PURE MATHEMATICS 212

Multivariable Calculus

ASSIGNMENT 2-SOLUTIONS

Question 1 [3 marks] The equation can be written as

$$4x^2 + 8x + y^2 - 2y - z^2 + 4z = 0$$

or

$$4(x+1)^2 + (y-1)^2 - (z-2)^2 = 4 + 1 - 4 = 1$$

This is a hyperboloid of one sheet centred at $(-1, 1, 2)$ with central axis parallel to the z-axis.

Question 2 [3 marks] By subtracting the two equations we eliminate z. The resulting equation is $x^2 + 2y^2 - 4 = 0$. This is the equation of a cylinder which contains the curve in question. It's projection is the ellipse $x^2 + 2y^2 = 4$ on the xy-plane, which contains the projection of the curve. In fact, this ellipse is the required projection. For any x between -4 and 4 the corresponding y value of the ellipse is $y = \pm\sqrt{\frac{4-x^2}{2}}$ and the z variable of the corresponding point of the curve is $z = \frac{4-x^2}{2}$.

Question 3 [4 marks]

(a) The vector equation is equivalent to the parametric equations

$$x = x(t) = 3\sin 2t, \qquad y = y(t) = 3\cos 2t.$$

This represents a circle of radius 3 with center (0,0).

(b) The vector-valued function may be expressed using the parametric equations

$$x = -2, \quad y = t, \quad z = t^2 - 1.$$

Substituting $t = y$ into the expression of z, we obtain $z = y^2 - 1$.

Thus, the curve is a parabola lying on the plane $x = -2$ with vertex at $(-2, 0, -1)$.

Question 4 [3 marks]

This amounts to showing that $\mathbf{r}'$ is a well-defined nowhere zero continuous vector-valued function. In fact, we have

$$\mathbf{r}'(t) = -2t\sin t^2\mathbf{i} + 2t\cos t^2\mathbf{j} - e^{-t}\mathbf{k}.$$

The three derivatives are all continuous functions on their natural domain $\mathbb{R}$ and the $\mathbf{k}$ component is different from 0. Therefore $\mathbf{r}$ is smooth.

Question 5 [3 marks]

Using the product rule for dot product, we have

$$\frac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{w}) + \frac{d\mathbf{u}}{dt}\cdot(\mathbf{v}\times\mathbf{w}).$$

Applying that rule for cross product, we conclude the desired formula.

Question 6 [4 marks]

(a)

$$\begin{aligned}
\int (t\sin t\mathbf{i} + \mathbf{j})dt &= \int t\sin t\,dt\mathbf{i} + \int dt\mathbf{j} \\
&= (-t\cos t + \int \cos t\,dt)\,\mathbf{i} + (t + C_1)\,\mathbf{j} \\
&= (-t\cos t + \sin t + C_2)\mathbf{i} + (t + C_1)\,\mathbf{j},
\end{aligned}$$

where C_1, C_2 are constants.

(b) According to the arc-length formula, the length is equal to

$$\int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t + 1}dt = \int_0^{2\pi} \sqrt{10}dt = 2\pi\sqrt{10}.$$