MATH101 ASSIGNMENT 5

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(1) (a) *Non-monotone*

$$\begin{aligned}
\text{Let } u_n &= \frac{n^2-n+1}{2-n^2}, \text{ then} \\
u_{n+1} &= \frac{(n+1)^2-(n+1)+1}{2-(n+1)^2} \\
&= \frac{n^2+n+1}{1-2n-n^2}, \text{ and} \\
\frac{u_{n+1}}{u_n} &= \frac{n^2+n+1}{1-2n-n^2} \cdot \frac{2-n^2}{n^2-n+1} \\
&= \frac{-n^4-n^3+n^2+2n+2}{-n^4-n^3+2n^2-3n+1}
\end{aligned}$$

Thus $u_{n+1} > u_n$ if and only if $2n^2-3n+1 < n^2+2n+2$. That is, $n^2-5n-1<0$. This is true if and only if $n \leq 5$. Consequently, the sequence is increasing when $2 \leq n < 6$ and decreasing when $n \geq 6$. Explicitly,

$$u_2 = -\frac{3}{2},\ \ u_3=-1,\ \ u_4 = -\frac{13}{14},\ \ u_5=-\frac{21}{23},\ \ u_6 = -\frac{31}{34},\ \ u_7=-\frac{43}{47},\ \ u_8=-\frac{57}{62}$$

Hence u_n is not monotonic since $u_2<u_3<u_4<u_5<u_6>u_7>u_8$.
Moreover, as $n \to \infty$, $u_n \to -1$. We show this by dividing out by n^2 and applying infinite limits such that

$$\begin{aligned}
\frac{n^2-n+1}{2-n^2} &= \frac{1-\frac{1}{n}+\frac{1}{n^2}}{\frac{2}{n^2}-1} \\
&\longrightarrow \frac{1-0-0}{0-1} = -1 \text{ as } n \longrightarrow \infty
\end{aligned}$$

(b) *Monotone increasing*

Let $u_n = \sqrt{2n^2-1}$, then

$$u_m - u_n = \sqrt{2m^2-1} - \sqrt{2n^2-1}$$

Since $2m^2-1>0$ and $2n^2-1>0$, $u_m-u_n>0$ if and only if $m>n$, showing that $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing. It is then plain to see that $u_n \to \infty$ as $n \to \infty$.

(c) *Monotone increasing*
We first note that u_n consists of two functions, namely n and $\sin\left(\frac{1}{n}\right)$. While the former is clearly increasing and unbounded above in $\mathbb{N}$, we must consider the oscillating sine function more carefully. We know that $\frac{1}{n} \to 0$ as $n \to \infty$. Since sine is continuous, then $\sin\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$. This suggests there are two counteracting effects as n tends to infinity. We also observe that $\sin\left(\frac{1}{n}\right)$ does not oscillate in $\mathbb{N}$ but is strictly decreasing. Due to the complexity of this composition, approximations are used to determine monotonicity.

$$u_1 \approx .84147, \quad u_2 \approx .95885, \quad u_3 \approx .98158, \quad u_4 \approx .98962, \quad u_5 \approx .99335$$

Furthermore, $u_{100} \approx .99998$. From these observations, we conclude that $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing since $u_1 < u_2 < ... < u_n < u_{n+1}$ ...
We also notice above that the sequence approaches 1 as n approaches infinity. This can be shown if we let $x = \frac{1}{n}$ such that

$$\begin{aligned} n\sin\left(\frac{1}{n}\right) &= \frac{1}{x}\sin x \\ &= \frac{\sin x}{x} \\ &\longrightarrow 1 \text{ as } x \longrightarrow 0 \end{aligned}$$

Therefore, $u_n \to 1$ as $n \to \infty$.

(d) *Monotone decreasing*

$$\begin{aligned} \text{Let } u_n &= \frac{n!}{n^n}, \text{ then} \\ u_{n+1} &= \frac{(n+1)!}{(n+1)^{n+1}} \text{ and} \\ \frac{u_{n+1}}{u_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \frac{n^n}{(n+1)^n} \end{aligned}$$

Thus, $u_{n+1} < u_n$ if and only if $n^n < (n+1)^n$. This is clearly true for all n so $(u_n)_{n\in\mathbb{N}}$ is monotonically decreasing.
To determine the behaviour of u_n as $n \to \infty$, we consider the following. Clearly, both $n!$ and n^n are strictly increasing and unbounded above (which might imply u_n also increases without bound). But given sufficiently large n, the denominator outgrows the numerator by an exponential magnitude that their ratio is infinitesimally close to zero. Explicitly, we observe that even for, say $n = 10$:

$$u_{10} = \frac{10!}{10^{10}} = \frac{567}{1562500} \approx .000363$$

Therefore, $u_n \to 0$ as $n \to \infty$

(e) *Monotone constant*
Since $\cos(2n\pi) = 1$ for all $n \in \mathbb{N}$, then

$$u_n = \frac{1}{\cos(2n\pi)} = \frac{1}{1} = 1$$

That is, $u_1 = u_2 = ... = u_n = ... = 1$. It then follows that as $n \to \infty$, $u_n = 1$.

(f) *Monotone decreasing*

$$\begin{aligned} \text{Let } u_n &= \sqrt{n+1} - \sqrt{n-1}, \text{ then} \\ u_{n+1} &= \sqrt{(n+1)+1} - \sqrt{(n+1)-1} = \sqrt{n+2} - \sqrt{n} \text{ and} \\ \frac{u_{n+1}}{u_n} &= \frac{\sqrt{n+2} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n-1}} \end{aligned}$$

Thus, $u_{n+1} < u_n$ if and only if $\sqrt{n+2} - \sqrt{n} < \sqrt{n+1} - \sqrt{n-1}$. We observe this to be true for all n so $(u_n)_{n \in \mathbb{N}}$ is monotonically decreasing.
Moreover, as $n \to \infty$, $u_n \to 0$. We show this by dividing by n and applying infinite limits such that

$$\begin{aligned} \sqrt{n+1} - \sqrt{n-1} &= \sqrt{1 + \frac{1}{n}} - \sqrt{1 - \frac{1}{n}} \\ &\longrightarrow \sqrt{1+0} - \sqrt{1-0} = 1 - 1 = 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

(2) *Zeno's Bug*
The bug does not get to eat. As the bug crawls to its destination, it only gets halfway there each minute before it begins to tire and slows down. Consequently, the distance the bug is required to travel can be described by the following sequence, $u_n = \{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\}$. Even though the sequence is monotonically decreasing, the bug will never reach its destination since u_n only tends to 0 as n tends to infinity.

(3) (a) *Convergent*

$$\frac{n^2}{n^4+1} < \frac{n^2+n^2}{n^4} = \frac{2n^2}{n^4} = \frac{2}{n^2}$$

Since $\sum \frac{1}{n^k}$ converges if $k > 1$, then $\sum \frac{1}{n^2}$ converges. So comparison with the convergent series $\sum \frac{2}{n^2}$ shows that

$$\sum_{n=0}^{\infty} \frac{n^2}{n^4+1} \quad \text{converges.}$$

(b) *Convergent*

$$\frac{n3^n}{5^n+1} < \frac{n3^n + n3^n + n3^n}{5^n+1} < \frac{n3^{n+1}}{5^n} < \frac{3^n \cdot 3^{n+1}}{5^n} = \frac{3^{2n+1}}{5^n} = \frac{3 \cdot (3^2)^n}{5^n}$$

That is,

$$\frac{n3^n}{5^n+1} \;<\; 3\left(\frac{9^n}{5^n}\right) \;=\; 3\left(\frac{9}{5}\right)^n \;=\; 3\;\frac{1}{(0.\dot{5})^n}$$

So comparison with the convergent geometric series $\sum \frac{1}{(0.\dot{5})^n}$ shows that

$$\sum_{n=1}^{\infty} \frac{n3^n}{5^n+1} \quad \text{converges.}$$

(c) *Convergent*
Since $|\cos n| \leq 1$, then

$$\frac{|\cos n|}{n^2-1} \;<\; \frac{1}{n^2-1}$$

We then expect $\sum \frac{1}{n^2-1}$ to converge since the nth term of the series is essentially $\frac{1}{n^2}$ for sufficiently large n. Thus we compare $\sum \frac{1}{n^2}$ with

$$\sum_{n=2}^{\infty} \frac{|\cos n|}{n^2-1} \quad \text{to show that the latter series converges.}$$

(d) *Divergent*

$$\frac{1}{\sqrt{2n^2+1}} \;>\; \frac{1}{\sqrt{2n^2+2n^2}} \;=\; \frac{1}{\sqrt{4n^2}} \;=\; \frac{1}{2}\cdot\frac{1}{n}$$

So comparison with the divergent harmonic series $\sum \frac{1}{n}$ shows that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}} \quad \text{diverges.}$$

(4) (a) *Divergent*

$$\begin{aligned}
\text{Let } u_n \quad &= \quad \frac{5^n}{n^3}, \;\; \text{then} \\
u_{n+1} \quad &= \quad \frac{5^{n+1}}{(n+1)^3}, \;\; \text{and} \\
\frac{u_{n+1}}{u_n} \quad &= \quad \frac{5^{n+1}}{(n+1)^3}\cdot\frac{n^3}{5^n} \;=\; \frac{5n^3}{(n+1)^3} \\
&= \quad \frac{5n^3}{n^3+3n^2+3n+1} \\
&= \quad \frac{5}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}} \\
&\longrightarrow \quad \frac{5}{1+0+0+0} \;=\; 5 \;\; \text{as} \;\; n \longrightarrow \infty
\end{aligned}$$

The series $\sum \frac{5^n}{n^3}$ diverges since the ratio test yields a number greater than 1.

(b) *Convergent*

$$\begin{aligned}
\text{Let } u_n &= \frac{7^n}{n!}, \text{ then} \\
u_{n+1} &= \frac{7^{n+1}}{(n+1)!}, \text{ and} \\
\frac{u_{n+1}}{u_n} &= \frac{7^{n+1}}{(n+1)!} \cdot \frac{n!}{7^n} = \frac{7}{n+1} = \frac{\frac{7}{n}}{1+\frac{1}{n}} \\
&\longrightarrow \frac{0}{1+0} = 0 \text{ as } n \longrightarrow \infty
\end{aligned}$$

The series $\sum \frac{7^n}{n!}$ converges since the ratio test yields a number less than 1.

(c) *Convergent*

$$\begin{aligned}
\text{Let } u_n &= \frac{2^n}{n^n}, \text{ then} \\
u_{n+1} &= \frac{2^{n+1}}{(n+1)^{n+1}}, \text{ and} \\
\frac{u_{n+1}}{u_n} &= \frac{2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n} = \frac{2n^n}{(n+1)^{n+1}}
\end{aligned}$$

Now by the same reasoning as in *Question 1(d)* we conclude that

$$\frac{n^n}{(n+1)^{n+1}} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Explicitly, for sufficiently large n, say $n = 100$:

$$u_n = \frac{100^{100}}{101^{101}} \approx .00366$$

The series $\sum \frac{2^n}{n^n}$ converges since the ratio test yields a number less than 1.

(d) *Convergent*

$$\begin{aligned}
\text{Let } u_n &= \sqrt{\frac{n}{n^3+1}}, \text{ then} \\
u_{n+1} &= \sqrt{\frac{n+1}{(n+1)^3+1}}, \text{ and} \\
\frac{u_{n+1}}{u_n} &= \sqrt{\frac{n+1}{n^3+3n^2+3n+2}} \cdot \sqrt{\frac{n^3+1}{n}} \\
&= \sqrt{\frac{\frac{1}{n^2}+\frac{1}{n^3}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}}} \cdot \sqrt{\frac{1+\frac{1}{n^3}}{\frac{1}{n^2}}} \longrightarrow 0 \text{ as } n \longrightarrow \infty
\end{aligned}$$

The series $\sum \sqrt{\frac{n}{n^3+1}}$ converges since the ratio test yields a number less than 1.