Tut. 6.

Find a unit vector $\vec{u}$ that is normal at $P(1,-2)$ to the level curve of $f(x,y) = 4x^2y$ through P

$\nabla f(x_0, y_0)$ is perpendicular to the level curve of $f(x,y)$ through (x_0, y_0)

$\nabla f(x,y) = \langle 8xy, 4x^2 \rangle = \langle -16, 4 \rangle$ at $(1,-2)$

$= 4 \langle -4, 1 \rangle$

$\| \nabla f(x_0, y_0) \| = 16 \sqrt{(-4)^2 + 1^2} = 16\sqrt{17}$

$$\frac{\nabla f(1,-2)}{\| \nabla f(1,-2) \|} = \left\langle -\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$$

$P(2,-3)$, $f(x,y) = 3x^2y - xy$

$\nabla f(2,-3) = \langle 6xy - y, 3x^2 - x \rangle =$

$= \langle -33, 10 \rangle$ at $(2,-3)$

$\| \nabla f(2,-3) \| = \sqrt{(-33)^2 + 10^2} = \sqrt{1189}$

$$\frac{\nabla f(2,-3)}{\| \nabla f(2,-3) \|} = \left\langle -\frac{33}{\sqrt{1189}}, \frac{10}{\sqrt{1189}} \right\rangle$$

Find all the points on the ellipsoid $2x^2+3y^2+4z^2=9$ at which the plane tangent the ellipsoid is parallel ~~p~~ to the plane

$$x-2y+3z=5$$

$$F(x,y,z)=2x^2+3y^2+4z^2-9=0$$

$$\vec{n}=\langle f_x, f_y, f_z\rangle=\langle 4x, 6y, 8z\rangle$$

must be parallel to $\langle 1,-2,3\rangle$

$4x=t \qquad x=\frac{t}{4}$

$6y=-2t \qquad y=-\frac{t}{3}$

$8z=3t \qquad z=\frac{3t}{8}$

$$2\,\frac{t^2}{16}+3\,\frac{t^2}{9}+4\,\frac{9t^2}{64}-9=0$$

$$t^2\left(\frac{6+16+3}{48}\right)=t^2\,\frac{25}{48}=9$$

$$t=\pm\frac{12\sqrt{3}}{5}$$

$$x=\pm\frac{3\sqrt{3}}{5},\quad y=\mp\frac{4\sqrt{3}}{5},\quad z=\pm\frac{9\sqrt{3}}{5}$$

answer: two points:

$$\left(\frac{3\sqrt{3}}{5}, -\frac{4\sqrt{3}}{5}, \frac{9\sqrt{3}}{5}\right) \text{ and } \left(-\frac{3\sqrt{3}}{5}, \frac{4\sqrt{3}}{5}, -\frac{9\sqrt{3}}{5}\right)$$

suppose $z = f(u)$ and $u = g(x, y)$

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{dz}{du} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix}$$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x}$$

$$(*)$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y}$$

Let $z = f(x^2 - y^2)$. Show that $y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0$

$z = f(u)$, $u = x^2 - y^2$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x} = \frac{dz}{du} \times 2x$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y} = \frac{dz}{du} \times (-2y)$$

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 2xy \frac{dz}{du} - 2xy \frac{dz}{du} = 0$$

Let $z = f(xy)$. Show that $x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 0$

$z = f(u)$, $u = xy$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x} = \frac{dz}{du} y$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y} = \frac{dz}{du} x$$

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = xy \frac{dz}{du} - xy \frac{dz}{du} = 0$$

Assume $F(x,y)=0$ defines y implicitly as a function of x. Show that $\frac{\partial y}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y}$

Denote $V(x,y) = F(x, y(x)) = 0$

$$\frac{\partial V}{\partial x} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} \quad \text{(chain rule)}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Find $\frac{dy}{dx}$

$x^2y^3 + \cos y = 0$

$F(x,y) = x^2y^3 + \cos y$

$\frac{\partial F}{\partial x} = 2xy^3$, $\frac{\partial F}{\partial y} = 3x^2y^2 - \sin y$

So $\frac{dy}{dx} = -\frac{2xy^3}{3x^2y^2 - \sin y}$

$e^{xy} + ye^y = 1$

$F(x,y) = e^{xy} + ye^y - 1 = 0$

$\frac{\partial F}{\partial x} = ye^{xy}$, $\frac{\partial F}{\partial y} = xe^{xy} + e^y + ye^y$

$$\frac{dy}{dx} = -\frac{ye^{xy}}{xe^{xy} + e^y + ye^y}$$