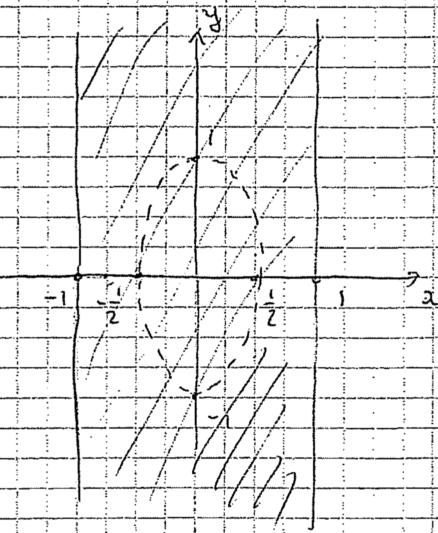
Tut 4

Find the region where the function is continuous.

① $f(x,y) = \dfrac{\sin^{-1} x}{4x^2+y^2-1}$

$|x| \le 1$

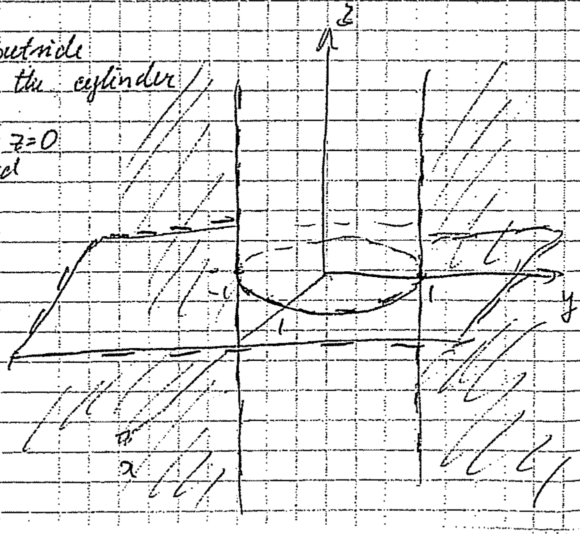
$4x^2+y^2 \ne 1$



② $f(x,y,z) = \dfrac{\ln(x^2+y^2-1)}{z}$

$x^2+y^2 \geqslant 1$ - outside of the cylinder

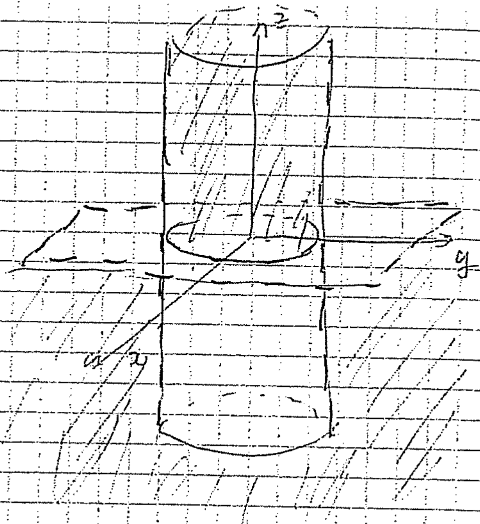
$z \ne 0$ - plane $z=0$ excluded



③ $f(x,y,z) = \ln \frac{1-x^2-y^2}{z}$

$x^2+y^2 \leq 1$ and $z \geq 0$

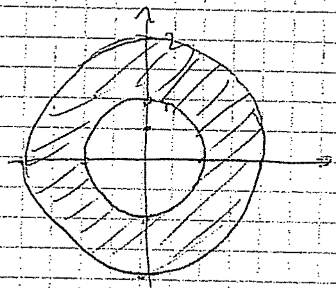
or $x^2+y^2 > 1$ and $z < 0$



④ $f(x,y) = \sqrt{(x^2+y^2-1)(4-x^2-y^2)}$

$x^2+y^2 \geq 1$
$x^2+y^2 \leq 4$

or $x^2+y^2 \leq 1$
$x^2+y^2 \geq 4$ $\emptyset$



Find the limit if it exists

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{2x^2+y^2}$

$(y=0)$ $\lim\limits_{\text{along } x\text{-axes}} \dfrac{x+y}{2x^2+y^2} = \lim\limits_{x\to 0} \dfrac{1}{2x} \to \infty \Rightarrow$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{2x^2+y^2}$ doesn't exist

(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{2x^2+y^2}$

$\lim\limits_{\text{along } x\text{-axi}} \dfrac{x^2}{2x^2} = \dfrac{1}{2}$ $(y=0)$

$\Rightarrow$ doesn't exist

$\lim\limits_{\text{al. } y\text{-ax.}} \dfrac{y^2}{y^2} = 1$ $(x=0)$

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^4}{2x^2+y^2} =$

$x = r\cos\varphi$

$y = r\sin\varphi$

$= \lim\limits_{r\to 0} \dfrac{r^3\cos^3\varphi + r^4\sin^4\varphi}{2r^2\cos^2\varphi + r^2\sin^2\varphi} =$

$= \lim\limits_{r=0} \dfrac{r(\cos^3\varphi + r\sin^4\varphi)}{2\cos^2\varphi + \sin^2\varphi} = 0$

④ $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} =$$

$x^2+y^2=z$

$z\to 0$ if $(x,y)\to(0,0)$

$$= \lim_{z\to 0} \frac{\sin z}{z} = 1$$

⑤ $$\lim_{(x,y)\to(0,0)} \frac{\ln(1+x^2+y^2)-(x^2+y^2)}{(x^2+y^2)^2} =$$

$x^2+y^2=z$

$z\to 0$ if $(x,y)\to(0,0)$

$$= \lim_{z\to 0} \frac{\ln(1+z)-z}{z^2} =$$

(We use the L'Hôpital's rule
the numerator and denominator
are diff-able and both $\to 0$ $(z\to 0)$)

$$= \lim_{z\to 0} \frac{(\ln(1+z)-z)'}{(z^2)'} = \lim_{z\to 0} \frac{\frac{1}{1+z}-1}{2z} =$$

$$= \lim_{z\to 0}\left(-\frac{1}{2(z+1)}\right) = -\frac{1}{2}$$

Find f_x, f_y, f_z

$$f(x,y,z) = z^2 \cos\frac{z}{xy}$$

$$f_x = -z^2 \sin\frac{z}{xy} \times \left(-\frac{z}{y}\frac{1}{x^2}\right) = \frac{z^3}{x^2y} \sin\frac{z}{xy}$$

$$f_y = -z^2 \sin\frac{z}{xy} \times \left(-\frac{z}{x}\frac{1}{y^2}\right) = \frac{z^3}{xy^2} \sin\frac{z}{xy}$$

$$f_z = 2z\cos\frac{z}{xy} + z^2\left(-\sin\frac{z}{xy}\right)\cdot\frac{1}{xy} =$$

$$= 2z\cos\frac{z}{xy} - \frac{z^2}{xy}\sin\frac{z}{xy}$$

Show that the $f(x,y)$ satisfies laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$z = x^2 - y^2 + 2xy$$

$$\frac{\partial z}{\partial x} = 2x + 2y, \quad \frac{\partial^2 z}{\partial x^2} = 2 \qquad 2 - 2 = 0$$

$$\frac{\partial z}{\partial y} = -2y + 2x, \quad \frac{\partial^2 z}{\partial y^2} = -2$$

Show that $u = \ln(x^2+y^2)$ and $v = 2\tan^{-1}\left(\frac{y}{x}\right)$

satisfy the Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\left(\tan^{-1} x\right)' = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{2\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}} = -\frac{2y}{x^2+y^2} = -\frac{\partial u}{\partial y} \checkmark$$

$$\frac{\partial v}{\partial y} = \frac{2\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{2x}{x^2+y^2} = \frac{\partial u}{\partial x} \checkmark$$