Sample Solutions for Tutorial 6

Question 1.

Take $a \in \mathbb{R}$. Then, for all $x \in \mathbb{R}$,

$$\cos x - \cos a = -2\sin(\tfrac{x+a}{2})\sin(\tfrac{x-a}{2})$$

Thus, taking $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \cos x$, we have

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{\cos x - \cos a}{x-a} \\ &= \lim_{x\to a}\frac{-2\sin(\frac{x+a}{2})\sin(\frac{x-a}{2})}{x-a} \\ &= -\lim_{x\to a}\sin(\tfrac{x+a}{2})\lim_{x\to a}\frac{\sin(\frac{x-a}{2})}{\frac{x-a}{2}} \\ &= -\lim_{x\to a}\sin(\tfrac{x+a}{2})\lim_{u\to 0}\frac{\sin u}{u} \qquad\qquad \text{where } u := \tfrac{x-a}{2} \\ &= -\sin a \end{aligned}$$

as the sine function is continuous and $\displaystyle\lim_{u\to 0}\frac{\sin u}{u} = 1$.

Question 2.

(a) f is the sum of the polynomial function given by $3x^5 - 6x^3$ and the cosine function. Since both of these are differentiable everywhere, so is f and its derivative is the sum of there derivatives. Thus, for every $x \in \mathbb{R}$,

$$f'(x) = 15x^4 - 18x^2 - \sin x.$$

(b) Since $g(x) = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x \ge 0 \end{cases}$, it is immediate that g is differentiable at a whenever $a \neq 0$, with

$$f'(a) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Now

$$g'(0) = \lim_{x\to 0}\frac{|x| - |0|}{x - 0} = \lim_{x\to 0}\frac{|x|}{x}$$

But

$$\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1$$

whereas

$$\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}(-1) = -1.$$

Since $\lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0} \neq \lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0}$, g is not differentiable at 0.

(c) Note that $h = \beta \circ \alpha$, with α, β given by

$$\begin{aligned} \alpha\colon [-1,1] &\longrightarrow [0,1], \quad x \longmapsto 1 - x^2 \\ \beta\colon [0,1] &\longrightarrow [0,1], \quad u \longmapsto \sqrt{u} \end{aligned}$$

Since α is a polynomial function, it is differentiable, with derivative is given by

$$\alpha'(x) = -2x.$$

β is differentiable at u when $u > 0$, with

$$\beta'(u) = \frac{1}{2\sqrt{u}} = \frac{1}{2\beta(u)}$$

Take $u \geq 0$. Then

$$\frac{\beta(u) - \beta(0)}{u - 0} = \frac{\sqrt{u}}{u} = \frac{1}{\sqrt{u}}$$
$$\rightarrow \infty \quad \text{as } u \rightarrow 0^+.$$

This shows that β is not differentiable at 0.
Since $\alpha(x) = 0$ if and only if $x = 0, l1,$we see that h is differentiable at every $a \in]0, 1[$, but not at $a = 0, 1$, and

$$\begin{aligned} h'(a) &= \beta'(\alpha(a))\alpha'(a) \\ &= \frac{1}{2\beta(\alpha(a))}(-2a) \\ &= \frac{-a}{\sqrt{1 - a^2}} \end{aligned}$$

(d) Since k is not continuous at $a \neq 0$, k cannot be differentiable at $a \neq 0$.

$$\begin{aligned} \frac{k(x) - k(0)}{x - 0} &= \frac{k(x)}{x} \\ &= \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise,} \end{cases} \\ &\longrightarrow 0 \quad \text{as } x \longrightarrow 0 \end{aligned}$$

Hence k is differentiable only at 0 and $k'(0) = 0$.

Question 4.

Recall that there is a best approximation of the form $p_1(x) = c_0 + c_1 x$ near $x = a$ to the function, f, if and only if f is differentiable at $x = a$, and then

$$p_1(x) = f'(a)(x - 1) + f(a).$$

This is then the equation of the line in the Cartesian plane which is tangent to the graph of f at the point with coordinates $(a, f(a))$.

$$\tan x := \frac{\sin x}{\cos x}$$

is differentiable everywhere on $] - \frac{\pi}{2}, \frac{\pi}{2}[$, with

$$\begin{aligned} \frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{d}{dx}\left(\sin x \frac{1}{\cos x}\right) \\ &= \cos x \frac{1}{\cos x} + \sin x(-\sin x)\left(\frac{-1}{\cos^2 x}\right) \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \end{aligned}$$

Now $\cos 0 = 1$ and $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so that the equation of the line tangent to the graph at the point $(0, 0)$ is $y = 1(x - 0) = 0$, or

$$x - y = 0,$$

and that of the line tangent to the graph at the point $(\frac{\pi}{4}, 1)$ is $y = \frac{1}{\frac{1}{2}}(x - \frac{\pi}{4}) + 1$, or

$$2x - y = \frac{\pi}{2} - 1$$