# Sample Solutions for Tutorial 3

**Question 1.**

$(S \circ T)(u,v) = S(T(u,v)) = S(u+2v, 2u+5v) = (3u+7v, u+2v)$
$(T \circ S)(x,y) = T(S(x,y)) = S(x+y, x) = (3x+y, 7x+2y)$
Thus, $S \circ T \neq T \circ S$. In particular, $(S \circ T)(1,0) = (3,1)$, but $(T \circ S)(1,0) = (3,7)$.

**Question 2.**

Recall that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation if and only if there are $a,b,c,d,e,f \in \mathbb{R}$ such that for all $(u,v) \in \mathbb{R}^2$

$$T(u,v) = (au + bv, cu + dv, eu + fv).$$

Now $(u,v)$ is on the given line in $\mathbb{R}^2$ if and only if $u = v$, and $(x,y,z)$ is on the given line in $\mathbb{R}^3$ if and only if $x = y = 0$.

Thus $T$ maps the given line in $\mathbb{R}^2$ to the given line in $\mathbb{R}^3$ if and only if $au + bu = cu + du =$ for all $u \in \mathbb{R}$, which is the case if and only if $b = -a$ and $d = -c$.

In order for the given line in $\mathbb{R}^2$ to be mapped *onto* the given line in $\mathbb{R}^3$, we must be able to solve $(e+f)u = z$ for all $z \in \mathbb{R}$, which is the case if and only if $e + f \neq 0$.

Thus we must have

$$T(u,v) = (a(u-v), c(u-v), eu + fv),$$

with $a, c, e, f \in \mathbb{R},\ f \neq -e$.

Two concrete examples:

| | | |
|---|---|---|
| (i) | $T(u,v) := (0,0,v)$ | $(a = c = e = 0, f = 1)$ |
| (ii) | $T(u,v) := (u-v, u-v, u+v)$ | $(a = c = e = f = 1)$ |

**Question 3.**

In order for $T$ to be a linear transformation, we must have

$$T(x,y,z) = (ax + by + cz, dx + ey + fz, gx + hy + iz)$$

for suitable $a,b,c,d,e,f,g,h,i \in \mathbb{R}$.

(i)

($\alpha$) $T(1,0,0) = (1,0,0) \iff a = 1, d = 0$ and $g = 0$.
($\beta$) $T(1,1,0) = (0,1,0) \iff a + b = 0, d + e = 1$ and $g + h = 0$. Using ($\alpha$), we see that $b = -1, e = 1$ and $h = 0$.
($\gamma$) $T(1,1,1) = (0,0,1) \iff a + b + c = 0, d + e + f = 0$ and $g + h + i = 1$. Using ($\alpha$) and ($\beta$), we see that $c = 0, f = -1$ and $i = 1$.

Thus $\quad T(x,y,z) = (x - y, y - z, z) \quad$ and observe that $T$ is uniquely determined by the given conditions.

(ii)

$$(\alpha)\ T(1,2,3) = (1,0,0) \iff \begin{cases} a + 2b + 3c = 1 & (\alpha 1) \\ d + 2e + 3f = 0 & (\alpha 2) \\ g + 2h + 3i = 0 & (\alpha 3) \end{cases}$$

$$(\beta)\ T(3,1,2) = (0,0,1) \iff \begin{cases} 3a + b + 2c = 0 & (\beta 1) \\ 3d + e + 2f = 0 & (\beta 2) \\ 3g + h + 2i = 1 & (\beta 3) \end{cases}$$

$$(\gamma)\ T(2,3,1) = (1,0,0) \iff \begin{cases} 2a + 3b + c = 0 & (\gamma 1) \\ 2d + 3e + f = 1 & (\gamma 2) \\ 2g + 3h + i = 0 & (\gamma 3) \end{cases}$$

Of course, we could attempt to solve these nine equations in nine unknowns directly, arriving either at a contradiction or finding solutions.

Your familiarity with matrix calculus from earlier courses provides an alternative. We encode the problem as solving the matrix equation

$$\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{bmatrix}\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

Notice that this is equivalent to

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix}\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

so that

$$\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix}^{-1}.$$

Using the Gauss-Jordan algorithm, we obtain

$$\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 2 & 3 & 1 & 0 & 1 & 0\\ 3 & 1 & 2 & 0 & 0 & 1\end{array} \quad \rightsquigarrow \quad \begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & -1 & -5 & -2 & 1 & 0\\ 0 & -5 & -7 & -3 & 0 & 1\end{array}$$

$$\rightsquigarrow \quad \begin{array}{ccc|ccc}1 & 0 & -7 & 1 & 2 & 0\\ 0 & 1 & 5 & 2 & -1 & 0\\ 0 & 0 & 18 & 7 & -5 & 1\end{array}$$

$$\rightsquigarrow \quad \begin{array}{ccc|ccc}1 & 0 & 0 & \frac{-5}{18} & \frac{1}{18} & \frac{7}{18}\\ 0 & 1 & 0 & \frac{1}{18} & \frac{7}{18} & \frac{-5}{18}\\ 0 & 0 & 1 & \frac{7}{18} & \frac{-5}{18} & \frac{1}{18}\end{array}$$

Thus

$$\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix} = \frac{1}{18}\begin{bmatrix}-5 & 1 & 7\\ 1 & 7 & -5\\ 7 & -5 & 1\end{bmatrix},$$

or

$$T(x, y, z) = \left(\frac{-5x + y + 7z}{18}, \frac{x + 7y - 5z}{18}, \frac{7x - 5y + z}{18}\right)$$

(iii) We represent the system of equations directly in matrix form.

$$\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

It is easy to see that $\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}$ is not an invertible matrix, for its *determinant* is 0, and, as you know from earlier courses, the determinant's being 0 is the single obstruction to a real matrix's having an inverse. (You'll see why these things are true later in this course!)

Thus, we cannot simply invert the matrix, and use the inverse to find a solution. Of course this does *not* mean that there is no solution, just that a particular technique is not available.

Note that the middle row of the left-hand matrix is the sum of its two other rows, but this is not true of the right-hand matrix. This means that there can be no solution, since the rows of a product of matrices consist of linear combinations of the left-most matrix, and these reserve linear relations.

If you find this explanation "too abstract", equating the middle column of the product on the left with the middle column of the matrix on the right yields

$$\begin{bmatrix} d+2e+f \\ d+2e+2f \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

so that $f = 1$ because of the first two rows, contradicting the third row. Thus, there is no solution.

(iv) Here the matrix representation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 1 & 0 \\ 2 & 0 & 1 \\ 4 & 5 & 9 \end{bmatrix}$$

There is, of course, no question of inverting the left-hand matrix, as it is not a "square matrix".

But note that the fourth row of the left-hand matrix is the sum of the first and third, and that this is not true of the right-hand matrix. This means that taking the first columns we obtain

$$\begin{bmatrix} a \\ 2b+2c \\ a+c \\ 2a+c \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \\ 4 \end{bmatrix}.$$

From the third and fourth rows we conclude that $a = 2$, which contradicts the first row. Hence there is no solution.

COMMENT. This exercise was intended to allow you to just compute, if you felt like it. It also illustrates the usefulness of the matrix techniques you've learnt before. Strictly speaking, you should not use these at this stage, since they have not been introduced yet. Moreover, you have not met rigorous general proofs of them. This shall be corrected later in this course.

## Question 3.

Consider the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfying

$$T(1,0,0) = (1,0,0), T(0,1,0) = (0,0,1) \quad \text{and} \quad T(0,0,1) = (1,0,1).$$

Then

$$\begin{aligned} T(x,y,z) &= xT(1,0,0) + yT(0,1,0) + zT(0,0,1) \\ &= x(1,0,0) + y(0,0,1) + z(1,0,1) \\ &= (x+z, 0, y+z). \end{aligned}$$

(i) $T(x,y,z) = (6,0,7)$ if and only if $x+z = 6, 0 = 0$ and $y+z = 7$, so that the general solution is clearly given by

$$(x,y,z) = (6+\lambda, 7+\lambda, -\lambda) = (6,7,0) + \lambda(1,1,-1) \qquad (\lambda \in \mathbb{R}).$$

(ii) $T(x,y,z) = (6,1,7)$ if and only if $x+z = 6, 0 = 1$ and $y+z = 7$, so that there is clearly no solution.