ME2115/TME2115

Mechanics of Machines

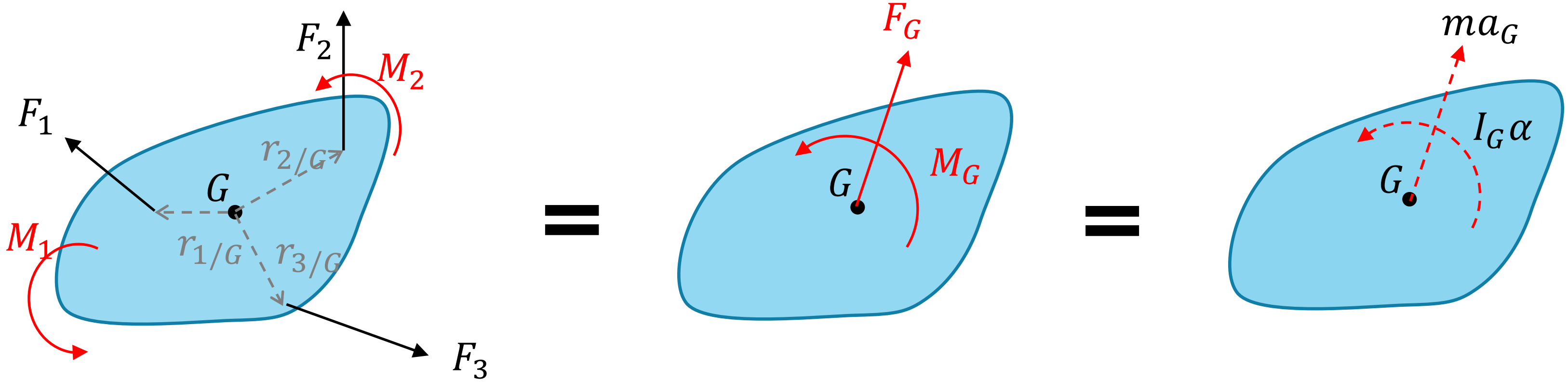
PRINCIPLE OF WORK AND ENERGY

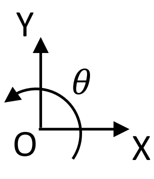
Learning Outcomes

- Able to recognize the *kinetic energy* of a rigid body undergoing general *2D planar motion*.
- Able to understand that the *work done* by *conservative forces* are path independent, and hence can be defined as *potential energy*.
- Able to recognize that *total energy* of system is *conserved* when only *conservative forces* are involved.
- Able to apply the *principle of work and energy* to solve dynamic problems.

Newton's 2nd Law - 2D planar motion

- Recall that the Newton's 2nd law of motion for a rigid body in 2D planar motion comprises of:





$$\vec{F}_G = m\vec{a}_G \text{ (translation motion)}$$

$$M_G = I_G\alpha \text{ (rotation motion)}$$

Alternative Principles from Newton's 2nd Law

- Consider the force equation for linear motion, i.e.

$$\vec{F} = m\vec{a}_G$$

If $\vec{F}$ is a function of time t, then

$$\vec{a}_G = \frac{d\vec{v}_G}{dt}$$

And if body mass is constant, then

$$\Rightarrow \vec{F}(t) = \frac{d}{dt}(m\vec{v}_G) = \frac{d\vec{L}}{dt}$$

$$\Rightarrow \int_{t_1}^{t_2} \vec{F}(t)dt = \vec{L}_2 - \vec{L}_1$$

Principle of impulse and momentum

If $\vec{F}$ is a function of spatial coordinates $\vec{x}$, then

$$\vec{a}_G = \frac{d\vec{v}_G}{dt} = \frac{d\vec{x}}{dt}\frac{d\vec{v}_G}{d\vec{x}} = \vec{v}_G\frac{d\vec{v}_G}{d\vec{x}}$$

$$\Rightarrow \vec{F}(\vec{x}) = m\vec{v}_G\frac{d\vec{v}_G}{d\vec{x}}$$

$$\Rightarrow \int_{x_1}^{x_2} \vec{F}(\vec{x}).\,d\vec{x} = \frac{1}{2}m|\vec{v}_2|^2 - \frac{1}{2}m|\vec{v}_1|^2$$

Principle of work and energy

Principle of Work and Energy – Translation

- Consider the force equation, which can be decomposed into its components as

$$\vec{F} = m\frac{d\vec{v}}{dt} \Rightarrow \begin{cases} F_x = m\dfrac{dv_x}{dt} \\ F_y = m\dfrac{dv_y}{dt} \end{cases}$$

- According to chain-rule of differentiation, the accelerations can be written as

$$\begin{cases} \dfrac{dv_x}{dt} = \dfrac{dv_x}{dx}\dfrac{dx}{dt} = v_x\dfrac{dv_x}{dx} \\ \\ \dfrac{dv_y}{dt} = \dfrac{dv_x}{dy}\dfrac{dy}{dt} = v_y\dfrac{dv_y}{dy} \end{cases} \Rightarrow \begin{cases} F_x = mv_x\dfrac{dv_x}{dx} \\ \\ F_y = mv_y\dfrac{dv_y}{dy} \end{cases}$$

Principle of Work and Energy – Translation…

- Finally, integrating the force equations from state 1 to state 2. For the force equation in *x*-direction, it is given by

$$\int_{x_1}^{x_2} F_x \, dx = \int_{v_{x,1}}^{v_{x,2}} m v_x \, dv_x$$

$$\Rightarrow U_{x,1\to 2} = \frac{1}{2} m \left(v_{x,2}\right)^2 - \frac{1}{2} m \left(v_{x,1}\right)^2$$

where

$U_{x,1\to 2}$ is the work done by the *x*-component of the resultant force.

- Similarly for the *y*-component, we have

$$U_{y,1\to 2} = \frac{1}{2} m \left(v_{y,2}\right)^2 - \frac{1}{2} m \left(v_{y,1}\right)^2$$

Principle of Work and Energy – Translation…

- Summing the two equations gives

$$U_{x,1\to2} + U_{y,1\to2} = \frac{1}{2}m\left[\left(v_{x,2}\right)^2 + \left(v_{y,2}\right)^2\right] - \frac{1}{2}m\left[\left(v_{x,1}\right)^2 + \left(v_{y,1}\right)^2\right]$$

$$\Rightarrow \boxed{U_{1\to2} = T_2 - T_1}$$

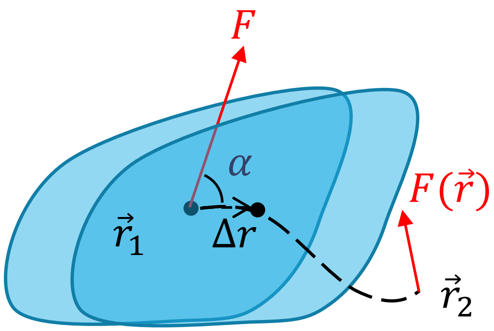
where

$U_{1\to2}$ is the work done by the forces that move the rigid body from state 1 to 2,

$T_1 = \frac{1}{2}m|\vec{v}_1|^2$, and $T_2 = \frac{1}{2}m|\vec{v}_2|^2$ are the translational kinetic energy of the rigid body at state 1 and 2, respectively.

- This is the *principle of work and energy* for rigid body in *translation motion*. It essentially states that the work done *by the external forces* would change the kinetic energy of the rigid body from that at state 1 to state 2.

Work Done by Force on Body under Translation



- Work done by a force for a infinitesimal displacement is

$$dU = \vec{F}.\Delta\vec{r}$$

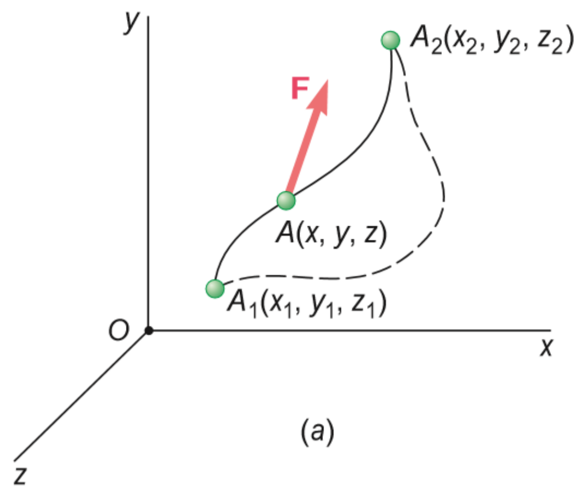
- And for finite displacement from $\vec{r}_1$ to $\vec{r}_2$, it becomes

$$U_{1\rightarrow 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}.d\vec{r}$$

Work of Conservative Forces

- A force (or moment) is said to be *conservative* if the work done by it $U_{1\rightarrow 2}$ is *independent of the path* it traces from point A_1 to A_2 (rotate from θ_1 to θ_2). That is,

$$U_{1\rightarrow 2} = \int_{Path\ 1} \vec{F}.d\vec{r} = \int_{Path\ 2} \vec{F}.d\vec{r}$$



- In this case, the *work done by* such a force can be expressed in terms of the change in the potential energy of the mechanism, i.e.

$$U_{1\rightarrow 2} = U_2 - U_1 = V_1 - V_2$$

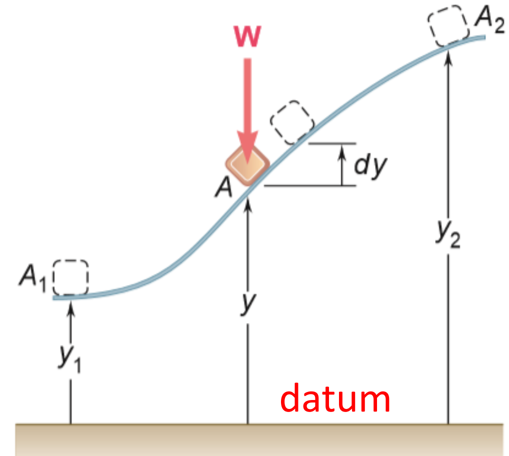
where V_1 and V_2 are the potential energies stored at state 1 and 2, respectively. Note that negative work done by the force gives rise to positive gains in potential energy.

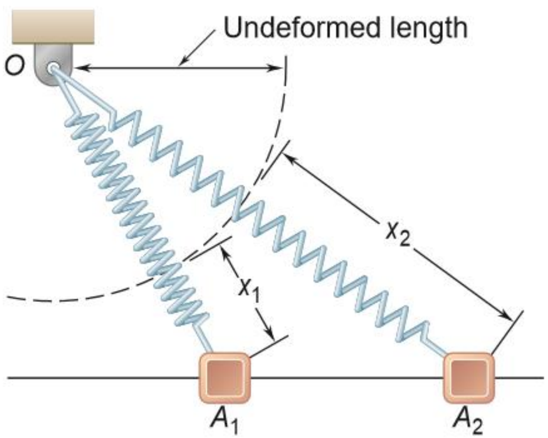
Work of Conservative Forces…

- Two common conservative forces are:
  - Gravitational force near earth surface

$$\vec{F} = -mg\vec{j}$$

$$V = mgy$$ (w.r.t. a given datum, which can be arbitrary defined)





Elastic force of linear spring

$$\vec{F} = -k\Delta x\vec{i'}$$

$$V = \frac{1}{2}k(\Delta x)^2$$ (w.r.t. undeformed length)

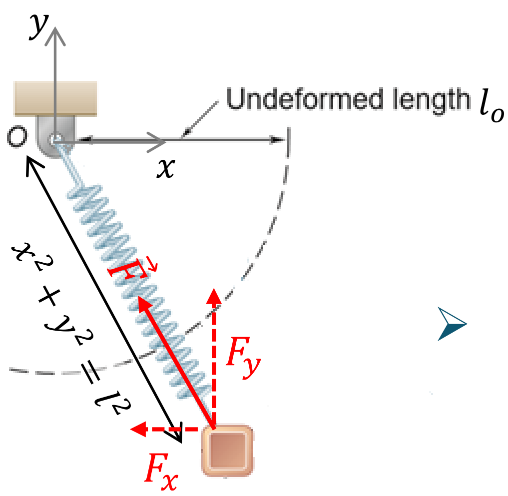
$$U_{1\rightarrow 2} = \frac{1}{2}k(x_1 - x_2)^2 \ (Wrong)$$

$$U_{1\rightarrow 2} = \frac{1}{2}k(x_1)^2 - \frac{1}{2}k(x_2)^2 (Correct)$$

Work of Conservative Forces...

- And the conservative force and the potential energy is also related by

$$\vec{F} = -\left(\frac{\partial V}{\partial x}\vec{\imath} + \frac{\partial V}{\partial y}\vec{\jmath} + \frac{\partial V}{\partial z}\vec{k}\right) = -\nabla V(x, y, z)$$



- Elastic potential energy of the spring is

$$V = \frac{1}{2}k(l - l_0)^2 = \frac{1}{2}k\left\{\sqrt{x^2 + y^2} - l_0\right\}^2$$

- Hence, the spring force in this coordinate system is given by

$$\vec{F} = -\left(\frac{\partial V}{\partial x}\vec{\imath} + \frac{\partial V}{\partial y}\vec{\jmath}\right) = -k(l - l_0)\left\{\left(\frac{x}{l}\right)\vec{\imath} + \left(\frac{y}{l}\right)\vec{\jmath}\right\}$$

$$= F_x\vec{\imath} + F_y\vec{\jmath}$$

- If only conservative forces exist in the system, then

$$V_1 - V_2 = T_2 - T_1 \;\Rightarrow\; \boxed{V_1 + T_1 = V_2 + T_2}$$

Conservation of Energy

Work of Non-conservative Forces

- In general, work done by forces that cannot be stored as energy are called *non-conservative* or *dissipative* forces. One obvious example is the frictional force, which is given by

$$\vec{F}_{fric} = \mu_k N$$

where N is the normal force and μ_k is the kinetic frictional coefficient.

- In this case, the work done is *always negative*, that is, the energy will be removed from the system, usually in terms of heat and sound energy. Hence, the work done is

$$U_{1\to 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{fric}.\,d\vec{r}$$

And for *constant frictional force*, it becomes $U_{1\to 2} = -F_{fric}.S$, where S is the distant covered by the force.

Principle of Work and Energy – Rotation

- Now, consider the moment equation, and using the same technique for the translational motion, gives

$$M_G = I_G\alpha^{\dagger} \Rightarrow M_G = I_G\omega\frac{d\omega}{d\theta}$$

$$\Rightarrow \int_{\theta_1}^{\theta_2} M_G\,d\theta = \int_{\omega_1}^{\omega_2} I_G\omega\,d\omega$$

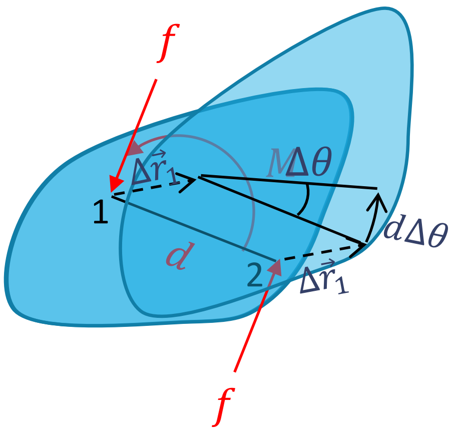
$$\Rightarrow U_{1\to2} = \frac{1}{2}I_G(\omega_2)^2 - \frac{1}{2}I_G(\omega_1)^2$$

$$\Rightarrow U_{1\to2} = T_2 - T_1$$

where the kinetic energy for the rotational motion is given by $T = \frac{1}{2}I_G\omega^2$.

† *This expression only works for rigid body in 2D motion. And the geometry of rigid body must be symmetrical about this plane.*

Work Done of Moment on Body under Rotation



- Work done by a moment for a infinitesimal rotation is

$$dU = -\vec{f}.\Delta\vec{r}_1 + \vec{f}.\left(\Delta\vec{r}_2 + \overrightarrow{d\Delta\theta}\right) = f d\Delta\theta = M\Delta\theta$$

- Hence, for finite rotation θ_1 to θ_2, it is

$$U_{1\to2} = \int_{\theta_1}^{\theta_2} M d\theta$$

And for *constant moment*, it becomes $U_{1\to2} = M\Delta\theta$.

- Torque due to linear torsional spring with stiffness k_θ is

$$M = -k_\theta \theta$$

Hence, its potential energy is given by

$$V = \frac{1}{2} k_\theta \theta^2$$

Principle of Work and Energy – 2D planar motion

- For a rigid body in 2D planar motion, the principle of work and energy is the sum of the translation and rotation motions, i.e.

$$\boxed{U_{1\to 2} = T_2 - T_1}$$

where

$$\boxed{U_{1\to 2} = \int \vec{F}.d\vec{r} + \int M d\theta}$$

is the total work done by the external forces and moments acting on the rigid body, and

the kinetic energy comprises two components, namely its translational and rotational parts,

$$\boxed{T = \frac{1}{2}m|\vec{v}_G|^2 + \frac{1}{2}I_G\omega^2}$$

Kinetic Energy of Rigid Body in General Motion

- The kinetic energy of a rigid body in general motion can also be derived as follows:
  - For a particle, its kinetic energy is

$$T = \frac{1}{2}\Delta m|\vec{v}|^2 = \frac{1}{2}\Delta m(\vec{v}.\vec{v})$$

  - For a system of particles, it becomes

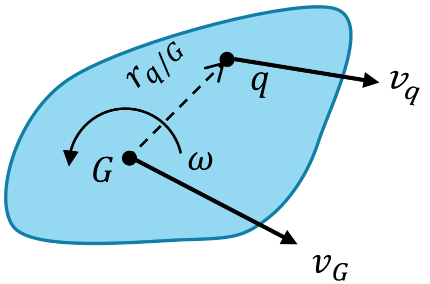
$$T = \sum_q \frac{1}{2}\Delta m_q\left(\vec{v}_q.\vec{v}_q\right)$$

  - For a rigid body, which resemble a system of particles in continuum body, it is

$$T = \int_B \frac{1}{2}\left(\vec{v}_q.\vec{v}_q\right)dm_q$$

Recall that for rigid body, the velocity at any point in the body is related to the velocity at the mass center as

$$\vec{v}_q = \vec{v}_G + \vec{\omega} \times \vec{r}_{q/G}$$



Kinetic Energy of Rigid Body in General Motion…

- Substitute this expression into the kinetic equation

$$T = \frac{1}{2}\int_B (\vec{v}_G + \vec{\omega} \times \vec{r}_{q/G}).(\vec{v}_G + \vec{\omega} \times \vec{r}_{q/G})dm_q$$

$$= \frac{1}{2}\int_B \{\vec{v}_G.\vec{v}_G + 2\vec{v}_G.(\vec{\omega} \times \vec{r}_{q/G}) + (\vec{\omega} \times \vec{r}_{q/G}).(\vec{\omega} \times \vec{r}_{q/G})\}dm_q$$

- Now, the first term is

$$\frac{1}{2}\int_B (\vec{v}_G.\vec{v}_G)dm_q = \frac{1}{2}|\vec{v}_G|^2 \int_B dm_q = \frac{1}{2}m|\vec{v}_G|^2 \quad \text{(translational energy)}$$

- The second term is

$$\frac{1}{2}\int_B 2\vec{v}_G.(\vec{\omega} \times \vec{r}_{q/G})dm_q = \frac{1}{2}\vec{v}_G.\vec{\omega} \times \left(\int_B \vec{r}_{q/G}dm_q\right) = 0$$

$$M\vec{r}_G = \int_B \vec{r}_q dm_q$$

$$\Rightarrow \left(\int_B dm_q\right)\vec{r}_G = \int_B \vec{r}_q dm_q$$

$$\Rightarrow \left(\int_B (\vec{r}_q - \vec{r}_G)dm_q\right)\vec{r}_G = 0$$

Kinetic Energy of Rigid Body in General Motion…

- The last term is

$$\frac{1}{2}\int_B (\vec{\omega}\times\vec{r}_{q/G}).(\vec{\omega}\times\vec{r}_{q/G})\,dm_q = \frac{1}{2}\left(I_x{\omega_x}^2 + I_y{\omega_y}^2 + I_z{\omega_z}^2\right) - 2\left(I_{xy}\omega_x\omega_y + I_{yz}\omega_y\omega_z + I_{zx}\omega_z\omega_x\right)$$

where

$$I_x = \int_B (y^2+z^2)dm_q\,;\;\; I_y = \int_B (x^2+z^2)dm_q\,;\;\; I_z = \int_B (y^2+x^2)dm_q$$

$$I_{xy} = \int_B xydm_q\,;\;\; I_{yz} = \int_B yzdm_q\,;\;\; I_{xz} = \int_B xzdm_q$$

- Now for *2D planar motion*, $\omega_x = \omega_y = 0$. And let $\omega_z = \omega$, $I_z = I_G$, this expression is reduced to

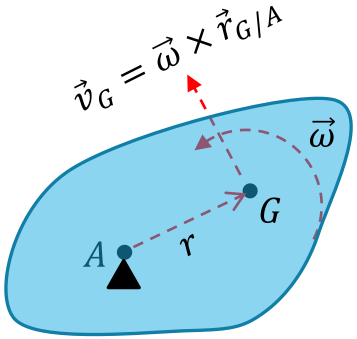
$$\frac{1}{2}\int_B (\vec{\omega}\times\vec{r}_{q/G}).(\vec{\omega}\times\vec{r}_{q/G})\,dm_q = \frac{1}{2}I_G\omega^2$$

➢ Hence, the kinetic energy of a rigid body in *2D planar motion* is

$$\boxed{T = \frac{1}{2}m|\vec{v}_G|^2 + \frac{1}{2}I_G\omega^2}$$

Kinetic Energy of Rigid Body – Fixed axis rotation

- For rigid body rotating about a ***fixed*** point *A*, its kinetic energy is given by



$$T = \frac{1}{2}m|\vec{v}_G|^2 + \frac{1}{2}I_G\omega^2$$

where

$$\vec{v}_G = \vec{\omega} \times \vec{r}_{G/A}$$

- And for 2D planar motion, $\vec{\omega}$ and $\vec{r}_{G/A}$ are always perpendicular. Hence,

$$|\vec{v}_G| = \left|\vec{\omega} \times \vec{r}_{G/A}\right| = |\vec{\omega}|\left|\vec{r}_{G/A}\right| sin(90^o) = \omega r$$

- Substitute this into the kinetic energy expression, gives

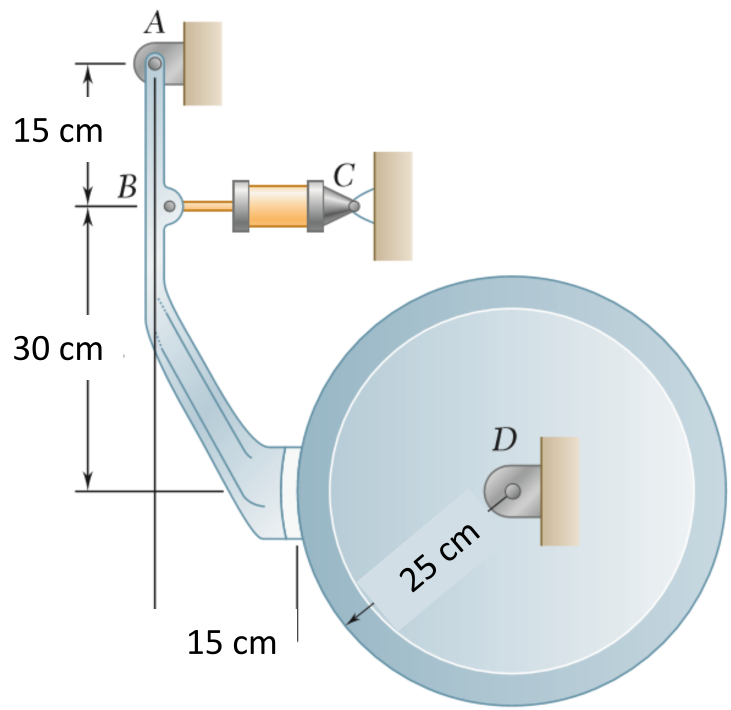
$$T = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}I_G\omega^2 = \frac{1}{2}[I_G + mr^2]\omega^2 \Rightarrow \boxed{T = \frac{1}{2}I_A\omega^2}$$

Parallel axis theorem

Note that this expression is also applicable if A is also the I.C.R.

Example 1

The 25 *cm* radius brake drum is attached to a larger flywheel which is not shown. The total mass the flywheel and drum is 20 *kg*, and its radius of gyration is 20 *cm*. And the coefficient of kinetic friction between the drum and the brake shoe is $\mu_k = 0.40$. Knowing that the initial angular velocity is 240 *rpm* clockwise, determine the force which must be exerted by the hydraulic cylinder if the system is to stop in 75 revolutions.



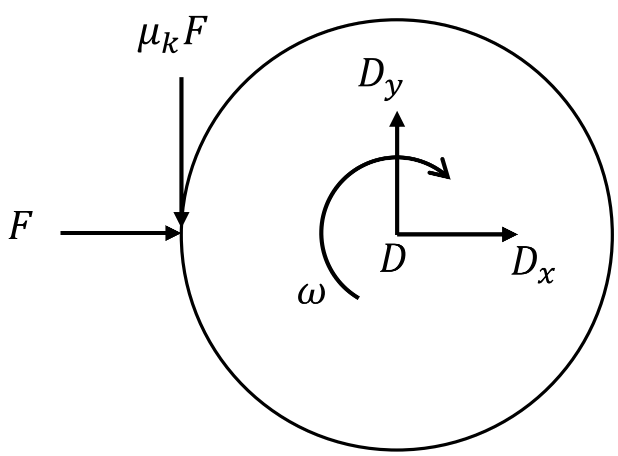
Example 1...

Based on the given information,

$$\omega_1 = 240\ rpm = 8\pi\ rad/s$$

$$I = mk^2 = (20)(0.2)^2 = 0.8\ kgm^2$$

$$75\ rev = 75(2\pi) = 150\pi\ rad$$



At initial state, the kinetic energy is

$$T_1 = \frac{1}{2} I {\omega_1}^2 = \frac{1}{2}(0.8)(8\pi)^2 = 252.7\ J$$

Work done by friction force,

$$U_{1\rightarrow 2} = -\big((\mu_k F)(DF)\big)(150\pi)$$

$$= -0.4(0.25)(150\pi)F = -47.124F$$

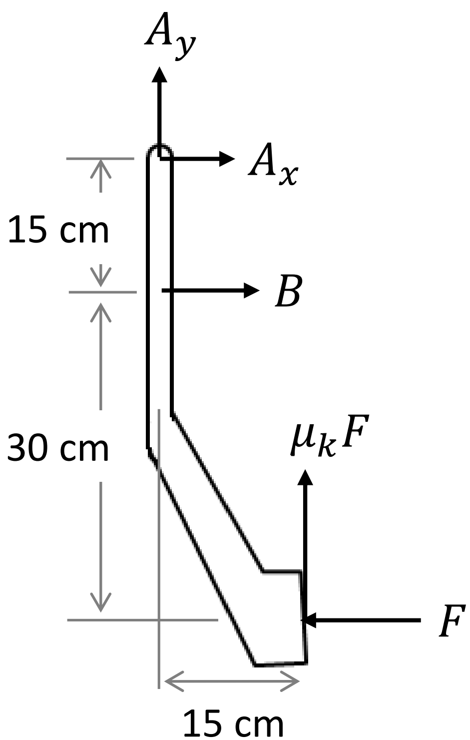
Example 1…

Applying the principle of work and energy, gives

$$T_1 + U_{1\to 2} = T_2 = 0$$

$$\Rightarrow\ 252.7 - 47.12F = 0$$

$$\Rightarrow F = 5.363\ N$$



Finally, consider the *FBD* of the braking level.

And taking moment about *A*, we have

$$(0.15)B - (0.45)F + (0.15)\mu_k F = 0$$

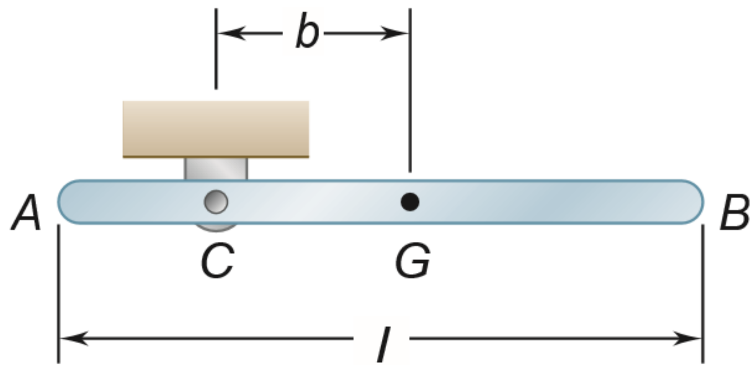
$$\Rightarrow 0.15B - (0.45)(5.363) + (0.15)(0.4)(5.363) = 0$$

$$\Rightarrow B = \mathbf{13.94}\ N$$

Example 2

A slender rod of length *l* is pivoted about a point *C* located at a distance *b* from its center *G*. It is released from rest in a horizontal position and swings freely.

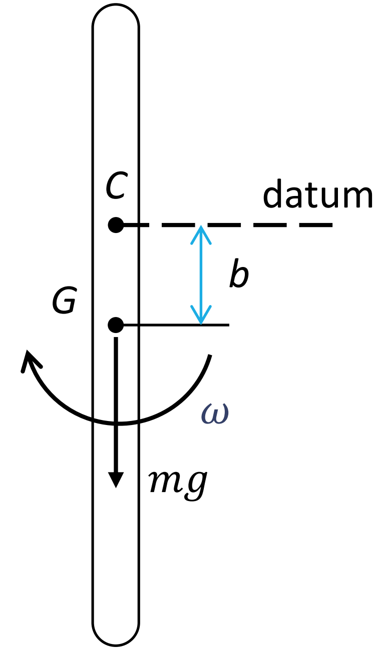
Determine the distance *b* for which the angular velocity of the rod as it passes through a vertical position is maximum, and its corresponding angular velocity.



Example 2...

(a) Consider the energy of the rod at its initial horizontal position, which is zero when the datum is defined at point *C*, i.e.

$$V_1 = T_1 = 0$$



Next, consider the energy of the rod when rotates to its vertical orientation,

$$V_2 = -mgb$$

$$T_2 = \frac{1}{2} I_C \omega^2 = \frac{1}{2}(I_G + mb^2)\omega^2$$

Applying the conservation of energy, we have

$$0 = -mgb + \frac{1}{2}\left[\frac{1}{12}ml^2 + mb^2\right]\omega^2 \Rightarrow \omega^2 = \frac{24gb}{12b^2 + l^2}$$

Example 2...

The maximum velocity occurs when $\frac{d\omega}{db} = 0$, which is equivalent to $\frac{d}{db}(\omega^2) = 2\omega\frac{d\omega}{db} = 0$, since ω cannot be zero. Hence,

$$\frac{d}{db}\left[\frac{24gb}{12b^2 + l^2}\right] = 0 \Rightarrow \frac{24g(12b^2 + l^2) - 24gb(24b)}{(12b^2 + l^2)^2} = 0$$

And considering only the numerator term,

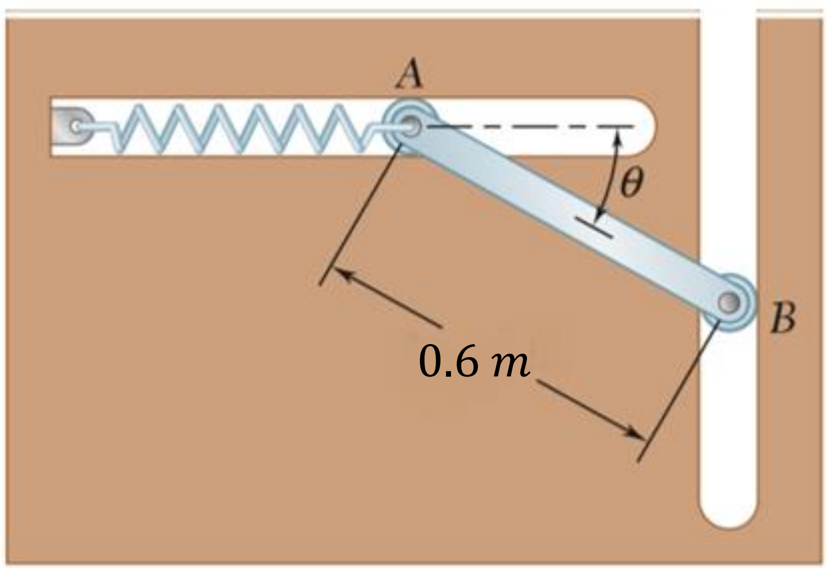
$$\Rightarrow (12b^2 + l^2) - b(24b) = 0 \Rightarrow b = \boldsymbol{\frac{l}{\sqrt{12}}}$$

And the corresponding angular speed is

$$\omega^2 = \frac{24g\left(\frac{l}{\sqrt{12}}\right)}{12\left(\frac{l}{\sqrt{12}}\right)^2 + l^2} = \sqrt{12}\frac{g}{l} \Rightarrow \omega = \boldsymbol{1.861\sqrt{\frac{g}{l}}}$$

Example 3

The ends of a 4 *kg* rod *AB* are constrained to move along slots cut in a vertical plate as shown. A spring of constant k = 500 N/m is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when $\theta = 0$, determine the angular velocity of the rod when $\theta = 30^o$.

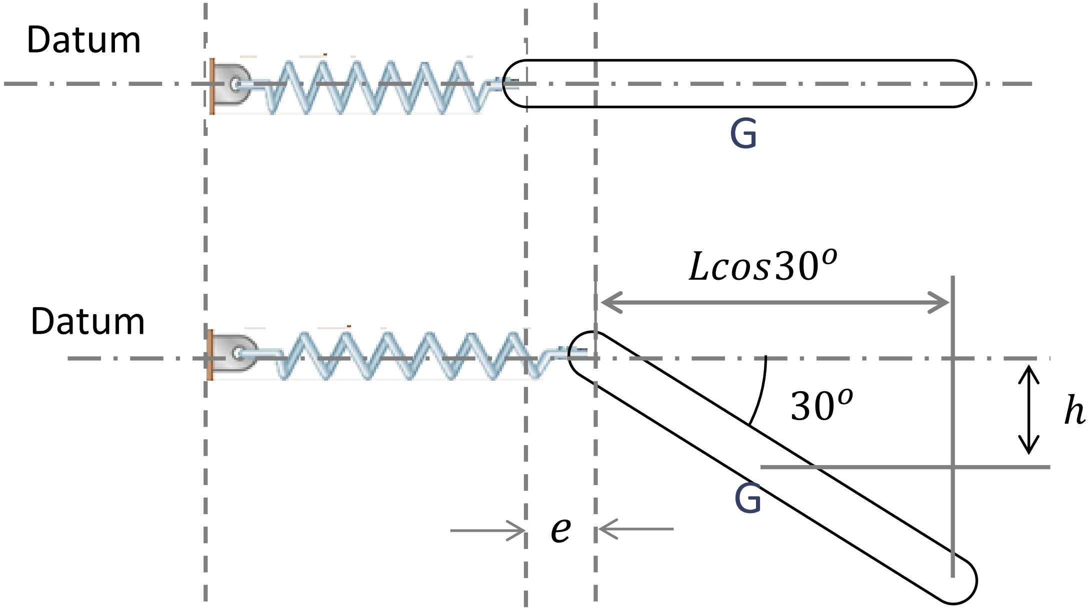


Example 3...

At initial horizontal position,

$$T_1 = V_1 = 0$$

At the final position,



Elongation of spring is,

$$e = L - Lcos30^o$$

$$= 0.60(1 - cos30^o)$$

$$= 0.08038\ m$$

Vertical position of G,

$$h = -\frac{L}{2}sin30^o$$

$$= -0.15\ \mathrm{m}$$

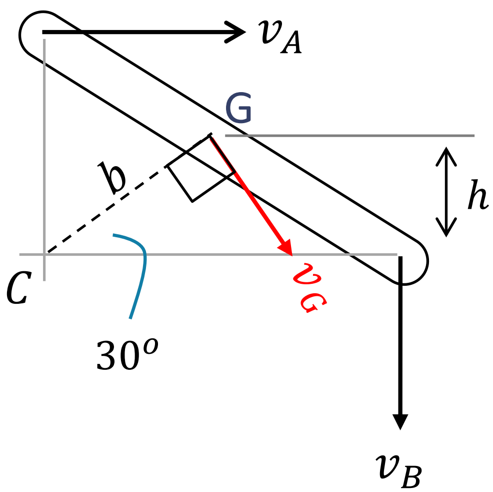
Example 3...

The potential energy at final state is

$$V_2 = \frac{1}{2}ke^2 + mgh$$

$$= \frac{1}{2}(500)(0.08038)^2 + (4)(9.81)(-0.15)$$

$$\Rightarrow \; V_2 = -4.271\,J$$



Using the instantaneous center of rotation,

$$\frac{h}{b} = sin30^o \Rightarrow b = 0.30\,m$$

$$v_G = b\omega = 0.3\omega$$

Example 3…

The kinetic energy is

$$T_2 = \frac{1}{2} m {v_G}^2 + \frac{1}{2} I_G \omega^2, \qquad or \qquad T_2 = \frac{1}{2} I_C \omega^2$$

Using either formulas, we get

$$T_2 = \frac{1}{2}\left(\frac{1}{12} mL^2 + mb^2\right)\omega^2 = \frac{1}{2}\left(\frac{1}{12}(4)(0.6)^2 + (4)(0.3)^2\right)\omega^2$$

$$\Rightarrow \; T_2 = 0.24\omega^2$$

Applying conservation of energy,

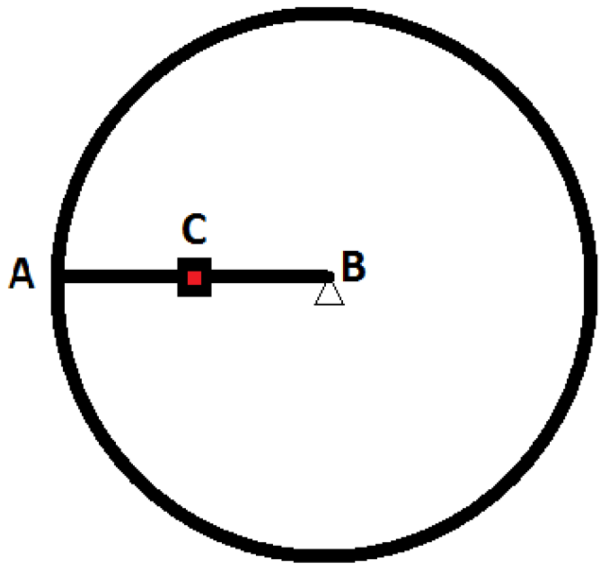
$$V_2 + T_2 = 0$$

$$\Rightarrow 0.24\omega^2 - 4.271 = 0$$

$$\Rightarrow \omega = \mathbf{4.22} \, rad/s$$

Example 4

A collar of mass 1 *kg* is rigidly attached at the mid-point of a slender rigid arm *AB* of mass 3 *kg* and length of 0.8 *m*. The arm is free to rotate about a pivot at *B*. The other end A of the rigid arm is also rigidly attached to a ring of mass 5 *kg* with the centre of the ring at *B*. The motion of the whole assembly is constrained within a vertical plane.



Knowing that the arm *AB* is released from rest from the position shown and the gravitational acceleration g = 9.81m/s^2, determine the angular velocity of the whole assembly when the arm *AB* has rotated through 90 degrees,

Example 4…

Let the datum be defined at B, which means that the total energy is zero at the initial rest position, i.e.

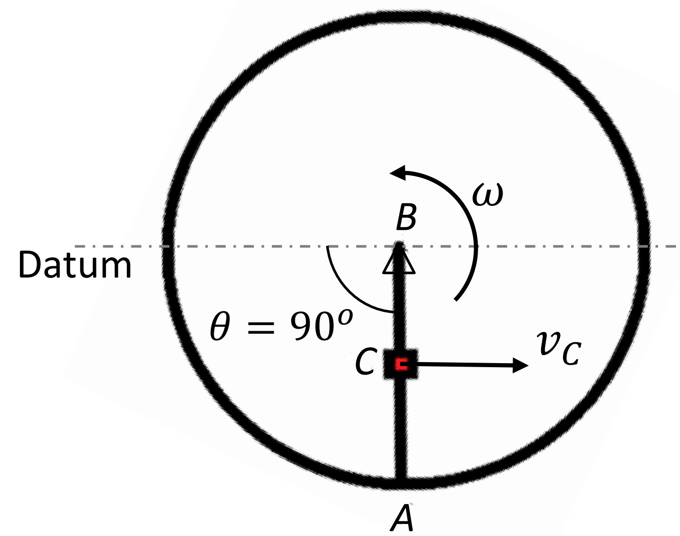
$$V_1 = T_1 = 0$$

When it is rotated by $\theta = 90^o$ to state 2, the gravitational potential energy becomes

$$V_2 = -m_c g \frac{R}{2} - m_{AB} g \frac{R}{2}$$

$$= -(1+3)(9.81)\frac{0.8}{2}$$

$$= -15.69\, J$$



Example 4…

Since the assembly is rotating about a fixed point, we have

$$v_C = \frac{R}{2}\omega = 0.4\omega$$

The total kinetic energy of the assembly is

$$T_{collar} = \frac{1}{2}m_c {v_C}^2 = \frac{1}{2}(1)(0.4\omega)^2 = 0.08\omega^2$$

$$T_{arm} = \frac{1}{2}m_{AB}{v_C}^2 + \frac{1}{2}I_{AB}\omega^2$$

$$= \frac{1}{2}(3)(0.4\omega)^2 + \frac{1}{2}\left(\frac{1}{12}(3)(0.8)^2\right)\omega^2 = 0.32\omega^2$$

$$T_{ring} = \frac{1}{2}I_R\omega^2 = \frac{1}{2}(m_R R^2)\omega^2$$

$$= \frac{1}{2}(5)(0.8^2)\omega^2 = 1.6\omega^2$$

Example 4...

Hence, the total kinetic energy of the body at state 2 is

$$T_2 = 0.08\omega^2 + 0.32\omega^2 + 1.6\omega^2 = 2.0\omega^2$$

Alternately, the kinetic energy at state 2 can be derived as

$$T_2 = \frac{1}{2} I_B \omega^2$$

where

$$I_B = \left(m_c \left(\frac{R}{2}\right)^2\right) + \left(\frac{1}{12} m_{AB} R^2 + m_{AB} \left(\frac{R}{2}\right)^2\right) + \left(m_R R^2\right) = 4.0$$

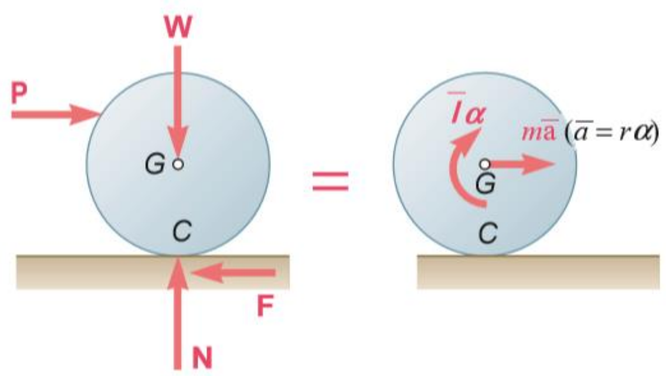
Finally, using conservation of energy, gives

$$V_1 + T_1 = T_2 + V_2 \Rightarrow 15.69 = 2\omega^2$$

$$\Rightarrow \omega = \mathbf{2.80}\ rad/s$$

Work Done under Rolling without Slipping

➢ Recall that when a circle body rotates, it can be



- Rolling without sliding:

$$F < \mu_s N, \quad \text{and} \quad a = r\alpha$$

- Rolling, sliding impending:

$$F = \mu_s N, \quad \text{and} \quad a = r\alpha$$

- Rotating and sliding:

$$F = \mu_k N, \quad \text{and} \quad a \text{ and } \alpha \text{ are independent.}$$

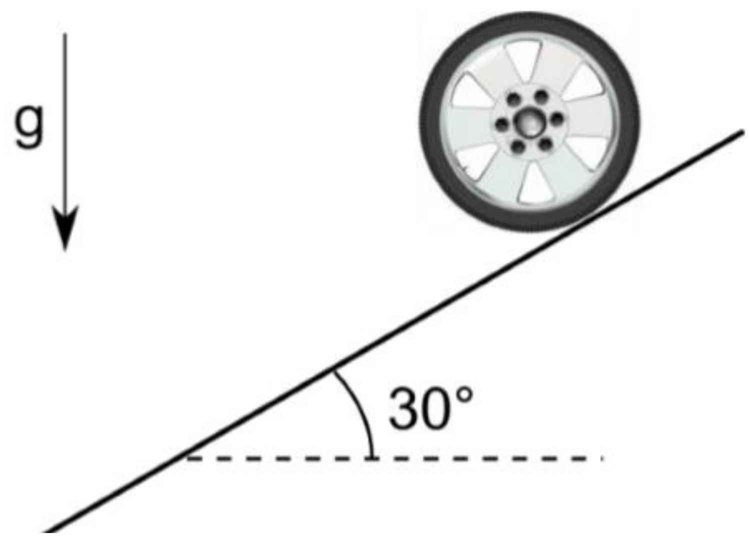
➢ But for a rolling body *without slipping*, the frictional force *does not* do any work because there is *no displacement* associated with it. In this case, the conservation of energy is still applicable †.

† *Does that means that the body will continue to roll forever? No, if we take into consideration the rolling frictional effect. In this module, this is assumed to be negligible.*

Example 5

An experiment was conducted by starting a wheel at rest and letting it rolls down a uniform rough surface which is inclined at 30° to the horizontal. It was observed that the wheel rolls without sliding, and it travels a distance of 24 *m* over a time period of 4 *s*.

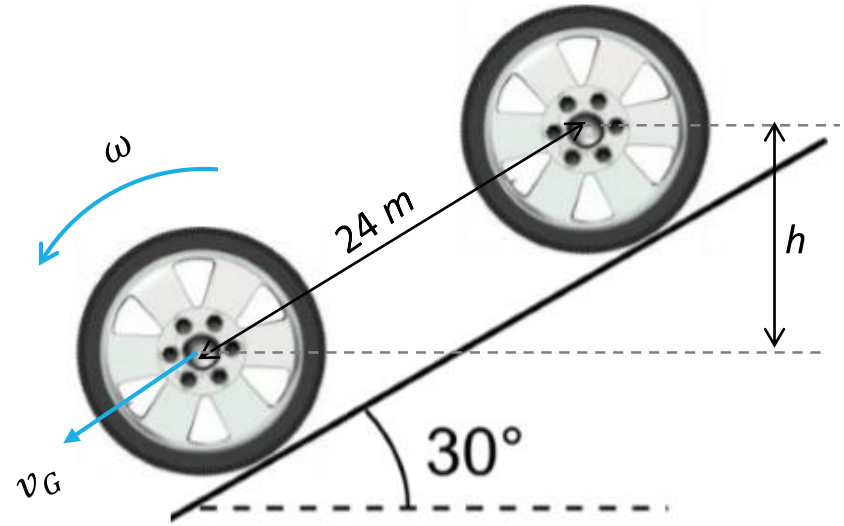
With this information, determine the centroidal moment of inertia of the wheel, given that its mass is 8 *kg* and its radius is 0.2 *m*. Take $g = 9.81\ m/s^2$.



Example 5…

Let the datum be defined at the initial rest position. This means that the potential and kinetic energies are zeros, i.e.

$$V_1 = T_1 = 0$$



At state 2, after it moved by 24 m, the gravitational potential energy is

$$V_2 = mgh = 8(9.81)(-24sin30)$$
$$= -941.76\, J$$

And its kinetic energy is

$$T_2 = \frac{1}{2}mv_G{}^2 + \frac{1}{2}I_G\omega^2 = \frac{1}{2}(8)v_G{}^2 + \frac{1}{2}I_G(\frac{v_G}{0.2})^2$$
$$= 4v_G{}^2 + 12.5I_G v_G{}^2$$

Due to rolling without slipping.

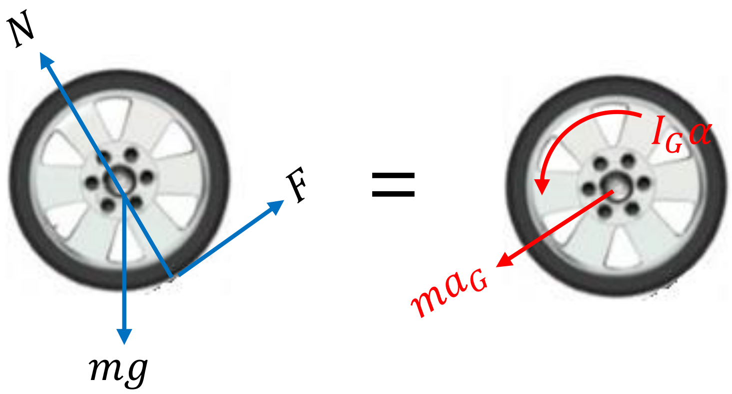
Example 5...

Applying the conservation of energy, gives

$$0 = V_2 + T_2$$

$$\Rightarrow \; -941.76 + (4{v_G}^2 + 12.5 I_G {v_G}^2) = 0$$

Now, consider the *FBD* of the wheel.



Taking moment about *G*, gives

$$Fr = I_G\alpha \Rightarrow \alpha = \frac{Fr}{I_G}$$

which is constant given that the surface has uniform roughness.

And since it is rolling without slipping, the linear acceleration would also be a constant value, given by

$$a_G = \alpha r$$

Example 5…

Using the kinematic formulas for a point under constant acceleration, we have

$$s = u_o t + \frac{1}{2} a_G t^2 \Rightarrow 24 = 0 + \frac{1}{2} a_G (4)^2$$

$$\Rightarrow a_G = 3\ m/s^2$$

And its velocity at state 2 would be

$$v_G = u_o + a_G t = 3(4) = 12\ m/s$$

Finally, substituting the velocity into conservation of energy, gives

$$\Rightarrow\ -941.76 + (4(12)^2 + 12.5 I_G (12)^2) = 0$$

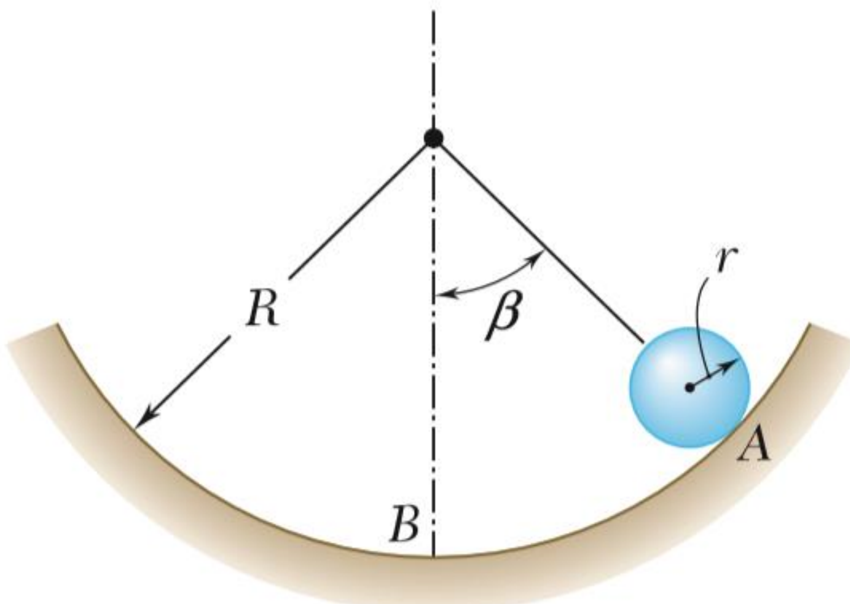
$$\Rightarrow I_G = \mathbf{0.203}\ kg.m^2$$

Example 6

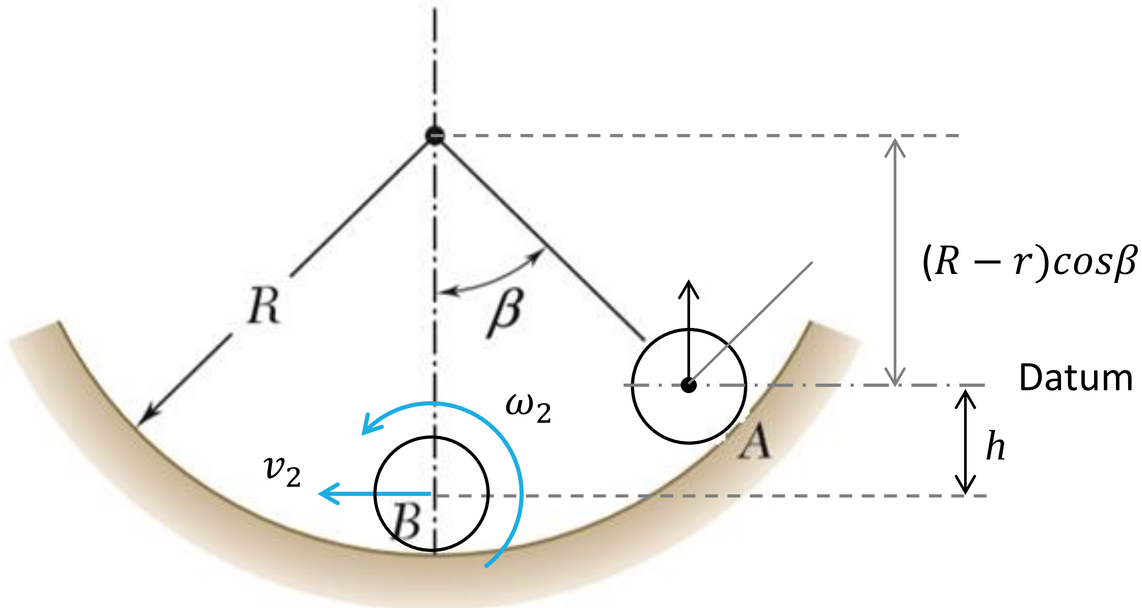
A sphere of mass *m* and radius *r* rolls without slipping inside a curved surface of radius *R*. Knowing that the sphere is released from rest in the position shown, derive an expression for:

(a) the linear velocity of the sphere as it passes through *B*, and

(b) the magnitude of the vertical reaction at that instant.



Example 6…



At initial position, total energy is zero.

At state 2, the gravitational potential energy is

$$V_2 = -mgh = -mg(R-r)(1-cos\beta)$$

Example 6...

And its kinetic energy is

$$T_2 = \frac{1}{2}m(v_2)^2 + \frac{1}{2}I_G(\omega_2)^2$$

$$= \frac{1}{2}m(v_2)^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v_2}{r}\right)^2 = \frac{7}{10}m(v_2)^2$$

Alternatively, it can also be calculated as

$$T_2 = \frac{1}{2}I_B(\omega_2)^2 = \frac{1}{2}I_B\left(\frac{v_2}{r}\right)^2$$

where I_B is the moment of inertia of the sphere about *B*, which is the *I.C.R*, and it is given by

$$I_B = I_G + mr^2 = \frac{7}{5}mr^2$$

Example 6...

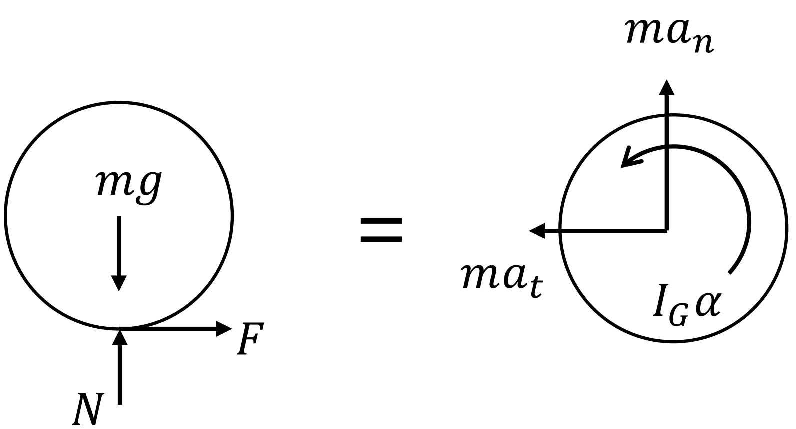
Finally, using the conservation of energy, we get

$$V_2 + T_2 = 0$$

$$\Rightarrow \frac{7}{10} m(v_2)^2 = mg(R - r)(1 - cos\beta)$$

$$\Rightarrow v_2 = \sqrt{\frac{10}{7} g(R - r)(1 - cos\beta)}$$

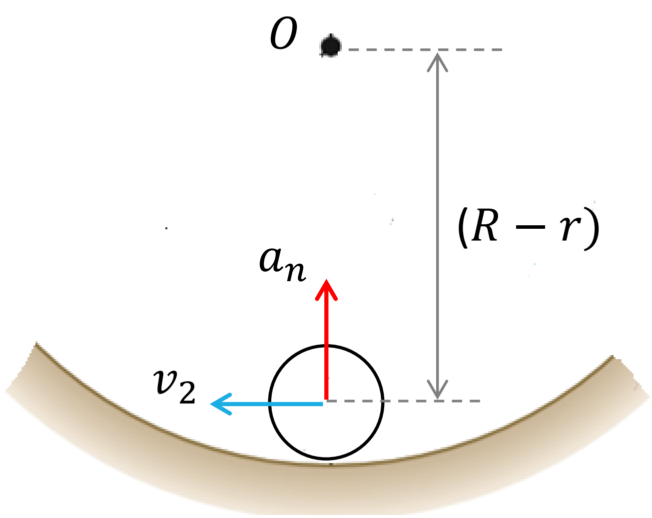
(b) The *FBD* of the sphere at state 2,



From the force equation in the vertical direction, we have

$$N - mg = ma_n \Rightarrow N = m(g + a_n)$$

Example 6...



And since the centre of the sphere is rotating about the centre of circular path, hence it will experience centripetal acceleration as

$$a_n = \frac{{v_2}^2}{R-r} = \frac{10}{7} g(1 - cos\beta)$$

Substituting this into the force equation, gives

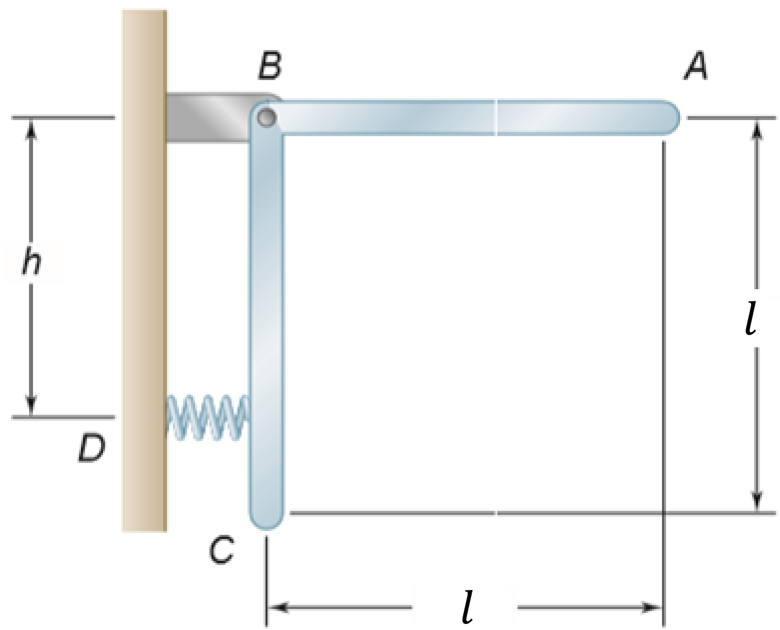
$$N = mg\left[1 + \frac{10}{7}(1 - cos\beta)\right]$$

Question 1

Two identical slender rods AB and BC, each mass m, are welded together to form an L-shaped assembly. The assembly is pressed against a spring at D by $1\ cm$, and then released from the position shown. Knowing that the maximum angle of rotation of the assembly in its subsequent motion is 90° counter-clockwise, determine:

(i) the spring constant of the spring in terms of the assembly parameters, and

(ii) the magnitude of angular velocity of the assembly as it passes through the position where rod AB forms an angle of 30° with the horizontal.



Answers:

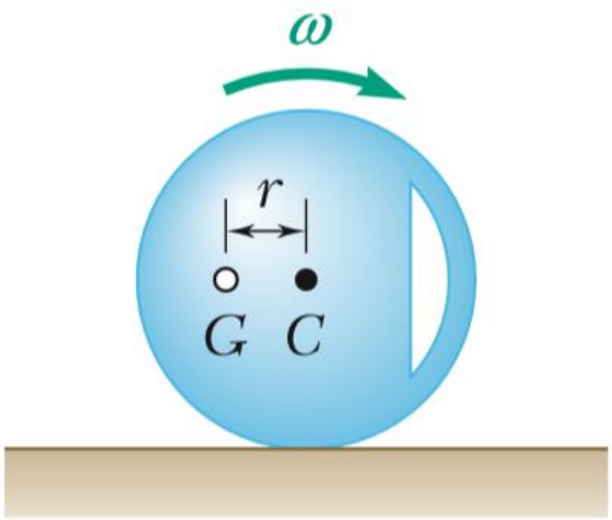
(i) $k = 20{,}000mgl$, and

(ii) $\omega = 1.431\sqrt{\frac{g}{l}}$

Question 2

The mass center G of a 3-*kg* wheel of radius R = 180 *mm* is located at a distance r = 60 *mm* from its geometric center C. The centroidal radius of gyration of the wheel is k = 120 *mm*. As the wheel rolls without sliding, its angular velocity is observed to vary. Knowing that ω = 8 *rad/s* in the position shown, determine the angular velocity of the wheel when the mass center G is directly above the geometric center C.

Answers: $\omega = 5.33$ *rad/s*.



Question 3

A point mass P (2 kg) is welded to a uniform disk at a location 0.3 m from the center C of the disk. The disk is placed on a slope, as shown in the figure, with the point mass directly above the center of the disk and released from rest. The disk rolls down the slope without sliding.

Determine the angular velocity of the disk after it has travelled a distance of 10 m along the slope. The disk has a mass of 1 kg, and a radius of 0.5 m. The acceleration due to gravity is $g = 9.81\ m/s^2$.

Answers: $\omega = 11.90\ rad/s.$