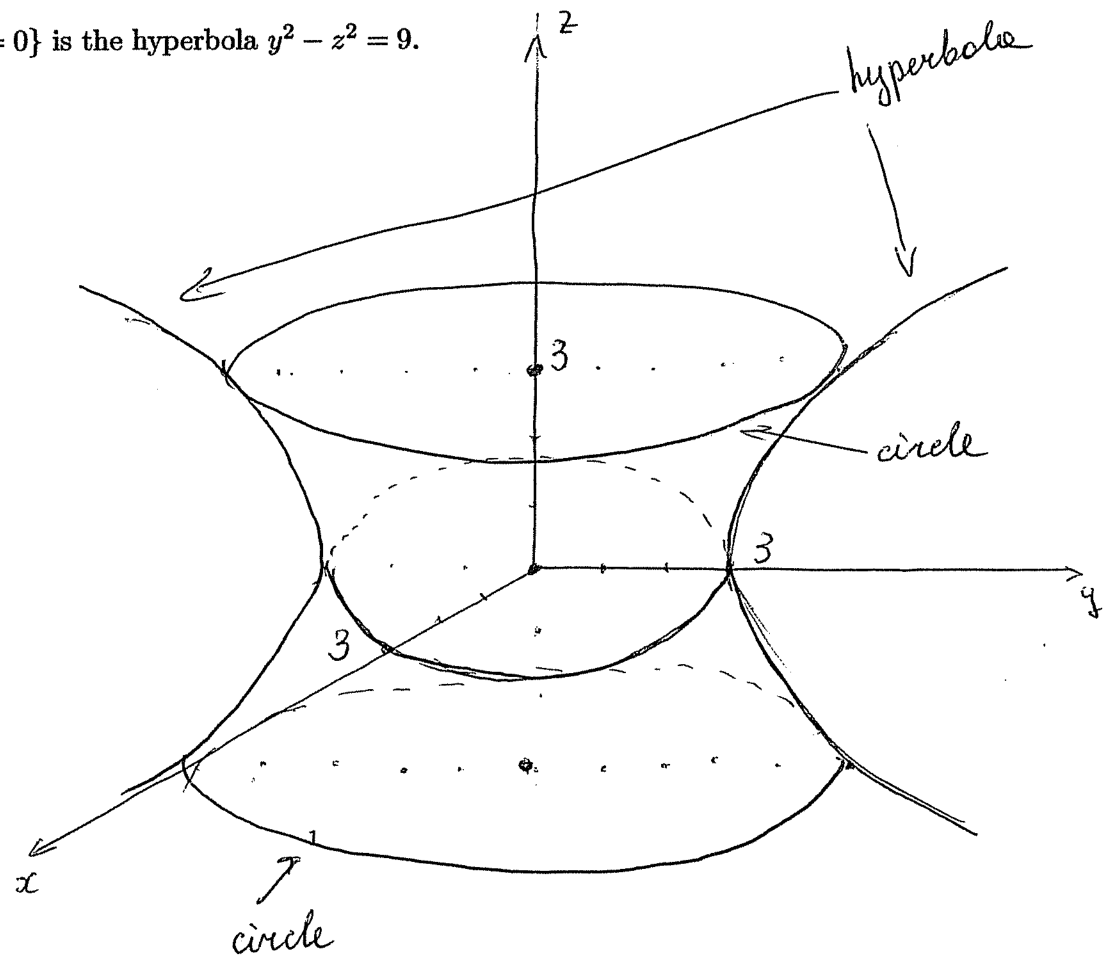
TUTORIAL 2, PMTH212

1. Name and sketch the surface $x^2 + y^2 - z^2 = 9$.

Solution. This is a hyperboloid of one sheet (two + signs and one - sign).
The traces on the horizontal planes $\{z = 0\}$ and $\{z = 4\}$ are circles of radius 3 and 5 respectively:

$$
\begin{aligned}
x^2 + y^2 - 0 &= 9, \qquad & x^2 + y^2 &= 9 \\
x^2 + y^2 - 16 &= 9, \qquad & x^2 + y^2 &= 25
\end{aligned}
$$

The trace on $\{x = 0\}$ is the hyperbola $y^2 - z^2 = 9$.

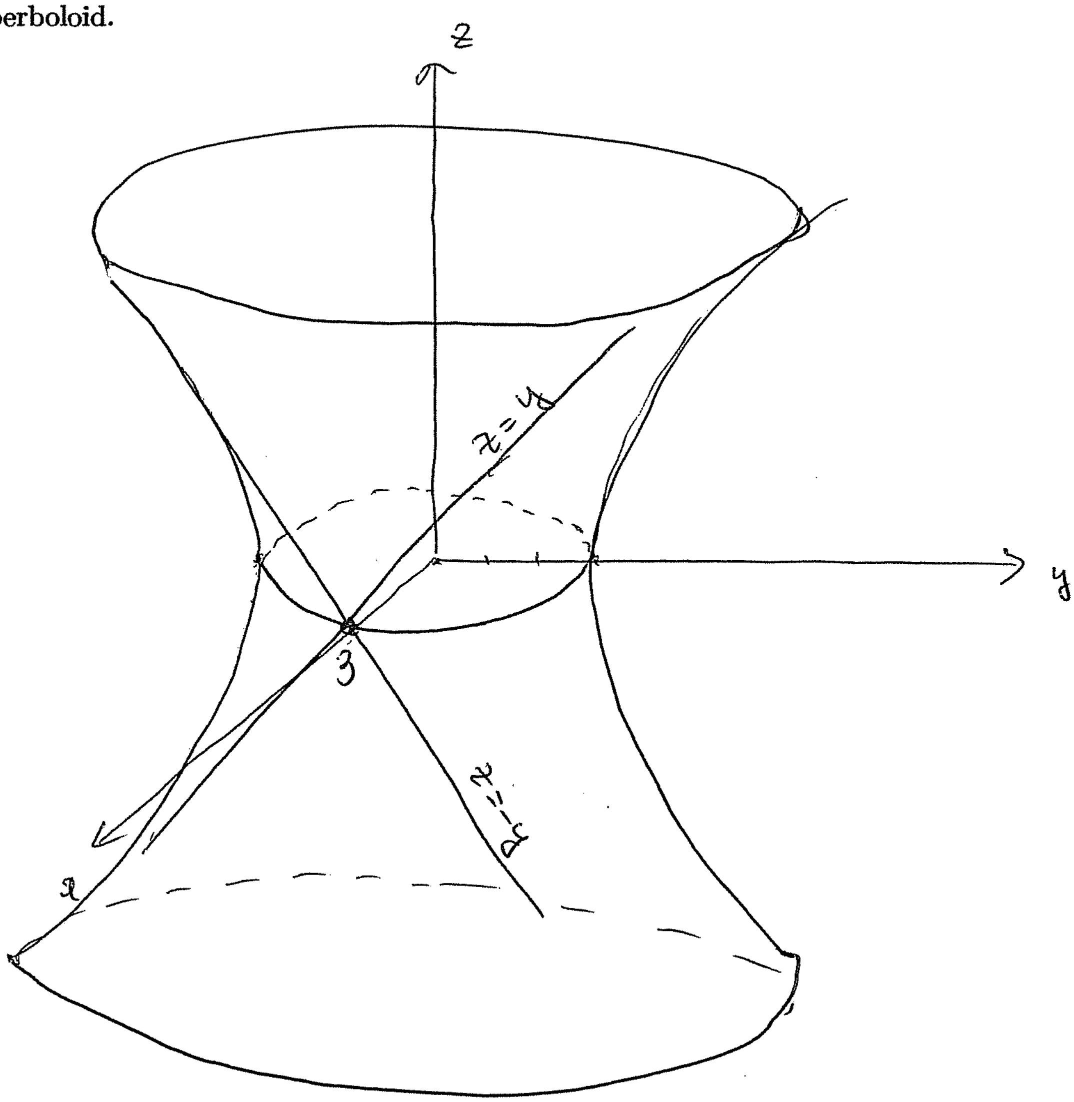


2. Find the trace of the surface $x^2 + y^2 - z^2 = 9$ on the plane $\{x = 0\}$.

Solution. For $x = 3$ we get

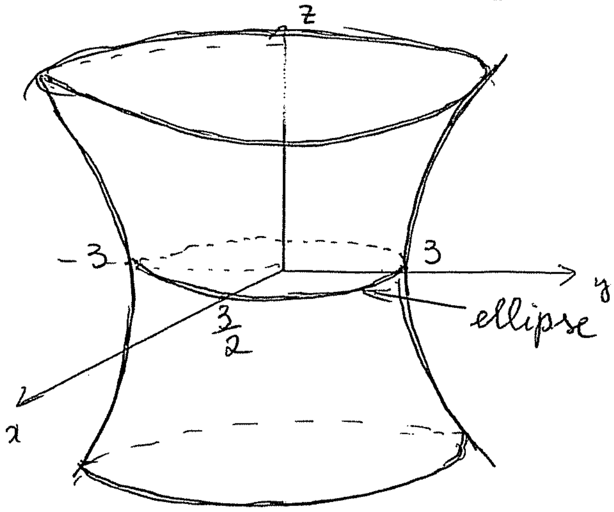
$$\begin{aligned} 3^2 + y^2 - z^2 &= 9 \\ y^2 - z^2 &= 0 \\ (y - z)(y + z) &= 0. \end{aligned}$$

The latter equation is satisfied if and only if either $y - z = 0$ or $y + z = 0$. This is a pair of straight lines that intersect at $(0, 0, 0)$. It is an interesting fact that through each point of any hyperboloid of one sheet there is a pair of straight lines that ere entirely contained in the hyperboloid.

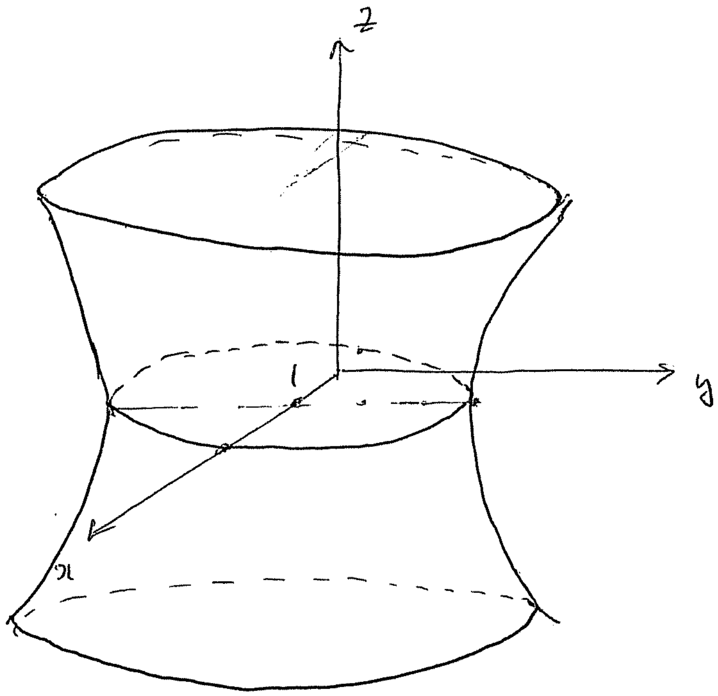


3. Sketch a.) $4x^2 + y^2 - z^2 = 9$ b.) $4x^2 - 8x + y^2 - z^2 = 5$.

Solution. a). This is a hyperboloid of one sheet. The trace on the plane $z = 0$ is the ellipse $4x^2 + y^2 = 9$ with vertices $x = 0, y = \pm 3$ and $y = 0, x = \pm\frac{3}{2}$.



b). By completing the square this equation becomes $4(x-1)^2 + y^2 - z^2 = 9$. This is the surface from a) shifted by one unit into x-direction. The centre is $(1, 0, 0)$.



4. Describe the graphs of the vector equations
a) $\vec{r} = 3\cos t\vec{i} + 3\sin t\vec{j} - \vec{k}$
b) $\vec{r} = 2t\vec{i} - t\vec{j} + (2+3t)\vec{k}$
c) $\vec{r} = t\vec{i} + t^2\vec{j} + 2\vec{k}$

Solution. a.) This is a circle of radius 3 in the plane $z = -1$. We have $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$
b.) This is a line with vector equation $\vec{r} = (0,0,2) + t\langle 2,-1,2\rangle$.
c.) This is a parabola in the plane $z = 2$. From $x = t, y = t^2$ we get $y = x^2$.

5. Find the natural domains for the functions $r(t)$. Determine, where it describes a smooth curve, i.e. where $r'(t)$ exists and is different from $\vec{0}$.
a.) $\vec{r} = e^t\vec{i} + (\ln t - 1)\vec{j} - \cos t\vec{k}$
b.) $\vec{r} = t^3\vec{i} + 3t^2\vec{j} + t^2\vec{k}$

Solution. a.) The natural domain is $\{t > 0\}$ for $\ln t$ to be well-defined. r' exists for $t > 0$and

$$r' = e^t\vec{i} + \frac{1}{x}\vec{j} + \sin t\vec{k}.$$

Since $e^t \neq 0$ we have $r' \neq \vec{0}$ and r is smooth.
b.) r is well-defined for any $t \in \mathbb{R}$ and has a derivative

$$r' = 3t^2\vec{i} + 6t\vec{j} + 2t^2\vec{k}.$$

For $t = 0$ this is $\vec{0}$ hence the curve is not smooth for $t = 0$.

6. Compute
a.) $\int(t^2\vec{i} - 2t\vec{j} + \frac{1}{t}\vec{k})dt.$
b.) $\int_0^1 \langle e^{-t}, te^t, 3t^2\rangle dt$

Solution. a.)

$$\int (t^2\vec{i} - 2t\vec{j} + \frac{1}{t}\vec{k})dt = \frac{t^3}{3}\vec{i} - t^2\vec{j} + \ln|t|\vec{k} + \vec{c}$$

b.)

$$\begin{aligned}\int_0^1 \langle e^{-t}, te^t, 3t^2\rangle dt &= \langle -e^{-t}, (t-1)e^t, t^3\rangle\big|_0^1 \\ &= \langle -e^{-1} + 1, 1, 1\rangle \\ &= \langle 1 - \frac{1}{e}, 1, 1\rangle\end{aligned}$$

7. Compute the arc-length of $r(t) = t^3\vec{i} - t\vec{j} + \frac{1}{2}\sqrt{6}t^2\vec{k}$ for $1 \leq t \leq 3$.

Solution. We have

$$\begin{aligned} L &= \int_1^3 \sqrt{(3t^2)^2 + 1 + (\sqrt{6}t)^2} dt \\ &= \int_1^3 \sqrt{9t^4 + 1 + 6t^2} dt \\ &= \int_1^3 (3t^2 + 1) dt \\ &= t^3 + t|_1^3 = 28 \end{aligned}$$