Chapter 11

The Determinant and the Trace

This chapter introduces two important functions defined on matrices: the *determinant of an $n \times n$ matrix* and the *trace* of an arbitrary matrix.

We show that the determinant is unique and derive its properties.

We shall also see that both the determinant and the trace are *invariant* in the sense that for $\underline{\mathbf{A}}, \underline{\mathbf{B}} \in \mathbf{M}(n; \mathbb{F})$ with $\underline{\mathbf{B}}$ invertible, $\underline{\mathbf{B}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{B}}$ has the same determinant and trace as $\underline{\mathbf{A}}$.

This fact means we can define the determinant and trace of a linear transformation $T\colon V \to V$ whenever V is finite dimensional.

11.1 The Determinant

Main Theorem. *There is a unique function $D\colon \mathbf{M}(n;\mathbb{F}) \to \mathbb{F}$ with the three properties below.*

D1 *D is linear in each row.*

D2 $D(\underline{\mathbf{A}}) = 0$ *whenever* $\operatorname{rk}\underline{\mathbf{A}} < n$.

D3 $D(\underline{\mathbf{1}}_n) = 1$

Definition 11.1. The unique function $D : \mathbf{M}(n;\mathbb{F}) \longrightarrow \mathbb{F}$ satisfying D1, D2 and D2 is called the *determinant* function. We write

$$\det\colon \mathbf{M}(n;\mathbb{F}) \longrightarrow \mathbb{F}, \quad \underline{\mathbf{A}} \longmapsto \det(\underline{\mathbf{A}}) = |\underline{\mathbf{A}}|.$$

Observation 11.2. Strictly speaking, there is not just one determinant function, but a whole family of determinant functions, one for each counting number n. However, it is customary to speak as if there were only one, and, in any case, there is little danger of confusion.

Our proof of Theorem 11.6 will make use of elementary row operations, formulated in terms of matrices in Section 10.2.1.

Observation 11.3. The third elementary row operation — swapping rows of matrix — is superfluous, since it can be replaced by a sequence of the other two:

Exchanging the i^{th} and j^{th} rows can be achieved by

1. adding the i^{th} row to the j^{th}, then
2. subtracting the j^{th} row from the i^{th}, then

3. adding the i^{th} row to the j^{th} and, finally,
4. multiplying the i^{th} row by -1.

In terms of the matrices just introduced,

$$\underline{\mathbf{S}}(i,j) = \underline{\mathbf{M}}(i,-1)\underline{\mathbf{A}}(j,i)\underline{\mathbf{A}}(i,(-1)j)\underline{\mathbf{A}}(i,j)$$

Hence we may dispense with our third elementary row operation, and the matrices $\underline{\mathbf{S}}(i,j)$.

Lemma 11.4. *If function $G\colon \mathbf{M}(n;\mathbb{F}) \to \mathbb{F}$ satisfies D1 and D2, then, for each $\underline{\mathbf{B}} \in \mathbf{M}(n;\mathbb{F})$,*

(i) $\quad G(\underline{\mathbf{M}}(i,\lambda)\underline{\mathbf{B}}) = \lambda G(\underline{\mathbf{B}})$

(ii) $\quad G(\underline{\mathbf{A}}(i,\mu j)\underline{\mathbf{B}}) = G(\underline{\mathbf{B}})$

Proof. **(i)** This follows immediately from D1.

(ii) Given the matrix $\underline{\mathbf{B}}$, let $\underline{\mathbf{B}}^{(i)}_{(j)}$ be obtained by replacing the i^{th} row of $\underline{\mathbf{B}}$ by its j^{th} row. Since it has two identical rows, $\text{rk}(\underline{\mathbf{B}}^{(i)}_{(j)}) < n$. Thus

$$\begin{aligned} G(\underline{\mathbf{A}}(i,\mu j)\underline{\mathbf{B}}) &= G(\underline{\mathbf{B}}) + \mu G(\underline{\mathbf{B}}^{(i)}_{(j)}) && \text{by D1} \\ &= G(\underline{\mathbf{B}}) && \text{since, by D2, } G(\underline{\mathbf{B}}^{(i)}_{(j)}) = 0. \end{aligned}$$

□

Before continuing, we prove a useful technical lemma, whose proof is essentially the Gauß-Jordan algorithm for transforming a matrix to reduced row echelon form.

Lemma 11.5. *The rank of $\underline{\mathbf{B}} \in \mathbf{M}(n;\mathbb{F})$ is n if and only if $\underline{\mathbf{B}}$ can be transformed into $\underline{\mathbf{1}}_n$ by means of elementary row operations.*

Proof. $\Leftarrow$: This follows immediately from Corollary 10.17 and the fact that $\text{rk}(\underline{\mathbf{1}}_n) = n$.

$\Rightarrow$: Let $\underline{\mathbf{B}} = [b_{ij}]_{n\times n}$ have rank n. Then, the columns of $\underline{\mathbf{B}}$ being linearly independent, no column of $\underline{\mathbf{B}}$ can be the zero column vector.

In particular, there is an i such that $b_{i1} \neq 0$.

Then $\underline{\mathbf{B}}' = \underline{\mathbf{M}}(i, \frac{1}{b_{1i}}1)\underline{\mathbf{S}}(1,i)\underline{\mathbf{B}}$ is of the form

$$\begin{bmatrix} 1 & \\ & * \end{bmatrix}.$$

Put $\underline{\mathbf{B}}_1 := \underline{\mathbf{A}}(n, -\hat{b}_{n1}1)\cdots\underline{\mathbf{A}}(2,-\hat{b}_{21}1)\underline{\mathbf{B}}'$, where $\hat{b}_{j1} = \begin{cases} b'_{11} & \text{if } j = i \\ b_{j1} & \text{otherwise} \end{cases}$.

Then $\underline{\mathbf{B}}_1$ is of the form

$$\begin{bmatrix} 1 & * & * \\ 0 & * & * \\ \vdots & * & * \\ 0 & * & * \end{bmatrix}.$$

Now suppose that we have transformed $\underline{\mathbf{B}}$ into $\underline{\mathbf{B}}_k = [c_{ij}]_{n\times n}$ for some $k < n$ by means of elementary row operations, where $\underline{\mathbf{B}}_k$ is of the form

$$\begin{bmatrix} \underline{\mathbf{1}}_k & * \\ \underline{\mathbf{0}} & * \end{bmatrix}.$$

Since $\mathrm{rk}(\underline{\mathbf{B}}_k) = \mathrm{rk}(\underline{\mathbf{B}}) = n$, $c_{i(k+1)} \neq 0$ for some $i > k$. For otherwise, the $(k+1)^{\mathrm{st}}$ column would be a linear combination of the first k columns.

Put $\underline{\mathbf{B}}_{k+1} := \underline{\mathbf{A}}(n, -\hat{c}_{n1}1)\cdots\underline{\mathbf{A}}(2, -\hat{c}_{21}1)\underline{\mathbf{B}}'$, where $\hat{c}_{j(k+1)} = \begin{cases} c_{(k+1)(k+1)} & \text{if } j = i \\ c_{j(k+1)} & \text{otherwise} \end{cases}.$

Then $\underline{\mathbf{B}}_{k+1}$ is of the form

$$\begin{bmatrix} \underline{\mathbf{1}}_{k+1} & * \\ 0 & * \end{bmatrix}.$$

Clearly $\underline{\mathbf{B}}_n = \underline{\mathbf{1}}_n$ as required. □

11.1.1 The Main Theorem on Determinants

Theorem 11.6. *There is a unique function $D : \mathbf{M}(n;\mathbb{F}) \to \mathbb{F}$ with the three properties below.*

D1 *D is linear in each row.*

D2 *$D(X) = 0$ whenever $\mathrm{rk}\, X < n$.*

D3 *$D(\underline{\mathbf{1}}_n) = 1$.*

Proof. **Uniqueness**: Let $F, G : \mathbf{M}(n;\mathbb{F}) \to \mathbb{F}$ satisfy D1, D2 and D3.

Choose $\underline{\mathbf{B}} \in \mathbf{M}(n;\mathbb{F})$.

If $\mathrm{rk}(\underline{\mathbf{B}}) < n$, then, by D2, $F(\underline{\mathbf{B}}) = G(\underline{\mathbf{B}}) = 0$.

If $\mathrm{rk}(\underline{\mathbf{B}}) = n$, then, by Lemma 11.5, we can transform $\underline{\mathbf{B}}$ into $\underline{\mathbf{1}}_n$ by means of elementary row operations.

In other words, there is a matrix $\underline{\mathbf{T}} \in \mathbf{M}(n;\mathbb{F})$ such that $\underline{\mathbf{T}}\,\underline{\mathbf{B}} = \underline{\mathbf{1}}_n$. In fact, $\underline{\mathbf{T}}$ is the product of finitely many matrices each of which is of the form $\underline{\mathbf{M}}(i, \lambda)$ or $\underline{\mathbf{A}}(i, \mu j)$.

By Lemma 11.4 $F(\underline{\mathbf{T}}\,\underline{\mathbf{B}}) = \Delta F(\underline{\mathbf{B}})$ and $G(\underline{\mathbf{T}}\,\underline{\mathbf{B}}) = \Delta G(\underline{\mathbf{B}})$, where Δ is the product of the λs (with repetition) which occur in the $\underline{\mathbf{M}}(i, \lambda)$s.

Then

$$\begin{aligned} \Delta F(\underline{\mathbf{B}}) &= F(\underline{\mathbf{T}}\,\underline{\mathbf{B}}) \\ &= F(\underline{\mathbf{1}}_n) \\ &= 1 \\ &= G(\underline{\mathbf{1}}_n) \\ &= G(\underline{\mathbf{T}}\,\underline{\mathbf{B}}) \\ &= \Delta G(\underline{\mathbf{B}}) \end{aligned}$$

Since $\Delta \neq 0$, we conclude that $F(\underline{\mathbf{B}}) = G(\underline{\mathbf{B}})$.

Existence: Take any of the usual definitions from your first year mathematics course, and verify D1, D2 and D3.

We do this for "expansion by the j^{th} column", which we define inductively.

Definition 11.7. For the 1×1 matrix $\underline{\mathbf{A}} = [a]$,

$$\det(\underline{\mathbf{A}}) := a.$$

For $n \geq 1$, take $\underline{\mathbf{A}} = [a_{ij}]_{(n+1)\times(n+1)}$ and let $\underline{\mathbf{A}}_{(i)(j)}$ be the $n \times n$ matrix obtained by deleting the i^{th} row and the j^{th} column from $\underline{\mathbf{A}}$, so that $\underline{\mathbf{A}}_{(i)(j)} = [x_{k\ell}]_{n\times n}$ where

$$x_{k\ell} = \begin{cases} a_{kl} & \text{if } 1 \leq k < i \text{ and } 1 \leq \ell < j \\ a_{k(\ell+1)} & \text{if } 1 \leq k < i \text{ and } j \leq \ell \leq n \\ a_{(k+1)\ell} & \text{if } i \leq k \leq n \text{ and } 1 \leq \ell < j \\ a_{(k+1)(\ell+1)} & \text{if } i \leq k \leq n \text{ and } j \leq \ell \leq n \end{cases}$$

Then

$$\det(\underline{\mathbf{A}}) := \sum_{i=1}^{n+1} (-1)^{i+j} a_{ij} \det(\underline{\mathbf{A}}_{(i)(j)}).$$

Example 11.8. The reader may find it useful to bear a concrete instance in mind:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) =$$

$$a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{21}\det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}\right) + a_{31}\det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right).$$

It remains to verify that D1, D2 and D3 are satisfied.

Clearly, this is true when $n = 1$.

Given $n > 1$, we make the inductive hypothesis that it is also true for all $r \leq n$.

D1 That $\det : \mathbf{M}(n+1;\mathbb{F}) \to \mathbb{F}$, $\underline{\mathbf{A}} \mapsto \det(\underline{\mathbf{A}})$ is linear in the k^{th} row of $\underline{\mathbf{A}}$, follows from that fact that each summand, $(-1)^{i+j} a_{ij} \det(\underline{\mathbf{A}}_{(i)(j)})$, in the definition of $\det(\underline{\mathbf{A}})$ is linear in the k^{th} row of $\underline{\mathbf{A}}$.

To see this, consider first the case $i \neq k$. Then $\det\colon \mathbf{M}(n;\mathbb{F}) \to \mathbb{F}$ enjoys this property, whereas a_{ij} does not depend on the k^{th} row of $\underline{\mathbf{A}}$. Hence $a_{ij}\det(\underline{\mathbf{A}}_{(i)(j)})$ depends linearly on the k^{th} row.

If, on the other hand, $i = k$, then, since the function $\mathbf{M}(n;\mathbb{F}) \to \mathbb{F}$, $[a_{k\ell}]_{n\times n} \mapsto a_{ij}$ is linear in the k^{th} row of $\underline{\mathbf{A}}$ and since $\underline{\mathbf{A}}_{(i)(j)}$ does not depend on the i^{th} row of $\underline{\mathbf{A}}$, $a_{ij}\det(\underline{\mathbf{A}}_{(i)(j)}$ depends linearly on the k^{th} row of $\underline{\mathbf{A}}$.

Thus $\det(\underline{\mathbf{A}})$ is the sum of functions each depending linearly on the k^{th} row of $\underline{\mathbf{A}}$, and hence itself depends linearly on the k^{th} row of $\underline{\mathbf{A}}$.

D2 If $\mathrm{rk}(\underline{\mathbf{A}}) < n$, then there is a row, say the p^{th}, of $\underline{\mathbf{A}}$ which is a linearly combination of the others.

We can find $\lambda_i \quad (i = 1, \ldots, n, i \neq p)$ such that for each j

$$a_{pj} = \sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i a_{ij}.$$

Thus, since we have already established linearity in each row, the determinant of $\underline{\mathbf{A}}$ is a linear combination of determinants of matrices with two identical rows.

It is therefore sufficient to show that the determinant of an $n \times n$ matrix with two identical rows is 0. We prove this using induction on n.

This is plainly true for $n = 2$.

So take $n > 2$ and assume the inductive hypothesis that the assertion is true for all $m \times m$ matrices with $m < n$.

Assume that rows p and $p+t$ are identical ($t \geq 1$). Then, since $\underline{\mathbf{A}}_{(i)(j)}$ has two identical rows whenever $i \neq p, p+t$, the inductive hypothesis implies that

$$\begin{aligned}\det(\underline{\mathbf{A}}) &= \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(\underline{\mathbf{A}}_{(i)(j)}) \\ &= (-1)^{p+j}a_{pj}\det(\underline{\mathbf{A}}_{(p)(j)}) + (-1)^{p+t+j}a_{(p+t)j}\det(\underline{\mathbf{A}}_{(p+t)(j)})\end{aligned}$$

As $a_{pj} = a_{(p+t)j}$ it only remains to investigate the relationship between $\det(\underline{\mathbf{A}}_{(p)(j)})$ and $\det(\underline{\mathbf{A}}_{(p+t)(j)})$ when the p^{th} and $(p+t)^{\text{h}}$ rows of $\underline{\mathbf{A}}$ are identical.

If $t = 1$, then $\underline{\mathbf{A}}_{(p)(j)}$ and $\underline{\mathbf{A}}_{(p+t)(j)}$ are identical, so their determinants agree. In this case

$$\det(\underline{\mathbf{A}}) = (-1)^{p+j}a_{pj}\det(\underline{\mathbf{A}}_{(p)(j)}) + (-1)^{p+1+j}a_{pj}\det(\underline{\mathbf{A}}_{(p)(j)}) = 0.$$

If, on the other hand, $t > 1$, then $\underline{\mathbf{A}}_{(p+t)(j)}$ can be obtained from $\underline{\mathbf{A}}_{(p)(j)}$ by interchanging rows $(t-1)$ times. We may assume as part of our inductive hypothesis that for $m \times m$ matrices with $m < n$, each such interchange alters the sign of the determinant. Then

$$\begin{aligned}\det(\underline{\mathbf{A}}) &= (-1)^{p+j}a_{pj}\det(\underline{\mathbf{A}}_{(p)(j)}) + (-1)^{p+t+j}a_{pj}(-1)^{t-1}\det(\underline{\mathbf{A}}_{(p)(j)}) \\ &= (-1)^{p+j}a_{pj}\det(\underline{\mathbf{A}}_{(p)(j)}(1 + (-1)^{2t-1}) \\ &= 0.\end{aligned}$$

D3: If $\underline{\mathbf{A}} = \underline{\mathbf{1}}_n$, then $a_{ij} = \delta_{ij}$ and the only non-zero summand in $\sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(\underline{\mathbf{1}}_{n\,(p)(j)})$ is $(-1)^{2j}\delta_{jj}\det(\underline{\mathbf{1}}_{n-1})$, and, plainly, this is 1. □

Observation 11.9. It follows from the uniqueness of the determinant function that the expansion by the j^{th} column is independent of the choice of j.

Corollary 11.10. *Given* $\underline{\mathbf{B}} \in \mathbf{M}(n;\mathbb{F})$, $\operatorname{rk}(\underline{\mathbf{B}}) = n$ *if and only if* $\det(\underline{\mathbf{B}}) \neq 0$.

Proof. By Lemma 11.5 and using the notation from the proof of Theorem 11.6, we see that $\underline{\mathbf{B}}$ is invertible if and only if $\operatorname{rk}(\underline{\mathbf{B}}) = n$ if and only if

$$\det(\underline{\mathbf{B}}) = \frac{1}{\Delta}.$$

□

Corollary 11.11. *Given* $\underline{\mathbf{A}}, \underline{\mathbf{B}} \in \mathbf{M}(n;\mathbb{F})$, $\det(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \det(\underline{\mathbf{A}})\det(\underline{\mathbf{B}})$.

Proof. By Lemma 10.16, $\operatorname{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) \leq \min\{\operatorname{rk}(\underline{\mathbf{A}}), \operatorname{rk}(\underline{\mathbf{B}})\}$. It follows that $\det(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = 0$ whenever $\det(\underline{\mathbf{A}}) = 0$.

Choose $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$ with $\det(\underline{\mathbf{A}}) \neq 0$, or, equivalently, $\operatorname{rk}(\underline{\mathbf{A}}) = n$. Define

$$F\colon \mathbf{M}(n;\mathbb{F}) \longrightarrow \mathbb{F}, \quad \underline{\mathbf{B}} \longmapsto \det(\underline{\mathbf{A}}\,\underline{\mathbf{B}}).$$

Since $L_{\underline{\mathbf{A}}}\colon \mathbf{M}(n;\mathbb{F}) \to \mathbf{M}(n;\mathbb{F})$, $\underline{\mathbf{B}} \longmapsto \underline{\mathbf{A}}\,\underline{\mathbf{B}}$ is a linear transformation, F satisfies D1.

Moreover, since this transformation is an isomorphism $\mathrm{rk}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \mathrm{rk}(\underline{\mathbf{B}})$. Thus $F(\underline{\mathbf{B}}) = 0$ whenever $\mathrm{rk}(\underline{\mathbf{B}}) < n$, showing that F satisfies D2.

Now $F(\underline{\mathbf{1}}_n) = \det(\underline{\mathbf{A}}\,\underline{\mathbf{1}}_n) = \det(\underline{\mathbf{A}})$, which is, in general, not 1.

However, since $\underline{\mathbf{A}}$ is invertible, it follows from Corollary 11.10 that $\det(\underline{\mathbf{A}}) \neq 0$.

Thus

$$\tilde{F} : \mathbf{M}(n;\mathbb{F}) \longrightarrow \mathbb{F}, \quad \underline{\mathbf{B}} \longmapsto \frac{1}{\det(\underline{\mathbf{A}})} F(\underline{\mathbf{B}}) = \frac{1}{\det(\underline{\mathbf{A}})} \det(\underline{\mathbf{A}}\,\underline{\mathbf{B}})$$

satisfies D1, D2 and D3.

By Theorem 11.6, $\tilde{F}(\underline{\mathbf{B}}) = \det(\underline{\mathbf{B}})$, that is,

$$\frac{1}{\det(\underline{\mathbf{A}})} \det(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \det(\underline{\mathbf{B}})$$

or, equivalently, $\det(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \det(\underline{\mathbf{A}}) \det(\underline{\mathbf{B}})$. □

Corollary 11.12. *If $\underline{\mathbf{A}}$ is invertible, then* $\det(\underline{\mathbf{A}}^{-1}) = (\det(\underline{\mathbf{A}}))^{-1}$.

Proof. Let $\underline{\mathbf{A}}$ be an invertible $n \times n$ matrix. Then $\underline{\mathbf{A}}^{-1}\underline{\mathbf{A}} = \underline{\mathbf{1}}_n$.

Hence, by Corollary 11.11 and by D3, $\det(\underline{\mathbf{A}}^{-1}) \det(\underline{\mathbf{A}}) = \det(\underline{\mathbf{1}}_n) = 1$. □

Corollary 11.13. *If the matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ represent the same endomorphism, $T : V \to V$, then* $\det(\underline{\mathbf{A}}) = \det(\underline{\mathbf{B}})$.

Proof. $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ represent the same endomorphism if and only if there is an invertible matrix, $\underline{\mathbf{C}}$, with $\underline{\mathbf{B}} = \underline{\mathbf{C}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{C}}$. The conclusion now follows by Corollaries 11.11 and 11.12. □

Corollary 11.13 allows us to define the deteminant of an endomorphism.

Definition 11.14. Let $T\colon V \to V$ be an endomorphism of the finitely generated vector space, V. The *determinant of* T, $\det(T)$, is the determinant of any matrix representing T.

11.2 Applications of the Determinant

Our discussion of the determinant focussed on its definition and principal properties, without regard to its applications. We now turn to two applications which require no further theory. We shall meet other applications later.

11.2.1 When Do $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ Comprise a Basis?

Suppose given the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in the vector space V.

By Theorem 8.1 and Definition 8.2, a necessary (but not sufficient) condition for these vectors to comprise a basis for V is that $\dim_{\mathbb{F}}(V) = n$.

If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V, then we can use the determinant to decide whether $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ comprise a basis for V.

We have seen that the choice of a basis for V is the choice of an isomorphism $V \to \mathbb{F}_{(n)}$, with the vector $\mathbf{v} \in V$ being mapped to its co-ordinate vector with respect to the chosen basis.

Let $\mathbf{c}_j$ be the co-orindate vector of $\mathbf{v}_j$ with respect to the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

The vector subspace of V generated by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is then mapped to the vector subspace of $\mathbb{F}_{(n)}$ generated by $\mathbf{c}_1, \ldots, \mathbf{c}_n$. But this is precisely the column space of the matrix $\underline{\mathbf{A}}$ whose j^{th} column is $\mathbf{c}_j$, and so $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V if and only if the column space of $\underline{\mathbf{A}}$ is $\mathbb{F}_{(n)}$.

This is the case if and only if $\mathrm{rk}(\underline{\mathbf{A}}) = n$, which is equivalent to $\det(\underline{\mathbf{A}}) \neq 0$.

The next theorem summarises the above.

Theorem 11.15. *Let $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for the vector space V. Take $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Let $\mathbf{c}_j$ be the co-ordinate vector of $\mathbf{v}_j$ with respect to $\mathcal{B}$ and $\underline{\mathbf{A}}$ be the matrix with $\mathbf{c}_j$ as j^{th} column. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V if and only if $\det(\underline{\mathbf{A}}) \neq 0$.*

Example 11.16. By the theory of linear differentiable functions with constant coefficients (cf. MATH102), the functions

$$\begin{aligned}
\sin\colon \mathbb{R} &\longrightarrow \mathbb{R}, \quad x \longmapsto \sin x \\
\cos\colon \mathbb{R} &\longrightarrow \mathbb{R}, \quad x \longmapsto \cos x
\end{aligned}$$

comprise a basis for the real vector space

$$V := \{f\colon \mathbb{R} \to \mathbb{R} \mid \tfrac{d^2 f}{dx^2} + f = 0\}$$

It is easy to see that

$$\begin{aligned}
f_1\colon \mathbb{R} &\longrightarrow \mathbb{R}, \quad x \longmapsto \cos(x + \tfrac{\pi}{4}) \\
f_2\colon \mathbb{R} &\longrightarrow \mathbb{R}, \quad x \longmapsto \cos(x + \tfrac{\pi}{3})
\end{aligned}$$

are vectors in V.

Since

$$\begin{aligned}
\cos(x + \tfrac{\pi}{4}) &= \cos x \cos \tfrac{\pi}{4} - \sin x \sin \tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}} \cos x - \tfrac{1}{\sqrt{2}} \sin x \\
\cos(x + \tfrac{\pi}{3}) &= \cos x \cos \tfrac{\pi}{3} - \sin x \sin \tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2} \cos x - \tfrac{1}{2} \sin x
\end{aligned}$$

the co-ordinate vectors of f_1 and f_2 with respect to our chosen basis, $\{\sin, \cos\}$, are

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

respectively. Since

$$\det\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}\right) = \frac{1}{\sqrt{2}}(-\frac{1}{2}) - (-\frac{1}{\sqrt{2}})\frac{\sqrt{3}}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}} \neq 0,$$

we see that $\{f_1, f_2\}$ is also a basis for V.

Observation 11.17. The determinant to decide whether given vectors form a basis depended on having available basis for the vector space in question. When there is an obvious basis, such as in the case of $V = \mathbb{F}^n$, or when there is a well-known basis, as in our example, this is convenient.

If there is no convenient basis at hand, it is usually easier to settle the question by other means.

11.2.2 Fitting Curves to Given Points

Take distinct points $(x_1, y_1), \ldots (x_n, y_n) \in \mathbb{F}^2$.

Can we find a function $f\colon \mathbb{F} \to \mathbb{F}$ with $f(x_j) = y_j \quad (1 \leq j \leq n)$?

This is equivalent to finding a function whose graph passes through the given points.

There is an obvious necessary condition, namely, that if $x_i = x_j$, then $y_i = y_j$, for if $f(x) = y$ and $f(x) = \tilde{y}$, Definition 1.9 forces $\tilde{y} = y$.

So we may restrict attention to the case $x_i = x_j$ if and only if $i = j$.

We can be more specific, by seeking a function which is easily computable, such as a polynomial function. We formulate our problem accordingly.

Is there a polynomial, $p(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} \in \mathbb{F}[t]$ with the function

$$f_p \colon \mathbb{F} \longrightarrow \mathbb{F}, \quad x \longmapsto a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

having the property that, for $1 \leq j \leq n$,

$$f_p(x_j) = y_j$$

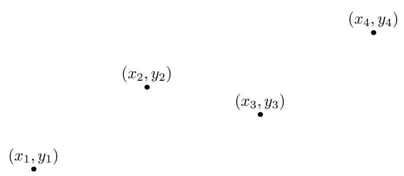
This condition is expressed by the system of linear equations

$$\begin{array}{ccccccccc} a_0x_1 & + & a_1x_1 & + & \cdots & + & a_{n-1}x_1^{n-1} & = & y_1 \\ \vdots & & & & & & \vdots & & \vdots \\ a_0x_n & + & a_1x_n & + & \cdots & + & a_{n-1}x_n^{n-1} & = & y_n \end{array}$$

As we are seeking $a_0, \ldots a_{n-1}$ which simultaneously satisfy these equations, we represent them by the matrix equation

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Example 11.18. Take $\mathbb{F} = \mathbb{R}$ and $(-1, -15), (0, 3), (1, -3), (2, 15) \in \mathbb{R}^2$.



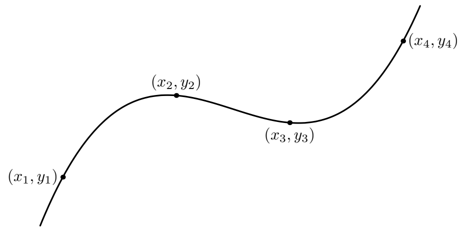
We seek $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in \mathbb{R}[t]$ satisfying

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -15 \\ 3 \\ -3 \\ 15 \end{bmatrix}$$

It is left to the reader to verify that the polynomial we obtain is $p(t) = t^3 - 12t^2 - 2t + 3$.

Our curve is thus the graph of

$$f_p \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^3 - 12x^2 - 2x + 3$$



11.3 The Trace

The trace of a matrix is, in a sense, complements the determinant. For while the determinant of a product of matrices is the product of the their determinants and there is no general relationship between the determinant of a sum of matrices and the individual determinants, the opposite is true of the trace: the trace of a sum of matrices is the sum of their traces, but there is no general relationship between the trace of a product of matricesand the individual traces.

Definition 11.19. The *trace* of an $n \times n$ matrix is the sum of its diagonal coefficients. We write

$$\mathrm{tr} : \mathbf{M}(n;\mathbb{F}) \longrightarrow \mathbb{F}, \quad \underline{\mathbf{A}} = [a_{ij}]_{n\times n} \longmapsto \mathrm{tr}(\underline{\mathbf{A}}) = \sum_{j=1}^{n} a_{jj}.$$

The central properties of the trace are contained in our next theorem.

Theorem 11.20. *Take* $\underline{\mathbf{A}}, \underline{\mathbf{B}} \in \mathbf{M}(n;\mathbb{F})$. *Then*

(i) $\mathrm{tr}(\underline{\mathbf{A}} + \underline{\mathbf{B}}) = \mathrm{tr}(\underline{\mathbf{A}}) + \mathrm{tr}(\underline{\mathbf{B}})$

(ii) $\mathrm{tr}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \mathrm{tr}(\underline{\mathbf{B}}\,\underline{\mathbf{A}})$.

A word of warning before we prove the theorem. It is important to avoid drawing the tempting, but false, conclusion from Theorem 11.20 (ii) that there is some regular relationship between $\mathrm{tr}(\underline{\mathbf{A}}\,\underline{\mathbf{B}})$ on the one hand, and $\mathrm{tr}(\underline{\mathbf{A}})$ and $\mathrm{tr}(\underline{\mathbf{B}})$ on the other.

Example 11.21. Put

$$\underline{\mathbf{A}} = \underline{\mathbf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Plainly $\mathrm{tr}(\underline{\mathbf{A}}) = \mathrm{tr}(\underline{\mathbf{B}}) = 0$, but $\mathrm{tr}\,\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = 2$.

Proof. [of Theorem 11.20] Take $\underline{\mathbf{A}} = [a_{ij}], \underline{\mathbf{B}} = [b_{ij}] \in \mathbf{M}(n;\mathbb{F})$.

(i) Since $\underline{\mathbf{A}} + \underline{\mathbf{B}} := [a_{ij} + b_{ij}]$,

$$\begin{aligned}\mathrm{tr}(\underline{\mathbf{A}} + \underline{\mathbf{B}}) &= \sum_{j=1}^{n}(a_{jj} + b_{jj}) \\ &= \sum_{j=1}^{n} a_{jj} + \sum_{j=1}^{n} b_{jj} \\ &= \mathrm{tr}(\underline{\mathbf{A}}) + \mathrm{tr}(\underline{\mathbf{B}}).\end{aligned}$$

(ii) Since $\underline{\mathbf{A}}\,\underline{\mathbf{B}} = [\sum_{j=1}^{n} a_{ij}b_{jk}]$ and $\underline{\mathbf{B}}\,\underline{\mathbf{A}} = [\sum_{j=1}^{n} b_{ij}a_{jk}]$,

$$\begin{aligned}
\operatorname{tr}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}b_{kj} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} b_{kj}a_{jk} \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} a_{jk}b_{kj} \\
&= \operatorname{tr}(\underline{\mathbf{B}}\,\underline{\mathbf{A}}).
\end{aligned}$$

□

One of the exercises extends Theorem 11.20(ii) slightly.

Corollary 11.22. *If* $\underline{\mathbf{C}}$ *is invertible, then* $\operatorname{tr}(\underline{\mathbf{C}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{C}}) = \operatorname{tr}\underline{\mathbf{A}}$.

Proof.

$$\begin{aligned}
\operatorname{tr}(\underline{\mathbf{C}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{C}}) &= \operatorname{tr}(\underline{\mathbf{C}}^{-1}(\underline{\mathbf{A}}\,\underline{\mathbf{C}})) \\
&= \operatorname{tr}((\underline{\mathbf{A}}\,\underline{\mathbf{C}})\underline{\mathbf{C}}^{-1}) \\
&= \operatorname{tr}(\underline{\mathbf{A}}(\underline{\mathbf{C}}\,\underline{\mathbf{C}}^{-1})) && \text{by Theorem 11.20} \\
&= \operatorname{tr}(\underline{\mathbf{A}}\,\underline{\mathbf{1}}_n) \\
&= \operatorname{tr}(\underline{\mathbf{A}}).
\end{aligned}$$

□

Corollary 11.23. *If the matrices* $\underline{\mathbf{A}}$ *and* $\underline{\mathbf{B}}$ *represent the same endomorphism* $T\colon V \to V$*, then* $\operatorname{tr}(\underline{\mathbf{A}}) = \operatorname{tr}(\underline{\mathbf{B}})$.

Corollary 11.23 allows us to define the trace of an endomorphism of finitely generated vector spaces.

Definition 11.24. Let $T\colon V \to V$ be an endomorphism of the finitely generated vector space V. The *trace of* T, $\operatorname{tr}(T)$ is defined to be the trace of any matrix representing T.

11.4 The Transpose of a Matrix

We introduce an important operation on matrices, whose true significance will only become apparent later.

Definition 11.25. The *transpose* of the $m \times n$ matrix $\underline{\mathbf{A}}$ is the $n \times m$ matrix $\underline{\mathbf{A}}^t$ obtained by interchanging each row with the corresponding column.

This defines a function

$$(\ \ )^t : \mathbf{M}(m \times n; \mathbb{F}) \longrightarrow \mathbf{M}(n \times m; \mathbb{F}), \quad [a_{ij}]_{m\times n} \longmapsto [x_{\nu\mu}]_{n\times n}$$

where $x_{\nu\mu} := a_{\mu\nu} \qquad (1 \leq \nu \leq n,\ 1 \leq \mu \leq n)$.

The next theorems summarises the basic properties of the transpose.

Theorem 11.26. *(i)* $(\ )^t : \mathbf{M}(m \times n; \mathbb{F}) \to \mathbf{M}(n \times m; \mathbb{F})$ *is a linear transformation.*

(ii) Given an $m \times n$ *matrix,* $\underline{\mathbf{A}}$, $(\underline{\mathbf{A}}^t)^t = \underline{\mathbf{A}}$.

(iii) Given an $m \times n$ *matrix,* $\underline{\mathbf{A}}$, *and an* $n \times p$ *matrix,* $\underline{\mathbf{B}}$, $(\underline{\mathbf{A}}\,\underline{\mathbf{B}})^t = \underline{\mathbf{B}}^t\underline{\mathbf{A}}^t$.

Proof. The assertions follow by direct calculations, which are left as exercises. □

Observation 11.27. It follows from Theorem 11.26(ii) that $(\ )^t : \mathbf{M}(m \times n; \mathbb{F}) \to \mathbf{M}(n \times m; \mathbb{F})$ is actually an isomorphism.

Theorem 11.28. *Let* $\underline{\mathbf{A}}$ *be an* $n \times n$ *matrix. Then*

(i) $\operatorname{tr}(\underline{\mathbf{A}}^t) = \operatorname{tr}(\underline{\mathbf{A}})$;

(ii) $\det(\underline{\mathbf{A}}^t) = \det(\underline{\mathbf{A}})$.

Proof. **(i)** : The assertion follows immediately from the fact that the diagonal of a matrix is unchanged by taking the transpose.

(ii): By the Main Theorem on Determinants (Theorem 11.6), there is a unique function, $D\colon \mathbf{M}(n; \mathbb{F}) \to \mathbb{F}$, which is linear in the rows of a matrix, which takes the value 0 on matrices whose rank is less than n, and which takes the value 1 on $\underline{\mathbf{1}}_n$.

Consider

$$F : \mathbf{M}(n; \mathbb{F}) \longrightarrow \mathbb{F}, \quad \underline{\mathbf{A}} \longmapsto \det\left(\underline{\mathbf{A}}^t\right).$$

Since the row and column ranks of a matrix agree, and since the identity matrix is its own transpose, D2 and D3 are clearly satisfied by F.

It remains only to establish D1 for F. Since the rows of $\underline{\mathbf{A}}^t$ are the columns of $\underline{\mathbf{A}}$, and since det is linear in each row, this is equivalent to proving that the determinant is linear in each column.

Note that the expansion of the determinant of $\underline{\mathbf{A}}$ by the j^{th} column of $\underline{\mathbf{A}}$, $\det(\underline{\mathbf{A}})$ is

$$\det(\underline{\mathbf{A}}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\left(\underline{\mathbf{A}}_{(i)(j)}\right),$$

is linear in the j^{th} column of $\underline{\mathbf{A}}$, as $\det(A_{(i)(j)})$ is independent of the j^{th} column of $\underline{\mathbf{A}}$. □

11.5 Exercises

Exercise 11.1. Let $\mathcal{P}_2$ denote the real vector space of all real polynomials of degree at most 2. Let $D : \mathcal{P}_2 \to \mathcal{P}_2$ be differentiation. Find the determinant of D.

Exercise 11.2. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for the vector space V. Take $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Let

$$\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

be the coordinate vector of $\mathbf{v}_j$ $(j = 1, \ldots, n)$ with respect to the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V if and only if $\det \underline{\mathbf{A}} \neq 0$, where $\underline{\mathbf{A}} := [a_{ij}]_{n \times n}$.

Exercise 11.3. Take $\underline{\mathbf{A}} \in \mathbf{M}(m \times n; \mathbb{F})$ and $\underline{\mathbf{B}} \in \mathbf{M}(n \times m; \mathbb{F})$. Prove that $\operatorname{tr}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \operatorname{tr}(\underline{\mathbf{B}}\,\underline{\mathbf{A}})$.

Exercise 11.4. Suppose that $\underline{\mathbf{A}} \in \mathbf{M}(n; \mathbb{F})$ can be written in the form

$$\underline{\mathbf{A}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}$$

Show that $\underline{\mathbf{A}}^{r+1} = (\operatorname{tr} \underline{\mathbf{A}})^r \, \underline{\mathbf{A}}$ for all $r \geq 1$, and find $\det(\underline{\mathbf{A}}^r)$.

Exercise 11.5. Prove Theorem 11.26.

Chapter 12

Eigenvalues and Eigenvectors

The direct sum of the vector spaces $V_1, \ldots, V_n$, $V_1 \oplus \cdots \oplus V_n$ is defined by

$$\bigoplus_{j=1}^{n} V_j = \{(\mathbf{v}_1, \ldots, \mathbf{v}_n) \mid \mathbf{v}_j \in V_j,\ j = 1, \ldots, n\}$$

with the vector spaces operations are defined "componentwise":

$$\begin{aligned}(\mathbf{v}_1, \ldots, \mathbf{v}_n) + (\mathbf{v}'_1, \ldots, \mathbf{v}'_n) &:= (\mathbf{v}_1 + \mathbf{v}'_1, \ldots, \mathbf{v}_n + \mathbf{v}'_n) \\ \lambda(\mathbf{v}_1, \ldots, \mathbf{v}_n) &:= (\lambda\mathbf{v}_1, \ldots, \lambda\mathbf{v}_n),\end{aligned}$$

It follows that $\mathbf{0}_{V_1\oplus\cdots\oplus V_n} = (\mathbf{0}_{V_1}, \ldots, \mathbf{0}_{V_n})$ and $-(\mathbf{v}_1, \ldots, \mathbf{v}_n) = (-\mathbf{v}_1, \ldots, -\mathbf{v}_n)$

Since $\mathbb{F}^n \cong \bigoplus_{j=1}^{n} \mathbb{F}$, we can reformulate the Classification Theorem for Finitely Generated Vector Spaces over the field $\mathbb{F}$ as stating that every such vector space is (up to isomorphism) a direct sum of copies of $\mathbb{F}$,

$$V \cong \bigoplus_{j=1}^{n} \mathbb{F},$$

where n is the dimension of the vector space in question. Moreover, $\mathbb{F}$ itself cannot be written as a direct sum of non-trivial vector spaces over $\mathbb{F}$.

What we have achieved is a decomposition of the finitely generated vector space, V, into finitely many components, which cannot be decomposed any further.

The direct sum construction also applies to linear transformations. The direct sum of the linear transformations $T_j \colon V_j \to W_j$ $(j = 1, \ldots, n)$ is

$$\bigoplus_{j=1}^{n} T_j : \bigoplus_{j=1}^{n} V_j \longrightarrow \bigoplus_{j=1}^{n} W_j$$

defined by

$$(T_1 \oplus \cdots \oplus T_n)(\mathbf{v}_1, \ldots \mathbf{v}_n) := (T_1(\mathbf{v}_1), \ldots, T_n(\mathbf{v}_n))$$

The verification that $(T_1 \oplus \cdots \oplus T_n)$ is a linear transformation is routine, and left to the reader.

Example 12.1. For

$$
\begin{aligned}
&R\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \longmapsto (2x+y, 3y)\\
&T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (u,v,w) \longmapsto u+v+w
\end{aligned}
$$

$$R \oplus T\colon \mathbb{R}^2 \oplus \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \oplus \mathbb{R}, \quad \big((x,y),(u,v,w)\big) \longmapsto \big((2x+y,3y), u+v+w\big)$$

We may identify $\mathbb{R}^2 \oplus \mathbb{R}^3$ with $\mathbb{R}^5$ and $\mathbb{R}^2 \oplus \mathbb{R}$ with $\mathbb{R}^3$, using the obvious isomorphisms

$$
\begin{aligned}
&\mathbb{R}^2 \oplus \mathbb{R}^3 \longrightarrow \mathbb{R}^5, \quad \big((x,y),(u,v,w)\big) \longmapsto (x,y,u,v,w)\\
&\mathbb{R}^2 \oplus \mathbb{R} \quad \longrightarrow \mathbb{R}^3, \quad \big((r,s),t\big) \longmapsto (r,s,t)
\end{aligned}
$$

Using these identifications, we may regard $R \oplus T$ as the linear transformation

$$\mathbb{R}^5 \longrightarrow \mathbb{R}^3, \quad (x,y,u,v,w) \longmapsto (2x+y, 3y, u+v+w)$$

The question arises:

> Given a finitely generated vector space V over $\mathbb{F}$, is every linear transformation $T\colon V \to V$ expressible as the direct sum of linear transformations $T_j\colon \mathbb{F} \to \mathbb{F} \quad (j = 1, \dots, \dim_{\mathbb{F}}(V))$?

This is the question we pursue here.

Let $\dim(V) = m$ and $\dim(W) = n$. Take endomorphisms $R : V \to V$ and $S : W \to W$.

Lemma 12.2. *Let $\{\mathbf{e}_1, \dots \mathbf{e}_m\}$ be a basis for V and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ for W. Then putting*

$$\mathbf{u}_i := \begin{cases} (\mathbf{e}_i, \mathbf{0}_W) & \text{if } i \leq m\\ (\mathbf{0}_V, \mathbf{f}_{i-m}) & \text{if } i > m \end{cases}$$

defines a basis, $\{\mathbf{u}_1, \dots, \mathbf{u}_{m+n}\}$, for $V \oplus W$.

Proof. Take $\mathbf{x} \in V \oplus W$.
There are unique $\mathbf{v} \in V, \mathbf{w} \in W$ with $\mathbf{x} = (\mathbf{v}, \mathbf{w})$, and unique $\alpha_i, \beta_j \in \mathbb{F}$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ with $\mathbf{v} = \sum_{i=1}^{m} \alpha_i \mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^{n} \beta_j \mathbf{f}_j$. Then

$$
\begin{aligned}
\mathbf{x} &= \Big(\sum_{i=1}^{m} \alpha_i \mathbf{e}_i, \sum_{j=1}^{n} \beta_j \mathbf{f}_j\Big)\\
&= \Big(\sum_{i=1}^{m} \alpha_i \mathbf{e}_i, \mathbf{0}_W\Big) + \Big(\mathbf{0}_V, \sum_{j=1}^{n} \beta_j \mathbf{f}_j\Big)\\
&= \sum_{i=1}^{m} \alpha_i (\mathbf{e}_i, \mathbf{0}_W) + \sum_{j=1}^{n} \beta_j (\mathbf{0}_V, \mathbf{f}_j)\\
&= \sum_{i=1}^{m+n} \lambda_i \mathbf{u}_i,
\end{aligned}
$$

where the coefficients $\lambda_i = \begin{cases} \alpha_i & \text{if } 1 \leq i \leq m\\ \beta_{i-m} & \text{if } m < i \leq m+n \end{cases}$ are uniquely determined. □

Corollary 12.3. *If the matrix of R with respect to $\{\mathbf{e}_i\}$ is $\underline{\mathbf{A}}$ and that of S with respect to $\{\mathbf{f}_j\}$ is $\underline{\mathbf{B}}$, then the matrix of $R \oplus S$ with respect to $\mathbf{u}_i$ is*

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$$

Proof. Exercise. □

Convention. The meaning of $\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$ needs clarification.

This should not be read here as a matrix of matrices, that is a 2×2 matrix, each of whose coefficients is itself a matrix, even though it is possible to do so sensibly. Rather, what is intended is that if $\underline{\mathbf{A}}$ is an $m \times n$ matrix and $\underline{\mathbf{B}}$ is a $p \times q$ matrix, then $\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$ is the $(m+q) \times (n+q)$ matrix obtained by copying the coefficients of $\underline{\mathbf{A}}$ into the top left, those of $\underline{\mathbf{B}}$ into the bottom right and placing 0s everywhere else.

Example 12.4. Take $\underline{\mathbf{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\underline{\mathbf{B}} = \begin{bmatrix} a & b & e \\ f & d & g \end{bmatrix}$. Then

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & a & b & e \\ 0 & 0 & f & d & g \end{bmatrix}$$

This can be generalised to any finite number of vector spaces, $V_1, \ldots, V_n$ and endomorphisms $T_j \colon V_j \to V_j$ $(j = 1, \ldots, k)$. In particular, if each V_j is 1-dimensional — equivalently, if each $V_j \cong \mathbb{F}$ — then the matrix, $[a_{ij}]_{n \times n}$, of $T = \oplus T_j : \oplus V_j \to \oplus V_j$ with respect to the canonically induced basis is a diagonal matrix: $a_{ij} = 0$, whenever $i \neq j$.

In other words, the endomorphism $T : V \to V$ is of the form $T_1 \oplus \cdots \oplus T_{\dim(V)}$, with each T_j a linear transformation $T_j \colon \mathbb{F} \to \mathbb{F}$ if and only if there is a basis for V with respect to which the matrix of T is a diagonal matrix.

Thus we may reformulate our question as:

> Is there a basis for V with respect to which the matrix of T is in diagonal form?

Note that if $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V with respect to which the matrix of $T : V \to V$ is in diagonal form, then

$$T(\mathbf{e}_j) = \lambda_j \mathbf{e}_j,$$

where λ_j is the j-th diagonal entry in the matrix of T with respect to $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

Definition 12.5. The scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of the endomorphism $T : V \to V$ if and only if there is a vector $\mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}_V$, such that

$$T(\mathbf{v}) = \lambda \mathbf{v}. \tag{12.1}$$

Such a vector $\mathbf{v}$ is an *eigenvector* for the eigenvalue λ. The *eigenspace* of λ, V_λ, is the set of all solutions of $T(\mathbf{v}) = \lambda \mathbf{v}$, that is

$$V_\lambda := \{\mathbf{x} \in V \mid T(\mathbf{x}) = \lambda \mathbf{x}\} \qquad (\lambda \in \mathbb{F}).$$

The next theorem, the main theorem of this section, summarises the preceding discussion.

Main Theorem. *$T : V \to V$ is the direct sum of endomorphisms $T_i : V_i \to V_i$, with $\dim(V_i) = 1$, if and only if V has a basis consisting of eigenvectors of V.*

We discuss related results of independent interest.

Theorem 12.6. *Take an endomorphism $T : V \to V$. Then for each $\lambda \in \mathbb{F}$*

$$V_\lambda := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda\mathbf{v}\}$$

is a vector subspace of V, and λ is an eigenvalue for T if and only if $V_\lambda \neq \{\mathbf{0}_V\}$.

Proof. Take $\mathbf{u}, \mathbf{v} \in V_\lambda$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned} T(\alpha\mathbf{u} + \beta\mathbf{v}) &= \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \\ &= \alpha\lambda\mathbf{u} + \beta\lambda\mathbf{v} \\ &= \lambda(\alpha\mathbf{u} + \beta\mathbf{v}). \end{aligned}$$

Thus, $\alpha\mathbf{u} + \beta\mathbf{v} \in V_\lambda$. □

Example 12.7. Let V be any vector space. Put $T = id_V$. Then $T(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in V$, so that 1 is the only possible eigenvalue and every non-zero vector is an eigenvector for 1.

Example 12.8. Let V be any vector space. Let T be the zero map $V \to V$, so that $T\colon \mathbf{v} \mapsto \mathbf{0}_V$. Then $T(\mathbf{v}) = \mathbf{0}_V = 0\mathbf{v}$ for every $\mathbf{v} \in V$. Plainly, 0 is the only possible eigenvalue and every non-zero vector is an eigenvector for 0.

Observation 12.9. The eigenvalue 0 plays a distinguished role, for, plainly, $V_0 = \ker(T)$. This establishes the following lemma.

Lemma 12.10. *Let $T : V \longrightarrow V$ be an endomorphism of the vector space V. Then 0 is an eigenvalue if and only if T is not injective.*

Example 12.11. Let V be the Euclidean plane, regarded as $\mathbb{R}^2$. Rotating the plane through an angle of θ (with $0 \leq \theta < 2\pi$) about the origin defines the linear transformation

$$T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longrightarrow (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Thus the real number λ is an eigenvalue for T_θ if and only if

$$(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) = (\lambda x, \lambda y)$$

for some $(x, y) \neq (0, 0)$.
Thus, λ is an eigenvalue for T_θ if and only if there are real $x, y \in \mathbb{R}$ with $x^2 + y^2 \neq 0$ such that

$$\begin{aligned} x\cos\theta - y\sin\theta &= \lambda x \\ x\sin\theta + y\cos\theta &= \lambda y \end{aligned}$$

Squaring and adding these equations we see that

$$x^2 + y^2 = \lambda^2(x^2 + y^2).$$

Since $x^2 + y^2 \neq 0$, $\lambda^2 = 1$ and so the only possible eigenvalues are -1 and 1

$\boldsymbol{\lambda = 1}$: Then

$$\begin{aligned} x\cos\theta - y\sin\theta &= x \\ x\sin\theta + y\cos\theta &= y. \end{aligned}$$

By elementary trigonometry, $\theta = 0$, since $(x, y) \neq (0, 0)$. Thus $T_0 = id_V$, and $V_1 = V$.

$\lambda = -\mathbf{1}$: Then

$$x \cos\theta - y \sin\theta = -x$$
$$x \sin\theta + y \cos\theta = -y.$$

By standard trigonometric arguments, $\theta = \pi$, since $(x, y) \neq (0, 0)$. Thus, $T_\pi(x, y) = (-x, -y)$ for all $(x, y) \in V$, and $V_{-1} = V$.

Furthermore, if $\theta \neq 0, \pi$, then T_θ has no real eigenvalues.

Example 12.12. Let V be the Euclidean plane, regarded as $\mathbb{R}^2$. Reflecting the plane in the x-axis defines the linear transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longrightarrow (x, -y).$$

Thus the real number λ is an eigenvalue for T if and only if $(x, -y) = (\lambda x, \lambda y)$ for some $(x, y) \neq (0, 0)$. In other words, λ is an eigenvalue for T if and only if there are real $x, y \in \mathbb{R}$ with $x^2 + y^2 \neq 0$ such that

$$x = \lambda x \qquad -y = \lambda y$$

Squaring and adding these equations we see that $x^2 + y^2 = \lambda^2(x^2 + y^2)$. Since $x^2 + y^2 \neq 0$, it follows that $\lambda^2 = 1$, so that the only possible eigenvalues are -1 and 1.

$\lambda = \mathbf{1}$: Then $x = x$ and $-y = y$, whence $y = 0$ and x is arbitrary.

Thus, $V_1 = \{(x, 0) \mid x \in \mathbb{R}\}$.

$\lambda = -\mathbf{1}$: Then $x = -x$ and $-y = -y$, whence $x = 0$ and y is arbitrary.

Thus $V_{-1} = \{(0, y) \mid y \in \mathbb{R}\}$.

We see that $\mathbb{R}^2 = V_1 \oplus V_{-1}$, and $\{(1, 0), (0, 1)\}$ is a basis consisting of eigenvectors for T.

Example 12.13. Let $V = \mathcal{C}^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable}\}$. Take

$$T : V \longrightarrow V, \quad f \longmapsto f'',$$

where f'' denotes the second derivative of f. Then -1 is an eigenvalue of T. Plainly, $f(t) = \cos t$ and $g(t) = \sin t$ are eigenvectors for -1 and it follows from the general theory of differential equations that they form a basis for V_{-1}. The details are left as an exercise.

Theorem 12.14. *Let $\lambda_1, \dots, \lambda_m$ be pairwise distinct eigenvalues for the endomorphism $T : V \to V$. If $\mathbf{v}_i$ is an eigenvector for $\lambda_i, (1 \leq i \leq m)$, then $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent.*

Proof. We prove the theorem by induction on m.

$m = 1$: Since an eigenvector cannot be the zero vector, the proposition is trivially true when $m = 1$.

$m > 1$: Suppose that the theorem is true for m.

Take eigenvectors $\mathbf{v}_i$ $(1 \leq i \leq m+1)$ for the pair-wise distinct eigenvalues λ_i of the endomorphism $T : V \to V$.

Then $\displaystyle\sum_{i=1}^{m+1} \alpha_i \mathbf{v}_i = \mathbf{0}_V$ if and only if $\displaystyle\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = -\alpha_{m+1}\mathbf{v}_{m+1}$, and, in that case

$$
\begin{aligned}
\sum_{i=1}^{m} \alpha_i \lambda_i \mathbf{v}_i &= \sum_{i=1}^{m} \alpha_i T(\mathbf{v}_i) && \text{as } \mathbf{e}_i \text{ is an eigenvector for } \lambda_i \\
&= T\Big(\sum_{i=1}^{m} \alpha_i \mathbf{v}_i\Big) && \text{as } T \text{ is a linear transformation} \\
&= T(-\alpha_{m+1}\mathbf{v}_{m+1}) && \\
&= -\alpha_{m+1} T(\mathbf{v}_{m+1}) && \text{as } T \text{ is a linear transformation} \\
&= -\alpha_{m+1}\lambda_{m+1}\mathbf{v}_{m+1} && \text{as } \mathbf{e}_{m+1} \text{ is an eigenvector for } \lambda_{m+1} \\
&= \lambda_{m+1}(-\alpha_{m+1}\mathbf{v}_{m+1}) && \\
&= \lambda_{m+1} \sum_{i=1}^{m} \alpha_i \mathbf{v}_i && \\
&= \sum_{i=1}^{m} \alpha_i \lambda_{m+1} \mathbf{v}_i &&
\end{aligned}
$$

Hence, $\sum_{i=1}^{m} \alpha_i(\lambda_i - \lambda_{m+1})\mathbf{v}_i = \mathbf{0}_V$

Since by the inductive hypothesis, $\mathbf{e}_1, \dots, \mathbf{e}_m$ are linearly independent, $\alpha_i(\lambda_i - \lambda_{m+1}) = 0$ for $1 \le i \le m$.

Since $\lambda_{m+1} \neq \lambda_i$ for $i < m+1$, it follows that $\alpha_i = 0$ for $i = 1, \dots, m$.

Then $\alpha_{m+1}\mathbf{v}_{m+1} = -\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = \mathbf{0}_V$.

Since $\mathbf{v}_{m+1}$ is an eigenvector to λ_{m+1}, $\mathbf{v}_{m+1} \neq \mathbf{0}_V$ and so $\alpha_{m+1} = 0$ as well. □

Corollary 12.15. *T has at most $\dim_{\mathbb{F}}(V)$ distinct eigenvalues.*

Corollary 12.16. *If $T : V \to V$ has n distinct eigenvalues, then V has a basis consisting of eigenvectors of T.*

Given the significance of eigenvalues and eigenvectors, it would be more than merely convenient to find a practical procedure for determining the eigenvalues of a given endomorphism.

When V is finite dimensional, each endomorphism $T : V \to V$ has an associated polynomial whose zeroes are precisely the eigenvalues of T, as we now show.

Recall that $\lambda \in \mathbb{F}$ is an eigenvalue for $T : V \to V$ if and only if the equation $T(\mathbf{v}) = \lambda\mathbf{v}$ has a non-zero solution, $\mathbf{v}$. Choose a basis for V. Let $\underline{\mathbf{A}}$ be the matrix of T and $\mathbf{x} \in \mathbb{F}_{(n)}$ the co-ordinate vector of $\mathbf{v}$ with respect to this basis.

Theorem 12.17. *λ is an eigenvalue for T if and only if $\det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) = 0$.*

Proof.

$$
\begin{aligned}
\lambda \text{ is and eigenvalue for } T \quad & \text{if and only if} \quad T(\mathbf{v}) = \lambda\mathbf{v} \text{ for some } \mathbf{v} \in V,\ \mathbf{v} \neq \mathbf{0}_V, \\
& \text{if and only if} \quad \underline{\mathbf{A}}\,\mathbf{x} = \lambda\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{F}_{(n)},\ \mathbf{x} \neq \mathbf{0}, \\
& \text{if and only if} \quad (\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n)\mathbf{x} = \mathbf{0} \text{ for some } \mathbf{x} \in \mathbb{F}_{(n)},\ \mathbf{x} \neq \mathbf{0}, \\
& \text{if and only if} \quad \operatorname{rk}(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) < n, \\
& \text{if and only if} \quad \det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) = 0.
\end{aligned}
$$

□

Since the determinant of $\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n$ is a polynomial function of λ, Theorem 12.17 provides for each endomorphism T a concrete polynomial in λ whose zeroes are precisely the eigenvalues the eigenvalues of T. This polynomial appears to be dependent on the basis chosen.

Fortunately, this is a case where appearances are deceptive. For if $\underline{\mathbf{B}}$ is the matrix of T with respect to another basis, then there is an invertible matrix $\underline{\mathbf{M}}$ such that $\underline{\mathbf{B}} = \underline{\mathbf{M}}\,\underline{\mathbf{A}}\,\underline{\mathbf{M}}^{-1}$. But then

$$\begin{aligned}
\det(\underline{\mathbf{B}} - \lambda\underline{\mathbf{1}}_n) &= \det(\underline{\mathbf{M}}\,\underline{\mathbf{A}}\,\underline{\mathbf{M}}^{-1} - \lambda\underline{\mathbf{M}}\,\underline{\mathbf{M}}^{-1}) \\
&= \det(\underline{\mathbf{M}}(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n)\underline{\mathbf{M}}^{-1}) \\
&= \det(\underline{\mathbf{M}})\det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n)\det(\underline{\mathbf{M}}^{-1}) \\
&= \det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) \qquad\qquad \text{as} \quad \det(\underline{\mathbf{M}}^{-1}) = (\det(\underline{\mathbf{M}}))^{-1}.
\end{aligned}$$

Thus the polynomial does not depend on the basis chosen.

Definition 12.18. Let T be an endomorphism of the n-dimensional vector space V. Let $\underline{\mathbf{A}}$ be a matrix representing T. Then

$$\chi_T(t) := \det(T - t\,id_V) = \det(\underline{\mathbf{A}} - t\,\underline{\mathbf{1}}_n) =: \chi_{\underline{\mathbf{A}}}(t)$$

is the *characteristic polynomial* of T and of $\underline{\mathbf{A}}$.

The eigenvalues are the zeroes of the characteristic polynomial, or, equivalently, the solutions of the *characteristic equation*, $\chi(t) = 0$.

We define eigenvalues, eigenvectors and eigenspaces for $n \times n$ matrices, by regarding the $n \times n$ matrix, $\underline{\mathbf{A}}$, over $\mathbb{F}$ as the linear transformation

$$T_{\underline{\mathbf{A}}} \colon \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(n)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\,\mathbf{x}$$

Observation 12.19. If $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$ has characteristic polynomial, $\xi_{\underline{\mathbf{A}}}(t) = b_0 + b_1 t + \cdots + b_n t^n$, then it follows form the definition of the characterstic function that

$$\begin{aligned}
b_0 &= \det(\underline{\mathbf{A}}) \\
b_{n-1} &= (-1)^{n-1}\operatorname{tr}(\underline{\mathbf{A}}) \\
b_n &= (-1)^n
\end{aligned}$$

Observation 12.20. The matrix of $T_{\underline{\mathbf{A}}}$ with respect to the standard basis of $\mathbb{F}_{(n)}$ is $\underline{\mathbf{A}}$ itself.

Definition 12.21. The *eigenvalues, eigenvectors and eigenspaces of* $\underline{\mathbf{A}}$ are those of the linear transformation

$$\underline{\mathbf{T}}_{\underline{\mathbf{A}}} : \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(n)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\mathbf{x}$$

Observation 12.22. In particular, if $\underline{\mathbf{A}}$ is an $n \times n$ matrix, then the eigenspace of the eigenvalue 0 is the null space of $\underline{\mathbf{A}}$. (Of course the null space (or kernel) of $\underline{\mathbf{A}}$ is trivial if and only if 0 is not an eigenvalue of $\underline{\mathbf{A}}$.)

We list further properties of eigenvalues.

Theorem 12.23. *Let λ be an eigenvalue of the matrix $\underline{\mathbf{A}}$.*

(i) λ^n is an eigenvalue of $\underline{\mathbf{A}}^n$ for any $n \in \mathbb{N}$.

(ii) If $\underline{\mathbf{A}}$ is invertible, then λ^n is an eigenvalue of $\underline{\mathbf{A}}^n$ for any $n \in \mathbb{Z}$.

(iii) λ is an eigenvalue of $\underline{\mathbf{A}}^t$.

Proof. **(i)** We adopt here the convention that $0^0 = 1$ and proceed by induction on n

$\mathbf{n = 0}$: Since $\underline{\mathbf{A}}^0 = \underline{\mathbf{1}}_n$ and $\lambda^0 = 1$, the statement is true for $n = 0$

$\mathbf{n > 0}$: Suppose that $\underline{\mathbf{A}}^n \, \mathbf{x} = \lambda^n \mathbf{x}$. Then

$$\underline{\mathbf{A}}^{n+1}\mathbf{x} = \underline{\mathbf{A}}(\underline{\mathbf{A}}^n\mathbf{x}) = \underline{\mathbf{A}}\lambda^n\mathbf{x} = \lambda^n\underline{\mathbf{A}}\,\mathbf{x} = \lambda^n\lambda\mathbf{x} = \lambda^{n+1}\mathbf{x}$$

(ii) Since we are dealing with finitely generated vector spaces, $\underline{\mathbf{A}}$ is invertible if and only if its null space is trivial, which is equivalent to 0's not being an eigenvalue of $\underline{\mathbf{A}}$. So

$$\underline{\mathbf{A}}\,\mathbf{x} = \lambda\mathbf{x} \quad \iff \quad \lambda^{-1}\mathbf{x} = \underline{\mathbf{A}}^{-1}\,\mathbf{x}.$$

The result now follows by applying Part (i) to $\underline{\mathbf{A}}^{-1}$

(iii) $\det(\underline{\mathbf{A}}^t - t\underline{\mathbf{1}}_n) = \det(\underline{\mathbf{A}}^t - t\underline{\mathbf{1}}_n^t) = \det((\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n)^t) = \det(\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n)$. □

Corollary 12.24. *Let λ be an eigenvalue of the endomorphism $T : V \to V$.*

(i) λ^n is an eigenvalue of T^n for any $n \in \mathbb{N}$, where T^n denotes the composition $T \circ \cdots \circ T$ with n terms.

(ii) If T is invertible, then λ^n is an eigenvalue of T^n for any $n \in \mathbb{Z}$

Example 12.25. We attempt to diagonalise the real matrix $\underline{\mathbf{A}} = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}$

We use elementary row operations on $\begin{bmatrix} 1-\lambda & 6 \\ 4 & 3-\lambda \end{bmatrix}$ to reduce it to a form from which we can read off the eigenvalues and readily determine the eigenvectors.

$$\begin{bmatrix} 1-\lambda & 6 \\ 4 & 3-\lambda \end{bmatrix}$$

Adding $(\lambda - 1)$ times the second row to four four times the first, then multiplying the second row by -1, we obtain

$$\begin{bmatrix} 0 & -(\lambda^2 - 4\lambda - 21) \\ 4 & 3-\lambda \end{bmatrix}$$

Because of the first column, this matrix has rank at least 1, no matter how we choose $\lambda \in \mathbb{R}$.

So, the only way its determinant can be 0, is if the second column is a multiple of the first.

By inspection, this occurs if and only if $\lambda^2 - 4\lambda - 21 = 0$.

Since $\lambda^2 - 4\lambda - 21 = (\lambda + 3)(\lambda - 7)$, the eigenvalues of $\underline{\mathbf{A}}$ are -3 and 7.

We substitute these values successively to obtain the corresponding eigenvectors.

$\lambda = -\mathbf{3}$: Our transformed matrix is

$$\begin{bmatrix} 0 & 0 \\ 4 & 6 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\underline{\mathbf{A}}$ for the eigenvalue -3 if and only if $2x+3y = 0$, so that $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ generates the eigenspace V_{-3}, and

$$\begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = -3\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \end{bmatrix} \tag{C1}$$

$\lambda = \mathbf{7}$: Our transformed matrix is

$$\begin{bmatrix} 0 & 0 \\ 4 & -4 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\underline{\mathbf{A}}$ to the eigenvalue 7 if and only if $x - y = 0$.
Hence, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ generates the eigenspace V_7, and

$$\begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \tag{C2}$$

We can combine C1 and C2 to obtain

$$\begin{aligned} \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} &= \begin{bmatrix} -9 & 7 \\ 6 & 7 \end{bmatrix} \\ &= \begin{bmatrix} (-3).3 & 7.1 \\ (-3).(-2) & 7.1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix} \qquad \text{by Section 9.5} \end{aligned}$$

We may thus regard $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ as a "change-of-basis" or "transition" matrix.

Since its inverse is $\frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ we see that the matrix corresponding to $\underline{\mathbf{A}}$ with respect to the basis for $\mathbb{R}_{(2)}$ consisting of the eigenvectors of $\underline{\mathbf{A}}$ is

$$\frac{1}{5} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix},$$

which is a diagonal matrix.

Emulating the above for the matrices

$$\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

illustrates not only what can be done, but also some of the difficulties that can arise.

Example 12.26. In particular, direct computation shows that the matrix $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$ has only one eigenvalue, namely 2, and that every eigenvector must be of the form $\begin{bmatrix} 2t \\ t \end{bmatrix}$. Hence there is no basis for $\mathbb{R}_{(2)}$ consisting of eigenvectors of our matrix, showing that the conclusion of Corollary 12.16 is not true without some condition being imposed.

This example illustrates what can go wrong when an $n \times n$ matrix has at least one eigenvalue, but does not have n distinct ones.

Observation 12.27. While our procedure for finding eigenvalues and eigenvectors is, in principle, quite simple, significant problem do arise.

An immediate one is finding the zeroes of a polynomial, or, equivalently, expressing a polynomial as the product of linear factors (factors of the form $(t - a)$. This is, in general, an unsolvable problem, even in the most familiar case, when the scalars are all complex numbers. In this case, the *Fundamental Theorem of Algebra* ensures that every polynomial can be factorised into linear factors, and *Cardano's formulæ*, dating from the 16th century, provide the factors when the polynomial in question has degree at most four. However, Lagrange, Abel and Galois proved in the 19th century, that no such general formula is possible for polynomials of degree at least five. This problem is studied in abstract algebra, where a proof is available using *Galois theory.*

The fact that exact solutions are only available in special cases means that in many practical situations, we are forced rely on *numerical methods* or other means to find sufficiently accurate approximations. This, in turn, leads to other interesting and important mathematical problems, such as finding efficient algorithms for the approximation and the question of the *stability* of the eigenvalues and eigenvectors when the coefficients are perturbed. Such questions are studied in courses on numerical methods and computer algebra.

12.1 The Cayley-Hamilton Theorem

If V is an n-dimensional vector space over $\mathbb{F}$ and $T\colon V \to V$ a linear transformation, then so is T^k for any $k \in \mathbb{N}$. Now the linear transformations $V \to V$ form a vector space $\mathrm{Hom}_{\mathbb{F}}(V,V)$ over $\mathbb{F}$ whose dimension is n^2. (To see this, recall that for a fixed basis, there is a bijection between $\mathrm{Hom}_{\mathbb{F}}(V,V)$ and $\mathbf{M}(n;\mathbb{F})$, which is actually a linear transformation, and hence an isomorphism: T corresponds to $\underline{\mathbf{A}}_T$.)

By Theorem 8.4, $id_V, T, T^2, \dots, T^{n^2}$ must be linearly dependent.

This means that there are $a_0, \dots a_{n^2} \in \mathbb{F}$, not all 0, with

$$a_0 id_V + a_1 T + \cdots + a_{n^2} T^{n^2} = 0.$$

The corresponding matrix version is that for any $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$ there are $a_0, \dots a_{n^2} \in \mathbb{F}$, not all 0, with

$$a_0 \underline{\mathbf{1}}_n + a_1 \underline{\mathbf{A}} + \cdots + a_{n^2} \underline{\mathbf{A}}^{n^2} = \underline{\mathbf{0}}_n.$$

We can express this by saying that every endomorphism of an n-dimensional vector space over $\mathbb{F}$ is a zero of of polynomial equation of degree at most n^2 over $\mathbb{F}$, or, equivalently, every $n \times n$ matrix over $\mathbb{F}$is a zero of of polynomial equation of degree at most n^2 over $\mathbb{F}$.

This immediately raises two questions:

1. Is this the best we can do, or is there a polynomial, p, of lower degree which also has T (resp. $\underline{\mathbf{A}}$) as a zero?
2. Given T (or $\underline{\mathbf{A}}$), determine the polynomial p explicitly.

If we let m_T (or $m_{\underline{\mathbf{A}}}$) be the lowest degree of any non-zero polynomial for which T (or $\underline{\mathbf{A}}$) is a zero, then what we have show is that if $\dim V = n$, then $m \leq n^2$.

The following example shows that, the best universal bound for $m$cannot be less than n. The Cayley-Hamilton Theorem (Theorem 12.32) then shows that T (or $\underline{\mathbf{A}}$) is always the zero of a specific polynomial of degree precisely n.

Example 12.28. Choose a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V. Take $T : V \longrightarrow V$ be defined by

$$T(\mathbf{e}_j) := \begin{cases} e_{j+1} & \text{if } j < n \\ e_1 & \text{if } j = n \end{cases}$$

It follows, successively, that $T(\mathbf{e}_1) = \mathbf{e}_2,\ T^2(\mathbf{e}_1),\ \ldots, T^{n-1}(\mathbf{e}_1) = \mathbf{e}_n$.

Let $p(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$ be a polynomial in $\mathbb{F}[t]$ for which $F(T) = 0$.

This means that $p(T)(\mathbf{v}) = \mathbf{0}_V$ for every $\mathbf{v} \in V$.

In particular, take $\mathbf{v} = \mathbf{e}_1$. Then

$$p(T)(\mathbf{v}) = a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + \cdots + a_{n-1}\mathbf{e}_n = \mathbf{0}_V.$$

Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V, the on;y possibility is that $a_0 = a_1 = \cdots = a_{n-1} = 0$.

This is the worst that can occur: If $T : V \to V$ is an endomorphism of an n-dimensional vector space, then it is a zero of a polynomial of specific degree n, the *characteristic polynomial*. We prove this in terms of matrices as the Cayley-Hamilton Theorem, which asserts that any $n \times n$ matrix satisfies its own *characteristic equation*.

As we need the construction of an $(n-1) \times (n-1)$ matrix from a given $n \times n$ matrix by deleting one row and one column, we our earlier definition.

Let $\underline{\mathbf{A}} = [a_{ij}]_{n\times n}$ be an $n \times n$ matrix. For $1 \le p, q \le n$ let $\underline{\mathbf{A}}_{(pq)} = [x_{ij}]_{(n-1)\times(n-1)}$ where

$$x_{ij} = \begin{cases} a_{ij} & i < p,\ j < q \\ a_{i(j+1)} & i < p,\ j \ge q \\ a_{(i+1)j} & i \ge p,\ j < q \\ a_{(i+1)(j+1)} & i \ge p,\ j \ge q \end{cases}.$$

Definition 12.29. Using the notation above, put

$$A_{ji} := (-1)^{i+j} \det\left(\underline{\mathbf{A}}_{(ij)}\right).$$

The *adjugate* of $\underline{\mathbf{A}}$ is the matrix

$$\operatorname{adj}\underline{\mathbf{A}} := [A_{ij}]_{n\times n}.$$

Lemma 12.30. *Given any $n \times n$ matrix $\underline{\mathbf{A}}$,*

$$(\operatorname{adj}\underline{\mathbf{A}})\underline{\mathbf{A}} = \underline{\mathbf{A}}(\operatorname{adj}\underline{\mathbf{A}}) = (\det\underline{\mathbf{A}})\underline{\mathbf{1}}_n.$$

Proof. The proof follows directly from the definition of matrix multiplication together with the definition and properties of the determinant function. □

We illustrate the Cayley-Hamilton Theorem with an example. Our proof of the theorem is a generalisation of this example.

Example 12.31. Take $\underline{\mathbf{A}} := \begin{bmatrix} c & b & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$$\chi_{\underline{\mathbf{A}}}(t) = \det\left(\begin{bmatrix} c-t & b & a \\ 1 & -t & 0 \\ 0 & 1 & -t \end{bmatrix}\right) = -t^3 + ct^2 + bt + a.$$

$$\underline{\mathbf{A}}^2 = \begin{bmatrix} c^2+b & cb+a & ca \\ c & b & a \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{A}}^3 = \begin{bmatrix} c^3+2bc+a & c^2b+b^2+ca & c^2a+ba \\ c^2+b & cb+a & ca \\ c & b & a \end{bmatrix}$$

$$
\begin{aligned}
\chi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}}) &= \begin{bmatrix} -c^3-2bc-a & -c^2b-b^2-ca & -c^2a-ba \\ -c^2-b & -cb-a & -ca \\ -c & -b & -a \end{bmatrix} \\
&\quad + \begin{bmatrix} c^3+cb & c^2b+ca & c^2a \\ c^2 & bc & ac \\ c & 0 & 0 \end{bmatrix} + \begin{bmatrix} bc & b^2 & ba \\ b & 0 & 0 \\ 0 & b & 0 \end{bmatrix} + \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Our proof makes use of the adjugate of $\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n$. We illustrate how this can be expressed as a polynomial in t with matrices as coefficients. Put $\underline{\mathbf{B}} := \underline{\mathbf{A}} - t\underline{\mathbf{1}}_3$, so that $\chi_{\underline{\mathbf{A}}}(t) = \det(\underline{\mathbf{B}})$.
We compute $\operatorname{adj}(\underline{\mathbf{B}}) = [x_{ij}]_{3\times 3}$, where

$$x_{ij} = (-1)^{i+j}\det(\underline{\mathbf{B}}_{(ji)}),$$

with $\underline{\mathbf{B}}_{(ji)}$ the 2×2 matrix obtained from $\underline{\mathbf{B}}$ by deleting its j^{th} row and i^{th} column.

$$
\begin{aligned}
x_{11} &= (-1)^{1+1}\det\left(\begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix}\right) &&= t^2 \\
x_{12} &= (-1)^{1+2}\det\left(\begin{bmatrix} b & a \\ 0 & -t \end{bmatrix}\right) &&= bt \\
x_{13} &= (-1)^{1+3}\det\left(\begin{bmatrix} b & a \\ -t & 0 \end{bmatrix}\right) &&= at \\
x_{21} &= (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix}\right) &&= t \\
x_{22} &= (-1)^{2+2}\det\left(\begin{bmatrix} c-t & a \\ 0 & -t \end{bmatrix}\right) &&= t^2-ct \\
x_{23} &= (-1)^{2+3}\det\left(\begin{bmatrix} c-t & a \\ 1 & 0 \end{bmatrix}\right) &&= a \\
x_{31} &= (-1)^{3+1}\det\left(\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\right) &&= 1 \\
x_{32} &= (-1)^{3+2}\det\left(\begin{bmatrix} c-t & b \\ 0 & 1 \end{bmatrix}\right) &&= t-c \\
x_{33} &= (-1)^{3+3}\det\left(\begin{bmatrix} c-t & b \\ 1 & -t \end{bmatrix}\right) &&= t^2-ct-b \\
\operatorname{adj}(\underline{\mathbf{B}}) &= \begin{bmatrix} t^2 & bt & at \\ t & t^2-ct & a \\ 1 & t-c & t^2-ct-b \end{bmatrix} \\
&= t^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & b & a \\ 1 & -c & 0 \\ 0 & 1 & -c \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 1 & 0 & -b \end{bmatrix}
\end{aligned}
$$

Theorem 12.32 (Cayley-Hamilton). *Let $\chi_{\underline{\mathbf{A}}}(t)$ be the characteristic polynomial of the $n\times n$ matrix $\underline{\mathbf{A}}$. Then $\chi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}}) = \underline{\mathbf{0}}_n$.*

Proof. Put $\underline{\mathbf{B}} := \underline{\mathbf{A}} - t\underline{\mathbf{1}}_n$.

By the definition of the determinant, there are $b_0, \dots, b_n \in \mathbb{F}$ with

$$\chi_{\underline{\mathbf{A}}}(t) = \det \underline{\mathbf{B}} = b_0 + b_1 t + \cdots + b_n t^n = \sum_{j=0}^{n-1} b_j t^j. \tag{i}$$

Since $\operatorname{adj}(\underline{\mathbf{B}}) := [x_{ij}]_{n\times n}$, with $x_{ij} := (-1)^{i+j} \det \underline{\mathbf{B}}_{(ji)}$ and $(-1)^{i+j} \det \underline{\mathbf{B}}_{(ji)}$ is a polynomial in t of degree at most $n-1$, there are $n \times n$ matrices $\underline{\mathbf{B}}_0, \dots, \underline{\mathbf{B}}_{n-1}$ with

$$\operatorname{adj} \underline{\mathbf{B}} \ = \ \underline{\mathbf{B}}_0 \ + \ \underline{\mathbf{B}}_1 t \ + \cdots + \ \underline{\mathbf{B}}_{n-1} t^{n-1} = \sum_{j=0}^{n-1} t^j \underline{\mathbf{B}}^j. \tag{ii}$$

Hence,

$$\begin{aligned}(\det \underline{\mathbf{B}})\underline{\mathbf{1}}_n &= \underline{\mathbf{B}} \operatorname{adj} \underline{\mathbf{B}} \\ &= (\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n) \operatorname{adj} \underline{\mathbf{B}} \\ &= \underline{\mathbf{A}} \operatorname{adj} \underline{\mathbf{B}} - t \operatorname{adj} \underline{\mathbf{B}} \end{aligned} \tag{iii}$$

and

$$\begin{aligned}\chi_{\underline{\mathbf{A}}}(t)\underline{\mathbf{1}}_n &= \sum_{j=0}^{n} b_j t^j \underline{\mathbf{1}}_n \\ &= (\det \underline{\mathbf{B}})\underline{\mathbf{1}}_n && \text{by (i)} \\ &= \underline{\mathbf{A}} \operatorname{adj} \underline{\mathbf{B}} - t \operatorname{adj} \underline{\mathbf{B}} && \text{by (iii)} \\ &= \underline{\mathbf{A}} \sum_{j=0}^{n-1} t^j \underline{\mathbf{B}}^j - t \sum_{j=0}^{n-1} t^j \underline{\mathbf{B}}^j && \text{by (ii)} \\ &= \underline{\mathbf{A}}\,\underline{\mathbf{B}}_0 + t(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_1 - \underline{\mathbf{B}}_0) + \cdots + t^{n-1}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_{n-1} - \underline{\mathbf{B}}_{n-2}) - t^n \underline{\mathbf{B}}_{n-1}. \end{aligned}$$

Thus

$$\begin{aligned}\chi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}}) &= \underline{\mathbf{A}}\,\underline{\mathbf{B}}_0 + \underline{\mathbf{A}}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_1 - \underline{\mathbf{B}}_0) + \cdots + \underline{\mathbf{A}}^{n-1}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_{n-1} - \underline{\mathbf{B}}_{n-2}) - \underline{\mathbf{A}}^n \underline{\mathbf{B}}_{n-1} \\ &= \underline{\mathbf{0}} \end{aligned}$$

□

By Observation 12.19, the characteristic polynomial of $\underline{\mathbf{A}} \in \mathbf{M}(n; \mathbb{F})$ is a polynomial over $\mathbb{F}$ of the form $(-1)^n(b_0 + b_1 t + \cdots + b_{n-1} t^{n-1} + t^n)$.

It is natural to ask whether every polynomial of this form is the characteristic polynomial of a matrix $\underline{\mathbf{A}} \in \mathbf{M}(n; \mathbb{F})$, or, equivalently, of an endomorphism $T\colon V \to V$, where $\dim_{\mathbb{F}}(V) = n$.

As suggested by Example 12.31, the answer is affirmative, as the following example shows.

Example 12.33. The $n \times n$ matrix

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & \cdots & -b_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

has characteristic polynomial $(-1)^n(b_0 + b_1 t + \cdots + b_{n-1} t^{n-1} + t^n)$.

The verification is left as an exercise.

This matrix is the *companion matrix* of the polynomial $b_0 + b_1 t + \cdots + b_{n-1} t^{n-1} + t^n$.

12.2 Discussion

While every $n \times n$ matrix is a zero of its characteristic polynomial, which has degree n, some matrices are zeroes of polynomial of lower degree. For example the zero matrix is a zero of the polynomial t, and the identity matrix is a zero of the polynomial $t - 1$.

We summarise a more complete analysis without providing proofs, since these require the introduction of concepts and techniques beyond the scope of these notes. They are investigated in abstract algebra.

If we restrict attention to polynomials whose the leading coefficient is 1, then there is a unique polynomial of lowest possible degree for which the matrix $\underline{\mathbf{A}}$ is a zero. This is the *minimm polynomial* of $\underline{\mathbf{A}}$, $\mu_{\underline{\mathbf{A}}}$. It divides every polynomial for which $\underline{\mathbf{A}}$ is a zero, and its zeroes are precisely the eigenvalues of $\underline{\mathbf{A}}$, that is, the zeroes of the characteristic polynomial of $\underline{\mathbf{A}}$. The main result on the minimum polynomial is that the matrix $\underline{\mathbf{A}}$ is diagonalisable if and only if

$$\mu_{\underline{\mathbf{A}}}(t) = (t - \lambda_1) \cdots (t - \lambda_m)$$

with $\lambda_i = \lambda_j$ if and only if $i = j$.

The field $\mathbb{F}$ is *algebraically closed* if and only if every polynomial in one indeterminate over $\mathbb{F}$ can be written as a product of linear factors. In such a case, every matrix, $\underline{\mathbf{A}}$ over $\mathbb{F}$ can be brought to *block diagonal form*, or *Jordan normal form*

$$\begin{bmatrix} \underline{\mathbf{A}}_{\lambda_1} & \underline{\mathbf{0}} & \cdots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{A}}_{\lambda_2} & \cdots & \\ \vdots & \vdots & \ddots & \end{bmatrix}$$

with each *Jordan block*, $\underline{\mathbf{A}}_{\lambda_j}$,of the form

$$\begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \vdots \end{bmatrix}.$$

Example 12.34. The minimum polynomial of the matrix in Example 2.4,

$$\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix},$$

is $(t-1)(t-2)$, which is of degree 2 and has two distinct zeroes. The block diagonal form comprises the two Jordan blocks

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} 3 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{A}}_1 = \begin{bmatrix} 1 \end{bmatrix}$$

The minimum polynomial of the matrix in Example 2.5,

$$\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix},$$

must divide its characteristic polynomial, $(t - 2)^2$. Hence it must be either $t - 2$ or $(t - 2)^2$. Since

$$\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the minimum polynomial cannot be $t - 2$. Hence it must be $(t - 2)^2$, which is of degree 2.

Since this fails to have two distinct zeroes, the block diagonal form has the single Jordan block

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The minimum polynomial of the matrix in Example 2.6,

$$\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix},$$

is $(t-2)^2+1$, which is of degree 2, which fails to have any real zeroes, but has two distinct complex zeroes. If we now regard is as a complex matrix, the block diagonal form comprises the two Jordan blocks

$$\underline{\mathbf{A}}_{2+i} = \begin{bmatrix} 2+i \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{A}}_{2-i} = \begin{bmatrix} 2-i \end{bmatrix}$$

where $i^2 = 1$.

The above shows that the matrix $\underline{\mathbf{A}}$ is diagonalisable if and only if each of its Jordan blocks is 1×1.

We turn to an an alternative formulation.

Definition 12.35. Let $T\colon V \to V$ be an endomorphism of the finitely generated vector space V (or, equivalently, take $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$) and λ an eigenvalue of T (or $\underline{\mathbf{A}}$).

The *algebraic multiplicity* of λ is $a \in \mathbb{N}$ if and only if $(t-\lambda)^a$ divides $\chi_T(t)$ (or $\chi_{\underline{\mathbf{A}}}(t)$), but $(t-\lambda)^{a+1}$ does not.

The *geometric multiplicity* of λ is $\dim(V_\lambda)$, that is to say, the number of linearly independent eigenvectors for the eigenvalue λ.

Example 12.36. Take $\underline{\mathbf{A}} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$.

Since $\chi_{\underline{\mathbf{A}}}(t) = (t-2)^2$, the algebraic multiplicity of the eigenvalue 2 is 2.

As we saw in Example 12.26, every eigenvector is of the form $\begin{bmatrix} 2t \\ t \end{bmatrix}$, showing that $\dim(V_2) = 1$, that is, the geometric multiplicity of 2 is 1.

We show that this is typical.

Lemma 12.37. *The geometric multiplicity of λ cannot exceed its algebraic multiplicity.*

Proof. Let λ be an eigenvalue of $T\colon V \to V$ with geometric multiplicity g.

Choose linearly independent eigenvectors $\mathbf{e}_1, \dots, \mathbf{e}_g$ for the eigenvalue λ. Extend this to a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_g, \dots \mathbf{e}_n\}$ of V.

The matrix, $\underline{\mathbf{A}}$, of T with respect to this basis is of the form

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 & * \\ 0 & \lambda & & \vdots & * \\ \vdots & & \ddots & 0 & * \\ 0 & \cdots & 0 & \lambda & * \\ \vdots & & & 0 & * \end{bmatrix}$$

This being the case, $(t-\lambda)^g$ must divide $\chi_{\underline{\mathbf{A}}}(t) = \chi_T(t)$. $\square$

Theorem 12.38. *An $n \times n$ matrix, is diagonalisable if and only if each eigenvalue has the same geometric and algebraic multiplicity, and the sum of these is n.*

Proof. A little thought shows that these conditions are necessary and sufficient to ensure that there is a basis consisting of eigenvectors. □

12.3 Exercises

Exercise 12.1. Given linear transformations $R : V \longrightarrow V'$ and $S : W \to W'$, let $\underline{\mathbf{A}}$, be the matrix of R with respect to the bases $\{\mathbf{e}_i\}$ for V and $\{\mathbf{e}_{k'}\}$ for V' and $\underline{\mathbf{B}}$ the matrix of S with respect to the bases $\{\mathbf{f}_j\}$ for W and $\{\mathbf{f}_{l'}\}$ for W'.

Show that

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$$

is the matrix of $R \oplus S$ with respect to $\{(\mathbf{e}_i, \mathbf{0}_W), (\mathbf{0}_V, \mathbf{f}_j)\}$ and $\{(\mathbf{e}'_k, \mathbf{0}_{W'}), (\mathbf{0}_{V'}, \mathbf{f}'_l)\}$

Exercise 12.2. Find the eigenvalues and eigenvectors of the following matrices:

(a) $$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -3 & 1 \\ 4 & -2 & -1 \end{bmatrix}$$

(b) $$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 12 & 1 & -3 \end{bmatrix}$$

Exercise 12.3. Find the real eigenvalues and eigenvectors of the following matrices.

(a) $\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

Exercise 12.4. Consider the real matrix

$$\underline{\mathbf{A}}_\varepsilon = \begin{bmatrix} 1+\varepsilon & 1 \\ 1 & 0 \end{bmatrix}.$$

Find its eigenvalues and eigenvectors as a function of $\varepsilon \geq 0$.

Exercise 12.5. Eigenvalues, eigenvectors and eigenspaces make sense in *any* vector space, not merely in finite dimensional vector spaces, and many problems can be formulated as eigenvalue problems. This exercise is devoted to examples of this.

Let $\mathcal{C}^\infty(\mathbb{R})$ denote the set of all smooth (that is, infinitely differentiable) real-valued functions defined on $\mathbb{R}$. Let

$$D\colon \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}), \quad f \longmapsto f'$$

be differentiation. In other words, $(D(f))(x) = f'(x)$ for all $f \in \mathcal{C}^\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

(a) Show that D is an endomorphism of the real vector space $\mathcal{C}^\infty(\mathbb{R})$. Find all of its eigenvalues and corresponding eigenvectors.

(b) Given $b \in \mathbb{R}$, show that

$$(D^2 + 2bD) : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}), \quad f \longmapsto f'' + 2bf'$$

defines an endomorphism. Find all of its eigenvalues and eigenvectors.

Exercise 12.6. Verify the Cayley-Hamilton Theorem for the following matrices.

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Exercise 12.7. Prove that the $n \times n$ matrix

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & \cdots & -b_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

has characteristic polynomial $(-1)^n(b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + t^n)$.

Chapter 13

Inner Product Spaces

The discussion and the theory developed so far have applied to vector spaces over any field. The only restriction made has been to vector spaces which are finitely generated. Even then, many of our results did not actually depend on this hypothesis.

On the other hand, we have frequently appealed to geometry to provide motivation, illustration or graphical representation of concepts and theorems, which meant restricting attention to $\mathbb{R}^n$ ($n \in \mathbb{N}$). This is hardly surprising, since $\mathbb{R}^n$ is not only the most familiar vector space, but also the locus of analytic geometry since Descartes.

We now turn our attention to formulating such informal discussion and heuristic arguments more rigorously. Specifically, we investigate the additional structure a vector space must support in order for us to be able to "do geometry", that is, to speak of *distances* and *angles*. [Recall that we have already discussed what we mean by a "line", a "plane", and so on, in any vector space.]

Surprisingly, measuring angles also provides a way of measuring distance. However, the converse is not true. We do not enter a discussion here of why, for this and related questions are discussed in detail in courses on functional analysis, which may be fruitfully thought of as the study of infinite dimensional real and complex vector spaces, requiring the additional ingredient of concepts from topology.

Our approach is to first briefly discuss making sense of the "length" of a vector, show how this permits us to define a notion of distance and to define continuity of functions between vector spaces. It follows that all linear transformations are continuous.

We then introduce the additional structure required to make sense of the notion of an "angle" between vectors and show how this allows us to speak of length, hence distance and hence continuity.

Our intuition is based on our experience with Euclidean space, which is a real vector space. The discussion actually applies to vector spaces over any sub-field of the field of complex numbers, although not for finite fields, or fields constructed from finite fields[1]. Even more, the proofs of the central results are simplest when we work over the complex numbers, and the more familiar cases are easy applications.

13.1 Normed Vector Spaces

We begin by examining the notion of *length*, or *magnitude*, of a vector.

(i) It should be clear that length should be a non-negative real number, which is 0 for, and only

[1]The reasons for this are beyond the scope of this course.

for, the zero vector.

(ii) If we scale a vector, its length is multiplied by the magnitude of the scaling factor.

(iii) The length of the sum of two vectors cannot exceed the sum of the lengths of the two vectors.

We mention here, without further explanation, that it is essentially the second condition which forces us to restrict ourselves to vector spaces over sub-fields of $\mathbb{C}$. So, unless otherwise specified, $\mathbb{F}$ henceforth denotes a sub-field of $\mathbb{C}$. This means, in particular, that $\mathbb{F}$ contains $\mathbb{Q}$, the field of rational numbers.

We now express the properties above formally, turning them into a definition.

Definition 13.1. A *norm* on the vector space V over the sub-field $\mathbb{F}$ of $\mathbb{C}$ is a function $\| \ \| : V \to \mathbb{R}_0^+$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$

N1 $\quad \|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}_V$

N2 $\quad \|\lambda\mathbf{u}\| = |\lambda|\,\|\mathbf{u}\|$

N3 $\quad \|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

A *normed vector space* is a vector space, V, over the field $\mathbb{F}$ ($\subseteq \mathbb{C}$), equipped with a norm, $\| \ \|$. It is denoted by $(V, \| \ \|)$, or simply by V when the norm is understood.

The vector, $\mathbf{v}$, in the normed vector space $(V, \| \ \|)$ is *normal* or *normalised*, or a *unit vector* if and only if $\|\mathbf{v}\| = 1$.

Example 13.2. The *absolute value* or *modulus* of a complex number defines a norm on any sub-field $\mathbb{F}$ of $\mathbb{C}$. The verification is left as an exercise.

Example 13.3. Let $\mathbb{F}$ be a sub-field of $\mathbb{C}$. Then

$$\| \ \|_{\mathbb{F}^n} : \mathbb{F}^n \longrightarrow \mathbb{R}_0^+, \quad (x_1, \dots, x_n) \longmapsto \big(\sum_{j=1}^{n} |x_j|^2\big)^{\frac{1}{2}}$$

defines a norm on $\mathbb{F}^n$, called the *Euclidean norm* on $\mathbb{F}$. The verification is left as an exercise.

Example 13.2 is just the case $n = 1$, and when $\mathbb{F} \subseteq \mathbb{R}$, we may replace $|x_j|^2$ by x_j^2.

Example 13.4. Recall that $G\colon \mathbb{R}^2 \to \mathbb{C}, \quad (x, y) \mapsto z := x + iy$ is an isomorphism of real vector spaces, and that

$$\| \ \|_{\mathbb{R}^2} = \sqrt{x^2 + y^2} = |x + iy| = \|G(x, y)\|_{\mathbb{C}^1}$$

Similarly,

$$G_n\colon \mathbb{R}^{2n} \longrightarrow \mathbb{C}^n, \quad (x_1, \dots, x_{2n}) \longmapsto (x_1 + ix_2, \dots, x_{2n-1} + ix_{2n}),$$

is an isomorphism of real vector spaces, with

$$\|(x_1, \dots, x_{2n})\|_{\mathbb{R}^{2n}} = \left(\sum_{k=1}^{2n} |x_k|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} = \|G_n(x_1, \dots, x_n)\|_{\mathbb{C}^n}.$$

Example 13.5. Let $V = \{f : [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$ denote the real vector space of all continuous real valued functions defined on the closed unit interval.

$$\| \ \|_1 : V \to \mathbb{R}_0^+, \quad f \mapsto \int_0^1 |f(t)|\, dt$$

defines a norm on V, called the $\mathcal{L}^1$ *norm* on V.

Observation 13.6. If $(V, \| \quad \|)$ is a non-trivial normed vector space then every $\mathbf{v} \in V$ in is a multiple of a unit vector. For if $\mathbf{v} \neq \mathbf{0}_V$, define $\mathbf{v}_u := \frac{\mathbf{v}}{\|\mathbf{v}\|}$. Then $\mathbf{v} = \|\mathbf{v}\|\mathbf{v}_u$. If, on the other hand, $\mathbf{v} = \mathbf{0}_V$, take any $\mathbf{w} \neq \mathbf{0}_V$ and define $\mathbf{u} := \frac{\mathbf{w}}{\|\mathbf{w}\|}$.

In either case, $\mathbf{v} = \mathbf{0}_V = 0\mathbf{v}_u = \|\mathbf{v}\|\mathbf{v}_u$.

The norm on a vector space can be used to measure of distance between any two elements of V. We first characterise what we mean by the *distance* between two points in a set.

(i) The distance between two points should be a non-negative real number, which is 0 if and only if the two points coincide.

(ii) The distance from one point to another is the same as the distance from the second to the first.

(iii) The distance between two points cannot exceed the sum of the distances of the first to any point plus the distance from that point to the second.

We mention here, without further explanation, that it these properties do not require any structure beyond being a set — in particular, there is no need to consider vector spaces. The study of sets equipped with a notion of distance between its points is the *theory of metric* spaces, a part of the study of topology.

We now express the properties above formally, turning them into a definition.

Definition 13.7. A *metric* (or *distance function*) on the set X is a function

$$d : X \times X \longrightarrow \mathbb{R}_0^+$$

such that for all $x, y, z \in X$

MS1 $d(x, y)$ if and only if $x = y$

MS2 $d(y, x) = d(x, y)$

MS3 $d(x, z) \leq d(x, y) + d(y, z)$.

A *metric space* comprises a set, X, equipped with a metric, d. We denote it by (X, d), writing only X when the metric is understood.

We now show that every normed vector space is a metric space in a natural way.

Definition 13.8. For the normed vector space, $(V, \| \quad \|)$, over the field $\mathbb{F}$, define

$$d_{\| \quad \|} : V \times V \longrightarrow \mathbb{R}_0^+, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \|\mathbf{u} - \mathbf{v}\|$$

Lemma 13.9. *Let $(V, \| \quad \|)$ be a normed vector space over the field $\mathbb{F}$. Then $(V, d_{\| \quad \|})$ is a metric space.*

Proof. Take $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Since $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in V$, $d(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\| \geq 0$ for all $\mathbf{u}, \mathbf{v} \in V$, showing that $d_{\| \ \|}$ is well defined. Moreover,

$$\begin{aligned}
d(\mathbf{u},\mathbf{v}) = 0 \quad & \text{if andf only if} \quad \|\mathbf{u}-\mathbf{v}\| & \\
& \text{if andf only if} \quad \mathbf{u}-\mathbf{v} = \mathbf{0}_V & \text{by } \mathbf{N1} \\
& \text{if andf only if} \quad \mathbf{u} = \mathbf{v}, & \text{verifying } \mathbf{MS1}. \\
d(\mathbf{v},\mathbf{u}) &:= \|\mathbf{v}-\mathbf{u}\| & \\
&= |-1|\|\mathbf{u}-\mathbf{v}\| & \text{by } \mathbf{N2} \\
&= \|\mathbf{u}-\mathbf{v}\| & \\
&=: d(\mathbf{u},\mathbf{v}), & \text{verifying } \mathbf{MS2}. \\
d(\mathbf{u},\mathbf{w}) &:= \|\mathbf{u}-\mathbf{w}\| & \\
&= \|\mathbf{u}-\mathbf{v}+\mathbf{v}-\mathbf{w}\| & \\
&\leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\| & \text{by } \mathbf{N3} \\
&=: d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}), & \text{verifying } \mathbf{MS3}.
\end{aligned}$$

□

Definition 13.10. If $(V, \| \ \|)$ is a normed vector space over the field $\mathbb{F}$, then $d_{\| \ \|}$ is the metric on V *induced by the norm* $\| \ \|$.

In particular, the Euclidean distance between points in $\mathbb{R}^n$ is the metric induced by the Euclidean norm. This allows us to reformulate the definition of continuity met in univariate and multivariate calculus in terms of metrics and so extend the notion of continuity to more general spaces.

Definition 13.11. Given metric spaces (X, d) and (Y, e), the function $f : X \to Y$ is *continuous* at $a \in X$ if and only if given any $\varepsilon > 0$ there is a $\delta > 0$ such that $e(f(x), f(a)) < \varepsilon$ whenever $d(x, a) < \delta$.

We do not pursue these ideas further here, but return to our main interest, the quest for the structure required to be able to sense of "angle" between two vectors.

13.2 Inner Products

Recall that if we take two points P and Q in the Cartesian plane, neither of which is the origin, O, with co-ordinates (x, y) and (u, v) respectively, then we can compute the cosine of the angle $\angle POQ$ directly from the co-ordinates. Suppose that the angle in question is θ. We express (x, y) in polar co-ordinates,

$$x = r\cos\alpha \quad \text{and} \quad y = r\sin\alpha$$

for uniquely determined $r > 0$ and $0 \leq \alpha < 2\pi$, so that, $r = \sqrt{x^2 + y^2}$. Then, without loss of generality,

$$u = s\cos(\alpha+\theta) \quad \text{and} \quad v = s\sin(\alpha+\theta),$$

so that $s = \sqrt{u^2 + v^2}$. Since $\theta = \alpha + \theta - \alpha$, it follows that

$$\begin{aligned}
\cos\theta &= \cos(\alpha+\theta)\cos(\alpha) + \sin(\alpha+\theta)\sin(\alpha) \\
&= \frac{u}{s}\frac{x}{r} + \frac{v}{s}\frac{y}{r} \\
&= \frac{ux + vy}{\sqrt{u^2+v^2}\sqrt{x^2+y^2}}.
\end{aligned}$$

Define

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad \big((u,v),(x,y)\big) \longmapsto ux + vy.$$

Then

$$\cos\theta = \frac{\langle\!\langle (u,v),(x,y)\rangle\!\rangle}{\sqrt{\langle\!\langle (u,v),(u,v)\rangle\!\rangle}\sqrt{\langle\!\langle (x,y),(x,y)\rangle\!\rangle}}. \tag{$*$}$$

and we can define the angle between (u,v) and (x,y) to be the unique angle $\theta \in [0,\pi]$ satisfying

$$\sqrt{\langle\!\langle (u,v),(u,v)\rangle\!\rangle}\sqrt{\langle\!\langle (x,y),(x,y)\rangle\!\rangle}\cos\theta = \langle\!\langle (u,v),(x,y)\rangle\!\rangle.$$

Since we can define the angle purely in terms of the function $\langle\!\langle\ ,\ \rangle\!\rangle$, its characteristic properties provide a basis for the definition of a general notion allowing the use of Equation $(*)$ to define an angle between two vectors in a real vector space. We first characterise $\langle\!\langle\ ,\ \rangle\!\rangle$.

Lemma 13.12. *Take $(x,y),(u,v),(r,s) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then*

(i) $\langle\!\langle (x,y),(x,y)\rangle\!\rangle \geq 0$ *with equality if and only if* $(x,y) = (0,0)$.

(ii) $\langle\!\langle (x,y),(u,v)\rangle\!\rangle = \langle\!\langle (u,v),(x,y)\rangle\!\rangle$

(iii) $\langle\!\langle \alpha(x,y),(u,v)\rangle\!\rangle = \alpha\langle\!\langle (x,y),(u,v)\rangle\!\rangle$

(iv) $\langle\!\langle (r,s)+(x,y),(u,v)\rangle\!\rangle = \langle\!\langle (r,s),(u,v)\rangle\!\rangle + \langle\!\langle (x,y),(u,v)\rangle\!\rangle$

Proof. The verifications are routine and left as an exercise. □

The function $\langle\!\langle\ ,\ \rangle\!\rangle$ just introduced leads naturally to

$$\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle} : \mathbb{R}^2 \longrightarrow \mathbb{R}^+_O, \quad (x,y) \longmapsto \sqrt{\langle\!\langle (x,y),(x,y)\rangle\!\rangle}.$$

Lemma 13.13. $\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle}$ *is a norm on* $\mathbb{R}^2$

Proof. It is routine to verify that $\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle}$ is well defined and that **N1** and **N2** hold. On the other hand, the verification of **N3** is not quite as trivial.
We leave these as an exercise, since we prove a more general version a little later. □

We extend the above discussion to complex vector spaces, using the results of Lemma 13.12 and Lemma 13.13 as a guide. One obvious generalisation, namely,

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \quad \big((u,v),(x,y)\big) \longmapsto ux + vy$$

will not do, for neither does Lemma 13.12 (i) hold, nor do we obtain a norm, since, by this definition, $\langle\!\langle (i,1),(i,1)\rangle\!\rangle = 0$.
If, on the other hand, we define

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \quad \big((u,v),(x,y)\big) \longmapsto u\overline{x} + v\overline{y},$$

then all our desired results hold, except for Lemma 13.12 (ii), which must be replaced by

(ii)′ $\langle\!\langle (x,y),(u,v)\rangle\!\rangle = \overline{\langle\!\langle (u,v),(x,y)\rangle\!\rangle}$

We take this as the model for our definition, and the characteristic properties serve as axioms.

Definition 13.14. Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. An *inner product* on the $\mathbb{F}$-vector space, V, is a function

$$\langle\!\langle\ ,\ \rangle\!\rangle : V \times V \longrightarrow \mathbb{F}$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{F}$

IP1 $\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle \geq 0$[2], with equality when, and only when, $\mathbf{u} = \mathbf{0}_V$;

IP2 $\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle = \overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}$;

IP3 $\langle\!\langle \lambda\mathbf{u}, \mathbf{v} \rangle\!\rangle = \lambda\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle$;

IP4 $\langle\!\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle\!\rangle = \langle\!\langle \mathbf{u}, \mathbf{w} \rangle\!\rangle + \langle\!\langle \mathbf{v}, \mathbf{w} \rangle\!\rangle$.

Observation 13.15. When $\mathbb{F}$ is a subfield of $\mathbb{R}$, condition **IP2** reduces to Lemma 13.12 (ii).

Example 13.16. Take $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$ ($n \in \mathbb{N} \setminus \{0\}$) Then

$$\langle\!\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle\!\rangle := \sum_{j=1}^{n} w_j \overline{z_j}$$

defines an inner product, called the *Euclidean inner product*. It (and its restriction to $\mathbb{F}^n$ for a subfield, $\mathbb{F}$, of $\mathbb{C}$) is frequently also referred to as the *standard inner product* on $\mathbb{C}^n$ or $\mathbb{F}^n$.

Example 13.17. Take $\mathbb{F} = \mathbb{C}$ and $V := \{f : [0,1] \to \mathbb{C} \mid f \text{ is continuous}\}$ Then

$$\langle\!\langle f, g \rangle\!\rangle := \int_0^1 f(t)\overline{g(t)}dt$$

defines an inner product on V. This V is usually denoted $\mathcal{L}^2([0,1])$. It is a focus of study in functional analysis as well as in measure and integration theory. You will also meet it and related spaces in statistics, the theory of differential equations and theoretical physics.

The verification that the inner product axioms hold requires a little of the theory of complex variables, namely, the fact that we may write $f(t)$ as $x(t) + iy(t)$ (where $i^2 = -1$) and then

$$\int_0^1 f(t)dt := \int_0^1 x(t)dt + i\int_0^1 y(t)dt.$$

The crucial properties of inner product spaces which enables the definition of the angle between two vectors is the Cauchy-Schwarz Inequality, which we establish next.

Theorem 13.18 (Cauchy-Schwarz Inequality). *Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space V over the subfield $\mathbb{F}$ of $\mathbb{C}$. Then for all $\mathbf{u}, \mathbf{v} \in V$*

$$|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle| \leq \sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}.$$

Proof. Take $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned} 0 &\leq \langle\!\langle \alpha\mathbf{u} - \beta\mathbf{v}, \alpha\mathbf{u} - \beta\mathbf{v} \rangle\!\rangle \\ &= \alpha\overline{\alpha}\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle - \alpha\overline{\beta}\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle - \beta\overline{\alpha}\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle + \beta\overline{\beta}\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \end{aligned}$$

Put $\alpha := \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle$ and $\beta := \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle$. Then, since $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \in \mathbb{R}$ and $z\overline{z} = |z|^2$,

$$\begin{aligned} 0 &\leq \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle - \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle - \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle\overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \\ &= \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle\left(\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle - |\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle|^2\right) \end{aligned}$$

Now $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}_V$, in which case $|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle| = 0 = \sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}$.
Otherwise, $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle > 0$, and so, $|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle|^2 \leq \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle$. □

[2]Here we use the convention that when we write $z \geq 0$ for $z \in \mathbb{C}$, we assert that z is, in fact, a real number.

It is this last result which allows us to define the *angle* between two vectors in an inner product space. We do this only for vector spaces when the scalars are real numbers in order to avoid questions about the meaning of "complex angles".

Definition 13.19. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space V over the subfield $\mathbb{F}$ of $\mathbb{R}$. Given $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}_V\}$, the *angle between* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $\angle\mathbf{uv}$, is the unique real number $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle}{\sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}}$$

We show that each inner product space is a normed vector space in a natural way.

Definition 13.20. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space, V, over the subfield, $\mathbb{F}$, of $\mathbb{C}$. Define

$$\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle} : V \longrightarrow \mathbb{F}, \quad \mathbf{v} \longmapsto \sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}$$

Theorem 13.21. *If* $\langle\!\langle\ ,\ \rangle\!\rangle$ *is an inner product on the vector space,* V*, over the field* $\mathbb{F}$*, then* $\|\ \|_{\langle\!\langle\ ,\ \rangle\!\rangle}$ *is a norm on* V*.*

Proof. Take $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$.

$$\begin{aligned}
\|\mathbf{u}\|_{\langle\!\langle\ ,\ \rangle\!\rangle} = 0 \ &\text{ if and only if } \ \|\mathbf{u}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} = 0 \\
&\text{ if and only if } \ \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle = 0 \\
&\text{ if and only if } \ \mathbf{u} = \mathbf{0}_V, && \text{by } \mathbf{IP1}\text{, verifying } \mathbf{N1}. \\
\|\lambda\mathbf{u}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} &= \langle\!\langle \lambda\mathbf{u}, \lambda\mathbf{u} \rangle\!\rangle \\
&= \lambda\overline{\lambda}\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle && \text{by } \mathbf{IP2} \text{ and } \mathbf{IP3} \\
&= |\lambda|^2 \|\mathbf{u}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle}, && \text{verifying } \mathbf{N2}. \\
\|\mathbf{u} + \mathbf{v}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} &:= \langle\!\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle\!\rangle \\
&= \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle + \overline{\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle} + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle && \text{by } \mathbf{IP2} \text{ and } \mathbf{IP4} \\
&= \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + 2\Re(\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle) + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \\
&\leq \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + 2|\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle| + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle && \text{as } \Re(z) \leq |z| \\
&\leq \langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle + 2\sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle} + \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle && \text{by Theorem 13.18} \\
&= \left(\sqrt{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle} + \sqrt{\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle}\right)^2 \\
&=: \left(\|\mathbf{u}\|_{\langle\!\langle\ ,\ \rangle\!\rangle} + \|\mathbf{v}\|_{\langle\!\langle\ ,\ \rangle\!\rangle}\right)^2, && \text{verifying } \mathbf{N3}.
\end{aligned}$$

□

13.3 Exercises

Exercise 13.1. Let $\mathbb{F}$ be a sub-field of $\mathbb{C}$. Verify that the function

$$\|\ \| : \mathbb{F} \longrightarrow \mathbb{R}^+_0, \quad z \longmapsto |z|,$$

where $|z|$ denotes the modulus of the complex number z, defines a norm on $\mathbb{F}$.

Exercise 13.2. Let $\mathbb{F}$ be a sub-field of $\mathbb{C}$. Take $\mathbb{F}^n$ with its standard vector space structure over $\mathbb{F}$. Verify that the following function defines a norm on $\mathbb{F}^n$.

$$\|\ \| : \mathbb{F}^n \longrightarrow \mathbb{R}^+_0, \quad (x_1, \ldots, x_n) \longmapsto \left(\sum_{j=1}^{n} |x|_j^2\right)^{\frac{1}{2}},$$

Exercise 13.3. Verify that multiplication of real numbers defines an inner product on $\mathbb{R}$.

Exercise 13.4. (a) $\mathbf{M}(m \times n; \mathbb{R})$ is a real vector space with respect to matrix addition and multiplication of a matrix by a constant. Show that

$$\langle\!\langle\ ,\ \rangle\!\rangle_M : \mathbf{M}(m \times n; \mathbb{R}) \times \mathbf{M}(m \times n; \mathbb{R}) \longrightarrow \mathbb{R}, \quad (\underline{\mathbf{A}}, \underline{\mathbf{B}}) \longmapsto \operatorname{tr}(\underline{\mathbf{A}}^t \underline{\mathbf{B}})$$

defines a real inner product on $\mathbf{M}(m \times n; \mathbb{R})$.

(b) Show that

$$\varphi : \mathbf{M}(m \times n; \mathbb{R}) \longrightarrow \mathbb{R}^{mn}, \quad [a_{ij}]_{m \times n} \longmapsto (x_{11}, \ldots, x_{mn}),$$

where $x_{(i-1)n+j} := a_{(i-1)n\,j}$ defines an isomorphism of real vector spaces.

(c) Show that for all $\underline{\mathbf{A}}, \underline{\mathbf{B}} \in \mathbf{M}(m \times n; \mathbb{R})$,

$$\langle\!\langle \underline{\mathbf{A}}, \underline{\mathbf{B}} \rangle\!\rangle_M = \langle\!\langle \varphi(\underline{\mathbf{A}}), \varphi(\underline{\mathbf{B}}) \rangle\!\rangle_{\mathbb{R}^{mn}},$$

where $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{R}^{mn}}$ is the Euclidean inner product on $\mathbb{R}^{mn}$. [Such an isomorphism, φ, is called a *linear isometry*.]

Exercise 13.5. Prove Lemma 13.12.

Exercise 13.6. Prove Lemma 13.13.

Chapter 14

Orthogonality

Definition 13.19 introduced the angle between two vectors in an inner product space, for vector spaces all of whose scalars are real numbers.

In particular, two (non-zero) vectors, $\mathbf{u}$ and $\mathbf{v}$, are perpendicular to each other, or *orthogonal* if the angle between them is a right angle. Since $\cos\frac{\pi}{2} = 0$, this is equivalent to $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = 0$.

Observe that this is expressed purely in terms of the inner product, without appeal to the notion of angle, so we have no need to restrict ourselves to real scalars. Hence we can define orthogonality in any inner product space.

Definition 14.1. Let $\langle\langle\ ,\ \rangle\rangle$ be an inner product on the vector space V over the subfield $\mathbb{F}$ of $\mathbb{C}$. The vectors $\mathbf{u}, \mathbf{v} \in V$ are said to be *orthogonal* if and only if $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = 0$.

Before investigating orthogonality in any detail, we provide a geometric application. Orthogonality generalises the notion of a right angle,which is central to Pythagoras' Theorem in geometry. We prove a generalised Pythagoras' Theorem.

Theorem 14.2 (**Pythagoras' Theorem**). *Let $\langle\langle\ ,\ \rangle\rangle$ be an inner product on V and $\|\ \ \|$ the norm it induces. If $\mathbf{u}, \mathbf{v} \in V$ are orthogonal, then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof. Since $\mathbf{u}, \mathbf{v}$ are orthogonal, $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = 0$. Thus

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &:= \langle\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle\rangle \\
&= \langle\langle \mathbf{u}, \mathbf{u} \rangle\rangle + \langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle + \langle\langle \mathbf{v}, \mathbf{u} \rangle\rangle + \langle\langle \mathbf{v}, \mathbf{v} \rangle\rangle \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 && \text{by orthogonality}
\end{aligned}$$

□

The next lemma is an immediate consequence of the axioms for inner products.

Lemma 14.3. *Let $(V, \langle\langle\ ,\ \rangle\rangle)$ be an inner product space. Then $\mathbf{0}_V$ is orthogonal to every $\mathbf{v} \in V$.*

Another elementary consequence is that non-zero orthogonal vectors must be linearly independent.

Theorem 14.4. *Let $(V, \langle\langle\ ,\ \rangle\rangle)$ be an inner product space. Take $\{\mathbf{v}_i \mid i \in I\} \subseteq V$ such that $\mathbf{v}_i \neq \mathbf{0}_V$ for all $i \in I$ and $\langle\langle \mathbf{v}_i, \mathbf{v}_j \rangle\rangle = 0$ unless $i = j$. Then $\{\mathbf{v}_i \mid i \in I\}$ is a set of linearly independent vectors.*

Proof. Take $\alpha_i \in \mathbb{F}$ $(i \in I)$ such that

$$\sum_{i \in I} \alpha_i \mathbf{v}_i = \mathbf{0}_V.$$

T For $j \in I$

$$\begin{aligned} 0 &= \langle\!\langle \sum_{i \in I} \alpha_i \mathbf{v}_i, \mathbf{v}_j \rangle\!\rangle \\ &= \sum_{i \in I} \alpha_i \langle\!\langle \mathbf{v}_i, \mathbf{v}_j \rangle\!\rangle \\ &= \alpha_j \langle\!\langle \mathbf{v}_j, \mathbf{v}_j \rangle\!\rangle && \text{as } \langle\!\langle \mathbf{v}_i, \mathbf{v}_j \rangle\!\rangle = 0 \text{ unless } i = j. \end{aligned}$$

Since $\mathbf{v}_j \neq \mathbf{0}_V$, $\langle\!\langle \mathbf{v}_j, \mathbf{v}_j \rangle\!\rangle \neq 0$, and so $\alpha_j = 0$. □

Unit vectors, that is, vectors whose length (norm) is 1, play a special rôle, especially when they are mutually orthogonal.

Definition 14.5. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space V. Then u_i $(i \in I$ are *orthonormal* if and only if $\langle\!\langle \mathbf{u}_i, \mathbf{u}_j \rangle\!\rangle = \delta_{ij}$, where is the *Kronecker delta*, defined by

$$\delta ij := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The basis $\mathcal{B} = \{ve_i \mid i \in I\}$ is an *orthonormal basis* if and only if the vectors in $\mathcal{B}$ are orthonormal.

Example 14.6. $V := \{f : [0, 2\pi] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a real vector space and

$$\langle\!\langle\ ,\ \rangle\!\rangle : V \times V \longrightarrow \mathbb{R}, \quad (f, g) \longmapsto \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(t)g(t)dt$$

defines an inner product on V.

For $n \in \mathbb{N} \setminus \{0\}$ define

$$\begin{aligned} c_n &: [0, 2\pi] \longrightarrow \mathbb{R}, \quad x \longmapsto \cos(nx) \\ s_n &: [0, 2\pi] \longrightarrow \mathbb{R}, \quad x \longmapsto \sin(nx) \end{aligned}$$

Then $\{c_n, s_n \mid n = 1, 2, \ldots\}$ is a set of orthonormal vectors in V.

The verification is a routine exercise in integrating trigonometric functions and left to the reader.

Orthonormal bases are particularly convenient for numerous purposes. For example, the coordinates of any vector with respect to an orthonormal basis can be computed directly, using only the inner product.

Theorem 14.7. *Let $\{\mathbf{e}_i \mid i \in I\}$ be an orthonormal basis for the inner product space V. Take $\mathbf{v} \in V$. Then*

$$\mathbf{v} = \sum_{i \in I} \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle \mathbf{e}_i.$$

Proof. Take $\mathbf{v} \in V$.

Since $\{\mathbf{e}_i \mid i \in I\}$ is a basis for V, there are uniquely determined $\alpha_i \in \mathbb{F}$ $(i \in I)$ with

$$\mathbf{v} = \sum \alpha_i \mathbf{e}_i.$$

Take $j \in I$. Then

$$\begin{aligned}
\langle\!\langle \mathbf{v}, \mathbf{e}_j \rangle\!\rangle &= \langle\!\langle \sum_{i\in I} \alpha_i \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle \\
&= \sum_{i\in I} \alpha_i \langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle \\
&= \alpha_j && \text{as } \langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle = \delta_{ij}.
\end{aligned}$$

□

Corollary 14.8. *If* $\{\mathbf{e}_i \mid i \in I\}$ *is an orthonormal basis for the inner product space* V *and* $\mathbf{v} \in V$,

$$\|\mathbf{v}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} = \sum_{i\in I} |\langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle|^2.$$

In particular, if $\mathbf{v} = \sum_{i\in I} \alpha_i \mathbf{e}_i$, *then*

$$\|\mathbf{v}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} = \sum_{i\in I} |\alpha_i|^2.$$

Proof. Take $\mathbf{v} \in V$.

$$\begin{aligned}
\|\mathbf{v}\|^2_{\langle\!\langle\ ,\ \rangle\!\rangle} &= \langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle \\
&= \langle\!\langle \sum_{i\in I} \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle \mathbf{e}_i, \sum_{j\in I} \langle\!\langle \mathbf{v}, \mathbf{e}_j \rangle\!\rangle \mathbf{e}_j \rangle\!\rangle && \text{by Theorem 14.7} \\
&= \sum_{i,j\in I} \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle \overline{\langle\!\langle \mathbf{v}, \mathbf{e}_j \rangle\!\rangle} \langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle \\
&= \sum_{i\in I} \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle \overline{\langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle} && \text{since } \langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle = \delta_{ij} \\
&= \sum_{i\in I} |\langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle|^2.
\end{aligned}$$

□

Example 14.9. Take $\mathbb{R}^n$ with the Eucldean inner product

$$\langle\!\langle\ ,\ \rangle\!\rangle_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \longmapsto \sum_{j=1}^n x_j y_j$$

As the reader no dout knows from Cartesian geomtery, that the *standard basis vectors*

$$\mathbf{e}_n = (0, \ldots, 0, 1)$$

form an orthonormal basis for $\mathbb{R}^n$,

$$\langle\!\langle (x_1, \ldots, x_n), \mathbf{e}_j \rangle\!\rangle = x_j$$

and

$$\|(x_1, \ldots, x_n)\|^2 = \sum_{j=1}^n x_j^2$$

This example shows that the above results generalise familiar ones from Cartesian geometry.

Since orthonormal bases are so useful and important, it is particularly satisfying that they can always be constructed. Given any basis whatsoever for an inner product space, there is an algorithm for constructing an orthonormal basis from it.

Theorem 14.10 (Gram-Schmidt Orthonormalisation). *Every finitely generated inner product space admits an orthonormal basis.*[1]

Proof. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be a inner product on the finitely generated vector space V over $\mathbb{F}$. Given a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of V, we construct an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ by means of a recursive procedure (algorithm), the *Gram-Schmidt procedure.*

Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a basis, $\mathbf{u}_1 \neq \mathbf{0}_V$. Put

$$\mathbf{e}_1 := \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}.$$

Suppose that orthonormal vectors $\mathbf{e}_1, \ldots \mathbf{e}_j$ have been constructed for $1 \leq j < m$ such that $\langle \mathbf{e}_1, \ldots, \mathbf{e}_j \rangle = \langle \mathbf{u}_1, \ldots, \mathbf{u}_j \rangle$.
Put

$$\mathbf{v}_{j+1} := \mathbf{u}_{j+1} - \sum_{i=1}^{j} \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_i \rangle\!\rangle \mathbf{e}_i.$$

For each $k \leq j$,

$$\begin{aligned}
\langle\!\langle \mathbf{v}_{j+1}, \mathbf{e}_k \rangle\!\rangle &= \langle\!\langle \mathbf{u}_{j+1} - \sum_{i=1}^{j} \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_i \rangle\!\rangle \mathbf{e}_i, \mathbf{e}_k \rangle\!\rangle \\
&= \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_k \rangle\!\rangle - \sum_{i=1}^{j} \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_i \rangle\!\rangle \langle\!\langle \mathbf{e}_i, \mathbf{e}_k \rangle\!\rangle \\
&= \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_k \rangle\!\rangle - \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_k \rangle\!\rangle = 0
\end{aligned}$$

Hence the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_j, \mathbf{v}_{j+1}$ are mutually orthogonal.

Moreover, $\mathbf{v}_{j+1} \neq \mathbf{0}_V$. For otherwise, by Equation **??**, $\mathbf{u}_{j+1} = \sum_{i=1}^{j} \langle\!\langle \mathbf{u}_{j+1}, \mathbf{e}_{ii} \rangle\!\rangle \mathbf{e}_i$, contradicting the linear independence of $\mathbf{u}_1, \ldots, \mathbf{u}_{j+1}$.
We may therefore put

$$\mathbf{e}_{j+1} := \frac{1}{\|\mathbf{v}_{j+1}\|} \mathbf{v}_{j+1}.$$

This clearly renders $\mathbf{e}_1, \ldots \mathbf{e}_j$ orthonormal, and hence, by Theorem (14.4), linearly independent. Thus, by Theorem (8.6), $\langle \mathbf{e}_1, \ldots, \mathbf{e}_{j+1} \rangle = \langle \mathbf{u}_1, \ldots \mathbf{u}_{j+1} \rangle$. In particular, $\{\mathbf{e}_1, \ldots \mathbf{e}_m\}$ is an orthonormal basis for V. □

Example 14.11. Consider $\mathbb{R}^2$ with the inner product

$$\langle\!\langle\ ,\ \rangle\!\rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \big((x, y), (u, v)\big) \longmapsto xu + 2xv + 2yu + 5yv.$$

and basis $\mathcal{B} = \{(1, 1), (0, 1)\}$.

[1] There is an extension of this to inner product spaces which are not finitely generated. You will meet such matters in measure and integration theory, and in functional analysis, for example. We do not pursue such matters further here.

We construct an orthonormal basis by applying the Gram-Schmid procedure to $\mathcal{B}$.

Before doing so, we verify that $\mathcal{B}$ is a basis for $\mathbb{R}^2$ and that $\langle\!\langle\ ,\ \rangle\!\rangle$ is an inner product.

Since $\mathcal{B}$ comprises two vectors and $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$, it follows from Theorem 8.6 that $\mathcal{B}$ is a basis for $\mathbb{R}^2$ if and only if $(1,1)$ and $(0,1)$ are linearly independent.

Choosing a basis for $\mathbb{R}^2$ is precisely choosing an isomorphism $\mathbb{R}^2 \to \mathbb{R}_{(2)}$, and so $(1,1)$ and $(0,1)$ are linearly independent if and only if $\mathbf{c}_1$ and $\mathbf{c}_2$, their co-ordinate vectors with respect to the chosen basis, are linearly independent.

This is true if and only if the matrix $\underline{\mathbf{A}}$, whose columns are $\mathbf{c}_1$ and $\mathbf{c}_2$, has rank 2, that is $\det(\underline{\mathbf{A}}) \neq 0$.

Since this does not depend on the choice of basis, we choose the standard basis for $\mathbb{R}^2$, $\{(1,0),(0,1)\}$.

The co-ordinate vectors are then $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ respectively, and $\det(\underline{\mathbf{A}}) = \det\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\right) = 1 \neq 0$.

We turn to verifying that $\langle\!\langle\ ,\ \rangle\!\rangle$ is a real inner product.

$$\langle\!\langle(x,y),(x,y)\rangle\!\rangle = x^2 + 4xy + 5y^2 = (x+2y)^2 + y^2 \geq 0 \quad (x,y) \in \mathbb{R}^2.$$

Moreover, $\langle\!\langle(x,y),(x,y)\rangle\!\rangle = 0$ if and only if $(x+2y)^2 + y^2 = 0$ if and only if $x = -2y$ and $y = 0$, that is, $(x,y) = (0,0)$.

$$\langle\!\langle(u,v),(x,y)\rangle\!\rangle = ux + 2uy + 2vx + 5vy = xu + 2xv + 2yu + 5yv = \langle\!\langle(x,y),(u,v)\rangle\!\rangle.$$

$$\begin{aligned}\langle\!\langle\alpha(x,y) + \beta(r,s),(u,v)\rangle\!\rangle &= \langle\!\langle(\alpha x + \beta r, \alpha y + \beta s),(u,v)\rangle\!\rangle \\ &= (\alpha x + \beta r)u + 2\alpha x + \beta r)v + 2(\alpha y + \beta s)u + 5(\alpha y + \beta s)v \\ &= \alpha(xu + 2xv + 2yu + 5yv) + \beta(ru + 2rv + 2su + 5sv) \\ &= \alpha\langle\!\langle(x,y),(u,v)\rangle\!\rangle + \beta\langle\!\langle(r,s),(u.v)\rangle\!\rangle\end{aligned}$$

We now apply the Gram-Schmid procedure.

As $\langle\!\langle(1,1),(1,1)\rangle\!\rangle = 10$, we put

$$\mathbf{e}_1 := \frac{1}{\sqrt{10}}(1,1).$$

As $\langle\!\langle\mathbf{e}_1,(0,1)\rangle\!\rangle = \frac{1}{\sqrt{10}}(2+5) = \frac{7}{\sqrt{10}}$, we put

$$\mathbf{e}_2^* := (0,1) - \langle\!\langle(0,1),\mathbf{e}_1\rangle\!\rangle\mathbf{e}_1 = (0,1) - \frac{7}{10}(1,1) = \frac{1}{10}(-7,3)$$

As $\langle\!\langle-7,3),(-7,3)\rangle\!\rangle = (-7+6)^2 + 3^2 = 10$ we put

$$\mathbf{e}_2 := \frac{1}{\sqrt{10}}(-7,3)$$

Our orthonormal basis is thus

$$\left\{\frac{1}{\sqrt{10}}(1,1), \frac{1}{\sqrt{10}}(-7,3)\right\}$$

Comment: We went to the trouble of verifying that $\mathcal{B}$ is a basis as illustration of how the theory developed earlier makes arguments simpler and avoids tedious, repetitive computations.

14.1 Orthogonal Complements

Definition 14.12. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the vector space V. The *orthogonal complement*, $S^{\perp}$, of the subset, S, of V, is the set of all vectors in V, that are orthogonal to every vector in S.

$$S^{\perp} := \{\mathbf{v} \in V \mid \langle\!\langle \mathbf{v}, \mathbf{x} \rangle\!\rangle = 0 \text{ for all } \mathbf{x} \in S\}$$

Theorem 14.13. *Let S be a subset of the inner product space $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$. Then*

(i) $S^{\perp}$ *is a vector subspace of* V.

(ii) If $S \subseteq T$, *then* $T^{\perp} \subseteq S^{\perp}$

(iii) $S^{\perp} = \langle S \rangle^{\perp}$

(iv) $\langle S \rangle \leq (S^{\perp})^{\perp}$

If, in addition, V is finitely generated,

(v) $V = \langle S \rangle \oplus \langle S \rangle^{\perp}$

(vi) $(S^{\perp})^{\perp} = \langle S \rangle$.

Proof. (i) Take $\mathbf{u}, \mathbf{v} \in S^{\perp}, \alpha, \beta \in \mathbb{F}$ and $\mathbf{x} \in S$. Then

$$\langle\!\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{x} \rangle\!\rangle = \alpha\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle + \beta\langle\!\langle \mathbf{v}, \mathbf{x} \rangle\!\rangle = 0,$$

whence $\alpha\mathbf{u} + \beta\mathbf{v} \in S^{\perp}$.

(ii) Take $\mathbf{v} \in T^{\perp}$ and $\mathbf{x} \in S$.

Since $S \subseteq T$, $\mathbf{x} \in T$, and so $\langle\!\langle \mathbf{v}, \mathbf{x} \rangle\!\rangle = 0$, whence $\mathbf{v} \in S^{\perp}$.

(iii) Since $S \subseteq \langle S \rangle$, it follows from (ii) that $\langle S \rangle^{\perp} \subseteq S^{\perp}$.

For the reverse inclusion, take $\mathbf{v} \in S^{\perp}$ and $\mathbf{x} \in \langle S \rangle$.

Then $\mathbf{x} = \alpha_1\mathbf{x}_1 + \cdots + \alpha_k\mathbf{x}_k$ for some $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in S$. Thus

$$\begin{aligned}
\langle\!\langle \mathbf{x}, \mathbf{v} \rangle\!\rangle &= \langle\!\langle \sum_{j=1}^{k} \alpha_j\mathbf{x}_j, \mathbf{v} \rangle\!\rangle \\
&= \sum_{j=1}^{k} \alpha_j \langle\!\langle \mathbf{x}_j, \mathbf{v} \rangle\!\rangle \\
&= \sum_{j=1}^{k} \alpha_j 0 \\
&= 0,
\end{aligned}$$

(iv) Take $\mathbf{x} \in \langle S \rangle$ and $\mathbf{v} \in S^{\perp} = \langle S \rangle^{\perp}$. Then $\langle\!\langle \mathbf{x}, \mathbf{v} \rangle\!\rangle = \langle\!\langle \mathbf{v}, \mathbf{x} \rangle\!\rangle = 0$, whence $\mathbf{x} \in (S^{\perp})^{\perp}$.

Suppose that V is finitely generated.

(v) For any subspace W of V, if $\mathbf{v} \in W \cap W^{\perp}$, then $\langle\!\langle \mathbf{v}, \mathbf{v} \rangle\!\rangle = 0$, whence $\mathbf{v} = \mathbf{0}_V$.

Hence $W \cap W^{\perp} = \{\mathbf{0}_V\}$.

It is therefore sufficient to show that $V = W + W^{\perp}$ and then take $W := \langle S \rangle$.

Let $\{\mathbf{e}, \dots, \mathbf{e}_k\}$ be an orthonormal basis for W and take $\mathbf{v} \in V$.

Put $\mathbf{x} := \langle\!\langle \mathbf{v}, \mathbf{e}_1 \rangle\!\rangle \mathbf{e}_1 + \cdots + \langle\!\langle \mathbf{v}, \mathbf{e}_k \rangle\!\rangle \mathbf{e}_k$ and $\mathbf{y} := \mathbf{v} - \mathbf{x}$.

Clearly, $\mathbf{x} \in W$ and $\mathbf{v} = \mathbf{x} + \mathbf{y}$.

It remains only to show that $\mathbf{y} \in W^{\perp}$.

$$\begin{aligned}
\langle\!\langle \mathbf{y}, \mathbf{e}_i \rangle\!\rangle &= \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle - \langle\!\langle \mathbf{x}, \mathbf{e}_i \rangle\!\rangle \\
&= \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle - \langle\!\langle \sum_{j=1}^{k} \langle\!\langle \mathbf{v}, \mathbf{e}_j \rangle\!\rangle \mathbf{e}_j, \mathbf{e}_i \rangle\!\rangle \\
&= \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle - \sum_{j=1}^{k} \langle\!\langle \mathbf{v}, \mathbf{e}_j \rangle\!\rangle \langle\!\langle \mathbf{e}_j, \mathbf{e}_i \rangle\!\rangle && \text{by bi-linearity} \\
&= \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle - \langle\!\langle \mathbf{v}, \mathbf{e}_i \rangle\!\rangle = 0 && \text{by orthonormality}
\end{aligned}$$

Thus, $\mathbf{y} \in W^{\perp}$.

(vi) Using (v) twice,

$$V = W \oplus W^{\perp} = W^{\perp} \oplus (W^{\perp})^{\perp}.$$

Since V is finitely generated, $W \cong (W^{\perp})^{\perp}$.

By (iv), $W \leq (W^{\perp})^{\perp}$.

Thus, $W = (W^{\perp})^{\perp}$ since V (and therefore also $(W^{\perp})^{\perp}$) is finitely generated. □

We show by example that (iv) and (v) need not hold if the vector space in question is not finitely generated.

Example 14.14. Take $V := \mathbb{R}[t]$, the real vector space of all real polynomials in one indeterminate. This is not finitely generated, since $\{t^n \mid n \in \mathbb{N}\}$ is an infinite set of linearly independent vectors. We use the inner product

$$\langle\!\langle\ ,\ \rangle\!\rangle \colon \mathbb{R}[t] \times \mathbb{R}[t] \longrightarrow \mathbb{R}, \quad (p, q) \longmapsto \int_0^1 p(x)q(x)\,dx$$

As subspace we take

$$W := \{p \mid p(0) = 0\}$$

Take any $h \in W^{\perp}$. Then

$$\begin{aligned}
\|th\|^2 &= \int_0^1 \big(xh(x)\big)^2\,dx \\
&= \int_0^1 h(x)x^2h(x)\,dx \\
&= \langle\!\langle h, t^2h \rangle\!\rangle \\
&= 0
\end{aligned}$$

since $t^2h \in W$ and $h \in W^{\perp}$.

Thus th is the 0 polynomial, whence h must be the zero polynomial.

Consequently $W^{\perp} = \{\mathbf{0}_V\}$.

It follows that $(W^{\perp})^{\perp} = V \neq W$ and also that $W + W^{\perp} = W \neq V$.

14.2 Orthogonal Transformations

When we studied vector spaces without considering any additional structure, linear transformations provide the means for comparing them, since linear transformations are precisely those functions between vector spaces which respect the vector space operations.

We have now specialised to subfields $\mathbb{F}$ of $\mathbb{C}$ in order to be able to introduce the notion of an inner product, which, as we have already seen, allows us to speak of angles and distances, thereby allowing us to "do" geometry.

It therefore behooves us to restrict ourselves to those linear transformations between inner product spaces, that respect the additional structure.

Definition 14.15. Let $(V, \langle\!\langle\ ,\ \rangle\!\rangle_V)$ and $(W, \langle\!\langle\ ,\ \rangle\!\rangle_W)$ be inner product spaces over $\mathbb{F}$.

The linear transformation $T : V \to W$ *preserves the inner product* if and only if for all $\mathbf{u}, \mathbf{v} \in V$

$$\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W = \langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle_V.$$

Theorem 14.16. *Let $T\colon V \to W$ be a linear transformation between finite dimensional inner product spaces $(V, \langle\!\langle\ ,\ \rangle\!\rangle_V), (W, \langle\!\langle\ ,\ \rangle\!\rangle_W)$. Then the following are equivalent.*

(a) T preserves the inner product.

(b) $\|T(\mathbf{u})\|_W = \|\mathbf{u}\|_V$ for all $\mathbf{u} \in V$.

(c) *If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for V, then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is an orthonormal basis for $im(T)$.*

Proof. (a) $\Rightarrow$ (b): Take $\mathbf{u} \in V$. Then

$$\|T(\mathbf{u})\|_W^2 = \langle\!\langle T(\mathbf{u}), T(\mathbf{u})\rangle\!\rangle_W = \langle\!\langle \mathbf{u}, \mathbf{u}\rangle\!\rangle_V = \|\mathbf{u}\|_V^2.$$

(b) $\Rightarrow$ (a): Take $\mathbf{u}, \mathbf{v} \in V$. Then

$$\begin{aligned}
\|\mathbf{u}\|_V^2 + \langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W + \langle\!\langle T(\mathbf{v}), T(\mathbf{u})\rangle\!\rangle_W + \|\mathbf{v}\|_V^2 &= \|T(\mathbf{u})\|_W^2 + \langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W + \langle\!\langle T(\mathbf{v}), T(\mathbf{u})\rangle\!\rangle_W + \|T(\mathbf{v})\|_W^2 \\
&= \|T(\mathbf{u}) + T(\mathbf{v})\|_W^2 \\
&= \|T(\mathbf{u} + \mathbf{v})\|_W^2 \\
&= \|\mathbf{u} + \mathbf{v}\|_V^2 \\
&= \|\mathbf{u}\|_V^2 + \langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle_V + \langle\!\langle \mathbf{v}, \mathbf{u}\rangle\!\rangle + \|\mathbf{v}\|_V^2,
\end{aligned}$$

Thus, the real parts of $\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W$ and $\langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle_V$ agree.

$$\begin{aligned}
\|\mathbf{u}\|_V^2 - i\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W + iIPT(\mathbf{v})T(\mathbf{u})_W + \|\mathbf{v}\|_V^2 &= \|T(\mathbf{u})\|_W^2 - i\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W + i\langle\!\langle T(\mathbf{v}), T(\mathbf{u})\rangle\!\rangle_W + \|T(\mathbf{v})\|_W^2 \\
&= \|T(\mathbf{u}) + iT(\mathbf{v})\|_W^2 \\
&= \|T(\mathbf{u} + i\mathbf{v})\|_W^2 \\
&= \|\mathbf{u} + ivv\|_V^2 \\
&= \|\mathbf{u}\|_V^2 - i\langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle_V + i\langle\!\langle \mathbf{v}, \mathbf{u}\rangle\!\rangle + \|\mathbf{v}\|_V^2.
\end{aligned}$$

Thus, the imaginary parts of $\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W$ and $\langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle_V$ also agree.

Hence $\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle_W = \langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle_V$.

(a) $\Rightarrow$ (c): Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis for V.

$$\langle\!\langle T(\mathbf{e}_i), T(\mathbf{e}_j)\rangle\!\rangle_W = \langle\!\langle \mathbf{e}_i, \mathbf{e}_j\rangle\!\rangle_V = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Thus, $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)\}$ is orthonormal

(c) $\Rightarrow$ (b) Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis for V and take $\mathbf{v} \in V$.
Then $\mathbf{v} = \langle\!\langle \mathbf{v}, \mathbf{e}_1\rangle\!\rangle_V \mathbf{e}_1 + \cdots + \langle\!\langle \mathbf{v}, \mathbf{e}_n\rangle\!\rangle_V \mathbf{e}_n$, and so $T(\mathbf{v}) = \langle\!\langle \mathbf{v}, \mathbf{e}_1\rangle\!\rangle_V T(\mathbf{e}_1) + \cdots + \langle\!\langle \mathbf{v}, \mathbf{e}_n\rangle\!\rangle_V T(\mathbf{e}_n)$.
Since $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ is an orthonormal basis for V, Theorem 14.7 that for all i

$$\langle\!\langle T(\mathbf{v}), T(\mathbf{e}_i)\rangle\!\rangle_W = \langle\!\langle \mathbf{v}, \mathbf{e}_i\rangle\!\rangle_V.$$

By Corollary 14.8,

$$T(\mathbf{v}) = \langle\!\langle T(\mathbf{v}), T(\mathbf{e}_1)\rangle\!\rangle_V T(\mathbf{e}_1) + \cdots + \langle\!\langle T(\mathbf{v}), T(\mathbf{e}_n)\rangle\!\rangle_V T(\mathbf{e}_n).$$

Thus, $\|T(\mathbf{v})\|_W^2 = \|\mathbf{v}\|_V^2$. □

Corollary 14.17. *If the linear tranformations, $T : V \longrightarrow W$, between the inner product spaces $(V, \langle\!\langle\ ,\ \rangle\!\rangle_V), (W, \langle\!\langle\ ,\ \rangle\!\rangle_W)$ preserves the inner product, then T is injective.*

Proof. $T(\mathbf{v}) = \mathbf{0}_W$ if and only if $\|T(\mathbf{v})\|_W = 0$ if and only if $\|\mathbf{v}\|_V = 0$ if and only if $=\mathbf{0}_V$. □

The most important case, particularly from the point of view of applications, is when $W = V$ and $\langle\!\langle\ ,\ \rangle\!\rangle_W = \langle\!\langle\ ,\ \rangle\!\rangle_V$.

Definition 14.18. Let $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$ be an inner product space. Then the linear transformation $T : V \longrightarrow V$ is said to be an *orthogonal transformation* with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ if and only if for all $\mathbf{u}, \mathbf{v} \in V$

$$\langle\!\langle T(\mathbf{u}), T(\mathbf{v})\rangle\!\rangle = \langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle.$$

Observation 14.19. Traditionally, orthogonal endomorphisms of a complex inner product space are called *unitary*.

Lemma 14.20. *Let $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$ be an inner product space. Then each orthogonal transformation $T : V \longrightarrow V$ is an isomorphism.*

Proof. If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an orthonormal basis for V, then $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ are orthonormal, hence linearly independent, and hence form a basis. □

14.3 Exercises

Exercise 14.1. For each of the following symmetric matrices, find an orthogonal matrix which diagonalises it.

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Chapter 15

Matrix Representations of Inner Products

Just matrices enables us to represent linear transformations and perform computations on them, so we now turn to representing inner products using matrices, with a view to allowing us to carry out concrete computations.

It is actually simpler to discuss general sesqui-linear forms and bi-linear forms, restricting to the appropriate special cases as required.

Take finitely generated vector spaces U and V over the sub-field $\mathbb{F}$ of $\mathbb{C}$ let $\beta : U \times V \to \mathbb{F}$ be a sesqui-linear function. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ be a basis for U and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ a basis for V.

Given $\mathbf{u} \in U$ and $\mathbf{v} \in V$, we have

$$\mathbf{u} = \sum_{i=1}^{m} x_i \mathbf{e}_i \qquad \text{and} \qquad \mathbf{v} = \sum_{j=1}^{n} y_j \mathbf{f}_j,$$

whence

$$\begin{aligned} \beta(\mathbf{u}, \mathbf{v}) &= \beta(\sum_{i=1}^{m} x_i \mathbf{e}_i, \sum_{j=1}^{n} y_j \mathbf{f}_j) \\ &= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} x_i \overline{y_j} \beta(\mathbf{e}_i, \mathbf{f}_j) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} \overline{y_j}, \end{aligned}$$

where $a_{ij} := \beta(\mathbf{e}_i, \mathbf{f}_j) \qquad (1 \leq i \leq m,\ 1 \leq j \leq n)$

Let

$$\underline{\mathbf{x}} := \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad \text{and} \qquad \underline{\mathbf{y}} \begin{bmatrix} y_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}$$

be the co-ordinate vectors of $\mathbf{u}$ and $\mathbf{v}$ with respect to the given bases, and put $\underline{\mathbf{A}} := [a_{ij}]_{m \times n}$ with $a_{ij} := \beta(\mathbf{e}_i, \mathbf{f}_j)$. Direct calculation shows that $\beta(\mathbf{u}, \mathbf{v}) = \underline{\mathbf{x}}^t \underline{\mathbf{A}}\, \overline{\underline{\mathbf{y}}}$.

Since our purposes require us to consider only the case $U = V$ and $\mathbf{f}_j = \underline{\mathbf{e}}_j$, we dispense with the greater generality for the rest of this chapter. But the importance and usefulness of the more general approach cannot be over-emphasised, for it marks the beginnings of tensor analysis, which has many applications in statistics, geometry, physics, chemistry and engineering.

Definition 15.1. Given a sesqui-linear form $\beta : V \times V \to \mathbb{F}$, the *matrix of* β *with respect to the basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *of* V is the matrix

$$\underline{\mathbf{A}} := \left[\beta(\mathbf{e}_i, \mathbf{e}_j)\right]_{n \times n}$$

Since an inner product is a sesqui-linear form, we can already deduce an important fact.

Theorem 15.2. *Let* $\langle\!\langle\ ,\ \rangle\!\rangle$ *be an inner product on* V. *Let* $\underline{\mathbf{A}}$ *be the matrix of* $\langle\!\langle\ ,\ \rangle\!\rangle$ *with respect to the basis* $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. *The* $\mathcal{B}$ *is an orthogonal basis if and only if* $\underline{\mathbf{A}}$ *is a diagonal matrix, and* $\mathcal{B}$ *is an orthonormal basis if and only if* $\underline{\mathbf{A}} = \underline{\mathbf{1}}_n$

The above discussion forms the basis of our computational techniques. We summarise the above discussion in our next theorem.

Theorem 15.3. *Let* $\{\mathbf{e}_i, \ldots, \mathbf{e}_n\}$ *be a basis for the vector space* V. *Let* $\underline{\mathbf{x}}$ *is the co-ordinate vector of* $\mathbf{u} \in V$ *and* $\underline{\mathbf{y}}$ *that of* $\mathbf{v} \in V$.
If $\beta : V \times V \to \mathbb{F}$ *is a sesqui-linear form on* V, *then*

$$\beta(\mathbf{u}, \mathbf{v}) = \underline{\mathbf{x}}^t \underline{\mathbf{A}}\, \overline{\underline{\mathbf{y}}}$$

and if β *is bi-linear, then*

$$\beta(\mathbf{u}, \mathbf{v}) = \underline{\mathbf{x}}^t \underline{\mathbf{A}}\, \underline{\mathbf{y}}.$$

We investigate the relationship between endomorphisms (or changes of basis) on the one hand, and sesqui-linear forms on the other.

Lemma 15.4. *Let* $\beta : V \times V \to \mathbb{F}$ *be sesqui-linear and* $T : V \to V$ *an endomorphism. Then*

$$\gamma : V \times V \longrightarrow \mathbb{F}, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \beta(T(\mathbf{u}), T(\mathbf{v}))$$

is a sesqui-linear form, denoted $\gamma = \beta \circ (T \times T)$.

Proof. Take $\lambda, \mu \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Then

$$\begin{aligned}
\gamma(\lambda\mathbf{u} + \mu\mathbf{v}, \mathbf{w}) &= \beta(T(\lambda\mathbf{u} + \mu\mathbf{v}), T(\mathbf{w})) \\
&= \beta(\lambda T(\mathbf{u}) + \mu T(\mathbf{v}), T(\mathbf{w})) \\
&= \lambda\beta(T(\mathbf{u}), T(\mathbf{w})) + \mu\beta(T(\mathbf{v}), T(\mathbf{w})) \\
&= \lambda\gamma(\mathbf{u}, \mathbf{w}) + \mu\gamma(\mathbf{v}, \mathbf{w}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma(\mathbf{u}, \lambda{+}\mu\mathbf{w}) &= \beta(T(\mathbf{u}), T(\lambda{+}\mu\mathbf{w})) \\
&= \beta(\mathbf{u}, \lambda T(\mathbf{v}) + \mu T(\mathbf{w})) \\
&= \overline{\lambda}\beta(T(\mathbf{u}), T(\mathbf{v})) + \overline{\mu}\beta(T(\mathbf{u}), T(\mathbf{w})) \\
&= \overline{\lambda}\gamma(\mathbf{u}, \mathbf{v}) + \overline{\mu}\gamma(\mathbf{u}, \mathbf{w}),
\end{aligned}$$

which shows that γ is sesqui-linear. □

Corollary 15.5. *Let* $\beta : V \times V \to \mathbb{F}$ *be bi-linear and* $T : V \to V$ *an endomorphism. Then* $\beta \circ (T \times T)$ *is a bi-linear form.*

Theorem 15.6. *Let $\beta, \gamma : V \times V \to \mathbb{F}$ be sesqui-linear forms and $T : V \to V$ an endomorphism. Choose a basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V. If the matrices of β, γ and T with respect to $\mathcal{B}$ are $\underline{\mathbf{B}}, \underline{\mathbf{C}}$ and $\underline{\mathbf{A}}$ respectively, then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \overline{\underline{\mathbf{A}}}.$$

Corollary 15.7. *Let $\beta, \gamma : V \times V \to \mathbb{F}$ be bi-linear forms and $T : V \to V$ an endomorphism. Choose a basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V. If the matrices of β, γ and T with respect to $\mathcal{B}$ are $\underline{\mathbf{B}}, \underline{\mathbf{C}}$ and $\underline{\mathbf{A}}$ respectively, then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \underline{\mathbf{A}}.$$

Corollary 15.8. *Let $\beta : V \times V \to \mathbb{F}$ be a sesqui-linear form. Choose bases $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\mathbb{C} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ for V. If the matrices of β with respect to $\mathcal{B}$ is $\underline{\mathbf{B}}$ and that with respect to $\mathcal{C}$ is $\underline{\mathbf{C}}$ then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \overline{\underline{\mathbf{A}}},$$

where $\underline{\mathbf{A}}$ is the "change of basis matrix" from the basis $\mathcal{C}$ to $\mathcal{B}$.

Proof. Recall that if $\mathbf{v} \in V$ has $\underline{\mathbf{x}}$ as co-ordinate vector with respect to $\mathcal{B}$ and $\underline{\mathbf{y}}$ with respect of $\mathcal{C}$, then $\underline{\mathbf{x}} = \underline{\mathbf{A}}\, \underline{\mathbf{y}}$. $\square$

Corollary 15.9. *Let $\beta : V \times V \to \mathbb{F}$ be a bi-linear form. Choose bases $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\mathcal{C} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ for V. If the matrices of β with respect to $\mathcal{B}$ is $\underline{\mathbf{B}}$ and that with respect to $\mathcal{C}$ is $\underline{\mathbf{C}}$ then*

$$\underline{\mathbf{C}} = \underline{\mathbf{A}}^t \underline{\mathbf{B}}\, \underline{\mathbf{A}},$$

where $\underline{\mathbf{A}}$ is the "change of basis matrix" from the basis $\mathcal{C}$ to $\mathcal{B}$.

In order to represent inner products on finitely generated spaces by matrices, we have only exploited the fact that an inner product on V is a sesqui-linear form on V. The other requirements impose conditions on the matrix representing our form.

Because $\langle\!\langle \mathbf{u}, \rangle\!\rangle = \overline{\langle\!\langle , \mathbf{u} \rangle\!\rangle}$ for all $\mathbf{u}, \mathbf{v} \in V$, we must have

$$a_{ji} := \langle\!\langle \mathbf{e}_j, \mathbf{e}_i \rangle\!\rangle = \overline{\langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle} =: a_{ij}$$

for any basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$. Thus, if $\underline{\mathbf{A}}$ is the matrix of the inner product, we must have

$$\underline{\mathbf{A}}^t = \overline{\underline{\mathbf{A}}}$$

Definition 15.10. The complex matrix $\underline{\mathbf{A}}$ is called *Hermitian* if and only if

$$\underline{\mathbf{A}}^t = \overline{\underline{\mathbf{A}}}.$$

Of course, if the $\mathbb{F}$ is a subset of $\mathbb{R}$, then the sesqui-liniearity becomes bi-linearity and $\langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle = \langle\!\langle \mathbf{v}, \mathbf{u} \rangle\!\rangle$ for all $\mathbf{u}, \mathbf{v} \in V$, so that if $\underline{\mathbf{A}}$ is the matrix of the inner product, then

$$\underline{\mathbf{A}}^t = \underline{\mathbf{A}}$$

Definition 15.11. The real matrix $\underline{\mathbf{A}}$ is called *symmetric* if and only if

$$\underline{\mathbf{A}}^t = \underline{\mathbf{A}}.$$

We summarise our discussion in the next theorem.

Theorem 15.12. *Any matrix representing a complex inner product must be Hermitian, and any matrix representing a real inner product must be symmetric.*

So far, we have not exploited the positive definiteness of inner products. This also has consequences for the matrix representation of inner products. It is again more convenient to present the discussion at the level of sesqui-linear forms, rather than restricting only to inner products.

First observe that for any inner product space V, given $\lambda \in \mathbb{F}$ and $\mathbf{x} \in V$, it follows from (IP1) and (IP2) that

$$\langle\!\langle \lambda\mathbf{x}, \lambda\mathbf{x} \rangle\!\rangle = \lambda\overline{\lambda}\langle\!\langle \mathbf{x}, \mathbf{x} \rangle\!\rangle = |\lambda|^2 \langle\!\langle \mathbf{x}, \mathbf{x} \rangle\!\rangle$$

In particular, given a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, we have $\mathbf{x} = \sum x_j\mathbf{e}_j$, for suitable x_j $(j = 1, \ldots, n)$, whence $\langle\!\langle \mathbf{x}, \mathbf{x} \rangle\!\rangle = \sum a_{ij}x_i\overline{x_j}$, with $a_{ij} := \langle\!\langle \mathbf{e}_i, \mathbf{e}_j \rangle\!\rangle$.

Thus we require that for all $x_1, \ldots, x_n$

$$\sum_{j=1}^{n} a_{ij}x_i\overline{x_j} \geq 0,$$

with equality if and only if each $x_j = 0$.

This last expression is a homogeneous quadratic polynomial in the co-ordinates of $\mathbf{x}$.

Given the importance of such functions, especially in the real case, we later devote a chapter to them.

15.1 Exercises

Exercise 15.1. Let V and W be finitely generated vector spaces over the subfield $\mathbb{F}$ of $\mathbb{C}$. Let $\gamma : W \times W \to \mathbb{F}$ be a bi-linear form on W and $T : V \to W$ a linear transformation.

Show that

$$\beta : V \times V \longrightarrow \mathbb{F}, \quad (\mathbf{u}, \mathbf{v}) \longmapsto \gamma(T(\mathbf{u}), T(\mathbf{v}))$$

defines a bi-linear form on V. [We write $\beta = \gamma \circ (T \times T)$.]

Show that if, instead, γ is sesqui-linear, then so is β.

Choose bases $\mathcal{B} = \{\mathbf{e}_i, \ldots, \mathbf{e}_n\}$ for V and $\mathcal{C} = \{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ for W. Let the matrix of T with respect to these bases be $\underline{\mathbf{A}}$. Let the matrix of γ with respect to $\mathcal{C}$ be $\underline{\mathbf{C}}$ and that of β with respect to $\mathcal{B}$ be $\underline{\mathbf{B}}$.

Show that if γ is bi-linear, then

$$\underline{\mathbf{B}} = \underline{\mathbf{A}}^t \underline{\mathbf{C}}\, \underline{\mathbf{A}},$$

and if γ is sesqui-linear, then

$$\underline{\mathbf{B}} = \underline{\mathbf{A}}^t \underline{\mathbf{C}}\, \overline{\underline{\mathbf{A}}}.$$

Exercise 15.2. Show that if $\underline{\mathbf{A}}$ is a complex Hermitian $n \times n$ matrix, and $\underline{\mathbf{B}}$ is any other complex $n \times n$ matrix, then $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \overline{\underline{\mathbf{B}}}$ is also Hermitian.

Show that if $\underline{\mathbf{A}}$ is a real symmetric $n \times n$ matrix, and $\underline{\mathbf{B}}$ is any other real $n \times n$ matrix, then $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \underline{\mathbf{B}}$ is also symmetric.

Chapter 16

Adjoint Linear Transformations

The presence of an inner product has significant consequences. In particular, it enables us to map a vector space into its dual space, and gives rise to the notion of the *adjoint* of a linear transformation, which is a cornerstone of several applications of linear algebra, such as to quantum mechanics.

To discuss the adjoint we first investigate the relation between a vector space and its dual in the presence of an inner product.

Recall that if V is a vector space over the field $\mathbb{F}$, then its dual space, V^*, is $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, the $\mathbb{F}$ vector space of all $\mathbb{F}$-linear transformations $V \to \mathbb{F}$. Such a linear transformation is often called a *1-form* or a *linear form*.

Lemma 16.1. *Let $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$ be an inner product space over $\mathbb{F}$. For each $\in V$*

$$\langle\!\langle\ , \mathbf{v}\rangle\!\rangle \colon V \longrightarrow \mathbb{F}, \quad \mathbf{x} \longmapsto \langle\!\langle \mathbf{x}, \mathbf{v}\rangle\!\rangle$$

is a linear transformation.

Proof. Take $\mathbf{x}, \mathbf{y} \in V$ and $\lambda, \mu \in \mathbb{F}$. By the definition of inner product,

$$\langle\!\langle \lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{v}\rangle\!\rangle = \lambda\langle\!\langle \mathbf{x}, \mathbf{v}\rangle\!\rangle + \mu\langle\!\langle \mathbf{y}, \mathbf{v}\rangle\!\rangle$$

□

We use Lemma 16.1 to embed V in V^*.

Lemma 16.2. *Let $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$ be an inner product space over $\mathbb{F}$. Then*

$$R \colon V \longrightarrow V^*, \quad \longmapsto \langle\!\langle\ , \mathbf{v}\rangle\!\rangle$$

is injective.

Proof. Take $\mathbf{u}, \mathbf{v} \in V$. Then $R(\mathbf{u}) = R(\mathbf{v})$ if and only if for all $\mathbf{x} \in V$, $\big(R(\mathbf{u})\big)(\mathbf{x}) = \big(R(\mathbf{v})(\mathbf{x})$, or, equivalently, $\langle\!\langle \mathbf{x}, \mathbf{u}\rangle\!\rangle = \langle\!\langle \mathbf{x}, \mathbf{v}\rangle\!\rangle$.

By the definition of inner product, this is equivalent to $\langle\!\langle \mathbf{x}, \mathbf{u} - \mathbf{v}\rangle\!\rangle = 0$ for all $\mathbf{x} \in V$.

In particular $\langle\!\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle\!\rangle = 0$, whence $\mathbf{u} = \mathbf{v}$ since $\langle\!\langle\ ,\ \rangle\!\rangle$ is an inner product.

□

16.1 Riesz Representation Theorem

For general vector spaces, there are no "natural" linear forms, that is linear transformations from the given space to the field of scalars.

The preceding discussion shows that for inner product spaces, there is a rich supply of them, at least one for each vector in V. The Riesz Representation Theorem states that under some additional conditions, these are the only linear forms. We prove the Riesz Representation Theorem for finitely generated inner product spaces.

Theorem 16.3 (**Riesz Representation Theorem**). *Let $(V, \langle\!\langle\ \ ,\ \ \rangle\!\rangle)$ be a finitely generated inner product space over $\mathbb{F}$. Given a linear transformation $\varphi\colon V \to \mathbb{F}$, there is a unique vector, $\mathbf{v}_\varphi \in V$, such that for all $\mathbf{x} \in V$*

$$\varphi(\mathbf{x}) = \langle\!\langle \mathbf{x}, \mathbf{v}_\varphi \rangle\!\rangle$$

Proof. We first establish the uniqueness of such a vector, $\mathbf{v}_\varphi$.

Suppose that $\langle\!\langle \mathbf{x}, \mathbf{u} \rangle\!\rangle = \langle\!\langle \mathbf{x}, \mathbf{v} \rangle\!\rangle$ for all $\mathbf{x} \in V$. Since this is equivalent to $\langle\!\langle \mathbf{x}, \mathbf{v} - \mathbf{u} \rangle\!\rangle = 0$ for all $\mathbf{x} \in V$, it follows that $\langle\!\langle \mathbf{v} - \mathbf{u}, \mathbf{v} - \mathbf{u} \rangle\!\rangle = 0$, whence $\mathbf{v} = \mathbf{u}$.

Next we show the existence of $\mathbf{v}_\varphi$.

Since $\mathrm{im}(\varphi)$ is a vector subspace of $\mathbb{F}$, either $\mathrm{im}(\varphi) = \{0\}$ or $\mathrm{im}(\varphi) = \mathbb{F}$.

In the former case, $\varphi(\mathbf{x}) = 0$ for all $\mathbf{x} \in V$, and so we may choose $\mathbf{v}_\varphi := \mathbf{0}_V$.

In the latter case, choose an orthonormal basis for $\ker(\varphi)$, say $\{\mathbf{e}_2, \ldots, \mathbf{e}_n\}$, and extend to an orthonormal basis, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of V.

Observe that since $\mathbf{e}_1 \notin \ker(\varphi)$, $\varphi(\mathbf{e}_1) \neq 0$.

Take $\mathbf{x} \in V$. By Theorem 14.7,

$$\mathbf{x} = \sum_{j=1}^{n} \langle\!\langle \mathbf{x}, \mathbf{e}_j \rangle\!\rangle \mathbf{e}_j,$$

so that

$$\begin{aligned}
\varphi(\mathbf{x}) &= \varphi\Big(\sum_{j=1}^{n} \langle\!\langle \mathbf{x}, \mathbf{e}_j \rangle\!\rangle \mathbf{e}_j\Big) \\
&= \sum_{j=1}^{n} \langle\!\langle \mathbf{x}, \mathbf{e}_j \rangle\!\rangle \varphi(\mathbf{e}_j) && \text{as } \varphi \text{ is linear} \\
&= \langle\!\langle \mathbf{x}, \mathbf{e}_1 \rangle\!\rangle \varphi(\mathbf{e}_1) && \text{as } \varphi(\mathbf{e}_j) = 0 \text{ for } j > 1 \\
&= \langle\!\langle \mathbf{x}, \overline{\varphi(\mathbf{e}_1)}\mathbf{e}_1 \rangle\!\rangle && \text{as } \varphi(\mathbf{e}_1) \in \mathbb{F}.
\end{aligned}$$

So $\mathbf{v}_\varphi := \overline{\varphi(\mathbf{e}_1)}\mathbf{e}_1$ clearly has the required property.

□

Corollary 16.4. *Let $(V, \langle\!\langle\ \ ,\ \ \rangle\!\rangle)$ be a finite dimensional inner product space over $\mathbb{F}$. Then the function*

$$R : V \longrightarrow \mathrm{Hom}(V, \mathbb{F}), \qquad \mathbf{v} \longmapsto \langle\!\langle\ \ , \mathbf{v} \rangle\!\rangle,$$

is an additive bijection, which is an isomorphism of vector spaces whenever $\mathbb{F} \subseteq \mathbb{R}$.

Proof. That R is well defined follows from the fact that for all $\mathbf{x}, \mathbf{y}, \mathbf{v} \in V,\ \alpha, \beta \in \mathbb{F}$, $\langle\!\langle \alpha\mathbf{x}+\beta\mathbf{y}, \mathbf{v}\rangle\!\rangle = \alpha\langle\!\langle \mathbf{x}, \mathbf{y}\rangle\!\rangle + \beta\langle\!\langle \mathbf{y}, \mathbf{v}\rangle\!\rangle$.

That R is bijective is a restatement of the Riesz Representation Theorem.

Finally, take $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$. Then, for any $\mathbf{x} \in V$,

$$\begin{aligned}\big(R(\alpha\mathbf{u}+\beta\mathbf{v})\big)(\mathbf{x}) &= \langle\!\langle \mathbf{x}, \alpha\mathbf{u}+\beta\mathbf{v}\rangle\!\rangle \\ &= \overline{\alpha}\langle\!\langle \mathbf{x}, \mathbf{u}\rangle\!\rangle + \overline{\beta}\langle\!\langle \mathbf{x}, \mathbf{v}\rangle\!\rangle. \\ &= \overline{\alpha}\big(R(\mathbf{u})\big)(\mathbf{x}) + \overline{\beta}\big(R(\mathbf{v})\big)(\mathbf{x}) \\ &= \big(\overline{\alpha}R(\mathbf{u}) + \overline{\beta}R(\mathbf{v})\big)(\mathbf{x})\end{aligned}$$

Thus $R(\alpha\mathbf{u}+\beta\mathbf{v}) = \overline{\alpha}R(\mathbf{u}) + \overline{\beta}R(\mathbf{v})$.

If, in fact, $\mathbb{F} \subseteq \mathbb{R}$, then $R(\alpha\mathbf{u}+\beta\mathbf{v}) = \alpha R(\mathbf{u}) + \beta R(\mathbf{v})$ □

Example 16.5. Our version of the Riesz Representation Theorem is not the original one. While the restriction to finitely generated inner product spaces is not necessary, some restriction (on either the inner product space, V, or on the class of linear forms, $\varphi : V \to \mathbb{F}$) is required, as we now show, recalling Example 14.14.

Take $V := \mathbb{R}[t]$ with the inner product

$$\langle\!\langle\ ,\ \rangle\!\rangle : V \times V \longrightarrow \mathbb{F}, \quad (f, g) \longrightarrow \int_0^1 f(x)g(x)\,dx$$

Take the linear transformation

$$\varphi : V \longrightarrow \mathbb{R}, \quad f \longmapsto f(0).$$

For any $h \in V$, $f := t^2h \in \ker(\varphi)$ and

$$\langle\!\langle f, h\rangle\!\rangle = \langle\!\langle t^2h, h\rangle\!\rangle = \int_0^1 x^2(h(x))^2dx = \langle\!\langle th, th\rangle\!\rangle.$$

Thus $\varphi(f) = \langle\!\langle f, h\rangle\!\rangle$ if and only if $\varphi(f) = \|th\|^2$.

Since $\varphi(f) = 0$, this is the case if and only if $th = 0$.

As t is not the zero polynomial, this implies that h is the zero polynomial. Then $\langle\!\langle p, h\rangle\!\rangle = 0$ for all $p \in V$.

But φ is not the zero transformation, since $\varphi(1) = 1$.

Hence there is no $\mathbf{v}_\varphi \in V$ with $\varphi(\mathbf{x}) = \langle\!\langle \mathbf{x}, \mathbf{v}_\varphi\rangle\!\rangle$ for all $\mathbf{x} \in V$.

We consider the effect of an endomorphism on the above.

Let $T : V \longrightarrow V$ be an endomorphism of the finitely generated inner product space $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$ and take $\mathbf{v} \in V$. Then

$$L_T^{\mathbf{v}} : V \longrightarrow \mathbb{F}, \quad \mathbf{x} \longmapsto \langle\!\langle T(\mathbf{x}), \mathbf{v}\rangle\!\rangle$$

is a linear form on V.

By the Riesz Representation Theorem, there is a unique $\mathbf{v}_{L_T^{\mathbf{v}}} \in V$ with $L_T^{\mathbf{v}}(\mathbf{x}) = \langle\!\langle \mathbf{x}, \mathbf{v}_{L_T^{\mathbf{v}}}\rangle\!\rangle$ for all $\mathbf{x} \in V$. In other words,

$$\langle\!\langle T(\mathbf{x}), \mathbf{v}\rangle\!\rangle = \langle\!\langle \mathbf{x}, \mathbf{v}_{L_T^{\mathbf{v}}}\rangle\!\rangle$$

for all $\mathbf{x} \in V$. Given a fixed endomorphism $T : V \longrightarrow V$ we obtain a function

$$T^* : V \longrightarrow V, \quad \mathbf{v} \longmapsto \mathbf{v}_{L_T^{\mathbf{v}}},$$

characterised by

$$\langle\langle T(\mathbf{x}), \mathbf{y} \rangle\rangle = \langle\langle \mathbf{x}, T^*(\mathbf{y}) \rangle\rangle \qquad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

Lemma 16.6. *For each endomorphism $T : V \longrightarrow V$,*

$$T^* : V \longrightarrow V, \quad \longmapsto \mathbf{v}_T^{\mathbf{v}}$$

is a linear transformation

Proof. Take $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{F}$. Then, for each $\mathbf{x} \in V$,

$$\begin{aligned}
\langle\langle \mathbf{x}, T^*(\alpha\mathbf{u} + \beta\mathbf{v} \rangle\rangle &= \langle\langle T(\mathbf{x}), \alpha\mathbf{u} + \beta\mathbf{v} \rangle\rangle \\
&= \overline{\alpha}\langle\langle T(\mathbf{x}), \mathbf{u} \rangle\rangle + \overline{\beta}\langle\langle T(\mathbf{x}), \mathbf{v} \rangle\rangle \\
&= \overline{\alpha}\langle\langle \mathbf{x}, T^*(\mathbf{u}) \rangle\rangle + \overline{\beta}\langle\langle \mathbf{x}, T^*(\mathbf{v}) \rangle\rangle \\
&= \langle\langle \mathbf{x}, \alpha T^*(\mathbf{u}) + \beta T^*(\mathbf{v}) \rangle\rangle.
\end{aligned}$$

By the uniqueness of $\mathbf{v}_{L_T^{\alpha\mathbf{u}+\beta\mathbf{v}}}$, $T^*(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T^*(\mathbf{u}) + \beta T^*(\mathbf{v})$. □

Definition 16.7. Given a linear transformation $T : V \longrightarrow V$, the linear transformation, $T^* : V \longrightarrow V$, is the *adjoint* of T.

Lemma 16.8.

$$id_V^* = id_V$$

and for $S, T : V \longrightarrow V$,

$$(S \circ T)^* = T^* \circ S^*.$$

Proof. The former statement is obvious.

As for the latter, take $\mathbf{v}, \mathbf{x} \in V$. Then

$$\begin{aligned}
\langle\langle \mathbf{x}, (S \circ T)^*(\mathbf{v}) \rangle\rangle &= \langle\langle (S \circ T)(\mathbf{x}), \mathbf{v} \rangle\rangle \\
&= \langle\langle S(T(\mathbf{x})), \mathbf{v} \rangle\rangle \\
&= \langle\langle T(\mathbf{x}), S^*(\mathbf{v}) \rangle\rangle \\
&= \langle\langle \mathbf{x}, T^*(S^*(\mathbf{v})) \rangle\rangle \\
&= \langle\langle \mathbf{x}, (T^* \circ S^*)(\mathbf{v}) \rangle\rangle.
\end{aligned}$$

By uniqueness, $(S \circ T)^*(\mathbf{v}) = (T^* \circ S^*)(\mathbf{v})$ for all $\mathbf{v} \in V$. Hence $(S \circ T)^* = T^* \circ S^*$. □

There is a useful and important relationship between subspaces invariant under an endomorphism and the adjoint of the endomorphism.

Theorem 16.9. *Let $T : V \to V$ be an endomorphism of the inner product space $(V, \langle\langle\ ,\ \rangle\rangle)$. If the subspace W of V is invariant under T, then $W^\perp$ is invariant under T^*.*

In other words, if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$, then $T^*(\mathbf{x}) \in W^\perp$ for all $\mathbf{x} \in W^\perp$.

Proof. Take $\mathbf{w} \in W$ and $\mathbf{x} \in W^{\perp}$. Then

$$\begin{aligned}\langle\!\langle \mathbf{w}, T^*(\mathbf{x})\rangle\!\rangle &= \langle\!\langle T(\mathbf{w}), \mathbf{x}\rangle\!\rangle \\ &= 0 && \text{since } T(\mathbf{w}) \in W \text{ and } \mathbf{x} \in W^{\perp}\end{aligned}$$

Thus $T^*(\mathbf{x}) \in W^{\perp}$. □

An important class of endomorphisms are those which agree with their adjoints.

Definition 16.10. The endomorphism $T : V \to V$ is *self-adjoint* if and only if $T^* = T$.

Being self-adjoint has significant consequences for the eigenvalues of an endomorphism.

Theorem 16.11. *The eigenvalues of a self-adjoint endomorphism are all real.*

Proof. Let $\mathbf{v} \neq \mathbf{0}_V$ be an eigenvector of the self-adjoint endomorphism T for the eigenvalue λ. Then

$$\lambda\langle\!\langle \mathbf{v}, \mathbf{v}\rangle\!\rangle = \langle\!\langle \lambda\mathbf{v}, \mathbf{v}\rangle\!\rangle = \langle\!\langle T(\mathbf{v}), \mathbf{v}\rangle\!\rangle = \langle\!\langle \mathbf{v}, T\rangle\!\rangle^*(\mathbf{v}) = \langle\!\langle \mathbf{v}, T(\mathbf{v})\rangle\!\rangle = \langle\!\langle \mathbf{v}, \lambda\mathbf{v}\rangle\!\rangle = \overline{\lambda}\langle\!\langle \mathbf{v}, \mathbf{v}\rangle\!\rangle.$$

Since $\mathbf{v} \neq \mathbf{0}_V$, it follows that $\lambda = \overline{\lambda}$, which is the case if and only if λ is real. □

Corollary 16.12. *Every self-adjoint endomorphism has eigenvalues, whenever the ground field contains all real numbers.*

Proof. By the Fundamental Theorem, the characteristic polynomial factors into linear factors over the complex numbers, so that

$$\chi_T(t) = \prod_{j=i}^{n}(t - \lambda_j).$$

The λ_j's are precisely the eigenvalues of T. Since these are all real, this is, in fact, a factorisation over the reals. Hence T has n real eigenvalues (with multiplicities). □

Corollary 16.13. *Eigenvectors to distinct eigenvalues of a self-adjoint endomorphism are mutually orthogonal.*

Proof. Let $T : V \to V$ be self-adjoint. Take $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}_V\}$ with $T(\mathbf{u}) = \lambda\mathbf{u}$ and $T(\mathbf{v}) = \mu\mathbf{v}$ with $\lambda \neq \mu$. Then

$$\begin{aligned}\lambda\langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle &= \langle\!\langle \lambda\mathbf{u}, \mathbf{v}\rangle\!\rangle \\ &= \langle\!\langle T(\mathbf{u}), \mathbf{v}\rangle\!\rangle \\ &= \langle\!\langle \mathbf{u}, T(\mathbf{v})\rangle\!\rangle && \text{as } T \text{ is self-adjoint} \\ &= \langle\!\langle \mathbf{u}, \mu\mathbf{v}\rangle\!\rangle \\ &= \mu\langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle && \text{as } \mu \in \mathbb{R}.\end{aligned}$$

Since $\lambda \neq \mu$, $\langle\!\langle \mathbf{u}, \mathbf{v}\rangle\!\rangle = 0$. □

We come to our main result on self-adjoint endomorphisms.

Theorem 16.14. *If $T\colon V \to V$ is a self-adjoint endomorphism of the finitely generated inner product space $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$, then V has an orthonormal basis of eigenvectors of T.*

Proof. We use induction on $\dim(V)$.

If $\dim(V) = 1$, then there is nothing to prove.

Suppose that the result holds for self-adjoint endomorphisms of inner product spaces of dimension less than n.

Let $\dim(V) = n$ and take a self-adjoint endomorphism $T : V \to V$.

By Corollary 16.12, T has an eigenvalue, say λ. Let $\mathbf{v} \neq \mathbf{0}_V$ be an eigenvector. Then $W := \mathbb{F}\mathbf{v}$ is a T-invariant subspace of V.

Since V is finitely generated, $V \cong W \oplus W^{\perp} = \mathbb{F}\mathbf{v} \oplus (\mathbb{F}\mathbf{v})^{\perp}$, whence $\dim W = n - 1$

By Theorem 16.9, $W^{\perp}$ is a T^*-invariant subspace of V.

Since T is self-adjoint, $W^{\perp}$ is an $(n-1)$–dimensional T-invariant subspace of V.

By the inductive hypothesis, $W^{\perp}$ has an orthonormal basis, $\{\mathbf{e}_2, \dots, \mathbf{e}_n\}$, consisting of eigenvectors of T.

Putting $\mathbf{e}_1 := \frac{\mathbf{v}}{\|\mathbf{v}\|}$, $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an orthonormal basis for V comprising eigenvectors of T,. □

Corollary 16.15. *If $\underline{\mathbf{A}}$ is a Hermitian $n \times n$ complex matrix, then there is a unitary matrix, $\underline{\mathbf{B}}$, such that $\overline{\underline{\mathbf{B}}}^t \underline{\mathbf{A}}\,\underline{\mathbf{B}}$ is a (real) diagonal matrix.*

Proof. Recall that if take $\mathbb{C}_{(n)}$ with the standard (Euclidean) inner product and regard the $n \times n$ complex matrix $\underline{\mathbf{A}}$ as the linear transformation

$$\mathbb{C}_{(n)} \longrightarrow \mathbb{C}_{(n)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\mathbf{x},$$

then $\underline{\mathbf{A}}$ is self-adjoint if and only if it is Hermitian.

In that case $\mathbb{C}_{(n)}$ has an orthonormal $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ basis comprising eigenvectors of $\underline{\mathbf{A}}$.

Let $\underline{\mathbf{B}}$ be the $n \times n$ complex matrix whose j^{th} column is $\mathbf{e}_j$.

Since $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is orthonormal, $\underline{\mathbf{B}}^t\overline{\underline{\mathbf{B}}} = \underline{\mathbf{1}}_n$.

Thus $\underline{\mathbf{B}}^{-1} = \overline{\underline{\mathbf{B}}}^t$, that is to say, $\underline{\mathbf{B}}$ is unitary.

Moreover, since for each j there is a $\lambda_j \in \mathbb{R}$ with $\underline{\mathbf{A}}\mathbf{e}_j = \lambda_j \mathbf{e}_j$, we have

$$\underline{\mathbf{A}}\,\underline{\mathbf{B}} = \underline{\mathbf{B}}\,\underline{\mathbf{diag}}(\lambda_1, \dots, \lambda_n),$$

where $\underline{\mathbf{diag}}(\lambda_1, \dots, \lambda_n)$ is the $n \times n$ matrix $\left[x_{ij}\right]_{n\times n}$ defined by

$$x_{ij} = \begin{cases} \lambda_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus $\overline{\underline{\mathbf{B}}}^t \underline{\mathbf{A}}\,\underline{\mathbf{B}} = \underline{\mathbf{B}}^{-1}\underline{\mathbf{A}}\,\underline{\mathbf{B}} = \underline{\mathbf{diag}}(\lambda_1, \dots, \lambda_n)$ is a (real) diagonal matrix. □

Corollary 16.16. *If $\underline{\mathbf{A}}$ is a symmetric $n \times n$ real matrix, then there is an orthogonal matrix, $\underline{\mathbf{B}}$, such that $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\,\underline{\mathbf{B}}$ is a diagonal matrix.*

16.2 Exercises

Exercise 16.1. Take $V := \mathcal{P}_3$, the set of all polynomials of degree at most 2 in the indeterminate t with real coefficients, with inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ given by

$$\langle\!\langle p, q \rangle\!\rangle := \int_0^1 p(x)q(x)dx$$

Consider the linear form

$$\varphi : V \longrightarrow \mathbb{R}, \quad p \longmapsto p(0)$$

Find the element of $\mathcal{P}_3$ which represents φ.

Exercise 16.2. Find an orthogonal matrix which diagonalises the real matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Exercise 16.3. Find a unitary matrix which diagonalises the complex matrix

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Chapter 17

Real Quadratic Forms

Recall from your study of real valued functions of n real variables that to find the extreme values of a sufficiently smooth function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad (x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$$

we first look at its *gradient*

$$\nabla(f)(x_1, \ldots, x_n) := (f_{x_1}(x_1, \ldots, x_n), \ldots, f_{x_n}(x_1, \ldots, x_n)),$$

where

$$f_{x_i} := \frac{\partial f}{\partial x_i}.$$

Because of the conditions we have imposed on f, a necessary — but not sufficient — condition for f to have an extreme value at a point in $\mathbb{R}^n$ is that the gradient be the zero vector at that point.

We then examine the *Hessian* of f,

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

whose properties provide sufficient — but not necessary — conditions for an extremum: f has a (local) minimum whenever the Hessian is "positive definite" and a (local) maximum whenever it is "negative definite".

This Hessian is an example of a *(real) quadratic form*, to whose study this chapter is devoted.

If $\beta\colon V \times V \to \mathbb{R}$ is a symmetric bi-linear form on V, a finitely generated real vector space, we can associate with it the real-valued function

$$q : V \longrightarrow \mathbb{R}, \quad \mathbf{v} \longmapsto \beta(\mathbf{v}, \mathbf{v}).$$

Take a basis, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, for V. Then $\mathbf{v} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n$ for suitable $x_1, \ldots, x_n \in \mathbb{R}$, and it follows from the bi-linearity of β that

$$q(\mathbf{v}) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \beta(\mathbf{e}_i, \mathbf{e}_j)$$

or, putting $a_{ij} := \beta(\mathbf{e}_i, \mathbf{e}_j)$,

$$q(\mathbf{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \tag{17.1}$$

In other words, $q(\mathbf{v})$ is a homogeneous quadratic polynomial in n variables, viz. the co-ordinates of $\mathbf{v}$. In particular, given $\mathbf{v} \in V$ and all $\lambda \in \mathbb{R}$,

$$q(\lambda\mathbf{v}) = \lambda^2 q(\mathbf{v}).$$

Definition 17.1. A *quadratic form* on the vector space, V, over the subfield, $\mathbb{F}$, of $\mathbb{R}$ is a function

$$q : V \longrightarrow \mathbb{F},$$

such that for all $\mathbf{x} \in V$ and $\lambda \in \mathbb{F}$

$$q(\lambda\mathbf{x}) = \lambda^2 q(\mathbf{x}),$$

and that there is a symmetric bi-linear form, $\beta : V \times V \to \mathbb{F}$, such that for all $\mathbf{x} \in V$

$$q(\mathbf{x}) = \beta(\mathbf{x}, \mathbf{x}).$$

Given a basis $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for V, the matrix $\underline{\mathbf{A}}_q := \big[\beta(\mathbf{e}_i, \mathbf{e}_j)\big]_{n\times n}$ is the *matrix of the quadratic from q with respect to the basis $\mathcal{B}$.*

Observation 17.2. The matrix $\underline{\mathbf{A}}_q$ is a symmetric matrix, since it is the matrix of the symmetric bi-linear form in the definition of a quadratic form.

We defined quadratic forms in terms of bi-linear forms. In fact, real quadratic forms and real symmetric bi-linear forms completely determine each other.

Theorem 17.3. *Given a quadratic form q on the vector space V over $\mathbb{F} \subseteq \mathbb{R}$, there is a unique bi-linear form, β, on V such that*

$$q(\mathbf{v}) = \beta(\mathbf{v}, \mathbf{v})$$

for all $\in V$.

Proof. As the existence of such a bi-linear form is ensured by the very definition of a quadratic form, it remains only to demonstrate its uniqueness.
But observe that

$$q(\mathbf{u} - \mathbf{v}) = \beta(\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = q(\mathbf{u}) - 2\beta(\mathbf{u}, \mathbf{v}) + q(\mathbf{v})$$

by the bi-linearity and symmetry of β and, similarly

$$q(\mathbf{u} + \mathbf{v}) = \beta(\mathbf{u} + , \mathbf{u} + \mathbf{v}) = q(\mathbf{u}) + 2\beta(\mathbf{u}, \mathbf{v}) + q(\mathbf{v}),$$

whence

$$\beta(\mathbf{u}, \mathbf{v}) = \frac{1}{4}\left(q(\mathbf{u} + \mathbf{v}) - q(\mathbf{u} - \mathbf{v})\right).$$

□

Quadratic forms are real valued functions. So we may ask about the values they take.

Definition 17.4. The real quadratic form $q : V \longrightarrow \mathbb{R}$ is be *positive (negative) semi-definite* if and only if $q(\mathbf{v}) \geq 0$ (resp. $q(\mathbf{v}) \leq 0$) for all $\mathbf{v} \in V$. It is *positive (negatgive) definite* if, in addition, $q(\mathbf{v}) = 0$ only for $=\mathbf{0}_V$.

Otherwise, it is *indefinite.*

Of course, the quadratic form derived from an inner product must be positive definite.

Example 17.5. We illustrate the above using quadratic forms $q : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

(i) $q(x, y, z) := x^2 + y^2$ is, plainly, positve semi-definite. It is not positive definite, since $q(0, 0, 1) = 0$.

(ii) $q(x, y, z) := x^2 + y^2 - z^2$ is idefinite, for $q(1, 0, 0) = 1$, whereas $q(0, 0, 1) = -1$

We shall see (Sylvester's Theorem), that these examples are typical, in that every quadratic form is equivalent to one like the ones above.

We consider two quadratic forms to be equivalent if there is an automorphism of the vector space such that one form is the composite of the other with the automorphism. Formally,

Definition 17.6. Two quadratic forms, q and $\tilde{q}$, on the finitely generated real vector space V are *equivalent* if and only if there is an isomorphism $\varphi : V \to V$ such that $\tilde{q} = q \circ q$.

We formulate this in terms of matrices.

Lemma 17.7. *Let $q : V \to \mathbb{R}$ be a quadratic form on the finitely generated real vector space V and $\varphi : V \to V$ a linear transformation. Then $q \circ \varphi : V \to \mathbb{R}$ is also a quadratic form.*

Moreover, if the matrix q with respect to a particular basis of V is $\underline{\mathbf{A}}$ and that of φ is $\underline{\mathbf{B}}$, then the matrix of $q \circ \varphi$ is $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \underline{\mathbf{B}}$.

Proof. By Theorem 17.3, it is sufficient to prove the corresponding result for the bi-linear form determined by q. But that is precisely the content of Corollary 15.5 and Theorem 15.6. □

Real quadratic forms are classified up to isomorphism by a triple of natural numbers, as the next theorem shows.

Theorem 17.8 (Sylvester's Theorem).

The real quadratic form $q : \mathbb{R}^n \to \mathbb{R}$ is equivalent to one or the form

$$\sum_{i=1}^{r-s} x_i^2 - \sum_{i=r-s+1}^{r} x_j^2$$

with $0 \leq s \leq r \leq n$.

Moreover, the triple of natural numbers (n, r, s) determines q up to isomorphism.

Proof. It is sufficient to prove that there is a basis for $\mathbb{R}^n$ with respect to which the matrix of q is a diagonal matrix all of whose entries are 0 or ± 1.

By Corollary 16.16 there is a basis, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, for $\mathbb{R}^n$ with respect to which the matrix of q is the diagonal matrix

$$\begin{bmatrix} d_1 & 0 & \cdots \\ 0 & d_2 & \\ \vdots & 0 & \ddots \end{bmatrix},$$

where we may assume that this basis has been so ordered that

$$\begin{array}{ll} d_i > 0 & \text{for } 1 \le i \le r-s \\ d_i < 0 & \text{for } r-s < i \le r \\ d_i = 0 & \text{for } r < i \le n \end{array}$$

Putting

$$\lambda_i := \begin{cases} \frac{1}{\sqrt{|d_i|}} & \text{for } i \le r \\ 1 & \text{for } i > r \end{cases}$$

it follows immediately that the matrix of q with respect to the basis $\left\{\frac{1}{\sqrt{\lambda_i}}\mathbf{e}_i \mid 1 \le i \le n\right\}$ has the form required. □

17.1 Exercises

Exercise 17.1. Let $\beta : V \times V \longrightarrow \mathbb{R}$ be a symmetric bi-linear form on the real vector space V and $q : V \longrightarrow \mathbb{R}$ a quadratic form on V. Show that for all $\mathbf{u}, \in V$

(a) $\beta_{q_\beta}(\mathbf{u}, \mathbf{v}) = \beta(\mathbf{u}, \mathbf{v})$

(b) $q_{\beta_q}(\mathbf{u}) = q(\mathbf{u})$

Exercise 17.2. Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the real vector space V and $\|\ \|$ the norm it induces. Decide whether

$$q : V \longrightarrow \mathbb{R}, \quad \mathbf{u} \longmapsto \|\mathbf{u}\|^2$$

defines a quadratic form on V.

Exercise 17.3. Let q be a positive definite quadratic form on the real vector space V. Prove that

$$\|\ \| : V \longrightarrow \mathbb{R}, \quad \mathbf{u} \longmapsto \sqrt{q(\mathbf{u})}$$

defines a norm on V.

Exercise 17.4. Classify each of the following bi-linear forms according to its definiteness property:

(a) $\beta : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad ((u,v,w),(x,y,z)) \longmapsto 2ux + uy - 2uz + vx + 3vy - vz - 2wx - wy + 3wz$

(b) $\beta : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad ((u,v,w),(x,y,z)) \longmapsto ux + 3uy + 3uz + 3vx + vy + vz + 3wx + wy + 2wz$

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