MATH102 ASSIGNMENT 4

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(1) (a)

$$\int_1^2 \frac{dx}{\sqrt{16-x^2}}, \text{ Let } x = 4\sin\theta,\ dx = 4\cos\theta\ d\theta$$

$$\text{when } x = 2 \to \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\text{when } x = 1 \to \theta = \sin^{-1}\left(\frac{1}{4}\right) = \alpha$$

$$\int_1^2 \frac{dx}{\sqrt{16-x^2}} = \int_\alpha^{\frac{\pi}{6}} \frac{4\cos\theta\ d\theta}{\sqrt{16-16\sin^2\theta}} = \int_\alpha^{\frac{\pi}{6}} \frac{4\cos\theta\ d\theta}{\sqrt{16\ (1-\sin^2\theta)}}$$

$$= \int_\alpha^{\frac{\pi}{6}} \frac{\cos\theta}{\sqrt{\cos^2\theta}}\ d\theta = \int_\alpha^{\frac{\pi}{6}} 1\ d\theta = \Big[\theta\Big]_\alpha^{\frac{\pi}{6}} = \frac{\pi}{6} - \sin^{-1}\left(\frac{1}{4}\right)$$

(b)

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{ds}{1+s^2}, \text{ Let } s = \tan x,\ ds = \sec^2 x\ dx$$

$$\text{when } s = \sqrt{3} \to x = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\text{when } s = \frac{1}{\sqrt{3}} \to x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{ds}{1+s^2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x\ dx}{1+\tan^2 x} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sec^2 x}\ dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\ dx = \Big[x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

(2) (a)

$$\int \frac{x^2}{1+x^6}\ dx, \text{ Let } u = x^3,\ \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$$

$$\int \frac{x^2}{1+x^6}\ dx = \frac{1}{3}\int \frac{1}{1+u^2}\ du = \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1} x^3 + C$$

(b)

$$\int \frac{x}{\sqrt{1-x^4}}\ dx, \text{ Let } u = x^2,\ \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^4}}\ dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\ du = \frac{1}{2}\sin^{-1} u + C \\ &= \frac{1}{2}\sin^{-1} x^2 + C\end{aligned}$$

(c)

$$\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{1+\cos^2\theta}\ d\theta, \text{ Let } u = \cos\theta,\ \frac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = -\frac{du}{\sin\theta}$$

$$\text{when } \theta = \frac{\pi}{4} \to u = \frac{1}{\sqrt{2}},\ \text{when } \theta = 0 \to u = 1$$

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{1+\cos^2\theta}\ d\theta &= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{1+u^2}\ du = \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{1+u^2}\ du = \Big[\tan^{-1} u\Big]_{\frac{1}{\sqrt{2}}}^1 \\ &= \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\end{aligned}$$

(3) (a)

$$\int x\cos x\ dx, \text{ Let } u = x,\ \frac{dv}{dx} = \cos x,\ \frac{du}{dx} = 1,\ v = \sin x$$

$$\begin{aligned}\int x\cos x\ dx &= x\sin x - \int \sin x\ (1)\ dx = x\sin x - (-\cos x) + C \\ &= x\sin x + \cos x + C\end{aligned}$$

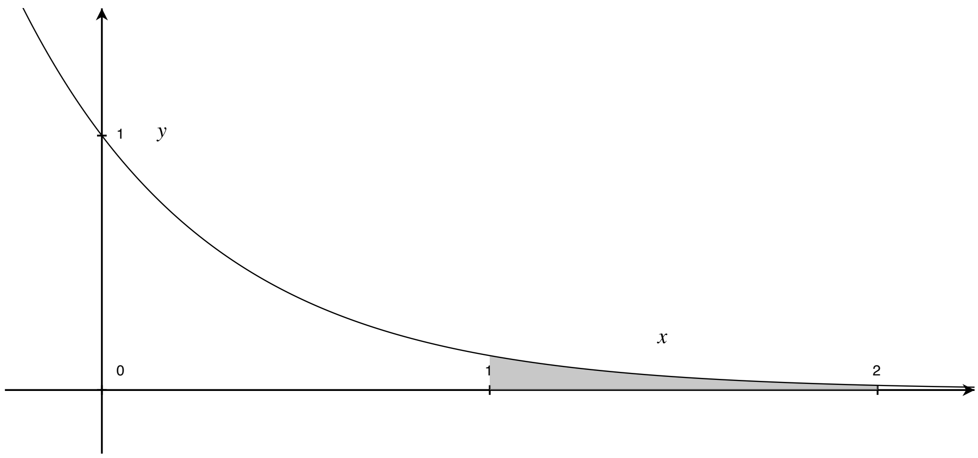
(b)

$$\int x^2\sin x\ dx, \text{ Let } u = x^2,\ \frac{dv}{dx} = \sin x,\ \frac{du}{dx} = 2x,\ v = -\cos x$$

$$\begin{aligned}\int x^2\sin x\ dx &= x^2(-\cos x) - \int(-\cos x)2x\ dx \\ &= -x^2\cos x + 2\int x\cos x\ dx \\ &= -x^2\cos x + 2\ (x\sin x + \cos x) + C \\ &= (2-x^2)\cos x + 2x\sin x + C\end{aligned}$$

(4) (a)

$$A = \int_1^2 e^{-2x}\ dx = \left[-\frac{1}{2}\ e^{-2x}\right]_1^2 = -\frac{1}{2}\left(e^{-4} - e^{-2}\right) = \frac{e^2-1}{2e^4}$$



(b)

$$V_x = \pi \int_1^2 \left(e^{-2x}\right)^2 \; dx = \pi \int_1^2 e^{-4x} \; dx = \pi \left[-\frac{1}{4} \; e^{-4x}\right]_1^2$$
$$= -\frac{\pi}{4} \left(e^{-8} - e^{-4}\right) = \frac{\pi}{4} \left(\frac{1}{e^4} - \frac{1}{e^8}\right) = \frac{\pi(e^4 - 1)}{4e^8}$$

(c)

$$V_y = 2\pi \int_1^2 xe^{-2x} \; dx, \text{ Let } u = x, \; \frac{dv}{dx} = e^{-2x}, \; \frac{du}{dx} = 1, \; v = -\frac{1}{2} \; e^{-2x}$$
$$= 2\pi \left[x\left(-\frac{1}{2} \; e^{-2x}\right) - \int -\frac{1}{2} \; e^{-2x} \; (1) \; dx\right]_1^2$$
$$= 2\pi \left[-\frac{1}{2} \; xe^{-2x} + \frac{1}{2} \int e^{-2x} \; dx\right]_1^2$$
$$= \pi \left[-xe^{-2x} + \int e^{-2x} \; dx\right]_1^2$$
$$= \pi \left[-xe^{-2x} - \frac{1}{2} \; e^{-2x}\right]_1^2$$
$$= \pi \left[\left(-2e^{-4} - \frac{1}{2} \; e^{-4}\right) - \left(-e^{-2} - \frac{1}{2} \; e^{-2}\right)\right]$$
$$= \pi \left(-\frac{5}{2e^4} + \frac{3}{2e^2}\right) = \pi \left(\frac{3e^2 - 5}{2e^4}\right)$$