# Sample Solutions for Tutorial 2

## Question 1.

We know that $\mathbb{Q}$ and $\mathbb{R}$ are fields with respect to the usual arithmetic operations, and that $\mathbb{Q} \subseteq \mathbb{R}$ is, in fact, a *sub-field* of $\mathbb{R}$.

We take $\mathbb{F} := \mathbb{Q}$, so that $0_{\mathbb{F}} = 0$ and $1_{\mathbb{F}} = 1$, $V := \mathbb{R}$ and for the operations we have for $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ and $\lambda \in \mathbb{Q}$

$$\begin{aligned} \mathbf{x} \boxplus \mathbf{y} &:= \mathbf{x} + \mathbf{y} \\ \lambda \boxdot \mathbf{x} &:= \lambda \mathbf{x} \end{aligned}$$

where the operations on the right hand side of the equations are the usual arithmetic ones. Finally, we put $\mathbf{0}_{\mathbb{R}} := 0$.

Then VS1 – VS4 simply (re-)state that the real numbers form an abelian group under addition, corresponding, as they do, to axioms A1 – A4 for fields. The other vector space axioms also follow from the field axioms for $\mathbb{R}$, as we demonstrate. Take $\lambda, \mu \in \mathbb{Q}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}$.

VS5 $1_{\mathbb{Q}} \boxdot \mathbf{x} := 1\mathbf{x} = \mathbf{x}$ by M2.

VS6 $\lambda \boxdot (\mathbf{x} \boxplus \mathbf{y}) := \lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$ by D.

VS7 $(\lambda + \mu) \boxdot \mathbf{x} := (\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$ by D.

VS8 $(\lambda\mu) \boxdot \mathbf{x} := (\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x}) =: \lambda \boxdot (\mu \boxdot \mathbf{x})$ by M1.

Comment. The most difficult parts of this question were probably the insight that there was something to prove and knowing how to express it.

But, having seen a solution, note that nowhere did we use any special features of $\mathbb{Q}$ and $\mathbb{R}$ other than the fact that $\mathbb{Q}$ is a sub-field of $\mathbb{R}$. Thus the same argument applies to show that if $\mathbb{F}$ is a sub-field of $\mathbb{K}$, then $\mathbb{K}$ forms a vector space over $\mathbb{F}$ with respect to the field operations. This fact is important in abstract algebra and number theory.

## Question 2.

The verification of the field axioms is a tedious, purely mechanical, finite procedure, best done over a few drinks while waiting for something more interesting on television. Here are some typical computations.

A1 There are $3^3 = 27$ equalities to check. Here is one. The intermediate steps are read off the table given:

$$\begin{aligned} (a + b) + c &= b + c = a \\ a + (b + c) &= a + a = a, \end{aligned}$$

showing that $(a + b) + c = a + (b + c)$.

A2 This can be seen to hold by inspection, with $a$ as $0_{\mathbb{F}}$: the row headed by $a$ shows that $a + x = x$ for all $x \in \mathbb{F}$, and the column headed by $a$ shows that $x + a = x$ for all $x \in \mathbb{F}$.

A3 It follows by inspection of the table that $-a = a$, $-b = c$ and $-c = b$.

A4 This follows by inspection from the symmetry of the table about its principal diagonal.

Similar arguments apply to multiplication, with $1_{\mathbb{F}} = b$. We read off that $b^{-1} = b$ and $c^{-1} = c$.

Finally, the distributivity of multiplication over addition requires $2 \times 2^3 = 54$ equalities to be verified. Here are two, with the intermediate steps read off the appropriate table.

$$\begin{aligned} a(b + c) &= aa = a \\ ab + ac &= a + a = a \end{aligned}$$

showing that $a(b + c) = ab + ac$.

$$\begin{aligned} (a + b)c &= bc = c \\ ac + bc &= a + c = c \end{aligned}$$

showing that $(a+b)c = ac + bc$.
[Given the quality current television programmes, I am confident you completed the verifications in one sitting!]

## Question 3.

(a) $\alpha \boxdot z \qquad (\alpha, z \in \mathbb{C})$
Then $(1+1) \boxdot i = 2 \boxdot i = 4i$, but $1 \boxdot i + 1 \boxdot i = 1^2 i + 1^2 i = 2i$, which shows that axiom VS7 does not hold. (But note that VS5, VS6 and VS8 do hold.)

(b) Note that $1_{\mathbb{F}} \boxdot (\beta, \gamma) := (1_{\mathbb{F}}\beta, 0) = (\beta, 0)$, which does not agree with $(\beta, \gamma)$ unless $\gamma = 0$, showing that axiom VS5 fails. (But note that VS6, VS7 and VS8 do hold.)

(c) Note that for all $\alpha \in \mathbb{F}_2, \alpha^2 = \alpha$. Hence $\alpha \boxdot (\beta, \gamma) = (\alpha\beta, \alpha\gamma)$ for all $(\beta, \gamma) \in V$, as $\alpha 0 = 0$. Moreover, as abelian group, $\mathbb{F}_4$ is just $(\mathbb{F}_2)^2$ and the vector space operations we have just defined coincide with the "usual" (component-wise) operations on $\mathbb{F}^2$ for any field $\mathbb{F}$.

Hence, this is, indeed, an example of a vector space, a familiar one, even if the operations were defined somewhat exotically.

(d) Let $\alpha = x + iy$ and $\beta = u + iv$, with $u, v, x, y \in \mathbb{R}$. Then $\Re(\alpha) = x, \Re(\beta) = u$ and $\Re(\alpha\beta) = \Re((xu - yv) + i(xv + yu)) = xu - yv$. So, taking $\alpha = \beta = i$ and $z = 1$, we have $(\alpha\beta) \boxdot 1 = -1.1 = -1$, but $\alpha \boxdot (\beta \boxdot z) = 0.(0.1) = 0$, which shows that axiom VS8 fails (But note that VS5, VS6 and VS7 do hold.)

(e) Since the product of two positive real numbers is again a positive real number, since the multiplicative inverse of a positive real number is again a positive real number, and since 1 is a positive real number, $\mathbb{R}^+$ is an abelian group under $\boxplus$ from the axioms for real numbers. Moreover,

$$\begin{aligned}
1 \boxdot x &:= x^1 = x \\
\lambda \boxdot (x \boxplus y) &:= (xy)^\lambda = x^\lambda y^\lambda =: (\lambda \boxdot x) \boxplus (\lambda \boxdot y) \\
(\lambda + \mu) \boxdot x &:= x^{\lambda+\mu} = x^\lambda x^\mu =: (\lambda \boxdot x) \boxplus (\mu \boxdot x) \\
(\lambda\mu) \boxdot x &:= x^{\lambda\mu} = (\lambda^\mu)^\lambda =: \lambda \boxdot (\mu \boxdot x),
\end{aligned}$$

the intermediate steps following from the standard properties of real numbers.

Thus, we do have a vector space.

## Question 4.

(i) Suppose $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$.
Then $(-\mathbf{u}) + (\mathbf{u} + \mathbf{v}) = (-\mathbf{u}) + (\mathbf{u} + \mathbf{w})$, by the existence of $-\mathbf{u}$.
Thus $((-\mathbf{u}) + \mathbf{u}) + \mathbf{v} = ((-\mathbf{u}) + \mathbf{u}) + \mathbf{w}$, by the associativity of vector addition.
Thus $\mathbf{0}_V + \mathbf{v} = \mathbf{0}_V + \mathbf{w}$, by the defining property of $-\mathbf{u}$.
Thus $\mathbf{v} = \mathbf{w}$, by the defining property of $\mathbf{0}_V$.

(ii) If $\mathbf{u} + \mathbf{x} = \mathbf{u} + \mathbf{y} = \mathbf{v}$, then, by (i), $\mathbf{x} = \mathbf{y}$.
Thus, given $\mathbf{u}, \mathbf{v}$, the equation $\mathbf{u} + \mathbf{x} = \mathbf{v}$ has at most one solution $\mathbf{x}$.
Now, if $\mathbf{u} + \mathbf{x} = \mathbf{v}$, then $(-\mathbf{u}) + (\mathbf{u} + \mathbf{x}) = -\mathbf{u} + \mathbf{v}$, by the existence of $-\mathbf{u}$.
Thus $((-\mathbf{u}) + \mathbf{u}) + \mathbf{x} = -\mathbf{u} + \mathbf{v}$, by associativity.
Thus $\mathbf{0}_V + \mathbf{x} = -\mathbf{u} + \mathbf{v}$, by the defining property of $-\mathbf{u}$.
Thus $\mathbf{x} = -\mathbf{u} + \mathbf{v}$, by the defining property of $\mathbf{0}_V$.
Hence, $-\mathbf{u} + \mathbf{v} = \mathbf{v} + (-\mathbf{u})$ is <u>the</u> solution.

(iii) Consider$-(-\mathbf{u})$.

$$\begin{aligned}
-(-\mathbf{u}) &= -(-\mathbf{u}) + \mathbf{0}_V && \text{by the defining property of } \mathbf{0}_V \\
&= -(-\mathbf{u}) + ((-\mathbf{u}) + \mathbf{u}) && \text{by the defining property of } -\mathbf{u} \\
&= (-(-\mathbf{u}) + (-\mathbf{u})) + \mathbf{u} && \text{by associativity} \\
&= \mathbf{0}_V + \mathbf{u} && \text{by the defining property of } -(-\mathbf{u}) \\
&= \mathbf{u} && \text{by the defining property of } \mathbf{0}_V.
\end{aligned}$$

(iv) $\mathbf{v} + \mathbf{0}_V = \mathbf{v} = 1.\mathbf{v} = (1+0).\mathbf{v} = 1.\mathbf{v} + 0\mathbf{v}$.
Hence, by (i), $0.\mathbf{v} = \mathbf{0}_V$.

(v) $\alpha\mathbf{u} + (-(\alpha\mathbf{u})) = \mathbf{0}_V = 0\mathbf{u} = (\alpha + (-\alpha))\mathbf{u} = \alpha\mathbf{u} + (-\alpha)\mathbf{u}$.
Hence, by (i), $(-\alpha)\mathbf{u} = -(\alpha\mathbf{u})$.
Note that $\alpha\mathbf{u} + \mathbf{0}_V = \alpha\mathbf{u} = \alpha(\mathbf{u} + \mathbf{0}_V) = \alpha\mathbf{u} + \alpha\mathbf{0}_V$.
Hence, by (i), $\alpha\mathbf{0}_V = \mathbf{0}_V$.
Thus, $\alpha\mathbf{u} + \alpha(-\mathbf{u}) = \alpha(\mathbf{u} + (-\mathbf{u})) = \alpha\mathbf{0}_V = \mathbf{0}_V = \alpha\mathbf{u} + (-(\alpha\mathbf{u}))$.
Hence, by (i), $\alpha(-\mathbf{u}) = -(\alpha\mathbf{u})$.

(vi) By (v) and (iii), $-\alpha(-\mathbf{u}) = \alpha(-(-\mathbf{u})) = \alpha\mathbf{u}$.

(vii) Suppose that $\alpha\mathbf{u} = \alpha\mathbf{v}$ and $\alpha \neq 0$.
Then, $\frac{1}{\alpha}(\alpha\mathbf{u}) = \frac{1}{\alpha}(\alpha\mathbf{v})$.
Thus, $(\frac{1}{\alpha}\alpha)\mathbf{u} = (\frac{1}{\alpha}\alpha)\mathbf{v}$.
Hence, $1.\mathbf{u} = 1.\mathbf{v}$, that is, $\mathbf{u} = \mathbf{v}$.
Thus, $\alpha\mathbf{u} = \alpha\mathbf{v}$ only if either $\alpha = 0$ or $\mathbf{u} = \mathbf{v}$.

Comments. Question 2, if done in complete detail (which you SHOULD do at least once in your life) should convince you that attempting problems using case-by-case verification is not only laborious, but also impracticable even with "small" problems: given the (Cayley) tables for an ostensible field with 13 elements, there are over 4,000 equalities to verify. This should make you receptive to more theoretical methods.

Question 3 established the independence of several of the vector space axioms.

Question 4, when specialised to the case $\mathbb{F} = V = \mathbb{R}$, contains proofs of results every child learns at school, but rarely, if ever, sees explained rigorously. Those who share my misgivings concerning just accepting things without sound reason, this exercise should be a release from low-level anxiety, or ease a disturbed conscience.