AMTH140 ASSIGNMENT 6

MARK VILLAR

(1)

$$\begin{aligned}
&f(0) = 1\\
&f(n) = 1 + \sum_{j=0}^{n-1} j f(j), \quad n \geq 1\\
&f(1) = 1 + 0 \cdot f(0) = 1 + 0 = 1\\
&f(2) = 1 + f(1) = 1 + 1 = 2\\
&f(3) = 1 + f(1) + 2f(2) = 1 + 1 + 4 = 6\\
&f(4) = 1 + f(1) + 2f(2) + 3f(3) = 1 + 1 + 4 + 18 = 24
\end{aligned}$$

(2) (a) No since one of the dependent functions a_k^2 is quadratic, not linear. However it is a constant coefficient recurrence relation since all coefficients c_k are constant.

$$\begin{aligned}
a_{k+1} &= -a_k^2 - k^2\\
a_n &= -a_{n-1}^2 - (n-1)^2\\
a_0 &= 1\\
a_1 &= -a_0^2 - (1-1)^2 = -1^2 - 0 = -1\\
a_2 &= -a_1^2 - (2-1)^2 = -(-1)^2 - 1^2 = -1 - 1 = -2\\
a_3 &= -a_2^2 - (3-1)^2 = -(-2)^2 - 2^2 = -4 - 4 = -8\\
a_4 &= -a_3^2 - (4-1)^2 = -(-8)^2 - 3^2 = -64 - 9 = -73
\end{aligned}$$

(b)

$$\begin{aligned}
&f(n) = 3f(n-1) \;\Rightarrow\; \lambda = 3\\
&f(1) = 3f(0) = 4 \;\Rightarrow\; f(0) = \frac{4}{3}\\
&f(n) = A3^n \;\Rightarrow\; f(0) = A3^0 \;\Rightarrow\; A = \frac{4}{3}\\
&f(n) = \frac{4}{3} \cdot 3^n = 4 \cdot 3^{n-1}, \quad n \geq 1
\end{aligned}$$

(3)

$$
\begin{aligned}
&a_{n+2}+a_{n+1}-12a_n=0 \;\Rightarrow\; \lambda^2+\lambda-12=0\\
&(\lambda+4)(\lambda-3)=0 \;\Rightarrow\; \lambda_1=-4,\; \lambda_2=3\\
&a_n=A\,(-4)^n+B\,3^n, \quad n\geq 0
\end{aligned}
$$

(4)

$$a_{n+2}-2a_{n+1}-63a_n=64n, \quad n\geq 0$$

(i)

$$
\begin{aligned}
&\lambda^2-2\lambda-63=0 \;\Rightarrow\; (\lambda-9)(\lambda+7)=0\\
&\lambda=-7,9 \;\Rightarrow\; u_n=A(-7)^n+B\,9^n
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&v_n=Cn+D \;\Rightarrow\; v_{n+2}-2vn+1-63v_n=64n\\
&\Rightarrow \big(C(n+2)+D\big)-2\big(C(n+1)+D\big)-63\big(Cn+D\big)=64n\\
&\Rightarrow Cn+2C+D-2Cn-2C-2D-63Cn-63D=64n\\
&\Rightarrow -64Cn-64D=64n \;\Rightarrow\; n=-Cn-D\\
&\Rightarrow\; C=-1,\; D=0 \;\Rightarrow v_n=-n
\end{aligned}
$$

(iii)

$$a_n=u_n+v_n=A(-7)^n+B\,9^n-n, \quad n\geq 0$$

(iv)

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\begin{aligned}
a_0&=A+B=0 \;\Rightarrow A=-B\\
a_1&=-7A+9B-1=7\\
&=-7A-9A-1=7\\
&\Rightarrow -16A=8,\; A=-\frac{1}{2},\; B=\frac{1}{2}\\
&\Rightarrow a_n=-\frac{1}{2}(-7)^n+\frac{1}{2}(9)^n-n, \quad n\geq 0
\end{aligned}
$$

(5)

$$b_{n+2}-4b_{n+1}+4b_n=8\times 2^n, \quad n\in\mathbb{N}$$

(i)

$$
\begin{aligned}
&\lambda^2-4\lambda+4=0 \;\Rightarrow\; (\lambda-2)^2=0\\
&\lambda=2,\; m=2 \;\Rightarrow\; u_n=(An+B)\,2^n
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&v_n = Cn^2\,2^n \;\Rightarrow\; v_{n+2} - 4v_{n+1} + 4v_n = 8 \times 2^n \\
&\Rightarrow \big(C(n+2)^2\,2^{n+2}\big) - 4\big(C(n+1)^2\,2^{n+1}\big) + 4\big(Cn^2\,2^n\big) = 8 \times 2^n \\
&\Rightarrow \big(C(n^2+4n+4)\,2^{n+2}\big) - 4\big(C(n^2+2n+1)\,2^{n+1}\big) + 4\big(Cn^2\,2^n\big) = 8 \times 2^n \\
&\Rightarrow Cn^2\,2^{n+2} + 4Cn\,2^{n+2} + 4C\,2^{n+2} - 4Cn^2\,2^{n+1} - 8Cn\,2^{n+1} - 4C\,2^{n+1} \\
&\qquad + 4Cn^2\,2^n = 8 \times 2^n \\
&\Rightarrow 4Cn^2\,2^n + 16Cn\,2^n + 16C\,2^n - 8Cn^2\,2^n - 16Cn\,2^n - 8C\,2^n + 4Cn^2\,2^n \\
&\qquad = 8 \times 2^n \\
&\Rightarrow \big(4Cn^2 + 16Cn + 16C - 8Cn^2 - 16Cn - 8C + 4Cn^2\big) \times 2^n = 8 \times 2^n \\
&\Rightarrow 8C \times 2^n = 8 \times 2^n \;\Rightarrow C = 1 \;\Rightarrow\; v_n = n^2\,2^n
\end{aligned}
$$

(iii)

$$
\begin{aligned}
b_n &= u_n + v_n = (An+B)\,2^n + n^2\,2^n \\
&= (n^2 + An + B)\,2^n, \quad n \in \mathbb{N}
\end{aligned}
$$