THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101/101A

PAPER TITLE: Algebra and Differential Calculus

PAPER NUMBER: First and Only

DATE: **Wednesday 15th November 2006** **TIME:** **2:00PM TO 5:00PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time.

NUMBER OF PAGES IN PAPER: SEVEN (7)

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE: $\boxed{1}$ X 6 LEAF A4 BOOKS $\boxed{1}$ X 12 LEAF A4 BOOKS

$\boxed{0}$ X ROUGH WORK BOOKS

GRAPH PAPER: **NIL** (NUMBER OF SHEETS)

POCKET CALCULATORS PERMITTED: **YES** (SILENT TYPE)

OTHER AIDS REQUIRED: **NIL**

INSTRUCTIONS FOR CANDIDATES:

- Candidates **may** make notes on this paper during the fifteen minutes reading time
- Questions are of equal value
- SECTION A: Answer all questions
- SECTION B: Only answer TWO (2) questions
- Candidates may retain their copy of this examination question paper

TEXTBOOKS OR NOTES PERMITTED: FIVE A4 sheets (10 pages if written on both sides) of HANDWRITTEN notes. No photocopies, no printed notes permitted.

SECTION A

You should attempt all questions in this section.

Question 1

(a) Find, if they exist, the supremum and infinum of the following sets. In each case state whether or not the supremum (infinum) is an element of the set.

(i) $\left\{x \in \mathbb{R} : \dfrac{1}{4-x^2} > 0\right\}$ (ii) $\left\{\dfrac{1}{n^3} : n = 1, 2, 3, \cdots\right\}$.

[3 marks]

(b) Prove by mathematical induction that for any counting number $n \geq 3$

$$\left(\frac{3}{2}\right)^n > n.$$

[3 marks]

(c) Let a be a positive real number and $a > 3$. Give a proof by contradiction of the following statement,

$$a^2 - a > 6.$$

[4 marks]

Question 2

(a) For $z = -1 - i$ express each of the following complex numbers in the form $x + iy$, with x and y real.

(i) $3i + 3z$ (ii) $\frac{z-i}{z-1}$ (iii) $\overline{z - 3i}$

(iv) $\left|\frac{z+2-i}{z}\right|$ (v) $\overline{z^2}$. *[7 marks]*

(b) Find, over $\mathbb{C}$, all solutions of the equation $z^3 = -i$, i.e. find (over the complex numbers) all possible values of $(-i)^{\frac{1}{3}}$. *[3 marks]*

Question 3 is on page 3.

Question 3

(a) For each of the following real valued functions find the natural domain and range

$$\text{(i)}\quad f(x) = \frac{-x}{x^2+2} \qquad\qquad \text{(ii)}\quad g(x) = \frac{1}{\sqrt{x^2-1}}.$$

[4 marks]

(b) Is $g(x)$ (of part (a), above) injective, surjective or bijective on its natural domain and range? Give brief reasons. What are (if they exist) $\sup g$ and $\inf g$? *[4 marks]*

(c) What value of k makes the following function continuous at $x = 1$?

$$f(x) = \begin{cases} 2+\cos(\pi x), & x < 1 \\ 3x + k, & x \geq 1. \end{cases}$$

[2 marks]

Question 4

(a) State which of the following sequences are monotone. Are they increasing or decreasing? Discuss the behaviour of each of the sequences as $n \longrightarrow \infty$.

$$\text{(i)}\quad \left(\frac{3}{4}\right)^n \qquad\qquad \text{(ii)}\quad \frac{\sin(n)}{n} \qquad\qquad \text{(iii)}\quad \frac{1+n}{n^2}.$$

[4 marks]

(b) Determine if the following infinite series are convergent or divergent.

$$\text{(i)}\quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+2}} \qquad\qquad \text{(ii)}\quad \sum_{n=1}^{\infty} \frac{2^n}{(n+1)!} \qquad\qquad \text{(iii)}\quad \sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^3}}.$$

[6 marks]

Question 5 is on page 4.

Question 5

(a) Differentiate the following functions, state clearly where each function is differentiable. If any of the functions is not differentiable at a point, explain (briefly) why it is not differentiable.

(i) $f(x) = \dfrac{1 - x^3}{1 + x^3}$ (ii) $f(x) = \tan(x)\sqrt{x+1}$.

[5 marks]

(b) Find $\dfrac{dy}{dx}$ for the following implicitly defined function,

$$x \ln(x^2 + y^2) + e^x + y = 0.$$

[5 marks]

Question 6

Consider the function, $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = 2x^3 + 9x^2 - 24x + 1.$$

(a) Determine the intervals on which f is (i) increasing or decreasing and (ii) concave up or concave down. *[4 marks]*

(b) Find all relative maxima and minima of the function f. What are the absolute maxima and minima for f on the interval $[-5, 1]$? *[3 marks]*

(c) Sketch the graph of f on the interval $[-5, 1]$. *[3 marks]*

Question 7 is on page 5.

Question 7

(a) Solve the following linear system,

$$
\begin{aligned}
x + y + z &= 1 \\
2x + 3y - 5z &= -7 \\
x - 2y + 4z &= 10.
\end{aligned}
$$

[4 marks]

(b) For

$$
A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix}
$$

state whether the following products and (or) sums are defined; calculate those which are defined.

(i) AB (ii) $A + 3C$ (iii) BC.

[3 marks]

(c) Find the eigenvalues of the matrix A, below

$$
A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
$$

[3 marks]

Question 8 is on page 6.

Question 8

Let P_1, P_2, P_3 and P_4 be the following four points in $\mathbb{R}^3$, $P_1(0,0,-1)$, $P_2(0,1,0)$, $P_3(1,1,0)$ and $P_4(0,0,1)$.

(a) Write down the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ in terms of the standard unit vectors i, j and k. *[2 marks]*

(b) Find the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\overrightarrow{P_1P_3}$. *[3 marks]*

(c) Find the area of the triangle formed by P_1, P_2 and P_3. *[3 marks]*

(d) Find the volume of the parallelepiped with sides given by the three vectors $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_4}$. *[2 marks]*

SECTION B.

You are to answer only TWO of the four questions in this section.

Question 9

(a) Prove formally that $f(x) = \frac{1}{x-1}$ is continuous on $\{x \in \mathbb{R} : x > 1\}$. *[4 marks]*

(b) Suppose that f is a function $\mathbb{R} \longrightarrow \mathbb{R}$ and that at all points $x_1, x_2 \in \mathbb{R}$ we have

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|.$$

Prove formally that f is continuous on its natural domain. *[6 marks]*

Question 10

(a) Show that $f(x) = \sin(x)$ satisfies the conditions of the Mean Value Theorem on an interval $[0, b]$, with $b < \infty$. Apply the Mean Value Theorem to f on the interval $[0, x], x < \infty$, to show that

$$-x \leq \sin(x) \leq x.$$

[6 marks]

(b) Suppose $f(x) = \frac{x^2-1}{x}$ on $[-1, 1]$. Show that there is no $c \in (-1, 1)$ such that $f'(c) = 0$ even though $f(-1) = f(1) = 0$. Explain why Rolle's Theorem fails in this case.

[4 marks]

Question 11

A wire of length 10 cm can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be a

(i) maximum?

(ii) minimum?

[10 marks]

Question 12

Consider the following Leslie matrix,

$$P = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

(a) Verify that the eigenvalues of P are $1, \frac{1}{2}(-1+i)$ and $\frac{1}{2}(-1-i)$ and show that 1 is the dominant eigenvalue. *[4 marks]*

(b) Find the asymptotic fractional population vector and hence the (long time) fraction of the population in each of the three age groups. *[6 marks]*