Sample Solutions for Tutorial 2

Question 1. Let $K := \inf(A), L := \sup(A), M := \inf(B)$ and $N := \sup(B)$.

$\mathbf{A \cup B}$: Take $x \in A \cup B$, so that $x \in A$ or $x \in B$.

If $x \in A$, then $x \leq L \leq \max\{L, N\}$.

Otherwise, $x \in B$, and so $x \leq N \leq \max\{L, N\}$.

Hence, $x \leq N \leq \max\{L, N\}$ for all $x \in A \cup B$.

Thus, $A \cup B$ is bounded above by $max\{L, N\}$.

Take $S < max\{L, N\}$. Then either $S < L$ or $S < N$.

In the former case, we can find $x \in A$, with $S < x \leq L$, whence S is not an upper bound for $A \cup B$, as $A \subseteq A \cup B$.

In the latter case, we can find $x \in B$, with $S < x \leq N$, whence S is not an upper bound for $A \cup B$, as $B \subseteq A \cup B$.

Hence is the least upper bound for (supremum of) $A \cup B$.

$\mathbf{A \cap B}$: Take $x \in A \cap B$, so that $x \in A$ and $x \in B$.

Since $x \in A$, we have $x \geq K$.

Since $x \in B$, we have $x \geq M$.

Since $x \geq K, M$, we have $x \geq \max\{K, M\}$.

Thus, $A \cap B$ is bounded below by $\max\{K, M\}$. Being a set of real numbers, that is bounded below, $A \cap B$ has an infimum and $\inf(A \cap B) \geq \max\{K, M\}$.

To see that equality need not hold, let $A := \{0, 2\}$ and $B := \{1, 2\}$.

Then $\inf(A) = 0$, $\inf(B) = 1$, so that $\max\{\inf(A), \inf(B)\} = \max\{0, 1\} = 1$.

On the other hand, $\inf(A \cap B) = \inf\{2\} = 2$.

Question 2. Plainly, the sum of two numbers is the same as the sum of their maximum and their minimum, so that for real numbers a, b,

$$\max\{a, b\} + \min\{a, b\} = a + b \tag{$*$}$$

Similarly, the absolute value of their difference is the smaller subtracted from the latter, or,

$$\max\{a, b\} - \min\{a, b\} = |a - b| \tag{$**$}$$

Adding $(**)$ to $(*)$ yields $2\max\{a, b\} = a + b + |a - b|$, or

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}.$$

Subtracting $(**)$ from $(*)$ yields $2\min\{a, b\} = a + b - |a - b|$, or

$$\min\{a, b\} = \frac{a + b - |a - b|}{2}.$$

Question 3.

(i): Take $n \in \mathbb{N}$. Then

$$\begin{aligned} \frac{1}{2^{n+1}} &= \frac{1}{2}\frac{1}{2^n} \\ &< \frac{1}{2^n} && \text{as } 0 < \tfrac{1}{2} < 1. \end{aligned}$$

Hence we can arrange the elements of A in strictly decreasing order as

$$A := \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$$

It follows immediately that 1 is the largest element of A, whence A is bounded above and has a supremum, 1, which is actually its maximum.

Since every element of A is positive, it follows immediately that A is bounded below by 0.

We now use the Principle of Mathematical Induction to show that, for every $n \in \mathbb{N}$, 2^n n.

$\mathbf{n = 0, 1}$: In these cases we have

$$2^0 = 1 > 0 \quad \text{and} \quad 2^1 = 2 > 1.$$

$\mathbf{n \geq 1}$: We make the inductive hypothesis that $2^n > n$. Then

$$\begin{aligned} 2^{n+1} &= 2.2^n \\ &> 2n && \text{by the Inductive Hypothesis} \\ &= n + n \\ &\geq n + 1 && \text{as } n \geq 1. \end{aligned}$$

This completes the proof by mathematical induction.

Take $a > 0$. Then $\frac{1}{a} > 0$. Since $2^n > n$, for every natural number n, 2^n grows without bound as n increases. Hence we can find a natural number, say k, with $2^k > \frac{1}{a}$, or, equivalently, $0 < \frac{1}{2^k} < a$.

Since $\frac{1}{2^k} \in A$, we have shown that a is not a lower bound for A. Thus $\sup(A) = 0$.

Since $0 \notin A$, A has an infimum, but not a minimum;

(ii): It follows from the Archimedean property of the real numbers that every integer n can be written in precisely in one of the forms $4k, 4k+1, 4k+2$ or $4k+3$, where k is itself an integer. Then there are four possibilities for $\cos(b\frac{pi}{2})$, namely:

$\mathbf{n = 4k}$: In this case $\cos(n\frac{\pi}{2}) = \cos(2k\pi) = \cos 0 = 1$ as $\cos(x + 2\pi) = \cos x$.

[$\mathbf{n = 4k + 1}$: In this case $\cos(n\frac{\pi}{2}) = \cos(2k\pi + \frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$ as $\cos(x + 2\pi) = \cos x$.

[$\mathbf{n = 4k + 2}$: In this case $\cos(n\frac{\pi}{2}) = \cos(2k\pi + \pi) = \cos\pi = -1$ as $\cos(x + 2\pi) = \cos x$.

[$\mathbf{n = 4k + 3}$: In this case $\cos(n\frac{\pi}{2}) = \cos(2k\pi + \frac{3\pi}{2}) = \cos\frac{3\pi}{2} = 0$ as $\cos(x + 2\pi) = \cos x$.

Thus $B = \{-1, 0, 1\}$ which is,plainly, bounded, with -1 as minimum and 1 as maximum.

(iii): In order for the inequality $\frac{x}{1+x} \geq 0$, we must have $1 + x \neq 0$, or, equivalently, $x \neq -1$.

If $x\ ne - 1$, then either $x < -1$ or $x > 1$.

In the former case,

$$\frac{x}{1+x} \leq 0 \iff x \leq 0(1+x) = 0,$$

which is clearly always the case, since $-1 < 0$.

In the latter case

$$\frac{x}{1+x} \geq 0 \iff x \geq 0(1+x) = 0,$$

which is only the case when $x \geq 0$.

Hence $C = \{x \in \mathbb{R} \mid x < -1 \text{ or } x \geq 0\}$, and this is plainly bounded neither below nor above.

(iv): Observe that for $x \neq -1$

$$\frac{x}{1+x} = \frac{1+x-1}{1+x} = \frac{1+x}{1+x} - \frac{1}{1+x} = 1 - \frac{1}{1+x}.$$

Since $x \geq 0$, $1 + x \geq 1 > 0$, whence $0 < \dfrac{1}{1+x} \leq 1$. Thus, $0 \leq 1 - \frac{1}{1+x} < 1$, or, equivalently,

$$0 \leq \frac{x}{1+x} < 1.$$

It follows immediately that D is bounded above by 1 and below by 0.

Moreover, since $\frac{0}{1+0} = 0$ and $0 \geq 0$, we see that $0 \in D$, whence $\min(D) = 0$.

Choose a real number, r, with $0 < r < 1$. Then $0 < 1 - r < 1$, whence $\frac{1}{1-r} > 1$.

Choose a real number, t, with $t > \frac{1}{1-r} - 1 > 0$.

Then $1+t > \frac{1}{1-r} > 1$, whence $0 < \frac{1}{1+t} < 1-r < 1$, or equivalently, $-1 < -(1-r) < -\frac{1}{1+t} < 0$.

Thus $0 < r = 1 - (1-r) < 1 - \frac{1}{1+t} = \frac{t}{1+t} < 1$.

Since $\frac{t}{1+t} \in D$, we see that r is not an upper bound for D.

We conclude that $\sup(D) = 1$, but$D$ has no maximum.

Question 4.

(i) $(2-i)(2i) = 2^2 - i^2 = 4-(-1) = 5 = 5+0i$.
Thus $|(2-i)(2i)| = |5+0i| = 5$ and $\overline{(2-i)(2i)} = \overline{5+i0} = 5-0i = 5$.

(ii) $(6+5i)(2-7i) = (12-(-35)+i(-12+10) = 47-2i$.
Thus $|(6+5i)(2-7i)| = \sqrt{(47^2+(-2)^2)} = \sqrt{2213}$ and $\overline{(6+5i)(2-7ii)} = 47+2i$.

(iii) $\dfrac{2-i}{1+2i} = \dfrac{(2-i)(1-2i)}{(1+2i)(1-2i)} = \dfrac{(2.1-(-1)(-2))+i(2(-2)+(-1)(1))}{1+22} = \dfrac{0-3i}{5} = 0-i\dfrac{3}{5}$.
Thus $|\dfrac{2-i}{1+2i}| - i\dfrac{3}{5}| = \dfrac{3}{5}$ and $\overline{\dfrac{2-i}{1+2i}} = \dfrac{3}{5}i$

(iv)
$$\begin{aligned}
\frac{1-3i}{(2+i)^2} + \frac{1+i^3}{1+i} &= \frac{(1-3i)(2-i)^2}{((2+i)(2-i))^2} + \frac{(1-i)(1-i)}{(1+i)(1-i)} \\
&= \frac{(1-3i)(5-2i)}{25} + \frac{2-2i}{2} \\
&= \frac{-1-17i}{25} + 1 - i \\
&= \frac{24}{25} - \frac{42}{25}i
\end{aligned}$$

Thus $|\dfrac{1-3i}{(2+i)^2} + \dfrac{1+i^3}{1+i} = |\dfrac{24}{25} - \dfrac{42}{25}i| = \dfrac{6}{25}|4-7i| = \dfrac{4\sqrt{65}}{5} = 4\sqrt{\dfrac{13}{5}}$ and

$\overline{\dfrac{1-3i}{(2+i)^2} + \dfrac{1+i^3}{1+i}} = \dfrac{24}{25} + \dfrac{42}{25}i$