The Principle of Mathematical Induction

The Principle of Mathematical Induction can provide a convenient strategy for proving particular statements of a particular form.

We concentrate here on a restricted form of this principle, but shall discuss more general forms at the end.

The Natural Numbers

We denote by $\mathbb{N}$ the set of all natural numbers, so that

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

The natural numbers have a natural ordering, $\leq$, as we all learnt at school.

The Principle of Mathematical Induction we use rests on an elementary property of $\mathbb{N}$, the set of all natural numbers, namely,

> Each natural number has a successor.

In other words,

Lemma 1. *Let A be a non-empty set of natural numbers. Then A has a first element.*

Before we turn to proving this, we explain what we mean by the *first element* (or *least element* or *minimum*) of A:

> $a \in A$ is the *first element* of A if and only if given any $x \in A$, $a \leq x$.

Proof. We prove the result by providing an algorithm for finding the first element of A.

Since $A \neq \emptyset$, we may choose a natural number $n \in A$.

We define a recursive procedure for finding the first element of A.

(1) Put $k = 0$.
(2) Ask the question *Is k an element of A?*
(3) If the answer to the question is "yes", then k is the first element of A.
(4) If the answer to the question is "no", then replace k by $k + 1$ and return to **(2)**.

The only possible problem that could conceivably arise is that the answer to the question is never "yes".

The condition that A be non-empty is precisely what is necessary and sufficient to avert this. We know that $n \in A$, so the answer to the question cannot be "no" more than n times. □

The form of the Principle of Mathematical Induction we use is, in reality, merely a reformulation of this.

Theorem 2 (**Principle of Mathematical Induction**)**.** *Let S be a subset of the set of all natural numbers, $\mathbb{N}$.*

Suppose that

(i) $0 \in S$ *and*

(ii) $n+1 \in S$ *whenever* $n \in S$.

Then $S = \mathbb{N}$.

Proof. We prove this indirectly (by contradiction).
Let A be the set of of natural numbers, that are not elements of S.
Suppose that $A \neq \emptyset$. Then, by Lemma 1, A has a first element, say m.
Since $x \in A$ if and only if $x \notin S$ for any natural number x, we see from (i) that $0 \notin A$.
Thus m must be of the form $n+1$ for some natural number n.
Since m is the first element in A and $n < m$, we must have $n \in S$.
But then, by (ii), $m = n+1 \in S$, contradicting the hypothesis that $m \notin S$
Hence $A = \emptyset$, or, equivalently, $S = \mathbb{N}$. □

Remark. (i) is often referred to as *anchoring the induction*, or *starting the induction*, and (ii) as *the inductive step*.

The Principle of Mathematical Induction is frequently used to prove that a proposition concerning the natural n, say $P(n)$, holds for every natural number. One strategy is to let S be the set of all natural number for which the proposition is true:

$$S := \{n \in \mathbb{N} \mid P(n) \text{ for which } P(n) \text{ is true}\}$$

If we can show that $P(0)$ is true, and that $P(n+1)$ is true whenever $P(n)$ is true, then we have shown that $0 \in S$ and that $n+1 \in S$ whenever $n \in S$, and so the Principle of Mathematical Induction ensures that $S = \mathbb{N}$, or, equivalently, that $P(n)$ is true for every natural number n.

Worked examples are in the lecture notes and sample solutions to the tutorial questions.

1. A Variation

A common enough situation is that a given proposition is true for all natural numbers other than a finite number of exceptions. If the largest exception is the natural number q, then, putting $k := q+1$, we have the statement of the form

$P(n)$ is true for every natural number $n \geq k$.

We reformulate this so as to be able to apply the Principle of Mathematical Induction, by letting $Q(m)$ be the statement $P(m+k)$. Then $Q(m)$ is true for every natural number m if and only if $P(n)$ is true for every natural number $n \geq k$.

This allows us to apply the Principle of Mathematical Induction to such situation by anchoring the induction at k instead of 0, and then continuing with the inductive step.

An example is provided by the statement

If a is a positive real number and n a natural number, then $(1+a)^n > 1+n$.

This is false for $n = 0, 1$, but true for all other natural numbers.

2. A Generalisation

A little thought shows that we can reformulate the Principle of Mathematical Induction in a manner equivalent to our original version, which, as we shall show below, can be applied in more general situations.

The reformulation affects only the inductive step. Rather than requiring that if a given natural number is in the set in question, so must be the next real number, we require that if every natural number less than a given natural number

Theorem 3 (**Principle of Mathematical Induction (Alternative Form)**). *Let S be a subset of the set of all natural numbers,* $\mathbb{N}$*.*

Suppose that

(i) $0 \in S$ *and*
(ii) $m \in S$ *whenever* $\{k \in \mathbb{N} \mid k < m\} \subseteq S$*.*

Then $S = \mathbb{N}$*.*

Proof. We prove this indirectly (by contradiction).

Let A be the set of of natural numbers, that are not elements of S.

Suppose that $A \neq \emptyset$. Then, by Lemma 1, A has a first element, say m.

Since $x \in A$ if and only if $x \notin S$ for any natural number x, $\{k \in \mathbb{N} \mid k < m\} \subseteq S$.

But then, by (ii), $m \in S$, contradicting the defining property of m.

Hence $A = \emptyset$, whence $S = \mathbb{N}$. □

We show that our original formulation of the Principle of Mathematical implies the Alternative Form.

Since Theorem 2 (i) and Theorem 3 (i) are the same condition, it only remains to show that Theorem 2 implies Theorem 3 (ii) as well.

Since $P(0)$ is true we need only consider $m > 0$. So suppose that $P(k)$ is true for every natural number, k, with $k < m$, where $m > 0$. Then $m - 1 \in \mathbb{N}$, and, since $m - 1 < m$, $P(m - 1)$ is true. It now follows from Theorem 2 (ii) that $P(m)$ is also true.

We now show that the Alternative Form the Alternative Form our original formulation of the Principle of Mathematical.

Since Theorem 2 (i) and Theorem 3 (i) are the same condition, it only remains to show that Theorem 3 implies Theorem 2 (ii) as well.

Take $n \in \mathbb{N}$ and suppose $P(n)$ is true.

Since $P(0)$ is true, $P(k)$ is true for every natural number, k, with $k < 1$. Hence, by Theorem 3 $P(1)$ is true.

Since $P(0)$ and $P(1)$ are both true, $P(k)$ is true for every natural number, k, with $k < 2$. Hence, by Theorem 3 $P(2)$ is true.

Continuing in this manner, we see that $P(k)$ is true for every natural number, k, with $k < n + 1$. Hence, by Theorem 3 $P(n + 1)$ is also true.

Thus our two formulations of the Principle of Mathematical Induction are equivalent when applied to the natural numbers. However, the Alternative Form is actually more general. To see this, we need to discuss well ordered sets.

3. Mathematical Induction for Well Ordered Sets

Definition 4. The ordering $\preceq$ on the set X is a *well ordering* if and only if for all $x, y, z \in X$

(i) $x \preceq x$;
(ii) if $x \preceq y$ and $y \preceq x$, then $y = x$;
(iii) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$;
(iv) every non-empty subset of X has a first element, that is, if $\emptyset \subset A \subseteq X$, then there is an $a \in A$ with $a \preceq x$ for every $x \in A$.

Example 5. The set of natural numbers, $\mathbb{N}$, is well ordered with respect to (the usual) $\leq$.

On the other hand $\mathbb{Z}$ is not well ordered with respect to $\leq$, since there is no first integer. It follows that neither $\mathbb{Q}$ nor $\mathbb{R}$ is well ordered with respect to $\leq$, since the non-empty subset Z does not have a first element.

In any well ordered set, the element n is called the *final* or *last element* if it is the maximum, that is to say, $x \preceq n$ for all $x \in X$, or, equivalently, $\{x \in X \mid n \prec x\} = \emptyset$.

In a well ordered set, every element other than the last element, has a successor, that is if p is not the last element of X then there is an element $q \in X$ with $p \prec q$ and $\{x \in X \mid p \prec x \prec q\} = \emptyset$.

Such an element in X is called a *successor element.* The first element of a well ordered set can never be a successor element.

In the case of $\mathbb{N}$, every element other than 0 is a successor element.

But this need not be the case. I know of no elementary example, so the reader needs — at this stage — to take in on trust that the theory of infinite sets provides important examples.

In such a set, our original formulation of the Principle of Mathematical induction fails, since the inductive step only proves that if $P(x)$ is true for some element of X, $P(y)$ is also true for the successor, y, of x. But it provides no way of dealing with elements which are not successor elements.

On the other hand, every element, m, of X, successor or not, is the first element of the non-empty subset $\{x \in X \mid \ell \preceq x\}$ of X, whose complement is $\{x \in X \mid x \prec m\}$.

We can now formulate a Principle of Mathematical Induction for an arbitrary well ordered set.

Theorem 6 (Principle of Mathematical Induction for a Well Ordered Set). *Let the set X be well ordered by $\simeq$. Let f be the first element of X.*

Let S be a subset of X.

Suppose that

(i) $f \in S$ *and*
(ii) $m \in S$ *whenever* $\{k \in X \mid k \prec m\} \subseteq S$.

Then $S = X$.

Plainly, our earlier formulations are a special case of this.