MATH101 ASSIGNMENT 9

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(1) We determine the following limits using the Bernoulli-de l'Hôpital Rule.

(a) Since $\ln x \to -\infty$ and $x - 1 \to -1$ as $x \to 0$, l'Hôpital's Rule cannot be applied. Consequently, the limit fails to exist as

$$\lim_{x\to 0} \frac{\ln x}{x-1} = \frac{-\infty}{-1} = \infty$$

(b) Since $(1+x)^p - 1 \to 0$ and $x \to 0$ as $x \to 0$, we have an indeterminate form $\frac{0}{0}$. If we let $f(x) = (1+x)^p - 1$ and $g(x) = x$, then f and g are both differentiable in $\mathbb{R}$ and $g'(x) \neq 0$, thus we can apply l'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0} \frac{(1+x)^p - 1}{x} &= \lim_{x\to 0} \frac{p(1+x)^{p-1}}{1} \\ &= p(1+0)^{p-1} = p\end{aligned}$$

(c) Let $y = \left(1 + \frac{4}{x}\right)^x$. Then

$$\begin{aligned}\ln y = \ln\left(1+\frac{4}{x}\right)^x &= x\ln\left(1+\frac{4}{x}\right) \\ &= \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}}\end{aligned}$$

Since $\ln\left(1+\frac{4}{x}\right) \to 0$ and $\frac{1}{x} \to 0$ as $x \to \infty$, we have indeterminate form $\frac{0}{0}$. As the conditions for applying l'Hôpital's Rule are met for $x > 0$, it follows that

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{4}{x}}\right)\left(-\frac{4}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{4}{1+\frac{4}{x}} = 4 \\ &= \lim_{x\to\infty} \ln y\end{aligned}$$

As the natural logarithm function is continuous,

$$\begin{aligned}\lim_{x\to\infty} \ln y = \ln\left(\lim_{x\to\infty} y\right) &= 4 \\ \Rightarrow \lim_{x\to\infty} y &= e^4 \\ \Rightarrow \lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x &= e^4\end{aligned}$$

(d) Since $\sin x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we have an indeterminate form $\frac{0}{0}$. If we let $j(x) = \sin x - x$ and $k(x) = x^3$, then j and k are both differentiable in $\mathbb{R}$ and $k'(x) \neq 0$ (except at $x = 0$). Thus we can apply l'Hôpital's Rule.

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$$

Since $\cos x - 1 \to 0$ and $3x^2 \to 0$ as $x \to 0$, we again have an indeterminate form and thus can apply l'Hôpital's Rule a second time.

$$\lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x}$$

This limit is indeterminate form $\frac{0}{0}$ once again so a third application of l'Hôpital's Rule is required.

$$\lim_{x \to 0} \frac{-\sin x}{6x} = \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

(2) For $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto -\frac{2x^2}{1+x^2}$
As f is differentiable everywhere and its domain has no endpoints, then

$$f'(x) = \frac{(1+x^2)(-4x) - (-2x^2)(2x)}{(1+x^2)^2} = \frac{-4x - 4x^3 + 4x^3}{(1+x^2)^2} = -\frac{4x}{(1+x^2)^2}$$

$$f''(x) = \frac{(1+x^2)^2(-4) - (-4x)(2)(1+x^2)(2x)}{(1+x^2)^4} = \frac{(1+x^2)(-4) + (8x)(2x)}{(1+x^2)^3}$$

$$= \frac{-4 - 4x^2 + 16x^2}{(1+x^2)^3} = \frac{12x^2 - 4}{(1+x^2)^3} = \frac{4(3x^2 - 1)}{(1+x^2)^3}$$

(a)

$$f'(x) = -\frac{4x}{(1+x^2)^2} = 0 \quad \text{if and only if} \quad x = 0$$

$$f''(0) = \frac{0 - 4}{(1+0)^3} = -\frac{4}{1} = -4 < 0$$

$$f(0) = -\frac{0}{1+0} = 0$$

Thus there is only one critical point at $x = 0$, which is an absolute maximum.

(b)

$$f'(x) = \begin{cases} < 0 & \text{for} \quad x > 0 \\ = 0 & \text{for} \quad x = 0 \\ > 0 & \text{for} \quad x < 0 \end{cases}$$

Thus f is increasing when $x < 0$ while it is decreasing when $x > 0$.

(c)

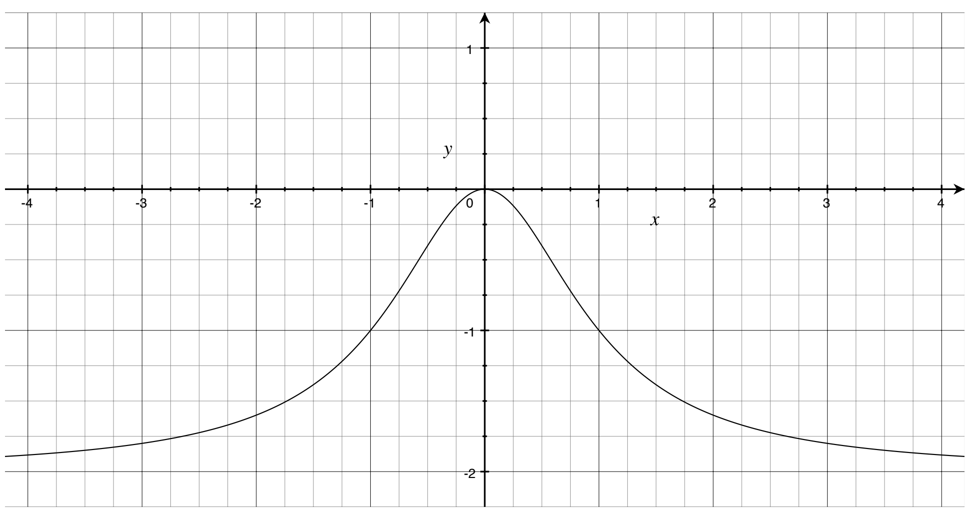
$$f''(x) = \begin{cases} < 0 & \text{for} \quad \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \\ = 0 & \text{for} \quad x = -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \\ > 0 & \text{for} \quad \left(-\infty, -\frac{\sqrt{3}}{3}\right) \quad \text{and} \quad \left(\frac{\sqrt{3}}{3}, \infty\right) \end{cases}$$

Thus f is concave down when $-\frac{\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3}$ while it is concave up when $x < -\frac{\sqrt{3}}{3}$ and $x > \frac{\sqrt{3}}{3}$. It has inflection points at $x = -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$ (which are approximately -0.58 and 0.58 respectively).

(d) Since $2x^2 \to \infty$ and $1 + x^2 \to \infty$ as $x \to \infty$, we have an indeterminate form $\frac{\infty}{\infty}$. As both functions are differentiable in $\mathbb{R}$, we apply l'Hôpital's Rule twice to evaluate the following limits,

$$\lim_{x\to\infty} -\frac{2x^2}{1+x^2} = -\lim_{x\to\infty} \frac{2x^2}{1+x^2} = -\lim_{x\to\infty} \frac{4x}{2x} = -\lim_{x\to\infty} \frac{4}{2} = -2$$
$$\lim_{x\to-\infty} -\frac{2x^2}{1+x^2} = -\lim_{x\to-\infty} \frac{2x^2}{1+x^2} = -\lim_{x\to-\infty} \frac{4x}{2x} = -\lim_{x\to-\infty} \frac{4}{2} = -2$$

Thus $f \to -2$ as $x \to \pm\infty$. Geometrically, f has an horizontal asymptote at $y = -2$. We sketch the graph of f below.



(3) For $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{-\frac{(x-\mu)^2}{\sigma^2}}$ with $\mu, \sigma \in \mathbb{R}$ and $\sigma > 0$
Since f is an exponential function it is differentiable everywhere. Moreover, its domain has no boundary points. If we define

$$u : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto -\frac{(x-\mu)^2}{\sigma^2}, \quad \text{then}$$

$$\begin{aligned}
f'(x) &= \frac{d}{du}\left(e^u\right)\cdot\frac{d}{dx}\left(u\right) = e^u\left(-\frac{2(x-\mu)}{\sigma^2}\right) = -\frac{2(x-\mu)\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^2}\\
&= -\frac{2(x-\mu)}{\sigma^2}f\\
f''(x) &= \frac{d}{dx}\left(\frac{-2(x-\mu)}{\sigma^2}\right)f + \left(\frac{-2(x-\mu)}{\sigma^2}\right)f' = -\frac{2}{\sigma^2}f - \frac{2(x-\mu)}{\sigma^2}f'\\
&= -\frac{2f}{\sigma^2} - \left(\frac{2(x-\mu)}{\sigma^2}\right)\left(-\frac{2(x-\mu)}{\sigma^2}f\right) = -\frac{2f}{\sigma^2} + \frac{4(x-\mu)^2}{\sigma^4}f\\
&= \frac{4(x-\mu)^2\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^4} - \frac{2\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^2}
\end{aligned}$$

(a)

$$\begin{aligned}
f'(x) &= -\frac{2(x-\mu)\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^2} \quad \text{if and only if} \quad x=\mu\\
f''(\mu) &= \frac{4(0)^2\,e^0}{\sigma^4} - \frac{2\,e^0}{\sigma^2} = -\frac{2}{\sigma^2} < 0\\
f(\mu) &= e^0 = 1
\end{aligned}$$

Thus there is only one critical point at $x=\mu$, which is an absolute maximum.

(b)

$$f'(x) = \begin{cases} <0 & \text{for} \quad x>\mu \\ =0 & \text{for} \quad x=\mu \\ >0 & \text{for} \quad x<\mu \end{cases}$$

Thus f is increasing when $x<\mu$ while it is decreasing when $x>\mu$.

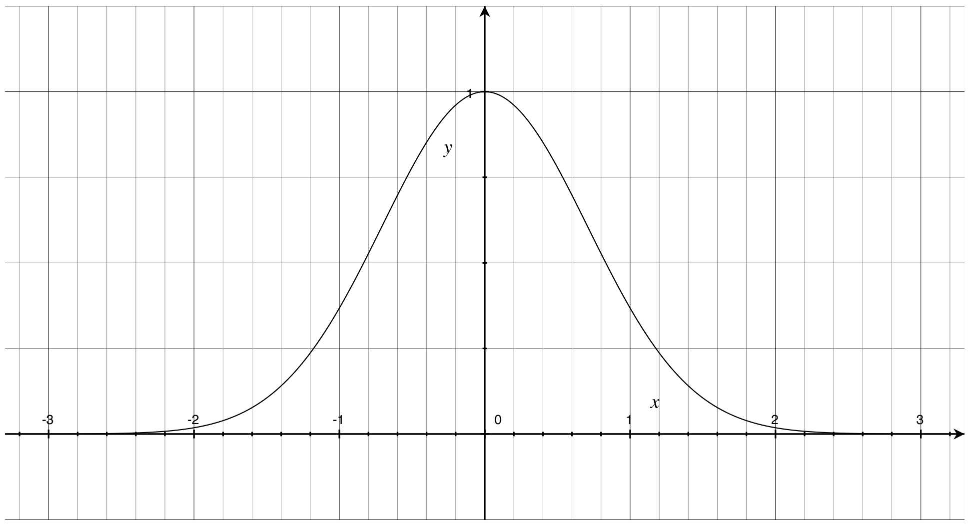
(c)

$$\begin{aligned}
\frac{4(x-\mu)^2\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^4} - \frac{2\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^2} &= 0\\
\frac{4(x-\mu)^2\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^4} &= \frac{2\,e^{-\frac{(x-\mu)^2}{\sigma^2}}}{\sigma^2}\\
\frac{2(x-\mu)^2}{\sigma^2} &= 1 \Rightarrow x = \mu \pm \frac{1}{\sqrt{2}}\,\sigma
\end{aligned}$$

$$f''(x) = \begin{cases} <0 & \text{for} \quad \left(\mu - \frac{1}{\sqrt{2}}\,\sigma,\ \mu + \frac{1}{\sqrt{2}}\,\sigma\right) \\ =0 & \text{for} \quad x = \mu - \frac{1}{\sqrt{2}}\,\sigma,\ \mu + \frac{1}{\sqrt{2}}\,\sigma \\ >0 & \text{for} \quad \left(-\infty,\ \mu - \frac{1}{\sqrt{2}}\,\sigma\right) \quad \text{and} \quad \left(\mu + \frac{1}{\sqrt{2}}\,\sigma,\ \infty\right) \end{cases}$$

Thus f is concave down when $\mu - \frac{1}{\sqrt{2}}\sigma < x < \mu + \frac{1}{\sqrt{2}}\sigma$ while it is concave up when $x < \mu - \frac{1}{\sqrt{2}}\sigma$ and $x > \mu + \frac{1}{\sqrt{2}}\sigma$.

(d) Since f is an exponential function $f(x) > 0$. Geometrically, f has an horizontal asymptote at $y = 0$. We sketch the graph of f for $\mu = 0$ and $\sigma = 1$.



Its inflection points occur at $x = -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (which are approximately -0.71 and 0.71 respectively).

(4) Let x be the length of one side of a square and r be the radius of a circle. Since a cube is made up of six identical square faces its surface area is $6x^2$. Meanwhile, the surface area of a sphere is given by $4\pi r^2$. If there is only enough silver to coat 1 square metre of surface area for both solids, we can write

$$A = 6x^2 + 4\pi r^2 \leq 1$$

Furthermore, the volume of the two solids can be expressed as

$$V = x^3 + \frac{4}{3}\pi r^3$$

Assuming we use the entire 1 square metre of silver, then

$$6x^2 + 4\pi r^2 = 1 \Rightarrow r = \pm\frac{1}{2}\sqrt{\frac{1-6x^2}{\pi}}$$

As $r \geq 0$ we can rule out the negative value of r. Moreover, since $1 - 6x^2 \geq 0$ then $0 \leq x \leq \frac{1}{\sqrt{6}}$. To determine the what values of x and r will maximise (minimise) V, we carry out the following algebraic steps.

$$V = x^3 + \frac{4}{3}\pi\left(\frac{1}{2}\sqrt{\frac{1-6x^2}{\pi}}\right)^3 = x^3 + \frac{(1-6x^2)^{\frac{3}{2}}}{6\sqrt{\pi}}$$

$$\begin{aligned}
V'(x) &= 3x^2 + \left(\frac{1}{6\sqrt{\pi}}\right)\left(\frac{3}{2}\right)\left(1-6x^2\right)^{\frac{1}{2}}(-12x) \\
&= 3x^2 - \frac{3x\sqrt{1-6x^2}}{\sqrt{\pi}} = 0 \text{ if and only if } x \in \left\{0,\ \pm\frac{1}{\sqrt{\pi+6}}\right\}
\end{aligned}$$

While it is clear to see that $V'(x) = 0$ when $x = 0$, the other roots required the following calculations.

$$\begin{aligned}
3x^2\sqrt{\pi} &= 3x\sqrt{1-6x^2} \\
x\sqrt{\pi} &= \sqrt{1-6x^2} \\
x^2\pi &= 1-6x^2 \\
1 &= (\pi+6)\,x^2 \\
x &= \pm\frac{1}{\sqrt{\pi+6}}
\end{aligned}$$

Since $0 \leq x \leq \frac{1}{\sqrt{6}}$ we can also rule out the negative value of x. We now perform the second derivative test on $x = 0, \frac{1}{\sqrt{\pi+6}}$.

$$\begin{aligned}
V''(x) &= 6x - \frac{3}{\sqrt{\pi}}\left(1-6x^2\right)^{\frac{1}{2}} - \frac{3x}{\sqrt{\pi}}\left(\frac{1}{2}\right)\left(1-6x^2\right)^{-\frac{1}{2}}(-12x) \\
&= 6x - \frac{3\sqrt{1-6x^2}}{\sqrt{\pi}} + \frac{18x^2}{\sqrt{\pi}\sqrt{1-6x^2}} \\
V''(0) &= 0 - \frac{3}{\sqrt{\pi}} + 0 = -\frac{3}{\sqrt{\pi}} < 0 \\
V''\left(\frac{1}{\sqrt{\pi+6}}\right) &= \frac{6}{\sqrt{\pi+6}} - \frac{3\sqrt{1-\frac{6}{\pi+6}}}{\sqrt{\pi}} + \frac{\frac{18}{\pi+6}}{\sqrt{\pi}\sqrt{1-\frac{6}{\pi+6}}} \\
&= \frac{6}{\sqrt{\pi+6}} - \frac{3\sqrt{\frac{\pi}{\pi+6}}}{\sqrt{\pi}} + \frac{18}{(\pi+6)\sqrt{\pi}\sqrt{\frac{\pi}{\pi+6}}} \\
&= \frac{6}{\sqrt{\pi+6}} - \frac{3}{\sqrt{\pi+6}} + \frac{18}{\pi\sqrt{\pi+6}} \\
&= \frac{3}{\sqrt{\pi+6}} + \frac{18}{\pi\sqrt{\pi+6}} = \frac{3\pi+18}{\sqrt{\pi}\sqrt{\pi+6}} > 0
\end{aligned}$$

Thus $x = 0$ is a relative maximum while $x = \frac{1}{\sqrt{\pi+6}}$ is a relative minimum. The dimensions required to maximise the total volume of the solids are therefore

$$x = 0, \quad r = \frac{1}{2}\sqrt{\frac{1-0}{\pi}} = \frac{1}{2\sqrt{\pi}}$$

Hence the maximum volume that the silvered sphere alone can hold is given by

$$V(0) = 0 + \frac{4}{3}\,\pi\left(\frac{1}{2\,\sqrt{\pi}}\right)^3 = \frac{1}{6\sqrt{\pi}}$$

To minimise the total volume the solids can hold, we can simply set the dimensions to $x = 0$ and $r = 0$ such that $V = 0$.

However, if we were considering only non-zero solutions (at least one solid is coated with silver) and as such $V > 0$, then the dimensions required for a minimum are

$$x = \frac{1}{\sqrt{\pi+6}}, \quad r = \frac{1}{2}\sqrt{\frac{\frac{\pi}{\pi+6}}{\pi}} = \frac{1}{2\,\sqrt{\pi+6}}$$

The volume given by these dimensions is given by

$$V\left(\frac{1}{\sqrt{\pi+6}}\right) = \left(\frac{1}{\sqrt{\pi+6}}\right)^3 + \frac{4}{3}\,\pi\left(\frac{1}{2\,\sqrt{\pi+6}}\right)^3 = \frac{1}{6\,\sqrt{\pi+6}}$$

This confirms our answer as $V(0) > V\left(\frac{1}{\sqrt{\pi+6}}\right)$.