

Solving Second order ODE with constant coefficients

Homogeneous case

First we solve homogeneous equations of the form

$$y'' + py' + qy = 0. \quad (1)$$

We start by looking at two particular cases:

1. $p = q = 0$, hence $y'' = 0$. By integration it follows $y' = C_1$ and $y = C_1x + C_2$. Any solution is a linear combination of the functions $y = x$ and $y = 1$.

2. If $q = 0$, but $p \neq 0$, we can reduce this second order equation to a first order equation by substituting $y'(x) = u(x)$. Then (1) takes the form

$$u' + pu = 0.$$

We find $u = Ae^{-px}$, where A is any constant. It follows

$$y' = Ae^{-px}.$$

Hence, $y = \frac{-A}{p}e^{-px} + B$, where B is any constant. Setting $C_1 = \frac{-A}{p}$, $C_2 = B$ yields

$$y = C_1 e^{-px} + C_2,$$

where C_1, C_2 are arbitrary constants.

Now we try to modify this trick to make it work for general q . Therefore we split $p = -m - n$ (Here the negative signs have been introduced to make the results compatible with traditional notation). Then (1) takes the form

$$(y'' - my') - n(y' - \frac{q}{n}y) = 0.$$

We want to choose m, n in such a way that $y'' - my'$ is the derivative of $y' - \frac{q}{n}y$, i.e. $\frac{q}{n} = m$. This is equivalent to the algebraic equations

$$-m - n = p \quad (2)$$

$$mn = q \quad (3)$$

From (2) we get $n = -p - m$. Then (3) reads

$$m^2 + pm + q = 0.$$

This quadratic equation is called the characteristic equation of (1). Notice for a solution m the corresponding n is the second solution of the characteristic equation.

First assume that the roots of the characteristic equation are real. Then we substitute $u(x) = y' - my$ and (1) becomes

$$u' - nu = 0,$$

hence $u = A e^{nx}$. Finally, we have to solve the (inhomogeneous) first order ODE

$$y' - my = A e^{nx}.$$

The integrating factor is e^{-mx} . Hence,

$$(y e^{-mx})' = A e^{(-m+n)x}.$$

For $m \neq n$, it follows

$$\begin{aligned} y e^{-mx} &= \frac{A}{-m+n} e^{(-m+n)x} + B \\ y &= \frac{A}{-m+n} e^{nx} + B e^{mx}. \end{aligned}$$

For $m = n$ (i.e. the characteristic equation has a double root), it follows

$$\begin{aligned} y e^{-mx} &= Ax + B \\ y &= Ax e^{mx} + B e^{mx}. \end{aligned}$$

If the roots m, n of the characteristic equation are different and real then the general solution of (1) is

$$y = C_1 e^{nx} + C_2 e^{mx},$$

where $C_1 = \frac{A}{-m+n}$ and $C_2 = B$ are arbitrary constants.

Consider now the case when m, n are complex. Then $m = \alpha + i\beta$ and $n = \alpha + i\beta$.

We will use the following facts:

1. A complex-valued function $y(x) = u(x) + i v(x)$ is a solution of (1) if and only if the real and imaginary parts $u(x)$ and $v(x)$ are solutions. This follows by direct verification.

2. Define the complex-valued function

$$y = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos \beta x + i \sin \beta x).$$

Note that this is consistent with the definition of the usual real exponential function and with Euler's formula. The so defined exponential function satisfies the usual properties such as $e^{a+b} = e^a e^b$ (for a, b complex) and $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. We will exploit the fact that

$$\frac{d}{dx} e^{(\alpha+i\beta)x} = (\alpha + i\beta) e^{(\alpha+i\beta)x}.$$

Indeed,

$$\begin{aligned} \frac{d}{dx} e^{(\alpha+i\beta)x} &= \frac{d}{dx} e^{\alpha x} (\cos \beta x + i \sin \beta x) \\ &= \alpha e^{\alpha x} \cos \beta x - \beta e^{\alpha x} \sin \beta x + i \alpha e^{\alpha x} \sin \beta x + i \beta e^{\alpha x} \cos \beta x \\ &= (\alpha + i \sin \beta) e^{\alpha x} \cos \beta x + (\alpha + i \sin \beta) e^{\alpha x} \sin \beta x \\ &= (\alpha + i \sin \beta) e^{\alpha x} (\cos \beta x + i \sin \beta) = (\alpha + i \sin \beta) e^{(\alpha+i\beta)x}. \end{aligned}$$

It follows

$$\int e^{(\alpha+i\beta)x} dx = \frac{e^{(\alpha+i\beta)x}}{\alpha + i\beta} + C.$$

Now we can repeat literally the procedure above for real roots and obtain the complex solutions

$$y = C_1 e^{mx} + C_2 e^{\bar{m}x}.$$

Though it is common in electronics to work with complex solutions, we would be interested in real solutions in mechanics. They occur if and only if the coefficients C_1 and C_2 are mutually conjugate. The real solutions are

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x,$$

where c_1, c_2 are real constants.

We summarise the recipe:

- Solve the characteristic equation $m^2 + pm + q = 0$.

- If the solutions m, n of the characteristic equation are real and distinct then the general solution of $y'' + py' + qy = 0$ is $y(x) = c_1 e^{mx} + c_2 e^{nx}$, where c_1, c_2 are arbitrary constants.
- If the characteristic equation has a single root m then the general solution of the differential equation is $y(x) = c_1 e^{mx} + c_2 x e^{mx}$, where c_1, c_2 are arbitrary constants.
- If the characteristic equation has complex roots $m = \alpha + i\beta$, $n = \alpha - i\beta$ then the general solution of the differential equation is $y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$, where c_1, c_2 are arbitrary constants.

Inhomogeneous case

We will solve the inhomogeneous linear second order equation

$$y'' + py' + qy = f(x). \quad (4)$$

for special $f(x)$. As before, the general solution will be the sum of a particular solution and the general solution of the corresponding homogeneous system.

The solutions of the homogeneous equation were spanned by functions $e^{kx} r(x)$ where k is a complex number and $r(x)$ is a (first or zero order) polynomial in x .

Define

$$\mathcal{P}^{k,N} = \{e^{kx} r(x) : \deg r \leq N\}.$$

$\mathcal{P}^{k,N}$ is a linear space of dimension $N+1$ (for fixed complex k). The functions $e^{kx} r(x)$ are called quasipolynomials. A quasipolynomial is a sum

$$c_0 e^{kx} + c_1 x e^{kx} + \dots + c_N x^N e^{kx}.$$

It is uniquely determined by the coefficients c_0, c_1, \dots, c_N . The monomials $\{e^{kx}, e^{kx} x, \dots, e^{kx} x^N\}$ are called the standard basis of $\mathcal{P}^{k,N}$. Any linear mapping on $\mathcal{P}^{k,N}$ is uniquely determined by its action on the standard basis.

Differentiation $\frac{d}{dx}$ is such a linear mapping (or operator) on $\mathcal{P}^{k,N}$:

$$\frac{d}{dx} e^{kx} r(x) = e^{kx} (kr(x) + r'(x)).$$

The linear operator $L = \frac{d^2}{dx^2} + p \frac{d}{dx} + q \text{id}$ can be associated with the equation (4).

We will solve (4) where $f(x)$ is a quasipolynomial. Since we are interested in just one particular solution we may look for a solution that is also a quasipolynomial.

Consider the standard basis $\{e^{kx}, e^{kx}x, \dots, e^{kx}x^N\}$ of $\mathcal{P}^{k,N}$. We compute

$$L(e^{kx}x^n) = Q(k)e^{kx}x^n + nL(k)e^{kx}x^{n-1} + n(n-1)e^{kx}x^{n-2},$$

where $Q(k) = k^2 + pk + q$ and $A(k) = 2k + p$. Observe that L does not increase the degree, but it decreases the degree by 1 if $Q(k) = 0$, i.e. k is a root of the characteristic equation and it decreases by 2 if $Q(k) = L(k) = 0$, i.e. k is a double root of the characteristic equation.

Solving the differential equation

$$Ly = f$$

with $f \in \mathcal{P}^{k,N}$ means to invert the action of L in the space of quasipolynomials. This can be done by solving a system of linear equations for the coefficients of y . This is called the method of undetermined coefficients.

If $Q(k) = 0$ the operator L decreases the degree, hence its inverse increases the degree. Hence if $Q(k) = 0$, but $A(k) \neq 0$ we need to look for a solution of degree bigger by 1 than the degree of f . If $Q(k) = 0$ and $A(k) = 0$ the operator L decreases the degree by 2, hence its inverse increases the degree by 2. Therefore, in this case, we need to look for a solution of degree bigger by 2 than the degree of f .

Below we write down the matrix of L with respect to the standard basis. It has the following (sparse) uppertriangular form

$$\begin{pmatrix} Q(k) & A(k) & 1 \cdot 2 & 0 & 0 & \cdots & 0 \\ 0 & Q(k) & 2A(k) & 2 \cdot 3 & 0 & \cdots & 0 \\ 0 & 0 & Q(k) & 3A(k) & 3 \cdot 4 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & (N-1)N \\ 0 & 0 & 0 & 0 & 0 & \ddots & NA(k) \\ 0 & 0 & 0 & 0 & 0 & \cdots & Q(k) \end{pmatrix}.$$

We see that L is invertible if and only if k is not a root of the characteristic polynomial. If $f = f_0 e^{kx} + f_1 x e^{kx} + \cdots + f_N x^N e^{kx}$ is the right hand side and $y = y_0 e^{kx} + y_1 x e^{kx} + \cdots + y_N x^N e^{kx}$ is the unknown solution then the coefficients y_0, y_1, \dots, y_N are solutions of the system

$$\begin{pmatrix}
Q(k) & A(k) & 1 \cdot 2 & 0 & 0 & \cdots & 0 \\
0 & Q(k) & 2A(k) & 2 \cdot 3 & 0 & \cdots & 0 \\
0 & 0 & Q(k) & 3A(k) & 3 \cdot 4 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & (N-1)N \\
0 & 0 & 0 & 0 & 0 & \ddots & NA(k) \\
0 & 0 & 0 & 0 & 0 & \cdots & Q(k)
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
\vdots \\
y_{N-2} \\
y_{N-1} \\
y_N
\end{pmatrix}
=
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots \\
f_{N-2} \\
f_{N-1} \\
f_N
\end{pmatrix}.
\tag{5}$$

It follows:

If $k \in \mathbb{R}$ is not a root of the characteristic polynomial of (4) then (4) with $f \in \mathcal{P}^{k,N}$ has a unique solution in $\mathcal{P}^{k,N}$. In particular, if $\pm i k$ (with $k \in \mathbb{R}$) is not a root of the characteristic polynomial of (4) then (4) with $f = \phi(x) \cos kx + \psi(x) \sin kx$, where ϕ, ψ are polynomials of degree $\leq N$, has a unique solution of the same form.

If k is one of two distinct roots of the characteristic polynomial then the last row of the linear system 5 reads

$$0 \cdot y_N = f_N.$$

It has only a solution if $f_N = 0$, i.e. f is a quasipolynomial of degree $\leq N-1$. Therefore (4) with $f \in \mathcal{P}^{k,N}$ has a solution in $f \in \mathcal{P}^{k,N+1}$, which is determined up to a solution $c e^{kx}$ of the homogeneous equation.

If k is a double root of the characteristic polynomial then $A(k) = 0$. Then the two last rows of the linear system 5 read

$$\begin{aligned}
0 \cdot y_{N-1} + 0 \cdot y_N &= f_{N-1} \\
0 \cdot y_N &= f_N.
\end{aligned}$$

It has only a solution if $f_{N-1} = f_N = 0$, i.e. f is a quasipolynomial of degree $\leq N-2$. Therefore (4) with $f \in \mathcal{P}^{k,N}$ has a solution in $f \in \mathcal{P}^{k,N+2}$, which is determined up to $(c_1 + c_2 x) e^{kx}$.

Before summarizing the results we notice the following superposition principle: Suppose $y_1(x)$ is a solution of $y'' + py' + qy = f_1$ and $y_2(x)$ is a solution of $y'' + py' + qy = f_2$. Then $y_1(x) + y_2(x)$ is a solution of $y'' + py' + qy = f_1 + f_2$.

Now the following recipe can be applied for problems

$$y'' + py' + qy = f(x).$$

1. Check if $f(x)$ is a sum of quasipolynomials. Determine the complex exponents k and the degrees N .
2. Split f according to the exponents k and solve the corresponding problems for each k separately.
3. Check if k is a root of the characteristic equation $m^2 + pm + q = 0$. If yes, check if k is a double root.
4. Look for a solution in $\mathcal{P}^{k,N}$ if k is not a root, in $\mathcal{P}^{k,N+1}$ if k is a simple root and in $\mathcal{P}^{k,N+2}$ if k is a double root. Use undetermined coefficients.

Example: $y'' - 3y' + 2y = \sin x$. The RHS is a sum of quasipolynomials $\sin x = \frac{1}{2i} e^{ix} - \frac{1}{2i} e^{-ix}$. The characteristic roots are $-1, -2$. Neither of them equals $\pm i$. So the solution is of the form $y(x) = c_1 e^{ix} + c_2 e^{-ix}$.

We have $y'(x) = i c_1 e^{ix} - i c_2 e^{-ix}$ and $y''(x) = -c_1 e^{ix} - c_2 e^{-ix}$. It follows

$$(-1 - 3i + 2)c_1 e^{ix} + (-1 + 3i + 2)c_2 e^{-ix} = \frac{1}{2i} e^{ix} - \frac{1}{2i} e^{-ix}.$$

Hence $c_1 = \frac{3-i}{20}$ and $c_2 = \frac{3+i}{20}$. We find

$$y = \frac{3}{10} \cos x + \frac{1}{10} \sin x.$$

Alternatively, we could look for a solution of the form $d_1 \sin x + d_2 \cos x$. This leads to the system

$$\begin{aligned} d_1 + 3d_2 &= 1 \\ -3d_1 + d_2 &= 0 \end{aligned}$$

with solution $d_1 = \frac{1}{10}$, $d_2 = \frac{3}{10}$.

Example: $y'' + y = \cos x$. Here $k = \pm i$ are roots of the characteristic equation. (The physical meaning of this circumstance is that the frequency of the applied force at the RHS equals to the frequency of the free oscillating system.) Hence the solution has the form $y = c_0 \cos x + d_0 \sin x + c_1 x \cos x + d_1 x \sin x$. Since $c_0 \cos x + d_0 \sin x$ are solutions of the homogeneous equation, c_0, d_0 are arbitrary. In order to determine c_1, d_1 we may temporarily assume $c_0 = d_0 = 0$.

We have

$$\begin{aligned}y &= c_1 x \cos x + d_2 x \sin x \\y' &= c_1 \cos x - c_1 x \sin x + d_2 \sin x + d_2 x \cos x \\y'' &= -2c_1 \sin x - c_1 x \cos x - 2d_2 \cos x - d_2 x \sin x\end{aligned}$$

hence, $2d_1 = 1$, $-2c_1 = 0$. So, $y(x) = \frac{1}{2}x \cos x$ is a particular solution and $y(x) = \frac{1}{2}x \cos x + A \sin x + B \cos x$ is the general solution.

The physical interpretation of this solution is the phenomenon of resonance. The amplitude builds up due to the fact that the force applied oscillates with the same frequency as the free system.