Sample Solutions for Tutorial 7

Question 1.

Observe that

$$\begin{array}{ccccccc} a_{11}x_1 & + \cdots + & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

if and only if

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

if and only if

$$x_1(a_{11}, \ldots, a_{m1}) + x_2(a_{12}, \ldots, a_{m2}) + \cdots + x_n(a_{1n}, \ldots, a_{mn}) = (b_1, \ldots, b_m)$$

if and only if $(b_1, \ldots, b_m)$ is in the sub-space of $\mathbb{F}^m$ generated by $\{(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn})\}$.

The solution is unique if and only if $(a_{11}, \ldots, a_{m1}), (a_{12}, \ldots, a_{m2}), \ldots, (a_{1n}, \ldots, a_{mn})$ are linearly independent, that is, if and only if they form a basis of the subspace they generate.

Question 2.

Let $T : V \to W$ have matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for V and $\{\mathbf{f}_1, \mathbf{f}_2\}$ for W.

Then $T(x\mathbf{e}_1 + y\mathbf{e}_2) = x\mathbf{f}_1$, whence

(i) T is not injective, because $\mathbf{e}_2 \in \ker(T)$ and

(ii) T is not surjective, because $\mathbf{f}_2 \notin \operatorname{im}(T)$.

Question 3.

Consider the equations

(i) $$x + 3y = a$$

(ii) $$2x + 4y = b$$

We apply "elementary row operations" to obtain equivalent pairs of equations. First, subtracting twice (i) from (ii), we obtain

(iii) $$x + 3y = a$$

(iv) $$-2y = b - 2a$$

Now adding three halves of (iv) to (iii), and dividing (iv) by -2, we obtain

(v) $$x = -2a + \frac{3}{2}b$$

(vi) $$y = a - \frac{1}{2}b$$

It now follows immediately from (v) and (vi) that

$$(1, 0) = -2(1, 2) + (3, 4) = \frac{3}{2}(2, 1) - \frac{1}{2}(4, 3)$$

and

$$(0, 1) = \frac{3}{2}(1, 2) - \frac{1}{2}(3, 4) = -2(2, 1) + (4, 3).$$

Thus

$$\{(1,0),(0,1)\} \subseteq \langle \mathcal{B} \rangle,$$

and

$$\{(1,0),(0,1)\} \subseteq \langle \mathcal{B}' \rangle,$$

whence

$$\mathbb{R}^2 = \langle (1,0),(0,1) \rangle \leq \langle \mathcal{B} \rangle \leq \mathbb{R}^2$$

and

$$\mathbb{R}^2 = \langle (1,0),(0,1) \rangle \leq \langle \mathcal{B}' \rangle \leq \mathbb{R}^2$$

Hence both $\mathcal{B}$ and $\mathcal{B}'$ generate $\mathbb{R}^2$.

Now $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$, and both $\mathcal{B}$ and $\mathcal{B}'$ comprise two elements. Hence each must be a basis for $\mathbb{R}^2$.

By the calculations above, the co-ordinates with respect to $\mathcal{B}$ of $u(2,1)+v(4,3) = (2u+4v, u+3v)$ are

$$x = -2(2u+4v) + \frac{3}{2}(u+3v) = -\frac{5}{2}u - \frac{7}{2}v$$

and

$$y = (2u+4v) - \frac{1}{2}(u+3v) = \frac{3}{2}u + \frac{5}{2}v$$

so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 & -7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

Since the inverse of $\frac{1}{2}\begin{bmatrix} -5 & -7 \\ 3 & 5 \end{bmatrix}$ is $\frac{1}{4}\begin{bmatrix} -5 & -7 \\ 3 & 5 \end{bmatrix}$, the matrix of T with respect to $\mathcal{B}'$ is

$$\frac{1}{8}\begin{bmatrix} -5 & -7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -5 & -7 \\ 3 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$