PURE MATHEMATICS 212

MULTIVARIABLE CALCULUS

SOLUTIONS TO ASSIGNMENT 3.

Question 1. [4 marks]

$$\begin{aligned}
T(t) &= \frac{r'(t)}{\|r'(t)\|} = \frac{e^t(\cos t - \sin t)\mathbf{i} + e^t(\sin t + \cos t)\mathbf{j} + e^t\mathbf{k}}{e^t\sqrt{3}} \\
&= \frac{(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + \mathbf{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \text{ at } t = 0. \\
T'(t) &= \left(\frac{-\sin t - \cos t}{\sqrt{3}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{3}}\right)\mathbf{j}
\end{aligned}$$

Let $t = 0, T'(0) = \frac{-1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j}$.

So, $N(0) = \dfrac{T'(0)}{\|T'(0)\|} = \dfrac{-1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$.

Question 2. [2 marks]

Using $\kappa(0) = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|^3}$, we have

$$\begin{aligned}
\mathbf{r}'(t) &= e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k} \\
\mathbf{r}''(t) &= e^t\mathbf{i} + e^{-t}\mathbf{j} \\
\kappa(0) &= \frac{\|\,(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (1\mathbf{i} + \mathbf{j})\,\|}{\|\mathbf{i} - \mathbf{j} + \mathbf{k}\|^3} \\
&= \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}
\end{aligned}$$

Question 3. [3 marks]

Proof. The easiest way is to consider the curve in 3-dimensional space $r(x) = x\mathbf{i} + f(x)\mathbf{j} + 0\mathbf{k}$. Then

$$\begin{aligned}
\mathbf{r}'(x) &= \mathbf{i} + f'(x)\mathbf{j} \\
\mathbf{r}''(x) &= f''(x)\mathbf{j} \\
\mathbf{r}'(x) \times \mathbf{r}''(x) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\mathbf{k} \\
\kappa(x) &= \frac{\|\mathbf{r}'(x) \times \mathbf{r}''(x)\|}{\|\mathbf{r}'(x)\|^3} = \frac{|f''|}{(1 + f'^2)^{\frac{3}{2}}}.
\end{aligned}$$

An alternative self-contained proof with no reference to Theorem 1 from Lecture 8 is a good exercise:

We have

$$\begin{aligned}
\mathbf{r}'(x) &= \mathbf{i} + f'(x)\mathbf{j} \\
T(x) &= \frac{1}{\sqrt{1+f'^2}}\mathbf{i} + \frac{f'}{\sqrt{1+f'^2}}\mathbf{j} \\
\frac{dT}{dx} &= \frac{f'f''}{(1+f'^2)^{3/2}}\mathbf{i} + \frac{f''}{(1+f'^2)^{3/2}}\mathbf{j}
\end{aligned}$$

Since $\frac{dx}{ds} = \frac{1}{\sqrt{1+f'^2}}$, the chain rule gives

$$\frac{dT}{ds} = \frac{dT}{dx}\frac{dx}{ds} = \frac{f'f''}{(1+f'^2)^2}\mathbf{i} + \frac{f''}{(1+f'^2)^2}\mathbf{j}$$

and consequently

$$\kappa(x) = \left\|\frac{dT}{ds}\right\| = \frac{|f''(x)|}{(1+f'^2(x))^{3/2}}.$$

Question 4. [3 marks]

(a) $f(t, t^2) = t + (t \cdot t^2)^{1/3} = 2t.$

(b) $f(x, x^2) = x + (x \cdot x^2)^{1/3} = 2x.$

(c) $f(2y^2, 4y) = 2y^2 + (8y^3)^{1/3} = 2y(y+1).$

Question 5. [2 marks]

$$g(u(x,y), v(x,y)) = v(x,y)\sin(u(x,y)v(x,y)) = \pi xy \sin(\pi x^5 y^7).$$

Question 6. [4 marks] a)

$$\begin{aligned}
f(x+y, x-y, x^2) &= x^2(x+y)(x-y) + x + y \\
&= x^4 - x^2y^2 + x + y
\end{aligned}$$

b)

$$\begin{aligned}
f(x+y, x-y, x^2) &= (xz)(xy)\frac{y}{x} + xy \\
&= xy^2z + xy \text{ for } x \neq 0
\end{aligned}$$

Question 7. [2 marks]

(a) Its level surfaces are all the planes with the same normal vector $\langle 3, -1, 2\rangle$.

(b) Its level surfaces are $m = z - x^2 - y^2$, i.e. $z = m + x^2 + y^2$. They are elliptic paraboloids with the same central axis (z-axis).