Sample Solutions for Tutorial 8

Question 1.

The function $f\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \ln(1+x)$ is continuous everywhere and differentiable on $\mathbb{R}^+$, with $f'(x) = \dfrac{1}{1+x}$

Take $x > 0$.

By the Mean Value Theorem, there is a c with $0 < c < x$ and

$$\frac{1}{1+c} = f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\ln(1+x)}{x}$$

As $0 < \frac{1}{1+c} < 1$,

$$\ln(1+x) < x$$

Question 2.

For $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{(x+1)^2}{x^2+1}$

$$\begin{aligned} f'(x) &= \frac{2(x+1)(x^2+1) - (x+1)^2 2x}{(x^2+1)^2} \\ &= 2\frac{1-x^2}{(x^2+1)^2} \\ &\begin{cases} < 0 & \text{for } x < -1 \text{ and for } x > 1 \\ = 0 & \text{for } x = \pm 1 \\ > 0 & \text{for } -1 < x < 1 \end{cases} \end{aligned}$$

Thus f is monotonically decreasing on $]-\infty, -1]$ and on $[1, \infty[$, whereas it is monotonically increasing on $[-1, 1]$. it has critical points at ± 1

$$\begin{aligned} f''(x) &= 2\frac{-2x(x^2+1)^4 - (1-x^2)(4x(x^2+1))}{(x^2+1)^4} \\ &= 4\frac{x(x^2-3)}{(x^2+1)^3} \\ &\begin{cases} < 0 & \text{for } x < -\sqrt{3} \text{ and for } 0 < x < \sqrt{3} \\ = 0 & \text{for } x = \pm 1 \\ > 0 & \text{for } -\sqrt{3} < x < 0 \text{ and for } x > \sqrt{3} \end{cases} \end{aligned}$$

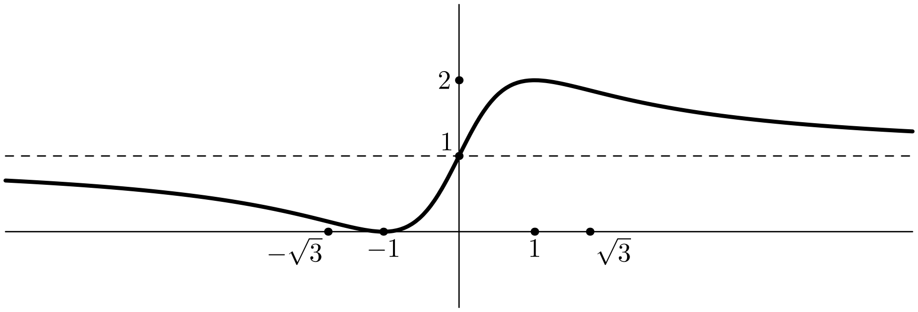
Thus f is concave down on $]-\infty, -\sqrt{3}]$ and on $[0, \sqrt{3}$, whereas it is concave up on $[-\sqrt{3}, 0]$ and on $[\sqrt{3}, \infty[$, and it has points of inflexion when $x = \pm\sqrt{3}$.

It follows that f has a local minimum of 0 at -1 and a local maximum of 2 at 1.

Since $f(x) \geq 0$ for all x, 0 is the absolute minimum value of f.

Since $f(x) \to 1$ as $x \to \pm\infty$, 2 is the absolute maximum value of f

Hence the graph of f is



Question 3.

It is immediate that $0 < x < 15$.
The area of the base of the container is $(30 - 2x)^2 = 4(15 - x)^2$ sq. cm.
Since its height is x cm., the volume, V is given by function

$$V\colon\ ]0, 15[\ \longrightarrow \mathbb{R}, \quad x \longmapsto 4x(15 - x)^2$$

Plainly, $V(x) > 0$ for all x, and $V(x) \to 0$ both as $x \to 0^+$ and as $x \to 15^-$.
The domain of V contains no boundary points.
Being a polynomial functional function, V is differentiable everywhere.
Thus extrema occur only where the derivative of V is 0.

$$\begin{aligned}
\frac{dV}{dx} &= 4\left(\frac{d}{dx}(x)(x-15)^2 + x\frac{d}{dx}(x-15)^2\right) \\
&= 4\Big((x-15)^2 + x.2.(x-15)\Big) \\
&= 12(x-15)(x-5) \\
&\begin{cases} > 0 & \text{for } 0 < x < 5 \\ = 0 & \text{for } x = 5 \\ < 0 & \text{for } 5 < x < 15 \end{cases}
\end{aligned}$$

Hence, the only extremum occurs when $x = 5$, and $V(5) = 2,000$ is a maximum.
Thus, the maximum capacity of the container is 2 litres, achieved when the small squares have sides of length 5 centimetres.