Chapter 7

More on Derivatives

We apply the theory from our earlier to draw graphs of functions and differentiate several important functions, refining the theory further as needed.

7.1 Graphing Functions

Given a function $f\colon \mathbb{R} \to \mathbb{R}$ it is important to know its image (range), whether it is injective (1–1), monotonic, where it has extrema, etc. Determining these features is equivalent to drawing the graph of f.

Differentiation provides a powerful tool for this task, as out theorems have already shown.

We illustrate this by returning to an earlier problem, graphing the function $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$.

The reader will be aware of the graph of this function. The problem can be solved without the use of calculus. A careful examination of the arguments used without resorting to differentiation reveals an unsatisfactory state of affairs:

(i) The geometric definition of a plane parabola is that it is the locus of all points in the plane whose perpendicular distance from a fixed line in the plane– the directrix of the parabola — is equal to its distance from a fixed point in the plane — the focus of the parabola.

(ii) Following Descartes, we can introduce a rectangular co-ordinate system, which allows us to represent each point in the plane uniquely by an ordered pair of real numbers, (x, y). In this system, a line comprises all points whose co-ordinates (x, y) satisfy an equation of the form $ax + by + c = 0$, with $a \neq$ or $b \neq 0$.

(iii) By choosing the Y-axis to be the line perpendicular to the directrix through the focus, and the X-axis to be the perpendicular bisector of the segment of the Y-axis between the focus and the directrix, the focus acquires the co-ordinates $(0, a)$ and the directrix has equation $y = a$ $(a > 0)$.

(iv) Calculation now shows that the point with co-ordinates (x, y) lies on the parabola if and only if $x^2 = 4ay$.

(v) We recognise the graph of $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ to be the parabola corresponding to $a = \frac{1}{4}$.

Such a procedure is clearly unsatisfactory, depending, as it does, on extraneous information, and relying on being able to recognise a function as having a graph with well-known geometric description. A function as simple as $g\colon \mathbb{R} \to \mathbb{R},\ \ x \mapsto x^3$ highlights the problem.

We revisit the study of the function $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ to indicate the specific problems arising and show how the theory we have developed solves them.

Recall that the arithmetic operations were originally defined only for the natural numbers, then extended to the integer, then to the rational numbers, and then to the real numbers. These gave rise, successively, to functions

$$\begin{aligned} f_{\mathbb{N}}\colon \mathbb{N} &\longrightarrow \mathbb{N}, \quad x \longmapsto x^2 \\ f_{\mathbb{Z}}\colon \mathbb{Z} &\longrightarrow \mathbb{Z}, \quad x \longmapsto x^2 \\ f_{\mathbb{Q}}\colon \mathbb{Q} &\longrightarrow \mathbb{Q}, \quad x \longmapsto x^2 \\ f_{\mathbb{R}}\colon \mathbb{R} &\longrightarrow \mathbb{R}, \quad x \longmapsto x^2 \end{aligned}$$

Of these functions only $f_{\mathbb{N}}$ is injective.

In each case, only positive numbers lie in the image, so that

$$\mathrm{im}(f_{\mathbb{N}}) \subseteq \mathbb{N}, \quad \mathrm{im}(f_{\mathbb{Z}}) \subseteq \mathbb{Z}_0^+ = \mathbb{N}, \quad \mathrm{im}(f_{\mathbb{Q}}) \subseteq \mathbb{Q}_0^+ \quad \text{and} \quad \mathrm{im}(f_{\mathbb{R}}) \subseteq \mathbb{R}_0^+$$

When we proved that there is no rational number, q, with $q^2 = 2$, we actually proved that the first three are proper inclusions:

$$\mathrm{im}(f_{\mathbb{N}}) \subset \mathbb{N}, \quad \mathrm{im}(f_{\mathbb{Z}}) \subset \mathbb{Z}_0^+, \quad \text{and} \quad \mathrm{im}(f_{\mathbb{Q}}) \subset \mathbb{Q}_0^+.$$

The argument using Euclidean geometry and Cartesian co-ordinate systems, shows that

$$\mathrm{im}(f_{\mathbb{R}}) = \mathbb{R}_0^+$$

We now apply the theory we have developed to prove this, using only properties of the function, without resorting to Euclidean geometry or Cartesian co-ordinates.

The function $f_{\mathbb{R}}\colon \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto x^2$ can be expressed as the product of the function

$$id_{\mathbb{R}}\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x$$

with itself.

Example 6.3 showed that $id_{\mathbb{R}}$ is differentiable. By Theorem 6.8, $f_{\mathbb{R}}$ is also differentiable, and so, by Theorem 6.7, it is continuous.

Since the domain of $f_{\mathbb{R}}$ is an interval of real numbers, the Intermediate Value Theorem now shows im($f_{\mathbb{R}}$ is also an interval of real numbers.

By Theorem 2.22, $\mathrm{im}(f_{\mathbb{R}}) \subseteq \mathbb{R}_0^+$.

Since $f_{\mathbb{R}}(0) = 0$, $0 \in \mathrm{im}(f_{\mathbb{R}})$

Since $n^2 \geq n$ for every $n \in \mathbb{N}$, $\mathrm{im}(f_{\mathbb{R}})$ is not bounded above.

Hence $\mathrm{im}(f_{\mathbb{R}}) = \{y \in \mathbb{R} \mid y \geq 0\} = \mathbb{R}_0^+$.

In particular, $f_{\mathbb{R}}$ has an absolute minimum at $x = 0$.

The question of other extrema (relative or absolute) arises.

By Theorem 6.16, extrema of a function can only occur at boundary points of its domain, or at points where the function is not differentiable, or at points where the derivative is 0.

Since the domain of $f_{\mathbb{R}} = \mathbb{R}$, it contains no boundary points.

Since $f'_{\mathbb{R}}(x) = 2x$, $f_{\mathbb{R}}$ is differentiable everywhere, and its derivative is 0 at, and only at, 0. Thus it has no extrema other than the absolute minimum at 0.

Moreover, since $f'_{\mathbb{R}}(x) < 0$ for $x < 0$ and $f'_{\mathbb{R}}(x) > 0$ for $x > 0$, $f_{\mathbb{R}}$ is decreasing when $x < 0$ and increasing when $x > 0$.

This leaves open several properties for the graph of $f_{\mathbb{R}}$, such as

We can refine our theory and techniques to decide the matter.

The derivative, f', of the function f provides information on whether f is increasing, decreasing or stationary.

Similarly, the derivative of f' provides information on whether f' is increasing, decreasing or stationary. This, in turn, provides information on the shape of the graph of f.

If the derivative of f' is positive, then f' is increasing. So if f is increasing, its rate of increase is increasing — the graph is rising more steeply, as in the second diagram above. If, on the other hand, he derivative of f' is negative and if f is increasing, then the graph is rising less steeply, as in the right-hand extreme of the first diagram. In other words, the second derivative provides information on how the graph is curved.

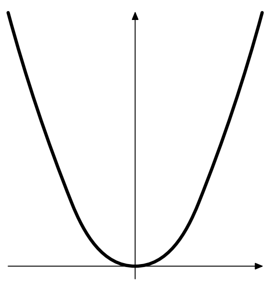
Definition 7.1. The *second derivative*, f'' of the function $f\colon \mathbb{R} \to \mathbb{R}$ is the derivative of the derivative of f. It is also denoted $\dfrac{d^2 f}{dx^2}$.

Definition 7.2. The function f is *concave up* whenever its derivative is increasing and it is *concave down* whenever its derivative is decreasing.

Definition 7.3. The function f has a *point of inflection* if the concavity of f changes at a.

Geometrically, the function f is concave up (resp. down) whenever its graph is "cupped up" (resp. "cupped down").

In the case of $f_{\mathbb{R}}$, we have $f''_{\mathbb{R}}(x) = 2$ for all $x \in \mathbb{R}$, whence the graph of $f_{\mathbb{R}}$ is becoming flatter as whenever$f_{\mathbb{R}}$ is increasing, and steeper as x increases whenever $f_{\mathbb{R}}$ is increasing. This allows us to sketch the graph of $f_{\mathbb{R}}$.



Graph of $y = x^2$

We went into such detail to sketch this graph as it illustrates several important ideas and techniques clearly in a simple setting. We extend these to more general situations.

The first extension concerns the second derivative.

Lemma 7.4 (Second Derivative Test). *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $f'(a) = 0$.*

(i) If $f''(a) < 0$, then f has a relative maximum at a.

(ii) If $f''(a) > 0$, then f has a relative maximum at a.

(iii) If $f''(a) = 0$, no conclusion can be drawn.

Proof. Let f be differentiable twice at a.
(i) Suppose that $f''(a) < 0$.
Then $f'(x)$ is strictly increasing on an interval containing a, say $]a-h, a+h[$.
Since $f'(a) = 0$, $f'(x) > 0$ for $a - h < x < a$ and $f'(x) < 0$ for $a < x < a + h$.
Hence, f is increasing on $]a-h, a[$ and decreasing on$]a, a+h[$.
Thus, f has a relative maximum at a.
(ii) The case of $f''(a) > 0$ is similar and left to the reader.
(iii) The functions $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$, $g\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^4$ and $h\colon \mathbb{R} \to \mathbb{R},\ x \mapsto -x^6$ all have first and second derivative 0 at 0.

The first has neither a maximum nor a minimum at 0, the second has a minimum at 0 and he third has a maximum at 0. □

Next we generalise the determination of $\operatorname{im}(f_{\mathbb{R}})$ to arbitrary polynomial functions.

Lemma 7.5. *Consider $p\colon \mathbb{R} \to \mathbb{R},\quad x \mapsto c_0 + c_1x + \ldots + c_nx^n$ with $n > 0$ and $c_n \neq 0$.*
If n is odd then p is surjective, so that $\operatorname{im}(p) = \mathbb{R}$.
If n is even, then $\operatorname{im}(p)$ is of the form $[a, \infty[$ when $c_n > 0$ and $]-\infty, a]$ when $c_n < 0$.

Proof. We consider the case $c_n > 0$, and leave the case $c_n < 0$ to the reader. Since p is a polynomial function, it is continuous.

By the Intermediate Value Theorem, $\operatorname{im}(p)$ is an interval of real numbers.

We consider the behaviour of $p(x)$ as $x \to \pm\infty$. If $n \neq 0$,

$$p(x) = x^n\left(\tfrac{c_0}{x^n} + \tfrac{c_1}{x^{n-1}} + \ldots + \tfrac{c_{n-1}}{x} + c_n\right)$$

Now $\displaystyle\lim_{x\to\pm\infty} \frac{c_j}{x^{n-j}} = 0$ for $0 \leq j < n$, whence

$$\lim_{x\to\pm\infty}\left(\tfrac{c_0}{x^n} + \tfrac{c_1}{x^{n-1}} + \ldots + \tfrac{c_{n-1}}{x} + c_n\right) = c_n.$$

Since $c_n \neq 0$, there is a $K \in \mathbb{R}$ such that if $|x| \geq K$,

$$\frac{c_n}{2} < \tfrac{c_0}{x^n} + \tfrac{c_1}{x^{n-1}} + \ldots + \tfrac{c_{n-1}}{x} + c_n < \frac{3c_n}{2},$$

and then

$$0 < \frac{c_n}{2}x^n < p(x) < \frac{3c_n}{2}x^n \qquad (*)$$

whenever $|x| \geq K$.

If n is odd, then $\frac{3c_n}{2}x^n \to -\infty$ as $x \to -\infty$ and $\frac{c_n}{2}x^n \to \infty$ as $x \to \infty$, whence, by the comparison test, $p(x) \to -\infty$ as $x \to -\infty$ and $p(x) \to \infty$ as $x \to \infty$.

Since $\mathrm{im}(p)$ is an interval of real numbers which is neither bounded above nor bounded below, we must have $\mathrm{im}(p) = \mathbb{R}$.

If n is even, then $\frac{c_n}{2}x^n \to \infty$ as $x \to \pm\infty$, whence, by the comparison test, $p(x) \to \pm\infty$ as $x \to \pm\infty$, showing that $\mathrm{im}(p)$ is not bounded above.

Since $p(x)$ is a polynomial function, it is differentiable everywhere, so its relative extrema occur at zeros of its derivative. Since it derivative is a polynomial function degree $n-1$, there are at most $n-1$ relative extrema. Let m be the minimum and M the maximum of $\{x \in \mathbb{R} \mid p'(x) = 0\}$.

Put $P := \min\{-K, m\}$ and $Q := \max\{K, M\}$.

Now consider p on the interval $[P, Q]$. By the Extreme Value Theorem, p has a minimum, say a, on $[P, Q]$. Moreover, $p(x)$ has no extrema on $\mathbb{R} \setminus [P, Q]$.

Thus, $\mathrm{im}(p) = [a, \infty[$. □

Example 7.6. We graph the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto 3x^4 - 4x^3 - 12x^2 + 7$

$f(x) \to \infty$ as $x \to \pm\infty$.

The domain of f is $\mathbb{R}$, which contains no boundary points.

Since f is a polynomial function, it is differentiable everywhere.

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2) = 0$ if and only if $x \in \{-1, 0, 2\}$.

$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$.

Then $f''(-1) = 36, f''(0) = -24, f''(2) = 72$

Thus f has relative minima at -1 and 2. It has a relative maximum at 0.

$f(-1) = 2, f(0) = 7, f(2) = -29$.

Hence f has an absolute minimum at 2.

$f'(x) < 0$ when $x < -1$ or $0 < x < 2$, and $f'(x) > 0$ when $-1 < x$ or $x > 2$.

Hence f is decreasing on $]-\infty, -1]$ and on $[0, 2]$. It is increasing on $[-1, 0]$ and on $[2, \infty[$.

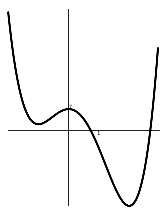
$f''(x) = 0$ if and only if $x = \frac{1 \pm \sqrt{7}}{3}$ and

$f''(x) > 0$ on $]-\infty, \frac{1-\sqrt{7}}{3}[$ and on $[\frac{1+\sqrt{7}}{3}, \infty[$.

$f''(x) < 0$ on $]\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}[$.

Hence f is concave on $]-\infty, \frac{1-\sqrt{7}}{3}[$ and on $[\frac{1+\sqrt{7}}{3}, \infty[$ and concave down on $]\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}[$.

W can now sketch the graph of f



7.2 Trigonometric Functions

The geometric definition of the sine, cosine, tangent, cosecant, secant and cotangent of an angle only applies to angles strictly between 0 and $\frac{\pi}{2}$ radians.

We have seen how to extend the definition of the first two to functions $\mathbb{R} \to \mathbb{R}$.

When we seek to do the same for the other four trigonometric functions, we strike a problem apparent from their definitions,

$$\tan\theta := \frac{\sin\theta}{\cos\theta} \qquad \sec\theta := \frac{1}{\cos\theta}$$

$$\cot\theta := \frac{\cos\theta}{\sin\theta} \qquad \csc\theta := \frac{1}{\sin\theta}$$

As they stand, the first two are only defined when $\cos\theta \neq 0$, and the last two when $\sin\theta \neq 0$.

This leaves us with two main options. The first is to define the trigonometric functions only where the above apply. This yields the four functions, *tangent*, *secant*, *cotangent* and *cosecant*, defined respectively by

$$\tan\colon \mathbb{R}\setminus\{(2n+1)\tfrac{\pi}{2} \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\sin x}{\cos x}$$

$$\sec\colon \mathbb{R}\setminus\{(2n+1)\tfrac{\pi}{2} \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{\cos x}$$

$$\cot\colon \mathbb{R}\setminus\{n\pi \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\cos x}{\sin x}$$

$$\csc\colon \mathbb{R}\setminus\{n\pi \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{\sin x}$$

The alternative is to seek to extend these definitions to all of $\mathbb{R}$ by specifying the values on the sets $\{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}$ and $\{n\pi \mid n \in \mathbb{Z}\}$ respectively.

While this is always possible, it serves little purpose, unless it leads to functions with useful properties. Since we are concerned with the differentiability of functions, any extension to all of $\mathbb{R}$ should result, if possible, in differentiable functions.

Now $\sin x \to \pm 1$ as $x \to (2n+1)\frac{\pi}{2}$, showing that $\tan x$, $\sec x \to \pm\infty$ as $x \to (2n+1)\frac{\pi}{2}$, and $\cos x \to \pm 1$ as $x \to n\pi$, showing that $\cot x$, $\csc x \to \pm\infty$ as $x \to n\pi$. This renders it impossible to extend the above definitions even to continuous functions $\mathbb{R} \to \mathbb{R}$.

Consequently, we use the restricted domains above for the trigonometric functions. It follows from the theory we've developed that each of the trigonometric functions is differentiable.

Lemma 7.7. *The trigonometric functions are differentiable, and*

(i) $\dfrac{d}{dx}(\tan x) = \sec^2 x = 1 + \tan^2 x$

(ii) $\dfrac{d}{dx}(\sec x) = \tan x \sec x$

(iii) $\dfrac{d}{dx}(\cot x) = -\csc^2 x = -1 - \cot^2 x$

(iv) $\frac{d}{dx}(\csc x) = -\cot x \sec x$

Proof.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{(\frac{d}{dx}\sin x)\cos x - \sin x \frac{d}{dx}\cos x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}
\end{aligned}$$

The other calculations follow a similar pattern and are left to the reader as exercises. □

We defer graphing the trigonometric functions until we have introduced the hyperbolic functions.

7.3 Odd and Even Functions

Odd and even functions have convenient properties which make them easier to work with. We explore some of these, and also show that every function can be written uniquely as he sum of an odd function and an even function.

Definition 7.8. The subset S of $\mathbb{R}$ is *symmetric* if and only if $x \in S \iff -x \in S$.

If X is a symmetric subset of $\mathbb{R}$, then the function $f\colon X \to \mathbb{R}$ is an *even function* if and only if $f(-x) = f(x)$ for every $x \in X$, and it is an *odd function* if and only if $f(-x) = -f(x)$ for every $x \in X$.

Example 7.9. (a) The function $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^n\ \ (n \in \mathbb{N})$ is an even function if and only if n is even and it is an odd function if and only if n is odd.
(b) $\cos x$ is an even function of x and $\sin x$ is an odd function of x.

Theorem 7.10. *Let X be a symmetric subset of $\mathbb{R}$.*

(i) A function is both even and odd if and only if it is the zero function.

(ii) The sum of two even (resp. odd) functions is even (resp. odd).

(iii) The product of two functions is even whenever the functions are either both odd or both eve, and their product is odd whenever one is even and the other odd.

(iv) Every function $X \to \mathbb{R}$ can be written uniquely as the sum of an odd function and an even function.

(v) The derivative of a differentiable odd (resp. even) function is an even (resp. odd) function.

Proof. (i) f is both odd and even if and only if $f(x) = f(-x) = -f(x)$, for every $x \in X$, which is the case if and only if $2f(x) = 0$ for all $x \in X$.
(ii) Follows directly from the definitions.
(iii) Follows directly from the definitions.
(iv) Given $f\colon X \to \mathbb{R}$, define

$$f_e\colon X \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{f(x) + f(-x)}{2}$$
$$f_o\colon X \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{f(x) - f(-x)}{2}$$

Plainly, $f(x) = f_e(x) + f_o(x)$ for every $x \in X$, $f_e(-x) = f_e(x)$ and $f_o(-x) = -f_o(x)$.

Now suppose given $g, h\colon X \to \mathbb{R}$ with g and odd function, h an even function, and $f(x) = g(x) + h(x)$ for every $x \in X$.

Take $x \in X$, by the above we have

$$g(x) + h(x) = f(x) = f_e(x) + f_o(x),$$

or, equivalently,

$$g(x) - f_e(x) = h(x) - f_o(x).$$

By (ii) the left-hand function is even and the right-hand function odd.

By (i), they are both the 0 function, so that $g = f_e$ and $h = f_o$.
(v) Suppose $f\colon X \to \mathbb{R}$ is differentiable.

If f is even,

$$f(x) + f(-x) = 0,$$

or

$$f(x) + f(u) = 0,$$

where $u := -x$. Since f is differentiable, the chain rule yields

$$0 = f'(x) + f'(u)\frac{du}{dx} = f'(x) - f'(-x)$$

The case when f is odd is left to the reader.

□

Definition 7.11. Given a function f, with symmetric domain, the functions f_e and f_o introduced in the proof of Theorem 7.10 are, respectively, the *even part* and the *odd part* of f.

Observation 7.12. Symmetry properties simplify the task of drawing the graphs of functions.

The graph of an even function $f\colon \mathbb{R} \to \mathbb{R}$ exhibits mirror symmetry in the y-axis, for if f is an even function, then $(x, y) \in Gr(f)$ if and only if $(-x, y) \in Gr(f)$.

The graph of an odd function $f\colon \mathbb{R} \to \mathbb{R}$ is symmetric through $(0, 0)$, for if f is an odd function, then $(x, y) \in Gr(f)$ if and only if $(-x, -y) \in Gr(f)$.

7.4 The Hyperbolic Functions

The exponential function, $\mathbb{R} \to \mathbb{R},\ x \mapsto e^x$ is neither odd nor even. The *hyperbolic functions* are derived from the even and odd part of the exponential function. Their names are motivated by their properties which parallel those of the trigonometric functions.

Definition 7.13. The *hyperbolic functions* are cosh, *hyperbolic cosine*, defined by

$$\cosh\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{e^x + e^{-x}}{2},$$

sinh, *hyperbolic sine*, defined by

$$\sinh\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{e^x - e^{-x}}{2},$$

tanh, *hyperbolic tangent*, defined by

$$\tanh\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\sinh x}{\cosh x},$$

sech, *hyperbolic secant*, defined by

$$\operatorname{sech}\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{\cosh x},$$

csch, *hyperbolic cosecant*, defined by

$$\operatorname{csch}\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{\sinh x},$$

and coth, *hyperbolic cotangent*, defined by

$$\coth\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\cosh x}{\sinh x}.$$

The next theorem summarises some basic properties of the hyperbolic functions.

Theorem 7.14. *Take $x, y \in \mathbb{R}$.*

(i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ *and* $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(iii) $\frac{d}{dx}(\cosh x) = \sinh x$, $\frac{d}{dx}(\sinh x) = \cosh x$, $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$, $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$, $\frac{d}{dx}(\operatorname{csch} x) = -\coth x \operatorname{csch} x$ *and* $\frac{d}{dx}(\cot x) = -\operatorname{csch}^2 x$

Proof. All the results follow from the definition by direct, routine calculation requiring no ingenuity. We illustrate some, leaving the rest to the reader.

(i) $\cosh^2 x = \dfrac{e^{2x} + 2 + e^{-2x}}{4}$ and $\sinh^2 x = \dfrac{e^{2x} - 2 + e^{-2x}}{4}$, whence $\cosh^2 x - \sinh^2 x = \dfrac{4}{4}$.

(iii) These follow from the facts, shown in MATH102, that $\dfrac{d}{dx}(e^x) = e^x$ and $\dfrac{d}{dx}(e^{-x}) = -e^{-x}$. □

Observation 7.15. The reader will have noticed similarities between the names and properties of the hyperbolic functions, and those of the trigonometric function, such as in the following table.

Trigonometric Functions	Hyperbolic Functions
$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\sinh(x+y) = \sinh x \cosh x + \cosh x \sinh x$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\cosh(x+y) = \cosh x \cosh x + \sinh x \sinh x$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sinh x) = \cosh x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cosh x) = \sinh x$

A thorough explanation is provided in the study of complex valued functions of a complex variable (for example in PMTH333), where it is demonstrated that the hyperbolic and trigonometric functions are determined by the complex exponential function.

7.5 Graphs of Trigonometric and Hyperbolic Functions

7.5.1 Sine and Cosine

Since $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$ for all $x \in \mathbb{R}$, it is enough to graph $\sin x$ and $\cos x$ for $a \le x < a + 2\pi$ with $a \in \mathbb{R}$.

Since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ for all $x \in \mathbb{R}$, $\sin x$ is an odd function of x, and $\cos x$ is an even function of x.

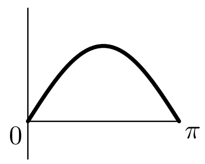
It is therefore enough to graph $\sin x$ and $\cos x$ for $x \ge 0$.

Combining these two observations, we see that it is enough to graph $\sin x$ and $\cos x$ for $x \in [0, \pi]$.

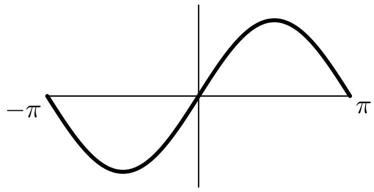
Consider $f\colon [0,\pi] \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x$.

- $f(x) \ge 0$ for all $x \in [0, \pi]$.
- The boundary points of the domain of f are 0 and π and $f(0) = f(\pi) =$. Hence 0 and π are absolute minima for f.
- Since $f'(x) = \cos x$, f is differentiable and its derivative is 0 in $[0, \pi]$ if and only if $x = \frac{\pi}{2}$.
- Since $f''(x) = -\cos x = -f(x)$, $f''(\frac{\pi}{2}) = -1$, f has a maximum at $\frac{\pi}{2}$.
- Since $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$ and $f'(x) < 0$ for $< \frac{\pi}{2} < x < \pi$, f is increasing on $[0, \frac{\pi}{2}]$ and decreasing on $[\frac{\pi}{2}, \pi]$.
- Since $f''(x) < 0$ for all $x \in [0, \pi]$, f is concave down.

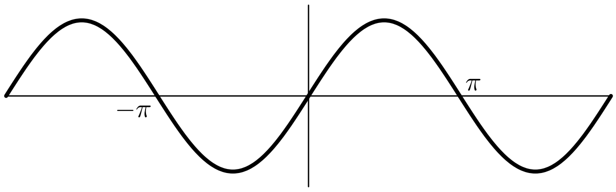
Thus the graph of f is.



Since $\sin x$ is an odd function of x, we can sketch the graph of $g\colon [-\pi, \pi] \to \mathbb{R},\ x \mapsto \sin x$.



Finally, since $\sin(x+2\pi) = \sin x$ for all $x \in$, we can sketch the graph of $\sin\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \sin x$.



We leave it to the reader to sketch the graph of the cosine function.

7.5.2 Secant and Cosecant

The secant and cosecant functions are defined as multiplicative inverses of cosine and sine respectively, wherever these are not 0, so that

$$\sec\colon \mathbb{R} \setminus \{(2n+1)\tfrac{\pi}{2} \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{\cos x}$$

$$\csc\colon \mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{\sin x}$$

Since $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$ for all $x \in \mathbb{R}$, $\csc(x+2\pi) = \csc x$ and $\sec(x+2\pi) = \sec x$ for all $x \in \mathbb{R}$, it is enough to graph $\csc x$ and $\sec x$ for $a \leq x < a + 2\pi$ with $a \in \mathbb{R}$.

Since $\csc(-x) = -\csc x$ and $\sec(-x) = \sec x$ for all $x \in \mathbb{R}$, $\csc x$ is an odd function of x, and $\sec x$ is an even function of x.

It is therefore enough to graph $\csc x$ and $\sec x$ for $x \geq 0$.

Combining these two observations, we see that it is enough to graph $\csc x$ on $]0,\pi[$ and $\sec x$ on $[0,\pi] \setminus \{\frac{\pi}{2}\}$.

Consider $f\colon [0,\pi] \setminus \{\frac{\pi}{2}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \sec x$.

- Since $0 < \cos x \leq 1$ for $x \in [0, \frac{\pi}{2}[$ and $-1 \leq \cos x < 0$ for $x \in]\frac{\pi}{2}, \pi]$, $f(x) \geq 1$ for $x \in [0, \frac{\pi}{2}[$ and $f(x) \leq -1$ for $x \in]\frac{\pi}{2}, \pi]$.

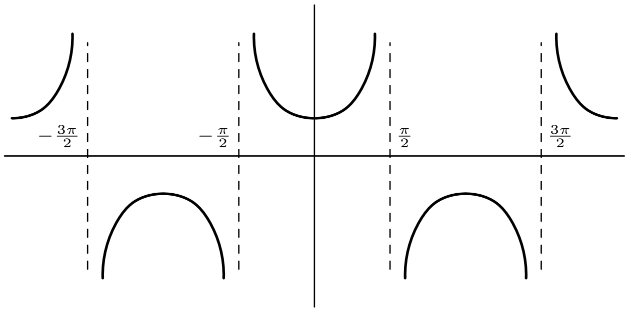
  $f(0) = 1$ and $f(\pi) = -1$.

  Since $\cos x \to 0^+$ as $x \to \frac{\pi}{2}^-$, $f(x) \to \infty$ as $x \to \frac{\pi}{2}^-$.

  Since $\cos x \to 0^-$ as $x \to \frac{\pi}{2}^+$, $f(x) \to -\infty$ as $x \to \frac{\pi}{2}^+$.
- The domain of f contains no boundary points.
- $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{\sin x}{\cos^2 x} = \tan x \sec x = f(x)\tan x$.

Hence f is differentiable and its derivative is 0 in $[0, \pi] \setminus \{\frac{\pi}{2}\}$ if and only if $x = 0, \pi$.

- $f''(x) = \dfrac{d}{dx}\left(f(x)\tan x\right) = (\tan^2 x + \sec^2 x)f(x) = (1 + 2\tan^2 x)f(x)$
  Since $f''(0) = 1$, f has a relative minimum at 0.
  Since $f''(\pi) = -1$, f has a relative maximum at 0.
- Since $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$ and $f'(x) < 0$ for $< \frac{\pi}{2} < x < \pi$, f is increasing on $[0, \frac{\pi}{2}[$ and decreasing on $]\frac{\pi}{2}, \pi]$.
- Since $f''(x) > 0$ for $0 \le x < \frac{\pi}{2}$, f is concave up on $[0, \frac{\pi}{2}[$.
  Since $f''(x) < 0$ for $0 \le x < \pi$, f is concave down on $]\frac{\pi}{2}, \pi]$.
  This allows us to sketch the graph of the secant function.



We leave it to the reader to sketch the graph of the cosecant function.

7.5.3 Tangent and Cotangent

We turn to the last two trigonometric functions,

$$\tan\colon \mathbb{R} \setminus \{(2n+1)\tfrac{\pi}{2} \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\sin x}{\cos x}$$

$$\cot\colon \mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\cos x}{\sin x}$$

Since $\sin(x + \pi) = -\sin x$ and $\cos(x + \pi) = -cosx$ for all $x \in \mathbb{R}$, $\tan(x + \pi) = \tan x$ and $\cot(x + \pi) = \cot x$ for all $x \in \mathbb{R}$, it is enough to graph $\csc x$ and $\sec x$ for $a \le x < a + \pi$ with $a \in \mathbb{R}$.

Since $\tan = \sin x \sec x$ and $\cot x = \cos x \csc x$ for all $x \in \mathbb{R}$, both $\tan x$ and $\cot x$ are odd functions of x, being the product of an even and an odd function. It is therefore enough to graph $\csc x$ and $\sec x$ for $x \ge 0$.

By the observations above, it is enough to graph $\tan x$ on $[0, \frac{\pi}{2}[$ and $\cot x$ on $]0, \frac{\pi}{2}]$.

Consider $f\colon [0, \frac{\pi}{2}[, \longrightarrow \mathbb{R}, \quad x \longmapsto \tan x$.

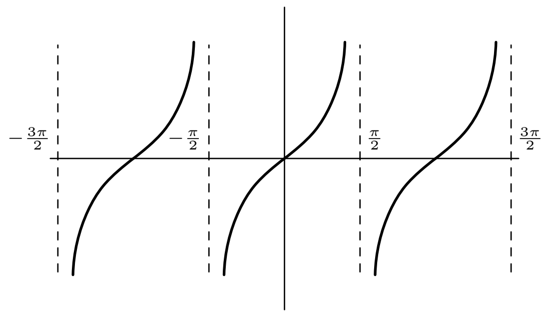
Since $\sin x, \cos x \ge 0$, $f(x) \ge 0$ for all $x \in [0, \frac{\pi}{2}[$.

Since $\lim\limits_{x \to \frac{\pi}{2}^+} \cos x = 0$ and $\lim\limits_{x \to \frac{\pi}{2}^+} \sin x =$, $f(x) = \tan x \to \infty$ as $x \to \frac{\pi}{2}^+$.

- The domain of f contains precisely on boundary point, 0.
  Since $\tan x$ is an odd function of x, $f(0) = 0$.
- $f'(x) = \dfrac{d}{dx}(\tan x) = 1 + \tan^2 x = 1 + \big(f(x)\big)^2$.
  Hence f is differentiable and $f'(x) \ge 1$ for all $x \in [0, \frac{\pi}{2}[$.

Thus f has an absolute minimum at 0, and no other extrema.

- $f''(x) = \dfrac{d}{dx}\left(1 + \left(f(x)\right)^2\right) = 2f(x)\left(1 + \left(f(x)\right)^2\right).$
  Plainly, $f''(x) \geq 0$, with equality if and only if $x = 0$.
- Since $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$, f is increasing on $[0, \frac{\pi}{2}[$.
- Since $f''(x) > 0$ for $0 < x < \frac{\pi}{2}$, f is concave up on $[0, \frac{\pi}{2}[$.
  This allows us to sketch the graph of the tangent function.



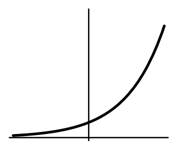
We leave it to the reader to sketch the graph of the cotangent function.

7.5.4 Hyperbolic Sine and Hyperbolic Cosine

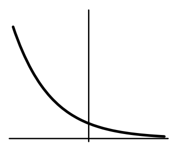
Hyperbolic sine and hyperbolic cosine are, respectively, the odd and even parts of the exponential function

$$\exp\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^x,$$

that graph of which (as shown in MATH102) is



The graph of the function $\mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{-x}$ is

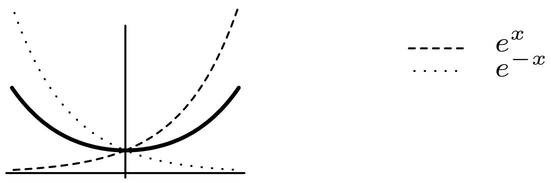


Consider $f\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \cosh x = \dfrac{e^x + e^{-x}}{2}$.

Since $e > 1$, $e^x \to \infty$ and $e^{-x} = \dfrac{1}{e^x} \to 0$ as $x \to \infty$. Thus $f(x) \to \infty$ as $x \to \infty$.

- The domain of f contains precisely on boundary point, 0, and $f(0) = 1$.
- $f'(x) = \dfrac{d}{dx}(\cosh x) = \sinh x = \frac{e^{-x}}{2}(e^{2x} - 1) = \frac{e^{-x}}{2}(e^x + 1)(e^x - 1).$
  Hence f is differentiable for all $x \in \mathbb{R}_0^+$.
  Since $e^x \geq 1$ for $x \geq 0$, $f'x \geq 0$, with equality if and only if $x = 0$.

- $f''(x) = \dfrac{d}{dx}(\sinh x) = \cosh x)$.
  Thus $f''(x) \geq 1$ for all $x \geq 0$.
  Thus f has an absolute minimum at 0, and no other extrema.
- Since $f'(x) > 0$ for all $x \in \mathbb{R}_0^+$, f is increasing.
- Since $f''(x) > 0$ for all $x \in \mathbb{R}_0^+$, f is concave up.
  This allows us to sketch the graph of the hyperbolic cosine function.

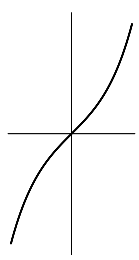


Consider $f \colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \quad x \longmapsto \sinh x = \dfrac{e^x - e^{-x}}{2}$.

Since $e > 1$, $e^x \to \infty$ and $e^{-x} = \dfrac{1}{e^x} \to 0$ as $x \to \infty$. Thus $f(x) \to \infty$ as $x \to \infty$.

Moreover, as we saw in the analysis of $\cosh x$, $\sinh x \geq 0$ for all $x \in \mathbb{R}_0^+$, with equality if and only if $x = 0$.

- The domain of f contains precisely on boundary point, 0, and $f(0) = 0$.
- $f'(x) = \dfrac{d}{dx}(\sinh x) = \cosh x$.
  Hence f is differentiable and $f'x \geq 1$ for all $x \in \mathbb{R}_0^+$.
- $f''(x) = \dfrac{d}{dx}(\cosh x) = \sinh x)$.
  Thus $f''(x) \geq 0$ for all $x \geq 0$.
  Thus f has an absolute minimum at 0, and no other extrema.
- Since $f'(x) > 0$ for all $x \in \mathbb{R}_0^+$, f is increasing.
- Since $f''(x) > 0$ for all $x \in \mathbb{R}_0^+$, f is concave up.
  This allows us to sketch the graph of the hyperbolic sine function.



7.5.5 Hyperbolic Tangent and Hyperbolic Cotangent

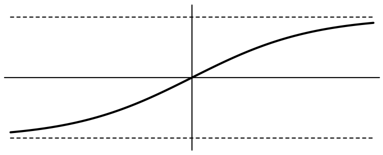
The hyperbolic tangent and hyperbolic cotangent functions are

$$\tanh \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\sinh x}{\cosh x}$$

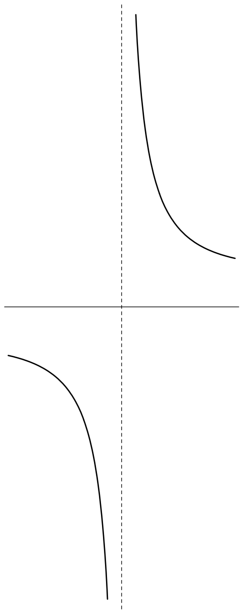
$$\coth \colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{\cosh x}{\sinh x}.$$

Plainly, both are odd functions. We sketch their graphs, leaving the reader to provide the details, following the procedure illustrated above.

The graph of hyperbolic tangent is



The graph of hyperbolic cotangent is



7.6 Implicit Functions

The standard way of presenting a function is in the form $f\colon X \to Y$.

However, it can happen that an explicit formulation in tis form is not readily available. This can occur in the natural sciences when we have sound reasons for believing two measurable quantities are functionally related, but we only have a relationship between the simultaneously measured values of the quantities.

We can partially overcome this difficulty by treating the two quantities as two separate, but related functions of a third variable, say

$$x\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto x(t) \qquad \text{and} \qquad y\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto y(t)$$

with the relation between them expressed in terms of some equation.

A simple example is provided by considering the two co-ordinates, (x, y) of a point particle in a parabolic orbit in the plane. Here we can regard both x and y as functions of t — heuristically, we may think of t as "time" — say $x = x(t)$ and $y = y(t)$, and we obtain a relation between $x(t)$ and $y(t)$ of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

with $a, b, c, d, e, f \in \mathbb{R}$.

This suggests our next definition.

Definition 7.16. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto F(x, y)$ be a function. Then the equation $F(x, y) = 0$ defines y *implicitly as function of* x if and only if there is a subset of $\mathbb{R}$, X, and a function $\psi\colon X \to \mathbb{R}$ such that for all $x \in X$

$$F(x, \psi(x)) = 0$$

Example 7.17. Consider $F\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto x^2 + y^2 - 1$.

Plainly, $F(x, y) = 0$ if and only if $x^2 + y^2 = 1$.

Take the function $\psi\colon [-1, 1] \longrightarrow \mathbb{R}, \quad x \longmapsto -\sqrt{1 - x^2}$.

Then $F(x, \psi(x)) = x^2 + \left(-\sqrt{1 - x^2}\right)^2 - 1 = x^2 + 1 - x^2 - 1 = 0$.

Observation 7.18. There is no suggestion that the function $\psi\colon X \to \mathbb{R}$ in Definition 7.16 is uniquely determined by the function $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Indeed, we could have replaced the function ψ in Example 7.17 by the function $\varphi\colon [-1, 1] \longrightarrow \mathbb{R}, \quad x \longmapsto \sqrt{1 - x^2}$.

While our definition of the derivative of a function and the techniques for differentiation have used the explicit formulation of functions, we can find the derivative of implicitly defined functions without first finding a function ψ as in Definition 7.16. To do so, we regard $F(x\psi(x)) = 0$ as an equation between two functions of x, and differentiate both sides, using the chain rule when differentiating the left-hand side, as we illustrate.

Example 7.19. Consider $F(x, y) = 6x^4 - 4x^3y^2 + y^5 - 3$ and suppose that $F(x, y) = 0$ defines y implicitly as a function of x. Then, using the addition, Leibniz (product) and chain rules,

$$\frac{d}{dx}\left(6x^4 - 4x^3y^2 + y^5 - 3\right) = 24x^3 - 12x^2y^2 - 8x^3y\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0,$$

or,

$$y(5y^3 - 8x^3)\frac{dy}{dx} + 12x^2(2x - y) = 0.$$

Thus, if $y(5y^3 - 8x^3) \neq 0$,

$$\frac{dy}{dx} = \frac{12x^2(2x - y)}{y(8x^3 - 5y^3)}.$$

We can apply this to find the the equation of tangent at the point $(1, 1)$ to the curve in the plane defined by the equation $F(x, y) = 0$:

$$y - 1 = \frac{dy}{dx}_{|(1,1)}(x - 1)$$

that is

$$y = 4x - 3$$

We leave it to the interested reader to solve the equation $F(x, y) = 0$ above to obtain an explicit expression $y = \psi(x)$ and use this to calculate the equation of the given tangent.

7.7 Bernoulli-de l'Hôpital

We have seen how differentiation can be applied to solve otherwise difficult problems, such as sketching graphs of functions. The first derivative detects where a function is increasing or decreasing, so its graph rising or falling. The second derivative detects the curvature of the graph.

In the process, we obtained an indication that differentiation plays a central role in solving optimisation problems. For optimisation problems can typically be expressed as finding extreme values (maxima or minima) of some function subject to certain constraints. We have only dealt with functions of a single variable, so our constraints consist of restricting the domain of the function. In multivariate calculus you will see how differentiation can be applied to more complex optimisation problems, where the functions depend on more than one independent variable, and the constraints can be more complex and subtle.

We do not pursue this further here. Instead we turn to another useful application of the derivative.

Given functions $f, g\colon \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, we have seen that if $f(x) \to \ell$ ad $g(x) \to m$ as $x \to a$, then $\dfrac{f(x)}{g(x)} \to \dfrac{\ell}{m}$ as $x \to a$ as long as $m \neq 0, \pm\infty$, where we allow $a, \ell = \pm\infty$.

What can we say when $m = 0$ or $m = \pm\infty$?

(i) Suppose $m = 0$. It is easy to show that

- if $\ell \pm \infty$, then $\dfrac{f(x)}{g(x)} \to \ell$ as $x \to a$;
- if $\ell \neq \pm\infty, 0$, then $\dfrac{f(x)}{g(x)} \to \pm\infty$ as $x \to a$, depending on whether $\ell > 0$ or $\ell < 0$.

(ii) Suppose $m = \pm\infty$. It is easy to show that if $\ell \neq \pm\infty$, then $\dfrac{f(x)}{g(x)} \to 0$ as $x \to a$

We leave the details to the reader, and turn our attention to the cases left over, namely when $\ell = m = 0$ and when $\ell, m = \pm\infty$.

Theorem 7.20 (Bernouli-de l'Hôpital's Rule). *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions with $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{gx} &= \lim_{x\to a} \frac{f(x) - f(a)}{gx - g(a)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{gx-g(a)}{x-a}} \\
&= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)}
\end{aligned}$$

□

While this form of Bernoulli-de l'Hôpital's Rule is not the most general form, it is extremely useful and can be adapted to the case $\ell, m = \pm\infty$, by replacing f, g by $\frac{1}{f}$ and $\frac{1}{g}$ respectively.

Example 7.21. We evaluate $\lim_{u\to 0^+} \frac{\ln(1+u)}{u}$

Putting $f(u) := \ln(1+u)$ and $g(u) := u$, we see that $f'(u) = \frac{1}{1+u}$ and $g'(u) = 1$, whence, by the Bernoulli-de l'Hopîtal Rule,

$$\lim_{u\to 0^+} \ln\left((1+u)^{\frac{1}{u}}\right) = \frac{f'(0)}{g'(0)} = 1,$$

We apply this to evaluate $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

Using the fact that e^v is a continuous function of v, we obtain

$$\lim_{u\to 0^+} (1+u)^{\frac{1}{u}} = \lim_{u\to 0^+} e^{\left(\ln\left((1+u)^{\frac{1}{u}}\right)\right)} = e^{\left(\lim_{u\to 0^+} \ln\left((1+u)^{\frac{1}{u}}\right)\right)} = e^1 = e.$$

Finally, putting $x := \frac{1}{u}$, we see that

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{u\to 0^+} (1+u)^{\frac{1}{u}} = e$$