Chapter 17

Real Quadratic Forms

Recall from your study of real valued functions of n real variables that to find the extreme values of a sufficiently smooth function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$$

we first look at its *gradient*

$$\nabla(f)(x_1, \ldots, x_n) := (f_{x_1}(x_1, \ldots, x_n), \ldots, f_{x_n}(x_1, \ldots, x_n)),$$

where

$$f_{x_i} := \frac{\partial f}{\partial x_i}.$$

Because of the conditions we have imposed on f, a necessary — but not sufficient — condition for f to have an extreme value at a point in $\mathbb{R}^n$ is that the gradient be the zero vector at that point.

We then examine the *Hessian* of f,

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

whose properties provide sufficient — but not necessary — conditions for an extremum: f has a (local) minimum whenever the Hessian is "positive definite" and a (local) maximum whenever it is "negative definite".

This Hessian is an example of a *(real) quadratic form*, to whose study this chapter is devoted.

If $\beta \colon V \times V \to \mathbb{R}$ is a symmetric bi-linear form on V, a finitely generated real vector space, we can associate with it the real-valued function

$$q : V \longrightarrow \mathbb{R}, \quad \mathbf{v} \longmapsto \beta(\mathbf{v}, \mathbf{v}).$$

Take a basis, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, for V. Then $\mathbf{v} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n$ for suitable $x_1, \ldots, x_n \in \mathbb{R}$, and it follows from the bi-linearity of β that

$$q(\mathbf{v}) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \beta(\mathbf{e}_i, \mathbf{e}_j)$$

or, putting $a_{ij} := \beta(\mathbf{e}_i, \mathbf{e}_j)$,

$$q(\mathbf{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \tag{17.1}$$

In other words, $q(\mathbf{v})$ is a homogeneous quadratic polynomial in n variables, viz. the co-ordinates of $\mathbf{v}$. In particular, given $\mathbf{v} \in V$ and all $\lambda \in \mathbb{R}$,

$$q(\lambda\mathbf{v}) = \lambda^2 q(\mathbf{v}).$$

Definition 17.1. A *quadratic form* on the vector space, V, over the subfield, $\mathbb{F}$, of $\mathbb{R}$ is a function

$$q : V \longrightarrow \mathbb{F},$$

such that for all $\mathbf{x} \in V$ and $\lambda \in \mathbb{F}$

$$q(\lambda\mathbf{x}) = \lambda^2 q(\mathbf{x}),$$

and that there is a symmetric bi-linear form, $\beta : V \times V \to \mathbb{F}$, such that for all $\mathbf{x} \in V$

$$q(\mathbf{x}) = \beta(\mathbf{x}, \mathbf{x}).$$

Given a basis $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for V, the matrix $\underline{\mathbf{A}}_q := \left[\beta(\mathbf{e}_i, \mathbf{e}_j)\right]_{n\times n}$ is the *matrix of the quadratic from q with respect to the basis $\mathcal{B}$.*

Observation 17.2. The matrix $\underline{\mathbf{A}}_q$ is a symmetric matrix, since it is the matrix of the symmetric bi-linear form in the definition of a quadratic form.

We defined quadratic forms in terms of bi-linear forms. In fact, real quadratic forms and real symmetric bi-linear forms completely determine each other.

Theorem 17.3. *Given a quadratic form q on the vector space V over $\mathbb{F} \subseteq \mathbb{R}$, there is a unique bi-linear form, β, on V such that*

$$q(\mathbf{v}) = \beta(\mathbf{v}, \mathbf{v})$$

for all $\in V$.

Proof. As the existence of such a bi-linear form is ensured by the very definition of a quadratic form, it remains only to demonstrate its uniqueness.
But observe that

$$q(\mathbf{u} - \mathbf{v}) = \beta(\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = q(\mathbf{u}) - 2\beta(\mathbf{u}, \mathbf{v}) + q(\mathbf{v})$$

by the bi-linearity and symmetry of β and, similarly

$$q(\mathbf{u} + \mathbf{v}) = \beta(\mathbf{u} + , \mathbf{u} + \mathbf{v}) = q(\mathbf{u}) + 2\beta(\mathbf{u}, \mathbf{v}) + q(\mathbf{v}),$$

whence

$$\beta(\mathbf{u}, \mathbf{v}) = \frac{1}{4}\left(q(\mathbf{u} + \mathbf{v}) - q(\mathbf{u} - \mathbf{v})\right).$$

□

Quadratic forms are real valued functions. So we may ask about the values they take.

Definition 17.4. The real quadratic form $q : V \longrightarrow \mathbb{R}$ is be *positive (negative) semi-definite* if and only if $q(\mathbf{v}) \geq 0$ (resp. $q(\mathbf{v}) \leq 0$) for all $\mathbf{v} \in V$. It is *positive (negatgive) definite* if, in addition, $q(\mathbf{v}) = 0$ only for $=\mathbf{0}_V$.

Otherwise, it is *indefinite*.

Of course, the quadratic form derived from an inner product must be positive definite.

Example 17.5. We illustrate the above using quadratic forms $q : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

(i) $q(x, y, z) := x^2 + y^2$ is, plainly, positve semi-definite. It is not positive definite, since $q(0, 0, 1) = 0$.

(ii) $q(x, y, z) := x^2 + y^2 - z^2$ is idefinite, for $q(1, 0, 0) = 1$, whereas $q(0, 0, 1) = -1$

We shall see (Sylvester's Theorem), that these examples are typical, in that every quadratic form is equivalent to one like the ones above.

We consider two quadratic forms to be equivalent if there is an automorphism of the vector space such that one form is the composite of the other with the automorphism. Formally,

Definition 17.6. Two quadratic forms, q and $\tilde{q}$, on the finitely generated real vector space V are *equivalent* if and only if there is an isomorphism $\varphi : V \to V$ such that $\tilde{q} = q \circ q$.

We formulate this in terms of matrices.

Lemma 17.7. *Let $q : V \to \mathbb{R}$ be a quadratic form on the finitely generated real vector space V and $\varphi : V \to V$ a linear transformation. Then $q \circ \varphi : V \to \mathbb{R}$ is also a quadratic form.*

Moreover, if the matrix q with respect to a particular basis of V is $\underline{\mathbf{A}}$ and that of φ is $\underline{\mathbf{B}}$, then the matrix of $q \circ \varphi$ is $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \underline{\mathbf{B}}$.

Proof. By Theorem 17.3, it is sufficient to prove the corresponding result for the bi-linear form determined by q. But that is precisely the content of Corollary 15.5 and Theorem 15.6. □

Real quadratic forms are classified up to isomorphism by a triple of natural numbers, as the next theorem shows.

Theorem 17.8 (Sylvester's Theorem).

The real quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ is equivalent to one or the form

$$\sum_{i=1}^{r-s} x_i^2 - \sum_{i=r-s+1}^{r} x_j^2$$

with $0 \leq s \leq r \leq n$.

Moreover, the triple of natural numbers (n, r, s) determines q up to isomorphism.

Proof. It is sufficient to prove that there is a basis for $\mathbb{R}^n$ with respect to which the matrix of q is a diagonal matrix all of whose entries are 0 or ± 1.

By Corollary 16.16 there is a basis, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, for $\mathbb{R}^n$ with respect to which the matrix of q is the diagonal matrix

$$\begin{bmatrix} d_1 & 0 & \cdots \\ 0 & d_2 & \\ \vdots & 0 & \ddots \end{bmatrix},$$

where we may assume that this basis has been so ordered that

$$\begin{aligned}
d_i &> 0 && \text{for } 1 \le i \le r-s \\
d_i &< 0 && \text{for } r-s < i \le r \\
d_i &= 0 && \text{for } r < i \le n
\end{aligned}$$

Putting

$$\lambda_i := \begin{cases} \frac{1}{\sqrt{|d_i|}} & \text{for } i \le r \\ 1 & \text{for } i > r \end{cases}$$

it follows immediately that the matrix of q with respect to the basis $\left\{\frac{1}{\sqrt{\lambda_i}}\mathbf{e}_i \mid 1 \le i \le n\right\}$ has the form required. □

17.1 Exercises

Exercise 17.1. Let $\beta : V \times V \longrightarrow \mathbb{R}$ be a symmetric bi-linear form on the real vector space V and $q : V \longrightarrow \mathbb{R}$ a quadratic form on V. Show that for all $\mathbf{u}, \in V$

(a) $\beta_{q_\beta}(\mathbf{u}, \mathbf{v}) = \beta(\mathbf{u}, \mathbf{v})$

(b) $q_{\beta_q}(\mathbf{u}) = q(\mathbf{u})$

Exercise 17.2. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the real vector space V and $\|\ \|$ the norm it induces. Decide whether

$$q : V \longrightarrow \mathbb{R}, \quad \mathbf{u} \longmapsto \|\mathbf{u}\|^2$$

defines a quadratic form on V.

Exercise 17.3. Let q be a positive definite quadratic form on the real vector space V. Prove that

$$\|\ \| : V \longrightarrow \mathbb{R}, \quad \mathbf{u} \longmapsto \sqrt{q(\mathbf{u})}$$

defines a norm on V.

Exercise 17.4. Classify each of the following bi-linear forms according to its definiteness property:

(a) $\beta : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad ((u,v,w),(x,y,z)) \longmapsto 2ux + uy - 2uz + vx + 3vy - vz - 2wx - wy + 3wz$

(b) $\beta : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad ((u,v,w),(x,y,z)) \longmapsto ux + 3uy + 3uz + 3vx + vy + vz + 3wx + wy + 2wz$

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