MATH101 ASSIGNMENT 12

MARK VILLAR

(1) After converting each matrix to triangular form we use the properties of triangular matrices to evaluate the following determinants.

(a)

$$\begin{array}{rrrrr} & 2 & 0 & -1 & 0 \\ R_2 + R_4 & 1 & 5 & 3 & 0 \\ & -2 & 3 & 1 & 0 \\ & 0 & 4 & 2 & -1 \end{array} \Rightarrow \begin{array}{rrrrr} & 2 & 0 & -1 & 0 \\ R_2 + 3R_1 & 7 & 5 & 0 & 0 \\ & -2 & 3 & 1 & 0 \\ & 0 & 4 & 2 & -1 \end{array}$$

$$\begin{array}{rrrrr} R_1 + R_3 & 0 & 3 & 0 & 0 \\ & 7 & 5 & 0 & 0 \\ & -2 & 3 & 1 & 0 \\ & 0 & 4 & 2 & -1 \end{array} \Rightarrow \begin{array}{rrrrr} R_1 - \frac{3}{5}R_2 & -\frac{21}{5} & 0 & 0 & 0 \\ & 7 & 5 & 0 & 0 \\ & -2 & 3 & 1 & 0 \\ & 0 & 4 & 2 & -1 \end{array}$$

Thus we have a lower triangular matrix with determinant equal to the product of its main diagonal elements. It follows that

$$\begin{vmatrix} 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ -2 & 3 & 1 & 0 \\ 0 & 4 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -\frac{21}{5} & 0 & 0 & 0 \\ 7 & 5 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 0 & 4 & 2 & -1 \end{vmatrix}$$

$$= -\frac{21}{5} \times 5 \times 1 \times (-1) = 21$$

(b)

$$\begin{array}{rrrrr} & -1 & 3 & 2 & 1 \\ & -7 & -6 & 0 & -2 \\ R_3 - 7R_1 & -3 & 0 & -16 & 0 \\ & 1 & 9 & -5 & 3 \end{array} \Rightarrow \begin{array}{rrrrr} R_1 + \frac{1}{2}R_2 & \frac{5}{2} & 0 & 2 & 0 \\ & -7 & -6 & 0 & -2 \\ & -3 & 0 & -16 & 0 \\ & 1 & 9 & -5 & 3 \end{array}$$

$$\begin{array}{rrrrr} R_1 + \frac{1}{8}R_3 & \frac{17}{8} & 0 & 0 & 0 \\ & -7 & -6 & 0 & -2 \\ & -3 & 0 & -16 & 0 \\ & 1 & 9 & -5 & 3 \end{array} \Rightarrow \begin{array}{rrrrr} & \frac{17}{8} & 0 & 0 & 0 \\ R_2 + \frac{2}{3}R_4 & -\frac{19}{3} & 0 & -\frac{10}{3} & 0 \\ & -3 & 0 & -16 & 0 \\ & 1 & 9 & -5 & 3 \end{array}$$

$$\Rightarrow \quad R_2 - \tfrac{5}{24}R_3 \quad \begin{matrix} \frac{17}{8} & 0 & 0 & 0 \\ -\frac{137}{24} & 0 & 0 & 0 \\ -3 & 0 & -16 & 0 \\ 1 & 9 & -5 & 3 \end{matrix}$$

Again we have a lower triangular matrix so

$$\begin{vmatrix} -1 & 3 & 2 & 1 \\ -7 & -6 & 0 & -2 \\ -10 & 21 & -2 & 7 \\ 1 & 9 & -5 & 3 \end{vmatrix} = \begin{vmatrix} \frac{17}{8} & 0 & 0 & 0 \\ -\frac{137}{24} & 0 & 0 & 0 \\ -3 & 0 & -16 & 0 \\ 1 & 9 & -5 & 3 \end{vmatrix} = 0$$

as one of the main diagonal elements is 0.

(c) Clearly we have an upper triangular matrix so its determinant is simply

$$\begin{vmatrix} -9 & 12 & 27 & 1 \\ 0 & 2 & 0 & 15 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -9 \times 2 \times 1 \times (-2) = 36$$

(2) Expansions on each row are shown below.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = 7 &= a(ej - fh) - b(dj - fg) + c(dh - eg) \\ &= -d(bj - ch) + e(aj - cg) - f(ah - bg) \\ &= g(bf - ce) - h(af - cd) + j(ae - bd) \end{aligned}$$

(a) Expanding on the third row,

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5j \end{vmatrix} &= 5g(bf - ce) - 5h(af - cd) + 5j(ae - bd) \\ &= 5\big(g(bf - ce) - h(af - cd) + j(ae - bd)\big) \\ &= 5 \times 7 = 35 \end{aligned}$$

(b) Expanding on the second row,

$$\begin{aligned} \begin{vmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & j \end{vmatrix} &= -(2d + a)(bj - ch) + (2e + b)(aj - cg) - (2f + c)(ah - bg) \\ &= 2(-d(bj - ch) + e(aj - cg) - f(ah - bg)) - a(bj - ch) \\ &\quad + b(aj - cg) - c(ah - bg) \\ &= 2 \times 7 - a(bj - ch) + b(aj - cg) - c(ah - bg) \end{aligned}$$

We then find the value of the three remaining terms by evaluating the following determinant. Expansions on the second and third rows are shown below.

$$\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & j \end{vmatrix} = -a(bj - ch) + b(aj - cg) - c(ah - bg)$$
$$= g(bc - bc) - h(ac - ac) + j(ab - ab) = 0$$

Thus,

$$\begin{vmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & j \end{vmatrix} = 2 \times 7 + 0 = 14$$

(c) We need only expand by the first row to confirm that the determinant remains the same. The other two row expansions, which are simply the negatives of the original expansions above, also confirm that the determinant does not change.

$$\begin{vmatrix} g & h & j \\ a & b & c \\ d & e & f \end{vmatrix} = g(bf - ce) - h(af - cd) + j(ae - bd) = 7$$
$$= -a(fh - ej) + b(fg - dj) - c(eg - dh)$$
$$= d(ch - bj) - e(cg - aj) + f(bg - ah)$$
$$= a(ej - fh) - b(dj - fg) + c(dh - eg)$$
$$= -d(bj - ch) + e(aj - cg) - f(ah - bg)$$

(3) (a)

$$\begin{vmatrix} 10 & 1 \\ -20 & 1 \end{vmatrix} = \frac{1}{10 \times 1 - (-20) \times 1} = \frac{1}{10 + 20} = \frac{1}{30} \neq 0$$

Since the matrix is square with determinant not equal to zero it is invertible. Its inverse is simply

$$\begin{pmatrix} 10 & 1 \\ -20 & 1 \end{pmatrix}^{-1} = \frac{1}{30}\begin{pmatrix} 1 & -1 \\ 20 & 10 \end{pmatrix} = \begin{pmatrix} \frac{1}{30} & -\frac{1}{30} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(b) Expanding on the second row,

$$\begin{vmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{vmatrix} = -1(0 \times 6 - 5 \times 2) + 1(1 \times 6 - 5 \times 3) + 0 = 10 - 9 = 1 \neq 0$$

This matrix is also invertible as it is square with determinant not equal to 0.

To find its inverse we simply need to calculate the adjoint as the determinant is 1. We transpose the matrix of cofactors below to find the adjoint.

$$\begin{aligned}\text{adj} &= \begin{pmatrix} 1\times 6-0\times 2 & -(1\times 6-0\times 3) & 1\times 2-1\times 3 \\ -(0\times 6-5\times 2) & 1\times 6-5\times 3 & -(1\times 2-0\times 3) \\ 0\times 0-5\times 1 & -(1\times 0-5\times 1) & 1\times 1-0\times 1 \end{pmatrix}^T \\ &= \begin{pmatrix} 6 & -6 & -1 \\ 10 & -9 & -2 \\ -5 & 5 & 1 \end{pmatrix}^T = \begin{pmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{pmatrix}\end{aligned}$$

Hence,

$$\begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{pmatrix}$$

(c) Using the following row eliminations we convert this matrix to lower triangular form as we did in Question 1.

$$R_2 - R_1, \quad R_3 + 2R_1, \quad R_2 + \frac{2}{3}R_3,$$
$$R_1 + \frac{3}{2}R_2, \quad R_1 - \frac{1}{3}R_3,$$
$$R_1 + \frac{5}{6}R_2, \quad R_1 + \frac{8}{9}R_3$$

Thus the product of its diagonal elements is our determinant.

$$\begin{aligned}\begin{vmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & -1 \\ -2 & -6 & 3 & 2 \\ 3 & 5 & 8 & -3 \end{vmatrix} &= \begin{vmatrix} -\frac{2}{3} & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 3 & 5 & 8 & -3 \end{vmatrix} \\ = -\frac{2}{3}\times(-2)\times 3\times(-3) = -12 \neq 0\end{aligned}$$

Hence the matrix is invertible and we calculate its inverse as follows.

$$\begin{aligned}c_{11} = \begin{vmatrix} 1 & -2 & -1 \\ -6 & 3 & 2 \\ 5 & 8 & -3 \end{vmatrix} &= 1(3\times(-3)-2\times 8)-(-2)(-6\times(-3)-2\times 5) \\ &\quad + (-1)(-6\times 8-3\times 5) = -25+16+63 = 54\end{aligned}$$

$$\begin{aligned}c_{12} = -\begin{vmatrix} 0 & -2 & -1 \\ -2 & 3 & 2 \\ 3 & 8 & -3 \end{vmatrix} &= -\big[0-(-2)(-2\times(-3)-2\times 3)+(-1)(-2\times 8-3\times 3)\big] \\ &= -\big[0+0+25\big] = -25\end{aligned}$$

$$c_{13} = \begin{vmatrix} 0 & 1 & -1 \\ -2 & -6 & 2 \\ 3 & 5 & -3 \end{vmatrix} = 0 - 1(-2 \times (-3) - 2 \times 3) + (-1)(-2 \times 5 - (-6) \times 3)$$
$$= 0 + 0 - 8 = -8$$

$$c_{14} = -\begin{vmatrix} 0 & 1 & -2 \\ -2 & -6 & 3 \\ 3 & 5 & 8 \end{vmatrix} = -\big[0 - 1(-2 \times 8 - 3 \times 3) + (-2)(-2 \times 5 - (-6) \times 3)\big]$$
$$= -\big[0 + 25 - 16\big] = -9$$

$$c_{21} = -\begin{vmatrix} 3 & 0 & -1 \\ -6 & 3 & 2 \\ 5 & 8 & -3 \end{vmatrix} = -\big[3(3 \times (-3) - 2 \times 8) - 0 + (-1)(-6 \times 8 - 3 \times 5)\big]$$
$$= -[-75 + 0 + 63] = 12$$

$$c_{22} = \begin{vmatrix} 1 & 0 & -1 \\ -2 & 3 & 2 \\ 3 & 8 & -3 \end{vmatrix} = 1(3 \times (-3) - 2 \times 8) - 0 + (-1)(-2 \times 8 - 3 \times 3)$$
$$= -25 + 0 + 25 = 0$$

$$c_{23} = -\begin{vmatrix} 1 & 3 & -1 \\ -2 & -6 & 2 \\ 3 & 5 & -3 \end{vmatrix} = -\big[1(-6 \times (-3) - 2 \times 5) - 3(-2 \times (-3) - 2 \times 3)$$
$$+ (-1)(-2 \times 5 - (-6) \times 3)\big] = -\big[8 + 0 - 8\big] = 0$$

$$c_{24} = \begin{vmatrix} 1 & 3 & 0 \\ -2 & -6 & 3 \\ 3 & 5 & 8 \end{vmatrix} = 1(-6 \times 8 - 3 \times 5) - 3(-2 \times 8 - 3 \times 3) + 0$$
$$= -63 + 75 + 0 = 12$$

$$c_{31} = \begin{vmatrix} 3 & 0 & -1 \\ 1 & -2 & -1 \\ 5 & 8 & -3 \end{vmatrix} = 3(-2 \times (-3) - (-1) \times 8) - 0 + (-1)(1 \times 8 - (-2) \times 5)$$
$$= 42 + 0 - 18 = 24$$

$$c_{32} = -\begin{vmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ 3 & 8 & -3 \end{vmatrix} = -\big[1(-2 \times (-3) - (-1) \times 8) - 0 + (-1)(0 \times 8 - (-2) \times 3)\big]$$
$$= -\big[14 + 0 - 6\big] = -8$$

$$c_{33} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 3 & 5 & -3 \end{vmatrix} = -0 + 1(1 \times (-3) - (-1) \times 3) - (-1)(1 \times 5 - 3 \times 3)$$
$$= 0 + 0 - 4 = -4$$

$$c_{34} = -\begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 3 & 5 & 8 \end{vmatrix} = -\big[1(1 \times 8 - (-2) \times 5) - 3(0 \times 8 - (-2) \times 3) + 0\big]$$
$$= -\big[18 - 18 + 0\big] = 0$$

$$c_{41} = -\begin{vmatrix} 3 & 0 & -1 \\ 1 & -2 & -1 \\ -6 & 3 & 2 \end{vmatrix} = -\big[3(-2 \times 2 - (-1) \times 3) - 0 + (-1)(1 \times 3 - (-2) \times (-6))\big]$$
$$= -[-3 + 0 + 9] = -6$$

$$c_{42} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ -2 & 3 & 2 \end{vmatrix} = 1(-2 \times 2 - (-1) \times 3) - 0 + (-1)(0 \times 3 - (-2) \times (-2))$$
$$= -1 + 0 + 4 = 3$$

$$c_{43} = -\begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ -2 & -6 & 2 \end{vmatrix} = -\big[0 + 1(1 \times 2 - (-1) \times (-2)) - (-1)(1 \times (-6) - 3 \times (-2))\big]$$
$$= -\big[0 + 0 + 0\big] = 0$$

$$c_{44} = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ -2 & -6 & 3 \end{vmatrix} = 1(1 \times 3 - (-2) \times (-6)) - 3(0 \times 3 - (-2) \times (-2)) + 0$$
$$= -9 + 12 + 0 = 3$$

Thus,

$$\text{adj} = \begin{pmatrix} 54 & -25 & -8 & -9 \\ 12 & 0 & 0 & 12 \\ 24 & -8 & -4 & 0 \\ -6 & 3 & 0 & 3 \end{pmatrix}^T = \begin{pmatrix} 54 & 12 & 24 & -6 \\ -25 & 0 & -8 & 3 \\ -8 & 0 & -4 & 0 \\ -9 & 12 & 0 & 3 \end{pmatrix}$$

We also confirm the determinant calculated earlier by expanding on the second row of our original matrix and using the corresponding cofactors above.

$$\begin{aligned} \det &= -0 \times c_{21} + 1 \times c_{22} - (-2) \times c_{23} + (-1) \times c_{24} \\ &= 0 \times 12 + 1 \times 0 + 2 \times 0 - 1 \times 12 \\ &= 0 + 0 + 0 - 12 = -12 \end{aligned}$$

Hence,

$$\begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & -1 \\ -2 & -6 & 3 & 2 \\ 3 & 5 & 8 & -3 \end{pmatrix}^{-1} = -\frac{1}{12}\begin{pmatrix} 54 & 12 & 24 & -6 \\ -25 & 0 & -8 & 3 \\ -8 & 0 & -4 & 0 \\ -9 & 12 & 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{9}{2} & -1 & -2 & \frac{1}{2} \\ \frac{25}{12} & 0 & \frac{2}{3} & -\frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{3}{4} & -1 & 0 & -\frac{1}{4} \end{pmatrix}$$

(4) (a)

Let $A = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{pmatrix}$

Since the characteristic equation for a square matrix A is $\det(A - \lambda I) = 0$,

$$(2-\lambda)^2 - 49 = 0 \Rightarrow 2 - \lambda = \pm 7$$
$$\Rightarrow \lambda = -5, 9$$

The associated eigenvectors are given below.

$\lambda_1 = -5$:

$$\begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x + 7y = -5x \Rightarrow 7x = -7y \Rightarrow x = -y$$
$$7x + 2y = -5y \Rightarrow 7x = -7y \Rightarrow x = -y$$
$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 9$:

$$\begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 9\begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x + 7y = 9x \Rightarrow 7x = 7y \Rightarrow x = y$$
$$7x + 2y = 9y \Rightarrow 7x = 7y \Rightarrow x = y$$
$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b)

Let $A = \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{pmatrix}$

By the characteristic equation of a square matrix,

$$\begin{aligned}(3-\lambda)(-1-\lambda)-(-2)\times 1 = -3-3\lambda+\lambda+\lambda^2+2 &= 0\\ \Rightarrow \lambda^2-2\lambda-1 &= 0\\ \Rightarrow (\lambda-1)^2-2 &= 0\\ \Rightarrow \lambda-1 &= \pm\sqrt{2}\\ \Rightarrow \lambda &= 1\pm\sqrt{2}\end{aligned}$$

The associated eigenvectors are as follows.

$\lambda_1 = 1-\sqrt{2}:$

$$\begin{pmatrix}3 & -2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = (1-\sqrt{2})\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{aligned}3x-2y = x-\sqrt{2}x &\Rightarrow 2x+\sqrt{2}x = 2y\\ &\Rightarrow x = \frac{2}{2+\sqrt{2}}\ y\\ &\Rightarrow x = (2-\sqrt{2})y\\ x-y = y-\sqrt{2}y &\Rightarrow x = 2y-\sqrt{2}y\\ &\Rightarrow x = (2-\sqrt{2})y\\ v_1 &= \begin{pmatrix}2-\sqrt{2}\\ 1\end{pmatrix}\end{aligned}$$

$\lambda_2 = 1+\sqrt{2}:$

$$\begin{pmatrix}3 & -2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = (1+\sqrt{2})\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{aligned}3x-2y = x+\sqrt{2}x &\Rightarrow 2x-\sqrt{2}x = 2y\\ &\Rightarrow x = \frac{2}{2-\sqrt{2}}\ y\\ &\Rightarrow x = (2+\sqrt{2})y\\ x-y = y+\sqrt{2}y &\Rightarrow x = 2y+\sqrt{2}y\\ &\Rightarrow x = (2+\sqrt{2})y\\ v_2 &= \begin{pmatrix}2+\sqrt{2}\\ 1\end{pmatrix}\end{aligned}$$

(5) If A is an invertible matrix, then $AA^{-1} = A^{-1}A = I$. To show that $\operatorname{adj} A$ is also invertible, we therefore require

$$\operatorname{adj} A\,(\operatorname{adj} A)^{-1} = (\operatorname{adj} A)^{-1}\operatorname{adj} A = I$$

By definition $\operatorname{adj} A := \det(A)A^{-1}$. It follows that

$$\begin{aligned} \det(A)A^{-1}\left(\det(A)A^{-1}\right)^{-1} &= \det(A)A^{-1}\det(A)^{-1}\left(A^{-1}\right)^{-1} \\ &= \det(A)A^{-1}\frac{1}{\det(A)}A \end{aligned}$$

$$\text{since } \det(A)^{-1} = \frac{1}{\det(A)} \text{ as } \det(A) \text{ is a scalar and}$$
$$\left(A^{-1}\right)^{-1} = A \text{ by the definition of inverse}$$

Hence,

$$\operatorname{adj} A\,(\operatorname{adj} A)^{-1} = \frac{\det(A)A^{-1}A}{\det(A)} = A^{-1}A = I$$

Similarly,

$$\begin{aligned} (\operatorname{adj} A)^{-1}\operatorname{adj} A &= \left(\det(A)A^{-1}\right)^{-1}\det(A)A^{-1} \\ &= \det(A)^{-1}\left(A^{-1}\right)^{-1}\det(A)A^{-1} \\ &= \frac{A\,\det(A)A^{-1}}{\det(A)} = AA^{-1} = I \end{aligned}$$

Thus we have shown that $\operatorname{adj} A$ is also an invertible matrix.