Sample Solutions for Tutorial 5

Question 1.

$$(2,2,2) \;=\; a(0,-1,1) + b(1,-1,3) \qquad\qquad (a,b \in \mathbb{R})$$

if and only if

$$\begin{array}{lrcl} \text{(i)} & b &=& 2 \\ \text{(ii)} & -a-b &=& 2 \\ \text{(iii)} & a+3b &=& 2 \end{array}$$

By (i), $b = 2$, whence, by (ii), $a = -4$. When we substitute these into (iii), we do, indeed, obtain equality.

Thus $(2,2,2) = -4(0,-1,1) + 2(1,-1,3)$.

$$(3,1,5) \;=\; a(0,-1,1) + b(1,-1,3) \qquad\qquad (a,b \in \mathbb{R})$$

if and only if

$$\begin{array}{lrcl} \text{(i)} & b &=& 3 \\ \text{(ii)} & -a-b &=& 1 \\ \text{(iii)} & a+3b &=& 5 \end{array}$$

Adding (ii) and (iii), we see that $\mu = 6$, contradicting (i). Thus $(3,1,5)$ is not a linear real combination of $(0,-1.1)$ and $(1,-1,3)$.

Question 2.

(a)

$$\lambda(4t^2 - t + 2) + \mu(2t^2{+}6t + 3) + \nu(-4t^2 + 10t + 2) = 0$$

if and only if

$$\begin{array}{lrcl} \text{(i)} & 2\lambda + \mu - 2\nu &=& 0 \\ \text{(ii)} & -\lambda + 6\mu + 10\nu &=& 0 \\ \text{(iii)} & 2\lambda + 3\mu + 2\nu &=& 0 \end{array}$$

Adding $2\times$ (ii) to (i) and to (iii)

$$\begin{array}{lrcl} \text{(iv)} & 13\mu + 18\nu &=& 0 \\ \text{(v)} & \lambda - 6\mu - 10\nu &=& 0 \\ \text{(vi)} & 15\mu + 22\nu &=& 0 \end{array}$$

Subtracting $15\times$ (iv) from $13\times$ (vi)

$$\begin{array}{lrcl} \text{(vii)} & 13\mu + 18\nu &=& 0 \\ \text{(viii)} & \lambda - 6\mu - 10\nu &=& 0 \\ \text{(ix)} & 16\nu &=& 0 \end{array}$$

if and only if $\lambda = \mu = \nu = 0$.

Hence the given polynomials are linearly independent.

(b)

$$\lambda(4t^2 - t + 2) + \mu(2t^2{+}6t + 3) + \nu(6t^2 + 5t + 5) = 0$$

if and only if

$$\begin{array}{lrcl} \text{(i)} & 2\lambda + \mu + 3\nu & = & 0 \\ \text{(ii)} & -\lambda + 6\mu + 5\nu & = & 0 \\ \text{(iii)} & 2\lambda + 3\mu + 5\nu & = & 0 \end{array}$$

Adding $2\times$ (ii) to (i) and to (ii)

$$\begin{array}{lrcl} \text{(iv)} & 13\mu + 13\nu & = & 0 \\ \text{(v)} & \lambda - 6\mu - 5\nu & = & 0 \\ \text{(vi)} & 15\mu + 15\nu & = & 0 \end{array}$$

if and only if $\lambda = \mu = -\nu$.

Thus the third polynomial is the sum of the first two, whence they are linearly dependent.

(c)

$$\lambda(t^2 + t + 23) + \mu(5t^2{-}t) + \nu(2) = 0$$

if and only if

$$\begin{array}{lrcl} \text{(i)} & \lambda + 5\mu & = & 0 \\ \text{(ii)} & \lambda - \mu & = & 0 \\ \text{(iii)} & 2\lambda + 2\nu & = & 0 \end{array}$$

Clearly, $\lambda = \mu = \nu = 0$ is the only solution, whence the polynomials are linearly independent.

(d) We are given four polynomials in $\mathcal{P}_2$. But $\mathcal{P}_2$is generated by the three polynomials $1, t$ and t^2. Hence any four polynomials in $\mathcal{P}_2$ must be linearly dependent.

Question 3.

(a) Suppose that $\alpha \cos(2x) + \beta \sin(x) + \gamma 7 = 0$ for all $x \in \mathbb{R}$. Then, by successively putting $x = 0, \frac{\pi}{4}, -\frac{5\pi}{4}$, we obtain

$$\begin{array}{lrcl} \text{(i)} & \alpha + 7\gamma & = & 0 \\ \text{(ii)} & \frac{1}{\sqrt{2}}\beta + 7\gamma & = & 0 \\ \text{(iii)} & \frac{-1}{\sqrt{2}}\beta + 7\gamma & = & 0 \end{array}$$

It follows from (ii) and (iii) that $\gamma = 0$, whence $\alpha = \beta = 0$, showing that the given functions are linearly independent.

(b) Suppose that $\alpha \ln(x^2 + 1) + \beta \sin(x) + \gamma e^x = 0$ for all $x \in \mathbb{R}$. Then, by successively putting $x = 0, \pi, -\frac{\pi}{2}$, we obtain

$$\begin{array}{lr} \text{(i)} & \gamma = 0 \\ \text{(ii)} & \alpha \ln(\pi^2 + 1)\gamma e^{\pi} = 0 \\ \text{(iii)} & \alpha \ln\left((\frac{\pi}{2})^2 + 1\right) + \beta + \gamma e^{\frac{\pi}{2}} = 0 \end{array}$$

It follows by substituting (i) in (ii) and the results in (iii) that $alpha = \beta = \gamma = 0$, showing that the given functions are linearly independent.