Q1

a) (i) $S = \{x \in \mathbb{N} / x^3 \geq \sqrt{5}\} = \{x \in \mathbb{N} / x^6 \geq 5\}$

Hence S has infimum 2 which belongs to S, but it has no supremum, since it is not bounded above.

(ii) $T = \{x \in \mathbb{R} \mid \sqrt{16 - x^4} > 0\} = \{x \in \mathbb{R} \mid x^4 < 2^4\} =]-2, 2[$

Thus $\inf(T) = -2$, $\sup(T) = 2$ and neither is an element of T.

b) Let $P(n)$ be the proposition $5^n > n^2$ $(n \in \mathbb{N} \setminus \{1\})$

Since $5^1 = 5 > 1 = 1^2$, $P(1)$ is true

Suppose that for some $n \in \mathbb{N} \setminus \{1\}$ $P(n)$ is true, i.e. $5^n > n^2$

Then $(n+1)^2 = n^2 + 2n + 1$

$\leq n^2 + 2n^2 + n^2$ as $n \geq 1$

$< 5n^2$

$< 5.5^n$ by the inductive hypothesis

$= 5^{n+1}$

So, by the PMI $5^n > n^2$ for all $n \geq 1$

c) $a^2 - 3a \leq -3 \iff a^2 - 3a + \frac{9}{4} \leq -3 + \frac{9}{4} = -\frac{3}{4}$

$\iff (a - 3/2)^2 \leq -3/4$

So if $a^2 - 3a \leq -3$, then $(a - 3/2)^2 \leq -3/4$

which is a contradiction, since the square of a real number cannot be negative

22

(a) $z = 2+3i$

(i) $4i - z = -2+i$

(ii) $\dfrac{z-2i}{z+1+i} = \dfrac{2+i}{3+4i} = \dfrac{2+i}{3+4i}\cdot\dfrac{3-4i}{3-4i} = \dfrac{10-5i}{25} = \dfrac{2-i}{5}$

(iii) $\overline{3z+2} = \overline{8+9i} = 8-9i$

(iv) $\left|\dfrac{z+2-i}{1+z+i}\right| = \left|\dfrac{4+2i}{3+4i}\right| = \dfrac{2\sqrt{5}}{5} = \dfrac{2}{\sqrt{5}}$

(v) $\dfrac{1}{z^2} = \dfrac{\bar{z}^2}{|z|^2} = \dfrac{-5+12i}{169}$

(b) Put $z = re^{i\theta}$ with $r>0$ and $0 \le \theta < 2\pi$

Since $i = e^{i\pi/2}$ and $z^4 = r^4 e^{i4\theta}$,

$z^4 = i \Leftrightarrow r^4 e^{i4\theta} = e^{i\pi/2}$

$\Leftrightarrow r^4 \cos 4\theta = 0$ with $r>0$

$\quad \& \; r^4 \sin 4\theta = 1$ and $0 \le 4\theta < 8\pi$

$\Leftrightarrow r=1$ and $4\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$

$\Leftrightarrow r=1$ and $\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

i.e. $z = \pm\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right), \pm\left(\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right)$

23

(a) (i) $f(x) = \frac{x}{\sqrt{1-x^2}}$ is well-defined $\Leftrightarrow$ $1-x^2>0$ $\Leftrightarrow$ $-1<x<1$

So consider $f:]-1, 1[\longrightarrow \mathbb{R}$, $x \longmapsto \frac{x}{\sqrt{1-x^2}}$ is cts

As $x \to 1^-$, $f(x) \longrightarrow \infty$

As $x \to -1^+$, $f(x) \longrightarrow -\infty$

So, since $im(f)$ is an interval, $im(f) = \mathbb{R}$.

(ii) $f(x) = \frac{x}{x^2+1}$ is well-defined for all $x \in \mathbb{R}$ as $x^2+1 \geq 1$

So consider $f: \mathbb{R} \to \mathbb{R}$, $x \longmapsto \frac{x}{x^2+1}$

As $f(-x) = -f(x)$, f is an odd function

$\lim_{x \to \pm\infty} f(x) = 0$

$$f'(x) = \frac{(x^2+1) - 2x^2}{x^2+1} = \frac{1-x^2}{1+x^2} \begin{cases} < 0 & \text{if } |x| > 1 \\ = 0 & \text{if } |x| = 1 \\ > 0 & \text{if } |x| < 1 \end{cases}$$

Thus $f(1) = \frac{1}{2}$ is the (global) maximum

and $f(-1) = -\frac{1}{2}$ the (global) minimum

So $im(f) = [-\frac{1}{2}, \frac{1}{2}]$

(b) Note that for $x \neq 0$ $f(\frac{1}{x}) = \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{x}{x^2+1} = f(x)$

Hence f is not injective

As $im(f) \neq \mathbb{R}$, f is not surjective

$\inf(f) = -\frac{1}{2}$, $\sup(f) = \frac{1}{2}$ by part (a) (ii)

(c) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \frac{\sin x^2}{x} = \lim_{x \to 0^+} x \frac{\sin(x^2)}{x^2} = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + k) = k = f(0)$

Hence f is continuous at 0 $\Leftrightarrow$ $k = 0$

24

(a) (i) $u_n := r^{n+1}$ $(r \in \mathbb{R})$

If $r = 0, 1$, then $(u_n)_{n\in\mathbb{N}}$ is a constant sequence

If $r < 0$, then $(u_n)_{n\in\mathbb{N}}$ is an alternating sequence

If $0 < r < 1$, then $(u_n)_{n\in\mathbb{N}}$ is monotonically decreasing

If $r > 1$, then $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing

$u_n \to \infty$ if $r > 1$

$u_n \to 0$ if $|r| < 1$

u_n oscillates without converging if $x \leq -1$

(ii) $u_n := \frac{n+1}{n^2+1}$. $u_0 = 1$, $u_1 = 1$

$$u_{n+1} - u_n = \frac{n+2}{n^2+2n+2} - \frac{n+1}{n^2+1} = \frac{(n+2)(n^2+1) - (n+1)(n^2+2n+2)}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{n^3+2n^2+n+2 - (n^3+3n^2+4n+2)}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{-(n^2+3n)}{(n^2+1)(n^2+2n+2)} < 0 \text{ for all } n \geq 1$$

Thus $(u_n)_{n\in\mathbb{N}}$ is monotonically decreasing for $n \geq 1$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}(1+\frac{1}{n})}{1+\frac{1}{n^2}} = \lim_{x\to 0^+} \frac{x(1+x)}{x^2+1} = 0$$

(b) (i) $u_n := \frac{5^n}{n!}$ So $\frac{u_{n+1}}{u_n} = \frac{5}{n+1} \to 0$ as $n \to \infty$.

Thus, by the ratio test, $\sum u_n$ converges.

(ii) $u_n := \frac{n}{\sqrt{n^2+1}} = \frac{\sqrt{n^2}}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$ for $n > 0$

Thus, $\lim_{n\to\infty} u_n = 1$, whence $\sum u_n$ cannot converge.

(iii) $u_n := \frac{2^n}{n^4+1}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} 2\,\frac{n^4+1}{(n+1)^4+1} = 2 > 1$$

So, by the ratio test, $\sum u_n$ diverges

2.5

(a) $f: \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto \sqrt{1+\cos x}$

$g: \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto e^{-x^2}$

$f'(x) = -\dfrac{\sin x}{\sqrt{1+\cos x}}$, so that f is differentiable whenever $\cos x \neq -1$ ie $x \neq (2n+1)\pi$ $(n \in \mathbb{Z})$

Now $f'((2n+1)\pi) := \lim\limits_{h\to 0} \dfrac{f((2n+1)\pi + h) - f((2n+1)\pi)}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{1-\cos h}}{h}$ as $f((2n+1)\pi) = 0$ and $\cos((2n+1)\pi + h) = -\cos h$

$=$

Now $\lim\limits_{h\to 0^-} \dfrac{\sqrt{1-\cos h}}{h} = -\lim\limits_{h\to 0^-} \sqrt{\dfrac{1-\cos h}{h^2}} = -\dfrac{1}{\sqrt{2}}$

but $\lim\limits_{h\to 0^+} \dfrac{\sqrt{1-\cos h}}{h} = \lim\limits_{h\to 0} \sqrt{\dfrac{1-\cos h}{h}} = \dfrac{1}{\sqrt{2}}$

Thus $\lim\limits_{h\to 0} \dfrac{f((2n+1)\pi + h) - f((2n+1)\pi)}{h}$ does not exist.

$g'(x) = -2xe^{-x^2}$, so that g is differentiable everywhere.

(b) Let y be a differentiable function of x with

$$xe^{-(x^2+y^2)} + x^2 + y^2 = 7$$

The $e^{-(x^2+y^2)} + x\left(-2x - 2y\dfrac{dy}{dx}\right)e^{-(x^2+y^2)} + 2x + 2y\dfrac{dy}{dx} = 0$

i.e $\left(-2xye^{-(x^2+y^2)} + 2y\right)\dfrac{dy}{dx} = -e^{-(x^2+y^2)} - 2x + 2x^2e^{-(x^2+y^2)}$

or $\dfrac{dy}{dx} = \dfrac{(2x^2-1)e^{-(x^2+y^2)} - 2x}{2y\left(1 - xe^{-(x^2+y^2)}\right)}$

(whenever this is defined)

26 $f : \mathbb{R} \to \mathbb{R},\ x \longmapsto x^4 - 4x^3 + 4x^2 + 1$

(a) (i) $f'(x) = 4x^3 - 12x^2 + 8x$

$= 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$

$\begin{cases} < 0 & \text{if } x < 0 \text{ or } 1 < x < 2 \\ = 0 & \text{if } x = 0, 1 \text{ or } 2 \\ > 0 & \text{if } 0 < x < 1 \text{ or } x > 2 \end{cases}$

so f is decreasing on $]-\infty, 0]$ and on $[1, 2]$

and increasing on $[0, 1]$ and on $[2, \infty[$

(iii) $f''(x) = 12x^2 - 24x + 8$

$= 12(x^2 - 2x + 1) - 4 = 12\left((x-1)^2 - \tfrac{1}{3}\right)$

$\begin{cases} < 0 & \text{if } 1 - \frac{1}{\sqrt{3}} < x < 1 + \frac{1}{\sqrt{3}} \\ = 0 & \text{if } x = 1 \pm \frac{1}{\sqrt{3}} \\ > 0 & \text{if } x < 1 - \frac{1}{\sqrt{3}} \text{ or } x > 1 + \frac{1}{\sqrt{3}} \end{cases}$

so f is concave up on $]-\infty, 1 - \frac{1}{\sqrt{3}}]$ and on $[1 + \frac{1}{\sqrt{3}}, \infty[$

and concave down on $[1 - \frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}]$

(b) (i) Boundary points $f(-1) = f(3) = 10$

(ii) where f is not diff'ble — no such points

(iii) where $f'(x) = 0$ $\quad x = 0 \quad f(0) = 1$

$x = 1 \quad f(1) = 2$

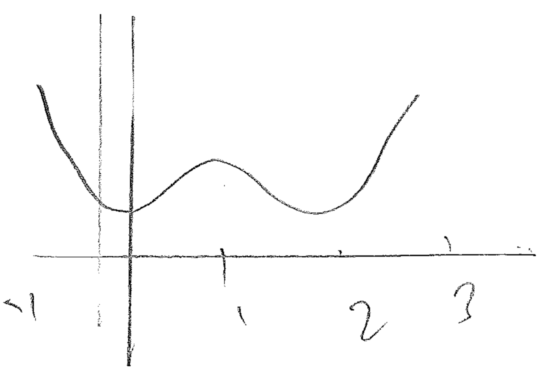
$x = 2 \quad f(2) = 1$

So abs max: 10 at $x = -1, 3$

abs min 1 at $x = 0, 2$

rel max 2 at $x = 1$

(c)



27

(a)

$$4x + 3y + z = 19 \quad \text{— (i)}$$
$$2x - y + 2z = 6 \quad \text{— (ii)}$$
$$x + y + z = 6 \quad \text{— (iii)}$$

$2\times$(iii) $-$ (ii) $\quad 3y = 6 \quad$ or $\quad y = 2$ — (iv)

Substitute in (i) $\quad 4x + z = 13$ — (v)

Substitute in (ii) $\quad x + z = 4$ — (vi)

(v) $-$ (vi) $\quad 3x = 9 \quad$ or $\quad x = 3$ — (vii)

Substitute in (vi): $\quad \therefore z = 1$

i.e. $\quad x = 3 \quad y = 2 \quad z = 1$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

(i) $AB = \begin{bmatrix} 3 & 3 \\ 0 & -2 \\ 2 & 2 \end{bmatrix}$

(ii) $2A + C = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 1 & -3 \\ 3 & 3 & 2 \end{bmatrix}$

(iii) BA is not defined as B is 3x2 & A is 3x3

(c) $\begin{vmatrix} 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \underset{=}{R_5 - R_4} \begin{vmatrix} 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 4$

Q8 $P: (-1,0,0)$, $Q: (1,0,0)$, $R: (1,1,1)$, $S: (2,1,2)$

(a) $\overrightarrow{PQ}: (2,0,0) \sim 2\underline{i}$

$\overrightarrow{PR}: (2,1,1) \sim 2\underline{i} + \underline{j} + \underline{k}$

(b) The orthogonal projection of $\overrightarrow{PQ}$ onto $\overrightarrow{PR}$ is

$$\frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\overrightarrow{PR} \cdot \overrightarrow{PR}} \cdot \overrightarrow{PR} = \frac{(2,0,0).(2,1,1)}{(2,1,1)(2,1,1)} \cdot (2,1,1)$$

$$= \frac{4}{4+1+1}(2,1,1) = \frac{2}{3}(2\underline{i} + \underline{j} + \underline{k})$$

$$= \left(\tfrac{4}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

(c) The area is half the length of $\overrightarrow{PQ} \times \overrightarrow{PR}$

Now $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = -2\underline{j} + 2\underline{k}$, with length $2\sqrt{2}$

So the area is $\sqrt{2}$ square units.

(d) $\overrightarrow{PS} = (3,1,2)$ The volume is the absolute value of

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} \times \overrightarrow{PS} = \left| \begin{vmatrix} 2 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} \right| = \left| 2 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right| = 2$$

So the volume is 2 cubic units

29

(a) For $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ take $a \in \mathbb{R}$ and $\varepsilon > 0$

Put $\delta := \sqrt{|a|^2 + \varepsilon} - |a|$

Suppose that $|x - a| < \delta$. Then $|x| < |a| + \delta$ and

$$
\begin{aligned}
|f(x) - f(a)| = |x^2 - a^2| &= |x-a|\,|x+a| \\
&\le |x-a|\,|2|a| + \delta| \\
&< \delta(2|a| + \delta) \\
&= (\sqrt{|a|^2+\varepsilon} - |a|)(\sqrt{|a|^2+\varepsilon} + |a|) \\
&= \varepsilon
\end{aligned}
$$

Thus f is continuous everywhere.

( ) Take $f, g: \mathbb{R} \to \mathbb{R}$. Suppose that for some $K > 0$ and all $u, v \in \mathbb{R}$

$$|f(u) - f(v)| \le K|g(u) - g(v)|$$

Suppose g is continuous at $a \in \mathbb{R}$

Take $\varepsilon > 0$. Then $\varepsilon/K > 0$

So, there is a $\delta > 0$ with $|g(x) - g(a)| < \varepsilon/K$ if $|x - a| < \delta$

Thus if $|x - a| < \delta$ then

$$
\begin{aligned}
|f(x) - f(a)| &< K|g(x) - g(a)| \\
&< K\,\varepsilon/K = \varepsilon
\end{aligned}
$$

So f is continuous at a.

21 (a) Consider $f: \mathbb{R}_0^+ \longrightarrow \mathbb{R}$, $x \longmapsto \ln(x+1)$

Take $c>0$. Then f is continuous on $[0,c]$

Moreover, since $f'(x) = \frac{1}{x+1}$, f is differentiable on $]0,c[$.

So we may apply the M.V.T.

For $x > 0$, there is a $\theta \in]0,1[$ with

$$f'(\theta x) = \frac{f(x)-f(0)}{x-0} = \frac{\ln(x+1)}{x}$$

i.e.

$$\frac{1}{1+\theta x} = \frac{\ln(x+1)}{x}$$

So, since $x > 0$,

$$x = (\ln(x+1))(1+\theta x)$$
$$< (\ln(x+1))(1+x) \quad \text{as } 0<\theta<1$$

i.e. $\quad \frac{x}{1+x} < \ln(x+1) \quad$ as $1+x>0$

(b) If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and $a<b$ are successive zeros of f, then, by MVT we can find a $c \in]a,b[$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$$

Q11

Let the length of each side of the equilateral triangle be x.
Its perimeter is then $3x$ and its area $\frac{x^2\sqrt{3}}{4}$
Let the length of each side of the square be y.
Its perimiter is then $4y$ and its area y^2
Thus $0 < x, y < L$ and
$$3x + 4y = L \quad \text{—— (i)}$$
The total area is
$$A = \frac{\sqrt{3}}{4}x^2 + y^2 \quad \text{—— (ii)}$$

From (i) $3 + 4\frac{dy}{dx} = 0$ —— (iii)
From (ii) $\frac{\sqrt{3}}{2}x + 2y\frac{dy}{dx} = \frac{dA}{dx}$ —— (iv)
For an extreme value, we must have $\frac{dA}{dx} = 0$
ie $\sqrt{3}x + 4y\frac{dy}{dx} = 0$ —— (v)
Then (iii) + (v) have a non-trivial solution if and only if $12y - 4\sqrt{3}x = 0$ ie $x = \sqrt{3}y$
So, from (i) $y = \frac{L}{3\sqrt{3}+4}$ & $x = \frac{\sqrt{3}L}{3\sqrt{3}+4}$

From (iii) $4\frac{d^2y}{dx^2} = 0$ —— (vi)
From (iv) $\frac{d^2A}{dx^2} = \frac{\sqrt{3}}{2} + 2\left(\frac{dy}{dx}\right)^2 + 2\frac{d^2y}{dx^2}$

So, by (iii) + (vi) $\frac{d^2A}{dx^2} = \frac{\sqrt{3}}{2} + \frac{9}{2} > 0$
Hence the total area is a minimum when
$$x = \frac{L\sqrt{3}}{3\sqrt{3}+4} \quad \text{and} \quad y = \frac{L}{3\sqrt{3}+4}$$

There is no maximum.

Q12

(a) Let $\underline{u} = (u_1, u_2, u_3)$ $\underline{v} = (v_1, v_2, v_3)$

Then $\underline{u}\times\underline{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$

So $|\underline{u}\times\underline{v}|^2 = (u_2v_3-u_3v_2)^2 + (u_3v_1-u_1v_3)^2 + (u_1v_2-u_2v_1)^2$

$$= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2$$
$$+ u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2$$
$$+ u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2$$

On the other hand

$$|\underline{u}|^2|\underline{v}|^2 = (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)$$
$$= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2$$
$$+ u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2$$
$$+ u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2$$

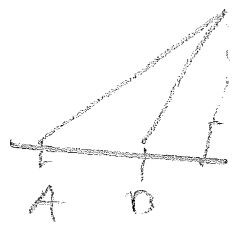
$$(\underline{u}\cdot\underline{v})^2 = (u_1v_1+u_2v_2+u_3v_3)^2$$
$$= u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$$
$$+ 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3$$

Thus $|\underline{u}|^2|\underline{v}|^2 - (\underline{u}\cdot\underline{v})^2 = u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2$

$$+ u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2$$
$$+ u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_2^2$$
$$= |\underline{u}\times\underline{v}|^2$$

(b) Let $P \in \mathbb{R}^3$ be any point and $A \neq B$ two distinct points in $\mathbb{R}^3$. Then the area of the triangle with vertices A, B, P is $\frac{1}{2}|\vec{PA}\times\vec{PB}|$

If the perpendicular distance from P to the line through A & B is d, then the area of the triangle is $\frac{1}{2}d.\,|\vec{AB}|$



ie $d = \dfrac{|\vec{PA}\times\vec{PB}|}{|\vec{AB}|}$