Sample Solutions for Tutorial 10

Question 1.

(a) Since every continuous function $[0,1] \to \mathbb{R}$ is integrable, $\beta : V \times V \to \mathbb{R}$ is a function. Take $f, g, h \in V$ and $\lambda, \mu \in \mathbb{R}$.

Since f is continuous, so is f^2. Moreover, $f^2(x) \geq 0$ for all $x \in [0,1]$, whence

$$\beta(f,f) := \int_0^1 (f(t))^2 dt \geq \int_0^1 0\, dt = 0.$$

Now suppose that $\beta(f,f) = 0$, that is $\int_0^1 (f(t))^2\, dt = 0$. Since the integrand is continuous and nowhere negative, $f(t) = 0$ for all $t \in [0,1]$, that is, $f = \mathbf{0}_V$.

This establishes (IP1), since $\beta(\mathbf{0}_V, \mathbf{0}_V) = \int_0^1 0\, dt = 0$.

$$\begin{aligned}
\beta(\lambda f + \mu g, h) &:= \int_0^1 (\lambda f(t) + \mu g(t))\, h(t)\, dt \\
&:= \lambda \int_0^1 f(t)h(t)\, dt + \mu \int_0^1 g(t)h(t)\, dt \qquad \text{(by properties of integration)} \\
&= \lambda\beta(f,h) + \mu\beta(g,h),
\end{aligned}$$

establishing (IP2) and (IP3)

We also have that

$$\beta(g,f) := \int_0^1 g(t)f(t)\, dt = \int_0^1 f(t)g(t)\, dt = \beta(f,g),$$

establishing (IP4), as V is a real vector space.

(b) Plainly $\beta : V \times V \to \mathbb{R}$ is a function.

Take $\begin{bmatrix} r \\ s \end{bmatrix}, \begin{bmatrix} t \\ u \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}_{(2)}$ and $\lambda, \mu \in \mathbb{R}$.

$\beta(\begin{bmatrix} r \\ s \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}) = \begin{bmatrix} r & s \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = r^2 + 4rs + 5s^2 = (r+2s)^2 + s^2 \geq 0$ for all $r, s \in \mathbb{R}$. Moreover, $(r+2s)^2 + s^2 = 0$ if and only if $r = s = 0$, establishing (IP1).

$$\begin{aligned}
\beta\left(\lambda \begin{bmatrix} r \\ s \end{bmatrix} + \mu \begin{bmatrix} t \\ u \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ w \end{bmatrix}\right) &= \left(\lambda\begin{bmatrix} r & s \end{bmatrix} + \mu\begin{bmatrix} t & u \end{bmatrix}\right) \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \\
&= \lambda \left(\begin{bmatrix} r & s \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}\right) + \mu \left(\begin{bmatrix} t & u \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}\right) \\
&= \lambda\beta\left(\begin{bmatrix} r \\ s \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ w \end{bmatrix}\right) + \mu\beta\left(\begin{bmatrix} t \\ u \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ w \end{bmatrix}\right),
\end{aligned}$$

establishing (IP2) and (IP3).

$$
\begin{aligned}
\beta\left(\begin{bmatrix} r \\ s \end{bmatrix}, \begin{bmatrix} t \\ u \end{bmatrix}\right) &:= \begin{bmatrix} r & s \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} \\
&= \begin{bmatrix} r \\ s \end{bmatrix}^t \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^t \begin{bmatrix} t \\ u \end{bmatrix} \\
&= \left(\begin{bmatrix} t \\ u \end{bmatrix}^t \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right)^t \\
&= \begin{bmatrix} t & u \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \quad \text{as this is just a } 1 \times 1 \text{ matrix} \\
&= \beta\left(\begin{bmatrix} t \\ u \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}\right),
\end{aligned}
$$

establishing (IP4), as V is a real vector space.

(c) It is plain that $\beta : \mathcal{P}_2 \times \mathcal{P}_2 \to \mathbb{R}$ is a function.

Take $p, q, r \in \mathcal{P}_2$ and $\lambda, \mu \in \mathbb{R}$. Suppose that $p = a + bt + ct^2$. Then

$$p(-1) = a - b + c, \qquad p(0) = a \qquad p(1) = a + b + c.$$

Thus

$$\beta(p, p) := (a - b + c)^2 + a^2 + (a + b + c)^2 \geq 0,$$

being the sum of squares of real numbers. Moreover, since

$$\beta(p, p) = 3a^2 + 2b^2 + 2c^2 + 4ac = 3(a + \frac{2}{3}c)^2 + 2b^2 + \frac{2}{3}c^2,$$

$\beta(p, p) = 0$ if and only if $a + \frac{2}{3}c = b = c = 0$, that is, $a = b = c = 0$, or $p = 0$, which verifies (IP1).

$$
\begin{aligned}
\beta(\lambda p + \mu q, r) &:= (\lambda p(-1) + \mu q(-1))\, r(-1) + (\lambda p(0) + \mu q(0))\, r(0) + (\lambda p(1) + \mu q(1))\, r(1) \\
&= \lambda\, (p(-1)r(-1) + p(0)r(0) + p(1)r(1)) + \mu\, (q(-1)r(-1) + q(0)r(0) + q(1)r(1)) \\
&= \lambda \beta(p.r) + \mu \beta(q, r),
\end{aligned}
$$

verifying (IP2) and (IP3).

$$\beta(q, p) = q(-1)p(-1) + q(0)p(0) + q(1)p(1) = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = \beta(p, q),$$

verifying (IP4) as $\mathcal{P}_2$ is a real vector space.

(d) It is plain that

$$\beta : \mathbf{M}(m \times n; \mathbb{R}) \times \mathbf{M}(m \times n; \mathbb{R}) \longrightarrow \mathbb{R}, \quad (\underline{\mathbf{A}}, \underline{\mathbf{B}}) \longmapsto \operatorname{tr}(\underline{\mathbf{A}}^t \underline{\mathbf{B}})$$

is a well defined function.

Take $\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{C}} \in \mathbf{M}(m \times n; \mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$.

Suppose that $\underline{\mathbf{A}} = [a_{ij}]_{m \times n}, \underline{\mathbf{B}} = [b_{ij}]_{m \times n} \in \mathbf{M}(m \times n; \mathbb{R})$. Then $\underline{\mathbf{A}}^t \underline{\mathbf{B}} = [c_{ij}]_{n \times n} \in \mathbf{M}(n; \mathbb{R})$, where

$$c_{ij} = \sum_{k=1}^{m} a_{ki} b_{kj}.$$

Then

$$\beta(\underline{\mathbf{A}}, \underline{\mathbf{A}}) := \operatorname{tr}(\underline{\mathbf{A}}^t \underline{\mathbf{A}}) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}^2 \right) \geq 0$$

being the sum of squares of real numbers. Moreover equality holds if and only if each $a_{ij}^2 = 0$, that is, each $a_{ij} = 0$, or $\underline{\mathbf{A}} = \underline{\mathbf{0}}_{m \times n}$. This establishes (IP1).

From the properties of matrix operations and the trace functions,

$$\begin{aligned}\beta(\lambda\underline{\mathbf{A}}+\mu\underline{\mathbf{B}},\underline{\mathbf{C}}) &:= \operatorname{tr}\Big((\lambda\underline{\mathbf{A}}+\mu\underline{\mathbf{B}})^t\underline{\mathbf{C}}\Big)\\ &= \operatorname{tr}(\lambda\underline{\mathbf{A}}^t\underline{\mathbf{C}}+\mu\underline{\mathbf{B}}^t\underline{\mathbf{C}})\\ &= \lambda\operatorname{tr}(\underline{\mathbf{A}}^t\underline{\mathbf{C}})+\mu\operatorname{tr}(\underline{\mathbf{B}}^t\underline{\mathbf{C}})\\ &= \lambda\beta(\underline{\mathbf{A}},\underline{\mathbf{C}})+\mu\beta(\underline{\mathbf{B}},\underline{\mathbf{C}}),\end{aligned}$$

establishing (IP2) and (IP3).

The properties of transposition and the trace function show that

$$\beta(\underline{\mathbf{B}},\underline{\mathbf{A}}) = \operatorname{tr}(\underline{\mathbf{B}}^t\underline{\mathbf{A}}) = \operatorname{tr}\big((\underline{\mathbf{A}}^t\underline{\mathbf{B}})^t\big) = \operatorname{tr}(\underline{\mathbf{A}}^t\underline{\mathbf{B}}) = \beta(\underline{\mathbf{A}},\underline{\mathbf{B}}),$$

establishing (IP4).

Question 2.

Take $\mathbf{u},\mathbf{v}\in V\setminus\{\mathbf{0}_V\}$ such that $\langle\!\langle\mathbf{u},\mathbf{v}\rangle\!\rangle = 0$. Then

$$\langle\!\langle\lambda\mathbf{u}+\mu\mathbf{v},\mathbf{u}\rangle\!\rangle = \lambda\langle\!\langle\mathbf{u},\mathbf{u}\rangle\!\rangle+\mu\langle\!\langle\mathbf{v},\mathbf{u}\rangle\!\rangle = \lambda\langle\!\langle\mathbf{u},\mathbf{u}\rangle\!\rangle,$$

so that if $\lambda\mathbf{u}+\mu\mathbf{v}=\mathbf{0}_V$, then $\lambda\langle\!\langle\mathbf{u},\mathbf{u}\rangle\!\rangle = \langle\!\langle\mathbf{0}_V,\mathbf{u}\rangle\!\rangle = 0$, whence $\lambda = 0$ as $\mathbf{u}\neq\mathbf{0}_V$. It follows that $\mu = 0$, since $\mathbf{v}\neq\mathbf{0}_V$.

Question 3.

Let $\beta : V\times V\to\mathbb{R}$ be a bowline form on V and $T : V\to V$ a linear transformation. Define

$$\gamma : V\times V\longrightarrow\mathbb{R},\quad (\mathbf{u},\mathbf{v})\longmapsto\beta\Big(T(\mathbf{u}),T(\mathbf{v})\Big).$$

Then, by the linearity of T and the bi-linearity of β,

$$\begin{aligned}\gamma(\lambda_1\mathbf{u}_1+\lambda_2\mathbf{u}_2,\mu_1\mathbf{v}_1+\mu_2\mathbf{v}_2) &:= \beta\Big(T(\lambda_1\mathbf{u}_1+\lambda_2\mathbf{u}_2),T(\mu_1\mathbf{v}_1+\mu_2\mathbf{v}_2)\Big)\\ &= \beta\Big(\lambda_1T(\mathbf{u}_1)+\lambda_2T(\mathbf{u}_2),\mu_1T(\mathbf{v}_1)+\mu_2T(\mathbf{v}_2)\Big)\\ &= \lambda_1\mu_1\beta\Big(T(\mathbf{u}_1),T(\mathbf{v}_1)\Big)+\lambda_1\mu_2\beta\Big(T(\mathbf{u}_1),T(\mathbf{v}_2)\Big)\\ &\qquad+\lambda_2\mu_1\beta\Big(T(\mathbf{u}_2),T(\mathbf{v}_1)\Big)+\lambda_2\mu_2\beta\Big(T(\mathbf{u}_2),T(\mathbf{v}_2)\Big)\\ &= \lambda_1\mu_1\gamma(\mathbf{u}_1,\mathbf{v}_1)+\lambda_1\mu_2\gamma(\mathbf{u}_1,\mathbf{v}_2)+\lambda_2\mu_1\gamma(\mathbf{u}_2,\mathbf{v}_1)+\lambda_2\mu_2\gamma(\mathbf{u}_2,\mathbf{v}_2),\end{aligned}$$

showing that γ is a linear form on v.

Now let $\{\mathbf{e}_1,\dots\mathbf{e}_n\}$ be a basis for V. Let

$$\mathbf{u}=\sum_{i=1}^n x_i\mathbf{e}_i\qquad\text{and}\qquad\mathbf{v}=\sum_{j=1}^n y_j\mathbf{e}_j.$$

Then

$$\beta(\mathbf{u},\mathbf{v}) = \beta(\sum_{i=1}^n x_i\mathbf{e}_i,\sum_{j=1}^n y_j\mathbf{e}_j) = \sum_{i=1}^n\sum_{i}^{j} x_iy_j\beta(\mathbf{e}_i,\mathbf{e}_j).$$

Then $\underline{\mathbf{A}} := [\beta(\mathbf{e}_i,\mathbf{e}_j)]_{n\times n}$ is the matrix of β with respect to the chosen matrix.

If we write $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ for the co-ordinate vectors of $\mathbf{u}$ and $\mathbf{v}$ (respectively) with respect to the given basis, then

$$\beta(\mathbf{u},\mathbf{v}) = \underline{\mathbf{x}}^t\underline{\mathbf{A}}\,\underline{\mathbf{y}}.$$

Let $\underline{\mathbf{B}}$ be the matrix of T with respect to the chosen basis. Then, if $\underline{\mathbf{C}}$ is the matrix of γ, we have

$$\underline{\mathbf{x}}^t\underline{\mathbf{C}}\,\underline{\mathbf{y}} = \gamma(\mathbf{u},\mathbf{v}) := \beta(T(\mathbf{u}),T(\mathbf{v})) = (\underline{\mathbf{B}}\,\underline{\mathbf{x}})^t\underline{\mathbf{A}}(\underline{\mathbf{B}}\,\underline{\mathbf{y}}) = \underline{\mathbf{x}}^t(\underline{\mathbf{B}}^t\underline{\mathbf{A}}\,\underline{\mathbf{B}})\underline{\mathbf{y}}.$$

Since this holds for all $\underline{\mathbf{x}},\underline{\mathbf{y}}\in\mathbb{R}_{(n)}$, we must have

$$\underline{\mathbf{C}} = \underline{\mathbf{B}}^t\underline{\mathbf{A}}\,\underline{\mathbf{B}}.$$

Question 4.

(a)

$$\begin{aligned} q(x,y) &= x^2 + 4xy + 5y^2 \\ &= (x+2y)^2 + y^2 \\ &\geq 0 \qquad\qquad\qquad\qquad \text{for all } x, y \in \mathbb{R} \end{aligned}$$

Moreover, $q(x,y) = 0$ if and only if $x + 2y = y = 0$, which is the case if and only if $x = y = 0$.

Hence this quadratic form is positive definite.

(b)

$$\begin{aligned} q(x,y,z) &= 4x^2 - 4xy + 5y^2 - 2yz + 3z^2 - 4zx \\ &= (2x - y - z)^2 + 4y^2 - 2yz + 2z^2 \\ &= (2x - y - z)^2 + (2y - \frac{1}{2}z)^2 + \frac{7}{4}z^2 \\ &\geq 0 \qquad\qquad\qquad\qquad \text{for all } x, y \in \mathbb{R} \end{aligned}$$

Moreover, $q(x,y,z) = 0$ if and only if $2x - y - z = 4y - zy = z = 0$, which is the case if and only if $x = y = z = 0$.

Hence this quadratic form is positive definite.

(c)

$$\begin{aligned} q(x,y,z) &= 2x^2 + 6xy + 4xz + 3y^2 + 6yz + 2z^2 \\ &= 2(x + \frac{3}{2}y + z)^2 - \frac{3}{2}y^2. \end{aligned}$$

Since $q(3,-2,0) = -6$ and $q(0,0,1) = 2$, this quadratic form is indefinite.

Question 5.

Take $\mathbf{x} \in V$. Since T is reflection in ℓ, the midpoint of the line segment joining $\mathbf{x}$ and $T(x)$ must lie on ℓ. But this midpoint, $\mathbf{v}$, is

$$\frac{T(\mathbf{x}) + \mathbf{x}}{2}.$$

So we must have $\mathbf{v} = \lambda \mathbf{u}$ for some $\lambda \in \mathbb{R}$. But $\mathbf{v}$ is the orthogonal projection of $\mathbf{x}$ onto ℓ. Thus

$$\lambda = \frac{\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle}{\|\mathbf{u}\|^2} = \frac{\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle}{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}$$

Hence

$$\frac{T(\mathbf{x}) + \mathbf{x}}{2} = \frac{\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle}{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\mathbf{u}, \qquad \text{or} \qquad T(\mathbf{x}) = \frac{2\langle\!\langle \mathbf{u}, \mathbf{x} \rangle\!\rangle}{\langle\!\langle \mathbf{u}, \mathbf{u} \rangle\!\rangle}\mathbf{u} - \mathbf{x}.$$