Tutorial 1

Question 1.

Find all real numbers x, y, z satisfying the following system of equations.

$$\begin{array}{rcrcrcl} x & + & 7y & + & 4z & = & 21 \\ 3x & - & 6y & + & 5z & = & 2 \\ 5x & + & y & - & 3z & = & 14 \end{array}$$

Question 2.

For each of the following matrices, $\underline{\mathbf{A}}$, below, find $\underline{\mathbf{A}}^n \quad (n \in \mathbb{N} \setminus \{0\})$.

(i) $\underline{\mathbf{A}} := \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$.

(ii) $\underline{\mathbf{A}} := \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$.

(iii) $\underline{\mathbf{A}} := \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$.

Question 3.

Prove that if $\underline{\mathbf{A}} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then, for all $n \in \mathbb{N}$,

$$\underline{\mathbf{A}}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Question 4.

Let $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ both satisfy the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0 \qquad (*)$$

Show that for all real numbers λ, μ, the function

$$h\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \lambda f(x) + \mu g(x)$$

also satisfies $(*)$.

Tutorial 2

Question 1.

Show that $\mathbb{R}$ is a vector space over $\mathbb{Q}$ with respect to the usual addition and multiplication of real numbers.

Question 2.

Take $X = \{a, b, c\}$, with all elements distinct.
Define binary operations, $+$ and $\cdot$, on X by

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

and

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

Prove that X is a field with respect to $+$ and $\cdot$. This field is usually written as$\mathbb{F}_3$.

Question 3.

Decide whether the following are vector spaces.

(a) Take $\mathbb{F} := \mathbb{C}$ and $V := \mathbb{C}$.
Let $\boxplus$ to be the usual addition of complex numbers, and define $\boxdot$ by

$$\alpha \boxdot z := \alpha^2 z \qquad (\alpha, z \in \mathbb{C}).$$

(b) Let $\mathbb{F}$ be any field and take $V := \mathbb{F}^2$.
Let $\boxplus$ be the usual (component-wise) addition of ordered pairs, and define $\boxdot$ by

$$\alpha \boxdot (\beta, \gamma) := (\alpha\beta, 0) \qquad (\alpha, \beta, \gamma \in \mathbb{F}).$$

(c) Take $\mathbb{F} := \mathbb{F}_2 = \{0, 1\}$ with operations $+$ and $\cdot$ defined by

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Let $V := (\mathbb{F}_2)^2 = \{(0,0), (0,1), (1,0), (1,1)\}$.
Define $\boxdot \colon \mathbb{F}_2 \times V \longrightarrow V$ by

$$\alpha \boxdot (\beta, \gamma) := \begin{cases} (\alpha\beta, \alpha\gamma) & \text{if } \gamma \neq 0 \\ (\alpha^2 & \end{cases}$$

Define $\boxplus \colon V \times V \longrightarrow V$ by

$\boxplus$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(0,0)$

(d) Take $\mathbb{F} := \mathbb{C}$ and $V := \mathbb{C}$.
Let $\boxplus$ be the usual addition of complex numbers, and define $\boxdot$ by

$$\alpha \boxdot z := \mathbb{R}e(\alpha)z \qquad (\alpha, z \in \mathbb{C}),$$

where $\mathbb{R}e(\alpha)$ denotes the real part of the complex number α.

(e) Take $\mathbb{F} := \mathbb{R}$ and $V := \mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$.
Define $\boxplus$ and $\boxdot$ by

$$x \boxplus y := xy \qquad (x, y \in \mathbb{R}^+)$$
$$\alpha \boxdot x := x^\alpha \qquad (\alpha \in \mathbb{R}, x \in \mathbb{R}^+)$$

Question 4.

Let V be a vector space over the field $\mathbb{F}$.
Take $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{F}$.
Prove each of the following statements.

(i) If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
(ii) The equation $\mathbf{u} + \mathbf{x} = \mathbf{v}$ has a unique solution, $\mathbf{x}$.
(iii) $-(-\mathbf{u}) = \mathbf{u}$.
(iv) $0\mathbf{v} = \mathbf{0}_V$.
(v) $-(\alpha\mathbf{u}) = (-\alpha)\mathbf{u} = \alpha(-\mathbf{u})$.
(vi) $(-\alpha)(-\mathbf{u}) = \alpha\mathbf{u}$.
(vii) If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{v}$.

Tutorial 3

Unless otherwise specified, we regard $\mathbb{R}^n$ as a vector space over $\mathbb{R}$ with the vector space operations defined component-wise.

Question 1.

Determine $T \circ S$ and $S \circ T$ for

$$T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (u,v) \longmapsto (u+2v, 2u+5v)$$
$$S\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \longmapsto (x+y, x)$$

Question 2.

Find all linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ which maps the line with equation $u = v$ onto the line with equations $x = y = 0$.

Question 3.

Find, if possible, linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the conditions specified.

(i) $T(1,0,0) = (1,0,0),\ T(1,1,0) = (0,1,0),\ T(1,1,1) = (0,0,1)$

(ii) $T(1,2,3) = (1,0,0),\ T(3,1,2) = (0,0,1),\ T(2,3,1) = (0,1,0)$

(iii) $T(1,2,1) = (1,0,0),\ T(1,2,2) = (1,1,0),\ T(0,0,1) = (0,0,0)$

(iv) $T(1,0,0) = (1,2,3),\ T(0,2,2) = (6,1,0),\ T(1,0,1) = (2,0,1),\ T(5,2,5) = (14,5,9)$

Where there is no solution, explain why not.

Question 3.

Take a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T(1,0,0) = (1,0,0), \quad T(0,1,0) = (0,0,1) \quad \text{and} \quad T(0,0,1) = (1,0,1).$$

Find all solutions $(x,y,z) \in \mathbb{R}^3$, of

(i) $T(x,y,z) = (6,0,7)$

(ii) $T(x,y,z) = (6,1,7)$.

Tutorial 4

Question 1.

Given a set X, let $\mathcal{F})(X)$ be the set of all real valued functions defined on X. This forms a real vector space with respect to "point-wise" operations: Given $f, g \in \mathcal{F})(X)$ and $\lambda \in \mathbb{R}$, $f+g$ and $\lambda.f$ are defined by

$$\begin{aligned} f+g\colon X &\longrightarrow \mathbb{R}, \quad x \longmapsto f(x)+g(x) \\ \lambda.f\colon X &\longrightarrow \mathbb{R}, \quad x \longmapsto \lambda f(x) \end{aligned}$$

Decide which of the following subsets of $\mathcal{F})(\mathbb{R})$ are vector subspaces.

(a) $\{f \in \mathcal{F})(\mathbb{R}) \mid f(x) \leq 0 \text{ for all } x \in \mathbb{R}\}$
(b) $\{f \in \mathcal{F})(\mathbb{R}) \mid f(7) = 0\}$
(c) $\{f \in \mathcal{F})(\mathbb{R}) \mid f(1) = 2\}$
(d) $\{f \in \mathcal{F})(\mathbb{R}) \mid \text{there are } a, b \in \mathbb{R} \text{ with } f(x) = a + b\sin x \text{ for all } x \in \mathbb{R}\}$
(e) $\mathcal{D}^n(\mathbb{R}) := \{f \in \mathcal{F})(\mathbb{R}) \mid f \text{ is } n \text{ times differentiable}\} \quad (n \in \{1, 2, \ldots\})$
(f) $\mathcal{C}^n(\mathbb{R}) := \{f \in \mathcal{F})(\mathbb{R}) \mid f \text{ is } n \text{ times continuously differentiable}\} \quad (n \in \{1, 2, \ldots\})$

Question 2.

Let $\mathbb{R}[t]$ denote the set of all polynomials with real coefficients, so that

$$\mathbb{R}[t] := \{a_0 + a_1 t + \cdots + a_m t^m \mid m \in \mathbb{N} \ \text{ and } \ a_j \in \mathbb{R} \ \text{ for } \ 0 \leq j \leq m\}$$

This forms a real vector space with respect to the usual addition of polynomials and multiplication of a polynomial by a fixed real number.

Show that for each $n \in \mathbb{N}$,

$$\mathcal{P}_n := \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$$

forms a vector subspace of $\mathbb{R}[t]$.

Question 3.

$\mathbf{M}(2;\mathbb{R})$, the set of all 2×2 matrices with real coefficients, is a real vector space with respect to the usual operations on real matrices.

Determine which of the following subsets of $\mathbf{M}(2;\mathbb{R})$ form vector subspaces.

(a) $\mathbf{M}(2;\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$

(b) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}(2;\mathbb{R}) \mid \ a+b+c+d = 0 \right\}$

(c) $\{\underline{\mathbf{A}} \in \mathbf{M}(2;\mathbb{R}) \mid \det(\underline{\mathbf{A}}) = 0\}$

Tutorial 5

Question 1.

Determine which of $(2, 2, 2)$ and $(3, 1, 5) \in \mathbb{R}^3$ are linear combinations of $\mathbf{u} := (0, -1, 1)$ and $\mathbf{v} := (1, -1, 3)$.

Question 2.

Using the notation in Question 2 of Tutorial 4, decide which of the following sets of elements of $\mathcal{P}_2$ are linearly independent.

(a) $\{4t^2 - t + 2, 2t^2 + 6t + 3, -4t^2 + 10t + 2\}$

(b) $\{4t^2 - t + 2, 2t^2 + 6t + 3, 6t^2 + 5t + 5\}$

(c) $\{t^2 + t + 23, 5t^2 - t, +2\}$

(d) $\{3t^2 + 3t + 1, t^2 + 6t + 3, 5t^2 + t + 2, -t^2 + 2t + 7\}$

Question 3.

Let $\mathcal{F}(\mathbb{R})$ be the set of all real valued functions defined on $\mathbb{R}$. This is a real vector space with respect to point-wise defined addition of functions and multiplication of a function by a real constant. Decide whether $\{f, g, h\}$ is a linearly independent set of elements of $\mathcal{F}(\mathbb{R})$ when f, g and h are defined by

(a) $f(x) := \cos 2x, \quad g(x) := \sin x, \quad h(x) := 7 \qquad (x \in \mathbb{R})$

(b) $f(x) := \ln(x^2 + 1), \quad g(x) := \sin x, \quad h(x) := e^x \qquad (x \in \mathbb{R})$

Tutorial 6

Question 1.

This question investigates findng the matrix representation of a linear transformation $T\colon V \longrightarrow W$. To do so, we specify bases $\{\mathbf{e}_j\}$ for V and $\{\mathbf{f}_i\}$ for W.

(a) Take $V = W = \mathbb{R}^2$ and $T = id_{_{\mathbb{R}^2}}$, so that $T(x,y) = (x,y)$ for all $(x,y) \in \mathbb{R}^2$.

Find the matrix $\underline{\mathbf{A}}_T$ in each of the following cases.

(i) $\mathbf{e}_1 := (1,0),\ \mathbf{e}_2 := (0,1)$ and $\mathbf{f}_1 := (1,0),\ \mathbf{f}_2 := (0,1)$

(ii) $\mathbf{e}_1 := (1,0),\ \mathbf{e}_2 := (0,1)$ and $\mathbf{f}_1 := (0,1),\ \mathbf{f}_2 := (1,0)$

(iii) $\mathbf{e}_1 := (1,2),\ \mathbf{e}_2 := (3,4)$ and $\mathbf{f}_1 := (1,0),\ \mathbf{f}_2 := (0,1)$

(iv) $\mathbf{e}_1 := (1,0),\ \mathbf{e}_2 := (0,1)$ and $\mathbf{f}_1 := (1,2),\ \mathbf{f}_2 := (3,4)$

(v) $\mathbf{e}_1 := (3,4),\ \mathbf{e}_2 := (1,2)$ and $\mathbf{f}_1 := (1,2),\ \mathbf{f}_2 := (3,4)$

(b) Let $\mathcal{P}_n$ be the set of all real polynomials in the indeterminate t of degree at most n. The polynomial $p \in \mathbb{R}[t]$ induces the function

$$f_p\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto p(x)$$

This allows us to define the derivative of p, $D(p)$, to be the polynomial q that $f_q(x) = f_p'(x)$ for all $x \in \mathbb{R}$.

Prove that the function

$$D\colon \mathcal{P}_3 \longrightarrow \mathcal{P}_2, \quad p \mapsto D(p)$$

is a linear transformation and find its matrix with respect to each of the following bases.

(i) $\mathbf{e}_1 := 1,\ \mathbf{e}_2 := t,\ \mathbf{e}_3 := t^2,\ \mathbf{e}_4 := t^3$ and $\mathbf{f}_1 := 1,\ \mathbf{f}_2 := t,\ \mathbf{f}_3 := t^2$

(ii) $\mathbf{e}_1 := 1,\ \mathbf{e}_2 := t,\ \mathbf{e}_3 := t^2,\ \mathbf{e}_4 := t^3$ and $\mathbf{f}_1 := 6,\ \mathbf{f}_2 := 6t,\ \mathbf{f}_3 := 3t^2$

(iii) $\mathbf{e}_1 := 1,\ \mathbf{e}_2 := 1+t,\ \mathbf{e}_3 := 1+t^2,\ \mathbf{e}_4 := 1+t+t^2+t^3$ and $\mathbf{f}_1 := 1,\ \mathbf{f}_2 := 1+t,\ \mathbf{f}_3 := 1+t+t^2$

Question 2.

Prove that a linear transformation between finitely generated vector spaces is an isomorphism if and only if every matrix representing it is invertible.

Tutorial 7

Question 1.

Consider the system of linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Prove that there is a solution $(x_1, \dots, x_n) \in \mathbb{F}^n$ if and only if $(b_1, \dots, b_m) \in \mathbb{F}^m$ is an element of the vector subspace of $\mathbb{F}^m$ generated by

$$\{(a_{11}, \dots a_{m1}), \dots, (a_{1n}, \dots, a_{mn})\}.$$

When is the solution unique?

Question 2.

Let $T \colon V \longrightarrow W$ be a linear transformation.

Prove that if the matrix of T with respect to some choice of bases is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then T is neither injective nor surjective.

Question 3.

Show that $\mathcal{B} := \{(1,2), (3,4)\}$ and $\mathcal{B}' := \{(2,1), (4,3)\}$ are bases for $\mathbb{R}^2$.

Suppose that the matrix with respect to $\mathcal{B}$ of the linear transformation $T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

What is the matrix of T with respect to $\mathcal{B}'$?

Tutorial 8

Question 1.

Evaluate the determinant of each of the following matrices.

(i) $\begin{bmatrix} 1 & 6 & 4 & 7 \\ 4 & 5 & 0 & 8 \\ 6 & 2 & 1 & 9 \\ 7 & 3 & 5 & 6 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 9 & 15 \end{bmatrix}$

(iii) $\begin{bmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{bmatrix}$

(iv) $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Question 2.

(a) Find the determinant and trace of each of the following matrices.

(i) $\begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$

(b) Find the determinant of each of the following matrices.

(i) $\begin{bmatrix} 4-\lambda & -3 \\ 1 & -\lambda \end{bmatrix}$

(ii) $\begin{bmatrix} 4-\lambda & -4 \\ 1 & -\lambda \end{bmatrix}$

(iii) $\begin{bmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{bmatrix}$

(c) Compare the corresponding results in parts (a) and (b).

Question 3.

Decide which of the following sets of vectors form a basis for $\mathbb{R}^3$.

(i) $\{(1,2,3),(6,5,4),(31,20,9)\}$

(ii) $\{(1,2,3),(1,4,9),(1,8,27)\}$

(iii) $\{(1,2,3),(2,3,1),(3,1,2)\}$

Tutorial 9

Question 1.

Show that eigenvectors for different eigenvalues of the same endomorphism must be linearly independent.

Question 2.

Consider the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where

(a) $T(x, y, z) = (-2x + 4y + 4z, -2x + 4y + 2z, -x + y + 3z)$
(b) $T(x, y, z) = (-2x + 8y, -2x + 6y, -x + 2y + 2z)$
(c) $T(x, y, z) = (-3x + 9y + 2z, -3x + 7y + 2z, -x + 2y + 2z)$

For each of the above

(i) find the matrix of T with respect to the standard basis for $\mathbb{R}^3$;
(ii) find the eigenvalues of T;
(iii) find the eigenvectors of T for each eigenvalue;
(iv) find, if possible, a basis for $\mathbb{R}^3$ consisting of eigenvectors of T;
(v) find the matrix of T with respect to this new basis for $\mathbb{R}^3$.

Question 3.

Let $\mathcal{C}^\infty(\mathbb{R})$ be the set of all infinitely differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. It is a real vector space with respect to point-wise addition of functions and point-wise multiplication of functions by real numbers.

Show that each of the following mappings, $T\colon \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})$, is a linear transformation and find all eigenvalues of each T, as well as their corresponding eigenvectors.

(i) $T(f) := \frac{df}{dx}$
(ii) $T(f) := \frac{d^2f}{dx^2}$
(iii) $T(f) := \frac{d^2f}{dx^2} - 4\frac{df}{dx}$

Tutorial 10

Question 1.

Show that in each of the cases below, $\beta\colon V \times V \longrightarrow \mathbb{R}$ defines an inner product on the real vector space V.

(a)

$$V := \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

$$\beta(f,g) := t\int_0^1 f(t)g(t)dt$$

(b)

$$V := \mathbb{R}_{(2)} = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \}$$

$$\beta(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}) := \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c)

$$V := \mathcal{P}_2,$$

$$\beta(p,q) := p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

(d)

$$V := \mathbf{M}(m \times n; \mathbb{R}),$$

$$\beta(\underline{\mathbf{A}}, \underline{\mathbf{B}}) := \operatorname{tr}(\underline{\mathbf{A}}^t \underline{\mathbf{B}})$$

Question 2.

Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the real vector space V.

Prove that if $\mathbf{u}, \mathbf{v} \neq \mathbf{0}_V$ and $\langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle = 0$ then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

Question 3.

Let β be a bilinear form on the finitely generated real vector space V.

Let $T\colon V \longrightarrow V$ be a linear transformation.

Show that

$$\gamma\colon V \times V \longrightarrow \mathbb{R}, \quad (\mathbf{x}, \mathbf{y}) \longmapsto \beta\left(T(\mathbf{x}), T(\mathbf{y})\right)$$

is also a bilinear form on V.

Choose a fixed basis for V.

Show that if $\underline{\mathbf{A}}$ is the matrix of β, and $\underline{\mathbf{B}}$ that of T, then $\underline{\mathbf{B}}^t \underline{\mathbf{A}}\, \underline{\mathbf{B}}$ is the matrix of γ.

Question 4. Classify each of the following real quadratic forms according to definiteness properties:

(a) $q(x,y) := x^2 + 4xy + 5y^2$

(b) $q(x,y,z) := 4x^2 - 4xy + 5y^2 - 2yz + 3z^2 - 4zx$

(c) $q(x,y,z) = 2x^2 + 3y^2 + 2z^2 + 6xy + 6yz + 4zx$

Question 5.

Let $\langle\!\langle\ ,\ \rangle\!\rangle$ be an inner product on the real vector space V.
Let $\mathbf{u} \in V$ be a fixed non-zero vector in V.
Let ℓ be the line determined by $\mathbf{u}$, so that $\ell = \{\lambda\mathbf{u} \mid \lambda \in \mathbb{R}\}$.
Show that if $T\colon V \to V$ is reflection in ℓ, then

$$T(\mathbf{x}) = \frac{2\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle}{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\mathbf{u} - \mathbf{x}.$$

Tutorial 11

Question 1.

Find the matrix of the inner product

$$\langle\!\langle\ ,\ \rangle\!\rangle : \mathcal{P}_2 \times \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad (p,q) \longmapsto p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

with respect to the basis $\{1, t, t^2\}$ of $\mathcal{P}_2$.

Do the same for

$$\langle\ \mid\ \rangle : \mathcal{P}_2 \times \mathcal{P}_2 \longrightarrow \mathbb{R}, \quad (p,q) \longmapsto \int_{-1}^{1} p(x)q(x)dx.$$

Question 2.

The matrix $\underline{\mathbf{B}} = [b_{ij}]_{n\times n} \in \mathbf{M}(n;\mathbb{R})$ is *orthogonal* if and only if $\underline{\mathbf{B}}^t\underline{\mathbf{B}} = \underline{\mathbf{1}}_n$ and it is *upper triangular* if and only if $b_{ij} = 0$ whenever $i > j$.

Prove that if $\underline{\mathbf{A}}$ is an invertible real $n \times n$ matrix, then there are an orthogonal matrix $\underline{\mathbf{Q}}$ and an upper triangular matrix $\underline{\mathbf{R}}$ such that

$$\underline{\mathbf{A}} = \underline{\mathbf{Q}}\,\underline{\mathbf{R}}.$$

Find an orthogonal matrix, $\underline{\mathbf{Q}}$, and an upper triangular matrix, $\underline{\mathbf{R}}$, such that

$$\underline{\mathbf{Q}}\,\underline{\mathbf{R}} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}.$$

Question 3.

Take the real vector space $V := \{\varphi : [0, 2\pi] \longrightarrow \mathbb{R} \mid \varphi \text{ is continuous }\}$.

Show that

$$\langle\!\langle \varphi, \psi \rangle\!\rangle := \frac{1}{\pi}\int_0^{2\pi} \varphi(x)\psi(x)dx$$

is an inner product on V.

For $n \in \mathbb{N} \setminus \{0\}$, define

$$\varphi_n(x) := \cos(nx)$$
$$\psi_n(x) := \sin(nx)$$

Show that $\{\varphi_n, \psi_n \mid n = 1, 2, \ldots\}$ is a family of orthonormal elements of $(V, \langle\!\langle\ ,\ \rangle\!\rangle)$.