Sample Solutions for Tutorial 1

Question 1.

We use *elimination of variables.*

(i)	$x + 7y + 4z = 21$	
(ii)	$3x - 6y + 5z = 2$	
(iii)	$5x + y - 3z = 14$	
(iv)	$-27y - 7z = -61$	[(ii) - 3(i)]
(v)	$-34y - 23z = -91$	[(iii) - 5(i)]
(vi)	$7y + 16z = 30$	[(iv)-(v)]
(vii)	$x - 12z = 9$	[(i)-(vi)]
(viii)	$y + 57z = 59$	[(v) + 5(vi)]
(ix)	$-383z = -383$	[(vi) - 7(viii)]
(x)	$z = 1$	[from (ix)]
(xi)	$y = 59 - 57z = 2$	[from (vii) and (x)]
(xii)	$x = -9 + 12z = 3$	[from (vii) and (x)]

Thus the (unique!) solution is $(x, y, z) = (1, 2, 3)$.

Comment. This solution is not particularly elegant, nor is it insightful — it provides no insight into why there are any solutions at all, or why there are not several solutions, beyond the fact that there was one and only one solution.

It is, however, the approach most likely to be tried by a "naïve" mathematician, who relies on direct computation. You should note that, even in a relatively simple example, you could need many computations, and you certainly need to keep track of what you have done.

By the end of this course on linear algebra, you should have an appreciation of the algorithm involved in this solution, its advantages and its limitations.

Question 2.

Take $a \in \mathbb{R}$.

(i) Take $\underline{\mathbf{A}} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$.

Then

$$\underline{\mathbf{A}}^2 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a.a + a.0 & a.1 + 1.a \\ 0.a + a.0 & 0.1 + a.a \end{bmatrix} = \begin{bmatrix} a^2 & a \\ 0 & a^2 \end{bmatrix},$$

$$\underline{\mathbf{A}}^3 = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^2 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} a^2 & a \\ 0 & a^2 \end{bmatrix} = \begin{bmatrix} a.a^2 + 1.0 & a.a + 1.a^2 \\ 0.a^2 + a.0 & 0.a + a.a^2 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$$

and, similarly,

$$\underline{\mathbf{A}}^4 = \begin{bmatrix} a^4 & 3a^3 \\ 0 & a^4 \end{bmatrix}$$

This invites the conjecture that for all $n \in \mathbb{N}$,

$$\underline{\mathbf{A}}^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}.$$

We use the Principle of Mathematical Induction to prove that our conjecture is true. We adopt the *notational* convention that $\underline{\mathbf{A}}^0 = \underline{\mathbf{1}}_k$ for any $k \times k$ matrix.

As the conjecture is true for $n = 0, 1, 2, 3$, it is sufficient to complete the "inductive step". So assume that for some $n \in \mathbb{N}$ we have

$$\underline{\mathbf{A}}^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}.$$

Then

$$\begin{aligned}\underline{\mathbf{A}}^{n+1} = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^{n} &= \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix} \\ &= \begin{bmatrix} a.a^n + 1.0 & a.na^{n-1} + 1.a^n \\ 0.a^n + a.0 & 0.na^{n-1} + a.a^n \end{bmatrix} \\ &= \begin{bmatrix} a^{n+1} & (n+1)a^n \\ 0 & a^{n+1} \end{bmatrix},\end{aligned}$$

which completes the proof by induction.

(ii) Take $\underline{\mathbf{A}} = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$. Then

$$\begin{aligned}\underline{\mathbf{A}}^2 &= \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2a & 1 \\ 0 & a^2 & 2a \\ 0 & 0 & a^2 \end{bmatrix}, \\ \underline{\mathbf{A}}^3 = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^2 &= \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} a^2 & 2a & 1 \\ 0 & a^2 & 2a \\ 0 & 0 & a^2 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 & 3a \\ 0 & a^3 & 3a^2 \\ 0 & 0 & a^3 \end{bmatrix}\end{aligned}$$

and, similarly,

$$\underline{\mathbf{A}}^4 = \begin{bmatrix} a^4 & 4a^3 & 6a^2 \\ 0 & a^4 & 4a^3 \\ 0 & 0 & a^4 \end{bmatrix}$$

This invites the conjecture that for all $n \in \mathbb{N}$,

$$\underline{\mathbf{A}}^n = \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & a^n & na^{n-1} \\ 0 & 0 & a^n \end{bmatrix}.$$

We use the Principle of Mathematical Induction to prove that our conjecture is true. We retain the convention that $\underline{\mathbf{A}}^0 = \underline{\mathbf{1}}_k$ for any $k \times k$ matrix.

As the conjecture is true for $n = 0, 1, 2, 3$, it is sufficient to complete the "inductive step". So assume that for some $n \in \mathbb{N}$ we have

$$\underline{\mathbf{A}}^n = \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & a^n & na^{n-1} \\ 0 & 0 & a^n \end{bmatrix}.$$

Then

$$\begin{aligned}\underline{\mathbf{A}}^{n+1} = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^{n} &= \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & a^n & na^{n-1} \\ 0 & 0 & a^n \end{bmatrix} \\ &= \begin{bmatrix} a.a^n + 1.0 + 0.0 & a.na^{n-1} + 1.a^n + 0.0 & a.\frac{n(n-1)}{2}a^{n-2} + 1.na^{n-1} + 0.a^n \\ 0.a^n + a.0 + 1.0 & 0.na^{n-1} + a.a^n + 1.0 & 0.\frac{n(n-1)}{2}a^{n-2} + a.na^{n-1} + 1.a^n \\ 0.a^n + 0.0 + a.0 & 0.na^{n-1} + 0.a^n + a.0 & 0.\frac{n(n-1)}{2}a^{n-2} + 0.na^{n-1} + a.a^n \end{bmatrix} \\ &= \begin{bmatrix} a^{n+1} & (n+1)a^n & \frac{(n+1)n}{2}a^{n-1} \\ 0 & a^{n+1} & (n+1)a^n \\ 0 & 0 & a^{n+1} \end{bmatrix},\end{aligned}$$

which completes the proof by induction.

(iii) Take $\underline{\mathbf{A}} = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$.

Then, as above,

$$\underline{\mathbf{A}}^2 = \begin{bmatrix} a^2 & 2a & 1 & 0 \\ 0 & a^2 & 2a & 1 \\ 0 & 0 & a^2 & 2a \\ 0 & 0 & 0 & a^2 \end{bmatrix},$$

$$\underline{\mathbf{A}}^3 = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^2 = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix} \begin{bmatrix} a^2 & 2a & 1 & 0 \\ 0 & a^2 & 2a & 1 \\ 0 & 0 & a^2 & 2a \\ 0 & 0 & 0 & a^2 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2 & 3a & 1 \\ 0 & a^3 & 3a^2 & 3a \\ 0 & 0 & a^3 & 3a^2 \\ 0 & 0 & 0 & a^3 \end{bmatrix}$$

and, similarly,

$$\underline{\mathbf{A}}^4 = \begin{bmatrix} a^4 & 4a^3 & 6a^2 & 4a \\ 0 & a^4 & 4a^3 & 6a^2 \\ 0 & 0 & a^4 & 4a^3 \\ 0 & 0 & 0 & a^4 \end{bmatrix}$$

This invites the conjecture that for all $n \in \mathbb{N}$,

$$\underline{\mathbf{A}}^n = \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} & \frac{n(n-1)(n-2)}{3.2}a^{n-3} \\ 0 & a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & 0 & a^n & na^{n-1} \\ 0 & 0 & 0 & a^n \end{bmatrix}.$$

We use the Principle of Mathematical Induction to prove that our conjecture is true.

As the conjecture is true for $n = 0, 1, 2, 3$, it is sufficient to complete the "inductive step". So assume that for some $n \in \mathbb{N}$ we have

$$\underline{\mathbf{A}}^n = \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} & \frac{n(n-1)(n-2)}{3.2}a^{n-3} \\ 0 & a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & 0 & a^n & na^{n-1} \\ 0 & 0 & 0 & a^n \end{bmatrix}.$$

Then

$$\underline{\mathbf{A}}^{n+1} = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^{n} = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix} \begin{bmatrix} a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} & \frac{n(n-1)(n-2)}{3.2}a^{n-3} \\ 0 & a^n & na^{n-1} & \frac{n(n-1)}{2}a^{n-2} \\ 0 & 0 & a^n & na^{n-1} \\ 0 & 0 & 0 & a^n \end{bmatrix}$$

$$= \begin{bmatrix} a^{n+1} & (n+1)a^n & (\frac{n(n-1)}{2}+n)a^n & (\frac{n(n-1)(n-2)}{3.2}+\frac{n(n-1)}{2})a^{n-2} \\ 0 & a^{n+1} & (n+1)a^n & (\frac{n(n-1)}{2}+n)a^{n-1} \\ 0 & 0 & a^{n+1} & (n+1)a^n \\ 0 & 0 & 0 & a^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} a^{n+1} & (n+1)a^n & \frac{(n+1)n}{2}a^{n-1} & \frac{(n+1)n(n-1)}{3.2}a^{n-2} \\ 0 & a^{n+1} & (n+1)a^n & \frac{(n+1)n}{2}a^{n-1} \\ 0 & 0 & a^{n+1} & (n+1)a^n \\ 0 & 0 & 0 & a^{n+1} \end{bmatrix},$$

which completes the proof by induction.

COMMENT. You might have observed an emerging pattern:

$$1 = \binom{n}{0}, \quad n = \binom{n}{1}, \quad \frac{n(n-1)}{2} = \binom{n}{2}, \quad \frac{n(n-1)(n-2)}{3.2} = \binom{n}{3}.$$

Thus, in Part (iii), we have

$$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & \binom{n}{1}a^{n-1} & \binom{n}{2}a^{n-2} & \binom{n}{3}a^{n-3} \\ 0 & a^n & \binom{n}{1}a^{n-1} & \binom{n}{2}a^{n-2} \\ 0 & 0 & a^n & \binom{n}{1}a^{n-1} \\ 0 & 0 & 0 & a^n \end{bmatrix}$$

Question 3.

Take $\underline{\mathbf{A}} := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Then

$$\underline{\mathbf{A}}^2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) & -\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) + \sin(\theta)\sin(\theta) & -\sin(\theta)\sin(\theta) + \cos(\theta)\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

$$\underline{\mathbf{A}}^3 = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta) & -\cos(\theta)\sin(2\theta) - \sin(\theta)\cos(2\theta) \\ \sin(\theta)\cos(2\theta) + \sin(\theta)\sin(2\theta) & -\sin(\theta)\sin(2\theta) + \cos(\theta)\cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(3\theta) & -\sin(3\theta) \\ \sin(3\theta) & \cos(3\theta) \end{bmatrix}$$

This suggests the conjecture that for all $n \in \mathbb{N}$,

$$\underline{\mathbf{A}}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

We use the Principle of Mathematical Induction to prove that our conjecture is true.

As the conjecture is true for $n = 0, 1, 2$, it is sufficient to complete the "inductive step". So assume that for some $n \in \mathbb{N}$ we have

$$\underline{\mathbf{A}}^n == \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}.$$

Then

$$\begin{aligned} \underline{\mathbf{A}}^{n+1} = \underline{\mathbf{A}}\,\underline{\mathbf{A}}^n &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta) & -\cos(\theta)\sin(n\theta) - \sin(\theta)\cos(n\theta) \\ \sin(\theta)\cos(n\theta) + \sin(\theta)\sin(n\theta) & -\sin(\theta)\sin(n\theta) + \cos(\theta)\cos(n\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos((n+1)\theta) & -\sin((n+1)\theta) \\ \sin((n+1)\theta) & \cos((n+1)\theta) \end{bmatrix}, \end{aligned}$$

completing the proof.

Question 4.

Take $f, g : \mathbb{R} \to \mathbb{R}$ with $\frac{d^2f}{dx^2} - 4\frac{df}{dx} + 3f = \frac{d^2g}{dx^2} - 4\frac{dg}{dx} + 3g = 0$, and $\lambda, \mu \in \mathbb{R}$. Define

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \lambda f(x) + \mu g(x),$$

so that we may think of h as $\lambda f + \mu g$. Then

$$\begin{aligned} \frac{d^2h}{dx^2} - 4\frac{dh}{dx} + 3h &= \frac{d^2}{dx^2}(\lambda f + \mu g) - 4\frac{d}{dx}(\lambda f + \mu g) + 3(\lambda f + \mu g) \\ &= \lambda\frac{d^2f}{dx^2} + \mu\frac{d^2g}{dx^2} - 4(\lambda\frac{df}{dx} + \mu\frac{dg}{dx}) + 3(\lambda f + \mu g) \\ &\qquad\qquad \text{by properties of differentiation} \\ &= \lambda(\frac{d^2f}{dx^2} - 4\frac{df}{dx} + 3f) + \mu(\frac{d^2g}{dx^2} - 4\frac{dg}{dx} + 3g) \\ &= \lambda 0 + \mu 0 \qquad \text{by hypothesis} \\ &= 0. \end{aligned}$$