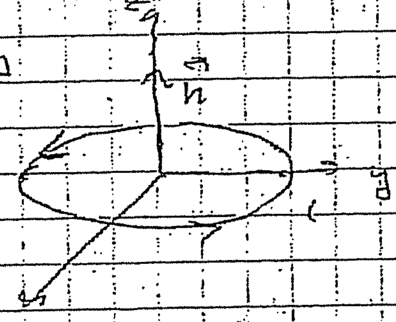
Tut 12

Stokes Th. $\displaystyle\int_C \vec{F}\,d\vec{r} = \iint_\sigma (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$

① $\vec{F}(x,y,z) = z^2\vec{i} + 2x\vec{j} - y^3\vec{k}$, C is the circle $x^2+y^2=1$ in the xy-plane with counterclockwise orientation looking down the positive z-axis



$$\int_C \vec{F}\,d\vec{r} = \iint_\sigma (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2x & -y^3 \end{vmatrix} =$$

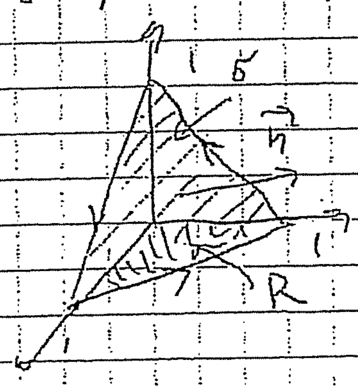
$$= (-3y^2-0)\vec{i} + (2z-0)\vec{j} + (2-0)\vec{k} = -3y^2\vec{i} + 2z\vec{j} + 2\vec{k}$$

$$\vec{n} = \langle 0, 0, 1 \rangle$$

$$\text{curl}\,\vec{F}\cdot\vec{n} = -3y^2\times 0 + 2z\times 0 + 2\times 1 = 2$$

$$\iint_\sigma 2\,ds = 2\iint_\sigma ds = 2\times\text{area of the circle} = 2\pi$$

② $\vec{F}(x,y,z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$; C is the triangle in the plane $x+y+z=1$ with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ with counterclockwise orientation looking from the first octant toward the origin



$$\int_C \vec{F}\cdot d\vec{r} = \iint_\sigma (\text{curl}\,\vec{F})\cdot\vec{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} =$$

$= -y\vec{i} - z\vec{j} - x\vec{k}$

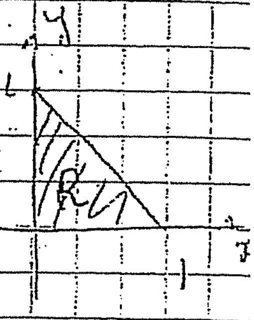
$G(x,y,z) = x+y+z-1=0$

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|} = \left\langle \frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3} \right\rangle$$

it is the correct direction.

$$(\operatorname{curl}\vec{F})\cdot\vec{n} = -\frac{y}{\sqrt3} - \frac{z}{\sqrt3} - \frac{x}{\sqrt3}$$

$$\iint_\sigma -\frac{y}{\sqrt3} - \frac{z}{\sqrt3} - \frac{x}{\sqrt3}\, dS = -\frac{1}{\sqrt3}\iint_R \left(x+y+(1-x-y)\right)\sqrt{z_x^2+z_y^2+1}\, dA$$

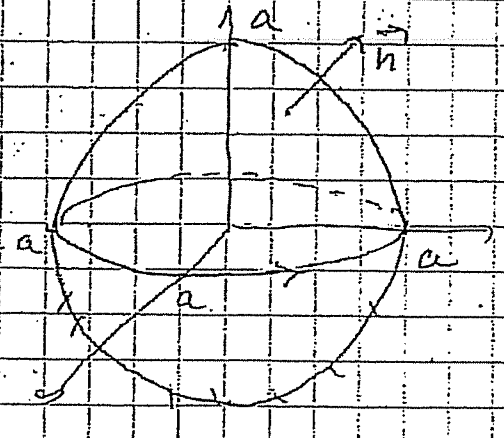


$$= -\frac{1}{\sqrt3}\iint_R 1\times\sqrt3\, dA = -\text{area } R = -\frac12$$

Verify the Stokes' Th by evaluating the line integral and the double integral.

① $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$, σ is the upper hemisphere $z=\sqrt{a^2-x^2-y^2}$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_\sigma (\operatorname{curl}\vec{F})\cdot\vec{n}\, dS$$



1). line integral

C: $x = a\cos\theta$ $\quad 0\le\theta\le 2\pi$
$y = a\sin\theta$
$z = 0$

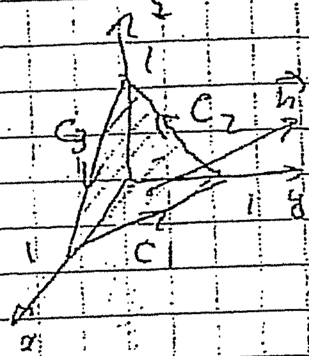
$$\int_C x\,dx + y\,dy + z\,dz = \int_0^{2\pi}\left(a\cos\theta(a\cos\theta)' + a\sin\theta(a\sin\theta)' + 0\right)d\theta$$

$$= \int_0^{2\pi}\left(-a^2\cos\theta\sin\theta + a^2\sin\theta\cos\theta\right)d\theta = 0$$

2). double integral

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \Rightarrow (\operatorname{curl}\vec{F})\cdot\vec{n} = 0 \Rightarrow \iint_\sigma (\operatorname{curl}\vec{F})\cdot\vec{n}\, dS = 0$$

(2) $\vec{F}(x,y,z) = (x-y)\vec{i} + (y-z)\vec{j} + (z-x)\vec{k}$, σ: the portion of the plane $x+y+z=1$ in the first octant



1) line integral

$$\int_C (x-y)\,dx + (y-z)\,dy + (z-x)\,dz =$$

$$= \int_{C_1} \dots + \int_{C_2} \dots + \int_{C_3} \dots, \text{ where}$$

C_1: $x = 1-t$, $y = t$, $z = 0$

C_2: $x = 0$, $y = 1-t$, $z = 0$ $\quad 0 \le t \le 1$

C_3: $x = t$, $y = 0$, $z = 1-t$

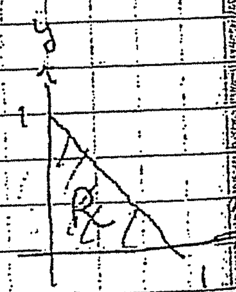
$$= \int_0^1 (1-t-t)(-1) + (t-0)\cdot 1 + 0 + 0 + (1-t-t)(-1) + t\cdot 1\, t + t\cdot 1 + 0 + (1-t-t)(-1)\, dt = \int_0^1 (9t-3)\,dt = \frac{9}{2} - 3 = \frac{3}{2}$$

2) double integral:

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$(\text{curl } \vec{F}) \cdot \vec{n} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$



$$\iint_\sigma \frac{3}{\sqrt{3}}\,dS = \iint_R \frac{3}{\sqrt{3}} \cdot \sqrt{z_x^2 + z_y^2 + 1}\,dA =$$

$$= \iint_R \frac{3}{\sqrt{3}} \sqrt{1+1+1}\,dA = 3 \iint_R dA = 3 \times \text{area of } R = \frac{3}{2}$$

Show that $\text{curl}(\nabla F) = 0$

$$F = F(x, y, z) : \mathbb{R}^3 \supset R \mapsto \mathbb{R}$$

$$\nabla F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}$$

$$\text{curl}(\nabla F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} =$$

$$= \left(\frac{\partial^2 F}{\partial y \partial z} - \frac{\partial^2 F}{\partial y \partial z}\right)\vec{i} + \left(\frac{\partial^2 F}{\partial x \partial z} - \frac{\partial^2 F}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial x \partial y}\right)\vec{k} =$$

$$= 0$$