# PMTH212 ASSIGNMENT 7

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(1) $f(x, y) = x^2 + xy - 2y - 2x + 1$

$$f_x(x, y) = 2x + y - 2 = 0 \;\Rightarrow\; y = 2 - 2x$$
$$f_y(x, y) = x - 2 = 0 \;\Rightarrow\; x = 2$$

We obtain only one solution when $x = 2,\; y = -2$. Hence there is a critical point at $(2, -2)$.

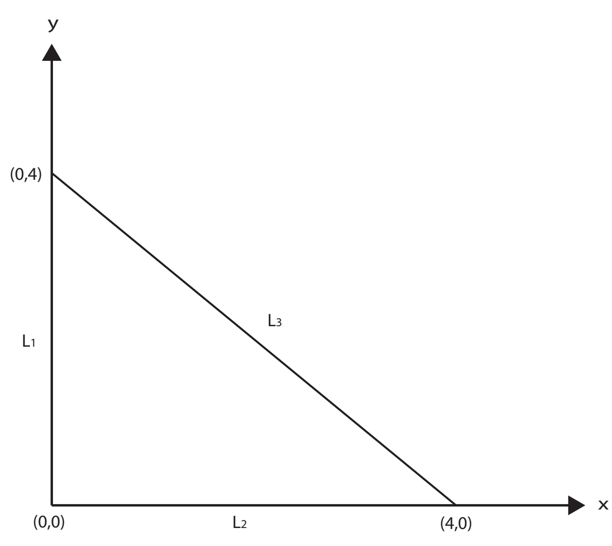
$$f_{xx}(x, y) = 0, \quad f_{yy}(x, y) = 0, \quad f_{xy}(x, y) = 1$$
$$D = f_{xx}f_{yy} - f_{xy}^2 = 2(0) - 1^2 = -1$$

Since $D < 0$ at $(2, -2)$, this critical point is a saddle point.

(2) $f(x, y) = xy - 2x$ on the triangular region $R$ with vertices $(0, 0), (0, 4), (4, 0)$

$$f_x = y - 2 = 0 \;\Rightarrow y = 2$$
$$f_y = x = 0 \;\Rightarrow x = 0$$

We obtain only one solution when $x = 0,\; y = 2$, with $f(0, 2) = 0$. Hence there is a critical point at $(0, 2)$. A sketch of the region $R$ divided into three line segments $L_1, L_2, L_3$, reveals $(0, 2)$ is a boundary point of $R$, lying on $L_1$.



$L_1$: $x = 0$ and $f(x,y) = f(0,y) = 0,\ 0 \le y \le 4$ (monotone)
$L_2$: $y = 0$ and $f(x,y) = f(x,0) = -2x,\ 0 \le x \le 4$ (monotone)
$L_3$: $y = 4 - x$ and $f(x,y) = f(x, 4-x) = 2x - x^2,\ 0 \le x \le 4$

On $L_1$, $f$ has no critical points and the value of $f$ is 0, regardless of $y$. In fact, $f$ is simply the $y$-axis on the closed interval $[0,4]$. Hence, the minimum and maximum are both 0, attained along the entire boundary line.

On $L_2$, $f$ has no critical points and the maximum is clearly 0 attained at $(0,0)$ while the minimum is $-8$ attained at $(4,0)$.

On $L_3$, $g(x) = 2x - x^2$, $g'(x) = 2 - 2x = 0 \ \Rightarrow\ x = 1,\ y = 3,\ g''(x) < 0$, $f(1,3) = 1$. Hence, we obtain a maximum of 1 at $(1,3)$.

Combining results from the three line segements, we find the maximum on the boundary is 1 and the minimum is $-8$.

Therefore the absolute maximum of $f$ over $R$ is 1 attained at $(1,3)$ while the absolute minimum is $-8$ attained at $(4,0)$.

(3) $f(x,y) = x^2 - y, \quad g(x,y) = x^2 + y^2 - 25 = 0, \quad \nabla f(x,y) = \lambda \nabla g(x,y)$

$$\nabla f(x,y) = 2x\vec{i} - \vec{j} \qquad \nabla g(x,y) = 2x\vec{i} + 2y\vec{j}$$

$$2x = \lambda 2x \ \Rightarrow \lambda = 1,\ x \ne 0 \qquad -1 = \lambda 2y \ \Rightarrow\ y = -\frac{1}{2}$$

$$x^2 + \left(-\frac{1}{2}\right)^2 = 25 \qquad x^2 = \frac{99}{4} \ \Rightarrow\ x = \pm\frac{3\sqrt{11}}{2}$$

Hence $\left(-\frac{3\sqrt{11}}{2}, -\frac{1}{2}\right)$ and $\left(\frac{3\sqrt{11}}{2}, -\frac{1}{2}\right)$ are critical points when $x \ne 0$.

When $x = 0$, we compute $y$ from the constraint and obtain the corresponding $\lambda$-values as follows.

$$y = \pm\sqrt{25 - x^2} = \pm\sqrt{25 - 0^2} = \pm 5$$
$$-1 = \lambda 2y = \lambda 2(5) \ \Rightarrow\ \lambda = -\frac{1}{10}$$
$$-1 = \lambda 2y = \lambda 2(-5) \ \Rightarrow\ \lambda = \frac{1}{10}$$

Substituting the $\lambda$-values back into $y = -\frac{1}{2\lambda}$, we obtain two more critical points at $(0,-5)$ and $(0,5)$.

| $(x,y)$ | $(0,-5)$ | $(0,5)$ | $(-\frac{3\sqrt{11}}{2},-\frac{1}{2})$ | $(\frac{3\sqrt{11}}{2},-\frac{1}{2})$ |
|---|---|---|---|---|
| $f(x,y)$ | $5$ | $-5$ | $25\frac{1}{4}$ | $25\frac{1}{4}$ |

We conclude from the table above that the maximum is $25\dfrac{1}{4}$ attained at $\left(\pm\dfrac{3\sqrt{11}}{2},-\dfrac{1}{2}\right)$ while the minimum is $-5$ attained at $(0,5)$.

(4) $D(x,y)=x^2+y^2, \quad g(x,y)=2x-4y-3=0, \quad \nabla D(x,y)=\lambda\nabla g(x,y)$

$\nabla D(x,y)=2x\vec{i}+2y\vec{j}$ $\qquad$ $\nabla g(x,y)=2\vec{i}-4\vec{j}$

$2x=2\lambda \;\Rightarrow x=\lambda$ $\qquad$ $2y=-4\lambda \;\Rightarrow\; y=-2\lambda=-2x$

$2x-4(-2x)-3=0 \;\Rightarrow 10x-3=0$ $\qquad$ $x=\dfrac{3}{10},\; y=-2\left(\dfrac{3}{10}\right)=-\dfrac{3}{5}$

$d(x,y)=\sqrt{x^2+y^2}$ $\qquad$ $d\left(\dfrac{3}{10},-\dfrac{3}{5}\right)=\sqrt{\dfrac{9}{100}+\dfrac{36}{100}}=\dfrac{3\sqrt{5}}{10}$

We use the second derivative test to confirm that the critical point is a minimum.

$$D_x=2x, \quad D_y=2y, \quad D_{xx}=2, \quad D_{yy}=2, \quad D_{xy}=0$$
$$T=D_{xx}D_{yy}-D_{xy}^2=2(2)-0^2=4>0$$

Since $D_{xx}>0$ and $T>0$, the critical point is a minimum. Hence the point on the line closest to the origin is $\left(\dfrac{3}{10},-\dfrac{3}{5}\right)$ with a distance of $\dfrac{3\sqrt{5}}{10}$.