TUTORIAL 1, PMTH212

Remark: The textbook by Anton contains more problems and worked out examples!

1. Let $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (1 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $P_4 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
a) Show that the quadrilateral $P_1P_2P_3P_4$ is a rhombus, i.e. all sides are of equal length.
b) Show that the diagonals P_1P_3 and P_2P_4 are perpendicular.

Solution. a). $|P_1P_2| = \sqrt{1^2 + 0^2} = 1$, $|P_2P_3| = \sqrt{(\sqrt{2}/2)^2 + (\sqrt{2}/2)^2} = 1$, $|P_3P_4| = \sqrt{(-1)^2 + 0^2} = 1$, $|P_4P_1| = \sqrt{(\sqrt{2}/2)^2 + (\sqrt{2}/2)^2} = 1$.

b). We show that the inner product of $\vec{P_1P_3} = \langle \frac{\sqrt{2}}{2} + 1, \frac{\sqrt{2}}{2} \rangle$ and $\vec{P_2P_4} = \langle \frac{\sqrt{2}}{2} - 1, \frac{\sqrt{2}}{2} \rangle$ vanishes. Indeed,

$$\left\langle \frac{\sqrt{2}}{2} + 1, \frac{\sqrt{2}}{2} \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2} - 1, \frac{\sqrt{2}}{2} \right\rangle = \frac{1}{2} - 1 + \frac{1}{2} = 0.$$

2. Find the distance of the point $P = (5, 4, 3)$
a). to the yz-plane
b). to the x-axis.

Solution. a). The orthogonal projection of P to the xy-plane is $Q = (0, 4, 3)$. The distance $|PQ| = 5$.

b). The orthogonal projection of P to the x-axis is $Q = (5, 0, 0)$. The distance $|PQ| = \sqrt{4^2 + 3^2} = 5$.

3. Let ℓ be the line $x - y + 1 = 0$ in $\mathbb{R}^2$.
a). Find the equation of the line parallel to ℓ and passing through $P = (1, 1)$.
b). Find the equation of the line perpendicular to ell and passing through $Q = (1, 0)$

Solution. a). The line has the same normal $\langle 1, -1 \rangle$ as ℓ. Hence, the equation is $x - y + a = 0$. Since P is on the line, we have $1 - 1 + a = 0$, hence $a = 0$. the equation is $x - y = 0$.
b). The normal of this line is perpendicular to $\langle 1, -1 \rangle$. We may choose, for instance, $\langle 1, 1 \rangle$. Hence, the equation is $x + y + b = 0$. Since Q is on the line, we have $1 + 0 + b = 0$, hence $b = -1$. the equation is $x + y - 1 = 0$.

4. Consider the lines $\ell_1 \colon 3x + 2y = 1$ and $\ell_2 \colon 6x + 4y = -2$
a). Show that the two lines are parallel.
b). Compute the distance between the two lines.

Solution. a.) The normal vector $\langle 3,2\rangle$ of ℓ_1 is a multiple of the normal vector $\langle 6,4\rangle$ of ℓ_2. Lines whose normals are parallel are parallel themselves.

b.) Let $P=(a,b)$ be a point on ℓ_1 and $Q=(c,d)$ be a point on ℓ_2. We have to choose P and Q in such a way that their distance becomes minimal. This can only happen if the connecting line ℓ_3 through PQ is perpendicular to both ell_1, ℓ_2. Therefore we choose Q to be the intersection point of the line through P, perpendicular to ℓ_1 with ℓ_2. The vector $\langle 2,-3\rangle$ is perpendicular to $\langle 3,2\rangle$ (notice that we just swapped the two coordinates and put a minus in front of the second one). The equation of a perpendicular line to ℓ_1 is

$$2x-3y=\alpha$$

where α needs to be determined from the condition that the line contains the point P. Since ℓ_1 contains P, we have 3a+2b=1, hence $b=\frac{1-3a}{2}$ and $\alpha=\frac{13a-3}{2}$ and the equation of ℓ_3 is

$$2x-3y=\frac{13a-3}{2}.$$

It intersection with ℓ_2 is the solution of the system

$$\begin{aligned}2x-3y&=\frac{13a-3}{2}\\6x+4y&=-2.\end{aligned}$$

Thus, $Q=(a-\frac{6}{13},\frac{5}{26}-\frac{3a}{2})$. Therefore, $\vec{PQ}=\langle -\frac{6}{13},-\frac{4}{13}\rangle$. We see that $\vec{PQ}$ does not depend on the choice of P, i.e the parameter a, which we would have expected. The desired distance is now

$$|PQ|=\sqrt{\frac{36}{169}+\frac{16}{149}}=\frac{\sqrt{52}}{13}.$$