Sample Solutions for Tutorial 4

Question 1.

Recall that for all $u, v \in \mathbb{R}$, $v^2 - u^2 = (v-u)(v+u)$. Consequently, if $v > u$ then $v^2 \geq u^2$ if and only if $v + u \geq 0$ and $v^2 \leq u^2$ if and only if $v + u \leq 0$.
(a) If $u, v \in \mathbb{R}_0^-$ and $u < v$, then $v + u < 0$, whence $f(u) > f(v)$, showing that f is monotonically decreasing.
(b) If $u, v \in \mathbb{R}_0^+$ and $u < v$, then $v + u > 0$, whence $g(u) < g(v)$, showing that g is monotonically increasing.
(c) If $u, v \in \mathbb{R}$ and $u < v$, then $v + u$ can be positive or negative. If $u = -2$ and $v = 1$, then $u < v$ and $h(u) = 4 > 1 = h(v)$. If $u = -1$ and $v = 2$, then $u < v$ and $h(u) = 1 > 4 = h(v)$. Hence h is not monotonic.

Question 2.

(a) Since $x^2 - 4 = (x-2)(x+2)$ for all x, $\dfrac{x^2-4}{x-2} = x + 2$ whenever $x \neq 2$.

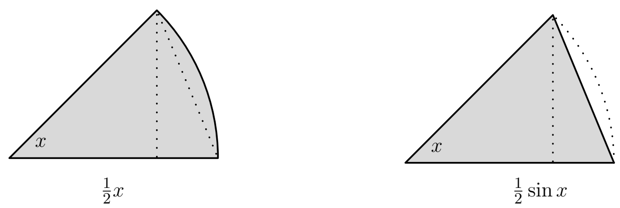
We conjecture that $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = 4$.
Given $\varepsilon > 0$, put $\delta := \varepsilon$.
Now $|x-2| < \delta$ if and only if $2 - \delta < x < 2 + \delta$.
In such a case $4 - \delta < x + 2 < 4 + \delta$, or equivalently, $|(x+2) - 4| < \delta = \varepsilon$
Hence, given $\varepsilon > 0$ there is a $\delta > 0$ with $\left|\frac{x^2-4}{x-2}\right| < \varepsilon$ whenever $|x-2| < \delta$ showing that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

(b) From the geometric definition of the cosine function, we conjecture that $\lim\limits_{x\to 0} \cos x = 1$.
To test this conjecture, we use properties of the trigonometric functions.
Recall that $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$.
Thus $\cos 2A - 1 = 2\sin^2 A$. Putting $A := \frac{x}{2}$, we see that

$$\cos x - 1 = -\frac{1}{2}\sin^2 \frac{x}{2}.$$

As indicated by the next diagrams, $\sin x \leq x$ for $0 < x < \frac{\pi}{2}$.



Thus

$$\begin{aligned}
|\cos x - 1| &= |2\sin^2 \frac{x}{2}| \\
&\leq |2\frac{x}{2}\sin\frac{x}{2}| && \text{by the above} \\
&\leq x && \text{as } |\sin y| \leq 1 \text{ for all } y
\end{aligned}$$

Hence, given $\varepsilon > 0$, choose $\delta; = \min\{\varepsilon, \frac{\pi}{4}\}$. If $|x| < \delta$, then $|\cos x - 1| \leq |x| \leq \delta \leq \varepsilon$, showing that $\lim\limits_{x\to 0} \cos x = 1$.

(c) If $x < 0$, then $|x| = -x$, whence $\frac{|x|}{x} = -1$.

If $x > 0$, then $|x| = x$, whence $\frac{|x|}{x} = 1$.

Hence we conjecture that $\lim_{x \to 0} \dfrac{|x|}{x}$ does not exist.

To verify this, suppose to the contrary, that $\lim_{x \to 0} \dfrac{|x|}{x} = \ell$.

Then, since $1 > 0$, there is a $\delta > 0$ such that if whenever $|x| < \delta$, then

$$\left| \frac{|x|}{x} - \ell \right| < 1$$

Put $u := -\frac{\delta}{2}$ and $v := \frac{\delta}{2}$. Then $|u|, |v| < \delta$, and so

$$\begin{aligned} 2 &= \left| \frac{|v|}{v} - \frac{|u|}{u} \right| \\ &\leq \left| \frac{|v|}{v} - \ell \right| + \left| \frac{|u|}{u} - \ell \right| \\ &< 1 + 1, \end{aligned}$$

which is a contradiction. Hence, $\lim_{x \to 0} \dfrac{|x|}{x}$ does not exist.

(d) We conjecture that $\lim_{x \to 0} \dfrac{1}{x^2 + 1} = \dfrac{1}{0 + 1} = 1$.

Suppose that $|x| < \delta$ for some $\delta > 0$.

Then $0 < x^2 = |x|^2 < \delta^2$.

Hence $1 < x^2 + 1 = |x|^2 < \delta^2 + 1$, so that $0 < \dfrac{1}{\delta^2 + 1} < \dfrac{1}{x^2 + 1}$, from which it follows that

$$0 < 1 - \frac{1}{x^2 + 1} < 1 - \frac{1}{\delta^2 + 1} = \frac{\delta^2}{\delta^2 + 1}$$

Observe that $\frac{\delta^2}{\delta^2+1} < 1$ for all real δ. So, given ε, put $r = \min\{\varepsilon, \frac{1}{2}\}$, so that $0 < r \leq \varepsilon$.

Since $\frac{\delta^2}{\delta^2+1} < r$ if and only if $\delta^2 < \frac{r}{1-r}$, choose $\delta := \sqrt{\frac{r}{1-r}}$. Then $\frac{\delta^2}{\delta^2+1} = r \leq \varepsilon$.

Hence, if $|x| < \delta$, then $|\frac{1}{x^2+1} < \frac{\delta^2}{\delta^2+1} = r \leq \varepsilon$, showing that $\lim_{x \to 0} \dfrac{1}{x^2 + 1} = 1$.