PURE MATHEMATICS 212

Multivariable Calculus

ASSIGNMENT 4-SOLUTIONS

Question 1 [4 marks]

(a) The natural domain of this function is $\{x \geq y\}$. The function is continuous there being the composition of continuous functions.

(b) This function is defined for all real x, y. It is the composition of continuous cosine and a continuous inner function that is the quotient of continuous functions with non-vanishing denominator.

Question 2 [4 marks]

(a) Set $x = r\cos\theta$ and $y = r\sin\theta$. Then this limit equals

$$\lim_{r\to 0} \frac{r^4(\cos^4\theta - 16\sin^4\theta)}{r^2(1+3\sin^2\theta)} = \lim_{r\to 0} r^2 \frac{(\cos^4\theta - 16\sin^4\theta)}{(1+3\sin^2\theta)} = 0$$

because r^2 tends to 0 and the second factor remains bounded (by 17).

(b) Again, using polar coordinates this is equivalent to

$$\lim_{r\to 0} \frac{\sin r^2}{r^2} = \lim_{r\to 0} \frac{2r\cos r^2}{2r} = \cos 0 = 1.$$

Here we have used l'Hôpital's rule.

Question 3 [4 marks]

(a) Let $y = mx$. Then

$$\frac{x^3y}{2x^6+y^2} = \frac{mx^4}{2x^6+m^2x^2} = \frac{mx^2}{2x^4+m^2}.$$

Thus as $(x, y) \to (0, 0)$ along the straight line $y = mx$, the limit is 0.

Now let $y = kx^2$. Then

$$\frac{x^3y}{2x^6+y^2} = \frac{kx^5}{2x^6+k^2x^4} = \frac{kx}{2x+k^2}.$$

Thus as $(x, y) \to (0, 0)$ along the parabola $y = kx^2$, the limit is 0.

(b) Let $y = x^3$ then

$$\frac{x^3y}{2x^6+y^2} = \frac{x^6}{2x^6+x^6} = \frac{1}{3}.$$

The limit along this curve is $\frac{1}{3} \neq 0$ hence the limit does not exist.

Question 4 [2 marks] This function is well-defined when $x \neq 0$ and $y \cos z > 0$. We have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= z \frac{2xy \cos z}{x^2 y \cos z} = \frac{2z}{x} \\
\frac{\partial f}{\partial y} &= z \frac{x^2 \cos z}{x^2 y \cos z} = \frac{z}{y} \\
\frac{\partial f}{\partial z} &= \ln(x^2 y \cos z) - z \frac{x^2 y \sin z}{x^2 y \cos z} = \ln(x^2 y \cos z) - z \tan z
\end{aligned}$$

Question 5 [4 marks]

(a) For $f(x, y) = \ln(x^2 + y^2)$, we have

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(x^2 + y^2) - 4x^2}{x^2 + y^2} = \frac{-2(x^2 - y^2)}{x^2 + y^2}.$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2(x^2 + y^2) - 4y^2}{x^2 + y^2} = \frac{2(x^2 - y^2)}{x^2 + y^2} = -\frac{\partial^2 f}{\partial x^2}.$$

Thus,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

(b) For $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$, we have

$$\frac{\partial u}{\partial x} = e^x \cos y,$$

$$\frac{\partial u}{\partial y} = -e^x \sin y,$$

$$\frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y},$$

$$\frac{\partial v}{\partial y} = e^x \cos y = \frac{\partial u}{\partial x}.$$

Question 6 [2 marks]

(a) Substituting $y = 0$, we have $f(x, 0) = (x^2)^{2/3} = x^{4/3}$. Taking derivative on x, we get $f_x(x, 0) = \frac{4}{3} x^{1/3}$. Substituting $x = 0$, we obtain $f_x(0, 0) = 0$.