Exponentials and Logarithms

1. Exponents

We begin by defining a^n for suitable a and n

If we restrict n to being a counting number, we can define exponentiation as *repeated multiplication* and this is then a well defined operation.

Informally, we define the n^{th} *power* of the real number, a, to be the real number obtained by taking n copies of a and multiplying them together:

$$a^n = \underbrace{a \cdots a}_{n \text{ copies}}$$

We formulate this in a recursive definition.

Definition 1.1. Let a be any real number. The n^{th} *power* of a, a^n, is given by

$$\begin{aligned} a^0 &:= a \\ a^{n+1} &:= a.a^n \qquad \text{for any counting number, } n. \end{aligned}$$

Lemma 1.2. *Take $a \in \mathbb{R}$ and $m, n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Then*

$$\begin{aligned} a^m a^n &= a^{m+n} \\ (a^m)^n &= a^{mn} \end{aligned}$$

Proof. The claims follow by repeated application of the associative law for multiplication.

$$\begin{aligned} a^m a^n &= \underbrace{a \cdots a}_{m \text{ copies}} \underbrace{a \cdots a}_{n \text{ copies}} \\ &= \underbrace{a \cdots a\, a \cdots a}_{m+n \text{ copies}} \\ &= a^{m+n} \\ (a^m)^n &= \underbrace{a^m \cdots a^m}_{n \text{ copies}} \\ &= \underbrace{\underbrace{a \cdots a}_{m \text{ copies}} \cdots \underbrace{a \cdots a}_{m \text{ copies}}}_{n \text{ copies}} \\ &= \underbrace{a \cdots a\, a \cdots a \cdots a \cdots a}_{mn \text{ copies}} \\ &= a^{mn} \end{aligned}$$

□

Remark 1.3. If we restrict attention to $a > 0$, then fr every counting number, n,

$$a^{n+1} \begin{cases} < a^n & \text{if } a < 1 \\ = a^n = 1 & \text{if } a = 1 \\ > a^n & \text{if } a > 1 \end{cases}$$

In particular, if $a > 0$ and $a \neq$, then

$$e : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{R}, \quad n \longmapsto a^n$$

is an injective function.

We now seek to extend the definition of the n^{th} power of the real number a to all natural numbers, n, not just counting numbers, while preserving Lemma 1.2.

In particular, we wish to have $a^m a^n = a^{m+n}$ continue to hold even if either m or n is 0. In other words, we want

$$a^n = a^{0+n} = a^0 a^n$$

to hold. But this is equivalent to

$$(a^0 - 1)a^n = 0,$$

from which we deduce that either $a^n = 0$ or $a^0 = 1$.

In the first case, we must have $a = 0$, and nothing can be inferred about a^0.

If, on the other hand, $a \neq 0$, then we must have $a^0 = 1$.

If $a \neq 0$, then $a^0 := 1$. We do not define 0^0.

This allows us to define a^n recursively for any non-zero real number, a, and any natural number:

Definition 1.4. Let a be a non-zero real number. Then

$$\begin{aligned} a^0 &:= 1 \\ a^{n+1} &:= a.a^n && \text{for any } n \in \mathbb{N} \end{aligned}$$

We verify that all of Lemma 1.2 remains true as long as $a \neq 0$.

$$\begin{aligned} a^m a^0 &= a^m 1 = a^m = a^{m+0} \\ \left(a^0\right)^n &= 1^n = 1 = a^0 = a^{0n} \\ (a^m)^0 &= 1 = a^0 = a^{m0} \end{aligned}$$

We next extend the definition to all integers. The problem is to define a^{-n} for $n \in \mathbb{N}$. We let Lemma 1.2 and Definition 1.4 guide us.

We want

$$1 = a^0 = a^{(-n)+n} = a^{-n}a^n$$

Since $a \neq 0$ and $n \in \mathbb{N}$, $a^n \neq 0$, whence the above equality implies a^{-n} must be $\dfrac{1}{a^n}$.

Definition 1.5. Given $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$,

$$a^{-n} := \frac{1}{a^n}$$

Remark 1.6. Since $(-(-n)) = n$ for every integer n, $\dfrac{1}{\frac{1}{x}} = x$ and $\left(\frac{1}{x}\right)^n = \frac{1}{x^n}$ or every non-zero real number, we see that Lemma 1.2 still holds.

We extend the definition to all rational numbers, the problem being to define $a^{\frac{1}{n}}$ where n is a counting number. We let Lemma 1.2 guide us again.

We want

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}n} = a^1 = a.$$

In other words, $a^{\frac{1}{n}}$ should solve the equation $x^n = a$.

Since this equation has no real solutions if $a < 0$ and n is even, we restrict attention to $a > 0$.

Definition 1.7. Given $a > 0$ and a counting number n, $a^{\frac{1}{n}}$ is the unique positive solution of the equation $x^n = a$.

We leave it to the reader to verify that Lemma 1.2 still holds.

The next theorem summarises our investigations.

Theorem 1.8. *Let a be any positive real number ad m, n rational numbers. Then*

$$\begin{aligned} a^(0) &:= 1 && (1) \\ a^1 &:= a && (2) \\ a^m a^n &= a^{m+n} && (3) \\ (a^m)^n &= a^{mn} && (4) \\ a^{-m} &:= \frac{1}{a^m} && (5) \\ a^{\frac{m}{n}} &:= \sqrt[n]{a^m} && \textit{if } n \neq 0 \quad (6) \end{aligned}$$

Moreover, if $a > 1$, then

$$f : \mathbb{Q} \longrightarrow \mathbb{R}^+, \quad x \longmapsto a^x$$

is monotonically strictly increasing (and hence injective).

Remark 1.9. We lose nothing by restricting attention to $a > 1$. For if $0 < a < 1$, then $a^x = b^{-x}$, where $b := \frac{1}{a} > 1$ and $1^x = 1$ for all $x \in \mathbb{Q}$ since we require a^x to be a real number.

Remark 1.10. It is, in fact, possible to extend the definition of a^x to all real numbers x, but this requires theory not yet developed in this course, on way of doing so uses the properties f the (Riemann) integral as developed in MATH102. We provide an outline of this below.

2. Logarithms

Given $a > 1$, the function

$$f\colon \mathbb{Q} \longrightarrow \mathbb{R}^+, \quad x \longmapsto a^x$$

is monotonically strictly increasing. Moreover, $a^n \to \infty$ as $n \to \infty$, and, as will be proved later, f is continuous. Thus $a^x \to \infty$ as $x \to \infty$ and $a^x \to 0$ as $x \to -\infty$.

Hence, by the Intermediate Value Theorem, $\operatorname{im}(f) = \mathbb{R}^+$.

Since f is bijective, it has an inverse, namely the *logarithm to the base a*.

Definition 2.1. Take $a > 1$ and $x \in \mathbb{R}^+$. Then u is the *logarithm of x to the base a*, written $u = \log_a x$, if and only if $x = a^u$.

Theorem 2.2. *Take $a > 1$, $x, y \in \mathbb{R}^+$ and $n \in \mathbb{R}$. Then*

$$\begin{aligned} \log_a 1 &:= 0 && (7)\\ \log_a &:= 1 && (8)\\ \log_a(xy) &= \log_a x + \log_a && (9)\\ \log_a(x^y) &= y \log_a x && (10) \end{aligned}$$

Proof. The proofs are direct consequences of the definitions and the corresponding results on exponents.

We illustrate this by proving (9).

Put $u := \log_a(x)$ and $v := \log_a(y$, so that $x = a^u$ and $y = a^v$. Then

$$xy = a^u a^v = a^{u+v},$$

from which it follows that

$$\log_a(xy) = u + v = \log_a x + \log_a y.$$

□

3. Natural Logarithms and the Exponential Function

We outline here the definitions and main results, omitting the proofs which involve the theory of (Riemann) integration.

Recall that for $n \in \mathbb{Z}$, the function

$$f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto x^n$$

is differentiable with Theorem 2.2

$$f'(x) = nx^{n-1}.$$

Using elementary properties and techniques of integration that for all $u, v \in \mathbb{R}^+$ and $w \in \mathbb{R}$,

$$\begin{aligned} L(1) &= 0\\ L(uv) &= L(u) + L(v)\\ L(u^w) &= wL(u)\\ L(\frac{1}{u}) & \ -L(u), \end{aligned}$$

which the reader is invited to compare with

Looked at another way, f is the derivative of

$$F\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x^{n+1}}{n+1},$$

as long as $n \neq -1$.

However, since

$$f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad a \longmapsto \frac{1}{x},$$

is a continuous function, by the Fundamental Theorem of Calculus

$$L\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \int_1^x \frac{1}{t} dt$$

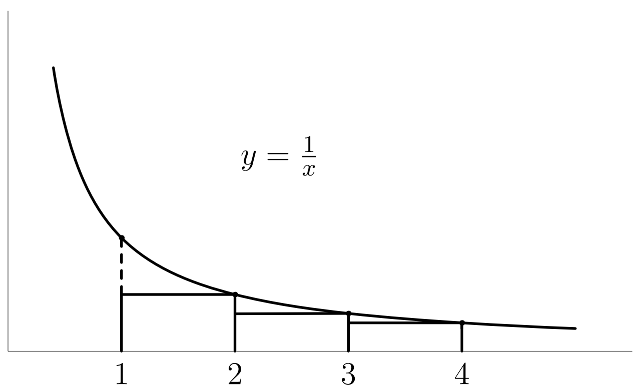
is a differentiable function with

$$L'(x) = \frac{1}{x} = f(x).$$

Since $L'(x) > 0$ for all x, L is monotonically strictly increasing, and hence injective.

Moreover, as indicated in the next diagram, if $n > 1$ is an integer,

$$\int_1^n \frac{1}{t} dt \ > \ \sum_{j=2}^{n} \frac{1}{j}.$$



Since $\sum \frac{1}{n}$ diverges, $L(x) \to \infty$ as $x \to \infty$. Since $L(\frac{1}{x}) = -L(x)$, $L(x) \to -\infty$ as $x \to 0$. Thus, $\mathrm{im}(L) = \mathbb{R}$.

Because L is bijective, it has an inverse,

$$E : \mathbb{R} \longrightarrow \mathbb{R}^+, \quad u \longmapsto x,$$

where $u = L(x)$.

It follows that for all $u, v, w \in \mathbb{R}$,

$$\begin{aligned} E(1) &= 0 \\ E(u+v) &= E(u)E(v) \\ E(uv) & \ (E(u))^v, \end{aligned} \tag{11}$$

which the reader is invited to compare with Theorem 1.8.

This motivates the next definition

Definition 3.1.

$$e := E(1)$$

Equivalently e is the (uniquely determined) real number such that

$$L(e) = \int_1^e \frac{1}{t} dt = 1$$

It then follows from Equation (11) and Theorem 1.8 that

$$E(x) = e^x,$$

and this allows us to define the *natural logarithm*.

Definition 3.2. The *natural logarithm* is the function

$$\ln\colon \mathbb{R}^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \ln x := \log_e x$$

and the *exponential function*

$$\exp\colon \mathbb{R} \longrightarrow \mathbb{R}^+, \quad x \longmapsto e^x$$

Since $L'(x) = \frac{1}{x} \neq 0$ for all $x \in \mathbb{R}^+$, E is differentiable and

$$E'(u) = \frac{1}{L'(E(u))} = \frac{1}{\frac{1}{E(u)}} = E(u).$$

Thus, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}e^x = e^x$$

Remark 3.3. The exponential function is one of the most important functions in mathematics.

One reason is that there is a generalisation of it to the theory of *Lie groups* and *Lie algebras*, which is at the heart of so much theoretical physics, particularly to quantum theory.

Another reason is that the exponential function in the case of functions of a complex variable encompassed the trigonometric functions, as we indicate.

Recall that each non-zero complex number $z = x + iy$ can be written in the form $r\cos\theta + i\sin\theta$, which is unique if we require that $0 \leq \theta < 2\pi$.

Thus $z = r\cos\theta + ir\sin\theta$. If we fix the modulus of z and treat i as just a constant, we may regard z as a function of θ and then

$$\frac{dz}{d\theta} = -r\sin\theta + ir\cos\theta = i(r\cos\theta + ir\sin\theta) = iz.$$

whence, from the theory of integration, we must have

$$z = Are^{i\theta}.$$

Using the fact that $z = r(\neq 0)$ when $\theta = 0$, we see that $A = 1$ and we have arrived at the *Argand form* of the complex number zm viz. $z = re^{i\theta}$. In particular, we obtain Euler's Formula

$$e^{i\pi} = -1.$$

We also obtain an "obvious" reason why

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

for this is just the equality

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

[The above is a heuristic argument. Formally rigorous justification is provided in MATH102 and in the study of functions of a complex variable.]