AMTH250
Assignment 5

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September 12, 2011

Question 1

(a) Solution to the system of linear equations (to 2 decimal places):

Parameter	Estimate
x_1	-28.28
x_2	20.00
x_3	10.00
x_4	-30.00
x_5	14.14
x_6	20.00
x_7	0.00
x_8	-30.00
x_9	7.07
x_{10}	25.00
x_{11}	20.00
x_{12}	-35.36
x_{13}	25.00

The estimated relative error is 2.312×10^{-15}.

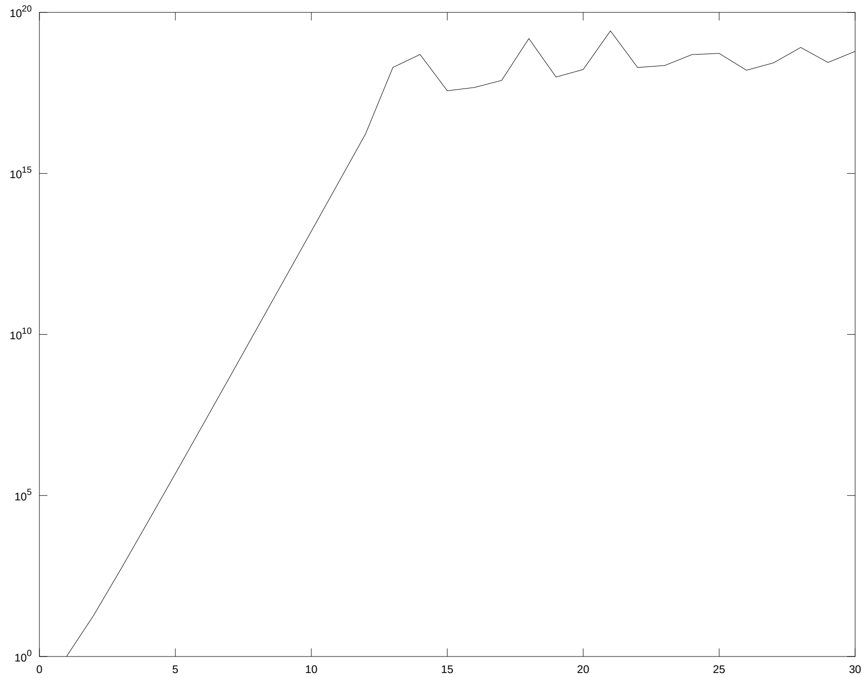
(b) The actual relative error is 1.382×10^{-16}. We have overestimated the error by a factor of 16.729.

(c) The *real* eigenvalues of the matrix are $\lambda_1 = -1.2728$, $\lambda_2 = 0.72297$ and $\lambda_3 = 0.51150$. Their corresponding eigenvectors are given below.

$$v_1 = \begin{bmatrix} 0.270745 \\ 0.105756 \\ -0.134606 \\ -0.282125 \\ 0.427441 \\ 0.450357 \\ -0.573209 \\ 0.261916 \\ -0.088249 \\ 0.085430 \\ -0.108734 \\ 0.086436 \\ -0.026892 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.217392 \\ 0.398094 \\ 0.287809 \\ -0.132159 \\ -0.324755 \\ 0.555261 \\ 0.401436 \\ 0.102629 \\ -0.138030 \\ 0.133001 \\ 0.096156 \\ -0.091204 \\ 0.232792 \end{bmatrix} \quad v_3 = \begin{bmatrix} -0.1703240 \\ 0.3917440 \\ 0.2003763 \\ -0.0946017 \\ -0.1814827 \\ 0.4788643 \\ 0.2449382 \\ -0.0017736 \\ -0.1661952 \\ 0.3427691 \\ 0.1753257 \\ -0.2955285 \\ 0.4277776 \end{bmatrix}$$

Question 2

(a) Graph of $\log_{10}$ of the condition number of $\mathbf{H}_n$ as a function of n for $n = 1, \ldots, 30$.

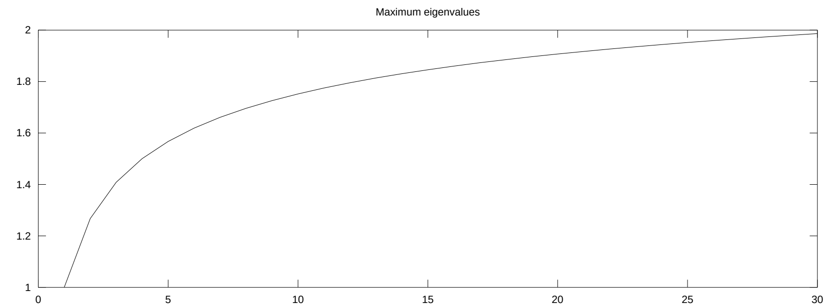


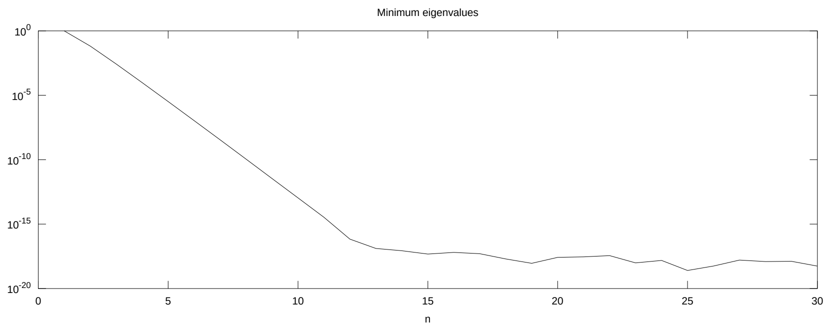
(b) The condition number is an increasing function of n for $n < 14$. For $n \geq 14$, the condition number no longer appears to depend on n.

(c) n can be as large as 8 to obtain a relative error less than 10^{-4}. When $n = 9$, the relative error is 1.0950×10^{-4}, which is slightly greater than 10^{-4}.

(d) Minimum and maximum eigenvalues of $\mathbf{H}_n$ for $n = 1, \ldots, 30$.

n	Minimum	Maximum
1	1.0000	1.0000
2	6.5741×10^{-2}	1.2676
3	2.6873×10^{-3}	1.4083
4	9.6702×10^{-5}	1.5002
5	3.2879×10^{-6}	1.5671
6	1.0828×10^{-7}	1.6189
7	3.4939×10^{-9}	1.6609
8	1.1115×10^{-10}	1.6959
9	3.4997×10^{-12}	1.7259
10	1.0930×10^{-13}	1.7519
11	3.4304×10^{-15}	1.7749
12	6.9144×10^{-17}	1.7954
13	1.2865×10^{-17}	1.8138
14	8.6284×10^{-18}	1.8306
15	4.8009×10^{-18}	1.8459
16	6.4139×10^{-18}	1.8600
17	5.0873×10^{-18}	1.8731
18	1.9835×10^{-18}	1.8852
19	9.0863×10^{-19}	1.8965
20	2.6251×10^{-18}	1.9071
21	2.8938×10^{-18}	1.9171
22	3.5883×10^{-18}	1.9265
23	9.7868×10^{-19}	1.9354
24	1.5220×10^{-18}	1.9438
25	2.5270×10^{-19}	1.9518
26	5.5925×10^{-19}	1.9594
27	1.6187×10^{-18}	1.9666
28	1.2497×10^{-18}	1.9735
29	1.2939×10^{-18}	1.9801
30	5.4938×10^{-19}	1.9865





Maximum eigenvalues as a function of n increases at a decreasing rate and appears to approach a limit of 2. For $n \leq 12$, minimum eigenvalues are a decreasing function of n, otherwise the dependence no longer seems to exist.

(e) The computed eigenvalues of $\mathbf{H}_n$ are real and positive for $n \leq 12$.

(f)

1.0000e+000	1.0000e+000	1.0000e+000
1.9281e+001	1.2676e+000	6.5741e-002
5.2406e+002	1.4083e+000	2.6873e-003
1.5514e+004	1.5002e+000	9.6702e-005
4.7661e+005	1.5671e+000	3.2879e-006
1.4951e+007	1.6189e+000	1.0828e-007
4.7537e+008	1.6609e+000	3.4939e-009
1.5258e+010	1.6959e+000	1.1115e-010
4.9315e+011	1.7259e+000	3.4997e-012
1.6025e+013	1.7519e+000	1.0930e-013
5.2124e+014	1.7749e+000	3.4304e-015
1.7033e+016	1.7954e+000	6.9144e-017

The larger the condition number, the greater the difference between the minimum and maximum eigenvalues. Hence, our solution is more sensitive to numerical errors as n increases.

(g) Our answer to (e) suggests that the Octave results for (a) and (d) are reliable for $n \leq 12$. However, if we look at the condition number in exponential notation $a \times 10^k$, the number of significant figures that should be believed is $15 - k$ (as a rule of thumb). We see from (f) that $n \leq 11$ is required to ensure $k < 15$ and thereby, not all the result can be ignored and the rule of thumb is satisfied.

Question 3

To formulate the objective function we first determined the machine costs, summarised below.

Product	Machine Cost ($)
A	4.65
B	4.50
C	5.00
D	4.80
E	1.95

We then denote the number of units of each product by x_i and maximise the following objective function

$$P = 5.35x_A + 4.5x_B + 5x_C + 4.7x_D + 3.05x_E$$

subject to the following constraints

$$\begin{aligned}
&15x_A + 8x_B + 8x_C + 12x_D + 9x_E \leq 4800 \\
&8x_A + 10x_B + 12x_C + 4x_D + 4x_E \leq 4800 \\
&6x_A + 9x_B + 10x_C + 12x_D \leq 4800 \\
&x_A, x_B, x_E \geq 0 \\
&x_C \geq 20 \\
&x_D \geq 30
\end{aligned}$$

The optimal solution to this linear programming problem is given below, with a maximum profit of $2469.

Product	Units
A	1
B	6
C	330
D	120
E	73

Appendix

1. (a)
```
%solves a system the 13x13 system of linear equations
%and estimates the error

k=sqrt(2)/2; %alpha

a=zeros(13,13); %initialise matrix and rhs to zeros
b=zeros(13,1);

a(1,2)=1; %1st equation
a(1,6)=-1;

a(2,3)=1; %2nd equation
b(2)=10;

a(3,1)=k;
a(3,4)=-1;
a(3,5)=-k;

a(4,1)=k;
a(4,3)=1;
a(4,5)=k;

a(5,4)=1;
a(5,8)=-1;

a(6,7)=1;

a(7,5)=k;
a(7,6)=1;
a(7,9)=-k
a(7,10)=-1;

a(8,5)=k;
a(8,7)=1;
a(8,9)=k;
b(8)=15;

a(9,10)=1;
a(9,13)=-1;

a(10,11)=1;
b(10)=20;
```

```
a(11,8)=1;
a(11,9)=k;
a(11,12)=-k;

a(12,9)=k;
a(12,11)=1;
a(12,12)=k;

a(13,12)=k;
a(13,13)=1;

x=a\b;
esterr=cond(a)*eps;
disp("Solution:"), disp(x)
disp("Estimated error:"), disp(esterr)
```

(b)

```
%compares estimated and actual errors for the 13x13 system
%of linear equations

x0=zeros(13,1);

x0(1)=-20*sqrt(2);
x0(2)=20;
x0(3)=10;
x0(4)=-30;
x0(5)=10*sqrt(2);
x0(6)=20;
x0(7)=0;
x0(8)=-30;
x0(9)=5*sqrt(2);
x0(10)=25;
x0(11)=20;
x0(12)=-25*sqrt(2);
x0(13)=25;

err=norm(x-x0)/norm(x0);
disp("Actual error:"), disp(err)
disp("Ratio estimate/actual:"), disp(esterr/err)
```

(c)
```
%computes the real eigenvalues of the matrix and their
%corresponding eigenvectors

eig(a)
[V,D]=eig(a)

k1=D(1,1)
v1=V(:,1)

k12=D(12,12)
v12=V(:,12)

k13=D(13,13)
v13=V(:,13)
```

2. (a)
```
%graphs the ln of the condition number of the Hilbert matrix
%as a function of n
chilb=zeros(30,1);
for n=1:30
chilb(n)=cond(hilb(n));
end
semilogy(chilb)
print('chilb.eps','-deps')
```

(c)
```
%computes the relative error in the solution for n=8,9
chilb(8)*eps
chilb(9)*eps
```

(d)
```
%finds the minimum and maximum eigenvalues of the
%Hilbert matrix for n=1,...,30
maxeig = zeros(30,1);
mineig = zeros(30,1);
for n = 1:30
eighilb = abs(eig(hilb(n)));
maxeig(n) = max(eighilb);
mineig(n) = min(eighilb);
end
subplot(2,1,1)
plot(maxeig)
title('Maximum eigenvalues')
subplot(2,1,2)
semilogy(mineig)
title('Minimum eigenvalues')
xlabel('n')
print('maxeig.eps','-deps')
```

(e)
```
t=1:1:12;
comp=[chilb(t) maxeig(t) mineig(t)]
```

3.
```
%calculates the machine costs by multiplying the time on each
%machine by the hourly rate of operating it
time=[15 8 6; 8 10 9; 8 12 10; 12 4 12; 9 4 0];
rate=[9;9;12];
cost=ones(5,1);
cost=1/60*time*rate

%solves profit maximisation problem
obj=[5.35 4.5 5 4.7 3.05]'
cnstr=[15 8 8 12 9; 8 10 12 4 4; 6 9 10 12 0]
rhs=[4800,4800,4800]
lb=[0 0 20 30 0]'
ub=[]
ctype="UUU"
vtype="IIIII"
ptype=-1
[x,opt]=glpk(obj,cnstr,rhs,lb,ub,ctype,vtype,ptype)
```