# UNIVERSITY OF NEW ENGLAND

**UNIT NAME:** MATH 101

**PAPER TITLE:** Algebra and Differential Calculus

**PAPER NUMBER:** First and Only

**DATE:** **Friday 19 June 2009** **TIME:** **9:30 AM TO 12:30 PM**

**TIME ALLOWED:** Three (3) hours plus fifteen minutes reading time

**NUMBER OF PAGES IN PAPER:** SEVEN (7)

**NUMBER OF QUESTIONS ON PAPER:** TWELVE (12)

**NUMBER OF QUESTIONS TO BE ANSWERED:** TEN (10)

**STATIONERY PER CANDIDATE:**

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| **1** | 6 LEAF A4 BOOKS | **0** | ROUGH WORK BOOK |
| **1** | 12 LEAF A4 BOOKS | **0** | GRAPH PAPER SHEETS |

**OTHER AIDS REQUIRED:** NIL

**POCKET CALCULATORS PERMITTED:** NO

**TEXTBOOKS OR NOTES PERMITTED: FIVE (5) A4 SHEETS (10 PAGES IF WRITTEN ON BOTH SIDES) OF HAND-WRITTEN NOTES. NO PHOTOCOPIES, NO PRINTED NOTES PERMITTED.**

**INSTRUCTIONS FOR CANDIDATES:**

- Candidates **MAY** make notes on this examination question paper during the fifteen minutes reading time
- Questions are of equal value
- **SECTION A: -** Answer **ALL** questions
- **SECTION B:** - Answer only **TWO (2)** of the **FOUR (4)** questions provided
- Candidates may retain their copy of this examination question paper

## SECTION A

## Attempt all eight questions in this section.

### Question 1 [10 *marks*]

(a) Find, if they exist, the infimum and supremum of each of the following sets of real numbers. In each case, state whether the set has a maximum or minimum.

(i) $\{x \in \mathbb{N} \mid x^3 \geq 9\}$

(ii) $\{x \in \mathbb{R} \mid \sqrt{x^2 - 16} > 0\}$

(b) Use the mathematical inductiomn to prove that for every counting number, $n$,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

### Question 2 [10 *marks*]

(a) Find all complex numbers, $z$, such that $z^3 = -1$.

(b) For $w = 3 - 4i$, express each of the following complex numbers in the form $x + yi$, with $x, y \in \mathbb{R}$.

(i) $iw + 4$

(ii) $\dfrac{w+2}{w+8i}$

(iii) $\overline{3w - 7}$

(iv) $\left|\dfrac{w + 6 - 2i}{w}\right|$

(v) $\dfrac{1}{w^2}$

**Question 3 is on page 3**

## Question 3 [10 *marks*]

(a) Determine the range of the following functions. State whether the functions are injective and/or surjective. What, if they exist, are $\sup(g)$ and $\inf(g)$? Justify your answers.

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x \sin x$$
$$g\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x^2}{x^2+1}$$

(b) Which value of $k$ renders the following function continuous?

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 3x^2 + 4x + 7 & \text{if } x \leq 0 \\ \dfrac{\sin x}{kx} & \text{if } x > 0 \end{cases}$$

## Question 4 [10 *marks*]

(a) Discuss the monotonicity of the following sequences and their behaviour as $n \to \infty$.

(i) $\left(\dfrac{n-1}{n^2}\right)_{n\in\mathbb{N}}$

(ii) $\left(\left(\dfrac{-1}{2}\right)^n\right)_{n\in\mathbb{N}}$

## Question 5 [10 *marks*]

(a) Determine where the following functions are differentiable and find their derivatives at those points.

(i) $f\colon [-2, \infty[ \longrightarrow \mathbb{R}, \quad x \longmapsto x\sqrt{x^3+8}$

(ii) $g\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} \dfrac{1}{1+x^2} & \text{if } x \leq 0 \\ \sqrt{1-x^2} & \text{if } 0 < x \leq 1 \\ -\sqrt{x-1} & \text{if } x > 1 \end{cases}$

**Question 6 is on page 4**

## Question 6 [10 *marks*]

Consider the function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto 2x^3 + 9x^2 - 24x + 15$$

(a) Determine the intervals on which $f$ is

(i) increasing,

(ii) decreasing.

(b) Find the relative extrema of $f$.

(c) What are the absolute extrema of $f$ on $[-5, 2]$?

(d) Sketch the graph of $f$ on the interval $[-5, 2]$.

## Question 7 [10 *marks*]

(a) Solve the following system of equations.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 4 \\ 2x & + & 3y & + & \mathbf{z} & = & 13 \\ x & + & y & + & 3z & = & 9 \end{array}$$

(b) Take

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Calculate the following matrices, where possible.

(i) $AB$

(ii) $A - 2C$

(iii) $BC$

**Question 8 is on page 5**

## Question 8 [10 *marks*]

Let $P_1, P_2, P_3$ and $P_4$ be points in $\mathbb{R}^3$ with co-ordinates $(0,0,-1), (0,1,0), (1,1,0)$ and $(0,0,1)$ respectively.

(a) Express the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ in terms of the standard unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

(b) Find the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\overrightarrow{P_1P_3}$.

(c) Find the area of the triangle with vertices $P_1, P_2$ and $P_3$.

(d) Find the volume of the parallelepiped with sides $\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_2}$.

**Section B is on page 6**

# SECTION B

**Answer any two of the four questions in this section.**

## Question 9 [10 *marks*]

(a) Prove that $f\colon ]1,\infty, \longrightarrow \mathbb{R}, \quad x \longmapsto \dfrac{1}{x-1}$ is continuous.

(b) Take functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$,

$$|f(b) - f(a)| \leq 2|g(b) - g(a)|.$$

Prove that if $g$ is continuous at $c \in \mathbb{R}$, then so is $f$.

## Question 10 [10 *marks*]

(a) Show that if $-\frac{\pi}{4} < x < \frac{\pi}{4}$, then

$$x + \frac{\pi}{4} - 1 < \tan x < 2(x + \frac{\pi}{4}) - 1$$

(b) Take the function

$$f\colon \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{x^2 - 1}{x}.$$

Show that even though $f(-1) = f(1)$, there is now $c \in [-1, 1]$ with $f'(c) = 0$. Explain this apparent counter-example to Rolle's Theorem.

## Question 11 [10 *marks*]

A wire of length 100 millimetres is used to form a circle, a square, or cut into two to form one of each. Determine the length of wire used for the circle if the sum of the areas enclosed by the two figures is the

(i) maximum,

(ii) minimum.

**Question 12 is on page 7**

## Question 12 [10 *marks*]

Consider the Leslie matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

(a) Verify that the eigenvalues are $\frac{1}{2}, \frac{1}{4}(-1 \pm i)$ and that $\frac{1}{2}$ is the dominant eigenvalue.

(b) Find the asymptotic fractional population vector and hence the (long term) fraction of the population in each of the three age groups.