THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: PMTH 213/213A

PAPER TITLE: Linear Algebra

PAPER NUMBER: First and Only

DATE: **Monday 20th November 2006** **TIME:** **9.30 AM TO 12.30 PM**

TIME ALLOWED: Three (3) hours plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: THREE (3)

NUMBER OF QUESTIONS ON PAPER: EIGHT (8)

NUMBER OF QUESTIONS TO BE ANSWERED: EIGHT (8)

STATIONERY PER CANDIDATE: **1** X 6 LEAF A4 BOOKS **1** X 12 LEAF A4 BOOKS

0 X ROUGH WORK BOOKS

GRAPH PAPER: **NIL** (NUMBER OF SHEETS)

POCKET CALCULATORS PERMITTED: **NO**

OTHER AIDS REQUIRED: **NIL**

INSTRUCTIONS FOR CANDIDATES:

- Candidates may make notes of this paper during the fifteen minutes reading time
- Questions are not of equal value
- Candidates may retain their copy of this examination question paper

TEXTBOOKS OR NOTES PERMITTED: FIVE (5) A4 sheets (10 pages) of handwritten notes. No photocopies allowed. No printed notes allowed.

THE UNIVERSITY CONSIDERS IMPROPER CONDUCT IN EXAMINATIONS TO BE A SERIOUS OFFENCE. PENALTIES FOR CHEATING ARE EXCLUSION FROM THE UNIVERSITY FOR ONE YEAR AND/OR CANCELLATION OF ANY CREDIT RECEIVED IN THE EXAMINATION FOR THAT UNIT.

Question 1: [10 *marks*]

Find all linear transformations, $T : \mathbb{R}^3 \to \mathbb{R}^2$, which map the plane determined by the equation $x - y - 2z = 0$ onto the line determined by the equation $u + v = 0$.

Question 2: [8 *marks*]

Determine whether the real quadratic form

$$Q(x, y, z) = x^2 + 8xy + 4y^2 + 8yz + 7z^2 + 6zx$$

is positive or negative definite, positive or negative semi-definite, or indefinite, justifying your answer.

Question 3: [10 *marks*]

Find all 2×2 complex matrices, $\underline{\mathbf{A}}$, with $\underline{\mathbf{A}}^2 = \underline{\mathbf{0}}_2$, where $\underline{\mathbf{0}}_2$ denotes the 2×2 zero matrix.

Question 4: [10 *marks*]

The matrix $\underline{\mathbf{A}}$ is *skew-symmetric* if $\underline{\mathbf{A}}^t = -\underline{\mathbf{A}}$, where $\underline{\mathbf{A}}^t$ denotes the transpose of $\underline{\mathbf{A}}$.
Show that the skew-symmetric $n \times n$ matrices form a vector subspace of the real vector space of all $n \times n$ matrices with real coefficients.
Show that if $\underline{\mathbf{A}}$ is a real skew-symmetric $n \times n$ matrix with n odd, then $\det(\underline{\mathbf{A}}) = 0$.

Question 5: [12 *marks*]

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for the vector space V and $T : V \longrightarrow V$ a linear transformation.

(a) Show that if $\mathbf{f}_1 := \mathbf{e}_1,\ \mathbf{f}_2 := 2\mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{f}_3 := 3\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$, then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is also a basis for V.

(b) Find the matrix, $\underline{\mathbf{B}}$, of T with respect to $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, given that its matrix with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

[**Question 6** *is on page 3.*]

Question 6: [10 *marks*]

Let V and W be finitely generated vector spaces over the field $\mathbb{F}$. Prove that the linear transformation, $T : V \to W$, is an isomorphism if and only if $\{T(\mathbf{e}_j) \mid j = 1, \ldots, n\}$ is a basis for W whenever $\{\mathbf{e}_j \mid j = 1, \ldots, n\}$ is a basis for V.

Question 7: [15 *marks*]

Given the symmetric matrix $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$, find

(a) its eigenvalues,

(b) bases for its eigenspaces,

(c) an orthogonal matrix, P, which diagonalises A, and

(d) $P^{-1}AP$.

Question 8: [25 *marks*]

Let $\mathcal{P}_2$ be the real vector space of all polynomials in t with real coefficients, whose degree is at most 2, so that $V = \{at^2 + bt + c \mid a, b, c \in \mathbb{R}\}$.

(a) Prove that

$$\langle\langle p, q \rangle\rangle := p(-1)q(-1) + p(0)q(0) + p(1)q(1) \qquad (p, q \in \mathcal{P}_2)$$

defines an inner product on $\mathcal{P}_2$.

(b) Apply the Gram-Schmidt procedure with respect to this inner product to find an orthonormal basis for the vector subspace of $\mathcal{P}_2$ generated by

$$\mathbf{v}_1 := t^2 + t, \ \mathbf{v}_2 := t^2 + 1 \ \text{ and } \ \mathbf{v}_3 := t + 1$$