MATH101 ASSIGNMENT 2

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(1) To prove $(A \cup B)' = A' \cap B'$ we divide the proof into three parts.
For $A, B \subseteq X$:
- Let $x \in (A \cup B)'$. This implies $x \notin (A \cup B)$ by definition of complement. Then, $x \notin A$ and $x \notin B$ because if x belonged to either A or B, then it would belong to their union.
By definition of $\notin$, $x \in A'$ and $x \in B'$.
Thus, $x \in (A' \cap B')$ by definition of intersection.
This proves $(A \cup B)' \subseteq A' \cap B'$.
- Let $x \in (A' \cap B')$. This implies $x \in A'$ and $x \in B'$ by definition of intersection.
By definition of complement, $x \notin A$ and $x \notin B$.
Hence, $x \notin (A \cup B)$ and consequently, $x \in (A \cup B)'$.
This then proves $A' \cap B' \subseteq (A \cup B)'$.
- Since $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$, then

$$(A \cup B)' = A' \cap B'$$

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(2) (a) $\inf(X_1) = 0 \notin X_1$ and $\sup(X_1) = 1 \in X_1$
- Assuming $n \in \mathbb{N}\backslash\{0\}$ we can see that every element of X_1 is of the form $\frac{1}{n^2}$. The elements are therefore arranged in strictly decreasing order, with the first element, 1, also being the largest. Hence X_1 is bounded above and has a supremum 1, which is contained in the set.
- We also observe that every element of X_1 is positive. It follows immediately that the set is bounded below by 0.
- To show that the infimum of the set is 0 we use the Principle of Mathematical Induction. Firstly, we show $n^2 \geq n$ for every $n \in \mathbb{N}\backslash\{0\}$.
$n = 1$:

$$1^2 = 1 \geq 1$$

$n > 1$: We make the inductive hypothesis that $n^2 \geq n$. Then

$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 2n + 1 \\ &= 3n + 1 \\ &> n + 1 \end{aligned}$$

- Assume $x > 0$. Then $\frac{1}{x} > 0$. Since $n^2 > n$ for every natural number n, n^2 grows without bound as n increases. Hence we can find a natural number, say m, with $m^2 > \frac{1}{x}$ or equivalently, $0 < \frac{1}{m^2} < x$.
- Since $\frac{1}{m^2} \in X_1$, we have shown that x is not a lower bound for X_1. Thus $\inf(X_1) = 0$. Finally, since $0 \notin X_1$, the infimum is not contained in the set.

(b) $\inf(X_2) = -1 \in X_2$ and $\sup(X_2) = 10 \notin X_2$
- X_2 is equivalent to the interval $[-1, 10)$. Thus, the set is bounded below with infimum 1 and bounded above with supremum 10. Since $x \geq -1$, the infimum is contained in the set. The supremum is not, however, since $x < 10$.

(c) $\inf(X_3) = 0 \notin X_3$ and X_3 has no supremum
- In order for $\frac{1}{1-x}$ to be defined, we must have $x \neq 1$. It follows that either $x < 1$ or $x > 1$.
- In the former case,

$$\frac{1}{1-x} > 0 \iff 1 > 0(1-x) = 0$$

which is clearly always true since $1 > 0$.
- In the latter case,

$$\frac{1}{1-x} < 0 \iff 1 < 0(1-x) = 0$$

which is a contradiction since $1 \not< 0$.
- Hence the inequality $\frac{1}{1-x} > 0$ is true if and only if $x < 1$. We can then express the set equivalently as $X_3 := \{\frac{1}{1-x} | x \in \mathbb{R} \text{ and } x < 1\}$
- Since $x < 1$, then $1 - x > 0$ and thus $\frac{1}{1-x} > 0$. It follows that X_3 is bounded below by 0 and that the infimum is not contained in the set.
- We also observe that $\frac{1}{1-x}$ grows without bound as x approaches 1 from the left. Thus the set has no supremum.

(3) (i) $(2-i)(3+4i) = 6 + 8i - 3i - 4i^2 = 10 + 5i$

(ii) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = (1-3) + (3i - i) = -2 + 2i$

(iii) $\frac{1-i}{2+3i} = \frac{1-i}{2+3i} \times \frac{2-3i}{2-3i} = \frac{2-3i-2i+3i^2}{4-9i^2} = \frac{2-3-5i}{13}$

$: = -\frac{1}{13} - \frac{5}{13}i$

(iv) $\frac{1}{2-3i} + \frac{1}{2+i} = \frac{2+i+2-3i}{(2-3i)(2+i)} = \frac{4-2i}{4+2i-6i-3i^2} = \frac{4-2i}{7-4i}$

$: = \frac{4-2i}{7-4i} \times \frac{7+4i}{7+4i} = \frac{28+16i-14i+8}{49+16} = \frac{36+2i}{65}$

$: = \frac{36}{65} + \frac{2}{65}i$

(4) Let $z = a + bi$ for $a, b \in \mathbb{R}$ such that $z^2 = a^2 - b^2 + 2abi = 6 - 8i$. Equating the real and imaginary parts give $a^2 - b^2 = 6, 2ab = -8$ and $b = -\frac{4}{a}$. Then,

$$a^2 - \left(\frac{-4}{a}\right)^2 = 6 \iff a^4 - 6a^2 - 16 = 0$$

Solving for a^2 using the quadratic formula gives

$$a^2 = \frac{6 \pm \sqrt{36 - 4 \times 1 \times (-16)}}{2}$$

$$a^2 = \frac{6 \pm \sqrt{100}}{2}$$

$$= 3 \pm 5$$

We could also have factorised the equation to give $(a^2 - 8)(a^2 + 2) = 0$. So there are two possible solutions, $a^2 = 8$ or $a^2 = -2$. But only $a^2 = 8$ is valid since a must be a real number. Therefore,

$$a = \pm 2\sqrt{2}$$

$$b = \mp \sqrt{2}$$

which means there are two solutions for z:

$$z = \pm 2\sqrt{2} \mp \sqrt{2}i$$

$$= \sqrt{2}(2 - i), \sqrt{2}(-2 + i)$$

(5) Let a be a non-zero complex number and m an integer such that

$$z^m = a$$

We express a, z and z^m in modulus-argument form,

$$\begin{aligned} a &= s(\cos\phi + i\sin\phi) \\ z &= r(\cos\theta + i\sin\theta) \\ z^m &= r^m(\cos(m\theta) + i\sin(m\theta)) \end{aligned}$$

with r, $s > 0$; $0 \leq \phi, \theta < 2\pi$; $r = |z|$ and $s = |a|$; $\phi = \arg(a)$ and $\theta = \arg(z)$.

Thus $z^m = a$ if and only if

$$\begin{aligned} r^m &= s \\ \cos(m\theta) &= \cos(\phi) \\ \sin(m\theta) &= \sin(\phi) \end{aligned}$$

with $0 \leq m\theta < 2m\pi$, whence

$$\begin{aligned} r &= s^{\frac{1}{m}} \\ m\theta &\equiv \phi \end{aligned}$$

As complex numbers have an infinite number of arguments which differ by integer multiples of 2π, we must have

$$m\theta = \phi + 2n\pi$$

with $0 \leq n < m$ where $n \in \mathbb{Z}$. This implies that

$$\theta \equiv \frac{\phi + 2n\pi}{m}$$

Hence we obtain the following expression for z

$$z = r\left(\cos\left(\frac{\phi + 2n\pi}{m}\right) + i\sin\left(\frac{\phi + 2n\pi}{m}\right)\right)$$

Now if we raise both sides of the equation $z = 1^{\frac{1}{m}}$ to the power m, it is clear to see that $z^m = 1$. So, in this case $a = 1$ and therefore $r = s = 1$. The complex number 1 can then be written in polar form as

$$1 = \cos\phi + i\sin\phi$$

In order for this to be true, ϕ must equal 0 or be an integer multiple of 2π. Equating this with our expression for θ, we obtain

$$z = \cos\left(\frac{0 + 2n\pi}{m}\right) + i\sin\left(\frac{0 + 2n\pi}{m}\right)$$

Therefore we have shown that

$$z = \cos\left(\frac{2n\pi}{m}\right) + i\sin\left(\frac{2n\pi}{m}\right)$$

whenever $z = 1^{\frac{1}{m}}$.

To find all third roots of unity, simply substitute $m = 3$ into the equation. We use $n = \{0, 1, .., m-1\}$.

$n = 0$:

$$\begin{aligned} z &= \cos(0) + i\sin(0) \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

$n = 1$:

$$\begin{aligned} z &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{aligned}$$

$n = 2$:

$$\begin{aligned} z &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\ &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$