UNIVERSITY OF NEW ENGLAND

NAME:______________________________

STUDENT NUMBER:____________________

UNIT NAME PMTH213

PAPER TITLE: Linear Algebra

PAPER NUMBER: First and Only

DATE: **Friday 18 November 2011** **TIME:** **1:45 PM TO 5:00 PM**

TIME ALLOWED: Three (3) hours and fifteen minutes

NUMBER OF PAGES IN PAPER: THREE (3)

NUMBER OF QUESTIONS ON PAPER: EIGHT (8)

NUMBER OF QUESTIONS TO BE ANSWERED: EIGHT (8)

STATIONERY PER CANDIDATE:

0	6 LEAF A4 BOOKS	**0**	ROUGH WORK BOOK	**0**	GENERAL PURPOSE ANSWER SHEET
1	12 LEAF A4 BOOKS	**0**	GRAPH PAPER SHEETS	**0**	SEE OTHER 'AIDS REQUIRED' BELOW

OTHER AIDS REQUIRED: **NIL**

POCKET CALCULATORS PERMITTED: **NO**

TEXTBOOKS OR NOTES PERMITTED: **FIVE (5) A4 SHEETS OF HANDWRITTEN DOUBLE SIDED NOTES (10 PAGES); NO PHOTOCOPIES; NO PRINTED PAGES. NO SCANNED PAGES**

INSTRUCTIONS FOR CANDIDATES:

- Candidates MAY NOT start writing until instructed to do so by the supervisor
- Please pay attention to the announcements and read all instructions carefully before commencing the paper
- Candidates MUST write their name and student number on the top of this page
- Questions are not of equal value
- Commence each question on a new page
- This examination question paper **MUST BE HANDED IN** with worked scripts. Failure to do so may result in the cancellation of all marks for this examination

REMEMBER TO WRITE YOUR NAME AND STUDENT NUMBER AT THE TOP OF THIS PAGE

Question 1

[10 marks]

Find all linear transformations

$$T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad (u, v, w) \longmapsto (x, y)$$

which map the u-w plane onto the line given by the equation $3y = 2x$.

Question 2

[10 marks]

Find a basis for the kernel and a basis for the image of the linear transformation

$$T\colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad (x, y, z) \longmapsto (x + y + 5z,\ 4x + 9y + 5z,\ 6x + 7y + 27z)$$

Question 3

[8 marks]

Find all rational 2×2 matrices, $\underline{\mathbf{A}}$ with $\underline{\mathbf{A}}^2 = \underline{\mathbf{0}}_2$, where $\underline{\mathbf{0}}_2$ denotes the 2×2 zero matrix.

Question 4

[10 marks]

Prove that the linear transformation $T\colon V \to W$ is an isomorphism if and only if T maps a basis of V to a basis of W, that is, $\{T(\mathbf{e}_j) \mid j \in J\}$ is a basis for W whenever $\{\mathbf{e}_j \mid j \in J\}$ is a basis for V.

Question 5

[12 marks]

Let $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ be a basis for the vector space V and $T\colon V \to V$ a linear transformation.

(a) Show that if $\mathbf{f}_1 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3,\ \mathbf{f}_2 = \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{f}_3 = \mathbf{e}_1$, then $\{\mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3\}$ is also a basis for V.

(b) Find the matrix, $\underline{\mathbf{B}}$, of the T with respect to the basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ given that its matrix with respect to $\{\mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3\}$ is

$$\underline{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(Question 6 is on page 3.)

Question 6

[10 marks]

(a) Let $\underline{\mathbf{A}} = [a_{ij}]_{m \times n}$ be an $m \times n$ matrix and $\underline{\mathbf{B}} = [b_{ij}]_{n \times m}$ an $n \times m$ matrix.
Prove that $\mathrm{tr}(\underline{\mathbf{A}}\,\underline{\mathbf{B}}) = \mathrm{tr}(\underline{\mathbf{B}}\,\underline{\mathbf{A}})$, where $\mathrm{tr}(\underline{\mathbf{X}})$ denotes the trace of the matrix $\underline{\mathbf{X}}$.

(b) Prove that if two matrices, $\underline{\mathbf{A}}$ and $\underline{\mathbf{C}}$, represent the linear transformation $T\colon V \to V$, then $\mathrm{tr}(\underline{\mathbf{A}}) = \mathrm{tr}(\underline{\mathbf{C}})$.

Question 7

[15 marks]

Given the symmetric matrix $\underline{\mathbf{A}} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$, find

(a) its eigenvalues,

(b) bases for its eigenspaces,

(c) an orthogonal matrix, $\underline{\mathbf{M}}$, which diagonalises it, and

(d) $\underline{\mathbf{M}}^{-1}\,\underline{\mathbf{A}}\,\underline{\mathbf{M}}$.

Question 8

[25 marks]

Let V be the real vector space of all continuous functions $f\colon [-1,1] \to \mathbb{R}$.

(a) Prove that

$$\langle\ ,\ \rangle\colon V \times V \longrightarrow \mathbb{R}, \quad (f,g) \longmapsto \int_{-1}^{1} f(t)g(t)\,dt$$

defines an inner product on V.

(b) Find an orthonormal basis for the subspace of V generated by the polynomial functions $\{x, x+x^2, x+x^2+x^3\}$.