MATH101 ASSIGNMENT 6

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(1) (a) f is differentiable *everywhere.*

$$\begin{aligned} f'(x) &= 6(7x^{6-1}) - 2(3x^{2-1}) \\ &= 42x^5 - 6x \end{aligned}$$

(b) g is differentiable *everywhere.*
Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(-2x) - (1-x^2)(2x)}{(x^2+1)^2} \\ &= \frac{-2x - 2x^3 - 2x + 2x^3}{(x^2+1)^2} \\ &= -\frac{4x}{(x^2+1)^2} \end{aligned}$$

(c) h is differentiable on $\mathbb{R}\backslash\{0\}$.
We first note that x^4+2x is continuous and differentiable while $\frac{x}{x+1}$ is undefined at $x=-1$. But the discontinuity at $x=-1$ is irrelevant, by definition of the piecewise function. So the only concern is at $x=0$. Recall that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Consider for x^4+2x,

$$\begin{aligned} &\lim_{h\to 0^-} \frac{\big((0+h)^4 + 2(0+h)\big) - \big(0^4 + 2(0)\big)}{h} \\ = &\lim_{h\to 0^-} \frac{h^4 + 2h - 0}{h} = \lim_{h\to 0^-} h^3 + 2 = 2 \end{aligned}$$

while for $\frac{x}{x+1}$,

$$\lim_{h\to 0^+} \frac{\frac{(0+h)}{(0+h)+1} - \frac{0}{0+1}}{h} = \lim_{h\to 0^+} \frac{1}{h+1} = 1$$

which disagree at $x=0$. Hence h is not differentiable at $x=0$.

Finally,

$$\begin{aligned} h'(x) &= \begin{cases} 4x^{4-1} + 2x^{1-1} & \text{if } \ x < 0 \\ \frac{(x+1)(1) - x(1)}{(x+1)^2} & \text{if } \ x > 0 \end{cases} \\ &= \begin{cases} 4x^3 + 2 & \text{if } \ x < 0 \\ \frac{1}{(x+1)^2} & \text{if } \ x > 0 \end{cases} \end{aligned}$$

(d) j is differentiable *everywhere*.
Applying the chain rule,

$$\begin{aligned} j'(x) &= \frac{1}{2}\left(x^2+1\right)^{\frac{1}{2}-1}(2x) \\ &= x(x^2+1)^{-\frac{1}{2}} \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

(2) Let $f(x) = \sqrt{x}$ for $x > 0$. Then by the formal definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{x + h + \sqrt{x}\sqrt{x+h} - \sqrt{x}\sqrt{x+h} - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

(3) (a) $f(1)$ is undefined so $f(x)$ and $f'(x)$ are non-differentiable at $x = 1$.
Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{1(1-x) - x(-1)}{(1-x)^2} \\ &= \frac{1 - x + x}{(1-x)^2} \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

Applying the quotient and chain rules,

$$\begin{aligned} f''(x) &= \frac{0(1-x)^2 - 1(2)(1-x)^{2-1}(-1)}{(1-x)^4} \\ &= \frac{0+2(1-x)}{(1-x)^4} \\ &= \frac{2(1-x)}{(1-x)^4} \\ &= \frac{2}{(1-x)^3} \end{aligned}$$

(b) g is differentiable *everywhere* and so is $g'(x)$.
Analagous to Question 1(d),

$$g'(u) = \frac{u}{\sqrt{u^2+1}}$$

Differentiating again gives

$$\begin{aligned} g''(u) &= u\left(-\frac{1}{2}\right)(u^2+1)^{-\frac{1}{2}-1}(2u) + 1(u^2+1)^{-\frac{1}{2}} \\ &= -u^2(u^2+1)^{-\frac{3}{2}} + (u^2+1)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{u^2+1}} - \frac{u^2}{\sqrt{(u^2+1)^3}} \end{aligned}$$

(c) $h(1)$ and $h(-1)$ are undefined so $h(x)$ and $h'(x)$ are non-differentiable at $x = \pm 1$. Differentiating yields

$$\begin{aligned} h'(x) &= -5(1-x^4)^{-5-1}(-4x^3) \\ &= 20x^3(1-x^4)^{-6} \\ &= \frac{20x^3}{(1-x^4)^6} \end{aligned}$$

Using the product and chain rules,

$$\begin{aligned} h''(x) &= 20x^3\big(-6(1-x^4)^{-6-1}(-4x^3)\big) + (1-x^4)^{-6}\big(3(20x^{3-1})\big) \\ &= 480x^6(1-x^4)^{-7} + 60x^2(1-x^4)^{-6} \\ &= \frac{480x^6}{(1-x^4)^7} + \frac{60x^2}{(1-x^4)^6} \end{aligned}$$

(d) j is differentiable *everywhere* and so is $j'(x)$.
Differentiating twice gives

$$\begin{aligned} j'(x) &= 7t^{7-1} - 4(5t^{4-1}) + 0 \\ &= 7t^6 - 20t^3 \\ j''(x) &= 6(7t^{6-1}) - 3(20t^{3-1}) \\ &= 42t^5 - 60t^2 \end{aligned}$$

(4) (a) The slope of the tangent line at x is

$$\frac{d}{dx}(x) = f'(x)$$

To find points where the tangent line to the graph of f is horizontal, we need to find where $f'(x) = 0$.

$$\begin{aligned} f'(x) &= \frac{(x^4+2)(0) - 1(4x^3)}{(x^4+2)^2} \\ &= -\frac{4x^3}{(x^4+2)^2} = 0 \end{aligned}$$

This is true if and only if $x = 0$. Thus, the only point where the tangent line is horizontal to the graph of f is at $(0, \frac{1}{2})$.

(b) The limits below show the behaviour of f at the following points.
$\mathbf{x = 0}$:

$$\begin{aligned} \lim_{x \to 0^-} \frac{1}{x^4+2} &= \frac{1}{2} \\ \lim_{x \to 0^+} \frac{1}{x^4+2} &= \frac{1}{2} \end{aligned}$$

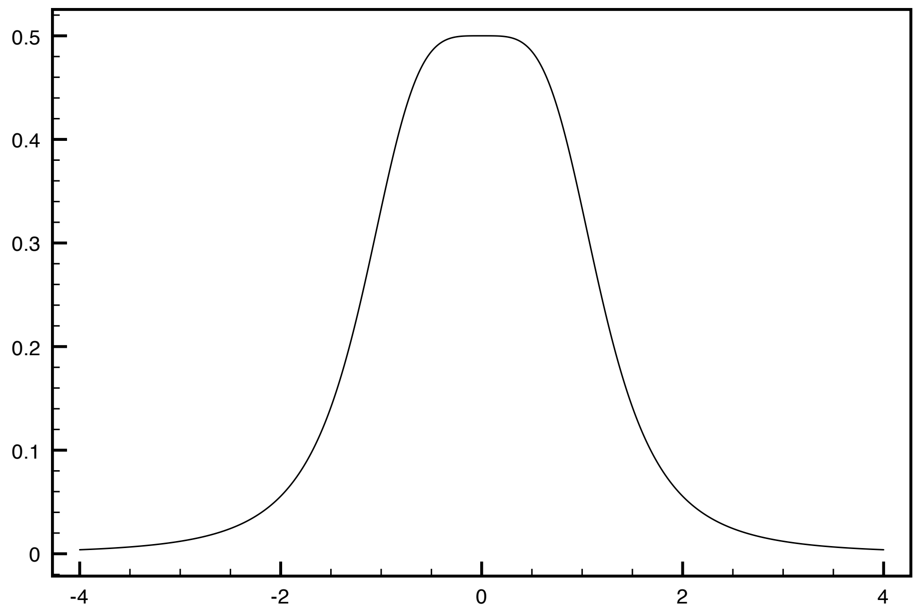
Thus, $f \to \frac{1}{2}$ as $x \to 0$ from the left and from the right.
$\mathbf{x = \pm\infty}$:

$$\begin{aligned} \lim_{x \to -\infty} \frac{1}{x^4+2} &= \lim_{x \to -\infty} \frac{\frac{1}{x^4}}{1+\frac{2}{x^4}} = \frac{0}{1+0} = 0 \\ \lim_{x \to +\infty} \frac{1}{x^4+2} &= \lim_{x \to +\infty} \frac{\frac{1}{x^4}}{1+\frac{2}{x^4}} = \frac{0}{1+0} = 0 \end{aligned}$$

The infinite limits show that $f \to 0$ as $x \to \pm\infty$.

(c) We sketch the graph of f using the information from parts (a) and (b). We also note the range of the f is $\{r \in \mathbb{R} \mid 0 < r \leq \frac{1}{2}\}$ for the following reasons.

$\inf(f) = 0$ since $x^4 \geq 0 \Rightarrow x^4 + 2 > 0 \Rightarrow \frac{1}{x^4+2} > 0$
$\max(f) = \frac{1}{2}$ since at $x = 0$, $x^4 + 2$ is smallest $\Rightarrow \frac{1}{x^4+2}$ is greatest



(5) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f(x) \neq 0$ for all $x \in \mathbb{R}$, then by the formal definition of the derivative,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{f(u)}\right) &= \lim_{h\to 0} \frac{\frac{1}{f(u+h)} - \frac{1}{f(u)}}{h} = \lim_{h\to 0} \frac{f(u) - f(u+h)}{h \cdot f(u+h) \cdot f(u)} \\
&= -\lim_{h\to 0} \frac{\frac{f(u+h)-f(u)}{h}}{f(u+h)\cdot f(u)} = -\frac{\lim_{h\to 0}\frac{f(u+h)-f(u)}{h}}{\lim_{h\to 0} f(u+h)\cdot f(u)} \\
&= -\frac{f'(u)}{f(u+0)\cdot f(u)} = -\frac{f'(u)}{f(u)\cdot f(u)} = -\frac{f'(u)}{[f(u)]^2}
\end{aligned}$$

As f is assumed differentiable at x, it is therefore continuous at x. This implies

$$\lim_{h\to 0} f(u+h) = f(u)$$

Meanwhile, since $f(u)$ does not involve h,

$$\lim_{h\to 0} f(u) = f(u)$$

Thus we have shown that $g : \mathbb{R} \to \mathbb{R}$, $u \mapsto \frac{1}{f(u)}$ is also differentiable for all $u \in \mathbb{R}$.