on* V*.*

Proof. Take $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$.

$$\begin{aligned}
\|\mathbf{u}\|_{\langle\!\langle\ ,\ \rangle\!\rangle} = 0 \ &\text{ if and only if } \ \|\mathbf{u}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} = 0 \\
&\text{ if and only if } \ \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle = 0 \\
&\text{ if and only if } \ \mathbf{u} = \mathbf{0}_V, && \text{by } \mathbf{IP1}\text{, verifying } \mathbf{N1}. \\
\|\lambda\mathbf{u}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} &= \langle\!\langle \lambda\mathbf{u}, \lambda\mathbf{u} \rangle\!\rangle \\
&= \lambda\overline{\lambda}\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle && \text{by } \mathbf{IP2} \text{ and } \mathbf{IP3} \\
&= |\lambda|^2\|\mathbf{u}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle}, && \text{verifying } \mathbf{N2}. \\
\|\mathbf{u}+\mathbf{v}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} &:= \langle\!\langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle\!\rangle \\
&= \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle + \overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle} + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle && \text{by } \mathbf{IP2} \text{ and } \mathbf{IP4} \\
&= \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + 2\Re(\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle) + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \\
&\leq \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + 2|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle| + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle && \text{as } \Re(z) \leq |z| \\
&\leq \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + 2\sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle} + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle && \text{by Theorem 13.18} \\
&= \left(\sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle} + \sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}\right)^2 \\
&=: \left(\|\mathbf{u}\|_{\langle\!\langle\ ,\ \rangle\!\rangle} + \|\mathbf{v}\|_{\langle\!\langle\ ,\ \rangle\!\rangle}\right)^2, && \text{verifying } \mathbf{N3}.
\end{aligned}$$

□

13.3 Exercises

Exercise 13.1. Let $\mathbb{F}$ be a sub-field of $\mathbb{C}$. Verify that the function

$$\|\ \| : \mathbb{F} \longrightarrow \mathbb{R}^+_0, \quad z \longmapsto |z|,$$

where $|z|$ denotes the modulus of the complex number z, defines a norm on $\mathbb{F}$.

Exercise 13.2. Let $\mathbb{F}$ be a sub-field of $\mathbb{C}$. Take $\mathbb{F}^n$ with its standard vector space structure over $\mathbb{F}$. Verify that the following function defines a norm on $\mathbb{F}^n$.

$$\|\ \| : \mathbb{F}^n \longrightarrow \mathbb{R}^+_0, \quad (x_1, \ldots, x_n) \longmapsto \left(\sum_{j=1}^{n} |x|^2_j\right)^{\frac{1}{2}},$$

Exercise 13.3. Verify that multiplication of real numbers defines an inner product on $\mathbb{R}$.

Exercise 13.4. (a) $\mathbf{M}(m \times n; \mathbb{R})$ is a real vector space with respect to matrix addition and multiplication of a matrix by a constant. Show that

$$\langle\!\langle\ ,\ \rangle\!\rangle_M : \mathbf{M}(m \times n; \mathbb{R}) \times \mathbf{M}(m \times n; \mathbb{R}) \longrightarrow \mathbb{R}, \quad (\underline{\mathbf{A}}, \underline{\mathbf{B}}) \longmapsto \operatorname{tr}(\underline{\mathbf{A}}^t \underline{\mathbf{B}})$$

defines a real inner product on $\mathbf{M}(m \times n; \mathbb{R})$.

(b) Show that

$$\varphi : \mathbf{M}(m \times n; \mathbb{R}) \longrightarrow \mathbb{R}^{mn}, \quad [a_{ij}]_{m \times n} \longmapsto (x_{11}, \dots, x_{mn}),$$

where $x_{(i-1)n+j} := a_{(i-1)n\,j}$ defines an isomorphism of real vector spaces.

(c) Show that for all $\underline{\mathbf{A}}, \underline{\mathbf{B}} \in \mathbf{M}(m \times n; \mathbb{R})$,

$$\langle\!\langle \underline{\mathbf{A}}, \underline{\mathbf{B}} \rangle\!\rangle_M = \langle\!\langle \varphi(\underline{\mathbf{A}}), \varphi(\underline{\mathbf{B}}) \rangle\!\rangle_{\mathbb{R}^{mn}},$$

where $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{R}^{mn}}$ is the Euclidean inner product on $\mathbb{R}^{mn}$. [Such an isomorphism, φ, is called a *linear isometry*.]

Exercise 13.5. Prove Lemma 13.12.

Exercise 13.6. Prove Lemma 13.13.