ME2115/TME2115

Mechanics of Machines

FORCED HARMONIC VIBRATION

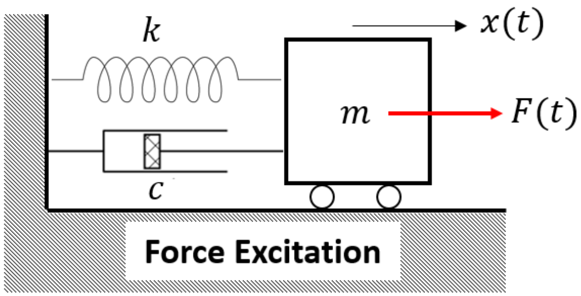
Learning Outcomes

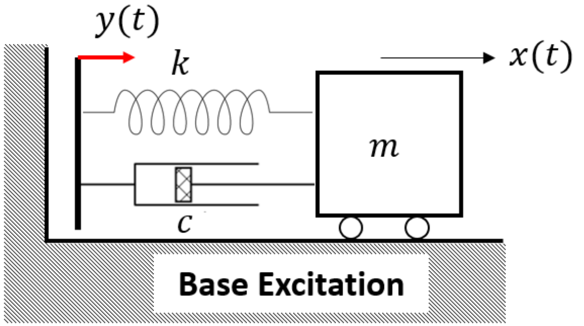
- Understand the different types of *1-DOF forced harmonic vibration systems* due to different source of excitations.
- Derive the equation of motion of 1-DOF forced vibration system, and hence solve for its *steady state frequency response*.
- Understand and apply the *concepts of transmissibility* in forced harmonic analyses.

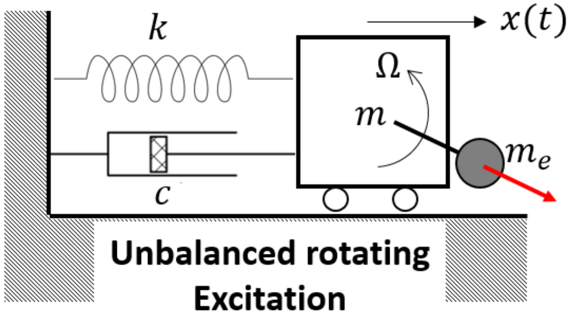
Forced Vibration

- A mechanical system is said to undergo forced vibration whenever *external energy* is supplied to the system during vibration, which can be due to:
  - *applied force*,
  - *imposed displacement excitation*, or
  - *unbalance rotating masses*.

- External excitation can be *periodic* or random in nature.
  - For periodic type of loading, we are interested in the *steady-state solution.*
  - For random excitation, the *transient solution* is important.







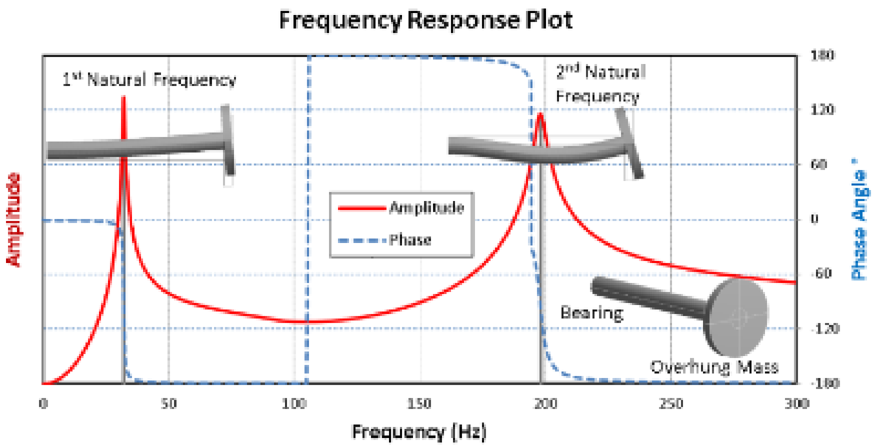
Forced Harmonic Vibration

- An important type of periodic loading is the *simple harmonic* excitation, i.e.

$$f(t) = F(\Omega)cos(\Omega t) \quad or \quad f(t) = F(\Omega)sin(\Omega t)$$

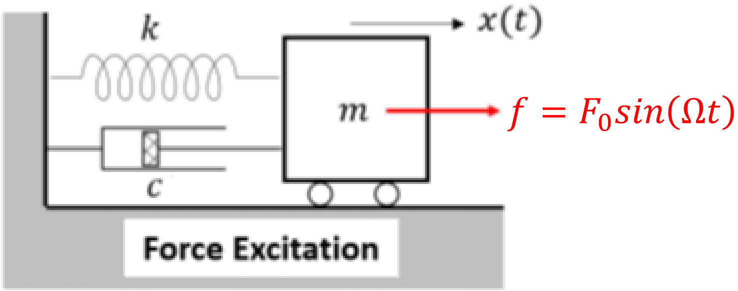
which is a sinusoidal function with *forced angular frequency Ω*.

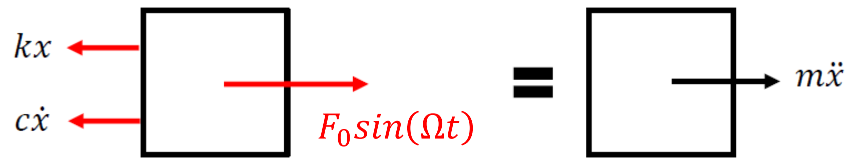
- One main objective of studying *forced harmonic vibration* is to understand the response of the system at different forcing frequency, which is called the frequency response function. This information is important for vibration control purposes.



- In a lightly damped system, its response would be very large if the excitation frequency coincides with the natural frequency of the system. This condition is called the *resonance*.

Equation of Motion – External force





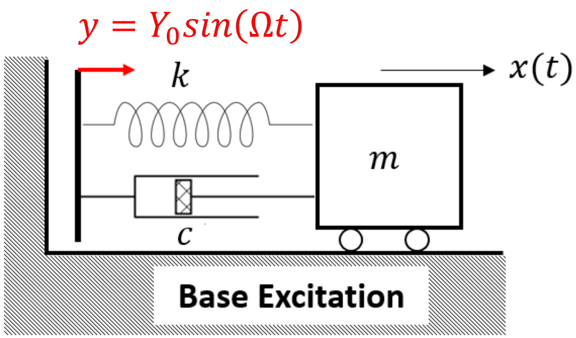
The *EOM* of the vibrating mass is

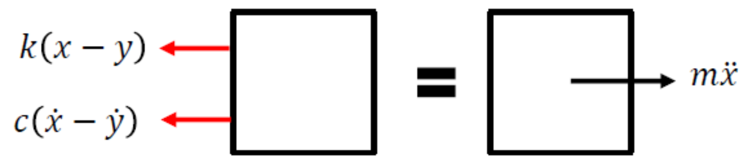
$$F_0 sin(\Omega t) - c\dot{x} - kx = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = F_0 sin(\Omega t)$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = \frac{F_0}{m} sin(\Omega t)$$

Equation of Motion – Base excitation





The *EOM* of the vibrating mass is

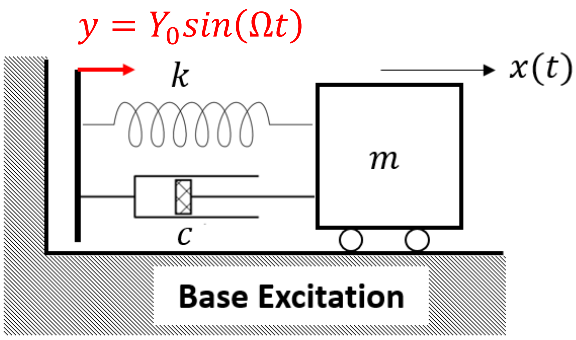
$$-c(\dot{x}-\dot{y}) - k(x-y) = m\ddot{x}$$

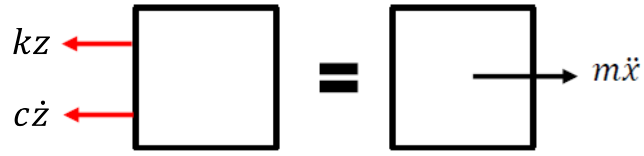
$$\Rightarrow m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = \underbrace{{\omega_n}^2 Y_0 sin(\Omega t) + 2\zeta\omega_n\Omega Y_0 cos(\Omega t)}_{\tilde{Y} sin(\Omega t + \tilde{\phi})}$$

where $\tilde{Y} = Y_0\omega_n\sqrt{(2\zeta\Omega)^2 + {\omega_n}^2}$, and $\tilde{\phi} = tan^{-1}\left(\frac{2\zeta\Omega}{\omega_n}\right)$

Equation of Motion – Base excitation…





where $z = x - y$

The *EOM* of the vibrating mass is

$$-c\dot{z} - kz = m\ddot{x}$$

$$\Rightarrow -c\dot{z} - kz = m\ddot{x} - m\ddot{y} + m\ddot{y} = m(\ddot{x} - \ddot{y}) + m\ddot{y}$$

$$\Rightarrow m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

$$\Rightarrow \ddot{z} + 2\zeta\omega_n\dot{z} + {\omega_n}^2 z = Y_o\Omega^2 sin(\Omega t)$$

Equation of Motion - Summary

- The *generic form of EOM* of the forced harmonic vibration is

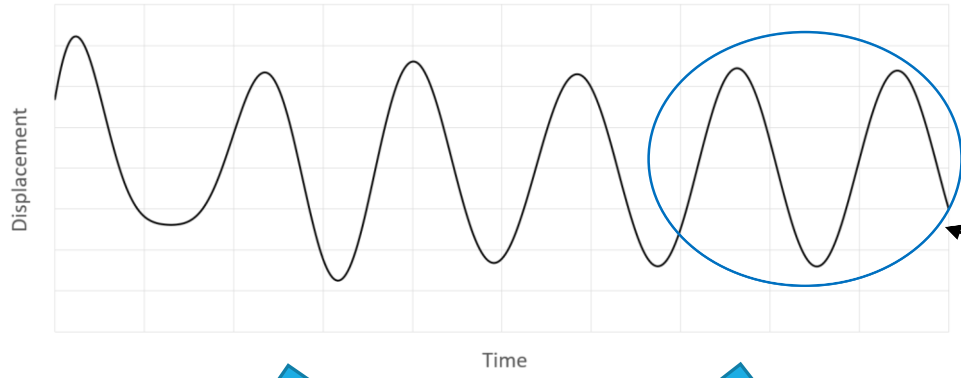
$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = f(\Omega t)$$

where $f(\Omega t)$ is a sinusoidal function with different amplitude expressions depending on the types of excitation, as summarized below,

Excitation Types	Excitation expression $f(\Omega t)$
External force	$\frac{F_o}{m} sin(\Omega t)$
Base excitation (absolute displacement)	$\tilde{Y} sin(\Omega t + \tilde{\phi})$ $where\ \tilde{Y} = Y_0\omega_n\sqrt{(2\zeta\Omega)^2 + {\omega_n}^2}$
Base excitation (relative displacement)	$Y_o\Omega^2 sin(\Omega t)$

Forced Harmonics Motion

➤ Time-history plot of a typical forced vibration motion.

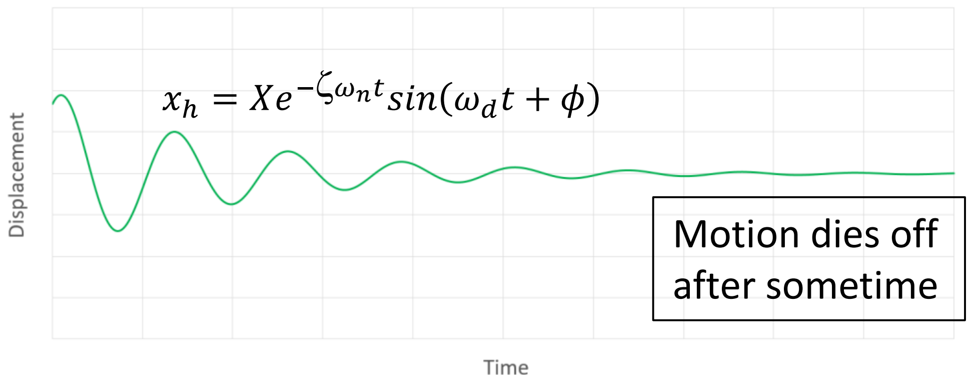


Waveform becomes identical after the damped free vibration dies off.

Homogenous solution

$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = 0$$

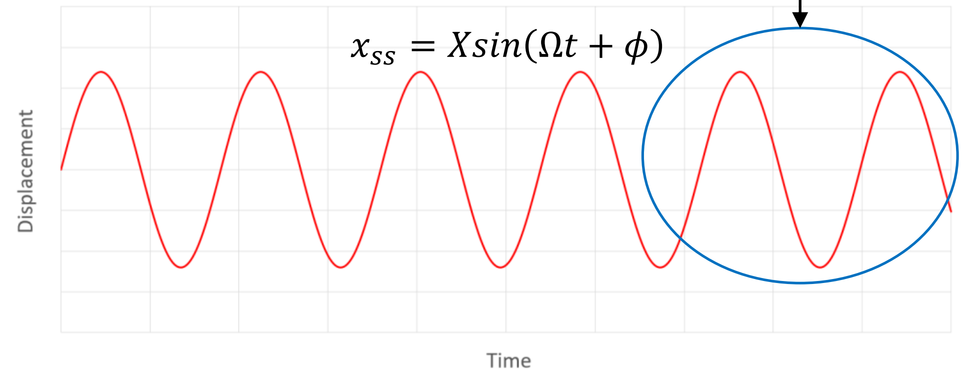
$$x_h = Xe^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$



Non-homogenous solution

$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = f(\Omega t)$$

$$x_{ss} = Xsin(\Omega t + \phi)$$

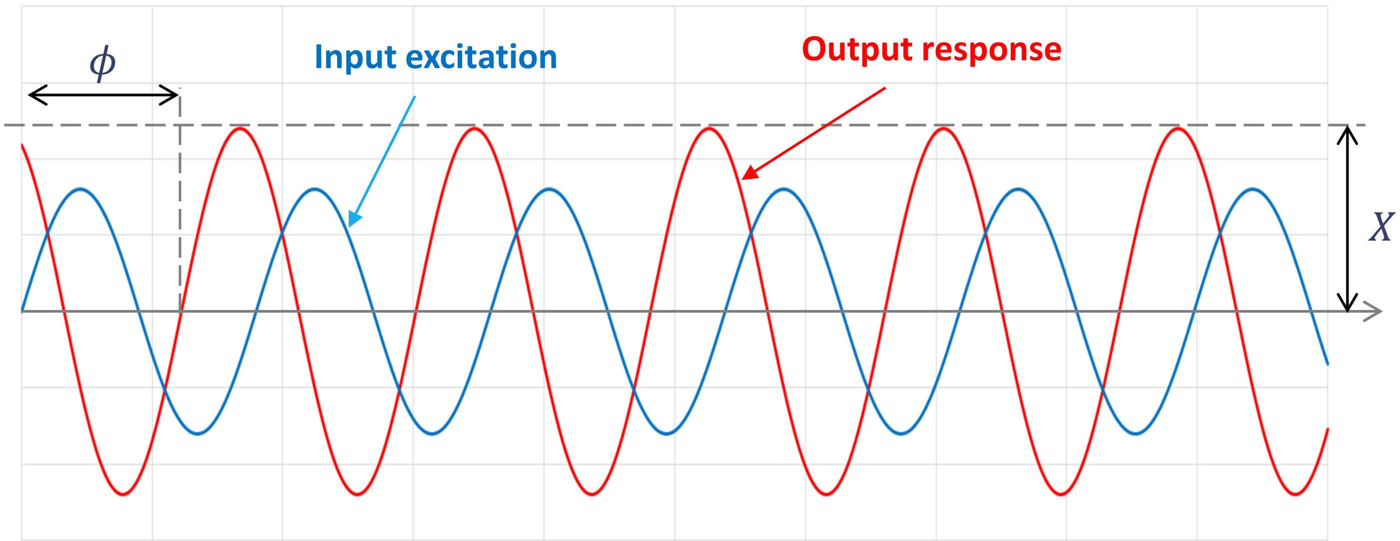


Forced Harmonics Motion…

- Hence, forced harmonic study is concerned with the *steady-state motion* of the system. The vibration occurs at the *same frequency as the forcing function*, i.e.

$$\boxed{x = X sin(\Omega t + \phi)}$$

where *X is the amplitude* of the vibration at *steady-state*, and *ϕ is the phase* difference between the motion of the mass and the excitation source (which arises due to the damping effect).



Forced Harmonic Vibration without Damping

- First, let us consider the *external force vibration without damping*, with *EOM*

$$\ddot{x} + {\omega_n}^2 x = \frac{F_o}{m} sin(\Omega t)$$

- The *steady state response*, which has to take the same sinusoidal form as the forcing term, can be expressed as

$$x = Xsin(\Omega t + \phi)$$

And the acceleration of the response is

$$\Rightarrow \ddot{x} = -\Omega^2 Xsin(\Omega t + \phi)$$

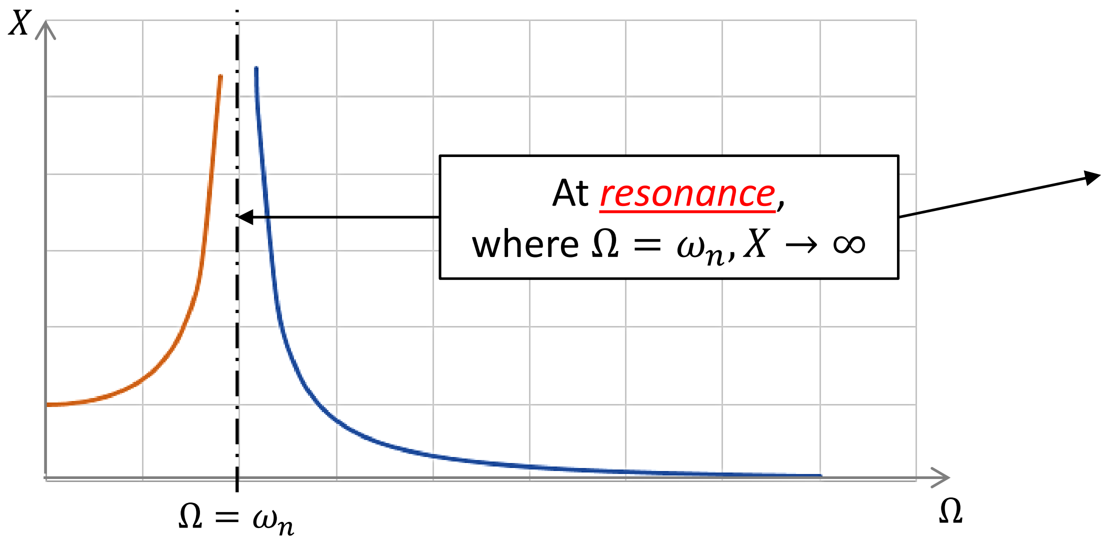
- Substituting the steady state solution into the *EOM*, gives

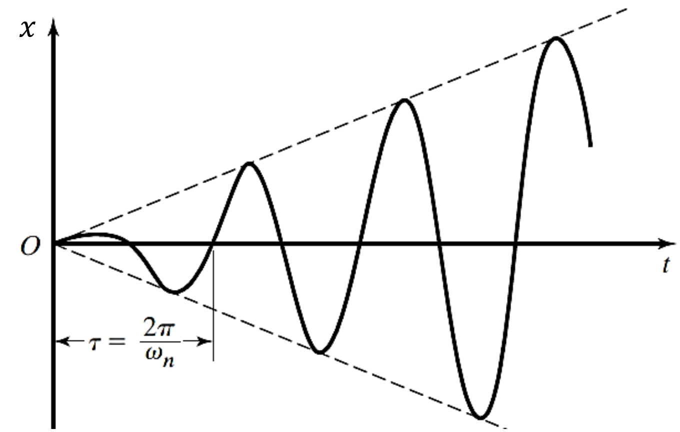
$$-\Omega^2 Xsin(\Omega t + \phi) + {\omega_n}^2 Xsin(\Omega t + \phi) = \frac{F_o}{m} sin(\Omega t)$$

$$\Rightarrow X({\omega_n}^2 - \Omega^2) sin(\Omega t + \phi) = \frac{F_o}{m} sin(\Omega t)$$

- Based on the sine function, it can be seen that the two sides of the equation can only match when the phase angle $\phi = 0$.
- And matching the amplitudes, gives

$$X({\omega_n}^2 - \Omega^2) = \frac{F_o}{m} \Rightarrow X = \frac{F_o/m}{{\omega_n}^2 - \Omega^2}$$



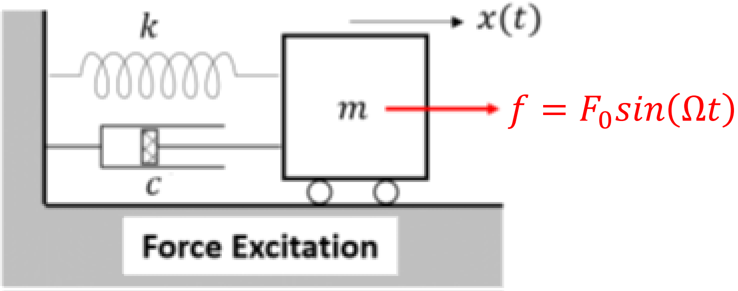


At resonance, the work done by the spring force continuously adds to the kinetic energy of mass, and hence its amplitude grows over time.

Damped Forced Harmonic Vibration

- Now, consider the *damped system under harmonic excitation*.

- The *EOM* is given by



$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = \frac{F_o}{m} sin(\Omega t)$$

And the steady-state solution is

$$x = Xsin(\Omega t + \phi)$$

$$\Rightarrow \dot{x} = \Omega Xcos(\Omega t + \phi);\ and\ \ \ddot{x} = -\Omega^2 Xsin(\Omega t + \phi)$$

- Substitute these expressions into the *EOM* gives

$$-\Omega^2 Xsin(\Omega t + \phi) + 2\zeta\omega_n \Omega Xcos(\Omega t + \phi) + {\omega_n}^2 Xsin(\Omega t + \phi) = \frac{F_o}{m} sin(\Omega t)$$

Damped Forced Harmonic Vibration…

- Now, let $\Omega t + \Omega = 0 \Rightarrow \Omega t = -\Omega$. Under this condition, the *EOM* becomes

$$2\zeta\omega_n\Omega X = -\frac{F_o}{m}sin(\Omega)$$

- Next, let $\Omega t + \Omega = \pi/2 \Rightarrow \Omega t = \pi/2 - \Omega$. With this, the *EOM* becomes

$$-\Omega^2 X + {\omega_n}^2 X = \frac{F_o}{m}sin(\pi/2 - \Omega) = \frac{F_o}{m}cos(\Omega)$$

- Summing the squares of the two equations, gives

$$(-\Omega^2 X + {\omega_n}^2 X)^2 + (2\zeta\omega_n\Omega X)^2 = \left(\frac{F_o}{m}\right)^2 \{sin^2(\Omega) + cos^2(\Omega)\}$$

$$\Rightarrow X^2\left(({\omega_n}^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2\right) = \left(\frac{F_o}{m}\right)^2$$

$$\Rightarrow X = \frac{\frac{F_o}{m}}{\sqrt{({\omega_n}^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2}}$$

Damped Forced Harmonic Vibration…

- This expression can be rewritten to

$$X = \frac{\frac{F_o}{m}}{\sqrt{({\omega_n}^2 - \Omega^2)^2 + (2\zeta\omega_n\Omega)^2}} = \frac{\frac{F_o}{m{\omega_n}^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

where $r = \frac{\Omega}{\omega_n}$ is referred to as the *frequency ratio*.

- And the numerator is essentially the static displacement due to the applied force F_o, as shown

$$\frac{F_o}{m{\omega_n}^2} = \frac{F_o}{m(k/m)} = \frac{F_o}{k} = \delta_o$$

- Hence, the displacement response becomes

$$X = \frac{\delta_o}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \Rightarrow \boxed{\frac{X}{\delta_o} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}}$$

where $\frac{X}{\delta_o}$ is the *amplification factor*, also known as the *displacement transmissibility*.

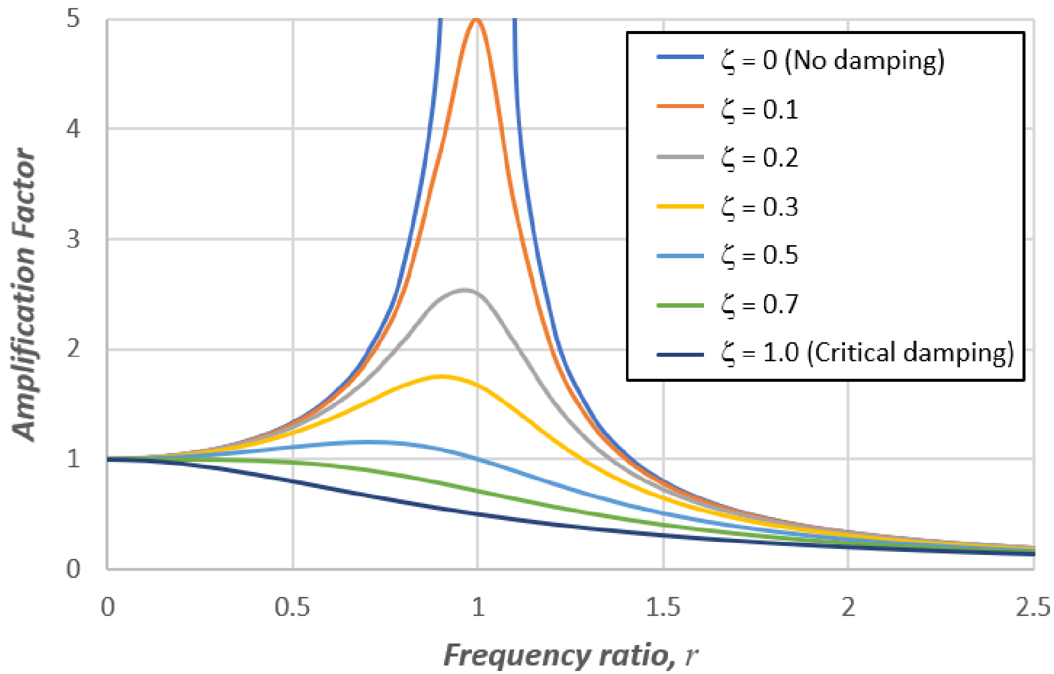
Damped Forced Harmonic Vibration…

- Damping reduces *A.F.* significantly, especially near resonance at $r \approx 1$.
- For $\zeta > 1/\sqrt{2}$, the *A.F.* decreases monotonically with increasing r.
- For $\zeta < 1/\sqrt{2}$, maximum *A.F.* occurs at

$$r^* = \sqrt{1 - 2\zeta^2}^{\dagger}$$

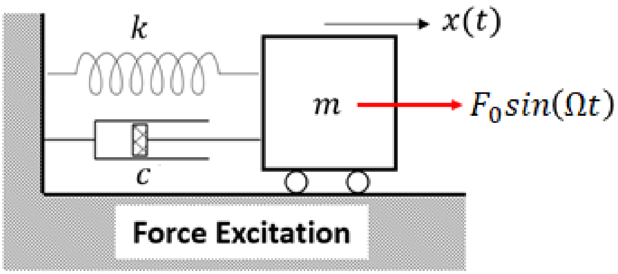
and the corresponding amplitude is

$$A.F. = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$



† *Taking the derivative of A.F. w.r.t. r, i.e.* $\frac{d(A.F.)}{dr} = 0$, *and then solving the equation gives* $r = \sqrt{1-2\zeta^2}$.

Damped Forced Harmonic Vibration…



- The spring and damper induce a resultant force on the wall, which is given by

$$F_{wall} = kx + c\dot{x}$$

- This is a *dynamic force*, and it is essential to evaluate this quantity so that one can ensure that the mounting wall is designed to withstand such load.

- Now, using $x = Xsin(\Omega t + \phi)$, the constraining force at the wall is

$$F_{wall} = kXsin(\Omega t + \phi) + c\Omega Xcos(\Omega t + \phi)$$

$$= \sqrt{k^2 + (c\Omega)^2}\, Xsin\big((\Omega t + \phi) + \varphi\big)$$

where $\varphi = tan^{-1}\left(\frac{c\Omega}{k}\right)$ is the phase different between the displacement and the force responses.

Damped Forced Harmonic Vibration…

- The magnitude of this force is

$$|F_{wall}| = \sqrt{k^2 + (c\Omega)^2}\, X = \frac{(F_o/k)\sqrt{k^2 + (c\Omega)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\Rightarrow \frac{|F_{wall}|}{F_o} = \sqrt{\frac{1 + \left(\frac{c}{k}\Omega\right)^2}{(1-r^2)^2 + (2\zeta r)^2}}$$

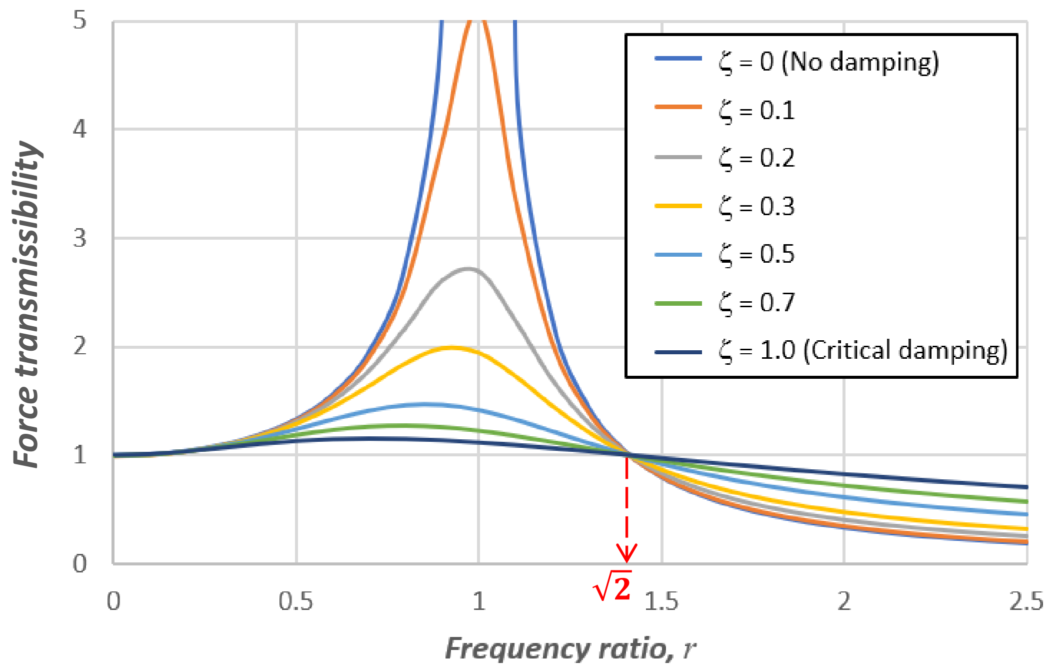
$$\left(\frac{c}{m}\right)\left(\frac{m}{k}\right)\Omega = (2\zeta\omega_n)\left(\frac{1}{{\omega_n}^2}\right)\Omega = 2\zeta r$$

$$= \boxed{\sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}}$$

- This ratio is called the *force transmissibility factor*, which measures the amplification of the dynamic force w.r.t. static applied force.

Damped Forced Harmonic Vibration…

- For $r < \sqrt{2}$, smaller damping ratio lead to larger values of transmissibility.
- But for $r > \sqrt{2}$, smaller damping ratio lead to smaller values of transmissibility. Nevertheless, dynamic force is smaller than the static force for all damping ratios.



- Again differentiating the transmissibility relation w.r.t. to r, and then equating it to zero, we can derive the frequency ratio r that give maximum transmissibility for different damping ratio,

$$r = \frac{1}{2\zeta}\left[\sqrt{1 + 8\zeta^2} - 1\right]^{1/2}$$

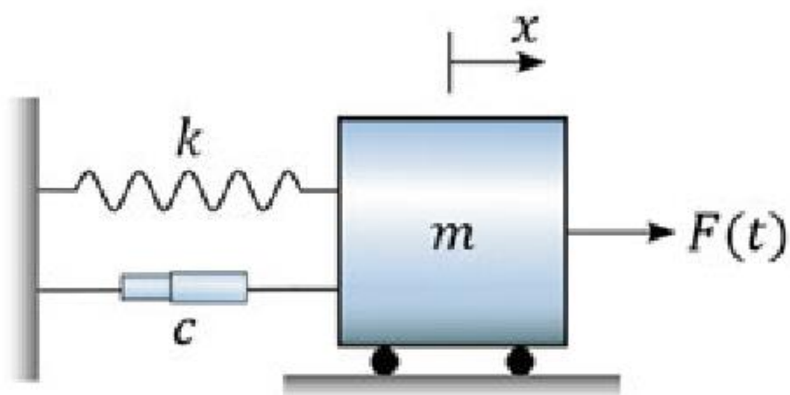
Example 1

A damped single-degree of freedom mass–spring system has mass $m = 10\ kg$, spring coefficient $k = 4000\ N/m$, and damping coefficient $c = 40\ N.s/m$. The amplitude of the harmonic forcing function $F_o = 60\ N$ and the forcing frequency $\Omega = 40\ rad/s$. Determine:

(i) the displacement amplification factor,

(ii) the force transmissibility to the wall, and

(iii) the amplitude of the force transmitted to the wall, and the individual supports for the spring and damper.



Example 1…

For this given system, the system parameters are:

Natural angular frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\ rad/s$$

Critical damping coefficient,

$$c_{cr} = 2m\omega_n = 2(10)(20) = 400\ Ns/m$$

Damping ratio,

$$\zeta = \frac{c}{c_{cr}} = \frac{40}{400} = 0.1$$

Frequency ratio,

$$r = \frac{\Omega}{\omega_n} = \frac{40}{20} = 2$$

Example 1…

(i) For the displacement amplification factor, it is

$$\frac{X}{\delta_o} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{1}{\sqrt{(1-(2)^2)^2 + (2(0.1)(2))^2}} = \mathbf{0.3304}$$

and the corresponding steady-state displacement is

$$X = 0.3304\left(\frac{F_o}{k}\right) = 0.3304\left(\frac{60}{4000}\right) = 4.956 \times 10^{-3}\ m$$

(ii) Using the force transmissibility factor, we get

$$\frac{F_{wall}}{F_0} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

$$\Rightarrow F_{wall} = (60)\sqrt{\frac{1+\big(2(0.1)(2)\big)^2}{(1-(2)^2)^2+\big(2(0.1)(2)\big)^2}} = \mathbf{21.35}\ N$$

Example 1…

(iii) Given that the steady-state displacement is given by, $x = X\sin(\Omega t + \phi)$, its velocity is given by

$$\dot{x} = X\Omega\cos(\Omega t + \phi) = X\Omega\sin\left(\Omega t + \phi + \frac{\pi}{2}\right)$$

For the spring force, it is

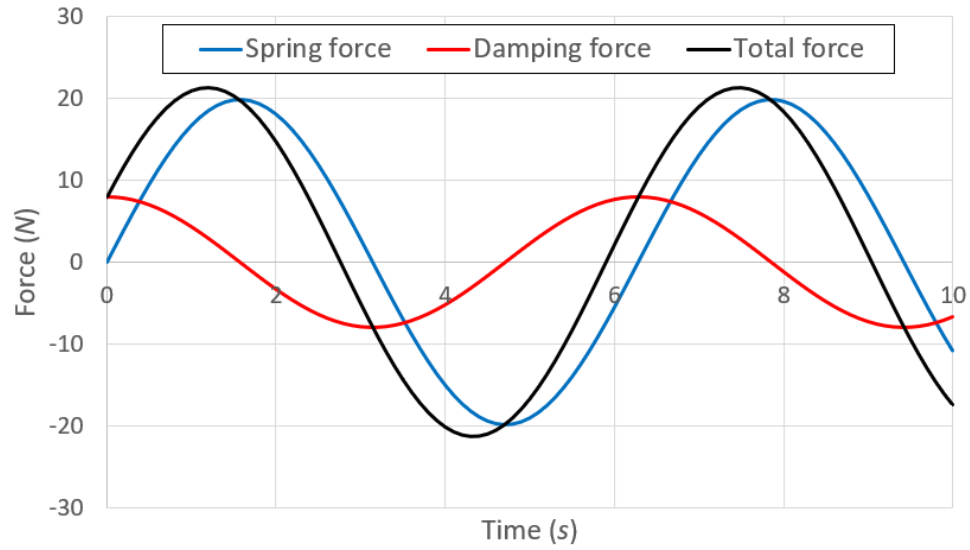
$$F_{spr} = kX$$

$$= (4000)(4.956 \times 10^{-3}) = \mathbf{19.82}\ N$$

As for the damping force, it is given by

$$F_{damp} = cX\Omega$$

$$= (40)(4.956 \times 10^{-3})(40) = \mathbf{7.93}\ N$$

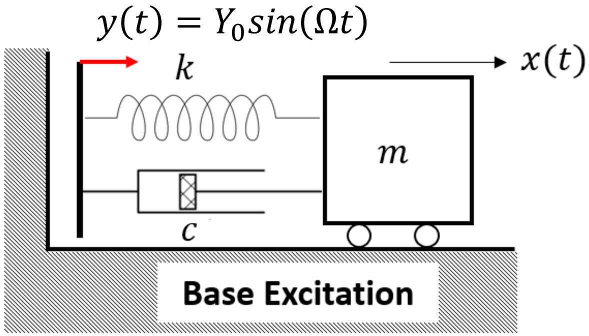


Force on the wall is

$$F_{wall} = \sqrt{{F_{spr}}^2 + {F_{damp}}^2} = 21.35\ N$$

Damped System under Base Excitation

- Consider a spring-mass-damper system that is subjected to base excitation as shown below.



- The *EOM* is given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \tilde{Y} sin(\Omega t + \tilde{\phi})$$

where

$$\tilde{Y} = Y_0\omega_n\sqrt{(2\zeta\Omega)^2 + \omega_n{}^2}, \qquad \text{and} \qquad \tilde{\phi} = tan^{-1}\left(\frac{2\zeta\Omega}{\omega_n}\right)$$

Damped System under Base Excitation…

- By following the procedures in slide 14, we can derived the steady-state response for this case, which is simply by replacing $\frac{F_o}{m}$ with $\tilde{Y}$ in slide 15, that is

$$X = \frac{\left(\tilde{Y}/{\omega_n}^2\right)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

- Now, substitute $\tilde{Y} = Y_0 \omega_n \sqrt{(2\zeta\Omega)^2 + {\omega_n}^2}$ into the above expression, gives

$$X = \frac{(Y_0/\omega_n)\sqrt{(2\zeta\Omega)^2 + {\omega_n}^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Rightarrow \boxed{\frac{X}{Y_0} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}}$$

- This is the *displacement transmissibility* factor for this base excitation case. It measures of the displacement sensitivity of a given system to base vibration.

- Notice that this expression is *identical* to the force transmissibility for the force excitation case. Hence, the interpretation of the plot is similar (refer to slide 19).

Damped System under Base Excitation…

- Another way of studying the displacement transmissibility is to look at the *relative motion* of the mass *x* to the base excitation *y*, i.e. $z = x - y$.

- In this case, the *EOM* used is

$$\ddot{z} + 2\zeta\omega_n\dot{z} + {\omega_n}^2 z = Y_o\Omega^2 sin(\Omega t)$$

- Again by substituting $\frac{F_o}{m}$ with $Y_o\Omega^2$, we have

$$Z = \frac{(Y_o\Omega^2/{\omega_n}^2)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\Rightarrow \boxed{\frac{Z}{Y_o} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$

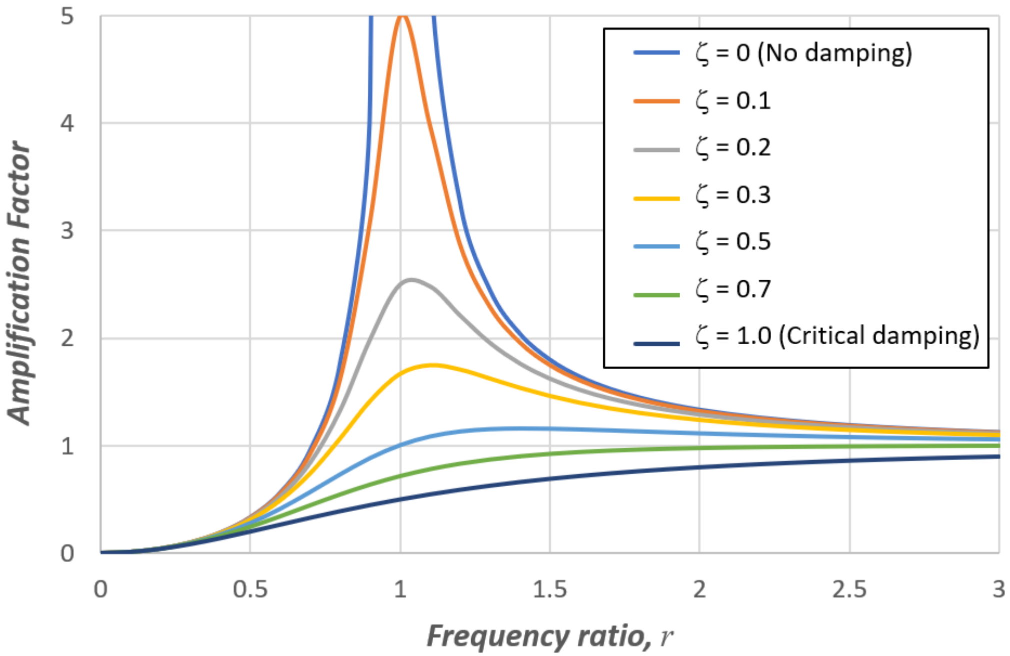
- This is an alternate form of *displacement transmissibility* for the based-excitation in terms of its relative motion.

Damped System under Base Excitation…

- Like the former cases, damping has significant effect on the amplitudes near resonance. Hence, it should be added deliberately to limit the relative motion.

- At high value of r, this ratio approach unity, and the effect of damping is negligible in this range.

- For $\zeta > \frac{1}{\sqrt{2}}$, the transmissibility grows monotonously from 0 to 1.



- For $\zeta < \frac{1}{\sqrt{2}}$, the maximum transmissibility occurs at $r = \frac{1}{\sqrt{1-2\zeta^2}}$, and the corresponding amplitude is $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$.

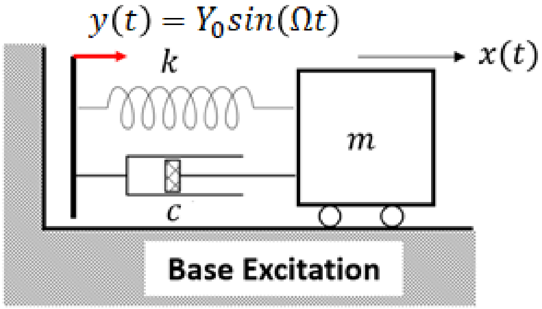
Damped System under Base Excitation…

- Next, consider the force that is transmitted to the base. This is again due to the reactions from the spring and dashpot attached to it, i.e.

$$F_{base} = k(x - y) + c(\dot{x} - \dot{y})$$

And from the *FBD* of the mass,

$$k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x} \Rightarrow F_{base} = -m\ddot{x}$$



- For steady-state response, $x = Xsin(\Omega t + \phi)$, then

$$\ddot{x} = -\Omega^2 Xsin(\Omega t + \phi)$$

- Hence, the magnitude of the force transmitted to the base is

$$F_{base} = m\Omega^2 X$$

Damped System under Base Excitation…

- Again, we normalize the amplitude of F_{base} w.r.t. a respresentative static force. In this case, we use kY_oas the reference force, which gives

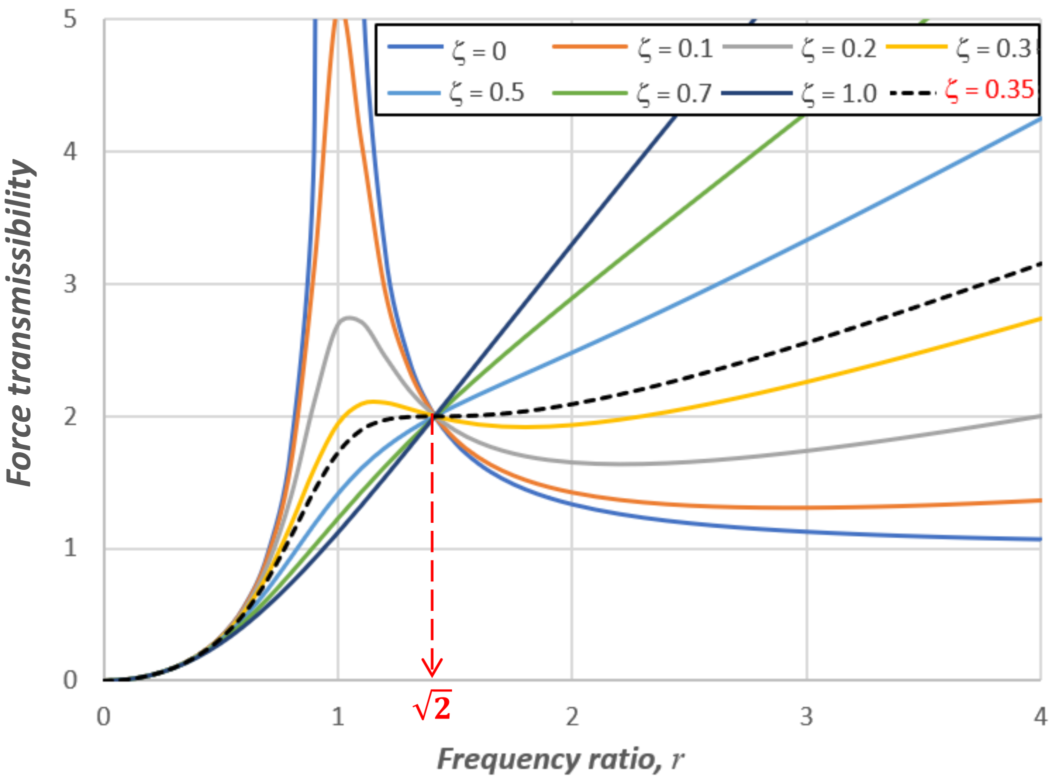
$$\frac{F_{Base}}{kY_o} = \frac{m\Omega^2 X}{kY_o} = \left(\frac{m}{k}\Omega^2\right)\left(\frac{X}{Y_o}\right) = r^2\left(\frac{X}{Y_o}\right)$$

displacement transmissibility

$$= \boxed{r^2\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}}$$

- For the based excitation case, this is the ***force transmissibility*** factor. It quantifies the amplification in the dynamic force applied on the base due to the vibrating mass w.r.t. the static force when the mass is displaced by Y_o.

Damped System under Base Excitation…

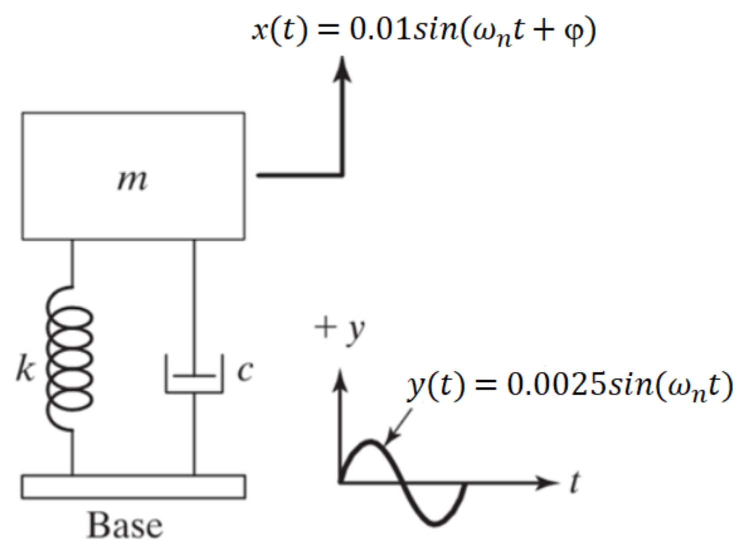


- Similarly, damping has significant effect on the force amplitude near the resonance. Hence, damping should be introduced purposefully to avoid dangerous level of force transmission.
- At $r = \sqrt{2}$, the force transmissibility is 2 for all damping ratio.

- Unlike the displacement transmissibility factors, which either converge to zero or unity as r increases, the force transmissibility increases with increasing r.
- This is especially significant when damping ratio $\zeta > 0.35$, which increases monotonously.

Example 2

A heavy machine, weighing $3000\ N$, is supported on a resilient foundation. The static deflection of the foundation due to the weight of the machine is found to be $7.5\ cm$. It is observed that the machine vibrates with an amplitude of $1.0\ cm$ when the base of the foundation is subjected to harmonic oscillation at the natural frequency of the system with an amplitude of $0.25\ cm$. Find



(i) the damping constant of the foundation,

(ii) the dynamic force amplitude on the base, and

(iii) the amplitude of the displacement of the machine relative to the base.

(iv) Suppose the absolute vibration of the machine should be less than $0.1\ cm$ when the base excitation frequency is doubled. Can the foundation used currently achieve this specification? If not, what should the damping coefficient be changed to, while keeping the stiffness the same?

Example 2...

(i) First, the stiffness of the foundation can be found from its static deflection

$$k = \frac{W}{\delta_{st}} = \frac{3000}{0.075} = 40{,}000\ N/m$$

When operating at its natural frequency, i.e. $r = 1$, the displacement transmissibility gives

$$\frac{X}{Y} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} \Rightarrow \frac{0.01}{0.0025} = \sqrt{\frac{1+(2\zeta(1))^2}{(1-(1)^2)^2+(2\zeta(1))^2}} \Rightarrow \zeta = 0.1291$$

Hence, the damping constant is then given by

$$c = \zeta c_{cr} = \zeta\left(2\sqrt{km}\right) = (0.1291)(2)\sqrt{40{,}000 \times \frac{3000}{9.81}} = \mathbf{903}\ Ns/m$$

Example 2…

(ii) The dynamic force amplitude on the base at $r = 1$ is

$$\frac{F}{kY} = r^2 \frac{X}{Y} = \frac{X}{Y}$$

$$\Rightarrow F = kY\left(\frac{X}{Y}\right) = kX = 40000 \times 0.01 = \mathbf{400}\ N$$

(iii) The amplitude of the relative displacement at $r = 1$ is

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{(1)^2}{\sqrt{(1 - 1^2)^2 + (2\zeta(1))^2}}$$

$$\Rightarrow Z = \frac{Y}{2\zeta} = \frac{0.0025}{2(0.1291)} = \mathbf{0.00968}\, m$$

Note that $z(t) = x(t) - y(t)$, but $Z \neq X - Y$. This is due to the phase angle difference between the input y and output x functions.

Example 2...

(iv) Under this operating condition, where r = 2, and $\zeta = 0.1291$, the displacement transmissibility relation gives

$$\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$= \sqrt{\frac{1 + (2\,(0.1291)(2))^2}{(1 - (0.1291)^2)^2 + (2\,(0.1291)(2))^2}} = \mathbf{0.370}$$

And the required displacement ratio is given by

$$\frac{X}{Y} = \frac{0.1}{0.25} = 0.4$$

Comparing the two values, it can be concluded that this foundation meets the requirement.

Example 2...

Alternatively, you can also solve the following inequality equation to determine the damping ratio required to meet the specification, i.e.

$$\sqrt{\frac{1+(2\zeta(2))^2}{(1-(2)^2)^2+(2\zeta(2))^2}} < 0.4$$

$$\Rightarrow \zeta < \mathbf{0.1809}$$

which also give the same conclusion since $\zeta = 0.1291.$

Note: Although the first approach is easy to use to check for the specification, it does not provide the required damping constant if the specification is not met. You will still have to solve the inequality equation to determine this information.

Example 3

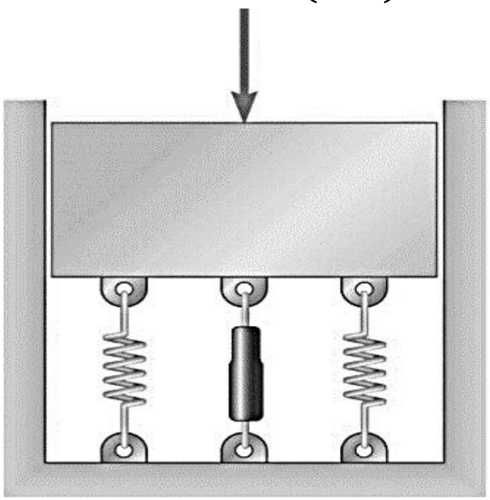
A $10\ kg$ machine is supported by two springs, each of constant $2000\ N/m$, and a damper with coefficient $40\ N.s/m$. It is subjected to a harmonic force of frequency $10\ rad/s$ and amplitude of $200\ N$.

(i) Determine the amplitude of the steady-state response of the mass.

(ii) Suppose the damping coefficient increases to $50\ N.s/m$, what is the steady-state amplitude?

(iii) And if the spring constant is also increased to $2500\ N/m$, what is the steady-state amplitude now?

$F = 200 sin(10t)$



Example 3…

(i) For example, the system parameters are:

$$k_{eq} = 2(2000) = 4000\ N/m \Rightarrow \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4000}{10}} = 20\ rad/s$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{40}{2(10)(20)} = 0.1$$

$$r = \frac{\Omega}{\omega_n} = \frac{10}{20} = 0.5$$

With these, amplitude the steady-state response is

$$X = \frac{\left(\frac{F_0}{k_{eq}}\right)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{\left(\frac{200}{4000}\right)}{\sqrt{(1-(0.5)^2)^2 + (2(0.1)(0.5))^2}} = \mathbf{0.0661}\ m$$

Example 3...

(ii) When the damping in increased to 50 $N.s/m$, the only the damping ratio is affected. which is now

$$\zeta = \frac{50}{400} = 0.125$$

With this new ratio, the amplitude becomes

$$X = \frac{\left(\frac{200}{4000}\right)}{\sqrt{(1-(0.5)^2)^2 + (2(0.125)(0.5))^2}} = \mathbf{0.0658}\ m$$

Example 3...

(iii) Now when the stiffness of the spring changes, all system parameters will be affected. Hence, you have to perform the entire calculations as in (i) again.

$$k_{eq} = 2(2500) = 5000\,\frac{N}{m} \Rightarrow \ \omega_n = \sqrt{\frac{5000}{10}} = 22.36\,rad/s$$

$$\zeta = \frac{50}{2(10)(22.36)} = 0.1118$$

$$r = \frac{10}{22.36} = 0.4472$$

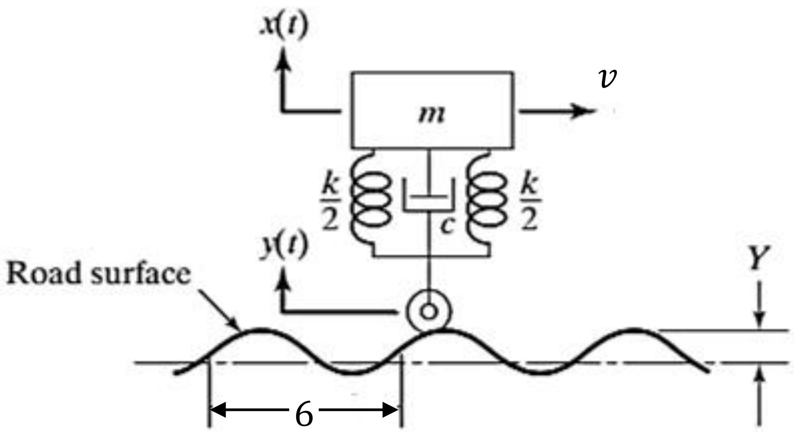
$$X = \frac{\left(\frac{200}{5000}\right)}{\sqrt{(1-(0.4472)^2)^2 + (2(0.1118)(0.4472))^2}} = \mathbf{0.0496}\,m$$

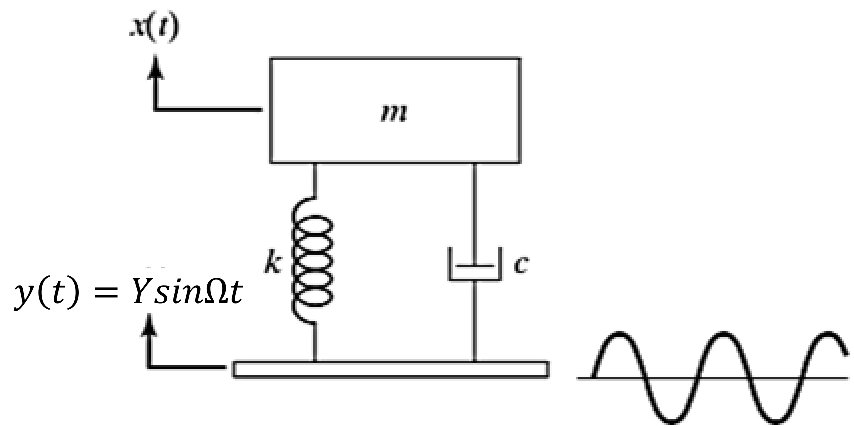
Example 4

A simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road, which varies sinusoidally with an amplitude of $Y = 0.05\ m$ and a wavelength of $6\ m$. The vehicle has a mass of $1200\ kg$. The suspension system has a spring constant of $400\ kN/m$, and a damping ratio of $\zeta = 0.5$.

(i) If the vehicle speed is $20\ km/h$, determine the displacement amplitude of the vehicle.

(ii) In order for the vehicle to oscillate at less than $4\ cm$, what is the minimum speed it should be travelling?





Example 4...

(i) The frequency of the base excitation is due to the modulation of the road roughness. It is given by

$$v = f\lambda$$

$$\Rightarrow f = \frac{v}{\lambda} = \left(\frac{20000}{3600}\right)\frac{1}{6} = 0.9259\ Hz$$

The natural frequency of the vehicle is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{400 \times 10^3}{1200}} = 2.906\ Hz$$

And hence the frequency ratio is

$$r = \frac{f}{f_n} = 0.3187$$

Example 4…

Using displacement transmissibility relation for base excitation case, gives

$$\frac{X}{Y_0} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$\Rightarrow X = (0.05)\sqrt{\frac{1 + \big(2(0.5)(0.3187)\big)^2}{(1 - (0.3187)^2)^2 + \big(2(0.5)(0.3187)\big)^2}} = \mathbf{0.05505}\, m$$

(ii) In this case, the displacement transmissibility factor has to satisfy

$$\frac{X}{Y_0} < \frac{0.04}{0.05} \Rightarrow \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} < 0.8$$

Example 4...

Solving the inequality equation, we get

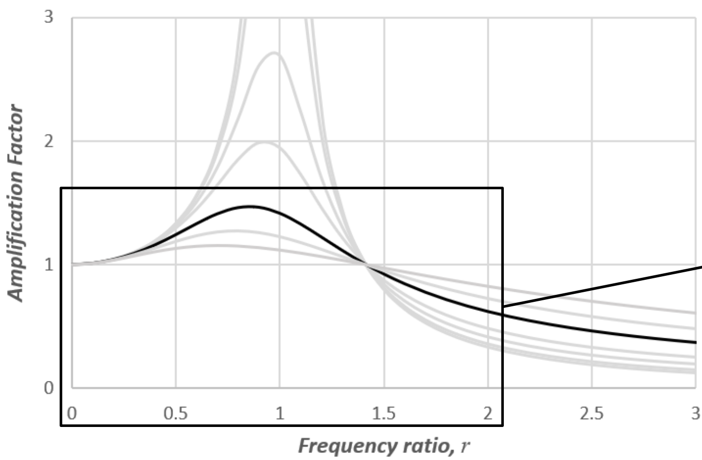
$$\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} < 0.8$$

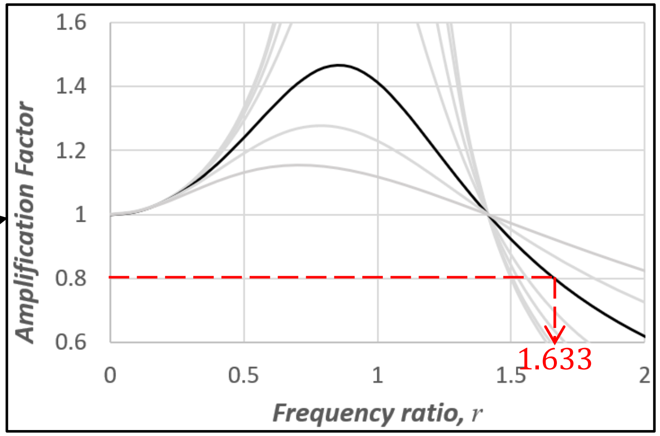
$$\Rightarrow 1+(2(0.5)r)^2 < (0.80)^2[(1-r^2)^2+(2(0.5)r)^2]$$

$$\Rightarrow r^4 - 2.5625r^2 - 0.5625 > 0$$

$$\Rightarrow r^2 = 2.766, \quad \text{or} \quad r^2 = -0.2034 \text{ (inadmissible)}$$

$$\Rightarrow r = 1.633$$





Example 4…

Based on this ratio, the base excitation frequency would be

$$\frac{f}{f_n} = 1.663$$

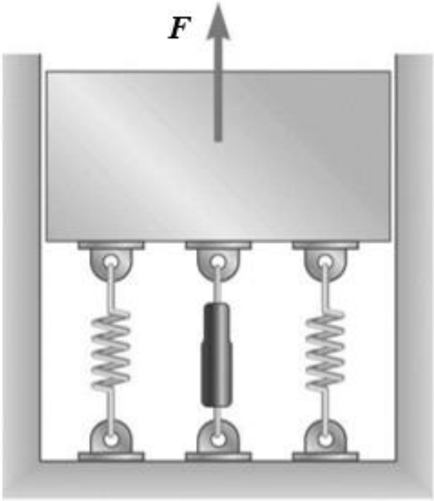
$$\Rightarrow f = 1.663(2.906) = 4.833\ Hz$$

This translate to the vehicle speed of

$$v = f\lambda = (4.833)(6) = \mathbf{29.0}\ m/s$$

Question 1

A machine of mass 100 *kg* is supported by two identical springs and a damper, as shown in the figure. It was noted that the springs were compressed by 1.0 *cm* when the mass was loaded onto them under static equilibrium condition. The damping constant of the damper is given by 1000 *Ns/m*. Use $g = 9.81\ m/s^2$ in your calculations.



(a) Based on the information given, determine the damped oscillation frequency, f_d of the system.

When the machine is in operation, the imbalance in the machine induced a vertical sinusoidal force F of amplitude 100 *N* at 4 *Hz*. Determine:

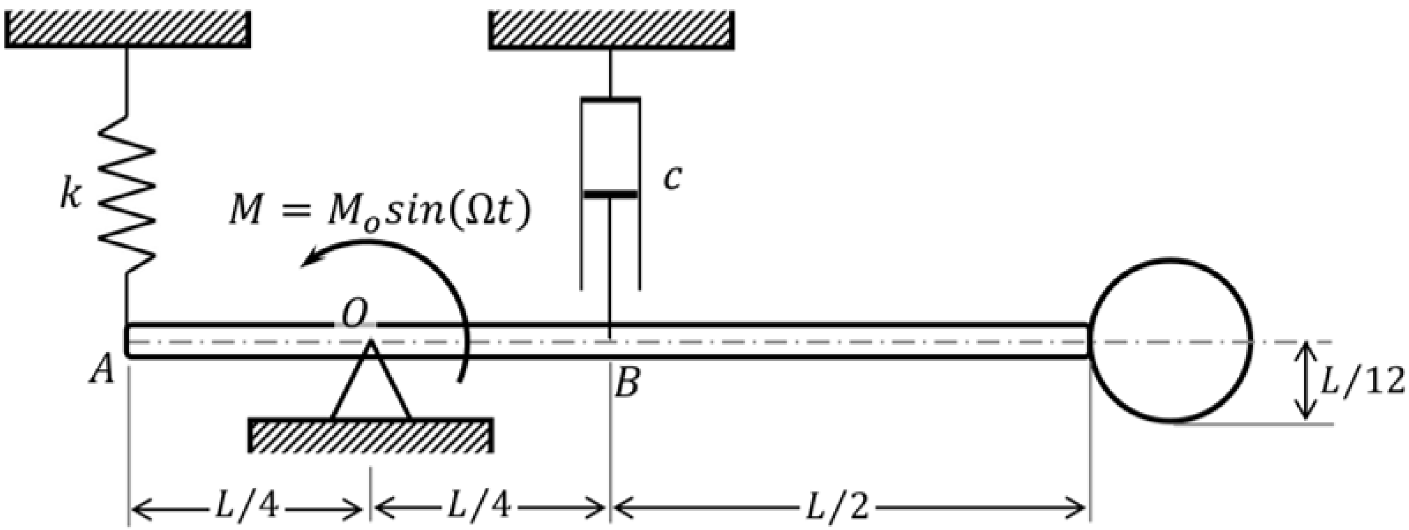
(b) The amplitude of the steady-state vibration of the machine.

(c) The maximum reaction forces induced on the base hinges that are attached to the damper, F_d.

Answers: (a) $f_d = 4.92\ Hz$; (b) $X = 2.32\ mm$, and (c) $F_d = 58.4\ N$

Question 2

A simplified hammering mechanism is shown in the figure, which comprises a hammer assembly that is pivoted at point *O*, with a linear spring at *A* and a viscous damper at *B*. The hammer assembly is made up of a slender rigid rod of length *L* and mass m, and a circular disk of radius $\frac{L}{12}$ and mass m that is welded to the rod at one end. Lastly, this system is driven by a sinusoidal varying moment $M = M_o sin(\Omega t)$ at *O*.



Question 2...

(a) Using the parallel axis theorem, show that the mass moment of inertia of the hammer assembly about the pivot point O is given by $I_O = 0.84375\, mL^2$.

(b) Sketch the free body diagram of the system with all the forces included, and derive its equation of motion for small angular displacement θ about the pivot point O.

(c) Suppose it is required that the damping ratio ζ for this system to be equal to 0.2. Determine the damping coefficient c (expressed in terms of m, L and k) that is needed for this condition to be satisfied.

(d) For this damping ratio $\zeta = 0.2$, determine the operating frequency Ω (again expressed in terms of m, L and k) that gives rise to maximum amplification factor, and the corresponding amplification factor.

Answers:

(b) $13.5m\ddot{\theta} + c\dot{\theta} + k\theta = \frac{16}{L^2} M_o sin(\Omega t)$

(c) $c = 1.470\sqrt{mk}$

(d) $\Omega = 0.2611\sqrt{k/m}\,; A.F = 2.55$