UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101

PAPER TITLE: Algebra and Differential Calculus

PAPER NUMBER: First and Only

DATE: **Wednesday 18 November 2009 TIME: 9:30 AM TO 12:30 PM**

TIME ALLOWED: THREE (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: FIVE (5)

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE:

0	6 LEAF A4 BOOKS	**0**	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	**0**	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: **NIL**

POCKET CALCULATORS PERMITTED: **NO**

TEXTBOOKS OR NOTES PERMITTED: FIVE (5) A4 SHEETS (10 PAGES IF WRITTEN ON BOTH SIDES) OF HAND-WRITTEN NOTES. NO PHOTOCOPIES. NO PRINTED NOTES PERMITTED.

INSTRUCTIONS FOR CANDIDATES:

- Candidates **MAY** make notes on this examination question paper during the fifteen minutes reading time
- Questions are of equal value
- **SECTION A: -** Answer **ALL** questions
- **SECTION B:** - Answer only **TWO (2)** of the **FOUR (4)** questions provided
- Candidates **MAY** retain their copy of this examination question paper

SECTION A

You must attempt all questions in this section.

Question 1 *[10 marks]*

(a) Assuming that a, b, c and d are positive numbers, prove that if $ad > bc$, $\frac{a}{b} > \frac{c}{d}$;

(b) Prove by mathematical induction that for any $n = 1, 2, \cdots$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

(c) Determine supremum and infimum (if they exist) of the following sets

$$(i) \quad \{n \in \mathbb{N} : n^2 > 4\} \qquad (ii) \quad \{x \in \mathbb{R} : x^3 > 8\}.$$

Question 2 *[10 marks]*

(a) Write each of the following complex numbers in the form $x + \mathrm{i}\, y$

(i) $(1 - i)(3 + 2i)$, (ii) $\frac{2+i}{1+i}$, (iii) $|3 - 2i|^2$,

(b) Find all complex numbers z that satisfy

$$z^2 = 1 + i.$$

Question 3 *[10 marks]*

(a) Find the natural domain X and the range Y of the functions defined by the following formulae

(i) $f(x) = \sin x + 1$, (ii) $f(x) = \dfrac{1}{x+1}$, (iii) $f(x) = \sqrt{x^2 - 1}$.

(b) Sketch the graph of the function $f : X \to \mathbb{R}$ from Part (a) (ii) $f(x) = \dfrac{1}{x+1}$. Decide whether this function is injective or surjective.

(c) Find a real number k that renders continuous the function

$$f : x \mapsto \begin{cases} \frac{k \sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Question 4 *[10 marks]*

(a) Determine which of the following sequences of real numbers $(u_n)_{n\in\mathbb{N}}$ is monotone and discuss the behaviour of u_n as $n \to \infty$.

(i) $u_n = \frac{n+1}{n^2+3n+2}$ (ii) $u_n = \frac{1}{2+(-1)^n}$.

(b) Determine which of the following series converge and which diverge, justifying your answer.

$$(i) \quad \sum_{n=1}^{\infty} \frac{3^n}{n!} \qquad (ii) \quad \sum_{n=1}^{\infty} \sqrt{\frac{2n^2+3}{n^4}} \qquad (iii) \quad \sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}.$$

Question 5 *[10 marks]*

(a) Find $\frac{dy}{dx}$ for the following implicitly defined functions:

(i) $xy^2 + x^2y = 1$ (ii) $y^3 \cos x + e^y = 0$.

(b) Differentiate the functions

(i) $f(x) = \dfrac{x}{x^4+1}$, (ii) $g(x) = x^3 \sin(x^2)$, (iii) $h(x) = \ln(1 + e^x)$.

Question 6 *[10 marks]*

Consider the function $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 1$.

(a) Determine the intervals on which f is (i) increasing or decreasing (ii) concave up or concave down.

(b) Find all the relative maxima and relative minima of f and the absolute maximum and the absolute minimum on $[-1, \frac{3}{2}]$.

(c) Sketch the graph of f on the interval $[-3, 3]$ (Choose an appropriate scale).

Question 7 *[10 marks]*

(a) Find all real numbers x, y, z such that

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ & - & y & + & 2z & = & 3 \\ 2x & + & y & + & z & = & 3. \end{array}$$

Question 7 (b) is on page 4

(b) For

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & -2 \\ -1 & 1 & -2 \end{pmatrix}.$$

state whether the following products and (or) sums are defined; calculate those which are defined:

(i) AB (ii) $A+B$ (iii) BC.

(c) Evaluate the determinant

$$\begin{vmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{vmatrix}.$$

Question 8 *[10 marks]*

(a) Find the inverse matrix of

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}.$$

Check your answer.

(b) Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

SECTION B is on page 5

SECTION B

You should attempt only TWO questions in this section.

Question 9 *[10 marks]*

(a) Let $f(x) = x^2$. Show that given any number $\varepsilon > 0$, there exists a number $\delta > 0$, such that if $|x-1| < \delta$, then $|f(x) - f(1)| < \varepsilon$.

(b) Let

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that $f(x)$ is continuous at $x = 0$.

Question 10 *[10 marks]*

(a) Prove that if $f(x)$ is continuous at $x = c$, then there is a constant $\delta > 0$, such that $f(x)$ is bounded in $(c-\delta, c+\delta)$.

(b) Use the Mean Value Theorem to show that $\sin x \leq x$ for $x \in [0, +\infty)$.

Question 11 *[10 marks]*

Find the largest possible area for a rectangle with vertices on the curve $x^2 + y^2 = 1$.

Question 12 *[10 marks]*

Prove $\ln(1+x) > x - \frac{x^2}{2}$ for all $x > 0$.