Test 8.

Evaluate the iterated integral

1) $\int_0^1 \int_{x^2}^{x} xy^2\,dy\,dx =$

$= \int_0^1 \left(x\frac{y^3}{3}\Big|_{x^2}^{x} \right) dx = \frac{1}{3}\int_0^1 x^4 - x^7 \; dx =$

$= \frac{1}{3}\left(\frac{x^5}{5} - \frac{x^8}{8} \right)\Big|_0^1 = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8} \right) = \frac{1}{40}$

2) $\int_0^x e^{x^2}\,dy\,dx = \int_0^1 y e^{x^2}\Big|_0^x \, dx = \int_0^1 x e^{x^2}\,dx =$

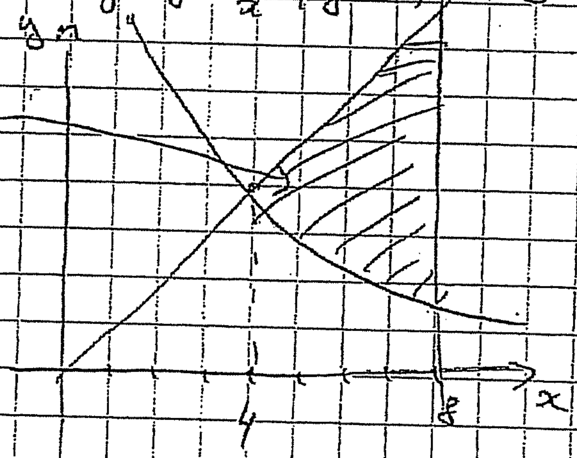
$= \frac{1}{2}\int e^t\,dt = \frac{1}{2} e^{x^2}\Big|_0^1 = \frac{1}{2}(e-1)$

$\boxed{\begin{aligned} t &= x^2 \\ dt &= 2x\,dx \end{aligned}}$

Calculate the double integral

1) $\iint_R x^2\,dA$ R: bounded by $y = \frac{16}{x}$, $y = x$, $x = 8$

type I region R

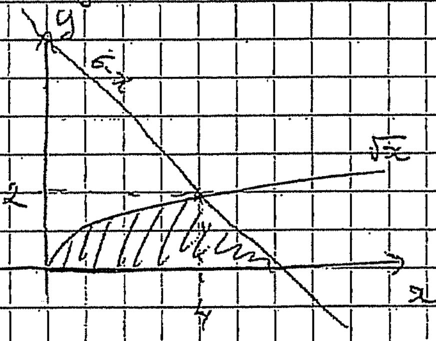


$\int_4^8 \int_{16/x}^{x} x^2\,dy\,dx =$

$\int_4^8 x^2 y\Big|_{\frac{16}{x}}^{x} dx = \int_4^8 (x^3 - 16x)\,dx =$

$= \frac{x^4}{4} - 8x^2\Big|_4^8 = \frac{8^4}{4} - 8^3 - \frac{4^4}{4} + 8\cdot 4^2 = 576$

(2) $\iint_R x\,y\,dA$ $\quad R:\ y=\sqrt{x},\ y=6-x,\ y=0$



type II region

$$\int_0^2\int_{y^2}^{6-y} xy\,dx\,dy =$$

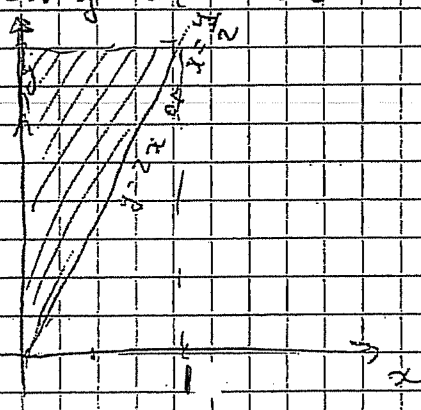
$$= \int_0^2 \frac{x^2}{2}\, y\,\Big|_{y^2}^{6-y} dy =$$

$$= \frac{1}{2}\int_0^2 \left((6-y)^2 y - y^4 \cdot y\right) dy =$$

$$= \frac{1}{2}\int_0^2 \left(36y - 12y^2 + y^3 - y^5\right) dy =$$

$$= \frac{1}{2}\left(18y^2 - 4y^3 + \frac{y^4}{4} - \frac{y^6}{6}\right)\Big|_0^2 = \frac{1}{2}\left(18\times 4 - 4\times 8 + 4 - \frac{32}{3}\right) = \frac{50}{3}$$

Evaluate integral by first reversing the order of integration



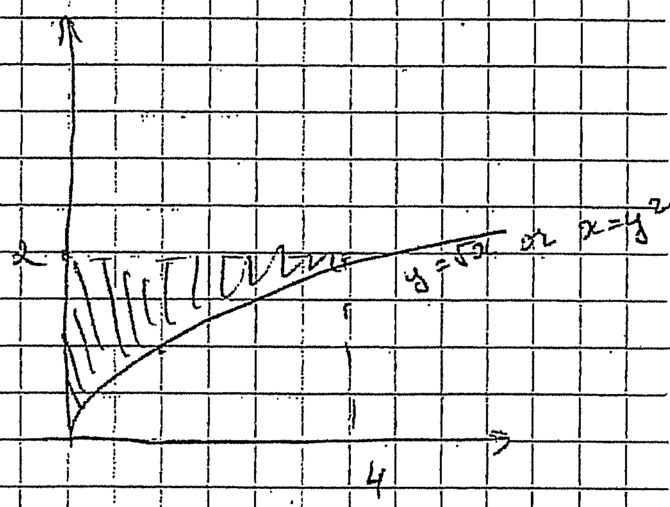
(1) $\int_0^2\int_{y/2}^1 \cos(x^2)\,dx\,dy$

$$= \int_0^1\int_0^{2x} \cos(x^2)\,dx\,dy =$$

$$= \int_0^1 y\cos(x^2)\Big|_0^{2x} dx = \int_0^1 2x\cos(x^2)\,dx =$$

$$= \int_0^1 \cos(x^2)\,d(x^2) = \sin(x^2)\Big|_0^1 = \sin 1$$

② $\int_0^4 \int_{\sqrt{x}}^2 \sin \pi y^3 \, dy \, dx$



$\int_0^2 \int_0^{y^2} \sin \pi y^3 \, dx \, dy =$

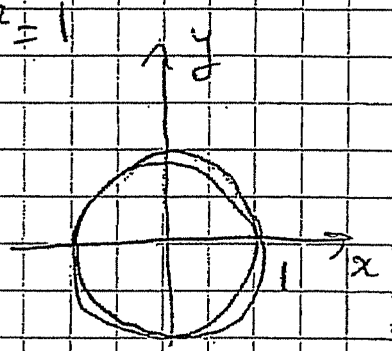
$\int_0^2 x \sin \pi y^3 \Big|_0^{y^2} dy =$

$\int_0^2 y^2 \sin \pi y^3 \, dy = -\frac{1}{3\pi} \cos \pi y^3 \Big|_0^2 = -\frac{1}{3\pi}(\cos 8\pi - \cos 0) = 0$

Use the polar coordinates to evaluate the double integral

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \times r \, dr \, d\theta$$

① $\iint_R e^{-(x^2+y^2)} dA$ R enclosed by the circle $x^2+y^2=1$



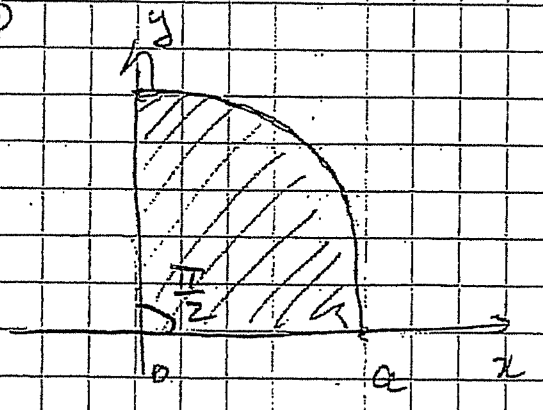
$\int_0^{2\pi} \int_0^1 e^{-r^2} \times r \, dr \, d\theta =$

$= -\frac{1}{2}\int_0^{2\pi} e^{-r^2}\Big|_0^1 d\theta = -\frac{1}{2}\int_0^{2\pi} (e^{-1} - 1) \, d\theta =$

$= \frac{1}{2}\theta(e^{-1} - 1)\Big|_0^{2\pi} = \pi\left(1 - \frac{1}{e}\right)$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$
$= r^2$

② $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{(1+x^2+y^2)^{3/2}} \quad , \ a>0$$



$$\int_0^{\pi/2} \int_0^a \frac{r}{(1+r^2)^{3/2}}\, dr\, d\theta =$$

$$\int_0^{\pi/2} \int_0^a \frac{1}{2}(1+r^2)^{-3/2}\, d(1+r^2) =$$

$$= \frac{1}{\cancel{2}} \int_0^{\pi/2} -\cancel{2}(1+r^2)^{-\frac{1}{2}}\Big|_0^a\, d\theta = -\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{1+a^2}} - 1\right) d\theta =$$

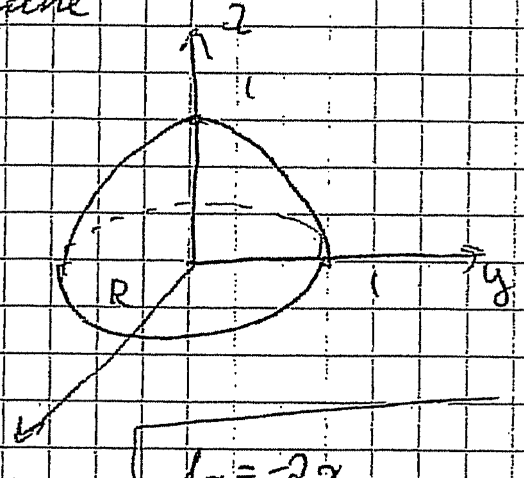
$$= \left(1 - \frac{1}{\sqrt{1+a^2}}\right)\theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{1+a^2}}\right)$$

Find the area of the given surface:

The portion of the paraboloid $z = 1 - x^2 - y^2$ that is above the xy-plane

$$\iint_R \sqrt{f_x^2 + f_y^2 + 1}\; dA$$

$R: \quad x^2 + y^2 \le 1$



$$\cancel{\int_0^{2\pi} \int_0^1 \sqrt{4x^2 + 4y^2 + 1}\; \cancel{dx\,dy}\, dx\, dv}$$

$$\begin{cases} f_x = -2x \\ f_y = -2y \end{cases}$$

$$\iint_R \sqrt{4x^2 + 4y^2 + 1}\; dA =$$

$$\int_0^{2\pi} \int_0^1 \sqrt{4r^2+1} \times r\; dr\, d\theta =$$

$4r^2 + 1 = t$

$8r\,dr = dt$

$$\iint \frac{1}{8}\sqrt{t}\,dt\,d\theta = \frac{1}{8}\int_0^{2\pi} \frac{2}{3}\,t^{3/2}\Big|\,d\theta =$$

$$= \frac{1}{12}\int_0^{2\pi} \left(4r^2+1\right)^{3/2}\Big|_0^1\,d\theta = \frac{1}{12}\int_0^{2\pi} \left(5\sqrt{5}-1\right)d\theta =$$

$$= \frac{1}{12}\left(5\sqrt{5}-1\right)\theta\Big|_0^{2\pi} = \frac{\pi}{6}\left(5\sqrt{5}-1\right)$$