Sample Solutions for Tutorial 9

Question 1.

Let $\lambda_1, \ldots, \lambda_m$ be pair-wise distinct eigenvalues of $T : V \to V$. Suppose that $\mathbf{v}_j$ is an eigenvector for λ_j $(j = 1, \ldots, m\}$. We use mathematical induction to show that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ must be linearly independent.

$\mathbf{m = 1}$: Since, by definition, $\mathbf{v}_1 \neq \mathbf{0}_V$, it is linearly independent.

$\mathbf{m \geq 1}$: We make the inductive hypothesis that if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are eigenvectors to the pair-wise distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ of $T : V \to V$, then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ are eigenvectors to the pair-wise distinct eigenvalues $\lambda_1, \ldots, \lambda_{m+1}$ of $T : V \to V$, and suppose that

$$\text{(i)} \qquad \sum_{j=1}^{m+1} \alpha_j \mathbf{v}_j = \mathbf{0}_V,$$

for some $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{F}$, so that

$$\text{(ii)} \qquad \alpha_{m+1}\mathbf{v}_{m+1} = -\sum_{j=1}^{m} \alpha_j \mathbf{v}_j$$

It follows from (i), that

$$\text{(iii)} \qquad \sum_{j=1}^{m} \alpha_j \lambda_j \mathbf{v}_j = \sum_{j=1}^{m} \alpha_j T(\mathbf{v}_j) = T(\sum_{j=1}^{m} \alpha_j \mathbf{v}_j) = T(\mathbf{0}_V) = \mathbf{0}_V,$$

whence

$$\text{(iv)} \qquad \alpha_{m+1}\lambda_{m+1}\mathbf{v}_{m+1} = -\sum_{j=1}^{m} \alpha_j \lambda_j \mathbf{v}_j$$

But from (ii)

$$\text{(v)} \qquad \alpha_{m+1}\lambda_{m+1}\mathbf{v}_{m+1} = -\sum_{j=1}^{m} \alpha_j \lambda_{m+1} \mathbf{v}_j.$$

Subtracting (v) from (iv), we see that

$$\sum_{j=1}^{m} \alpha_j (\lambda_j - \lambda_{m+1}) \mathbf{v}_j = \mathbf{0}_V.$$

But by the inductive hypothesis $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent, so that

$$\alpha_j(\lambda_j - \lambda_{m+1}) = 0 \qquad\qquad (j = 1, \ldots, m),$$

whence $\alpha_j = 0$ for $j = 1, \ldots, m$, since $\lambda_j \neq \lambda_{m+1}$ for $j \neq m+1$. Thus $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ are linearly independent, which completes the proof by induction.

Question 2.

We observe that if $k \neq 0$, then $\underline{\mathbf{A}}\mathbf{x} = \lambda\mathbf{x}$ if and only if $k\underline{\mathbf{A}}\mathbf{x} = k\lambda\mathbf{x}$.

(a) Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (2x - z, 2y - z, \frac{-x-y+4z}{2})$. Then matrix of T with respect to the standard basis for $\mathbb{R}^3$ is

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ \frac{-1}{2} & \frac{-1}{2} & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -1 & -1 & 4 \end{bmatrix} = \frac{1}{2}\underline{\mathbf{B}},$$

so that λ is an eigenvalue for $\underline{\mathbf{A}}$ if and only if $\mu = 2\lambda$ is an eigenvalue for $\underline{\mathbf{B}}$, and the eigenvectors of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ clearly coincide. We therefore apply elementary row operations to $\underline{\mathbf{B}} - \mu\underline{\mathbf{1}}_3$.

$$\begin{bmatrix} 4-\mu & 0 & -2 \\ 0 & 4-\mu & -2 \\ -1 & -1 & 4-\mu \end{bmatrix} \quad \underset{\rightsquigarrow}{R_1 + (4-\mu)R_3} \quad \begin{bmatrix} 0 & \mu-4 & \mu^2-8\mu+14 \\ 0 & 4-\mu & -2 \\ -1 & -1 & 4-\mu \end{bmatrix}$$

$$\underset{\rightsquigarrow}{R_1 + R_2} \quad \begin{bmatrix} 0 & 0 & \mu^2-8\mu+12 \\ 0 & 4-\mu & -2 \\ -1 & -1 & 4-\mu \end{bmatrix}$$

Since $\mu^2 - 8\mu + 12 = (\mu-2)(\mu-6)$, the eigenvalues of $\underline{\mathbf{B}}$ are $\mu = 2, 4, 6$ whence those of $\underline{\mathbf{A}}$ are $\lambda = 1, 2, 3$. We determine the eigenvectors, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\lambda = \mathbf{1}$: The defining equations for eigenvectors are

$$-2y + 2z = 0, \; 2y - 2z = 0 \;\text{ and }\; -x - y + 2z = 0,$$

so that the corresponding eigenvectors are

$$r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (r \in \mathbb{R}).$$

$\lambda = \mathbf{2}$: The defining equations for eigenvectors are

$$-2z = 0, \; -2z = 0 \;\text{ and }\; -x - y = 0,$$

so that the corresponding eigenvectors are

$$s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad (s \in \mathbb{R}).$$

$\lambda = \mathbf{3}$: The defining equations for eigenvectors are

$$2y + 2z = 0, \; -2y - 2z = 0 \;\text{ and }\; -x - y - 2z = 0,$$

so that the corresponding eigenvectors are

$$t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad (t \in \mathbb{R}).$$

Since there are three distinct eigenvalues, $\{(1,1,1), (1,-1,0), (1,1,-1)\}$ is a basis for $\mathbb{R}^3$ and the matrix of T with respect to this basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$, $(x,y,z) \mapsto (\frac{7x+3y-2z}{4}, \frac{-3x+9y+2z}{4}, \frac{-2x-2y-8z}{4})$. Then matrix of T with respect to the standard basis for $\mathbb{R}^3$ is

$$\underline{\mathbf{A}} = \frac{1}{4} \begin{bmatrix} 7 & 3 & -2 \\ -3 & 9 & 2 \\ -2 & 2 & 8 \end{bmatrix}.$$

Put $\underline{\mathbf{B}} := 4\underline{\mathbf{A}}$, so that λ is an eigenvalue for $\underline{\mathbf{A}}$ if and only if $\mu = 4\lambda$ is an eigenvalue for $\underline{\mathbf{B}}$, and the eigenvectors of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ clearly coincide. We therefore apply elementary row operations to $\underline{\mathbf{B}} - \mu\underline{\mathbf{1}}_3$.

$$\begin{bmatrix} 7-\mu & 3 & -2 \\ -3 & 9-\mu & 2 \\ -2 & 2 & 8-\mu \end{bmatrix} \quad \begin{matrix} R_2 - R_3 \\ \rightsquigarrow \end{matrix} \quad \begin{bmatrix} 7-\mu & 3 & -2 \\ -3 & 9-\mu & 2 \\ 1 & \mu-7 & 6-\mu \end{bmatrix}$$

$$\begin{matrix} R_1 + (\mu-7)R_3 \\ \rightsquigarrow \\ R_2 + 3R_3 \end{matrix} \quad \begin{bmatrix} 0 & \mathbf{u}^2 - 14\mu + 52 & -\mu^2 + 13\mu - 44 \\ 0 & 2 - 12\mu & 20 - 3\mu \\ 1 & \mu - 7 & 6 - \mu \end{bmatrix}$$

$$\begin{matrix} R_1 + R_1 \\ \rightsquigarrow \end{matrix} \quad \begin{bmatrix} 0 & \mathbf{u}^2 - 16\mu + 64 & -\mu^2 + 16\mu - 64 \\ 0 & 2 - 12\mu & 20 - 3\mu \\ 1 & \mu - 7 & 6 - \mu \end{bmatrix}$$

Since

$$\det\left(\begin{bmatrix} (\mu-8)^2 & -(\mu-8)^2 \\ 2\mu - 12 & 20 - 3\mu \end{bmatrix}\right) = -(\mu-8)^3,$$

the only eigenvalue of $\underline{\mathbf{B}}$ is $\mu = 8$, hence of $\underline{\mathbf{A}}$, $\lambda = 2$.

Now the defining equations for eigenvectors are

$$4y - 4z = 0 \qquad \text{and} \qquad x + y - 2z = 0$$

Hence, the eigenvectors are

$$r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (r \in \mathbb{R}).$$

Since any two eigenvectors are linearly dependent, there is no basis for $\mathbb{R}^3$ consisting of eigenvectors for T, and hence no basis with respect to which the matrix of T is in diagonal form.

(c) Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (\frac{3x+y}{2}, \frac{-x+5y}{2}, \frac{-x+y+4z}{2})$. Then matrix of T with respect to the standard basis for $\mathbb{R}^3$ is

$$\underline{\mathbf{A}} = \frac{1}{2}\begin{bmatrix} 3 & 1 & 0 \\ -1 & 5 & 0 \\ -1 & 1 & 4 \end{bmatrix}.$$

Put $\underline{\mathbf{B}} := 2\underline{\mathbf{A}}$, so that λ is an eigenvalue for $\underline{\mathbf{A}}$ if and only if $\mu = 2\lambda$ is an eigenvalue for $\underline{\mathbf{B}}$, and the eigenvectors of $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ clearly coincide. We therefore apply elementary row operations to $\underline{\mathbf{B}} - \mu\underline{\mathbf{1}}_3$.

$$\begin{bmatrix} 3-\mu & 1 & 0 \\ -1 & 5-\mu & 0 \\ -1 & 1 & 4-\mu \end{bmatrix} \quad \begin{matrix} R_1 + (3-\mu)R_2 \\ \rightsquigarrow \\ R_3 - R_2 \end{matrix} \quad \begin{bmatrix} 0 & (\mu-4)^2 & 0 \\ -1 & 5-\mu & 0 \\ 0 & \mu-4 & 4-\mu \end{bmatrix}$$

Thus the only eigenvalue of $\underline{\mathbf{B}}$ is $\mu = 4$, hence of $\underline{\mathbf{A}}$, $\lambda = 2$.

Now the defining equation for eigenvectors is

$$-x + yy - 4z = 0$$

Hence, the eigenvectors are

$$r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (r, s \in \mathbb{R}).$$

Since any three eigenvectors are linearly dependent, there is no basis for $\mathbb{R}^3$ consisting of eigenvectors for T, and hence no basis with respect to which the matrix of T is in diagonal form.

Question 3.

(i) Take $T : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty,\ f \mapsto \frac{df}{dx}$. Then $\lambda \in \mathbb{R}$ is an eigenvalue if and only if the ordinary differential equation

$$\text{(a)} \qquad \frac{df}{dx} = \lambda f$$

has a non-trivial solution.

Now given $\lambda \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R},\ x \mapsto e^{\lambda x}$ is a non-trivial solution of (a).

Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary solution of (a), and define

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longrightarrow \frac{h(x)}{e^{\lambda x}}$$

Since $e^{\lambda x} \neq 0$ for all $\lambda, x \in \mathbb{R}$, φ is everywhere differentiable, and

$$\varphi'(x) = \frac{h'(x)e^{\lambda x} - h(x)\lambda e^{\lambda x}}{e^{2\lambda x}} = 0,$$

as $h'(x) = \lambda h(x)$.

Thus, by the Mean Value Theorem of Calculus, there is an $A \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\varphi(x) = A$. Hence $h(x) = Ae^{\lambda x}$ for all $Rx \in \mathbb{R}$.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of T, and h is an eigenvector for λ if and only if $h(x) = Ae^{\lambda x}$ for some $A \in \mathbb{R}$.

(ii) Take $T : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty,\ f \mapsto \frac{d^2f}{dx2}$. Then $\lambda \in \mathbb{R}$ is an eigenvalue if and only if the ordinary differential equation

$$\text{(b)} \qquad \frac{d^2f}{dx^2} = \lambda f$$

has a non-trivial solution.

We consider three cases separately.

$\lambda = \mathbf{0}$: It follows from two successive applications of the Fundamental Theorem of Calculus that there are $A, B \in \mathbb{R}$ with

$$f(x) = Ax + B$$

for all $x \in \mathbb{R}$.

$\lambda > \mathbf{0}$: Then $\lambda = k^2\mathrm{t}$ for some $k \in \mathbb{R}^+$.

Let f be a solution of (b) and put $g := \frac{df}{dx} - kf$.

Then

$$\frac{dg}{dx} = \frac{d^2f}{dx^2} - k\frac{df}{dx} = -kg$$

as $\frac{d^2f}{dx^2} = k^2 f$ by hypothesis.

Hence, by (i), $g(x) = A_1e^{-kx}$ for some $A_1 \in \mathbb{R}$, so that

$$\frac{df}{dx} - kf =_1 e^{-kx},$$

or, equivalently,

$$\frac{df}{dx}e^{-kx} - ke^{-kx}f = A_1e^{-2kx},$$

that is to say,

$$\frac{d}{dx}(e^{-kx}f(x)) = A_1e^{-2kx},$$

whence, by the Fundamental Theorem of Calculus, $e^{-kx}f(x) = \frac{-A_1}{2k}e^{-2kx} + B$, for some $B \in \mathbb{R}$. So, putting $\underline{\mathbf{A}} := \frac{-A_1}{2k}$, we see that

$$f(x) = Ae^{-kx} + Be^{kx}$$

for all $x \in \mathbb{R}$.

$\lambda < \mathbf{0}$: Then $\lambda = -k^2\mathrm{t}$ for some $k \in \mathbb{R}^+$.

Clearly $\cos_k : \mathbb{R} \to \mathbb{R},\ x \mapsto \cos(kx)$ is one solution.

Let f be a solution of (b) and put $\varphi := \frac{f}{\cos_k}$.

[Notice that this introduces some "singularities": if $x = (2n+1)\frac{\pi}{2}$ for some integer n, then g is not defined. In other words, we have a function

$$\varphi : \mathbb{R} \setminus \{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\} \to \mathbb{R},$$

instead of $\varphi : \mathbb{R} \to \mathbb{R}$.

Our strategy is to solve the equation on $\mathbb{R} \setminus \{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}$. It follows from general "topological" considerations that a solution on this set has at most one extension to a solution on $\mathbb{R}$, and we shall easily see that our solutions do, indeed, have such an extension.]

We have $f(x) = \varphi(x)\cos(kx)$, so that

$$\begin{aligned} f'(x) &= \varphi'(x)\cos(kx) - k\varphi(x)\sin(kx) \\ f''(x) &= \varphi''(x)\cos(kx) - 2k\varphi'(x)\sin(kx) - k^2\varphi(x)\cos(kx). \end{aligned}$$

But $f'' = -k^2 f$, whence

$$\varphi''(x)\cos(kx) - 2k\varphi'(x)\sin(kx) = 0.$$

Thus

$$\varphi''(x)\cos^2(kx) - 2k\varphi'(x)\sin(kx)\cos(kx) = 0,$$

or, equivalently,

$$\frac{d}{dx}\left(\cos^2(kx)\varphi'(x)\right) = 0.$$

So, by the Fundamental Theorem of Calculus,

$$\cos^2(kx)\varphi'(kx) = A_1,$$

for some $A_1 \in \mathbb{R}$. Hence

$$\varphi'(x) = A_1\sec^2(kx) = \frac{A_1}{k}\frac{d}{dx}(\tan(kx)),$$

whence, by the Fundamental Theorem of Calculus, $\varphi(x) = A\tan(kx) + B$, for $A := \frac{A_1}{k}$ and some $B \in \mathbb{R}$. Thus

$$f(x) = \cos(kx)\varphi(x) = A\sin(kx) + B\cos(kx),$$

which is clearly well defined for all $x \in \mathbb{R}$ and satisfies (b) on all of $\mathbb{R}$.

Summarising, we have shown that every real number is an eigenvalue of

$$T : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty, \quad f \longmapsto f'',$$

and that a basis for the eigenspace of λ is

$$\begin{array}{ll} \{x, 1\} & \text{if } \lambda = 0 \\ \{e^{kx}, e^{-kx}\} & \text{if } \lambda = k^2 \text{ for some } k > 0 \\ \{\cos(kx), \sin(kx)\} & \text{if } \lambda = -k^2 \text{ for some } k > 0 \end{array}$$

(iii) Take $T : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty,\ f \mapsto f'' - 4f'$. Then $\lambda \in \mathbb{R}$ is an eigenvalue if and only if the ordinary differential equation $f'' - 4f' = \lambda f$, or, equivalently,

$$\text{(c)} \qquad \frac{d^2 f}{dx^2} - 4\frac{df}{dx} + 4f = (\lambda + 4)f$$

has a non-trivial solution.

But (c) is equivalent to

$$e^{-2x}f''(x) - 4e^{-2x}f'(x) + 4e^{-2x}f(x) = (\lambda + 4)e^{-2x}f(x),$$

which, in turn, is equivalent to

$$\text{(d)} \qquad \frac{d^2}{dx^2}\left(e^{-2x}f(x)\right) = \mu e^{-2x}f(x),$$

where $\mu := \lambda + 4$.

Putting $g(x) := e^{-2x}f(x)$, (d) becomes

$$g'' = \mu g,$$

so that by (iii)

$$
\begin{aligned}
g(x) &= A\cos(kx) + B\sin(kx) && \text{if } \mu = -k^2 \text{ for some } k > 0\\
g(x) &= Ax + B && \text{if } \mu = 0\\
g(x) &= Ae^{kx} + Be^{-kx} && \text{if } \mu = k^2 \text{ for some } k > 0
\end{aligned}
$$

But $\mu = \lambda + 4$ and $f(x) = e^{2x}g(x)$, so

$$
\begin{aligned}
f(x) &= e^{2x}\left(A\cos(kx) + B\sin(kx)\right) && \text{if } \lambda = 4 - k^2 \text{ for some } k > 0\\
f(x) &= e^{-2x}(Ax + B) && \text{if } \lambda = 4\\
f(x) &= Ae^{(2+k)x} + Be^{(2-k)x} && \text{if } \lambda = 4 + k^2 \text{ for some } k > 0
\end{aligned}
$$

COMMENT. Compare this last problem with the examples in the chapter in your notes titled *Introductory Examples.*