Chapter 8

Classification of Finitely Generated Vector Spaces

Recall that two vector space over a given field are equivalent (as vector spaces) if and only if they are isomorphic. This raises the classification problem for vector spaces:

> Given vector spaces V and W over the field $\mathbb{F}$, decide whether $V \cong W$.

This problem has an elegant solution. We can assign to each vector space, V, over the field $\mathbb{F}$ a numerical invariant, its *dimension*, $\dim_{\mathbb{F}}(V)$, which solves the classification problem completely.

Main Theorem (Classification Theorem). *Given vector spaces V, W over the field $\mathbb{F}$, $V \cong W$ if and only if $\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(W)$.*

This chapter is devoted to introducing the necessary concepts and proving the Classification Theorem for finitely generated vector spaces.[1]

Theorem 8.1. *If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ are bases for the vector space V, then $m = n$.*

This theorem, whose proof we defer briefly, justifies the next definition.

Definition 8.2. Let V be a finitely generated vector space over $\mathbb{F}$. The *dimension of V over $\mathbb{F}$*, $\dim_{\mathbb{F}} V$, is the number of vectors in a basis for V.

We derive Theorem 8.1 as a corollary to another theorem, which we first illustrate with an explicit example.

Example 8.3. Let $U = \langle \mathbf{u}_{1,}, \mathbf{u}_2 \rangle$ be a vector subspace of V. Consider $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in U$, where

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 + \mathbf{u}_2 \\ \mathbf{v}_2 &= -\mathbf{u}_1 + \mathbf{u}_2 \\ \mathbf{v}_3 &= \mathbf{u}_1 \end{aligned}$$

We add suitable multiples of $\mathbf{v}_1$ to $\mathbf{v}_2$ and $\mathbf{v}_3$ to eliminate $\mathbf{u}_1$, obtaining

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= 2\mathbf{u}_2 \\ \mathbf{v}_1 - \mathbf{v}_3 &= \mathbf{u}_2 \end{aligned}$$

[1] The theorem actually holds for all vector spaces. Since the general case uses the Axiom of Choice, we omit it.

It follows that

$$\mathbf{v}_1 + \mathbf{v}_2 = 2(\mathbf{v}_1 - \mathbf{v}_3),$$

or

$$\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}_V$$

Thus $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

Our next theorem generalises this, and our method of proof is based upon the above calculation.

Theorem 8.4. *Let U be a vector subspace of the vector space V. If U can be generated by a set of n vectors, then any set of more than n vectors from U is linearly dependent.*

Proof. We prove the theorem by induction on n.

$\mathbf{n = 1}$: In this case $U = \langle \mathbf{u} \rangle$.

Take $\mathbf{v}_1, \ldots, \mathbf{v}_m \in U$ for some $m > 1$.

Then there are $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that $\mathbf{v}_i = \lambda_i \mathbf{u}$ for each $i \in \{1, \ldots, m\}$.

If $\lambda_i = 0$, then $\mathbf{v}_i = \mathbf{0}_V$, whence $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

If no $\lambda_i = 0$, then

$$\lambda_2 \mathbf{v}_1 - \lambda_1 \mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_m = \lambda_2\lambda_1 \mathbf{u} - \lambda_1\lambda_2 \mathbf{u} + \mathbf{0}_V + \cdots \mathbf{0}_V = \mathbf{0}_V,$$

showing that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent.

$\mathbf{n > 1}$: We make the inductive hypothesis that if a vector subspace, S, of V can be generated by $n-1$ vectors, then every set of more than $n-1$ vectors in S must be linearly dependent.

Let $U := \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$ be a vector subspace of V and put $S := \langle \mathbf{u}_2, \ldots, \mathbf{u}_n \rangle$.

Take $\mathbf{v}_1, \ldots, \mathbf{v}_m \in U$ with $m > n$. Then

$$\mathbf{v}_i = \sum_{j=1}^{n} \lambda_{ij} \mathbf{u}_i$$

with $\lambda_{ij} \in \mathbb{F}$ $(i = 1, \ldots, m \quad j = 1, \ldots, n)$

If $\lambda_{i1} = 0$ for each i, then $\mathbf{v}_1, \ldots, \mathbf{v}_m \in S$ and we have $m > n > n-1$ vectors inthe vector subspace S of V generated by $n-1$ vectors.

By the inductive hypothesis, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ must be linearly dependent.

Otherwise $\lambda_{i1} \neq 0$ for some i. Renumbering the vectors if necessary, we may assume that $\lambda_{11} \neq 0$. Then, for each $i > 1$,

$$\lambda_{11}\mathbf{v}_i - \lambda_{i1}\mathbf{v}_1 = \sum_{j=1}^{n} (\lambda_{11}\lambda_{ij} - \lambda_{i1}\lambda_{1j})\mathbf{u}_j$$

Putting $\mathbf{w}_i := \lambda_{11}\mathbf{v}_i - \lambda_{i1}\mathbf{v}_1$, we obtain $m-1$ vectors, $\mathbf{w}_2, \ldots, \mathbf{w}_m$, in S.

Since S is generated by $n-1$ vectors and $m-1 > n-1$, it follows from the inductive hypothesis, $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1}$ are linearly dependent. Hence there are $\alpha_2, \ldots \alpha_m \in \mathbb{F}$, not all 0, such that

$$\alpha_2 \mathbf{w}_2 + \cdots + \alpha_m \mathbf{w}_m = \mathbf{0}_V.$$

Putting $\alpha_1 := -\alpha_2\lambda_{21} - \ldots - \alpha_m\lambda_{m1}$, we have

$$\alpha_1 \mathbf{v}_1 + \alpha_2\lambda_{11}\mathbf{v}_2 + \cdots + \alpha_m\lambda_{11}\mathbf{v}_m = \mathbf{0}_V$$

Since $\lambda_{11} \neq 0$, at least one $\alpha_i\lambda_{11} \neq 0$, and so $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent. $\square$

Corollary 8.5 (Theorem 8.1). *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be bases for the vector space V. Then $m = n$.*

Proof. Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form a basis for V, we have $V = \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$.

Since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ form a basis for V, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are linearly independent.

Hence, by Theorem 8.4, $m \leq n$.

Reversing the roles of the $\mathbf{u}$'s and the $\mathbf{v}$'s, it follows that $n \leq m$.

Thus $m = n$. □

Theorem 8.1 justifies the definition of the *dimension* of the vector space V,$\dim_{\mathbb{F}} V$ as the number of vectors in a basis for V, because this number depends only on the vector space and not on the choice of basis.

We deduce two more theorems as corollaries.

Theorem 8.6. *Let V be a vector space of dimension n. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, they must generate V (and hence form a basis for V).*

Proof. Take any $\mathbf{x} \in V$.

Since $\dim V = n$, V is generated by a set of n vectors.

Hence, by Theorem 8.4, the $n + 1$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{x}$ must be linearly dependent.

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent, it follows from Theorem 7.7 that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Thus $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle = V$. □

Theorem 8.7. *Let V be a vector space of dimension n. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ generate V, they must be linearly independent (and hence form a basis for V).*

Proof. Since $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle = V$, some subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ containing $q \leq n$ vectors must be linearly independent and still generate V, thus forming a basis for V.

But $\dim V = n$. By Theorem 8.4, $q = n$.

Thus, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ must be linearly independent. □

Our next result is a refinement of Theorem 7.10.

Lemma 8.8. *Let V and W be finite dimensional vector spaces, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a basis for V and $T \colon V \to W$ a linear transformation.*

(i) T is injective if and only if $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent.

(ii) T is surjective if and only if $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ generate W.

(iii) T is an isomorphism if and only if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.

Proof. **(i)** $\Rightarrow$: Suppose that T is injective and that $\lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n) = \mathbf{0}_W$.

By the linearity of T, $T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = \mathbf{0}_W$.

Thus, by the injectivity of T, $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}_V$.

But $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Hence $\lambda_1 = \cdots = \lambda_n = 0$, showing that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent.

(i) $\Leftarrow$: Suppose that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent and take $\mathbf{x} \in V$.

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a basis for V, $\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{v}_j$, for uniquely determined $x_1, \ldots, x_n \in \mathbb{F}$.

Since T is a linear transformation, $T(\mathbf{x}) = \sum_{j=1}^{n} x_j T(\mathbf{v}_j)$.

Hence, $\mathbf{x} \in \ker(T)$ only if

$$\sum_{j=1}^{n} x_j T(\mathbf{v}_j) = \mathbf{0}_W$$

Since $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent, $x_1 = \cdots = x_n = 0$.

Thus $\mathbf{x} = \mathbf{0}_V$, showing that T is injective.

(ii) $\Rightarrow$: Suppose that T is surjective and take $\mathbf{y} \in W$.

By the surjectivity of T, $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x} \in V$.

Since $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle = V$, $\mathbf{x} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n$.

Since T is a linear transformation, $\mathbf{y} = T(\mathbf{x}) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n)$.

Thus $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ generate W.

(ii) $\Leftarrow$: Suppose that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ generate W.

Take $\mathbf{y} \in W$.

Then $\mathbf{y} = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_m T(\mathbf{v}_m)$ since $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)$ generate W.

Since T is a linear transformation, $\mathbf{y} = T(\mathbf{x})$ for $\mathbf{x} = \lambda_1 \mathbf{e}_1 + \cdots + \lambda_m \mathbf{e}_m$.

Thus T is surjective.

(iii) Exercise. □

Corollary 8.9. *Let $T\colon V \to W$ be a linear transformation.*

(a) If T is injective, then $\dim(V) \leq \dim(W)$.

(b) If T is surjective, then $\dim(V) \geq \dim(W)$.

Proof. Exercise. □

Corollary 8.10. *For the endomorphism, $T\colon V \to V$, of the finitely generated vector space, V, the following are equivalent.*

(i) T is injective.

(ii) T is surjective.

(iii) T is an isomorphism.

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V, and consider $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$.

By Lemma 8.7, T is injective if and only if $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent.

By Theorem 8.8, since $dim(V) = n$, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ are linearly independent if and only if they generate V.

By Lemma 8.7, $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ generate V if and only if T is surjective.

This shows that (i) and (ii) are equivalent.

Hence, each of (i) and (ii) is equivalent to T's being a bijective linear transformation.

By Theorem 5.10, this is equivalent to T's being an isomorphism. □

The following example show that Corollary 8.10 does not hold when V is not finitely generated.

Example 8.11. Put $V := \mathbb{F}[t]$, the vector space of all polynomials in the indeterminate t with coefficients in $\mathbb{F}$.

$$T\colon V \longrightarrow V, \quad \sum_{j=0}^{n} a_j t^j \longmapsto \sum_{j=0}^{n} a_j t^{j+1}$$

is an injective endomorphism which is not surjective.

Corollary 8.12. *$\mathbb{F}^m \cong \mathbb{F}^n$ if and only if $m = n$.*

Proof. Plainly, only the "only if" part requires proof.

Let $T\colon \mathbb{F}^m \to \mathbb{F}^n$ be an isomorphism.

The standard basis for for $\mathbb{F}^m$, $\{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$, is a basis for $\mathbb{F}^m$

By Corollary 8.8(iii), the m vectors $T(1, 0, \dots, 0), \dots, T(0, \dots, 0, 1)$ comprise a basis for $\mathbb{F}^n$.

Since the standard basis for $\mathbb{F}^n$ contains n vectors, it follows from Theorem 8.1 that $m = n$. □

Theorem 8.13. *Let V be a finitely generated vector space over $\mathbb{F}$. Then $\dim_{\mathbb{F}} V = n$ if and only if V is isomorphic with $\mathbb{F}^n$.*

Proof. $\Rightarrow$: Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis for V. Define $T\colon V \to \mathbb{F}^n$ by

$$T(\mathbf{x}) := (x_1, \dots, x_n) \quad \text{if} \quad \mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

T is well defined and bijective because $\mathbf{e}_1, \dots, \mathbf{e}_n$ is a basis for V.

It remains only to show that T is a linear transformation.

Take $\mathbf{x} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n, \mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n \in V$ and $\lambda, \mu \in \mathbb{F}$. Then

$$\begin{aligned}
T(\lambda\mathbf{x} + \mu\mathbf{y}) &= T(\lambda \sum_{i=1}^{n} x_i\mathbf{e}_i + \mu \sum_{i=1}^{n} y_i\mathbf{e}_i) \\
&= T(\sum_{i=1}^{n} (\lambda x_i + \mu y_i)\mathbf{e}_i) \\
&= (\lambda x_1 + \mu y_1, \dots, \lambda x_n + \mu y_n) \\
&= \lambda(x_1, \dots, x_n) + \mu(y_1, \dots, y_n) \\
&= \lambda T(\mathbf{x}) + \mu T(\mathbf{y})
\end{aligned}$$

$\Leftarrow$: Let $T : \mathbb{F}^n \longrightarrow V$ be an isomorphism and $\mathbf{e}_1, \dots, \mathbf{e}_n$ the standard basis for $\mathbb{F}^n$.

By Lemma 8.8, $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)\}$ is a basis for V, whence $\dim_{\mathbb{F}}(V) = n$. □

Observation 8.14. Theorem 8.13 provides us with a complete answer to the question "When are two finitely generated vector spaces over $\mathbb{F}$ isomorphic?".

But its significance does not end there. The proof of the theorem makes it clear that if V is a finitely generated vector space over $\mathbb{F}$, then there are as many different isomorphisms $V \cong \mathbb{F}^n$ as their are choices of a basis for V.

Choosing a basis for V is the same as choosing an isomorphism $V \cong \mathbb{F}^n$, $(n = \dim V)$.

We reformulate Theorem 8.13 in terms of direct sums.

Theorem 8.15. *Let V be a finitely generated vector space over $\mathbb{F}$. Then*

$$V \cong \mathbb{F} \oplus \cdots \oplus \mathbb{F},$$

where the number of copies of $\mathbb{F}$ in the direct sum is precisely the dimension of V.

Proof. By Theorem 8.13, it suffices to show that the dimension of $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$ is the number of copies of $\mathbb{F}$ in the direct sum. But $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ is plainly a basis for $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$. (It is the standard basis for $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$.) □

The Classification Theorem follows immediately.

Theorem 8.16. *Every finitely generated vector space over the field $\mathbb{F}$ is isomorphic with one of the form $\mathcal{F}(X, \mathbb{F}) = \{f : X \to \mathbb{F} \mid f \text{ is a function}\}$.*

Proof. We present the essential idea for a proof, leaving the details as an exercise for the reader.

Let $X = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for the vector space V over $\mathbb{F}$.

Then the function

$$T : \{f : X \to \mathbb{F} \mid f \text{ is a function}\} \longrightarrow V, \quad f \longmapsto \sum_{j=1}^{n} f(\mathbf{e}_j)\mathbf{e}_j$$

is an isomorphism of vector spaces over $\mathbb{F}$.

□

8.1 The Universal Property of a Basis

We have shown that every finitely generated vector space has a basis, and that the number of vectors in a basis for a fixed vector space is independent of the choice of basis. This enabled a complete classification of finitely generated vector spaces over a fixed field up to isomorphism in terms of a single intrinsic numerical invariant, the *dimension*, which is the number of vectors in any basis for V.

We showed in Example 3.13, that for every natural number n, $\mathbb{F}^n$ admits a standard vector space structure, and we have seen in this chapter that every finitely generated vector space is isomorphic to precisely one such vector space, namely $\mathbb{F}^{\dim V}$, with the choice of a basis providing being the same as the choice of an isomorphism.

Hence, up to isomorphism, finitely generated vector spaces over a field are in bijection with $\mathbb{N}$, the set of all natural numbers.

While this is already sufficient to justify the importance of bases, they have another property with far-reaching consequences. Specifically, a basis does not only determine a vector space up to isomorphism, it also determines completely all linear transformations defined on a vector space.

Theorem 8.17 (Universal Property of a Basis). *Let $\mathcal{B}$ be a basis for the vector space V over the field $\mathbb{F}$.*

Given any vector space W over $\mathbb{F}$ and any function $f : \mathcal{B} \to W$, there is a unique linear transformation $T : V \to W$ with $T(\mathbf{e}) = f(\mathbf{e})$ for every $\mathbf{e} \in \mathcal{B}$.

This is expressed diagrammatical by

$$\begin{tikzcd} V \arrow[r, dashed, "\exists!\, T"] & W \\ \mathcal{B} \arrow[u, "i^V_{\mathcal{B}}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad (*)$$

Proof. The commutativity of $(*)$ is equivalent to $f = T \circ i^V_{\mathcal{B}}$.
Given $\mathbf{e} \in \mathcal{B}$, we must then have

$$T(\mathbf{e}) = T(i^V_{\mathcal{B}}(\mathbf{e})) = (T \circ i^V_{\mathcal{B}})(\mathbf{e}) = f(\mathbf{e})$$

This in turn, forces the definition of T, for given $\mathbf{v} \in V$, there are unique $x_1, \dots, x_n \in \mathbb{F}$ with

$$\mathbf{v} = \sum_{j=1}^{n} x_j \mathbf{e}_j \qquad (\mathbf{e}_j \in \mathcal{B})$$

Hence, in order for T to be a linear transformation, we must have

$$T(\mathbf{v}) = \sum_{j=1}^{n} x_j T(\mathbf{e}_j) = \sum_{j=1}^{n} x_j f(\mathbf{e}_j)$$

It remains to verify that

$$T\colon V \longrightarrow W, \quad \sum_{j=1}^{n} x_j \mathbf{e}_j \longmapsto \sum_{j=1}^{n} x_j f(\mathbf{e}_j)$$

— the only possible definition of T — does, indeed, define a linear transformation.

Since $\mathcal{B}$ is a basis for V, each $\mathbf{v} \in V$ can be written uniquely as $\sum_{j=1}^{n} x_j \mathbf{e}_j$ with $\mathbf{e}_j \in \mathcal{B}$. This uniquely determines $\sum_{j=1}^{n} x_j f(\mathbf{e}_j)$, showing that T is a function.

Take $\mathbf{v} = \sum_{j=1}^{n} x_j \mathbf{e}_j, \mathbf{v}' = \sum_{j=1}^{n} x'_j \mathbf{e}_j \in V$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned}
T(\mathbf{v} + \mathbf{v}') &= T(\sum_{j=1}^{n} x_j \mathbf{e}_j + \sum_{j=1}^{n} x'_j \mathbf{e}_j) \\
&= T(\sum_{j=1}^{n} (x_j + x'_j) \mathbf{e}_j) \\
&= \sum_{j=1}^{n} (x_j + x'_j) f(\mathbf{e}_j) \\
&= \sum_{j=1}^{n} x_j f(\mathbf{e}_j) + \sum_{j=1}^{n} x'_j f(\mathbf{e}_j) \\
&= T(\mathbf{v}) + T(\mathbf{v}')
\end{aligned}$$

and

$$\begin{aligned}
T(\alpha\mathbf{v}) &= T(\alpha\sum_{j=1}^{n} x_j\mathbf{e}_j) \\
&= T(\sum_{j=1}^{n} \alpha x_j\mathbf{e}_j) \\
&= \sum_{j=1}^{n} \alpha x_j T(\mathbf{e}_j) \\
&= \sum_{j=1}^{n} \alpha x_j f(\mathbf{e}_j) \\
&= \alpha\sum_{j=1}^{n} x_j f(\mathbf{e}_j) \\
&= \alpha T(\mathbf{v})
\end{aligned}$$

□

Observation 8.18. The significance of Theorem 8.17 cannot be overstated.

It means, in particular, that every linear transformation, T, defined on V is completely determined by the values it takes on any basis, and that we can assign any value to any of the vectors in a basis — we are *free* to choose the values of T on the basis in any way whatsoever. We have complete control over T.

This is particularly useful when $\mathbb{F}$ is an infinite field, for then it reduces an in principle infinite calculation to a finite one.

8.2 Exercises

Exercise 8.1. Let $\mathcal{C}^\infty(\mathbb{R})$ be the real vector space of all smooth — that is, infinitely differentiable — real-valued functions of a real variable. Put

$$V := \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \frac{d^2 f}{dx^2} + f = 0\}.$$

Prove that V is a real vector space, and that it is isomorphic to $\mathcal{P}_1$, the real vector space of all real polynomials of degree less than two.

Exercise 8.2. Let V and W be vector spaces over the field $\mathbb{F}$, $\mathcal{B}$ be a basis for V, $\mathcal{C}$ a basis for W and $\varphi : \mathcal{B} \to \mathcal{C}$ a function.

By Exercise 7.6, there is a uniquely determined linear transformation $T : V \longrightarrow W$ such that $T(\mathbf{v}) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{B}$.

Prove that this T is an isomorphism if and only if φ is bijective.

Exercise 8.3. Let V be a finitely generated vector space over $\mathbb{F}$ and W a vector subspace of V. Prove that if $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V)$ then $W = V$.

Exercise 8.4. Prove that every finitely generated vector space over $\mathbb{F}$ is (isomorphic with one) of the form

$$\mathcal{F}(X, \mathbb{F}) := \{f \colon X \to \mathbb{F} \mid f \text{ is a function}\}$$