MATH102 ASSIGNMENT 8

MARK VILLAR

(1) (a)

$$\begin{aligned} \mathcal{F}(y) &= \text{mes}\{0 < \sqrt[3]{x} \le y\} \\ &= \text{mes}\{0 < x \le y^3\} \end{aligned}$$

$$\mathcal{F}(y) = \begin{cases} y^3 \text{ for } & 0 \le y < 2 \\ 0 \text{ for } & y < 0 \\ 8 \text{ for } & y > 2 \end{cases}$$

$$\rho(y) = \mathcal{F}'(y) = \begin{cases} 0 & \text{for } \ y < 0 \ \text{ or } \ y > 2 \\ 3y^2 & \text{for } \ 0 \le y < 2 \end{cases}$$

(b)

$$\begin{aligned} \int_0^2 y\rho(y)\ dy &= \int_0^2 3y^3\ dy = \left[\frac{3}{4}y^4\right]_0^2 \\ &= \frac{3}{4}\left(2^4 - 0\right) = 12 \end{aligned}$$

$$\begin{aligned} \int_0^8 f(x)\ dx &= \int_0^8 x^{\frac{1}{3}}\ dx = \left[\frac{3}{4}x^{\frac{4}{3}}\right]_0^8 \\ &= \frac{3}{4}\left(8^{\frac{4}{3}} - 0\right) = 12 \end{aligned}$$

$$\int_0^2 y\rho(y)\ dy = \int_0^8 f(x)\ dx = 12$$

(2) Since $\mathcal{F}(x) = |x|$ has a derivative except at $x = 0$ and

$$d|x| = \begin{cases} -dx & \text{for } \ x < 0 \\ \ \ dx & \text{for } \ x > 0 \end{cases}$$

then,

$$\begin{aligned} \int_{-1}^1 x^3\ d|x| &= -\int_{-1}^0 x^3\ dx + \int_0^1 x^3\ dx \\ &= -\frac{1}{4}\left[x^4\right]_{-1}^0 + \frac{1}{4}\left[x^4\right]_0^1 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\big(0-1\big) + \frac{1}{4}\big(1-0\big) \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{aligned}
$$

(3)

$$
\Gamma(x) = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\ dt = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\ dt
$$

Using the substitutions $t = u^2 \ \Rightarrow\ dt = 2u\ du,\ \ t^{-\frac{1}{2}} = u^{-1}$ and $\int_0^\infty e^{-x^2}\ dx = \sqrt{\pi}$,

$$
\begin{aligned}
\int_0^\infty t^{-\frac{1}{2}} e^{-t}\ dt &= \int_0^\infty \frac{1}{u}\ e^{-u^2} \cdot 2u\ du \\
&= 2\int_0^\infty e^{-u^2}\ du \\
&= 2\left(\frac{1}{2}\sqrt{\pi}\right) = \sqrt{\pi}
\end{aligned}
$$

(4) (a)

$$
E(X) = \sum_{i=0}^{2} x_i\ P(X=x) = \frac{1}{3}\Big(0+1+2\Big) = \frac{1}{3} + \frac{2}{3} = 1
$$

(b)

$$
\begin{aligned}
E(X^2) &= \sum_{i=0}^{2} x_i^2\ P(X=x) = \frac{1}{3}\Big(0+1+4\Big) = \frac{1}{3} + \frac{4}{3} = \frac{5}{3} \\
V(X) &= E(X^2) - \big[E(X)\big]^2 \\
&= \frac{5}{3} - 1 = \frac{2}{3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
E(X^4) &= \sum_{i=0}^{2} x_i^4\ P(X=x) = \frac{1}{3}\Big(0+1+16\Big) = \frac{1}{3} + \frac{16}{3} = \frac{17}{3} \\
V(X^2) &= E(X^4) - \big[E(X^2)\big]^2 \\
&= \frac{17}{3} - \frac{25}{9} = \frac{26}{9}
\end{aligned}
$$

(5) (a)

$$
\begin{aligned}
f(x) = \mathcal{F}'(x) &= \frac{1}{2}\ e^{-\frac{x}{2}},\ x > 0 \\
\mathcal{F}(x) &= \int_0^{x_0} \frac{1}{2}\ e^{-\frac{x}{2}} = P(0 < X < x_0) = P(X < x_0) \\
&= 1 - P(X > x_0) = 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}
$$

Solving for x_0 we find,

$$
\begin{aligned}
\frac{1}{2}\int_0^{x_0} e^{-\frac{x}{2}}\ dx = \frac{1}{2} \ &\Rightarrow\ \int_0^{x_0} e^{-\frac{x}{2}}\ dx = 1 \\
\Big[-2e^{-\frac{x}{2}}\Big]_0^{x_0} = 1 \ &\Rightarrow\ -2\Big[e^{-\frac{x}{2}}\Big]_0^{x_0} = 1 \\
\Big[e^{-\frac{x}{2}}\Big]_0^{x_0} = -\frac{1}{2} \ &\Rightarrow\ e^{-\frac{x_0}{2}} - 1 = -\frac{1}{2} \\
e^{-\frac{x_0}{2}} = \frac{1}{2} \ &\Rightarrow\ -\frac{x_0}{2} = \log\frac{1}{2} \\
-\frac{x_0}{2} = -\log 2 \ &\Rightarrow\ x_0 = 2\log 2 = \log 4
\end{aligned}
$$

(b) $P(X > \log 4) = \frac{1}{2}$ implies that the probability of randomly selecting a number greater $\log 4$ is a half.