UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101

PAPER TITLE: Algebra & Differential Calculus

PAPER NUMBER: First and Only

DATE: **Wednesday 12 November 2008** **TIME:** **9:30 AM TO 12:30 PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: FIVE (5)

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE:

1	6 LEAF A4 BOOKS	**0**	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	**0**	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: NIL

POCKET CALCULATORS PERMITTED: YES **(SILENT TYPE)**

TEXTBOOKS OR NOTES PERMITTED: **FIVE (5) A4 sheets of handwritten double sided notes (10 pages). No photocopies. No printed pages.**

<u>INSTRUCTIONS FOR CANDIDATES</u>:

- Candidates MAY make notes on this examination question paper during the fifteen minutes reading time
- **SECTION A: -** Answer all questions
- **SECTION B: -** answer only TWO (2) of the FOUR (4) questions provided
- Questions are of equal value
- Candidates may retain this examination question paper

SECTION A

You should attempt all questions in this section.

Question 1 *[10 marks]*

(a) Assuming that a is a negative number and $b < a$ prove that $b^2 > ab$;

(b) Prove by the mathematical induction that for any $n = 1, 2, \cdots,$

$$2^n \geq n + 1.$$

(c) Determine supremum and infimum (if exist) of the following sets

$$(i) \quad \{n \in \mathbb{N} : n^3 > 10\} \qquad (ii) \quad \{x \in \mathbb{R} : x^3 < 8\}.$$

Question 2 *[10 marks]*

(a) For $z = 1 + 3\,\mathrm{i}$, write each of the following complex numbers in the form $x + \mathrm{i}\,y$

(i) $\bar{z}$, (ii) $|z|$, (iii) $|z|^2$, (iv) $\dfrac{1}{\bar{z}^2}$, (v) $\dfrac{z^2 + \mathrm{i}}{\bar{z}}$.

(b) Find all complex numbers z that satisfy

$$z^5 = 1.$$

Question 3 *[10 marks]*

(a) Find the natural domain X and the range Y of the functions defined by the following formulae

(i) $f(x) = x^4 + 1$, (ii) $f(x) = \dfrac{1}{x - 1}$, (iii) $f(x) = \sqrt{x^5 + 1}$.

(b) Sketch the graph of the function $f : X \to \mathbb{R}$ from Part (a) (ii) $f(x) = \dfrac{1}{x-1}$. Decide whether this function is injective or surjective.

(c) Find a real number k that renders continuous the function

$$f : x \mapsto \begin{cases} e^{x-1}, & x < 1 \\ \cos(\pi x) + k, & x \geq 1. \end{cases}$$

Question 4 *[10 marks]*

(a) Determine which of the following sequences of real numbers $(u_n)_{n\in\mathbb{N}}$ is monotone and discuss the behaviour of u_n as $n \to \infty$.

(i) $u_n = \frac{1}{n} - \frac{1}{n+1}$ (ii) $u_n = (-1)^n$.

(b) Determine which of the following series converge and which diverge, justifying your answer.

$$(i) \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad (ii) \quad \sum_{n=1}^{\infty} \frac{2n+4}{n} \qquad (iii) \quad \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}.$$

Question 5 *[10 marks]*

(a) Find all points at which each of the following functions is well-defined and differentiable, and find the derivatives when they exist. When the function fails to be differentiable at the point, explain why.

(i) $f(x) = \dfrac{1}{\cos x + 1}$ (ii) $g(x) = |x-1| + x^3$.

(b) Differentiate the functions

(i) $f(x) = \dfrac{2x+1}{x^2+2}$, (ii) $g(x) = e^{x^2}\cos x$, (iii) $h(x) = \sin(\cos x)$.

Question 6 *[10 marks]*

Consider the function $f(x) = x^3 - 3x^2 - 9x + 6$.

(a) Determine the intervals on which f is (i) increasing or decreasing (ii) concave up or concave down.

(b) Find all the relative maxima and relative minima of f and the absolute maximum and the absolute minimum on $[-4, 4]$.

(c) Sketch the graph of f on the interval $[-4, 4]$ (Choose an appropriate scale).

Question 7 is on page 4

Question 7 *[10 marks]*

(a) Find all real numbers x, y, z such that

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ x & - & y & + & 2z & = & 0 \\ x & + & y & + & z & = & 2. \end{array}$$

(b) For

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -2 \\ -1 & 1 & -2 \end{pmatrix}.$$

state whether the following products and (or) sums are defined; calculate those which are defined:

(i) BA (ii) $A + B$ (iii) CB.

(c) Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 2 \\ 1 & -3 & -4 & -6 \\ 99 & 1 & 0 & 16 \end{vmatrix}.$$

Question 8 *[10 marks]*

(a) Find the inverse matrix of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

Check your answer.

(b) Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.$$

SECTION B

You should attempt only TWO questions in this section.

Question 9 *[10 marks]*

(a) Let $f(x) = x^3$. Find $\delta > 0$ such that $|x - 1| < \delta$ guarantees $|f(x) - f(1)| < 0.01$.

(b) Let

$$f(x) = \begin{cases} \sin(\pi x), & \text{if } x \geq 1; \\ x^2 - 1, & \text{if } x < 1. \end{cases}$$

Prove that $f(x)$ is continuous at $x = 1$.

Question 10 *[10 marks]*

(a) Prove that if $\lim_{x\to c} f(x)$ exists, then there is a constant $\delta > 0$, such that $f(x)$ is bounded in $(c, c + \delta)$.

(b) Prove that if $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$.

Question 11 *[10 marks]*

Find the largest possible area for a rectangle with base on the x-axis and upper vertices on the curve $y = \sqrt{1 - x^2}$.

Question 12 *[10 marks]*

Prove $e^x > x + 1$ for all $x > 0$.