PMTH212 ASSIGNMENT 3

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(1)

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\begin{aligned}
&x(t) = e^t \cos t, \quad y(t) = e^t \sin t, \quad z(t) = e^t, \quad t = 0 \\
&x'(t) = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t) \;\Rightarrow\; x'(0) = 1 \\
&y'(t) = e^t \sin t + e^t \cos t = e^t(\cos t + \sin t) \;\Rightarrow\; y'(0) = 1 \\
&z'(t) = e^t \;\Rightarrow\; z'(0) = 1 \\
&||\mathbf{r}'(t)|| = \sqrt{e^{2t}(1 - 2\sin t\cos t) + e^{2t}(1 + 2\sin t\cos t) + e^{2t}} \\
&\qquad\quad = \sqrt{e^{2t} + e^{2t} + e^{2t}} = \sqrt{3e^{2t}} = \sqrt{3}e^t \\
&||\mathbf{r}'(0)|| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}e^0 = \sqrt{3} \\
&\mathbf{T}(t) = \frac{\cos t - \sin t}{\sqrt{3}}\mathbf{i} + \frac{\cos t + \sin t}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \\
&\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{||\mathbf{r}'(0)||} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \\
&\mathbf{T}'(t) = \frac{-\sin t - \cos t}{\sqrt{3}}\mathbf{i} + \frac{-\sin t + \cos t}{\sqrt{3}}\mathbf{j} + 0\mathbf{k} \\
&\qquad = -\frac{\cos t + \sin t}{\sqrt{3}}\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{3}}\mathbf{j} \\
&\mathbf{T}'(0) = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} \\
&||\mathbf{T}'(t)|| = \sqrt{\frac{1 + 2\sin t\cos t}{3} + \frac{1 - 2\sin t\cos t}{3}} = \sqrt{\frac{2}{3}} \\
&\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{||\mathbf{T}'(0)||} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}
\end{aligned}
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(2)

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\begin{aligned}
\mathbf{r}(t) &= e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} \\
\mathbf{r}'(t) &= e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k} \\
\mathbf{r}''(t) &= e^t\mathbf{i} + e^{-t}\mathbf{j}
\end{aligned}
$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -e^{-t} & 1 \\ e^t & e^{-t} & 0 \end{vmatrix} = (0 - e^{-t})\mathbf{i} - (0 - e^t)\mathbf{j} + (1+1)\mathbf{k}$$
$$= -e^{-t}\mathbf{i} + e^t\mathbf{j} + 2\mathbf{k}$$

$$||\mathbf{r}'(t)|| = \sqrt{e^{2t} + e^{-2t} + 1^2} = \sqrt{1+1+1} = \sqrt{3} \quad \text{for} \quad t = 0$$
$$||\mathbf{r}'(t) \times \mathbf{r}''(t)|| = \sqrt{e^{-2t} + e^{2t} + 2^2} = \sqrt{1+1+4} = \sqrt{6} \quad \text{for} \quad t = 0$$
$$\kappa(0) = \frac{\sqrt{6}}{(\sqrt{3})^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$$

(3) We assume that $\mathbf{r}(x)$ is smooth and show that $\mathbf{r}''(x)$ and $\mathbf{T}'(x)$ exist.

$$\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j}$$
$$\mathbf{r}'(x) = \mathbf{i} + \frac{dy}{dx}\mathbf{j} \ \Rightarrow \ \mathbf{r}''(x) = \frac{d^2y}{dx^2}\mathbf{j}$$
$$||\mathbf{r}'(x)|| = \sqrt{1^2 + \left(\frac{dy}{dx}\right)^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$$
$$\mathbf{T}(x) = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\mathbf{i} + \frac{dy}{dx}\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\mathbf{j}$$
$$\mathbf{T}'(x) = -\frac{\frac{dy}{dx}\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}\mathbf{i} + \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}\mathbf{j}$$

Hence,

$$\kappa(x) = \frac{||\mathbf{r}'(x) \times \mathbf{r}''(x)||}{||\mathbf{r}'(x)||^3}$$
$$\mathbf{r}'(x) \times \mathbf{r}''(x) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{dy}{dx} & 0 \\ 0 & \frac{d^2y}{dx^2} & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \frac{d^2y}{dx^2}\mathbf{k}$$
$$||\mathbf{r}'(x) \times \mathbf{r}''(x)|| = \sqrt{\left(\frac{d^2y}{dx^2}\right)^2} = \left|\frac{d^2y}{dx^2}\right|$$
$$\kappa(x) = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}\right]^3} = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

(4) $f(x,y) = x + (xy)^{\frac{1}{3}}$
   (a) $f(t,t^2) = t + (t \cdot t^2)^{\frac{1}{3}} = t + t = 2t$
   (b) $f(x,x^2) = t + (x \cdot x^2)^{\frac{1}{3}} = x + x = 2x$
   (c) $f(2y^2, 4y) = 2y^2 + (2y^2 \cdot 4y)^{\frac{1}{3}} = 2y^2 + (8y^3)^{\frac{1}{3}} = 2y^2 + 2y = 2y(y+1)$

(5) $g(u(x,y), v(x,y)) = \pi xy \sin(x^4 y^6 \pi xy) = \pi xy \sin(\pi x^5 y^7)$

(6) $f(x,y,z) = zxy + x$
   (a) $f(x+y, x-y, x^2) = x^2(x+y)(x-y) + (x+y) = x^2(x^2-y^2) + x + y$
   (b) $f(xy, y/x, xz) = xz \cdot xy \cdot \dfrac{y}{x} + xy = xy^2z + xy = xy(yz+1)$

(7) (a) $f(x,y,z) = 3x - y + 2z = k$

$\mathbf{k=0}$: $3x - y + 2z = 0$ is a plane with the following three points $(0,0,0), (1,1,-1), (-1,-1,1)$ lying on the plane.

$\mathbf{k=1}$: $3x - y + 2z = 1$ is a plane parallel to $3x - y + 2z = 0$ which intersects the z-axis at $(0,0,1/2)$.

For any $k,$ the level surface is the plane $3x - y + 2z = k,$ parallel to $3x - y + 2z = 0$ which intersects the z-axis at $(0,0,k/2)$.

   (b) $f(x,y,z) = z - x^2 - y^2 = k$

$\mathbf{k=0}$: $z = x^2 + y^2$ is an elliptic paraboloid with centre $(0,0,0)$.

$\mathbf{k=1}$: $z - 1 = x^2 + y^2$ is an elliptic paraboloid with centre $(0,0,1)$.

For any $k,$ the level surface is an elliptic paraboloid with centre $(0,0,k)$.