UNIVERSITY OF NEW ENGLAND

UNIT NAME: PMTH 213

PAPER TITLE: Linear Algebra

PAPER NUMBER: First and Only

DATE: **Thursday 12 November 2009** **TIME:** **2:00 PM TO 5:00 PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: THREE (3)

NUMBER OF QUESTIONS ON PAPER: EIGHT (8)

NUMBER OF QUESTIONS TO BE ANSWERED: EIGHT (8)

STATIONERY PER CANDIDATE:

0	6 LEAF A4 BOOKS	0	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	0	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: NIL

POCKET CALCULATORS PERMITTED: NO

TEXTBOOKS OR NOTES PERMITTED: FIVE (5) A4 SHEETS OF HANDWRITTEN DOUBLE SIDED NOTES (10 PAGES); NO PHOTOCOPIES; NO PRINTED PAGES.

INSTRUCTIONS FOR CANDIDATES:

- Candidates may make notes on this examination question paper during the fifteen minutes reading time
- Answer all questions
- Questions **ARE NOT** of equal value
- Candidates may retain their copy of this examination question paper

Question 1 [10 *marks*]

Find all linear transformations, $T\colon \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (u, v)$ which map the xy-plane onto the line determined by the equation $v = u$.

Question 2 [10 *marks*]

Find the complete set of eigenvalues of a 2×2 complex matrix, $\underline{\mathbf{A}}$, satisfying

$$\underline{\mathbf{A}}^3 = \underline{\mathbf{A}}^2 + 6\underline{\mathbf{A}},$$

carefully justifying your answer.

Question 3 [10 *marks*]

The matrix $\underline{\mathbf{A}}$ is *symmetric* if and only if $\underline{\mathbf{A}}^t = \underline{\mathbf{A}}$, and *skew-symmetric* if and only if $\underline{\mathbf{A}}^t = -\underline{\mathbf{A}}$, where $\underline{\mathbf{A}}^t$ denotes the transpose of $\underline{\mathbf{A}}$.
Let $\mathbf{M}_S(n;\mathbb{R})$ denote the set of all real symmetric $n \times n$ matrices, and $\mathbf{M}_A(n;\mathbb{R})$ the set of all skew-symmetric ones.
Show that $\mathbf{M}_S(n;\mathbb{R})$ and $\mathbf{M}_A(n;\mathbb{R})$ form vector subspaces of $\mathbf{M}(n;\mathbb{R})$, the real vector space of all real $n \times n$ matrices.
Show that $\mathbf{M}(n;\mathbb{R}) = \mathbf{M}_S(n;\mathbb{R}) \oplus \mathbf{M}_A(n;\mathbb{R})$.

Question 4 [8 *marks*]

Find a basis for the kernel and a basis for the image of the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad (x, y, z) \longmapsto (x + 3y + 5z, x + 2y + 3z, 2x + 9y + 16z)$$

Question 5 [12 *marks*]

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for the vector space V, and $T : V \to V$ a linear transformation.
Show that if $\mathbf{f}_1 := \mathbf{e}_2 + \mathbf{e}_3, \mathbf{f}_2 := \mathbf{e}_1 + \mathbf{e}_3, \mathbf{f}_3 := \mathbf{e}_1 + \mathbf{e}_2$, then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is also a basis for V.
Find the matrix, $\underline{\mathbf{B}}$, of T with respect to $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, given that its matrix with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Question 6 is on page 3

Question 6 [10 *marks*]

The matrix, $\underline{\mathbf{A}}$, is *normal* if and only if it commutes with its adjoint, $\underline{\mathbf{A}}^*$, that is, $\underline{\mathbf{A}}\,\underline{\mathbf{A}}^* = \underline{\mathbf{A}}^*\underline{\mathbf{A}}$ and it is *orthogonal* if and only its adjoint is its inverse, that is, $\underline{\mathbf{A}}^{-1} = \underline{\mathbf{A}}^*$.

(a) Prove that if $\underline{\mathbf{A}}$ is invertible, so is $\underline{\mathbf{A}}^*$ and $(\underline{\mathbf{A}}^*)^{-1} = (\underline{\mathbf{A}}^{-1})^*$.

(b) Prove that if $\underline{\mathbf{A}}$ is invertible and normal, then $\underline{\mathbf{B}} := \underline{\mathbf{A}}^*\underline{\mathbf{A}}^{-1}$ is orthogonal.

Question 7 [15 *marks*]

Given the real symmetric matrix $\underline{\mathbf{A}} = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$, find

(a) its eigenvalues;

(b) bases for its eigenspaces;

(c) an orthogonal matrix, $\underline{\mathbf{P}}$, which diagonalises it;

(d) $\underline{\mathbf{P}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{P}}$.

Question 8 [25 *marks*]

Let V be the real vector space of all 2×2 matrices with real coefficients, so that

$$V := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

(a) Prove that

$$\langle\!\langle \underline{\mathbf{A}}, \underline{\mathbf{B}} \rangle\!\rangle := \operatorname{tr}\left(\underline{\mathbf{A}}^t\,\underline{\mathbf{B}}\right)$$

defines an inner product on V, where $\underline{\mathbf{X}}^t$ denotes the transpose of the matrix $\underline{\mathbf{X}}$, and $\operatorname{tr}(\underline{\mathbf{X}})$ its trace.

(b) Apply the Gram-Schmidt procedure to construct an orthonormal basis with respect to this inner product for the subspace generated by

$$\mathbf{v}_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v}_2 := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{v}_3 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$