Sample Solutions for Tutorial 1

Question 1.

(a) If we let $P(x)$ be x **is a positive integer divisible by 4 and 6** and $Q(x)$ be x **is a positive integer divisible by 24**, then the proposition is of the form

$$(\forall x)\big(P(x) \Rightarrow Q(x)\big).$$

The negation is then

$$(\exists x)\neg\big(P(x) \Rightarrow Q(x)\big), \qquad \text{which is} \qquad (\exists x)\big(P(x) \wedge \neg Q(x)\big),$$

or, in English,

There is a positive integer is divisible by 4 and by 6, which is not divisible by 24.

The converse is

$$(\forall x)\big(Q(x) \Rightarrow P(x)\big),$$

or, in English,

If a positive integer is divisible by 24, then it is divisible by 4 and by 6.

(b) If we write $P(x)$ for x **is a prime number**, $Q(x)$ for x **is an odd number**, then the first proposition is

$$(\forall x)\big(P(x) \Rightarrow Q(x)\big).$$

Its negation is thus

$$(\exists x)\big(P(x) \wedge \neg Q(x)\big),$$

or, in English,

There is a prime number which is not odd.

If we write $C(x)$ for x **is a clever person**, $D(x)$ for x **does dumb things**, then the second proposition is

$$(\exists x)\big(C(x) \wedge D(x)\big).$$

Its negation is thus

$$(\forall x)\neg\big(C(x) \wedge D(x)\big), \qquad \text{which is} \qquad (\exists x)\ \big((\neg C(x)) \wedge (\neg D(x))\big)$$

or, in English,

No clever person does dumb things.

Question 2.

Let a be a real number and suppose that for every real number, b,

$$(a+b)^2 = a^2 + b^2.$$

Then, since 1 is a real number, we must have

$$\begin{aligned} a^2 + 1 &= (a+1)^2 \\ &= a^2 + 2a + 1 \end{aligned}$$

Subtracting $a^2 + 1$ from both sides of the equation, we see that $2a = 0$, whence $aa = 0$ as $2 \neq 0$.

Question 3.

Let $P(n)$ be the proposition $\sum_{j=1}^{n} j^3 = \left(\frac{1}{2}1(1+1)\right)^2$

$n = 1$:

$$\sum_{j=1}^{1} j^3 = 1^3 = 1 = \left(\frac{1}{2} \times 2\right)^2 = \left(\frac{1}{2}1(1+1)\right)^2.$$

Hence $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is,

$$\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2.$$

Then

$$\begin{aligned}
\sum_{j=1}^{n+1} j^3 &= \left(\sum_{j=1}^{n} j^3\right) + (n+1)^3 \\
&= \left(\frac{1}{2}n(n+1)\right)^2 + (n+1)^3 && \text{by the Inductive Hypothesis} \\
&= (n+1)^2\left((\frac{1}{2}n)^2 + (n+1)\right) \\
&= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{1}{2}(n+1)\big((n+1)+1\big)\right)^2
\end{aligned}$$

Hence $P(n+1)$ is true whenever $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, n. That is to say, $\sum_{j=1}^{1} j^3 = \left(\frac{1}{2}1(1+1)\right)^2$ for every counting number, n.

Question 4.

Let a be a positive real number, and $P(n)$ the proposition $(1+a)^n \geq 1 + na$.

$n = 1$: Then

$$(1+a)^1 = 1 + a = 1 + 1.a,$$

showing that $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is,

$$(1+a)^n \geq 1 + na.$$

Then

$$\begin{aligned}
(1+a)^{n+1} &= (1+a)(1+a)^n \\
&\geq (1+a)(1+na) && \text{by the Inductive Hypothesis and the fact that } 1+a>0 \\
&= 1 + (n+1)a + na^2 \\
&> 1 + (n+1)a && \textbf{as } na^2 > 0
\end{aligned}$$

Hence $P(n+1)$ is true whenever $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, n. That is to say, $(1+a)^n \geq 1 + na$. for every counting number.

Question 5.

Let $P(n)$ the proposition that if n is a counting number, then $3^{2n} - 1$ is divisible by 8.

$n = 1$: Then

$$3^{2\times 1} - 1 = 9 - 1 = 8,$$

so that $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is, there is a counting number k, such that

$$3^{2n} - 1 = 8k.$$

Then

$$\begin{aligned} 3^{2(n+1)} - 1 &= 3^{2n+2} - 1 \\ &= 9.3^{2n} - 1 \\ &= 9.3^{2n} - 9 + 8 \\ &= 9(3^{2n} - 1) + 8 \\ &= 9 \times 8k + 8 && \text{by the Inductive Hypothesis} \\ &= 8 \times (9k + 1), \end{aligned}$$

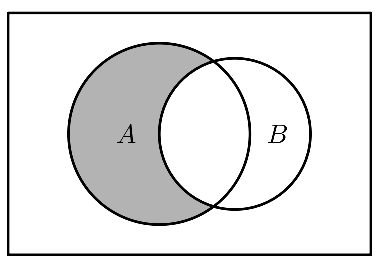
which is plainly divisible by 8.

Hence $P(n+1)$ is true whenever $P(n)$ is true.

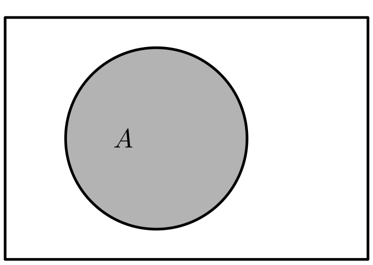
By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, n. That is to say, if n is a counting number, $3^{2n} - 1$ is divisible by 8.

Question 6.

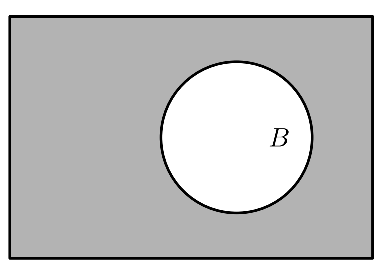
We illustrate, successively $A \setminus B, A, B'$ and $A \cap B'$



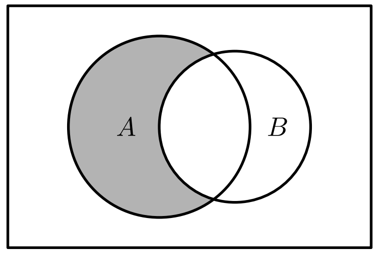
$A \setminus B$



A



B'



$A \cap B'$

Since the shaded areas in the first and last diagrams agree, the sets they depict coincide.