PMTH212 ASSIGNMENT 8

MARK VILLAR

(1)

$$\int_0^1 \int_0^x e^{x^2}\ dy\ dx = \int_0^1 e^{x^2} y \Big|_0^x\ dx = \int_0^1 x e^{x^2}\ dx$$
$$= \frac{1}{2} e^{x^2} \Big|_0^1 = \frac{1}{2}(e-1)$$

(2) (a) R is a type I region enclosed between $y = x^2$ and $y = \sqrt{x}$ and bounded by the vertical lines $x = 0$ and $x = 1$.

$$\int_0^1 \int_{x^2}^{\sqrt{x}} x + y\ dy\ dx = \int_0^1 xy + \frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}}\ dx = \int_0^1 x^{3/2} + \frac{x}{2} - x^3 - \frac{x^4}{2}\ dx$$
$$= \frac{2}{5}x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\Big|_0^1 = \frac{2}{5} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10} = \frac{3}{10}$$

(b) R is a type I region enclosed between $y = x$ and $y = 0$ and bounded by the vertical lines $x = 0$ and $x = \pi$.

$$\int_0^\pi \int_0^x x\cos y\ dy\ dx = \int_0^\pi x \sin y \Big|_0^x\ dx = \int_0^\pi x \sin x\ dx$$

$$\int_0^\pi x\sin x\ dx = -x\cos x \Big|_0^\pi + \int_0^\pi \cos x\ dx \quad \text{(integration by parts)}$$
$$= -\pi(-1) - 0 + \sin x \Big|_0^\pi = \pi + 0 - 0 = \pi$$

(3) (a)

$$\frac{y}{2} \le x \le 1 \quad \text{and} \quad 0 \le y \le 2 \ \Rightarrow\ y \le 2x \le 2$$
$$\Rightarrow\ 0 \le y \le 2x \le 2 \ \Rightarrow\ 0 \le \frac{y}{2} \le x \le 1$$
$$\Rightarrow\ 0 \le y \le 2x \quad \text{and} \quad 0 \le x \le 1$$

$$\int_0^2 \int_{y/2}^1 \cos x^2 \; dx \; dy = \int_{y/2}^1 \int_0^2 \cos x^2 \; dy \; dx = \int_0^1 \int_0^{2x} \cos x^2 \; dy \; dx$$
$$= \int_0^1 \cos(x^2) y \Big|_0^{2x} dx = \int_0^1 2x \cos x^2 = \sin x^2 \Big|_0^1$$
$$= \sin 1 - 0 = \sin 1$$

(b)

$$1 \le x \le 3 \;\text{ and }\; 0 \le y \le \ln x \;\Rightarrow\; 0 \le \ln x \le \ln 3$$
$$\Rightarrow\; 0 \le y \le \ln x \le \ln 3 \;\Rightarrow\; 1 \le \; e^y \le x \le 3$$
$$\Rightarrow\; 0 \le y \le \ln 3 \;\text{ and }\; e^y \le x \le 3$$

$$\int_1^3 \int_0^{\ln x} x \; dy \; dx = \int_0^{\ln x} \int_1^3 x \; dx \; dy = \int_0^{\ln 3} \int_{e^y}^3 x \; dx \; dy$$
$$= \int_0^{\ln 3} \frac{1}{2} x^2 \Big|_{e^y}^3 dy = \int_0^{\ln 3} \frac{1}{2}\left(9 - e^{2y}\right) \; dy$$
$$= \frac{1}{2}\left(9y - \frac{1}{2}e^{2y}\right)\Bigg|_0^{\ln 3} = \frac{1}{2}\left(9 \ln 3 - \frac{1}{2}e^{\ln 9} - 0 + \frac{1}{2}\right)$$
$$= \frac{1}{2}\left(9 \ln 3 - \frac{9}{2} + \frac{1}{2}\right) = \frac{9}{2}\ln 3 - 2$$

(4) (a) The region R in the first quadrant within the circle $x^2 + y^2 = 9$ is described by the simple polar region $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 3$. Hence,

$$\int\int_R (9 - x^2 - y^2)^{1/2} \; dA = \int_0^{\frac{\pi}{2}} \int_0^3 \left(9 - r^2 \cos^2\theta - r^2 \sin^2\theta\right)^{\frac{1}{2}} r \; dr \; d\theta$$
$$= \int_0^{\frac{\pi}{2}} \int_0^3 \left(9 - r^2\right)^{\frac{1}{2}} r \; dr \; d\theta = \int_0^{\frac{\pi}{2}} -\frac{1}{3}\left(9 - r^2\right)^{\frac{3}{2}} \Big|_0^3 d\theta$$
$$= \int_0^{\frac{\pi}{2}} 9 \; d\theta = 9\theta \Big|_0^{\frac{\pi}{2}} = \frac{9\pi}{2}$$

(b) We express the region of integration R enclosed by the circle $x^2 + y^2 = 4$ as a simple polar region.

$$R: \quad -\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}, \quad -2 \le y \le 2$$
$$R: \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2$$

Thus,

$$\begin{aligned}
\int_{-2}^{2}\int_{-(4-y^2)^{1/2}}^{(4-y^2)^{1/2}} e^{-(x^2+y^2)}\ dx\ dy &= \int_0^{2\pi}\int_0^2 e^{-(r^2\cos^2\theta+r^2\sin^2\theta)} r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 e^{-r^2} r\ dr\ d\theta = \int_0^{2\pi} -\frac{e^{-r^2}}{2}\bigg|_0^2\ d\theta \\
&= \int_0^{2\pi} -\frac{e^{-4}}{2}+\frac{1}{2}\ d\theta = \frac{1-e^{-4}}{2}\theta\ \bigg|_0^{2\pi} \\
&= \pi\left(1-\frac{1}{e^4}\right)
\end{aligned}$$

(5) The surface area of the portion of $2x + 2y + z = 8$ in the first octant that is cut off by the three coordinate planes is given by