Chapter 6

Deriving Vector Spaces from Given Ones

Now that we know what a vector space is and have met some concrete examples to show that the concept is not an empty one, we investigate the problem of deriving vector spaces from given ones.

6.1 Vector Subspaces

One possibility is to start with a vector space, say V, and to consider subsets of V. When is a subset U of V form a vector space in its own right?

It is tempting to call a subset U of V that is a vector space in its own right a *vector subspace of* V. The defects of such a definition are readily illustrated by concrete examples, which ultimately suggest a more useful definition

Example 6.1. Take the set of real numbers $\mathbb{R}$ with its usual structure as $\mathbb{R}$ vector space. Then the set of rational numbers, $\mathbb{Q}$, clearly forms a subset.

Being a field, it is a vector space in its own right. But this only makes $\mathbb{Q}$ vector space over $\mathbb{Q}$, and **not** a vector space over $\mathbb{R}$ with respect to the usual addition and multiplication, for multiplying together a real number and a rational number need not result in a rational number.

In fact, there is no way whatsoever to make $\mathbb{Q}$ a vector space over $\mathbb{R}$, because, as we shall see later, a vector space over $\mathbb{R}$ has either precisely one element, or has at least as many elements as $\mathbb{R}$. But it is a basic result from set theory, that this is false of $\mathbb{Q}$.

The salient feature this example was that the fields underlying the two vector space structures in question are different. So we need to insist that the two vector spaces have a common field of scalars.

But even this is not enough, as we now show.

Example 6.2. Our example requires the following elementary facts from basic number theory.

1. Any two integers, say x and y, which are not both 0, have a *greatest common divisor*[1] denoted by $\gcd(x, y)$. This is defined to be the (uniquely determined) positive integer d such
   (a) d divides both x and y and
   (b) if the integer c divides both x and y, then c divides d.

[1] It is also called their *highest common factor* and denoted by $\mathrm{hcf}(x, y)$.

2. The integers x and y are said to be *relatively prime* if and only if their greatest common divisor is 1.

3. Given any integers, x and y, not both 0, with $\gcd(x, y) = d$, there are (uniquely determined) relatively prime integers u and v with $x = du$ and $y = dv$.

Put $V := \{(x, y) \in \mathbb{Z}^2 \mid y > 0 \text{ and } \gcd(x, y) = 1\}$, and define

$$\boxplus : V \times V \longrightarrow V$$
$$\boxdot : \mathbb{Q} \times V \longrightarrow V$$

by putting

$$(u, v) \boxplus (x, y) := (r, s) \qquad \text{where } rvy = s(uy + vx), \text{ and } \gcd(r, s) = 1$$
$$\frac{p}{q} \boxdot (x, y) := (r, s) \qquad \text{where } rqy = spx \text{ and } \gcd(r, s) = 1.$$

Then V becomes a vector space over $\mathbb{Q}$. We discuss two ways of seeing this to be true.

The first is to verify the vector space axioms by direct computation. This is to be recommended to those who are still wary of abstract methods and feel more comfortable with brute-force computation.

Those who have overcome such masochistic fixations should, instead, observe that if we rewrite $(u, v) \in V$ as $\frac{u}{v}$, then we recognise that V essentially consists of the set of all rational numbers, written in *reduced form*, and that $\boxplus$ and $\boxdot$ are just the usual addition and multiplication of rational numbers re-written in the form appropriate to V. Thus the statement: "V is a vector space over $\mathbb{Q}$" is plainly just a restatement of the fact that every field is a vector space over itself![2] [Strictly speaking, we have shown that V is isomorphic to $\mathbb{Q}$ as vector space over $\mathbb{Q}$.]

Now it is obvious that $V \subseteq \mathbb{Q}^2 := \{(x, y) \mid x, y \in \mathbb{Q}\}$, which is a vector space over $\mathbb{Q}$ with respect to the customary componentwise definition of the vector space operations.

But the two vector space structures, have nothing to do with each other, beyond having a common field of scalars.

Thus, for a meaningful and useful notion of a vector subspace we require not merely that the subset form a vector space in its own right over the same field, but that this vector space structure be precisely that derived from the ambient vector space.

One way of ensuring this is to insist that the inclusion of the subspace be a linear transformation.

Definition 6.3. The subset U of the vector space V over the field $\mathbb{F}$ is a *vector subspace of* V if and only if the inclusion

$$i_U^V : U \longrightarrow V, \quad \mathbf{u} \longmapsto \mathbf{u}$$

is a linear transformation.

We write $U \leq V$ to denote that U is a vector subspace of V.

Theorem 6.4. *Let U be a subset of the $\mathbb{F}$-vector space V. Then the following are equivalent.*

(i) U is a vector subspace of V.

(ii) (a) Given $\mathbf{u}, \mathbf{u}' \in U$, $\mathbf{u} + \mathbf{u}' \in U$.

(b) Given $\mathbf{u} \in U$ and $\lambda \in \mathbb{F}$, $\lambda\mathbf{u} \in U$.

[2] Score one more for abstract methods!

(iii) Given $\mathbf{u}, \mathbf{u}' \in U$ *and* $\lambda, \mu \in \mathbb{F}$, $\lambda\mathbf{u} + \mu\mathbf{u}' \in U$.

Proof. Since (ii) and (iii) are plainly equivalent, we confine ourselves to proving their equivalence to (i).

So assume that U is a vector subspace of V, and take $\mathbf{u}, \mathbf{u}' \in U$ and $\lambda, \mu \in \mathbb{F}$. Then

$$\begin{aligned}\lambda\mathbf{u} + \mu\mathbf{u}' &= \lambda i_U^V(\mathbf{u}) + \mu i_U^V(\mathbf{u}') \\ &= i_U^V(\lambda\mathbf{u} + \mu\mathbf{u}').\end{aligned}$$

But then, $\lambda\mathbf{u} + \mu\mathbf{u}'$ must be in the domain, U, of i_U^V, as this coincides with its range. Thus $\lambda\mathbf{u} + \mu\mathbf{u}' \in U$, establishing (iii).

For the converse, note that if $\lambda\mathbf{u} + \mu\mathbf{u}' \in U$ for all $\mathbf{u}, \mathbf{u}' \in U$ and $\lambda, \mu \in \mathbb{F}$, then

$$i_U^V(\lambda\mathbf{u} + \mu\mathbf{u}') = \lambda\mathbf{u} + \mu\mathbf{u}' = \lambda i_U^V(\mathbf{u}) + \mu i_U^V(\mathbf{u}').$$

□

The most significant single example of a vector subspace requires a definition.

Definition 6.5. Let V be a vector space over $\mathbb{F}$ and take $\mathbf{v} \in V$. Then

$$\mathbb{F}\mathbf{v} \ := \ \{\lambda\mathbf{v} \mid \lambda \in \mathbb{F}\}.$$

Lemma 6.6. *Take a vector space,* V*, over* $\mathbb{F}$ *and* $\mathbf{v} \in V$*. Then* $\mathbb{F}\mathbf{v}$ *is a vector subspace of* V*.*

Proof. Take $\mathbf{u}, \mathbf{u}' \in \mathbb{F}\mathbf{v}$ and $\lambda, \mu \in \mathbb{F}$.

Then there are $\alpha, \beta \in \mathbb{F}$ with $\mathbf{u} = \alpha\mathbf{v}$ and $\mathbf{u}' = \beta\mathbf{v}$.

Thus

$$\lambda\mathbf{u} + \beta\mathbf{u}' = \lambda(\alpha\mathbf{v}) + \mu(\beta\mathbf{v}) = (\lambda\alpha)\mathbf{v} + (\mu\beta)\mathbf{v} = \nu\mathbf{v},$$

where $\nu := \lambda\alpha + \mu\beta \in \mathbb{F}$. □

The vector subspace $\mathbb{F}\mathbf{v}$ of the vector space V generalises the notion of a line through the origin in $\mathbb{R}^n$, for such a line is determined uniquely by a point on it (other than the origin) with co-ordinates, say $(a_1, \ldots, a_n)$. Then the line is the set

$$\begin{aligned}\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_\imath = \lambda a_\imath (\imath = 1, \ldots n) \text{for some } \lambda \in \mathbb{R}\} \\ = \mathbb{R}((a_1, \ldots, a_n)).\end{aligned}$$

This is just the familiar *parametric representation* of the given line.

Given a family of vector subspaces of a fixed vector space V, their intersection is again a vector subspace of V.

Theorem 6.7. *Let* V *be a vector space and suppose that for each* $\gamma \in \Gamma$*,* W_γ *is a vector subspace of* V*. Then*

$$W := \bigcap_{\gamma \in \Gamma} W_\gamma$$

is a vector subspace of V*.*

Proof. Take $\mathbf{u}, \mathbf{v} \in W$ and $\lambda \in \mathbb{F}$. Then $\mathbf{u}, \mathbf{v} \in W_\gamma$ for each $\gamma \in \Gamma$.

Since each W_γ is a vector subspace of V, it follows that $\mathbf{u} + \mathbf{v} \in W_\gamma$ and $\lambda\mathbf{u} \in W_\gamma$ for each $\gamma \in \Gamma$.

Thus both $\mathbf{u} + \mathbf{v} \in W$ and $\lambda\mathbf{u} \in W$. □

While the intersection of vector subspaces of a given vector space is again a vector subspace, the same is not true of the union of vector subspaces, as the next example shows.

Example 6.8. Consider $\mathbb{R}^2$ as vector space over $\mathbb{R}$.

Then $U := \{(x,0) \mid x \in \mathbb{R}\}$ and $V := \{(0,y) \mid y \in \mathbb{R}\}$ are both vector subspaces of $\mathbb{R}^2$.

But $U \cup V = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$ is not a vector subspace of $\mathbb{R}^2$, for while $(1,0), (0,1) \in U \cup V$, $(1,0) + (0,1) = (1,1) \notin U \cup V$.

However, given vector subspaces U, W of the vector space V, the subset of V comprising those vectors in V that can be written as the sum of a vector from U and one from W is, in fact, a vector subspace of V.

Definition 6.9. Let U and W be vector subspaces of the vector subspace V. Then their *sum*, $U + W$, is defined by

$$U + W \ := \ \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\}.$$

Lemma 6.10. *Let U and W be vector subspaces of the vector subspace V. Then $U + W$ is a vector subspace of V.*

Proof. Take $\mathbf{u}_1 + \mathbf{w}_1, \mathbf{u}_2 + \mathbf{w}_2 \in U + W$ and $\lambda \in \mathbb{F}$. Then

$$(\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{u} + \mathbf{w},$$

where $\mathbf{u} := \mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{w} := \mathbf{w}_1 + \mathbf{w}_2 \in W$, and

$$\lambda(\mathbf{u}_1 + \mathbf{w}_1) = \lambda\mathbf{u}_1 + \lambda\mathbf{w}_1 = \mathbf{u}' + \mathbf{w}',$$

where $\mathbf{u}' := \lambda\mathbf{u}_1 \in U$ and $\mathbf{w}' = \lambda\mathbf{w}_1 \in W$. $\square$

This construction can be generalised. We define for any subset S of the vector space V, the vector subspace of V *generated* by that subset.

Definition 6.11. Let S be a subset of the vector space V. The *vector subspace of V generated by S*, $\langle S \rangle$, is the *smallest* vector subspace of V containing S, in the sense that

(i) $\langle S \rangle$ is a vector subspace of V containing S and

(ii) given any vector subspace, W, of V, if $S \subseteq W$ then $\langle S \rangle \subseteq W$.

The elements of S are called *generators*, and S a *generating set* for $\langle S \rangle$.

We also write $\langle \mathbf{v}_1, \ldots \rangle$ when $S := \{\mathbf{v}_1, \ldots\}$.

Theorem 6.7 is the key to proving that there is such a vector subspace of V

Theorem 6.12. *Let S be a subset of the vector space V. Then the vector subspace of V generated by S is the intersection of all vector subspaces U of V with $S \subseteq U$.*

Proof. Let $\mathfrak{A} := \{W \leq V \mid S \subseteq W\}$.

Then $\mathfrak{A} \neq \emptyset$, as $V \in \mathfrak{A}$. Put $U := \bigcap_{W \in \mathfrak{A}} W$.

So, by Theorem 6.7, U is a vector subspace of V, and clearly $S \subseteq U$.

If $W \leq V$ and $S \subseteq W$, then $W \in \mathfrak{A}$ and so $U = \bigcap_{X \in \mathfrak{A}} X \subseteq W$. $\square$

Observation 6.13. If U and W are vector subspaces of V then $U + W = \langle U \cup W \rangle$.

A linear transformation $T : V \to W$ naturally determines both a vector subspace of V and one of W, as we now show.

Theorem 6.14. *Let $T : V \longrightarrow W$ be a linear transformation. Then*

(i) $\ker(T)$ *is a vector subspace of* V*, and*

(ii) $\operatorname{im}(T)$ *is a vector subspace of* W*.*

Proof. (i). Take $\mathbf{u}, \mathbf{v} \in \ker(T)$ and $\lambda, \mu \in \mathbb{F}$. Then

$$\begin{aligned} T(\lambda\mathbf{u} + \mu\mathbf{v}) &= \lambda T(\mathbf{u}) + \mu T(\mathbf{v}) && \text{as } T \text{ is linear} \\ &= \lambda \mathbf{0}_W + \mu \mathbf{0}_W && \text{as } \mathbf{u}, \mathbf{v} \in \ker(T)) \\ &= \mathbf{0}_W \end{aligned}$$

(ii). Take $T(\mathbf{u}), T(\mathbf{v}) \in \operatorname{im}(T)$ and $\lambda, \mu \in \mathbb{F}$. Then

$$\lambda T(\mathbf{u}) + \mu T(\mathbf{v}) = T(\lambda\mathbf{u} + \mu\mathbf{v}) \quad \text{as } T \text{ is linear.}$$

□

6.2 Direct Sums

Another way of constructing vector spaces from given ones is to take as the underlying set of vectors the cartesian product of the sets of given vectors and to define the operations component-wise.

Definition 6.15. Let V and W be vector spaces over $\mathbb{F}$. We define a new vector space over $\mathbb{F}$, $V \oplus W$, called the *direct sum of V and W*, by putting

$$\begin{aligned} V \oplus W &:= \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in V, \mathbf{w} \in W\} \\ (\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) &:= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2) && (\mathbf{v}_1, \mathbf{v}_2 \in V,\ \mathbf{w}_1, \mathbf{w}_2 \in W) \\ \lambda.(\mathbf{v}, \mathbf{w}) &:= (\lambda\mathbf{v}, \lambda\mathbf{w}) && (\lambda \in \mathbb{F},\ \mathbf{v} \in V,\ \mathbf{w} \in W) \end{aligned}$$

Theorem 6.16. *Let V and W be vector spaces over $\mathbb{F}$. Then $V \oplus W$ is a vector space over $\mathbb{F}$ with respect to the operations just defined, with $\mathbf{0}_{V \oplus W} = (\mathbf{0}_V, \mathbf{0}_W)$ and $-(\mathbf{v}, \mathbf{w}) = (-\mathbf{v}, -\mathbf{w})$.*

Proof. The proof is routine verification and is left to the reader □

The direct sum of given vector spaces over the same field is a new vector space constructed from two given ones. This raises the question: When is a given vector space "in effect" non-trivial direct sum of vector spaces?

Definition 6.17. The vector space V is a *(non-trivial) direct sum* if and only if there are non-zero vector spaces U and W such that V and $U \oplus W$ are isomorphic.

6.2.1 Internal Direct Sum

Of particular importance is the case when U and W in Definition 6.17 may be chosen to be subspaces of V.

Theorem 6.18. *Let U, W be non-trivial vector subspaces of the vector space V. If $U + W = V$, where $U \cap W = \{\mathbf{0}_V\}$, then V is isomorphic to $U \oplus W$.*

Proof. Suppose that $V = U + W$ and that $U \cap W = \{\mathbf{0}_V\}$.

Then each $\mathbf{v} \in V$ can be written uniquely as $\mathbf{u} + \mathbf{w} \in U + W$. For if $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ with $\mathbf{u}, \mathbf{u}' \in U$ and $\mathbf{w}, \mathbf{w}' \in W$, then $\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w}$. But $\mathbf{u} - \mathbf{u}' \in U$ and $\mathbf{w}' - \mathbf{w} \in W$, so that $\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} \in U \cap W = \{\mathbf{0}_V\}$, from which we deduce that $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w}' = \mathbf{w}$.

Thus

$$T : U \oplus W \longrightarrow V, \quad (\mathbf{u}, \mathbf{w}) \longmapsto \mathbf{u} + \mathbf{w}$$

is a bijection.

Moreover, it is plainly a linear transformation.

It is, therefore, an isomorphism. $\square$

Definition 6.19. The vector space V is called the *internal direct sum of the subspaces* U, W if and only if $V \cong U \oplus W$. We then write $V = U \oplus W$.

The importance of this notion is difficult to overstate. For one thing it is one of the keys to our programme of classifying (up to isomorphism) vector spaces over a given field. We shall see that a non-trivial vector space, V, can be *decomposed* as the internal direct sum of vector subspaces, V_j, each of which is isomorphic with $\mathbb{F}$.

6.3 Quotient Spaces

Another important construction is that of a *quotient vector space.* While its true significance will not be apparent until you have have studied more mathematics, we introduce its construction here to demonstrate that new constructions are possible even with the limited theory we have already developed, and to illustrate some of the interrelationships with the concepts introduced.

Let U be a vector subspace of the vector space V over the field $\mathbb{F}$.

Define a relation, $\sim$, on V by

$$\mathbf{v} \sim \mathbf{v}' \quad \text{if and only if} \quad \mathbf{v}' - \mathbf{v} \in U$$

It is easy to see that $\sim$ defines an equivalence relation on V.

Let $[\mathbf{v}]$ denote the $\sim$-equivalence class containing $\mathbf{v}$, and put

$$V/U := \{[\mathbf{v}] \mid \mathbf{v} \in V\}.$$

We have the natural function

$$\eta : V \longrightarrow V/U, \quad \mathbf{v} \longmapsto [\mathbf{v}].$$

Define vector space operations on V/U by

$$\begin{aligned} [\mathbf{v}] + [\mathbf{v}'] &:= [\mathbf{v} + \mathbf{v}'] \\ \lambda[\mathbf{v}] &:= [\lambda \mathbf{v}] \end{aligned}$$

for all $[\mathbf{v}], [\mathbf{v}'] \in V/U$ and $\lambda \in \mathbb{F}$.

It is easy to verify that these definitions do, in fact, render V/U a vector space over $\mathbb{F}$, and, moreover, $\eta : V \longrightarrow V/U$ is then a linear transformation whose kernel is precisely U. (In fact, this is the only way of defining a vector space structure on V/U with respect to which η is a linear transformation.)

Definition 6.20. If U is a vector subspace of V, then the *quotient space of* V *modulo* U is V/U with the vector space operations just defined.

6.4 $\mathrm{Hom}_{\mathbb{F}}(V,W)$ and the Dual of a Vector Space

Given vector spaces V and W over, the same field $\mathbb{F}$, we write $\mathrm{Hom}_{\mathbb{F}}(V,W)$ for the set of all linear transformations from V to W.

Definition 6.21. Let V and W be vector spaces over the field $\mathbb{F}$. Then the vector space $\mathrm{Hom}_{\mathbb{F}}(V,W)$ is the set $\{T : V \longrightarrow W \mid T \text{ is a linear transformation}\}$ together with the operations

$$\begin{aligned} \boxplus : \mathrm{Hom}_{\mathbb{F}}(V,W) \times \mathrm{Hom}_{\mathbb{F}}(V,W) &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(V,W), \quad (S,T) \longmapsto S \boxplus T : V \to W \\ \boxdot : \mathbb{F} \times \mathrm{Hom}_{\mathbb{F}}(V,W) &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(V,W), \quad (\lambda,T) \longmapsto \lambda \boxdot T : V \to W \end{aligned}$$

where $S \boxplus T \colon \mathbf{v} \mapsto S(\mathbf{v}) + T(\mathbf{v})$ and $\lambda \boxdot T \colon \mathbf{v} \mapsto \lambda\big(T(\mathbf{v})\big)$, with $\mathbf{0}_{\mathrm{Hom}_{\mathbb{F}}(V,W)} \colon V \to W,\ \mathbf{v} \mapsto \mathbf{0}_W$ as zero vector.

Theorem 6.22. *If V and W are vector spaces over the field $\mathbb{F}$, then $\mathrm{Hom}_{\mathbb{F}}(V,W)$, as defined in Definition 6.21, is also a vector space over $\mathbb{F}$.*

The proof consists of no more than routine verification that the operations are well defined and that the axioms for a vector space over $\mathbb{F}$ all hold. We reproduce details here for the benefit of those readers to whom proofs have not lost their novelty. (The details left to the reader to complete follow the same patterns as the details provided.)

Proof. Given $S, T \in \mathrm{Hom}_{\mathbb{F}}(V,W)$ and $\lambda \in \mathbb{F}$, we first show that $S \boxplus T$ and $\lambda \boxdot T$ are again in $\mathrm{Hom}\, V, W$, which is to say, that they are, in fact, linear transformations.

So take $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned} (S \boxplus T)(\alpha\mathbf{u} + \beta\mathbf{v}) &:= S(\alpha\mathbf{u} + \beta\mathbf{v}) + T(\alpha\mathbf{u} + \beta\mathbf{v}) \\ &= (\alpha S(\mathbf{u}) + \beta S(\mathbf{v})) + (\alpha T(\mathbf{u}) + \beta T(\mathbf{v})) && \text{as } S, T \text{ are linear} \\ &= \alpha(S(\mathbf{u}) + T(\mathbf{u})) + \beta(S(\mathbf{v}) + T(\mathbf{v})) && \text{as } W \text{ is an } \mathbb{F}\text{-vector space} \\ &=: \alpha\big((S \boxplus T)\big)(\mathbf{u}) + \beta\big((S \boxplus T)\big)(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} (\lambda \boxdot T)(\alpha\mathbf{u} + \beta\mathbf{v}) &:= \lambda T(\alpha\mathbf{u} + \beta\mathbf{v}) \\ &= \lambda(\alpha T(\mathbf{u}) + \beta T(\mathbf{v})) && \text{as } T \text{ is linear} \\ &= (\alpha(\lambda T(\mathbf{u})) + \beta(\lambda T(\mathbf{v})) && \text{as } W \text{ is an } \mathbb{F}\text{-vector space} \\ &= \alpha\big((\lambda \boxdot T)\big)(\mathbf{u}) + \beta\big((\lambda \boxdot T)\big)(\mathbf{v}) \end{aligned}$$

We next verify some of the vector space axioms for $\mathrm{Hom}_{\mathbb{F}}(V,W)$, leaving the remaining ones as an exercise for the reader.

Take $R, S, T \in \mathrm{Hom}_{\mathbb{F}}(V,W)$, $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$. Then

VS1

$$\begin{aligned} \big(R \boxplus (S \boxplus T)\big)(\mathbf{v}) &:= R(\mathbf{v}) + (S \boxplus T)(\mathbf{v}) \\ &:= R(\mathbf{v}) + (S(\mathbf{v}) + T(\mathbf{v})) \\ &= (R(\mathbf{v}) + S(\mathbf{v})) + T(\mathbf{v})) && \text{as } W \text{ is a vector space} \\ &=: (R \boxplus S)(\mathbf{v}) + T(\mathbf{v}) \\ &=: \big((R \boxplus S) \boxplus T\big)(\mathbf{v}) \end{aligned}$$

VS2 Exercise.

VS3 Define $-T$ by

$$(-T) : V \longrightarrow W,\ \mathbf{v} \longmapsto -(T(\mathbf{v}))$$

(To see that, in fact, $(-T) \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ is left as an exercise.) Then

$$\begin{aligned}((-T) \boxplus T)(\mathbf{v}) &:= (-T)(\mathbf{v}) + T(\mathbf{v})\\ &:= -T(\mathbf{v}) + T(\mathbf{v})\\ &= \mathbf{0}_W && \text{as } W \text{ is a vector space}\\ &=: \mathbf{0}_{\mathrm{Hom}_{\mathbb{F}}(V,W)}(\mathbf{v})\end{aligned}$$

VS4 Exercise.

VS5 Exercise.

VS6

$$\begin{aligned}\big(\lambda \boxdot (R \boxplus T)\big)(\mathbf{v}) &:= \lambda\big((R \boxplus T)\big)(\mathbf{v})\\ &= \lambda\big(R(\mathbf{v}) + T(\mathbf{v})\big)\\ &= \lambda R(\mathbf{v}) + \lambda T(\mathbf{v}) && \text{as } W \text{ is a vector space over } \mathbb{F}\\ &=: \big(\lambda \boxdot R \boxplus \lambda \boxdot T\big)(\mathbf{v})\end{aligned}$$

VS7 Exercise.

VS8 Exercise.

□

Observation 6.23. $\mathcal{L}(V, W)$ and $\mathrm{Hom}(V, W)$ are two common notations for $\mathrm{Hom}_{\mathbb{F}}(V, W)$.

In the special case that $W = \mathbb{F}$, $\mathrm{Hom}_{\mathbb{F}}(V, W)$ is called the *dual* space of V.

Definition 6.24. The *dual space* of the vector V over the field $\mathbb{F}$ is $\mathrm{Hom}(V, \mathbb{F})$,the elements of which are called *linear forms (on V)*. We sometimes write V^* for $\mathbb{F}$ is $\mathrm{Hom}(V, \mathbb{F})$.

We investigate the effect of composition of linear transformations on the vector space $\mathrm{Hom}(V, W)$.

Theorem 6.25. *Take vector spaces U, V, W, X, $\alpha \in \mathbb{F}$, and linear transformations $Q\colon W \to X$, $R, S\colon V \to W$, $T\colon U \to V$.*

(i) $(R \boxplus S) \circ T = (R \circ T) \boxplus (S \circ T)$

(ii) $Q \circ (R \boxplus S) = (Q \circ R) \boxplus (Q \circ S)$

(iii) $(\alpha \boxdot S) \circ T = \alpha \boxdot (S \circ T) = S \circ (\alpha \boxdot T)$

Proof. Take $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

$$\begin{aligned}\big((R \boxplus S) \circ T\big)(\mathbf{u}) &:= (R \boxplus S)(T(\mathbf{u}))\\ &:= R(T(\mathbf{u})) + S(T(\mathbf{u}))\\ &=: (R \circ T)(\mathbf{u}) + (S \circ T)(\mathbf{u})\\ &=: \big((R \circ T) \boxplus (S \circ T)\big)(\mathbf{u})\end{aligned}$$

$$
\begin{aligned}
\big(Q \circ (R \boxplus S)\big)(\mathbf{v}) &:= Q\big((R \boxplus S)(\mathbf{v})\big) \\
&:= Q\big((R(\mathbf{v}) + S(\mathbf{v})\big) \\
&= Q\big((R(\mathbf{v})\big) + Q\big(S(\mathbf{v})\big) && \text{as } Q \text{ is a linear transformation} \\
&=: (Q \circ R)(\mathbf{v}) + (Q \circ S)(\mathbf{v}) \\
&=: \big((Q \circ R) \boxplus (Q \circ S)\big)(\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\big((\alpha \boxdot S) \circ T\big)(\mathbf{u}) &:= (\alpha \boxdot S)(T\mathbf{u}))) \\
&:= \alpha S(T(\mathbf{u})) \\
&=: \alpha(S \circ T)(\mathbf{u}) \\
&=: \big(\alpha \boxdot (S \circ T)\big)(\mathbf{u}) \\
\big(S \circ (\alpha \boxdot T)\big)(\mathbf{u}) &:= S\big(\alpha \boxdot T)(\mathbf{u})\big) \\
&:= S(\alpha T(\mathbf{u})) \\
&= \alpha S(T(\mathbf{u})) && \text{as } S \text{ is a linear transformation} \\
&=: \alpha(S \circ T)(\mathbf{u} \\
&=: \big(\alpha \boxdot (S \circ T)\big)(\mathbf{u})
\end{aligned}
$$

□

Corollary 6.26. *The linear transformations* $Q\colon W \to X$*, and* $T\colon U \to V$ *induce linear transformations*

$$
\begin{aligned}
Q_*\colon\ \mathrm{Hom}(V, W) &\longrightarrow \mathrm{Hom}(V, X), & R &\longmapsto Q \circ R \\
T^*\colon\ \mathrm{Hom}(V, W) &\longrightarrow \mathrm{Hom}(U, W), & S &\longmapsto S \circ T
\end{aligned}
$$

Proof. The corollary is just a restatement of Theorem 6.25. □

6.5 Exercises

Exercise 6.1. Let $\mathcal{C}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of all real valued functions defined on $\mathbb{R}$. Then $\mathcal{C}(\mathbb{R})$ is a real vector space with respect to point-wise operations. Decide which of the following subsets of $\mathcal{C}(\mathbb{R})$, are, in fact, vector subspaces.

(a) $\mathcal{C}^0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous }\}$

(b) $\mathcal{C}^r(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid \frac{d^r f}{dx^r} \text{ is continuous }\}$ $(r \in \mathbb{N} \setminus \{0\})$.

(c) $\big(\mathcal{F}(\mathbb{R})\big)_{(x_0)} := \{f : \mathbb{R} \to \mathbb{R} \mid f(x_0) = 0\}$, where x_0 is a fixed real number.

(d) $\big(\mathcal{F}(\mathbb{R})\big)_{0,1} := \{f : \mathbb{R} \to \mathbb{R} \mid f(0)f(1) = 0\}$.

Exercise 6.2. Let V and W be vector spaces over the field $\mathbb{F}$. Let

$$\mathfrak{Set}(V, W) := \{f : V \to W\}$$

denote the set of all functions from V to W. Prove that $\mathfrak{Set}(V, W)$ forms a vector space over $\mathbb{F}$ with respect to point-wise operations and that

$$\mathrm{Hom}_{\mathbb{F}}(V, W) := \{f : V \to W \mid f \text{ is an } \mathbb{F}\text{-linear transformation}\}$$

is a vector subspace of $\mathfrak{Set}(V, W)$.

Exercise 6.3. Find all vector subspaces of $\mathbb{C}^2$, when

(a) $\mathbb{C}^2$ is taken as a vector space over $\mathbb{C}$.

(b) $\mathbb{C}^2$ is taken as a vector space over $\mathbb{R}$ in the usual manner.

Exercise 6.4. Prove Theorem 6.16: If V and W are vector spaces over $\mathbb{F}$, then $V \oplus W$ is a vector space over $\mathbb{F}$ with respect to the operations defined by

$$\begin{aligned}(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) &:= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2)\\ \lambda(\mathbf{v}, \mathbf{w}) &:= (\lambda\mathbf{v}, \lambda\mathbf{w})\end{aligned}$$

for all $\lambda \in \mathbb{F}$, $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}$ and $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in W$.

Exercise 6.5. Prove that U and W are vector subspaces of V then

$$U + W = \langle U \cup W \rangle.$$

Exercise 6.6. Let U be a vector subspace of the vector space V over the field $\mathbb{F}$.

Define a relation $\sim$ on V by

$$\mathbf{v} \sim \mathbf{v}' \quad \text{if and only if} \quad \mathbf{v}' - \mathbf{v} \in U$$

a. Prove that $\sim$ defines an equivalence relation on V.

Let $[\mathbf{v}]$ denote the $\sim$-equivalence class containing $\mathbf{v}$, and put

$$V/U := \{[\mathbf{v}] \mid \mathbf{v} \in V\}.$$

We have the natural function

$$\eta : V \longrightarrow V/U, \quad \mathbf{v} \longmapsto [\mathbf{v}].$$

Define

$$\begin{aligned}+ : V/U \times V/U &\longrightarrow V/U, \quad ([\mathbf{v}], [\mathbf{v}']) \longmapsto [\mathbf{v} + \mathbf{v}']\\ \cdot : \mathbb{F} \times V/U &\longrightarrow V/U, \quad (\lambda, []) \longmapsto [\lambda].\end{aligned}$$

b. Prove the following statements.

(i) These definitions render V/U a vector space over $\mathbb{F}$.

(ii) $\eta : V \longrightarrow V/U$ is a linear transformation.

(iii) $\ker(\eta) = U$.

c. Prove that if W is any vector space over $\mathbb{F}$ and $T : V \to W$ is any linear transformation with $\ker(T) \subseteq U$, then there is a uniquely determined linear transformation $\tilde{T} : V/U \longrightarrow W$ such that $T = \tilde{T} \circ \eta$.[3] In diagrammatic form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & Y \\ X/\sim \arrow[ur, dashed, "\exists!\, \tilde{f}"'] & \end{tikzcd}$$

Exercise 6.7. Complete the proof of Theorem 6.22

[3]This is an example of a *universal property*. You will meet universal properties if you pursue further studies in mathematics, especially in *category theory*.