AMTH250 Assignment 5

Due: 12th September

Question 1 *[5 marks]*

Consider the following system of linear equations:

$$\begin{aligned}
x_2 &= x_6 \\
x_3 &= 10 \\
\alpha x_1 &= x_4 + \alpha x_5 \\
\alpha x_1 + x_3 + \alpha x_5 &= 0 \\
x_4 &= x_8 \\
x_7 &= 0 \\
\alpha x_5 + x_6 &= \alpha x_9 + x_{10} \\
\alpha x_5 + x_7 + \alpha x_9 &= 15 \\
x_{10} &= x_{13} \\
x_{11} &= 20 \\
x_8 + \alpha x_9 &= \alpha x_{12} \\
\alpha x_9 + x_{11} + \alpha x_{12} &= 0 \\
x_{13} + \alpha x_{12} &= 0
\end{aligned}$$

where $\alpha = \sqrt{2}/2$.

(a) Solve these equations in Octave and estimate the error in your solution.

(b) The exact solution is

$$\begin{array}{lllll}
x_1 = -20\sqrt{2}, & x_2 = 20, & x_3 = 10, & x_4 = -30, & x_5 = 10\sqrt{2}, \\
x_6 = 20, & x_7 = 0, & x_8 = -30, & x_9 = 5\sqrt{2}, & x_{10} = 25, \\
x_{11} = 20, & x_{12} = -25\sqrt{2}, & x_{13} = 25 & &
\end{array}$$

Determine the error in the Octave solution. How does this compare to the estimated error?

(c) Find the *real* eigenvalues of the matrix and their corresponding eigenvectors.

Question 2 *[6 marks]*

The **Hilbert matrix** is the $n \times n$ matrix

$$\mathbf{H}_n = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

These matrices are notoriously ill-conditioned and are often used to test numerical algorithms. The Octave command `hilb(n)` returns the $n \times n$ Hilbert matrix.

(a) Plot a graph of $\log_{10}$ of the condition number of the Hilbert matrix as a function of n for $n = 1, \ldots, 30$.

(b) How does the condition number change with n?

(c) Suppose we solve a linear system $\mathbf{H}_n\mathbf{x} = \mathbf{b}$. How large can n be if we require a relative error in the solution of less than 10^{-4}?

(d) Find the largest and smallest magnitude eigenvalues of $\mathbf{H}_n$ for $n = 1, \ldots, 30$. How do these change with n?

(e) The eigenvalues of $\mathbf{H}_n$ are known to be real and positive. For what range of n is this true of the computed eigenvalues of $\mathbf{H}_n$.

(f) Produce a table of the condition number, largest magnitude eigenvalue and smallest magnitude eigenvalue of $\mathbf{H}_n$ for $n = 1, \ldots, 12$. What is the relationship between these three numbers?

(g) For what range of n are the results of **(a)** and **(d)** reliable?

Question 3 *[5 marks]*

A plant manager is planning a week's production to manufacture five products A, B, C, D and E. Products can be produced in any combination, except that the plant has already accepted an order for 20 units of product C and 30 units of product D; so least these amounts of the two products must be made. The manufacture of each of the five products requires time on three machines M_1, M_2 and M_3. Each machine is available for 80 hours per week.

Machines M_1 and M_2 cost \$9 per hour to operate, machine M_3 costs \$12 per hour to operate. There is no standby costs for a machine not being used. The table below gives the processing time (minutes), the selling price and material costs for one unit of each product.

Product	Required time on machine (mins)			Selling Price ($)	Materials Cost ($)
	M_1	M_2	M_3		
A	15	8	6	12.00	2.00
B	8	10	9	11.00	2.00
C	8	12	10	12.00	2.00
D	12	4	12	10.50	1.00
E	9	4	0	6.00	1.00

How many units of each product should be produced to maximize profit from the plant?

Hints on the Assignment

Question 1

Before starting this question make sure you know how use script files in Octave.

(a) Here we have a system of 13 equations in 13 unknowns, so we have to construct a 13×13 matrix and a 13 dimensional vector for the right-hand-side. It is best to enter the problem into Octave using a script file containing the matrix and right hand side. This allows us to easily make corrections by editing the script file.

The script file will look like:

```
k = sqrt(2)/2;      // alpha

a = zeros(13,13);   // initialize matrix and rhs to zeros
b = zeros(13,1);

a(1,2) = 1;         // 1st equation
a(1,6) = -1;

a(2,3) = 1;         // 2nd equation
b(2)   = 10;

.......
```

We just have to enter the non-zero components of the matrix `a` and the right-hand-side `b`. Note that the first equation

$$x_2 = x_6$$

is written

$$x_2 - x_6 = 0$$

so `a(1,2) = 1` and `a(1,6) = -1`, etc.

Once the script has been executed, we can solve the linear system by `x = a\b`.

The error in the error in solution is estimated from the condition number of the matrix.

Question 2

(a) You will have to write a `for` loop to produce data for the graph, e.g.

```
chilb =zeros(30);
for n = 1:30
  chilb(n) = cond(hilb(n));
end
```

Then `semilogy(chilb)` will produce the graph (n.b. if `plot` etc. are given a single vector argument they will plot the components of the vector against the indices of the vector.)

(b) Be precise. Simply saying the condition number increases with n is not enough.

(d) You will need to write a `for` loop to produce the data as in **(a)**.

(f) You don't need to produce a LaTeX table, an Octave matrix containing the data will do.

(g) Look at the graphs in **(a)** and **(d)** in light of your answer to **(e)**.

Question 3

There are two steps to solving the assignment question:

1. Formulate the problem as a linear programming problem. This is the difficult part.
2. Solve the linear programming problem in Octave. This step is pretty mechanical once the problem is formulated correctly.

To formulate the linear programming problem follow the steps given in the notes:

1. Identify the Variables:
   There are just 5 variables, the amount of each product to produce.
2. Write down the objective:
   The objective is to maximize the profit. For each of the 5 products, the profit per unit is the selling cost minus the materials cost minus the machine costs. The machines costs are determined by the time on each machine and the cost of operating the machine.
3. Write down the constraints:
   Besides the constraints on the minimum of each product produced, the only constraints are that each machine is limited to 80 hours operation. The time each machine is in operation is determined by the number of units of each product produced.

Check your answer makes sense!