AMTH250 Assignment 2

Due: 15th August

Reproduce the following *[2 marks for each question]*:

Question 1

The reduced cubic equation $y^3 + 3py + 2q = 0$ has one real and two complex solutions when $D = q^2 + p^3 > 0$. These are given by Cardano's formula as

$$y_1 = u + v, \quad y_2 = -\frac{u+v}{2} + \frac{i}{2}\sqrt{3}(u-v), \quad y_3 = -\frac{u+v}{2} - \frac{i}{2}\sqrt{3}(u-v)$$

where

$$u = \sqrt[3]{-q + \sqrt{q^2 + p^3}}, \qquad v = \sqrt[3]{-q - \sqrt{q^2 + p^3}}$$

Question 2

An $n \times n$ matrix $\mathbf{A}$ is *non-singular* if it satisfies any one of the following equivalent conditions:

1. $\mathbf{A}$ has an inverse, i.e. there is a matrix $\mathbf{A}^{-1}$ such that
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$
2. $\det \mathbf{A} \neq 0$.
3. $\operatorname{rank} \mathbf{A} = n$. (The rank of a matrix is the maximum number of linearly independent rows or columns of the matrix.)
4. For every non-zero vector $\mathbf{z}$, $\mathbf{A}\mathbf{z} \neq 0$.

Question 3

The *gamma function* $\Gamma(x)$ is defined by

$$\begin{aligned}\Gamma(x) &\equiv \lim_{n\to\infty} \prod_{\nu=0}^{n-1} \frac{n!\, n^{x-1}}{x+\nu} \\ &= \lim_{n\to\infty} \frac{n!\, n^{x-1}}{x(x+1)\cdots(x+n-1)} \\ &= \int_0^\infty e^{-t} t^{x-1}\, dt\end{aligned}$$

The integral definition is only valid for $x > 0$.

Question 4

Given an n-vector $\mathbf{a}$, we can annihilate **all** of its entries below the kth position, provided that $a_k \neq 0$, by the following transformation:

$$\mathbf{M}_k\mathbf{a} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & -m_{k+1} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -m_n & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $m_i = a_i/a_k$, $i = k+1, \dots, n$.

Question 5

The absolute and relative error are defined by

$$\text{Absolute error} = \text{Approximate value} - \text{True value} \tag{1}$$

$$\text{Relative error} = \frac{\text{Absolute error}}{\text{True value}} \tag{2}$$

A useful way to think of relative error is via the expression

$$\text{Approximate value} = \text{True value} \times (1 + \text{Relative error}) \tag{3}$$

Question 6

Differentiating the differential equation

$$\frac{dy}{dt} = f(t, y)$$

gives

$$\frac{d^2y}{dt^2} = \frac{d}{dt}f(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f$$

and we have the Taylor series approximation

$$\begin{aligned} y(t+h) &\approx y(t) + y'(t)h + \frac{1}{2}y''(t)h^2 \\ &= y(t) + hf(t,y) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)f(t,y)\right) \end{aligned}$$

Question 7

No Pivoting		Partial Pivoting		Complete Pivoting	
Error	Residual	Error	Residual	Error	Residual
8.964	2.07×10^{-14}	0.156	2.53×10^{-16}	0.164	2.53×10^{-16}
1.426	2.77×10^{-15}	0.113	2.04×10^{-16}	0.175	2.93×10^{-16}
0.883	3.60×10^{-16}	0.080	2.97×10^{-16}	0.036	3.48×10^{-16}

Table 1: Errors and residuals for 3 random 100×100 matrices.

Question 8

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<tr><th colspan="2">Payoff ($)</th></tr>
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