# Sample Solutions for Tutorial 11

## Question 1.

Take $p, q \in \mathcal{P}_2$, say $p = a + bt + ct^2$ and $q = d + et + ft^2$, with $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
\langle\langle p, q\rangle\rangle &:= p(-1)q(-1) + p(0)q(0) + p(1)q(1) \\
&= (a - b + c)(d - e + f) + ad + (a + b + c)(d + e + f) \\
&= 3ad + 2af + 2be + 2cd + 2cf
\end{aligned}$$

Hence

$$\langle\langle 1, 1\rangle\rangle = 3, \langle\langle 1, t\rangle\rangle = 0, \langle\langle 1, t^2\rangle\rangle = 2, \langle\langle t, t\rangle\rangle = 2, \langle\langle t, t^2\rangle\rangle = 0, \langle\langle t^2, t^2\rangle\rangle = 2,$$

so that the matrix of $\langle\langle\ ,\ \rangle\rangle$ with respect to the basis $\{1, t, t^2\}$ is

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

[Note that we have proved earlier that $\langle\langle\ ,\ \rangle\rangle$ is an inner product on $\mathcal{P}_2$.

Now consider $\langle p \mid q\rangle := \int_{-1}^{1} p(x)q(x)\,dx$, which is also an inner product, the verification being left as an exercise. Since for any $k \in \mathbb{N}$,

$$\int_{-1}^{1} x^k\,dx = \frac{1 - (-1)^k}{k + 1} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{2}{k+1} & \text{if } k \text{ is even} \end{cases},$$

$$\langle 1 \mid 1\rangle = 2, \quad \langle 1 \mid t\rangle = 0, \quad \langle 1 \mid t^2\rangle = \frac{2}{3}, \quad \langle t \mid t\rangle = \frac{2}{3}, \quad \langle t \mid t^2\rangle = 0, \quad \langle t^2 \mid t^2\rangle = \frac{2}{5}$$

Hence the matrix of $\langle\ \mid\ \rangle$ with respect to $\{1, t, t^2\}$ is

$$\begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix}$$

## Question 2.

Let $\underline{\mathbf{A}}$ be a real invertible $n \times n$ matrix. Let $\mathbf{c}_j \in \mathbb{R}_{(n)}$ be the $j$-th column of $\underline{\mathbf{A}}$.

Since $\underline{\mathbf{A}}$ is invertible, $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is a basis for $\mathbb{R}_{(n)}$.

Applying the Gram-Schmidt procedure with respect to the Euclidean inner product, we obtain an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

Let $\underline{\mathbf{Q}}$ be the matrix whose $j$-th column is $\mathbf{e}_j$. Let $\underline{\mathbf{Q}}^t\underline{\mathbf{Q}} = [x_{ij}]$. Then, by the definition of matrix multiplication, $x_{ij} = \langle\langle \mathbf{e}_i, \mathbf{e}_j\rangle\rangle$, where $\langle\langle\ ,\ \rangle\rangle$ is the Euclidean inner product on $\mathbb{R}_{(n)}$. Hence, by the orthogonality of $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$,

$$x_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

whence $\underline{\mathbf{Q}}$ is an orthogonal matrix.

Furthermore, recall from the Gram-Schmidt procedure, that

$$\mathbf{e}_j = \frac{\mathbf{e}_j^*}{\|\mathbf{e}_j^*\|},$$

where

$$\mathbf{e}_j^* := \mathbf{c}_j - \sum_{i<j} \langle\langle \mathbf{c}_j, \mathbf{e}_i\rangle\rangle \mathbf{e}_i.$$

So, putting

$$r_{ij} := \begin{cases} \langle\langle \mathbf{c}_j, \mathbf{e}_i\rangle\rangle & \text{if } i < j \\ \|\mathbf{e}_j^*\| & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

We have

$$\mathbf{c}_j = \sum_{i=1}^{j} r_{ij}\mathbf{e}_i = \sum_{i=1}^{n} r_{ij}\mathbf{e}_i.$$

If $\underline{\mathbf{R}} := [r_{ij}]_{n\times n}$, then $\underline{\mathbf{R}}$is upper triangular and $\underline{\mathbf{A}} = \underline{\mathbf{Q}}\,\underline{\mathbf{R}}$.

For $\underline{\mathbf{A}} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{c}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$\langle\!\langle \mathbf{c}_1, \mathbf{c}_1 \rangle\!\rangle = \langle\!\langle \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \rangle\!\rangle = 3.3 + 4.4 = 25,$$

whence

$$\mathbf{e}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \qquad \text{and} \qquad r_{11} = 5$$

$$\begin{aligned} \mathbf{e}_2^* &= \mathbf{c}_2 - \langle\!\langle \mathbf{c}_2, \mathbf{e}_1 \rangle\!\rangle \mathbf{e}_1 \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \langle\!\langle \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \rangle\!\rangle \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \end{aligned}$$

Now

$$\|\mathbf{e}_2^*\|^2 = \left(\frac{-4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = 1,$$

whence

$$\mathbf{e}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \qquad r_{12} = 3 \qquad \text{and} \qquad r_{22} = 1.$$

Thus

$$\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$$

expresses the given matrix as an orthogonal matrix multiplied (on the right) by an upper triangular one.

## Question 3.

Recall the trigonometric identities[1]

$$\begin{aligned} \sin A \sin B &= \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big) \\ \sin A \cos B &= \frac{1}{2}\big(\sin(A+B) + \sin(A-B)\big) \\ \cos A \cos B &= \frac{1}{2}\big(\cos(A-B) + \cos(A+B)\big) \end{aligned}$$

[1] These all follow from $\cos(x+y) = \cos x \cos y - \sin x \sin$ and $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

It follows that for $m, n \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned}
\int_0^{2\pi} \sin(mx)\sin(nx)\,dx &= \frac{1}{2}\int_0^{2\pi} (\cos((m-n)x) - \cos((m+n)x))\,dx \\
&= \begin{cases} \frac{1}{2}\left[x - \frac{\sin((2n)x)}{2n}\right]_0^{2\pi} & \text{if } m = n \\ \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_0^{2\pi} & \text{if } m \neq n \end{cases} \\
&= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\
\int_0^{2\pi} \sin(mx)\cos(nx)\,dx &= \frac{1}{2}\int_0^{2\pi} (\sin((m+n)x) + \sin((m-n)x))\,dx \\
&= \begin{cases} \frac{1}{2}\left[\frac{\cos((2n)x)}{2n}\right]_0^{2\pi} & \text{if } m = n \\ \frac{1}{2}\left[-\frac{\cos((m-n)x)}{m-n} - \frac{\cos((m+n)x)}{m+n}\right]_0^{2\pi} & \text{if } m \neq n \end{cases} \\
&= 0 \qquad \text{for all } m, n \\
\int_0^{2\pi} \cos(mx)\cos(nx)\,dx &= \frac{1}{2}\int_0^{2\pi} (\cos((m-n)x) + \cos((m+n)x))\,dx \\
&= \begin{cases} \frac{1}{2}\left[x + \frac{\sin((2n)x)}{2n}\right]_0^{2\pi} & \text{if } m = n \\ \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} + \frac{\sin((m+n)x)}{m+n}\right]_0^{2\pi} & \text{if } m \neq n \end{cases} \\
&= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\end{aligned}$$

Thus, putting $V := \{\varphi : [0, 2\pi] \to \mathbb{R} \mid \varphi \text{ is continuous}\}$ and defining

$$\langle\!\langle\ ,\ \rangle\!\rangle : V \times V \longrightarrow \mathbb{R}, \quad (f, g) \longmapsto \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)\,dx,$$

we see that

$$\langle\!\langle \sin(mx), \sin(nx) \rangle\!\rangle = \langle\!\langle \cos(mx), \cos(nx) \rangle\!\rangle = \begin{cases} 1 & \text{if } \ m = n \\ 0 & \text{if } \ m \neq n \end{cases}$$

and

$$\langle\!\langle \cos(mx), \sin(nx) \rangle\!\rangle = 0 \qquad \text{for all } m, n.$$

Thus $\{\sin(mx), \cos(nx) \mid m.n = 1, 2, 3, \dots\}$ defines a set of orthonormal vectors in $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$.