TUTORIAL 3, PMTH212

The formula for the unit tangent vector of a smooth curve $\vec{r}(t)$ is $\vec{T}(t) = \frac{\vec{r'}(t)}{||\vec{r'}(t)||}$.

1. Find $\vec{T}(t)$ at the given point.

(a) $\vec{r}(t) = (t^2-1)\vec{i} + t\vec{j}$ at $t = 1$.

Solution.

$$\begin{aligned} \vec{r'}(t) &= 2t\vec{i} + \vec{j}, \\ \vec{r'}(1) &= 2\vec{i} + \vec{j}, \\ ||\vec{r'}(1)|| &= \sqrt{2^2+1^2} = \sqrt{5}, \\ \vec{T}(1) &= \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}. \end{aligned}$$

(b) $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ at $t = 0$.

Solution.

$$\begin{aligned} &x'(t) = e^t\cos t - e^t \sin t = e^t(\cos t - \sin t), \quad x'(0) = 1; \\ &y'(t) = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t), \quad y'(0) = 1; \\ &z'(t) = e^t, \quad z'(0) = 1. \end{aligned}$$

$$||\vec{r'}(0)|| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\vec{T}(0) = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}.$$

The formula for curvature is $\kappa(t) = \frac{||\vec{r'}(t) \times \vec{r''}(t)||}{||\vec{r'}(t)||^3}$.

2. Find $\kappa(t)$ at the given point.

(a) $\vec{r}(t) = e^t\vec{i} + e^{-t}\vec{j} + t\vec{k}$ at $t = 0$.

Solution.

$$\begin{aligned} \vec{r'}(t) &= e^t\vec{i} - e^{-t}\vec{j} + \vec{k}, \\ \vec{r''}(t) &= e^t\vec{i} + e^{-t}\vec{j}. \end{aligned}$$

$$
\begin{aligned}
\vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t & -e^{-t} & 1 \\ e^t & e^{-t} & 0 \end{vmatrix} = \\
&= \begin{vmatrix} -e^{-t} & 1 \\ e^{-t} & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} e^t & \vec{i} \\ e^t & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{vmatrix} \vec{k} \\
&= -e^{-t}\vec{i} + e^t\vec{j} + 2\vec{k}. \\
||\vec{r}'(t)|| &= \sqrt{e^{2t} + e^{-2t} + 1} = \sqrt{3} \quad \text{at} \quad t = 0, \\
||\vec{r}'(t) \times \vec{r}''(t)|| &= \sqrt{e^{-2t} + e^{2t} + 4} = \sqrt{6} \quad \text{at} \quad t = 0, \\
\kappa(0) &= \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}.
\end{aligned}
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(b) $\vec{r}(t) = 3\cos t\,\vec{i} + 4\sin t\,\vec{j} + t\vec{k}$, at $t = \frac{\pi}{2}$

Solution.

$$
\begin{aligned}
\vec{r}'(t) &= -3\sin t\,\vec{i} + 4\cos t\,\vec{j} + \vec{k}, \\
\vec{r}''(t) &= -3\cos t\,\vec{i} - 4\sin t\,\vec{j}. \\
\vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin t & 4\cos t & 1 \\ -3\cos t & -4\sin t & 0 \end{vmatrix} = \\
&= \begin{vmatrix} 4\cos t & 1 \\ -4\sin t & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -3\sin t & 1 \\ -3\cos t & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -3\sin t & 4\cos t \\ -3\cos t & -4\sin t \end{vmatrix} \vec{k} \\
&= 4\sin t\,\vec{i} - 3\cos t\,\vec{j} - 12(\sin^2 t + \cos^2 t)\vec{k} \\
&= 4\sin t\,\vec{i} - 3\cos t\,\vec{j} - 12\vec{k}. \\
||\vec{r}'(t)|| &= \sqrt{9\sin^2 t + 16\cos^2 t + 1} = \sqrt{10} \quad \text{at} \quad t = \frac{\pi}{2}, \\
||\vec{r}'(t) \times \vec{r}''(t)|| &= \sqrt{16\sin^2 t + 9\cos^2 t + 144} = \sqrt{160} \quad \text{at} \quad t = \frac{\pi}{2}, \\
\kappa(\frac{\pi}{2}) &= \frac{\sqrt{160}}{10\sqrt{10}} = \frac{4}{10} = \frac{2}{5}.
\end{aligned}
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3.

(a) Let $g(x) = x\sin x$, find $g(\frac{x}{y})$, $g(xy)$ and $g(x - y)$.

Solution.

$$g\left(\frac{x}{y}\right) = \frac{x}{y}\sin\frac{x}{y},$$
$$g(xy) = xy\sin(xy),$$
$$g(x-y) = (x-y)\sin(x-y).$$

(b) Let $f(x,y) = x^2 - y^2$, find $f(x-y, x+y)$.

Solution.

$$f(x-y, x+y) = (x-y)^2 - (x+y)^2 = -4xy.$$

(c) Let $F(x,y) = xe^{xy}$, $g(x) = x^3$, $h(x) = 3y+1$. Find $F(g(x), h(x))$.

Solution.

$$F(g(x), h(x)) = x^3 e^{x^3(3y+1)}.$$

4. Sketch the level surfaces for the function
a) $f(x,y,z) = 4x - 2y + z$, b) $f(x,y,z) = 4x^2 + y^2 + 4z^2$.

Solution. a.) The level surfaces are the surfaces given by equations $4x - 2y + z = k$ for $k \in \mathbb{R}$. For $k = 4$ we get three points defining a plane: $(0,0,4), (0,-2,0), (1,0,0)$. The level surfaces are all the planes parallel to this one, intersecting the 0z-axis at $(0,0,k)$.

b.) The level surfaces are the surfaces given by equations $4x^2 + y^2 + 4z^2 = k$ for $k \in \mathbb{R}$.
There are no solutions for $k < 0$,
for $k = 0$ $4x^2 + y^2 + 4z^2 = 0$, the level surface is the point (x,y,z),
for $k > 0$ $4x^2 + y^2 + 4z^2 = k$, the level surface is an ellipsoid. See the figure for $k = 1$ and $k = 4$.