Chapter 12

Eigenvalues and Eigenvectors

The direct sum of the vector spaces $V_1, \ldots, V_n$, $V_1 \oplus \cdots \oplus V_n$ is defined by

$$\bigoplus_{j=1}^{n} V_j = \{(\mathbf{v}_1, \ldots, \mathbf{v}_n) \mid \mathbf{v}_j \in V_j,\ j = 1, \ldots, n\}$$

with the vector spaces operations are defined "componentwise":

$$\begin{aligned}(\mathbf{v}_1, \ldots, \mathbf{v}_n) + (\mathbf{v}'_1, \ldots, \mathbf{v}'_n) &:= (\mathbf{v}_1 + \mathbf{v}'_1, \ldots, \mathbf{v}_n + \mathbf{v}'_n)\\ \lambda(\mathbf{v}_1, \ldots, \mathbf{v}_n) &:= (\lambda\mathbf{v}_1, \ldots, \lambda\mathbf{v}_n),\end{aligned}$$

It follows that $\mathbf{0}_{V_1 \oplus \cdots \oplus V_n} = (\mathbf{0}_{V_1}, \ldots, \mathbf{0}_{V_n})$ and $-(\mathbf{v}_1, \ldots, \mathbf{v}_n) = (-\mathbf{v}_1, \ldots, -\mathbf{v}_n)$

Since $\mathbb{F}^n \cong \bigoplus_{j=1}^{n} \mathbb{F}$, we can reformulate the Classification Theorem for Finitely Generated Vector Spaces over the field $\mathbb{F}$ as stating that every such vector space is (up to isomorphism) a direct sum of copies of $\mathbb{F}$,

$$V \cong \bigoplus_{j=1}^{n} \mathbb{F},$$

where n is the dimension of the vector space in question. Moreover, $\mathbb{F}$ itself cannot be written as a direct sum of non-trivial vector spaces over $\mathbb{F}$.

What we have achieved is a decomposition of the finitely generated vector space, V, into finitely many components, which cannot be decomposed any further.

The direct sum construction also applies to linear transformations. The direct sum of the linear transformations $T_j \colon V_j \to W_j$ $(j = 1, \ldots, n)$ is

$$\bigoplus_{j=1}^{n} T_j : \bigoplus_{j=1}^{n} V_j \longrightarrow \bigoplus_{j=1}^{n} W_j$$

defined by

$$(T_1 \oplus \cdots \oplus T_n)(\mathbf{v}_1, \ldots \mathbf{v}_n) := (T_1(\mathbf{v}_1), \ldots, T_n(\mathbf{v}_n))$$

The verification that $(T_1 \oplus \cdots \oplus T_n)$ is a linear transformation is routine, and left to the reader.

Example 12.1. For

$$R\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \longmapsto (2x+y, 3y)$$
$$T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (u,v,w) \longmapsto u+v+w$$

$$R \oplus T\colon \mathbb{R}^2 \oplus \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \oplus \mathbb{R}, \quad \big((x,y),(u,v,w)\big) \longmapsto \big((2x+y,3y), u+v+w\big)$$

We may identify $\mathbb{R}^2 \oplus \mathbb{R}^3$ with $\mathbb{R}^5$ and $\mathbb{R}^2 \oplus \mathbb{R}$ with $\mathbb{R}^3$, using the obvious isomorphisms

$$\mathbb{R}^2 \oplus \mathbb{R}^3 \longrightarrow \mathbb{R}^5, \quad \big((x,y),(u,v,w)\big) \longmapsto (x,y,u,v,w)$$
$$\mathbb{R}^2 \oplus \mathbb{R} \longrightarrow \mathbb{R}^3, \quad \big((r,s),t\big) \longmapsto (r,s,t)$$

Using these identifications, we may regard $R \oplus T$ as the linear transformation

$$\mathbb{R}^5 \longrightarrow \mathbb{R}^3, \quad (x,y,u,v,w) \longmapsto (2x+y, 3y, u+v+w)$$

The question arises:

> Given a finitely generated vector space V over $\mathbb{F}$, is every linear transformation $T\colon V \to V$ expressible as the direct sum of linear transformations $T_j\colon \mathbb{F} \to \mathbb{F} \quad (j = 1, \dots, \dim_{\mathbb{F}}(V))$?

This is the question we pursue here.

Let $\dim(V) = m$ and $\dim(W) = n$. Take endomorphisms $R : V \to V$ and $S : W \to W$.

Lemma 12.2. *Let $\{\mathbf{e}_1, \dots \mathbf{e}_m\}$ be a basis for V and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ for W. Then putting*

$$\mathbf{u}_i := \begin{cases} (\mathbf{e}_i, \mathbf{0}_W) & \text{if } i \le m \\ (\mathbf{0}_V, \mathbf{f}_{i-m}) & \text{if } i > m \end{cases}$$

defines a basis, $\{\mathbf{u}_1, \dots, \mathbf{u}_{m+n}\}$, for $V \oplus W$.

Proof. Take $\mathbf{x} \in V \oplus W$.
There are unique $\mathbf{v} \in V, \mathbf{w} \in W$ with $\mathbf{x} = (\mathbf{v}, \mathbf{w})$, and unique $\alpha_i, \beta_j \in \mathbb{F}$ $(1 \le i \le m,\ 1 \le j \le n)$ with $\mathbf{v} = \sum_{i=1}^{m} \alpha_i \mathbf{e}_i$ and $\mathbf{w} = \sum_{j=1}^{n} \beta_j \mathbf{f}_j$. Then

$$\begin{aligned}
\mathbf{x} &= \Big(\sum_{i=1}^{m} \alpha_i \mathbf{e}_i, \sum_{j=1}^{n} \beta_j \mathbf{f}_j\Big) \\
&= \Big(\sum_{i=1}^{m} \alpha_i \mathbf{e}_i, \mathbf{0}_W\Big) + \Big(\mathbf{0}_V, \sum_{j=1}^{n} \beta_j \mathbf{f}_j\Big) \\
&= \sum_{i=1}^{m} \alpha_i (\mathbf{e}_i, \mathbf{0}_W) + \sum_{j=1}^{n} \beta_j (\mathbf{0}_V, \mathbf{f}_j) \\
&= \sum_{i=1}^{m+n} \lambda_i \mathbf{u}_i,
\end{aligned}$$

where the coefficients $\lambda_i = \begin{cases} \alpha_i & \text{if } 1 \le i \le m \\ \beta_{i-m} & \text{if } m < i \le m+n \end{cases}$ are uniquely determined. □

Corollary 12.3. *If the matrix of R with respect to $\{\mathbf{e}_i\}$ is $\underline{\mathbf{A}}$ and that of S with respect to $\{\mathbf{f}_j\}$ is $\underline{\mathbf{B}}$, then the matrix of $R \oplus S$ with respect to $\mathbf{u}_i$ is*

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$$

Proof. Exercise. □

Convention. The meaning of $\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$ needs clarification.

This should not be read here as a matrix of matrices, that is a 2×2 matrix, each of whose coefficients is itself a matrix, even though it is possible to do so sensibly. Rather, what is intended is that if $\underline{\mathbf{A}}$ is an $m \times n$ matrix and $\underline{\mathbf{B}}$ is a $p \times q$ matrix, then $\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$ is the $(m+q) \times (n+q)$ matrix obtained by copying the coefficients of $\underline{\mathbf{A}}$ into the top left, those of $\underline{\mathbf{B}}$ into the bottom right and placing 0s everywhere else.

Example 12.4. Take $\underline{\mathbf{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\underline{\mathbf{B}} = \begin{bmatrix} a & b & e \\ f & d & g \end{bmatrix}$. Then

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & a & b & e \\ 0 & 0 & f & d & g \end{bmatrix}$$

This can be generalised to any finite number of vector spaces, $V_1, \dots, V_n$ and endomorphisms $T_j \colon V_j \to V_j$ $(j = 1, \dots, k)$. In particular, if each V_j is 1-dimensional — equivalently, if each $V_j \cong \mathbb{F}$ — then the matrix, $[a_{ij}]_{n \times n}$, of $T = \oplus T_j : \oplus V_j \to \oplus V_j$ with respect to the canonically induced basis is a diagonal matrix: $a_{ij} = 0$, whenever $i \neq j$.

In other words, the endomorphism $T : V \to V$ is of the form $T_1 \oplus \cdots \oplus T_{\dim(V)}$, with each T_j a linear transformation $T_j \colon \mathbb{F} \to \mathbb{F}$ if and only if there is a basis for V with respect to which the matrix of T is a diagonal matrix.

Thus we may reformulate our question as:

Is there a basis for V with respect to which the matrix of T is in diagonal form?

Note that if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for V with respect to which the matrix of $T : V \to V$ is in diagonal form, then

$$T(\mathbf{e}_j) = \lambda_j \mathbf{e}_j,$$

where λ_j is the j-th diagonal entry in the matrix of T with respect to $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

Definition 12.5. The scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of the endomorphism $T : V \to V$ if and only if there is a vector $\mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}_V$, such that

$$T(\mathbf{v}) = \lambda \mathbf{v}. \tag{12.1}$$

Such a vector $\mathbf{v}$ is an *eigenvector* for the eigenvalue λ. The *eigenspace* of λ, V_λ, is the set of all solutions of $T(\mathbf{v}) = \lambda \mathbf{v}$, that is

$$V_\lambda := \{\mathbf{x} \in V \mid T(\mathbf{x}) = \lambda \mathbf{x}\} \qquad (\lambda \in \mathbb{F}).$$

The next theorem, the main theorem of this section, summarises the preceding discussion.

Main Theorem. *$T : V \to V$ is the direct sum of endomorphisms $T_i : V_i \to V_i$, with $\dim(V_i) = 1$, if and only if V has a basis consisting of eigenvectors of V.*

We discuss related results of independent interest.

Theorem 12.6. *Take an endomorphism $T : V \to V$. Then for each $\lambda \in \mathbb{F}$*

$$V_\lambda := \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda\mathbf{v}\}$$

is a vector subspace of V, and λ is an eigenvalue for T if and only if $V_\lambda \neq \{\mathbf{0}_V\}$.

Proof. Take $\mathbf{u}, \mathbf{v} \in V_\lambda$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned} T(\alpha\mathbf{u} + \beta\mathbf{v}) &= \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \\ &= \alpha\lambda\mathbf{u} + \beta\lambda\mathbf{v} \\ &= \lambda(\alpha\mathbf{u} + \beta\mathbf{v}). \end{aligned}$$

Thus, $\alpha\mathbf{u} + \beta\mathbf{v} \in V_\lambda$. □

Example 12.7. Let V be any vector space. Put $T = id_V$. Then $T(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in V$, so that 1 is the only possible eigenvalue and every non-zero vector is an eigenvector for 1.

Example 12.8. Let V be any vector space. Let T be the zero map $V \to V$, so that $T\colon \mathbf{v} \mapsto \mathbf{0}_V$. Then $T(\mathbf{v}) = \mathbf{0}_V = 0\mathbf{v}$ for every $\mathbf{v} \in V$. Plainly, 0 is the only possible eigenvalue and every non-zero vector is an eigenvector for 0.

Observation 12.9. The eigenvalue 0 plays a distinguished role, for, plainly, $V_0 = \ker(T)$. This establishes the following lemma.

Lemma 12.10. *Let $T : V \longrightarrow V$ be an endomorphism of the vector space V. Then 0 is an eigenvalue if and only if T is not injective.*

Example 12.11. Let V be the Euclidean plane, regarded as $\mathbb{R}^2$. Rotating the plane through an angle of θ (with $0 \leq \theta < 2\pi$) about the origin defines the linear transformation

$$T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longrightarrow (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Thus the real number λ is an eigenvalue for T_θ if and only if

$$(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) = (\lambda x, \lambda y)$$

for some $(x, y) \neq (0, 0)$.
Thus, λ is an eigenvalue for T_θ if and only if there are real $x, y \in \mathbb{R}$ with $x^2 + y^2 \neq 0$ such that

$$\begin{aligned} x\cos\theta - y\sin\theta &= \lambda x \\ x\sin\theta + y\cos\theta &= \lambda y \end{aligned}$$

Squaring and adding these equations we see that

$$x^2 + y^2 = \lambda^2(x^2 + y^2).$$

Since $x^2 + y^2 \neq 0$, $\lambda^2 = 1$ and so the only possible eigenvalues are -1 and 1

$\lambda = \mathbf{1}$: Then

$$\begin{aligned} x\cos\theta - y\sin\theta &= x \\ x\sin\theta + y\cos\theta &= y. \end{aligned}$$

By elementary trigonometry, $\theta = 0$, since $(x, y) \neq (0, 0)$. Thus $T_0 = id_V$, and $V_1 = V$.

$\lambda = \mathbf{-1}$: Then

$$x \cos\theta - y \sin\theta = -x$$
$$x \sin\theta + y \cos\theta = -y.$$

By standard trigonometric arguments, $\theta = \pi$, since $(x, y) \neq (0, 0)$. Thus, $T_\pi(x, y) = (-x, -y)$ for all $(x, y) \in V$, and $V_{-1} = V$.

Furthermore, if $\theta \neq 0, \pi$, then T_θ has no real eigenvalues.

Example 12.12. Let V be the Euclidean plane, regarded as $\mathbb{R}^2$. Reflecting the plane in the x-axis defines the linear transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longrightarrow (x, -y).$$

Thus the real number λ is an eigenvalue for T if and only if $(x, -y) = (\lambda x, \lambda y)$ for some $(x, y) \neq (0, 0)$. In other words, λ is an eigenvalue for T if and only if there are real $x, y \in \mathbb{R}$ with $x^2 + y^2 \neq 0$ such that

$$x = \lambda x \qquad -y = \lambda y$$

Squaring and adding these equations we see that $x^2 + y^2 = \lambda^2(x^2 + y^2)$. Since $x^2 + y^2 \neq 0$, it follows that $\lambda^2 = 1$, so that the only possible eigenvalues are -1 and 1.

$\lambda = \mathbf{1}$: Then $x = x$ and $-y = y$, whence $y = 0$ and x is arbitrary.

Thus, $V_1 = \{(x, 0) \mid x \in \mathbb{R}\}$.

$\lambda = \mathbf{-1}$: Then $x = -x$ and $-y = -y$, whence $x = 0$ and y is arbitrary.

Thus $V_{-1} = \{(0, y) \mid y \in \mathbb{R}\}$.

We see that $\mathbb{R}^2 = V_1 \oplus V_{-1}$, and $\{(1, 0), (0, 1)\}$ is a basis consisting of eigenvectors for T.

Example 12.13. Let $V = \mathcal{C}^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable}\}$. Take

$$T : V \longrightarrow V, \quad f \longmapsto f'',$$

where f'' denotes the second derivative of f. Then -1 is an eigenvalue of T. Plainly, $f(t) = \cos t$ and $g(t) = \sin t$ are eigenvectors for -1 and it follows from the general theory of differential equations that they form a basis for V_{-1}. The details are left as an exercise.

Theorem 12.14. *Let $\lambda_1, \ldots, \lambda_m$ be pairwise distinct eigenvalues for the endomorphism $T : V \to V$. If $\mathbf{v}_i$ is an eigenvector for λ_i, $(1 \leq i \leq m)$, then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent.*

Proof. We prove the theorem by induction on m.

$m = 1$: Since an eigenvector cannot be the zero vector, the proposition is trivially true when $m = 1$.

$m > 1$: Suppose that the theorem is true for m.

Take eigenvectors $\mathbf{v}_i$ $(1 \leq i \leq m+1)$ for the pair-wise distinct eigenvalues λ_i of the endomorphism $T : V \to V$.

Then $\displaystyle\sum_{i=1}^{m+1} \alpha_i \mathbf{v}_i = \mathbf{0}_V$ if and only if $\displaystyle\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = -\alpha_{m+1}\mathbf{v}_{m+1}$, and, in that case

$$
\begin{aligned}
\sum_{i=1}^{m} \alpha_i \lambda_i \mathbf{v}_i &= \sum_{i=1}^{m} \alpha_i T(\mathbf{v}_i) && \text{as } \mathbf{e}_i \text{ is an eigenvector for } \lambda_i \\
&= T(\sum_{i=1}^{m} \alpha_i \mathbf{v}_i) && \text{as } T \text{ is a linear transformation} \\
&= T(-\alpha_{m+1}\mathbf{v}_{m+1}) \\
&= -\alpha_{m+1} T(\mathbf{v}_{m+1}) && \text{as } T \text{ is a linear transformation} \\
&= -\alpha_{m+1}\lambda_{m+1}\mathbf{v}_{m+1} && \text{as } \mathbf{e}_{m+1} \text{ is an eigenvector for } \lambda_{m+1} \\
&= \lambda_{m+1}(-\alpha_{m+1}\mathbf{v}_{m+1}) \\
&= \lambda_{m+1} \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \\
&= \sum_{i=1}^{m} \alpha_i \lambda_{m+1} \mathbf{v}_i
\end{aligned}
$$

Hence, $\sum_{i=1}^{m} \alpha_i(\lambda_i - \lambda_{m+1})\mathbf{v}_i = \mathbf{0}_V$

Since by the inductive hypothesis, $\mathbf{e}_1, \ldots, \mathbf{e}_m$ are linearly independent, $\alpha_i(\lambda_i - \lambda_{m+1}) = 0$ for $1 \leq i \leq m$.

Since $\lambda_{m+1} \neq \lambda_i$ for $i < m+1$, it follows that $\alpha_i = 0$ for $i = 1, \ldots, m$.

Then $\alpha_{m+1}\mathbf{v}_{m+1} = -\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = \mathbf{0}_V$.

Since $\mathbf{v}_{m+1}$ is an eigenvector to λ_{m+1}, $\mathbf{v}_{m+1} \neq \mathbf{0}_V$ and so $\alpha_{m+1} = 0$ as well. □

Corollary 12.15. *T has at most $\dim_{\mathbb{F}}(V)$ distinct eigenvalues.*

Corollary 12.16. *If $T : V \to V$ has n distinct eigenvalues, then V has a basis consisting of eigenvectors of T.*

Given the significance of eigenvalues and eigenvectors, it would be more than merely convenient to find a practical procedure for determining the eigenvalues of a given endomorphism.

When V is finite dimensional, each endomorphism $T : V \to V$ has an associated polynomial whose zeroes are precisely the eigenvalues of T, as we now show.

Recall that $\lambda \in \mathbb{F}$ is an eigenvalue for $T : V \to V$ if and only if the equation $T(\mathbf{v}) = \lambda\mathbf{v}$ has a non-zero solution, $\mathbf{v}$. Choose a basis for V. Let $\underline{\mathbf{A}}$ be the matrix of T and $\mathbf{x} \in \mathbb{F}_{(n)}$ the co-ordinate vector of $\mathbf{v}$ with respect to this basis.

Theorem 12.17. *λ is an eigenvalue for T if and only if $\det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) = 0$.*

Proof.

$$
\begin{aligned}
\lambda \text{ is and eigenvalue for } T \quad &\text{if and only if} \quad T(\mathbf{v}) = \lambda\mathbf{v} \text{ for some } \mathbf{v} \in V,\ \mathbf{v} \neq \mathbf{0}_V, \\
&\text{if and only if} \quad \underline{\mathbf{A}}\,\mathbf{x} = \lambda\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{F}_{(n)},\ \mathbf{x} \neq \mathbf{0}, \\
&\text{if and only if} \quad (\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n)\mathbf{x} = \mathbf{0} \text{ for some } \mathbf{x} \in \mathbb{F}_{(n)},\ \mathbf{x} \neq \mathbf{0}, \\
&\text{if and only if} \quad \operatorname{rk}(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) < n, \\
&\text{if and only if} \quad \det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) = 0.
\end{aligned}
$$

□

Since the determinant of $\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n$ is a polynomial function of λ, Theorem 12.17 provides for each endomorphism T a concrete polynomial in λ whose zeroes are precisely the eigenvalues the eigenvalues of T. This polynomial appears to be dependent on the basis chosen.

Fortunately, this is a case where appearances are deceptive. For if $\underline{\mathbf{B}}$ is the matrix of T with respect to another basis, then there is an invertible matrix $\underline{\mathbf{M}}$ such that $\underline{\mathbf{B}} = \underline{\mathbf{M}}\,\underline{\mathbf{A}}\,\underline{\mathbf{M}}^{-1}$. But then

$$\begin{aligned}
\det(\underline{\mathbf{B}} - \lambda\underline{\mathbf{1}}_n) &= \det(\underline{\mathbf{M}}\,\underline{\mathbf{A}}\,\underline{\mathbf{M}}^{-1} - \lambda\underline{\mathbf{M}}\,\underline{\mathbf{M}}^{-1}) \\
&= \det(\underline{\mathbf{M}}(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n)\underline{\mathbf{M}}^{-1}) \\
&= \det(\underline{\mathbf{M}})\det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n)\det(\underline{\mathbf{M}}^{-1}) \\
&= \det(\underline{\mathbf{A}} - \lambda\underline{\mathbf{1}}_n) \qquad\qquad \text{as} \quad \det(\underline{\mathbf{M}}^{-1}) = (\det(\underline{\mathbf{M}}))^{-1}.
\end{aligned}$$

Thus the polynomial does not depend on the basis chosen.

Definition 12.18. Let T be an endomorphism of the n-dimensional vector space V. Let $\underline{\mathbf{A}}$ be a matrix representing T. Then

$$\chi_T(t) := \det(T - t\, id_V) = \det(\underline{\mathbf{A}} - t\,\underline{\mathbf{1}}_n) =: \chi_{\underline{\mathbf{A}}}(t)$$

is the *characteristic polynomial* of T and of $\underline{\mathbf{A}}$.

The eigenvalues are the zeroes of the characteristic polynomial, or, equivalently, the solutions of the *characteristic equation*, $\chi(t) = 0$.

We define eigenvalues, eigenvectors and eigenspaces for $n \times n$ matrices, by regarding the $n \times n$ matrix, $\underline{\mathbf{A}}$, over $\mathbb{F}$ as the linear transformation

$$T_{\underline{\mathbf{A}}} : \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(n)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\,\mathbf{x}$$

Observation 12.19. If $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$ has characteristic polynomial, $\xi_{\underline{\mathbf{A}}}(t) = b_0 + b_1 t + \cdots + b_n t^n$, then it follows form the definition of the characterstic function that

$$\begin{aligned}
b_0 &= \det(\underline{\mathbf{A}}) \\
b_{n-1} &= (-1)^{n-1}\operatorname{tr}(\underline{\mathbf{A}}) \\
b_n &= (-1)^n
\end{aligned}$$

Observation 12.20. The matrix of $T_{\underline{\mathbf{A}}}$ with respect to the standard basis of $\mathbb{F}_{(n)}$ is $\underline{\mathbf{A}}$ itself.

Definition 12.21. The *eigenvalues, eigenvectors and eigenspaces of* $\underline{\mathbf{A}}$ are those of the linear transformation

$$\underline{\mathbf{T}}_{\underline{\mathbf{A}}} : \mathbb{F}_{(n)} \longrightarrow \mathbb{F}_{(n)}, \quad \mathbf{x} \longmapsto \underline{\mathbf{A}}\mathbf{x}$$

Observation 12.22. In particular, if $\underline{\mathbf{A}}$ is an $n \times n$ matrix, then the eigenspace of the eigenvalue 0 is the null space of $\underline{\mathbf{A}}$. (Of course the null space (or kernel) of $\underline{\mathbf{A}}$ is trivial if and only if 0 is not an eigenvalue of $\underline{\mathbf{A}}$.)

We list further properties of eigenvalues.

Theorem 12.23. *Let λ be an eigenvalue of the matrix $\underline{\mathbf{A}}$.*

(i) λ^n is an eigenvalue of $\underline{\mathbf{A}}^n$ for any $n \in \mathbb{N}$.

(ii) If $\underline{\mathbf{A}}$ is invertible, then λ^n is an eigenvalue of $\underline{\mathbf{A}}^n$ for any $n \in \mathbb{Z}$.

(iii) λ is an eigenvalue of $\underline{\mathbf{A}}^t$.

Proof. **(i)** We adopt here the convention that $0^0 = 1$ and proceed by induction on n

$\mathbf{n = 0}$: Since $\underline{\mathbf{A}}^0 = \underline{\mathbf{1}}_n$ and $\lambda^0 = 1$, the statement is true for $n = 0$

$\mathbf{n > 0}$: Suppose that $\underline{\mathbf{A}}^n\,\mathbf{x} = \lambda^n\mathbf{x}$. Then

$$\underline{\mathbf{A}}^{n+1}\mathbf{x} = \underline{\mathbf{A}}(\underline{\mathbf{A}}^n\mathbf{x}) = \underline{\mathbf{A}}\lambda^n\mathbf{x} = \lambda^n\underline{\mathbf{A}}\,\mathbf{x} = \lambda^n\lambda\mathbf{x} = \lambda^{n+1}\mathbf{x}$$

(ii) Since we are dealing with finitely generated vector spaces, $\underline{\mathbf{A}}$ is invertible if and only if its null space is trivial, which is equivalent to 0's not being an eigenvalue of $\underline{\mathbf{A}}$. So

$$\underline{\mathbf{A}}\,\mathbf{x} = \lambda\mathbf{x} \quad\iff\quad \lambda^{-1}\mathbf{x} = \underline{\mathbf{A}}^{-1}\,\mathbf{x}.$$

The result now follows by applying Part (i) to $\underline{\mathbf{A}}^{-1}$

(iii) $\det(\underline{\mathbf{A}}^t - t\underline{\mathbf{1}}_n) = \det(\underline{\mathbf{A}}^t - t\underline{\mathbf{1}}_n^t) = \det((\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n)^t) = \det(\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n)$. □

Corollary 12.24. *Let λ be an eigenvalue of the endomorphism $T : V \to V$.*

(i) λ^n is an eigenvalue of T^n for any $n \in \mathbb{N}$, where T^n denotes the composition $T \circ \cdots \circ T$ with n terms.

(ii) If T is invertible, then λ^n is an eigenvalue of T^n for any $n \in \mathbb{Z}$

Example 12.25. We attempt to diagonalise the real matrix $\underline{\mathbf{A}} = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}$

We use elementary row operations on $\begin{bmatrix} 1-\lambda & 6 \\ 4 & 3-\lambda \end{bmatrix}$ to reduce it to a form from which we can read off the eigenvalues and readily determine the eigenvectors.

$$\begin{bmatrix} 1-\lambda & 6 \\ 4 & 3-\lambda \end{bmatrix}$$

Adding $(\lambda - 1)$ times the second row to four four times the first, then multiplying the second row by -1, we obtain

$$\begin{bmatrix} 0 & -(\lambda^2 - 4\lambda - 21) \\ 4 & 3-\lambda \end{bmatrix}$$

Because of the first column, this matrix has rank at least 1, no matter how we choose $\lambda \in \mathbb{R}$.

So, the only way its determinant can be 0, is if the second column is a multiple of the first.

By inspection, this occurs if and only if $\lambda^2 - 4\lambda - 21 = 0$.

Since $\lambda^2 - 4\lambda - 21 = (\lambda + 3)(\lambda - 7)$, the eigenvalues of $\underline{\mathbf{A}}$ are -3 and 7.

We substitute these values successively to obtain the corresponding eigenvectors.

$\lambda = -\mathbf{3}$: Our transformed matrix is

$$\begin{bmatrix} 0 & 0 \\ 4 & 6 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\underline{\mathbf{A}}$ for the eigenvalue -3 if and only if $2x+3y = 0$, so that $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ generates the eigenspace V_{-3}, and

$$\begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = -3\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \end{bmatrix} \tag{C1}$$

$\lambda = \mathbf{7}$: Our transformed matrix is

$$\begin{bmatrix} 0 & 0 \\ 4 & -4 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of $\underline{\mathbf{A}}$ to the eigenvalue 7 if and only if $x - y = 0$.

Hence, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ generates the eigenspace V_7, and

$$\begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \tag{C2}$$

We can combine C1 and C2 to obtain

$$\begin{aligned} \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} &= \begin{bmatrix} -9 & 7 \\ 6 & 7 \end{bmatrix} \\ &= \begin{bmatrix} (-3).3 & 7.1 \\ (-3).(-2) & 7.1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix} && \text{by Section 9.5} \end{aligned}$$

We may thus regard $\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ as a "change-of-basis" or "transition" matrix.

Since its inverse is $\frac{1}{5} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ we see that the matrix corresponding to $\underline{\mathbf{A}}$ with respect to the basis for $\mathbb{R}_{(2)}$ consisting of the eigenvectors of $\underline{\mathbf{A}}$ is

$$\frac{1}{5} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix},$$

which is a diagonal matrix.

Emulating the above for the matrices

$$\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

illustrates not only what can be done, but also some of the difficulties that can arise.

Example 12.26. In particular, direct computation shows that the matrix $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$ has only one eigenvalue, namely 2, and that every eigenvector must be of the form $\begin{bmatrix} 2t \\ t \end{bmatrix}$. Hence there is no basis for $\mathbb{R}_{(2)}$ consisting of eigenvectors of our matrix, showing that the conclusion of Corollary 12.16 is not true without some condition being imposed.

This example illustrates what can go wrong when an $n \times n$ matrix has at least one eigenvalue, but does not have n distinct ones.

Observation 12.27. While our procedure for finding eigenvalues and eigenvectors is, in principle, quite simple, significant problem do arise.

An immediate one is finding the zeroes of a polynomial, or, equivalently, expressing a polynomial as the product of linear factors (factors of the form $(t - a)$. This is, in general, an unsolvable problem, even in the most familiar case, when the scalars are all complex numbers. In this case, the *Fundamental Theorem of Algebra* ensures that every polynomial can be factorised into linear factors, and *Cardano's formulæ*, dating from the 16th century, provide the factors when the polynomial in question has degree at most four. However, Lagrange, Abel and Galois proved in the 19th century, that no such general formula is possible for polynomials of degree at least five. This problem is studied in abstract algebra, where a proof is available using *Galois theory.*

The fact that exact solutions are only available in special cases means that in many practical situations, we are forced rely on *numerical methods* or other means to find sufficiently accurate approximations. This, in turn, leads to other interesting and important mathematical problems, such as finding efficient algorithms for the approximation and the question of the *stability* of the eigenvalues and eigenvectors when the coefficients are perturbed. Such questions are studied in courses on numerical methods and computer algebra.

12.1 The Cayley-Hamilton Theorem

If V is an n-dimensional vector space over $\mathbb{F}$ and $T\colon V \to V$ a linear transformation, then so is T^k for any $k \in \mathbb{N}$. Now the linear transformations $V \to V$ form a vector space $\mathrm{Hom}_{\mathbb{F}}(V, V)$ over $\mathbb{F}$ whose dimension is n^2. (To see this, recall that for a fixed basis, there is a bijection between $\mathrm{Hom}_{\mathbb{F}}(V, V)$ and $\mathbf{M}(n; \mathbb{F})$, which is actually a linear transformation, and hence an isomorphism: T corresponds to $\underline{\mathbf{A}}_T$.)

By Theorem 8.4, $id_V, T, T^2, \ldots, T^{n^2}$ must be linearly dependent.

This means that there are $a_0, \ldots a_{n^2} \in \mathbb{F}$, not all 0, with

$$a_0 id_V + a_1 T + \cdots + a_{n^2} T^{n^2} = 0.$$

The corresponding matrix version is that for any $\underline{\mathbf{A}} \in \mathbf{M}(n; \mathbb{F})$ there are $a_0, \ldots a_{n^2} \in \mathbb{F}$, not all 0, with

$$a_0 \underline{\mathbf{1}}_n + a_1 \underline{\mathbf{A}} + \cdots + a_{n^2} \underline{\mathbf{A}}^{n^2} = \underline{\mathbf{0}}_n.$$

We can express this by saying that every endomorphism of an n-dimensional vector space over $\mathbb{F}$ is a zero of of polynomial equation of degree at most n^2 over $\mathbb{F}$, or, equivalently, every $n \times n$ matrix over $\mathbb{F}$is a zero of of polynomial equation of degree at most n^2 over $\mathbb{F}$.

This immediately raises two questions:

1. Is this the best we can do, or is there a polynomial, p, of lower degree which also has T (resp. $\underline{\mathbf{A}}$) as a zero?
2. Given T (or $\underline{\mathbf{A}}$), determine the polynomial p explicitly.

If we let m_T (or $m_{\underline{\mathbf{A}}}$) be the lowest degree of any non-zero polynomial for which T (or $\underline{\mathbf{A}}$) is a zero, then what we have show is that if $\dim V = n$, then $m \leq n^2$.

The following example shows that, the best universal bound for $m$cannot be less than n. The Cayley-Hamilton Theorem (Theorem 12.32) then shows that T (or $\underline{\mathbf{A}}$) is always the zero of a specific polynomial of degree precisely n.

Example 12.28. Choose a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V. Take $T : V \longrightarrow V$ be defined by

$$T(\mathbf{e}_j) := \begin{cases} e_{j+1} & \text{if } j < n \\ e_1 & \text{if } j = n \end{cases}$$

It follows, successively, that $T(\mathbf{e}_1) = \mathbf{e}_2,\ T^2(\mathbf{e}_1),\ \ldots, T^{n-1}(\mathbf{e}_1) = \mathbf{e}_n$.

Let $p(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1}$ be a polynomial in $\mathbb{F}[t]$ for which $F(T) = 0$.

This means that $p(T)(\mathbf{v}) = \mathbf{0}_V$ for every $\mathbf{v} \in V$.

In particular, take $\mathbf{v} = \mathbf{e}_1$. Then

$$p(T)(\mathbf{v}) = a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + \cdots + a_{n-1}\mathbf{e}_n = \mathbf{0}_V.$$

Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V, the on;y possibility is that $a_0 = a_1 = \cdots = a_{n-1} = 0$.

This is the worst that can occur: If $T : V \to V$ is an endomorphism of an n-dimensional vector space, then it is a zero of a polynomial of specific degree n, the *characteristic polynomial*. We prove this in terms of matrices as the Cayley-Hamilton Theorem, which asserts that any $n \times n$ matrix satisfies its own *characteristic equation*.

As we need the construction of an $(n-1) \times (n-1)$ matrix from a given $n \times n$ matrix by deleting one row and one column, we our earlier definition.

Let $\underline{\mathbf{A}} = [a_{ij}]_{n\times n}$ be an $n \times n$ matrix. For $1 \le p, q \le n$ let $\underline{\mathbf{A}}_{(pq)} = [x_{ij}]_{(n-1)\times(n-1)}$ where

$$x_{ij} = \begin{cases} a_{ij} & i < p,\ j < q \\ a_{i(j+1)} & i < p,\ j \ge q \\ a_{(i+1)j} & i \ge p,\ j < q \\ a_{(i+1)(j+1)} & i \ge p,\ j \ge q \end{cases}.$$

Definition 12.29. Using the notation above, put

$$A_{ji} := (-1)^{i+j} \det\left(\underline{\mathbf{A}}_{(ij)}\right).$$

The *adjugate* of $\underline{\mathbf{A}}$ is the matrix

$$\operatorname{adj}\underline{\mathbf{A}} := [A_{ij}]_{n\times n}.$$

Lemma 12.30. *Given any $n \times n$ matrix $\underline{\mathbf{A}}$,*

$$(\operatorname{adj}\underline{\mathbf{A}})\underline{\mathbf{A}} = \underline{\mathbf{A}}(\operatorname{adj}\underline{\mathbf{A}}) = (\det\underline{\mathbf{A}})\underline{\mathbf{1}}_n.$$

Proof. The proof follows directly from the definition of matrix multiplication together with the definition and properties of the determinant function. □

We illustrate the Cayley-Hamilton Theorem with an example. Our proof of the theorem is a generalisation of this example.

Example 12.31. Take $\underline{\mathbf{A}} := \begin{bmatrix} c & b & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$$\chi_{\underline{\mathbf{A}}}(t) = \det\left(\begin{bmatrix} c-t & b & a \\ 1 & -t & 0 \\ 0 & 1 & -t \end{bmatrix}\right) = -t^3 + ct^2 + bt + a.$$

$$\underline{\mathbf{A}}^2 = \begin{bmatrix} c^2+b & cb+a & ca \\ c & b & a \\ 1 & 0 & 0 \end{bmatrix} \quad\text{and}\quad \underline{\mathbf{A}}^3 = \begin{bmatrix} c^3+2bc+a & c^2b+b^2+ca & c^2a+ba \\ c^2+b & cb+a & ca \\ c & b & a \end{bmatrix}$$

$$\begin{aligned}
\chi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}}) &= \begin{bmatrix} -c^3-2bc-a & -c^2b-b^2-ca & -c^2a-ba \\ -c^2-b & -cb-a & -ca \\ -c & -b & -a \end{bmatrix} \\
&\quad + \begin{bmatrix} c^3+cb & c^2b+ca & c^2a \\ c^2 & bc & ac \\ c & 0 & 0 \end{bmatrix} + \begin{bmatrix} bc & b^2 & ba \\ b & 0 & 0 \\ 0 & b & 0 \end{bmatrix} + \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Our proof makes use of the adjugate of $\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n$. We illustrate how this can be expressed as a polynomial in t with matrices as coefficients. Put $\underline{\mathbf{B}} := \underline{\mathbf{A}} - t\underline{\mathbf{1}}_3$, so that $\chi_{\underline{\mathbf{A}}}(t) = \det(\underline{\mathbf{B}})$.

We compute $\operatorname{adj}(\underline{\mathbf{B}}) = [x_{ij}]_{3\times 3}$, where

$$x_{ij} = (-1)^{i+j}\det(\underline{\mathbf{B}}_{(ji)}),$$

with $\underline{\mathbf{B}}_{(ji)}$ the 2×2 matrix obtained from $\underline{\mathbf{B}}$ by deleting its j^{th} row and i^{th} column.

$$\begin{aligned}
x_{11} &= (-1)^{1+1}\det\left(\begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix}\right) &&= t^2 \\
x_{12} &= (-1)^{1+2}\det\left(\begin{bmatrix} b & a \\ 0 & -t \end{bmatrix}\right) &&= bt \\
x_{13} &= (-1)^{1+3}\det\left(\begin{bmatrix} b & a \\ -t & 0 \end{bmatrix}\right) &&= at \\
x_{21} &= (-1)^{2+1}\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix}\right) &&= t \\
x_{22} &= (-1)^{2+2}\det\left(\begin{bmatrix} c-t & a \\ 0 & -t \end{bmatrix}\right) &&= t^2-ct \\
x_{23} &= (-1)^{2+3}\det\left(\begin{bmatrix} c-t & a \\ 1 & 0 \end{bmatrix}\right) &&= a \\
x_{31} &= (-1)^{3+1}\det\left(\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\right) &&= 1 \\
x_{32} &= (-1)^{3+2}\det\left(\begin{bmatrix} c-t & b \\ 0 & 1 \end{bmatrix}\right) &&= t-c \\
x_{33} &= (-1)^{3+3}\det\left(\begin{bmatrix} c-t & b \\ 1 & -t \end{bmatrix}\right) &&= t^2-ct-b \\
\operatorname{adj}(\underline{\mathbf{B}}) &= \begin{bmatrix} t^2 & bt & at \\ t & t^2-ct & a \\ 1 & t-c & t^2-ct-b \end{bmatrix} \\
&= t^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & b & a \\ 1 & -c & 0 \\ 0 & 1 & -c \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 1 & 0 & -b \end{bmatrix}
\end{aligned}$$

Theorem 12.32 (Cayley-Hamilton). *Let $\chi_{\underline{\mathbf{A}}}(t)$ be the characteristic polynomial of the $n\times n$ matrix $\underline{\mathbf{A}}$. Then $\chi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}}) = \underline{\mathbf{0}}_n$.*

Proof. Put $\underline{\mathbf{B}} := \underline{\mathbf{A}} - t\underline{\mathbf{1}}_n$.

By the definition of the determinant, there are $b_0, \dots, b_n \in \mathbb{F}$ with

$$\chi_{\underline{\mathbf{A}}}(t) = \det \underline{\mathbf{B}} = b_0 + b_1 t + \cdots + b_n t^n = \sum_{j=0}^{n-1} b_j t^j. \tag{i}$$

Since $\operatorname{adj}(\underline{\mathbf{B}}) := [x_{ij}]_{n\times n}$, with $x_{ij} := (-1)^{i+j} \det \underline{\mathbf{B}}_{(ji)}$ and $(-1)^{i+j} \det \underline{\mathbf{B}}_{(ji)}$ is a polynomial in t of degree at most $n-1$, there are $n \times n$ matrices $\underline{\mathbf{B}}_0, \dots, \underline{\mathbf{B}}_{n-1}$ with

$$\operatorname{adj} \underline{\mathbf{B}} \ = \ \underline{\mathbf{B}}_0 \ + \ \underline{\mathbf{B}}_1 t \ + \cdots + \ \underline{\mathbf{B}}_{n-1} t^{n-1} = \sum_{j=0}^{n-1} t^j \underline{\mathbf{B}}^j. \tag{ii}$$

Hence,

$$\begin{aligned}(\det \underline{\mathbf{B}})\underline{\mathbf{1}}_n &= \underline{\mathbf{B}} \operatorname{adj} \underline{\mathbf{B}} \\ &= (\underline{\mathbf{A}} - t\underline{\mathbf{1}}_n) \operatorname{adj} \underline{\mathbf{B}} \\ &= \underline{\mathbf{A}} \operatorname{adj} \underline{\mathbf{B}} - t \operatorname{adj} \underline{\mathbf{B}} \end{aligned} \tag{iii}$$

and

$$\begin{aligned}\chi_{\underline{\mathbf{A}}}(t)\underline{\mathbf{1}}_n &= \sum_{j=0}^{n} b_j t^j \underline{\mathbf{1}}_n \\ &= (\det \underline{\mathbf{B}})\underline{\mathbf{1}}_n && \text{by (i)} \\ &= \underline{\mathbf{A}} \operatorname{adj} \underline{\mathbf{B}} - t \operatorname{adj} \underline{\mathbf{B}} && \text{by (iii)} \\ &= \underline{\mathbf{A}} \sum_{j=0}^{n-1} t^j \underline{\mathbf{B}}^j - t \sum_{j=0}^{n-1} t^j \underline{\mathbf{B}}^j && \text{by (ii)} \\ &= \underline{\mathbf{A}}\,\underline{\mathbf{B}}_0 + t(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_1 - \underline{\mathbf{B}}_0) + \cdots + t^{n-1}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_{n-1} - \underline{\mathbf{B}}_{n-2}) - t^n \underline{\mathbf{B}}_{n-1}. \end{aligned}$$

Thus

$$\begin{aligned}\chi_{\underline{\mathbf{A}}}(\underline{\mathbf{A}}) &= \underline{\mathbf{A}}\,\underline{\mathbf{B}}_0 + \underline{\mathbf{A}}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_1 - \underline{\mathbf{B}}_0) + \cdots + \underline{\mathbf{A}}^{n-1}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}_{n-1} - \underline{\mathbf{B}}_{n-2}) - \underline{\mathbf{A}}^n \underline{\mathbf{B}}_{n-1} \\ &= \underline{\mathbf{0}} \end{aligned}$$

□

By Observation 12.19, the characteristic polynomial of $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$ is a polynomial over $\mathbb{F}$ of the form $(-1)^n(b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + t^n)$.

It is natural to ask whether every polynomial of this form is the characteristic polynomial of a matrix $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F})$, or, equivalently, of an endomorphism $T\colon V \to V$, where $\dim_{\mathbb{F}}(V) = n$.

As suggested by Example 12.31, the answer is affirmative, as the following example shows.

Example 12.33. The $n \times n$ matrix

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & \cdots & -b_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

has characteristic polynomial $(-1)^n(b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + t^n)$.

The verification is left as an exercise.

This matrix is the *companion matrix* of the polynomial $b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + t^n$.

12.2 Discussion

While every $n \times n$ matrix is a zero of its characteristic polynomial, which has degree n, some matrices are zeroes of polynomial of lower degree. For example the zero matrix is a zero of the polynomial t, and the identity matrix is a zero of the polynomial $t - 1$.

We summarise a more complete analysis without providing proofs, since these require the introduction of concepts and techniques beyond the scope of these notes. They are investigated in abstract algebra.

If we restrict attention to polynomials whose the leading coefficient is 1, then there is a unique polynomial of lowest possible degree for which the matrix $\underline{\mathbf{A}}$ is a zero. This is the *minimm polynomial* of $\underline{\mathbf{A}}$, $\mu_{\underline{\mathbf{A}}}$. It divides every polynomial for which $\underline{\mathbf{A}}$ is a zero, and its zeroes are precisely the eigenvalues of $\underline{\mathbf{A}}$, that is, the zeroes of the characteristic polynomial of $\underline{\mathbf{A}}$. The main result on the minimum polynomial is that the matrix $\underline{\mathbf{A}}$ is diagonalisable if and only if

$$\mu_{\underline{\mathbf{A}}}(t) = (t - \lambda_1) \cdots (t - \lambda_m)$$

with $\lambda_i = \lambda_j$ if and only if $i = j$.

The field $\mathbb{F}$ is *algebraically closed* if and only if every polynomial in one indeterminate over $\mathbb{F}$ can be written as a product of linear factors. In such a case, every matrix, $\underline{\mathbf{A}}$ over $\mathbb{F}$ can be brought to *block diagonal form*, or *Jordan normal form*

$$\begin{bmatrix} \underline{\mathbf{A}}_{\lambda_1} & \underline{\mathbf{0}} & \cdots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{A}}_{\lambda_2} & \cdots & \\ \vdots & \vdots & \ddots & \end{bmatrix}$$

with each *Jordan block*, $\underline{\mathbf{A}}_{\lambda_j}$,of the form

$$\begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \vdots \end{bmatrix}.$$

Example 12.34. The minimum polynomial of the matrix in Example 2.4,

$$\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix},$$

is $(t-1)(t-2)$, which is of degree 2 and has two distinct zeroes. The block diagonal form comprises the two Jordan blocks

$$\underline{\mathbf{A}}_3 = \begin{bmatrix} 3 \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{A}}_1 = \begin{bmatrix} 1 \end{bmatrix}$$

The minimum polynomial of the matrix in Example 2.5,

$$\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix},$$

must divide its characteristic polynomial, $(t-2)^2$. Hence it must be either $t - 2$ or $(t-2)^2$. Since

$$\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the minimum polynomial cannot be $t - 2$. Hence it must be $(t-2)^2$, which is of degree 2.

Since this fails to have two distinct zeroes, the block diagonal form has the single Jordan block

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The minimum polynomial of the matrix in Example 2.6,

$$\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix},$$

is $(t-2)^2+1$, which is of degree 2, which fails to have any real zeroes, but has two distinct complex zeroes. If we now regard is as a complex matrix, the block diagonal form comprises the two Jordan blocks

$$\underline{\mathbf{A}}_{2+i} = \begin{bmatrix} 2+i \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{A}}_{2-i} = \begin{bmatrix} 2-i \end{bmatrix}$$

where $i^2 = 1$.

The above shows that the matrix $\underline{\mathbf{A}}$ is diagonalisable if and only if each of its Jordan blocks is 1×1.

We turn to an an alternative formulation.

Definition 12.35. Let $T\colon V \to V$ be an endomorphism of the finitely generated vector space V (or, equivalently, take $\underline{\mathbf{A}} \in \mathbf{M}(n;\mathbb{F}))$ and λ an eigenvalue of T (or $\underline{\mathbf{A}}$).

The *algebraic multiplicity* of λ is $a \in \mathbb{N}$ if and only if $(t-\lambda)^a$ divides $\chi_T(t)$ (or $\chi_{\underline{\mathbf{A}}}(t)$), but $(t-\lambda)^{a+1}$ does not.

The *geometric multiplicity* of λ is $\dim(V_\lambda)$, that is to say, the number of linearly independent eigenvectors for the eigenvalue λ.

Example 12.36. Take $\underline{\mathbf{A}} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$.

Since $\chi_{\underline{\mathbf{A}}}(t) = (t-2)^2$, the algebraic multiplicity of the eigenvalue 2 is 2.

As we saw in Example 12.26, every eigenvector is of the form $\begin{bmatrix} 2t \\ t \end{bmatrix}$, showing that $\dim(V_2) = 1$, that is, the geometric multiplicity of 2 is 1.

We show that this is typical.

Lemma 12.37. *The geometric multiplicity of λ cannot exceed its algebraic multiplicity.*

Proof. Let λ be an eigenvalue of $T\colon V \to V$ with geometric multiplicity g.

Choose linearly independent eigenvectors $\mathbf{e}_1, \dots, \mathbf{e}_g$ for the eigenvalue λ. Extend this to a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_g, \dots \mathbf{e}_n\}$ of V.

The matrix, $\underline{\mathbf{A}}$, of T with respect to this basis is of the form

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 & * \\ 0 & \lambda & & \vdots & * \\ \vdots & & \ddots & 0 & * \\ 0 & \cdots & 0 & \lambda & * \\ \vdots & & & 0 & * \end{bmatrix}$$

This being the case, $(t-\lambda)^g$ must divide $\chi_{\underline{\mathbf{A}}}(t) = \chi_T(t)$. $\square$

Theorem 12.38. *An $n \times n$ matrix, is diagonalisable if and only if each eigenvalue has the same geometric and algebraic multiplicity, and the sum of these is n.*

Proof. A little thought shows that these conditions are necessary and sufficient to ensure that there is a basis consisting of eigenvectors. $\square$

12.3 Exercises

Exercise 12.1. Given linear transformations $R : V \longrightarrow V'$ and $S : W \to W'$, let $\underline{\mathbf{A}}$, be the matrix of R with respect to the bases $\{\mathbf{e}_i\}$ for V and $\{\mathbf{e}_{k'}\}$ for V' and $\underline{\mathbf{B}}$ the matrix of S with respect to the bases $\{\mathbf{f}_j\}$ for W and $\{\mathbf{f}_{l'}\}$ for W'.

Show that

$$\begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{B}} \end{bmatrix}$$

is the matrix of $R \oplus S$ with respect to $\{(\mathbf{e}_i, \mathbf{0}_W), (\mathbf{0}_V, \mathbf{f}_j)\}$ and $\{(\mathbf{e}'_k, \mathbf{0}_{W'}), (\mathbf{0}_{V'}, \mathbf{f}'_l)\}$

Exercise 12.2. Find the eigenvalues and eigenvectors of the following matrices:

(a) $$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -3 & 1 \\ 4 & -2 & -1 \end{bmatrix}$$

(b) $$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 12 & 1 & -3 \end{bmatrix}$$

Exercise 12.3. Find the real eigenvalues and eigenvectors of the following matrices.

(a) $$\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Exercise 12.4. Consider the real matrix

$$\underline{\mathbf{A}}_\varepsilon = \begin{bmatrix} 1+\varepsilon & 1 \\ 1 & 0 \end{bmatrix}.$$

Find its eigenvalues and eigenvectors as a function of $\varepsilon \geq 0$.

Exercise 12.5. Eigenvalues, eigenvectors and eigenspaces make sense in *any* vector space, not merely in finite dimensional vector spaces, and many problems can be formulated as eigenvalue problems. This exercise is devoted to examples of this.

Let $\mathcal{C}^\infty(\mathbb{R})$ denote the set of all smooth (that is, infinitely differentiable) real-valued functions defined on $\mathbb{R}$. Let

$$D\colon \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}), \quad f \longmapsto f'$$

be differentiation. In other words, $(D(f))(x) = f'(x)$ for all $f \in \mathcal{C}^\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

(a) Show that D is an endomorphism of the real vector space $\mathcal{C}^\infty(\mathbb{R})$. Find all of its eigenvalues and corresponding eigenvectors.

(b) Given $b \in \mathbb{R}$, show that

$$(D^2 + 2bD) : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}), \quad f \longmapsto f'' + 2bf'$$

defines an endomorphism. Find all of its eigenvalues and eigenvectors.

Exercise 12.6. Verify the Cayley-Hamilton Theorem for the following matrices.

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Exercise 12.7. Prove that the $n \times n$ matrix

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & \cdots & -b_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

has characteristic polynomial $(-1)^n(b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + t^n)$.