MATH102 ASSIGNMENT 9

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(1) (a)

$$\int_0^\infty cxe^{-\frac{x^2}{2}}\ dx = 1 \ \Rightarrow\ c\int_0^\infty xe^{-\frac{x^2}{2}}\ dx = 1$$

$$\text{Let } u = -\frac{x^2}{2} \ \Rightarrow\ \frac{du}{dx} = -x,\ dx = -\frac{du}{x}$$

$$\Rightarrow c\int_0^\infty xe^{-\frac{x^2}{2}}\ dx = c\int_0^\infty xe^{u}\ \left(-\frac{du}{x}\right) = 1$$

$$\Rightarrow -c\int_0^\infty e^u\ du = -c\left[e^{-\frac{x^2}{2}}\right]_0^\infty = -c\ \big(0-1\big) = 1$$

$$\Rightarrow\ c = 1$$

(b)

$$E(X) = \int_0^\infty x\cdot xe^{-\frac{x^2}{2}}\ dx$$

$$\text{Let } u = x \ \Rightarrow\ \frac{du}{dx} = 1,\ \frac{dv}{dx} = xe^{-\frac{x^2}{2}} \ \Rightarrow\ v = -e^{-\frac{x^2}{2}}$$

$$\Rightarrow \int_0^\infty x\cdot xe^{-\frac{x^2}{2}}\ dx = \left[-xe^{-\frac{x^2}{2}}\right]_0^\infty + \int_0^\infty e^{-\frac{x^2}{2}}\ dx$$

$$\text{Since } \int_{-\infty}^\infty e^{-\frac{x^2}{2\sigma^2}}\ dx = \sqrt{2\pi\sigma} \ \Rightarrow\ \int_0^\infty e^{-\frac{x^2}{2\sigma^2}}\ dx = \sqrt{\frac{\pi\sigma}{2}}$$

Then by substituting $w = \dfrac{x}{\sqrt{2}\sigma}$ and letting $\sigma = 1$

$$\int_{-\infty}^\infty e^{-w^2}\ dw = \sqrt{\pi} \ \Rightarrow\ \int_0^\infty e^{-w^2}\ dw = \sqrt{\frac{\pi}{2}}$$

Thus $E(X) = \big(0-0\big) + \sqrt{\dfrac{\pi}{2}} = \sqrt{\dfrac{\pi}{2}}$

(2) (a)

$$\sum_x P(X=x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = e^0 = 1$$

(b) For $\lambda = 1$

$$P(X=x) = \frac{e^{-1} \cdot 1^x}{x!} = \frac{1}{e}\ \frac{1}{x!}$$
$$E(X) = \sum_{x=0}^{\infty} xP(X=x) = \sum_{x=0}^{\infty} x\ \frac{1}{e}\ \frac{1}{x!} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{x}{x!} = \frac{1}{e} \cdot e = 1$$

(c)

$$P(X=0) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} > \frac{1}{2}$$
$$\Rightarrow\ -\lambda > \log \frac{1}{2}$$
$$\Rightarrow\ -\lambda > -\log 2$$
$$\Rightarrow \lambda < \log 2$$

Therefore the distribution is right skewed when λ is less than $\log 2$.

(3) (a) When $\lambda = 0.9$

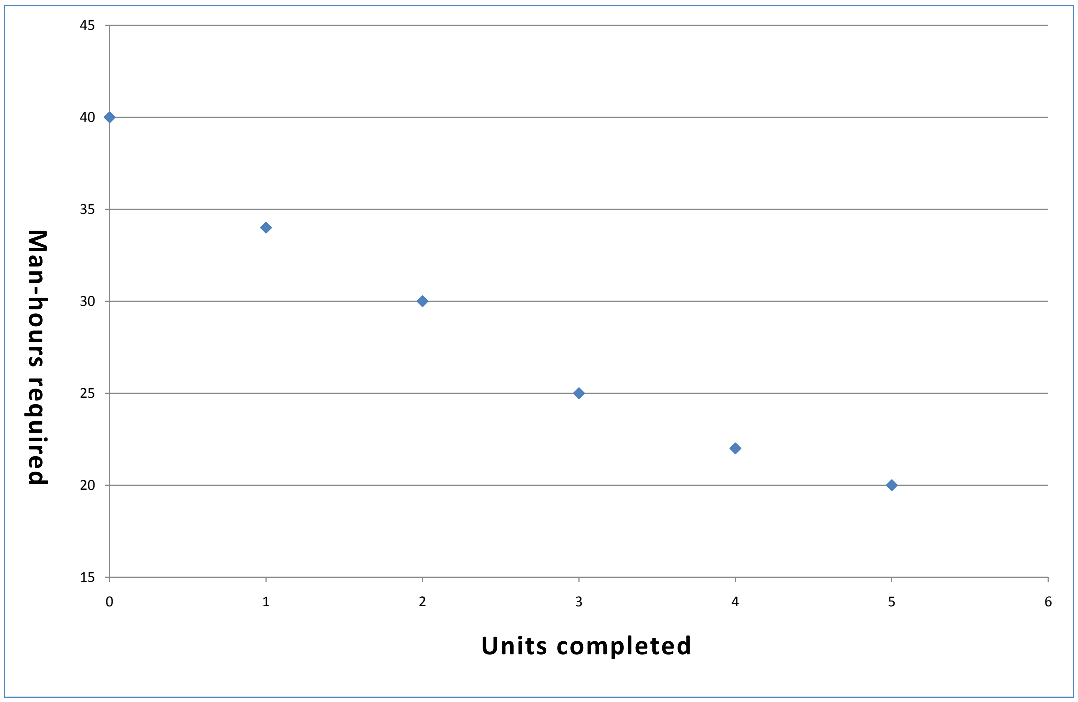
$$E(X) = \sum_{x=0}^{\infty} xP(X=x) = \sum_{x=0}^{\infty} \frac{0.9^x e^{-0.9} x}{x!} = \sum_{x=1}^{\infty} \frac{0.9^x e^{-0.9} x}{x!}$$
$$= \sum_{x=1}^{\infty} \frac{0.9^x e^{-0.9}}{(x-1)!} = e^{-0.9} \sum_{x=1}^{\infty} \frac{0.9 \cdot 0.9^{x-1}}{(x-1)!} = 0.9\ e^{-0.9} \sum_{x=1}^{\infty} \frac{0.9^{x-1}}{(x-1)!}$$
$$= 0.9\ e^{-0.9} \sum_{m=0}^{\infty} \frac{0.9^m}{m!} = 0.9\ e^{-0.9}\ e^{0.9} = 0.9 = \lambda$$

(b)

$$E(X=0) = nP(X=0) = 365 \times \frac{0.9^0 e^{-0.9}}{0!} = 365\ e^{-0.9} = 148.4$$

(4) (a)

X-Y Plot



(b)

$$\sum_{i=1}^{6} x_i y_i = 0 \times 40 + 1 \times 34 + 2 \times 30 + 3 \times 25 + 4 \times 22 + 5 \times 20 = 357$$

$$\begin{aligned}\sum_{i=1}^{6} x_i \sum_{i=1}^{6} y_i &= \big(0+1+2+3+4+5\big)\big(40+34+30+25+22+20\big) \\ &= 15 \times 171 = 2565 \\ \operatorname{cov}(x,y) &= \frac{1}{5}\left(357 - \frac{2565}{6}\right) = -14.1\end{aligned}$$

$$\overline{x} = \frac{1}{6}\sum_{i=1}^{6} x_i = \frac{1}{6}\big(15\big) = 2.5 \qquad \overline{y} = \frac{1}{6}\sum_{i=1}^{6} y_i = \frac{1}{6}\big(171\big) = 28.5$$

$$s_x^2 = \frac{1}{5}\sum_{j=1}^{6}(x_j - \overline{x})^2 = \frac{1}{5}\left[(0-2.5)^2 + ... + (5-2.5)^2\right] = 3.5$$

$$s_y^2 = \frac{1}{5}\sum_{j=1}^{6}(y_j - \overline{y})^2 = \frac{1}{5}\left[(40-28.5)^2 + ... + (20-28.5)^2\right] = 58.3$$

$$\rho(x,y) = -\frac{14.1}{\sqrt{3.5}\sqrt{58.3}} = -0.9871$$

(c) Yes the two variables are related since $\text{cov}(x,y) \neq 0$. In fact, there is a strong linear negative association between X and Y since $\rho(x,y)$ is close to -1.