THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: MATH 101/101A

PAPER TITLE: Algebra & Differential Calculus

PAPER NUMBER: First and Only

DATE: **Friday 15th June 2007** **TIME:** **9:30AM TO 12:30AM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: FIVE (5)

006301

NUMBER OF QUESTIONS ON PAPER: TWELVE (12)

NUMBER OF QUESTIONS TO BE ANSWERED: TEN (10)

STATIONERY PER CANDIDATE:

1	6 LEAF A4 BOOKS	0	ROUGH WORK BOOK
1	12 LEAF A4 BOOKS	0	GRAPH PAPER SHEETS

OTHER AIDS REQUIRED: **NIL**

POCKET CALCULATORS PERMITTED: **YES** **(SILENT TYPE)**

TEXTBOOKS OR NOTES PERMITTED: ***FIVE (5) A4 SHEETS (10 PAGES IF WRITTEN ON BOTH SIDES) OF HANDWRITTEN NOTES. NO PHOTOCOPIES, NO PRINTED NOTES PERMITTED.***

INSTRUCTIONS FOR CANDIDATES:

- Candidates **may** make notes on this paper during the fifteen minutes reading time
- Questions are of equal value
- **Section A:** Answer ALL questions
- **Section B:** Answer any TWO (2) questions
- Candidates may retain this examination question paper

SECTION A

You should attempt all questions in this section.

Question 1 *[10 marks]*

(a) Assuming that a is a positive number and $b > a$ prove that $b^2 > a^2$

(b) Prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

(c) Determine supremum and infimum (if they exist) of the following sets

$$(i) \quad \{n \in \mathbb{N} : n^2 > 6\} \qquad (ii) \quad \{x \in \mathbb{R} : 81 - x^4 \geq 0\}.$$

Question 2 *[10 marks]*

(a) For $z = 2 + 3\,\mathrm{i}$, write each of the following complex numbers in the form $x + \mathrm{i}\,y$

(i) $\bar{z}$, (ii) $|z|$, (iii) z^2, (iv) $\dfrac{1}{z}$, (v) $\dfrac{z^2 + \mathrm{i} - 1}{z}$.

(b) Find all complex numbers z that satisfy

$$z^5 = -1.$$

Question 3 *[10 marks]*

(a) Find the natural domain X and the range Y of the functions defined by the following formulae

(i) $f(x) = 1 - x$, (ii) $f(x) = \dfrac{1}{9 - x}$, (iii) $f(x) = \sqrt{4 + x}$.

(b) Sketch the graph of the function $f : X \to \mathbb{R}$ from Part (a) (ii). Investigate whether this function is injective or surjective.

(c) Find a real number k that renders continuous the function

$$f : x \mapsto \begin{cases} 2x + 3 & x < 0 \\ 5x + k & x \geq 0. \end{cases}$$

Question 4 is on Page 3

Question 4 *[10 marks]*

(a) Determine which of the following sequences of real numbers $(u_n)_{n\in\mathbb{N}}$ is monotone and discuss the behaviour of u_n as $n \to \infty$.

(i) $u_n = \dfrac{n}{n+1}$ (ii) $u_n = \dfrac{1}{(-3)^n}$.

(b) Determine which of the following series converge and which diverge, justifying your answer.

$$(i) \quad \sum_{n=1}^{\infty} \frac{n}{2^n} \qquad (ii) \quad \sum_{n=1}^{\infty} \frac{2^n}{n} \qquad (iii) \quad \sum_{n=1}^{\infty} \frac{n}{n^4+1}.$$

Question 5 *[10 marks]*

(a) Find all points at which each of the following functions is well-defined and differentiable, and compute its derivative. Where the function fails to be differentiable, explain why.

(i) $f(x) = \dfrac{1}{\cos^2 x}$ (ii) $g(x) = |x|$.

(b) Differentiate the functions

(i) $f(x) = \dfrac{x+1}{x^2+1}$, (ii) $g(x) = x^3 \sin x$, (iii) $h(x) = \sin(x^3)$.

Question 6 *[10 marks]*

Consider the function $f(x) = x^3 - 3x^2 - 24x + 3$.

(a) Determine the intervals on which f is (i) increasing or decreasing (ii) concave up or concave down.

(b) Find all the relative maxima and relative minima of f and the absolute maximum and the absolute minimum on $[-3, 5]$.

(c) Sketch the graph of f on the interval $[-3, 5]$ (Choose an appropriate scale of the y-axis).

Question 7 *[10 marks]*

(a) Find all real numbers x, y, z such that

$$\begin{array}{rcrcrcr} 4x & + & 3y & + & z & = & 13 \\ 2x & - & y & + & 2z & = & 6 \\ x & + & y & + & z & = & 6. \end{array}$$

Question 7(b) is on Page 4

Question 7 continued

(b) For

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix}.$$

state whether the following are defined; calculate those which are defined:

(i) AB (ii) $A - C$ (iii) BC.

(c) Evaluate the determinant

$$\begin{vmatrix} 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 \end{vmatrix}.$$

Question 8 *[10 marks]*

(a) Find the inverse matrix of

$$A = \begin{pmatrix} 4 & 5 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Check your answer.

(b) Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Section B is on Page 5

SECTION B

You should attempt only TWO questions in this section.

Question 9 [10 marks]

(a) Let $f(x) = x^2$. Find $\delta > 0$ such that $|x - 1| < \delta$ implies $|f(x) - f(1)| < 0.01$.

(b) Give an example of a function defined on $\mathbb{R}$ that is **not** continuous at $x = 0$. Justify your answer.

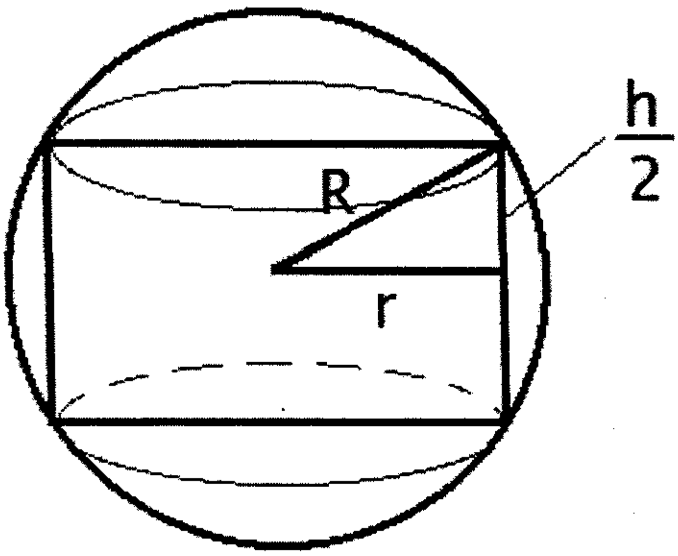
Question 10 *[10 marks]*

(a) Prove that a function f that is differentiable at $x = c$ must be continuous at $x = c$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that between any two zeros of f, its derivative f' must also have a zero.

Question 11 [10 marks]

A cylinder has been cut out of a ball of given radius R. Find height h and radius r of the cylinder (as functions of R) if the cylinder has maximal possible volume. Hints: The picture below shows that R, r and $\frac{h}{2}$ are the sides of a right triangle. The volume of a cylinder can be computed by the formula $V = \pi r^2 h$.



Question 12 *[10 marks]*

Find a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ of degree 3 such that $p(0) = 0$, $p(1) = 1$, $p(-1) = -1$, $p(2) = 8$.