AMTH250 Assignment 8

Due: 7th November

Question 1 *[3 marks]*

Evaluate the following integrals using `quad`:

$$\begin{aligned}
&\text{(a)} \quad I_1 = \int_0^\infty \frac{x^3}{e^x - 1}\, dx \\
&\text{(b)} \quad I_2 = \int_{-1}^1 \ln(1+x)\ln(1-x)\, dx \\
&\text{(c)} \quad I_3 = \int_0^\pi (\tan(\sin x) - \sin(\tan x))\, dx
\end{aligned}$$

Do you think the results are reliable?

Question 2 *[3 marks]*

The intensity of diffracted light near a straight edge is determined by the *Fresnel integrals*

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$$

and

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

Use `quad` to evaluate these integrals for enough values of x to draw smooth plots of $C(x)$ and $S(x)$ over the range $-10 \le x \le 10$. Also plot $C(x)$ against $S(x)$ — the result is the *Cornu* or *Euler spiral*.

Question 3 *[3 marks]*

Use Monte-Carlo methods to

(a) Estimate the volume of the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{16} \le 1.$$

(b) Estimate the double integral

$$\iint_\Omega e^{-\sqrt{x^2+y^2}}\,dx\,dy$$

over the semicircular region Ω defined by

$$x^2 + y^2 \leq 1, \quad x \geq 0.$$

Question 4 *[3 marks]*

The initial value problem

$$\frac{dy}{dt} = -2ty, \qquad y(0) = 1$$

has the solution

$$y(t) = e^{-t^2}$$

Apply Euler's method with various step-sizes h to this problem.

(a) Solve the initial value problem on the interval $[0, 1]$. As a measure of error, compare the computed and exact values at $t = 1$. Plot and characterize the error as a function of the step-size h.

(b) The exact solution decays to 0 as $t \to \infty$. A numerical method for this problem is *unstable* if it does not decay to 0 as $t \to \infty$. Determine the range of step-sizes h for which Euler's method is unstable for this problem.

Question 5 *[4 marks]*

The system of differential equations

$$\begin{aligned}
\frac{dy_1}{dt} &= \sigma(y_2 - y_1) \\
\frac{dy_2}{dt} &= ry_1 - y_2 - y_1y_3 \\
\frac{dy_3}{dt} &= y_1y_2 - by_3
\end{aligned}$$

was used by Lorenz as a crude model of atmospheric circulation.

Take $\sigma = 10$, $b = 8/3$ and $r = 28$ and initial values $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = 0$ and use `lsode` to solve the differential equation from $t = 0$ to $t = 100$. Plot (a) each component individually, (b) phase plots of each pair of components, and (c) a 3D plot of the curve $(y_1(t), y_2(t), y_3(t))$.

Now experiment by changing the initial conditions by a tiny amount. Describe how the graphs above and the final values $(y_1(100), y_2(100), y_3(100))$ change.

Notes on the Assignment

Question 2

You will need a `for` loop to generate the vectors of values of $C(x)$ and $S(x)$. A step of 0.01 gives reasonable results, so try something like

```
x = -10:0.01:10;
.....
for i = 1:length(x)
  % evaluate C(x(i)) & S(x(i))
end
```

Questions 3

It is easy to get these wrong. Some things to watch out for:

1. First identify the range of the random numbers for each dimension, e.g. $y \in [0, 2]$.
2. Use rand correctly, e.g. for $y \in [0, 2]$ use `2*rand(1,1)` not `rand(0,2)`.
3. Make sure you account for the volume of the region in which you generate the random numbers.

If in doubt, study the examples in the notes carefully making sure you understand each step in the computation. Beware that the lines

```
    x = rand(1,4);     % x = point in unit cube
    if (norm(x) <= 1) % point x lies in sphere
```

in the example of §1.6.3 are quite specific to that particular problem.

Questions 4

This is the same differential equation that was used as an example in the notes.

(a) Try step sizes $h = 1/2, 1/4, 1/8, \ldots$. You will need a `for` loop to generate the data. Start with something like

```
N = 16;
h = zeros(1,N);
y1 = zeros(1,N);   % values of y at t=1
for k = 1:N
  hk = 2^-k;
  h(k) = hk;
  y =euler(f, 0, 1, hk, 2^k); % 2^k steps needed to reach t=1
  y1(k) = y(end);             % y(end) is the last element of y
end
```

(b) The example at the top of p25 of the notes shows Euler's method is unstable for $h = 1$.

Question 5

1. See the example in the notes on the Lotka-Volterra equations for how to solve this sort of problem in Octave.

2. See §3.4 of the Octave notes for plotting 3D curves.

3. The initial values are `[0 1 0]`. For a small change in initial values, try adding `1e-10*randn(1,3)` for example.