Sample Solutions for Tutorial 7

Question 1.

(a) Take $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^2e^x$

Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) \\ &= 2xe^x + x^2e^x \\ &= x(x+2)e^x \end{aligned}$$

(b) Take $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Then $f(x) = \dfrac{e^{2x}-1}{e^{2x}+1} = 1 - 2\dfrac{1}{e^{2x}+1}$, whence,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1) - 2\frac{d}{dx}\left(\frac{1}{e^{2x}+1}\right) \\ &= 0 - 2\frac{-1}{(e^{2x}+1)^2}\frac{d}{dx}(e^{2x}+1) \\ &= \frac{4e^{2x}}{(e^{2x}+1)^2} \\ &= \left(\frac{2}{e^x + e^{-x}}\right)^2 \end{aligned}$$

(c) $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad y \longmapsto \ln\left(y + \sqrt{y^2+1}\right)$

Then

$$\begin{aligned} f'(y) &= \frac{d}{dy}\ln\left(y + \sqrt{y^2+1}\right) \\ &= \frac{1}{y+\sqrt{y^2+1}}\frac{d}{dy}(y + \sqrt{y^2+1}) \\ &= \frac{1}{y+\sqrt{y^2+1}}\left(1 + \frac{1}{2\sqrt{y^2+1}}\frac{d}{dy}(y^2+1)\right) \\ &= \frac{1}{y+\sqrt{y^2+1}}\left(1 + \frac{2y}{2\sqrt{y^2+1}}\right) \\ &= \frac{1}{\sqrt{y^2+1}} \end{aligned}$$

Question 2.

(a) Given that $x^2 + y^2 = 1$ we must have $-1 \leq x \leq 1$. Hence we take $X = [-1, 1]$. Differentiating both sides of the above equation with respect to x, we obtain

$$2x + 2y\frac{dy}{dx} = 0,$$

so that $\dfrac{dy}{dx} = \dfrac{-x}{y}$as long as $y \neq 0$, that is, $x \neq \pm 1$.

(b) Differentiating both sides of the equation $xy^2 + y\sin(xy) + e^x = 0$ with respect to x, we obtain

$$y^2 + 2xy\frac{dy}{dx} + \frac{dy}{dx}\sin(xy) + y\cos(xy)(y + x\frac{dy}{dx}) + e^x = 0,$$

or

$$\frac{dy}{dx} = -\frac{e^x + y^2(1 + \cos(xy))}{\sin(xy) + xy(2 + \cos(xy))}$$

as long as $\sin(xy) + xy(2 + \cos(xy)) \neq 0$

(c) Differentiating both sides of the equation $e^{2x} - 2ye^x - 1 = 0$ with respect to x, we obtain

$$2e^{2x} - 2\frac{dy}{dx}e^x - 2ye^x = 0,$$

or

$$\frac{dy}{dx} = e^x - y$$

as $e^x \neq 0$ for all $x \in R$.

Question 3.

Take $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^3 - 6x^2 + 3x - 7$.
Since f is a polynomial function, it is differentiable everywhere, and hence it satisfies the hypotheses of the Mean Value Theorem on $[0, 6]$.
Now $\dfrac{f(6) - f(0)}{6 - 0} = \frac{6^3 - 6.6^2 + 3.6}{6} = 3$
As $f'(x) = 3x^2 - 12x + 3$, $\;f'(c) = \dfrac{f(6) - f(0)}{6 - 0} = 3$ if and only if $3c^2 - 12c + 3 = 3$, or, equivalently, $c(c - 4) = 0$.
Thus $f^c = \dfrac{f(6) - f(0)}{6 - 0}$ for $c \in [0, 6]$ if and only if $c = 0, 4$.

Question 4.

Given $f\colon \mathbb{R} \to \mathbb{R}$ with $f'(x) = 0$ for all $x \in \mathbb{R}$, take $a, b \in \mathbb{R}$ with $a < b$.
Since $f'(x) = 0$ for all $x \in \mathbb{R}$, f satisfies the hypotheses of the Mean Value Theorem.
Thus, there is a $c \in [a, b]$ with $f(b) - f(a) = f'(c)(b - a)$.
But $f'(c) = 0$, whence $f(b) = f(a)$, showing that f is constant.

Take $f\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{x}{|x|} = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$

Since $f(x) = -1$ for all $x < 0$, we have $f'(x) = 0$ for all $x < 0$.
Similarly, since $f(x) = 1$ for all $x > 0$, we have $f'(x) = 0$ for all $x > 0$.
Hence $f'(x) = 0$ for all $x \in \mathrm{dom}(f)$, even though f is not constant.