PMTH212 ASSIGNMENT 2

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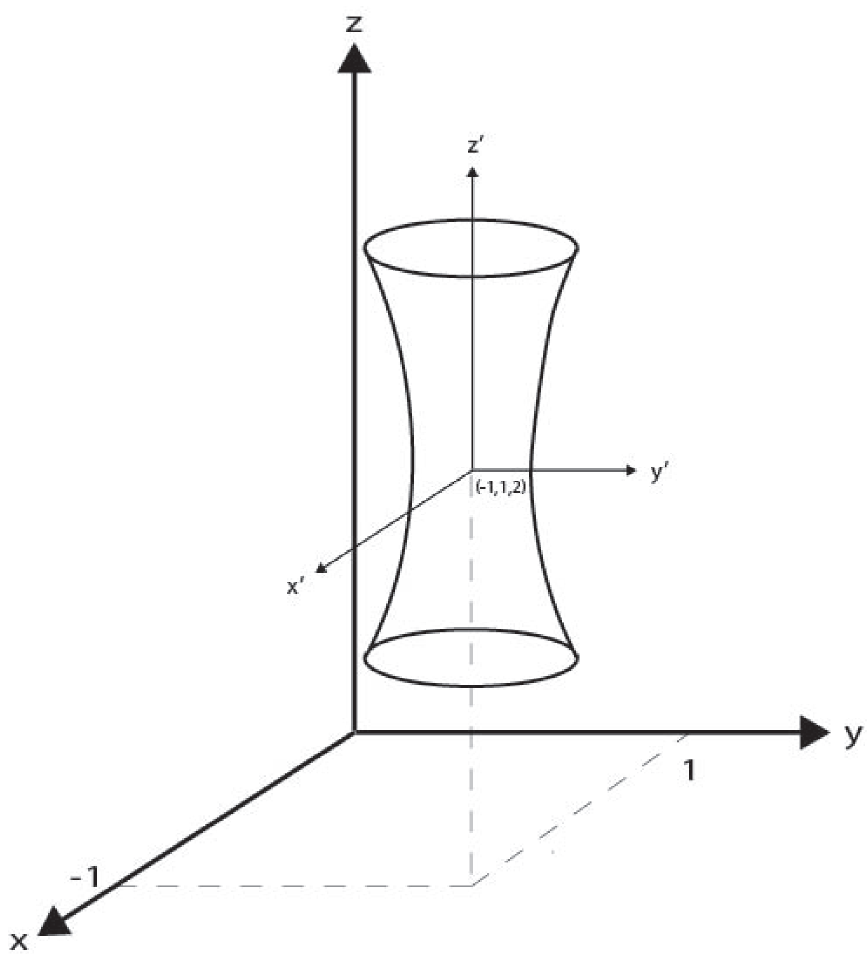
(1) The surface is a hyperboloid of 1 sheet with centre $(-1, 1, 2)$.

$$\begin{aligned} z^2 - 4z &= 4x^2 + 8x + y^2 - 2y \\ (z-2)^2 - 4 &= 4(x^2 + 2x) + (y-1)^2 - 1 \\ (z-2)^2 &= 4\big[(x+1)^2 - 1\big] + (y-1)^2 + 3 \\ (z-2)^2 &= 4(x+1)^2 + (y-1)^2 - 1 \\ 1 &= 4(x+1)^2 + (y-1)^2 - (z-2)^2 \end{aligned}$$



(2) The orthogonal projection onto the xy-plane of the curve of intersection is given by

$$z = 4 - x^2 - y^2 = y^2 \ \Rightarrow\ x^2 + 2y^2 = 4$$

Since this equation does not contain z the projection is a cylindrical surface.

(3) (a) The graph is a circle with radius 3 and centre $(0, 0)$.

$$\begin{aligned} &x = 3\sin 2t, \quad y = 3\cos 2t \\ &x^2 + y^2 = 9\sin^2 2t + 9\cos^2 2t = 9 \\ &(x-0)^2 + (y-0)^2 = 3^2 \end{aligned}$$

(b) The graph is a parabola with vertex $(y, z) = (0, -1)$ which lies on the plane $x = -2$.

$$x = -2, \quad y = t, \quad z = t^2 - 1 \ \Rightarrow\ z = y^2 - 1$$

(4) $\mathbf{r}$ is a smooth curve of the parameter t since $\mathbf{r}'(t)$ exists and is continuous and $\mathbf{r}'(t) \neq 0$ for all t. We examine the components of $\mathbf{r}'(t)$ to confirm this.

$$\begin{aligned} &x(t) = \cos t^2, \quad y(t) = \sin t^2, \quad z(t) = e^{-t} \\ &x'(t) = -2t\sin t^2, \quad y'(t) = 2t\cos t^2, \quad z'(t) = -e^{-t} \\ &\mathbf{r}'(t) = (-2t\sin t^2)\mathbf{i} + (2t\cos t^2)\mathbf{j} + (-e^{-t})\mathbf{k} \end{aligned}$$

Since $-2t\sin t^2,\ 2t\cos t^2,\ -e^{-t}$ are all continuous functions of t, then $\mathbf{r}'(t)$ is also continuous. Moreover,

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = 4t^2 + e^{-2t} > 0 \text{ for all } t$$

Hence, there is no value of t for which all three components are zero.

(5)

$$\begin{aligned} \frac{d}{dt}\big[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\big] &= \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot\frac{d(\mathbf{v}\times\mathbf{w})}{dt} \\ &= \frac{d\mathbf{u}}{dt}\cdot\big[\mathbf{v}\times\mathbf{w}\big] + \mathbf{u}\cdot\left[\mathbf{v}\times\frac{d\mathbf{w}}{dt} + \frac{d\mathbf{v}}{dt}\times\mathbf{w}\right] \\ &= \frac{d\mathbf{u}}{dt}\cdot\big[\mathbf{v}\times\mathbf{w}\big] + \mathbf{u}\cdot\left[\frac{d\mathbf{v}}{dt}\times\mathbf{w}\right] + \mathbf{u}\cdot\left[\mathbf{v}\times\frac{d\mathbf{w}}{dt}\right] \end{aligned}$$

(6) (a) Using integration by parts: $u = t, \quad \dfrac{dv}{dt} = \sin t, \quad \dfrac{du}{dt} = 1, \quad v = -\cos t$

$$\int t\sin t\ dt = -t\cos t - \int(-\cos t)\ dt = -t\cos t + \sin t + c$$

$$\begin{aligned} \int \big[(t\sin t)\mathbf{i} + \mathbf{j})\big]dt \ &= \int(t\sin t)\mathbf{i}\ dt\ + \int\mathbf{j}\ dt \\ &= (\sin t - t\cos t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} \\ &= (\sin t - t\cos t)\mathbf{i} + t\mathbf{j} + \vec{C} \end{aligned}$$

(b)

$$
\begin{aligned}
x(t) &= 3\cos t, \quad y(t) = 3\sin t, \quad z(t) = t \\
x'(t) &= -3\sin t, \quad y'(t) = 3\cos t, \quad z'(t) = 1 \\
L &= \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 1}\ dt \\
&= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t + 1}\ dt \\
&= \int_0^{2\pi} \sqrt{9+1}\ dt = \left[\sqrt{10}\ t\right]_0^{2\pi} = 2\pi\sqrt{10}
\end{aligned}
$$