ME2115/TME2115

Mechanics of Machines

FREE VIBRATION WITHOUT DAMPING

Learning Outcomes

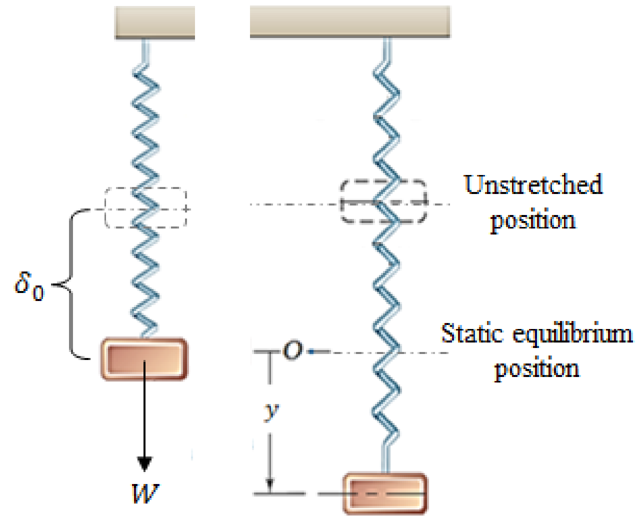
- Able to recognize the basic characteristic of a *free vibration without damping*, namely its *simple harmonics motion*.
- Able to relate the *system parameters*, such as mass and stiffness, to the system vibration characteristic, that is its *natural frequency*.
- Able to derive the *equation of motion* using *Newton's 2nd law of motion* and *principle of conservation of energy*, and then solve for a *single-DOF* free vibration problem.

Vibration of Spring-Mass System

- Consider a mass m attached to a massless spring of constant k. At static equilibrium, we have

$$W = k\delta_0$$

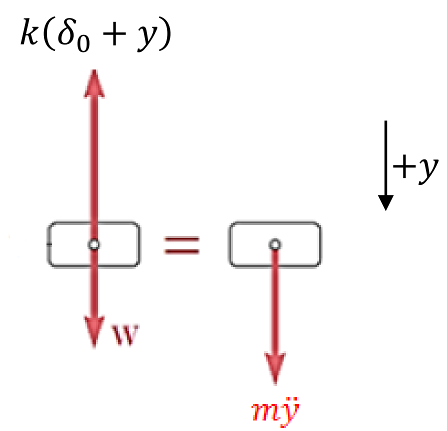
where δ_0 is the elongation of the spring.



- Suppose the mass is set into oscillation with y (+ve downwards) displacement from the equilibrium point. Applying the Newton's 2nd law of motion to the mass, gives

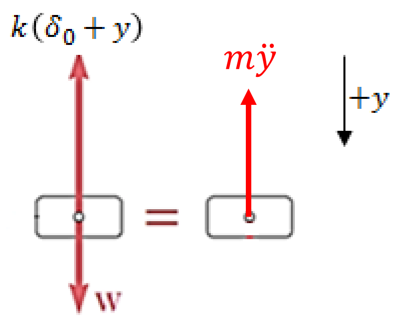
$$W - k(\delta_0 + y) = m\ddot{y} \Rightarrow (W - k\delta_0) - ky = m\ddot{y}$$

$$\underbrace{\Rightarrow m\ddot{y} + ky = 0, \qquad or \qquad \ddot{y} + \left(\frac{k}{m}\right)y = 0}_{\textbf{Equation of motion (EOM)}}$$



Vibration of Spring-Mass System…

- Now, suppose if we assume the mass to be accelerating in the opposite direction. In this case, the *EOM* becomes



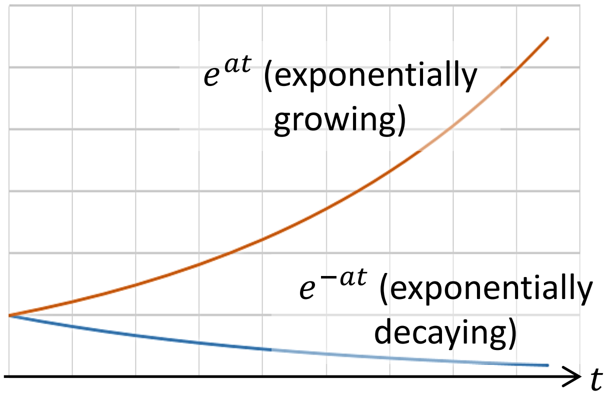
$$W - k(\delta_0 + y) = -m\ddot{y}$$

$$\Rightarrow m\ddot{y} - ky = 0, \qquad or \qquad \ddot{y} - \left(\frac{k}{m}\right)y = 0$$

The solution to this equation is

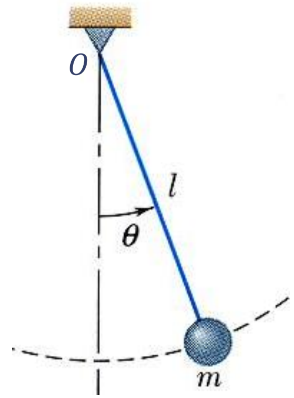
$$y = C_1 e^{-\sqrt{\frac{k}{m}}t} + C_2 e^{\sqrt{\frac{k}{m}}t}$$

which is an exponential function. This is obviously inconsistent with the expected behavior of a vibration system, which an *oscillation motion*.



For a free vibrating system, the *displacement* and *acceleration* have to be defined in the *same direction* in the free body diagram.

Simple Pendulum with Small Amplitude



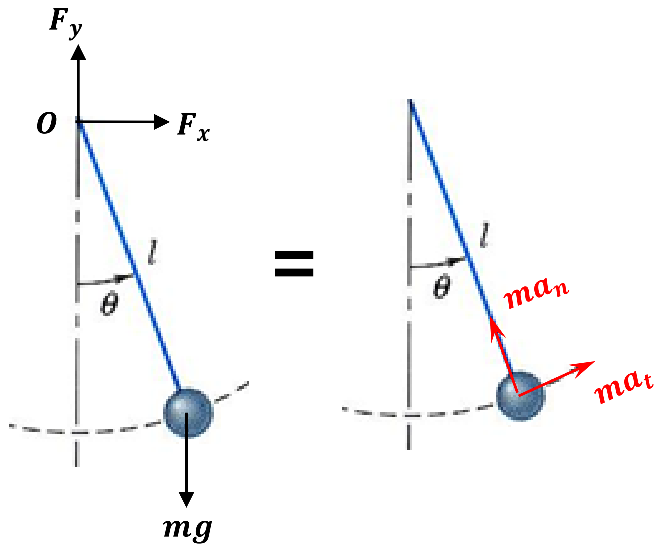
- Consider a simple pendulum with a point mass m attached to the end of a long cable of length l.

- The *EOM* of the mass can be derived by taking moment about *O,*

$$-mgl\sin\theta = ma_t l$$

$$\Rightarrow -mgl\sin\theta = m\left(l\ddot{\theta}\right)l = (ml^2)\ddot{\theta}$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad or \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$



- And for <u>*small amplitude of oscillation*</u>, i.e. <u>$\sin\theta \approx \theta$</u>, it becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

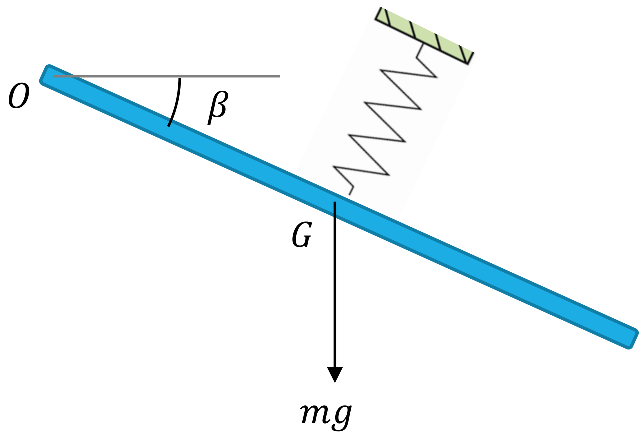
Static Equilibrium Analysis

- Notice that for the spring mass system, the static equilibrium analysis (i.e. $W = k\delta_0$) is done and used to arrive at the *EOM*. But why is it not needed for the pendulum case?
- Static equilibrium analysis is needed when there are more than one potential energy source in the system.
  - For the spring-mass system, they are the elastic potential of the mechanical spring, and gravitational potential energies.
  - For the pendulum, there is only gravitational potential energy.
- Static equilibrium analysis provides the equation that will remove the *redundant forces/moments* from the *EOM* equation, as they do not contribute to the vibration of the system about its equilibrium state.

Static Equilibrium Analysis…

- In most of the cases used in the study, the initial spring force seems to always balance the weight of the mass. Is it always true?

- Consider the following case, where the rod pivoted at *O* is at an angle β at its equilibrium state, and the string attached at the midpoint of the rod is also perpendicular to it.



Taking moment about *O*,

$$mg\frac{l}{2}cos\beta - k\delta_o\frac{l}{2} = 0$$

$$\Rightarrow k\delta_o = mgcos\beta$$

The initial spring force only balance ***partial of the weight*** of the rod. Hence, both the gravity and spring force will contribute to the vibration of the system.

Simple Harmonic Motion

- The *EOMs* for both the spring-mass and simple pendulum can be written in a more generic form as

$$\boxed{\ddot{u} + {\omega_n}^2 u = 0}$$

where

u denotes the *displacement of the degree of freedom* of the system, and

ω_n is referred to as the *natural angular frequency*.

For the spring-mass system, the natural angular frequency is

$$\omega_n = \sqrt{\frac{k}{m}}$$

and for pendulum, it is

$$\omega_n = \sqrt{\frac{g}{l}}$$

- The motion defined by this *EOM* is called the *simple harmonic motion (SHM).*

Simple Harmonic Motion…

- The two fundamental solutions that satisfy the differential equation are the $sin(\omega_n t)$ and $cos(\omega_n t)$.

- Suppose the solution is given by $u = C_1 sin(\lambda t)$, then

$$\ddot{u} = -\lambda^2 C_1 sin(\lambda t)$$

- Substitute these into the *EOM*, gives

$$-\lambda^2 C_1 sin(\lambda t) + {\omega_n}^2 C_1 sin(\lambda t) = 0$$

$$\Rightarrow (-\lambda^2 + {\omega_n}^2) C_1 sin(\lambda t) = 0$$

Cannot be zero for all *t.*

This must be zero, which implies that $\lambda = \omega_n$

- Similarly, it can be shown that $u = C_2 cos(\omega_n t)$ is also a possible solution to the generic *EOM* for free vibration without damping.

Simple Harmonic Motion…

- Hence, the general solution is the linear combination of the two functions, i.e.

$$u = C_1 sin(\omega_n t) + C_2 cos(\omega_n t)$$

which can be rewritten as

$$u = A sin(\omega_n t + \phi)^{\dagger}$$

where

$A = \sqrt{{C_1}^2 + {C_2}^2}$ is the *amplitude* of vibration, and

$\phi = tan^{-1}\left(\frac{C_2}{C_1}\right)$ is its *phase* with respect to the *initial conditions* of the system.

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† Derivation of the two equivalent forms is given by

$$u = C_1 sin(\omega_n t) + C_2 cos(\omega_n t)$$

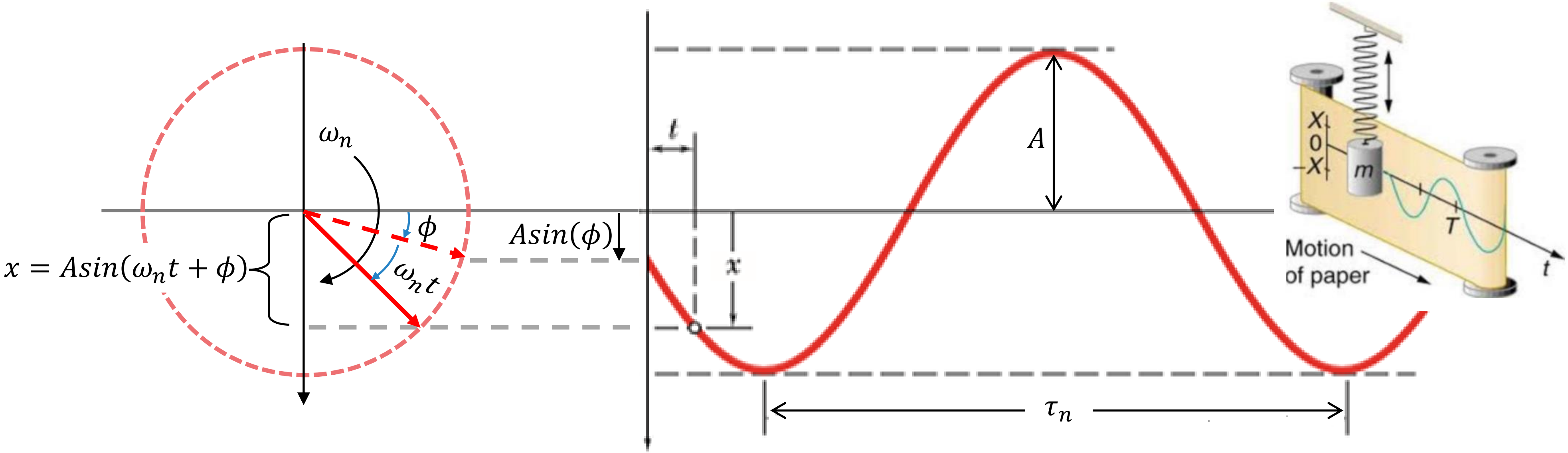
$$u = A sin(\omega_n t + \phi) = (A cos\phi) sin(\omega_n t) + (A sin\phi) cos(\omega_n t)$$

$$\Rightarrow C_1 = A cos\phi, \quad \text{and} \quad C_2 = A sin\phi$$

Hence,

$$A = \sqrt{{C_1}^2 + {C_2}^2}, \quad \text{and} \quad \phi = tan^{-1}\left(\frac{C_2}{C_1}\right)$$

Simple Harmonic Motion…



- Period of oscillation,

$$\tau_n \omega_n = 2\pi \Rightarrow \boxed{\tau_n = \frac{2\pi}{\omega_n}}$$

- Natural frequency (usually in *cycles per seconds* or *Hz*),

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} \Rightarrow \boxed{\omega_n = 2\pi f_n}$$

Simple Harmonic Motion…

- Given the displacement as

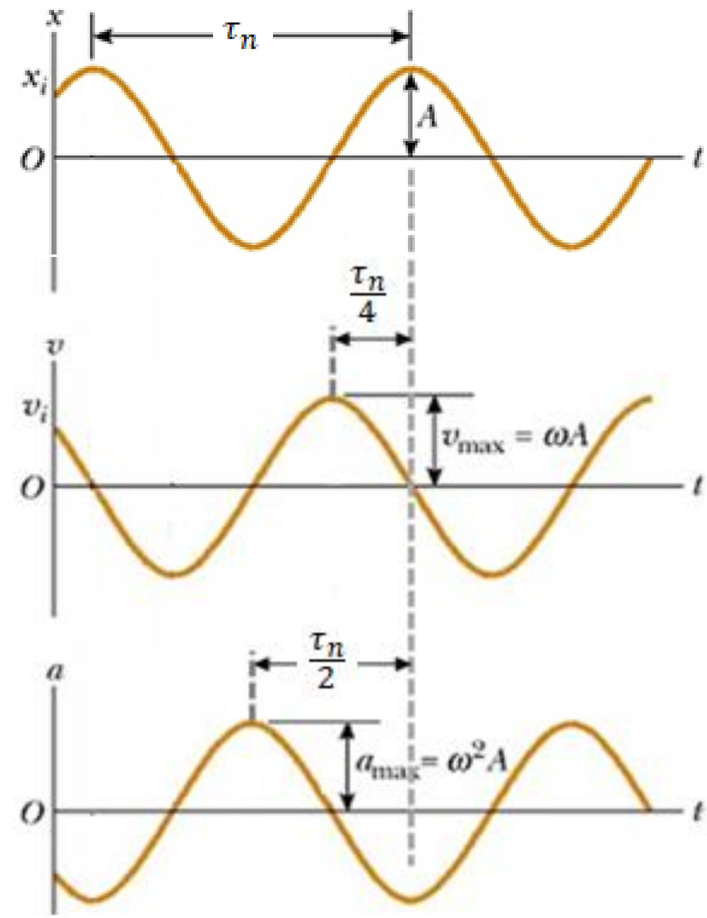
$$x = A\sin(\omega_n t + \phi)$$

- Its velocity and acceleration are

$$\dot{x} = \omega_n A\cos(\omega_n t + \phi) = A\omega_n \sin\left((\omega_n t + \phi) + \frac{\pi}{2}\right)$$

$$\ddot{x} = -\omega_n{}^2 A\sin(\omega_n t + \phi) = A\omega_n{}^2 \sin\big((\omega_n t + \phi) + \pi\big)$$

- Comparing the three plots:
  - The velocity and acceleration amplitudes are scaled by ω_n and $\omega_n{}^2$ w.r.t. to displacement amplitude, respectively.
  - Velocity and acceleration curves were shifted by ¼ and ½ of the period w.r.t. the displacement curve, respectively.



Simple Harmonic Motion…

- The simple harmonic motion is complete when the amplitude and phase angle are defined, which depends on how the system is set into oscillation.
- Suppose the *initial conditions*, i.e. the displacement and velocity of the mass at *time = 0*, are given, i.e.

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

- Imposing these conditions, gives

$$x_0 = Asin(\omega_n(0) + \phi) \Rightarrow x_0 = Asin\phi$$

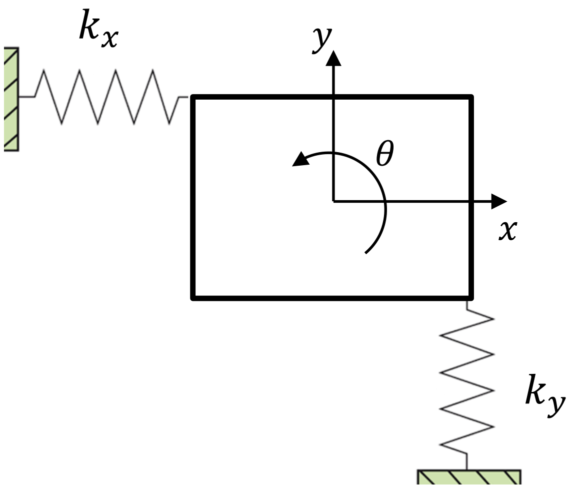
$$v_0 = \omega_n Acos(\omega_n(0) + \phi) = A\omega_n cos\phi \Rightarrow \frac{v_0}{\omega_n} = Acos\phi$$

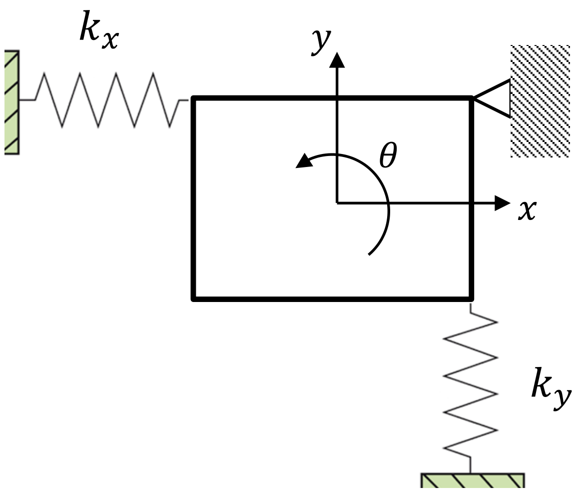
- Finally, solving these two equations for A and ϕ, we have

$$\boxed{A = \sqrt{{x_0}^2 + \left(\frac{v_0}{\omega_n}\right)^2}}\,; and\, \boxed{\phi = tan^{-1}\left(\frac{x_0\omega_n}{v_0}\right)}$$

Free Vibration of Rigid Bodies

- Recall that a rigid body can *translate* and *rotate*. In 2D plane motion, this gives rise to *three DOFs*. Hence, it also has potentially *three modes* of vibrations.



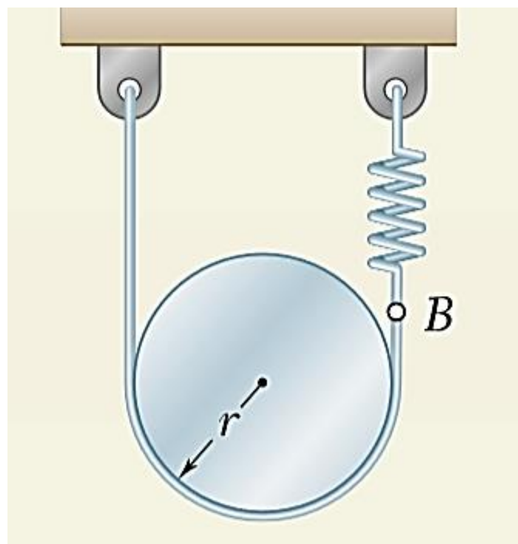


- However, the general 3 *DOFs* problem can be reduced to *single-DOF* system when the rigid body is constrained in its motion.

  Hence, you need to identify the *kinematic constraint* of the system.

Example 1

A cylinder of mass m and radius r is suspended from a looped cord as shown. One end of the cord is attached to a rigid support, while the other end is attached to a spring of constant k. This system is set into small oscillation, and assuming that the cylinder does not slip during the vibration. Determine the natural frequency of vibration of the cylinder.



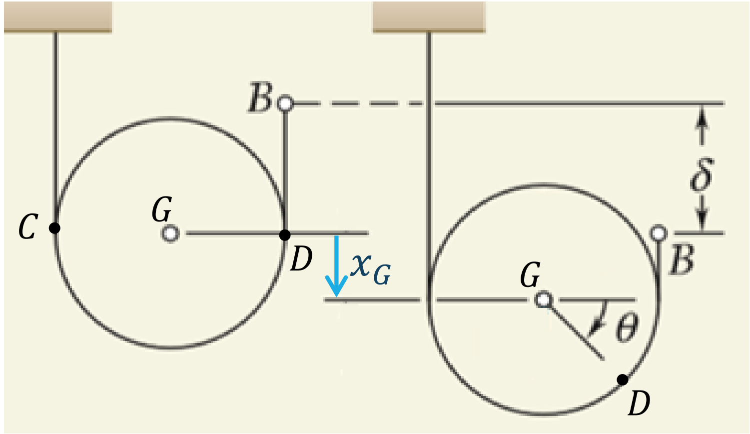
Example 1…

Given that the cylinder rolls on the cord without slipping. This is similar to the situation in which the cylinder is rolling without slipping on a vertical wall. Hence, we have

$$x_G = r\theta,$$

and

$$v_G = r\dot{\theta}, \qquad a_G = r\ddot{\theta}$$



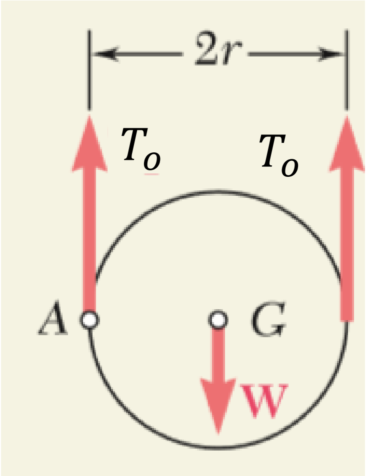
Another point of interest is displacement of *B*, which defines the spring force attached to it. This displacement is equal to the distant travelled by point *D*, which is

$$\delta = \underbrace{x_G}_{\text{Translational motion}} + \underbrace{r\theta}_{\text{Rotational motion}} = 2r\theta$$

where δ is the displacement at *B* from its *equilibrium position* during the oscillation.

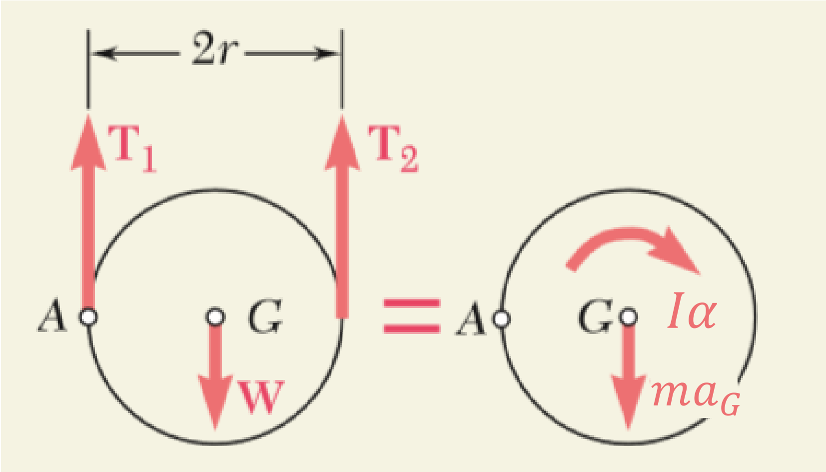
Example 1…

At static equilibrium position, the initial tension in the cable must hold the weigh of the cylinder, i.e.



$$2T_o = mg \Rightarrow T_o = \frac{1}{2}mg$$

Now, consider the *FBD* of the cylinder



Taking moment about point *A*, we derive the angular *EOM* as

$$mgr - T_2(2r) = ma_G r + I_G\alpha$$

$$\Rightarrow mgr - 2rT_2 = m(r\ddot{\theta})r + \frac{1}{2}mr^2\ddot{\theta}$$

$$\Rightarrow \frac{3}{2}mr^2\ddot{\theta} - mgr + 2rT_2 = 0$$

Example 1…

Now the tension in the spring during oscillation is given by

$$T_2 = k\delta + T_o = k(2r\theta\ ) + \frac{1}{2}mg$$

Substituting T_2 expression into the above *EOM* gives

$$\frac{3}{2}mr^2\ddot{\theta} - mgr + 2r\left(k(2r\theta\ ) + \frac{1}{2}mg\right) = 0$$

The moment due to self-weigh and the initial spring force cancels each other in the *EOM*.

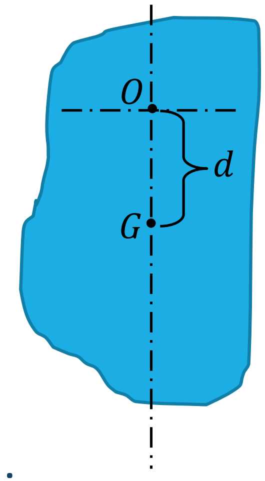
$$\Rightarrow \ddot{\theta} + \left[\frac{8}{3}\left(\frac{k}{m}\right)\right]\theta = 0$$

This resemble the generic EOM of a spring-mass system, and hence its natural frequency is given by

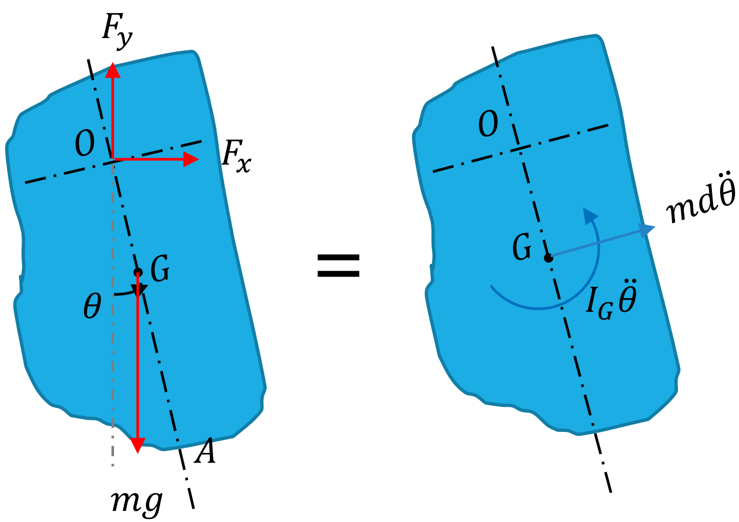
$$\omega_n = \sqrt{\boldsymbol{\frac{8k}{3m}}}$$

Free Oscillation of a Rigid Body

- Consider a rigid body of mass m that is hinged at O. And the centre of mass G is at a distant of d below the pivot point. Determine the natural frequency of this generalize pendulum for small oscillation amplitude.



Consider the FBD of the rigid body



Taking moment about O, the *EOM* is

$$-mgd\sin\theta = I_G\ddot{\theta} + \left(md\ddot{\theta}\right)d$$

$$\Rightarrow (I_G + md^2)\ddot{\theta} + (mgd)\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left(\frac{mgd}{I_G + md^2}\right)\theta = 0$$

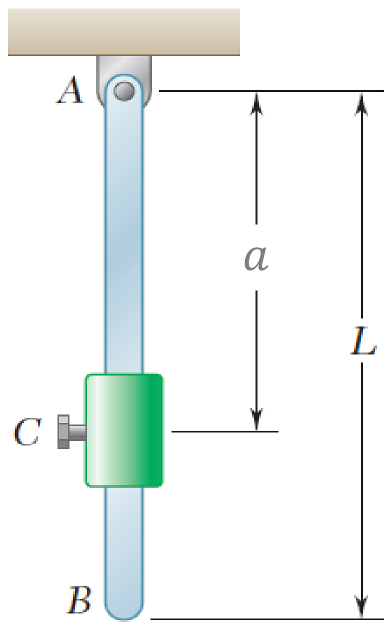
$$\boxed{\Rightarrow \omega_n = \sqrt{\frac{mgd}{I_O}}}$$

Example 2

A small collar weighing 1 *kg* is rigidly attached to a 3 *kg* uniform rod of length L = 1 *m*. Determine

(a) the distance a to maximize the frequency of oscillation when the rod is given a small initial displacement,

(b) the corresponding period of oscillation.

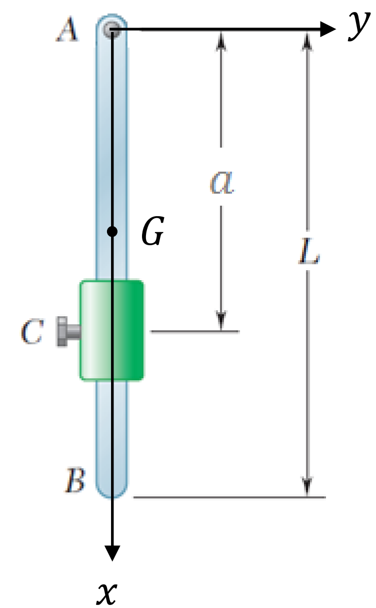


Example 2…

First, one should recognize that this is a *free oscillation a rigid body under gravity*. Hence, we can use the natural frequency expression directly, i.e.

$$\omega_n = \sqrt{\frac{mgd}{I_A}}$$

where d is the vertical distant from pivot point *A* to the *C.G.* of the system, and I_A is mass moment of inertia about *A*.



The mass of the system is

$$m_t = m_c + m_r = 4\ kg$$

And *C.G.* of the system is

$$x_G = \frac{m_c a + m_r\left(\frac{L}{2}\right)}{m_t} = \frac{(1)a + (3)\left(\frac{1}{2}\right)}{4} = \frac{a}{4} + \frac{3}{8}$$

Example 2…

The moment of inertia of the system about *A* is

$$I_A = \left[\frac{1}{12} m_r L^2 + m_r \left(\frac{L}{2}\right)^2\right] + [m_c a^2]$$

(collar is assumed to be point mass)

Substituting the masses into the expression, gives

$$I_A = \left[\frac{1}{12}(3)(1)^2 + (3)\left(\frac{1}{2}\right)^2\right] + [(1)a^2] = 1 + a^2$$

Hence, the natural frequency is expressed as

$$\omega_n = \sqrt{\frac{mgd}{I_A}} = \sqrt{\frac{4g\left(\frac{a}{4} + \frac{3}{8}\right)}{1 + a^2}}; \text{ or } \omega_n{}^2 = \frac{g(a + 1.5)}{1 + a^2}$$

Example 2…

(a) To maximize the frequency for varying a distant, it is the same as maximizing ${\omega_n}^2$. Hence, we do

$$\frac{d}{da}\left({\omega_n}^2\right) = \frac{d}{da}\left[\frac{g(a+1.5)}{1+a^2}\right]$$

$$\Rightarrow 2\omega_n \frac{d\omega_n}{da} = \left[\frac{g}{1+a^2} - \frac{2ag(a+1.5)}{(1+a^2)^2}\right]$$

$$\Rightarrow \frac{d\omega_n}{da} = \frac{g}{2\omega_n}\left[\frac{1}{1+a^2} - \frac{2a(a+1.5)}{(1+a^2)^2}\right] = \frac{g}{2\omega_n}\left[\frac{1-3a-a^2}{(1+a^2)^2}\right]$$

Finally, the maximum occurs when $\frac{d\omega_n}{da} = 0$, which implies that the numerator must be zero, i.e.

$$1 - 3a - a^2 = 0$$

$$\Rightarrow a = \mathbf{0.303}, \qquad \text{or} \qquad a = -3.303 \text{ (inadmissible)}$$

Example 2…

(b) At this given collar position, the natural frequency is

$$\omega_n = \sqrt{\frac{g(a+1.5)}{1+a^2}} = \sqrt{\frac{9.81(0.303+1.5)}{1+(0.303)^2}} = 4.03\, rad/s$$

And hence the period is

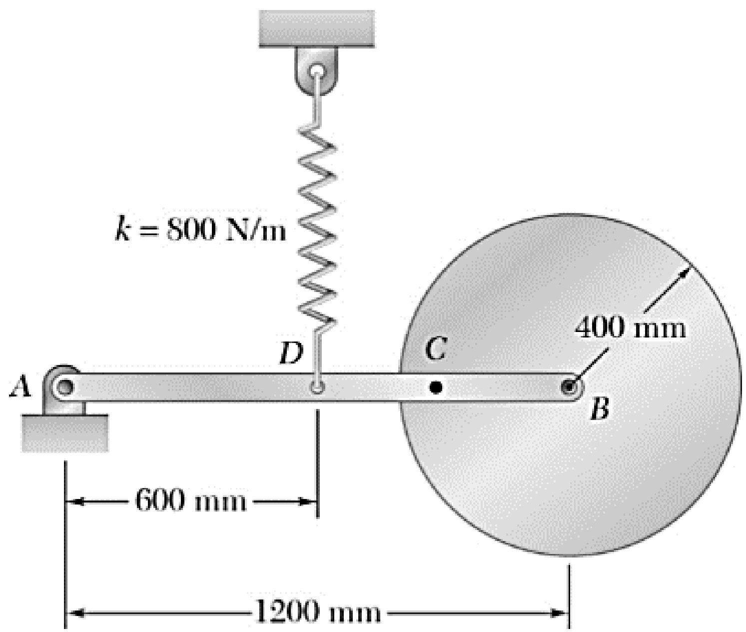
$$\tau_n = \frac{2\pi}{\omega_n} = \mathbf{1.56}\, s$$

Example 3

A 8 *kg* uniform rod *AB* is hinged to a fixed support at *A* and is attached by means of pins *B* and *C* to a 12 *kg* disk of radius 400 *mm*. A spring with a spring constant of 800 *N/m* attached at *D* holds the rod at rest in the position shown. If point *B* is moved downwards 25 *mm* and release, determine

(a) the period of vibration,

(b) the maximum velocity at point *B*.

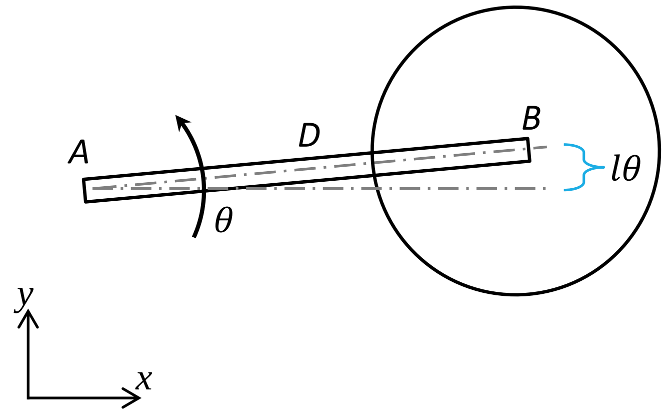


Example 3...

For small angle rotation, the following kinematic relations exist

$$y_B = l\theta, \qquad \text{and} \qquad y_D = \frac{l}{2}\theta$$

$$\ddot{y}_B = l\ddot{\theta}, \qquad \text{and} \qquad \ddot{y}_D = \frac{l}{2}\ddot{\theta}$$

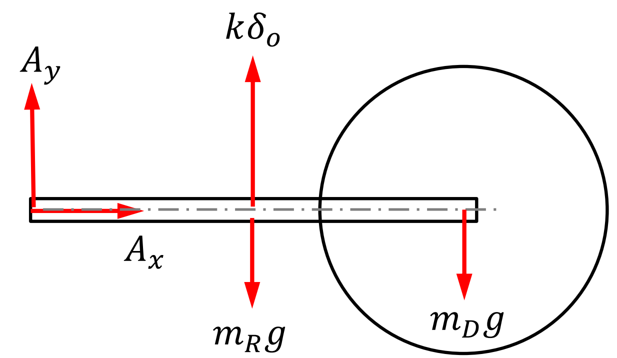


where θ is small angular rotation about A.

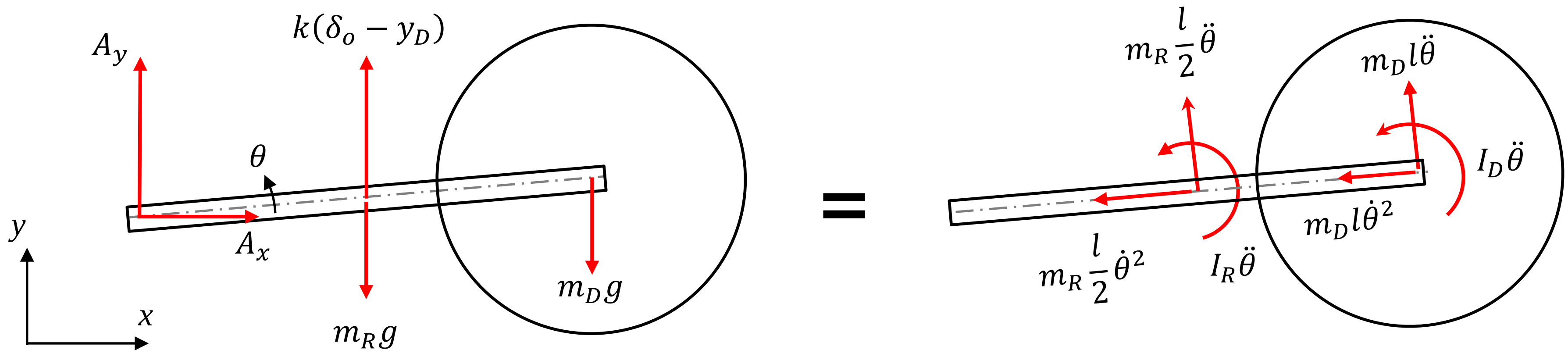
At the static equilibrium, taking moment about A, gives

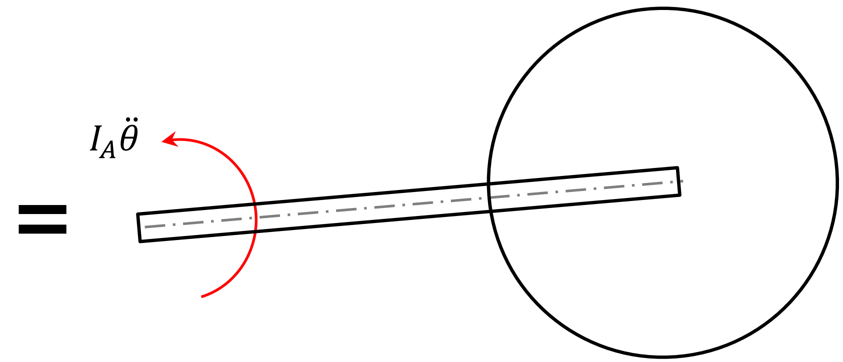
$$k\delta_0\left(\frac{l}{2}\right) - m_R g\left(\frac{l}{2}\right) - m_D g l = 0$$

$$\Rightarrow \; k\delta_0 - m_R g - 2m_D g = 0$$



Example 3…





Taking moment about *A* is

$$k(\delta_o - y_D)\left(\frac{l}{2}cos\theta\right) - m_R g\left(\frac{l}{2}cos\theta\right) - m_D g(lcos\theta) = \underbrace{I_R\ddot{\theta} + m_R\left(\frac{l}{2}\right)^2\ddot{\theta} + I_D\ddot{\theta} + m_D l^2\ddot{\theta}}_{I_A\ddot{\theta}}$$

Example 3...

Applying the kinematic relations, gives

$$k\left(\delta_o - \frac{l}{2}\theta\right)\left(\frac{l}{2}cos\theta\right) - m_R g\left(\frac{l}{2}cos\theta\right) - m_D g(lcos\theta) = I_A\ddot{\theta}$$

And using the small angle assumption ($cos\theta \approx 1$), gives

$$k\left(\delta_o - \frac{l}{2}\theta\right)\left(\frac{l}{2}\right) - m_R g\left(\frac{l}{2}\right) - m_D g l = I_A\ddot{\theta}$$

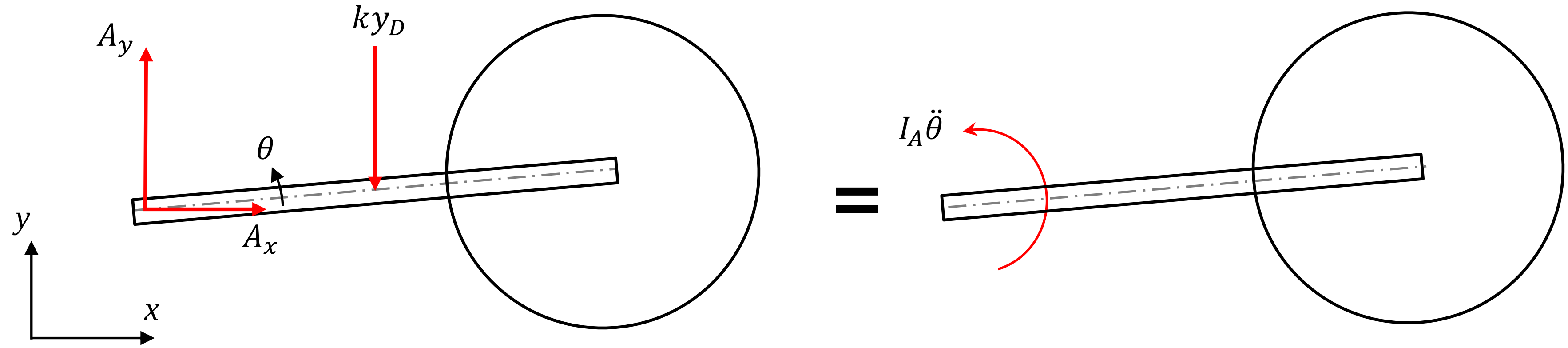
Equal to zero from static equilibrium

$$\Rightarrow I_A\ddot{\theta} + \left[k\left(\frac{l}{2}\right)^2\right]\theta - [k\delta_0 - m_R g - 2m_D g]\left(\frac{l}{2}\right) = 0$$

$$\Rightarrow I_A\ddot{\theta} + \frac{kl^2}{4}\theta = 0$$

Example 3…

Suppose you know that the moment effect of initial spring force is totally balanced by weight of the system, then we can skip the static equilibrium analysis. In this case, the initial spring force and weights must be removed from the *FBD* of the system.



Taking moment about *A* is

$$-ky_D\left(\frac{l}{2}cos\theta\right) = I_A\ddot{\theta} \Rightarrow I_A\ddot{\theta} + \frac{kl^2}{4}\theta = 0$$

Example 3…

The moment of inertia of the rod and disk are

$$I_R = \frac{1}{12} m_R l^2 = 0.960\ kgm^2, \quad \text{and} \quad I_D = \frac{1}{2} m_D R^2 = 0.960\ kgm^2$$

And hence,

$$I_A = \left(0.960 + \left(\frac{1.2}{2}\right)^2 (8)\right) + (0.960 + (1.2)^2(12)) = 22.08\ kgm^2$$

Substitute these values into the *EOM* gives,

$$22.08\ddot{\theta} + 800\left(\frac{1.2}{2}\right)^2 \theta = 0 \Rightarrow \ddot{\theta} + 13.04\theta = 0$$

The natural frequency and period are

$$\omega_n = \sqrt{13.04} = 3.611\ rad/s; \ \text{and} \ \tau_n = \frac{2\pi}{\omega_n} = \mathbf{1.740}\ s$$

Example 3…

(b) The oscillation of the system is given by

$$\theta = \Theta sin(\omega_n t + \phi)$$

And the same simple harmonic motion can be expected for the vertical displacement at point *B* since $y_B = l\theta$, i.e.

$$y_B = Y_B sin(\omega_n t + \phi) = 0.025 sin(\omega_n t + \phi)$$

Hence, the velocity of *B* is

$$\dot{y}_B = 0.025\omega_n cos(\omega_n t + \phi)$$

where the maximum velocity is

$$(\dot{y}_B)_{max} = 0.025(3.611) = \mathbf{0.0903}\ m/s$$

Principle of Conservation of Energy

- For an *undamped* vibration system, where the external forces on the rigid bodies are *conservative forces*, we can employ the principle of conservation of energy to derive the equation of motion and/or its natural frequency of the system.

- Based on this principle, sum of the kinetic and potential energies must be conserved at all instances,

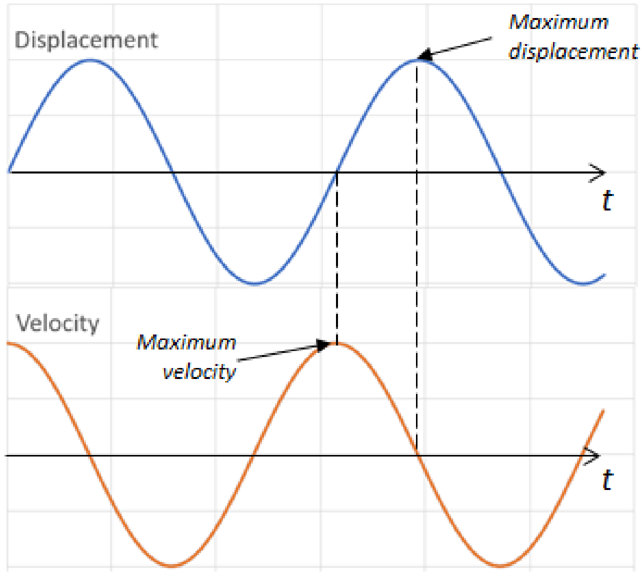
$$T + V = constant$$

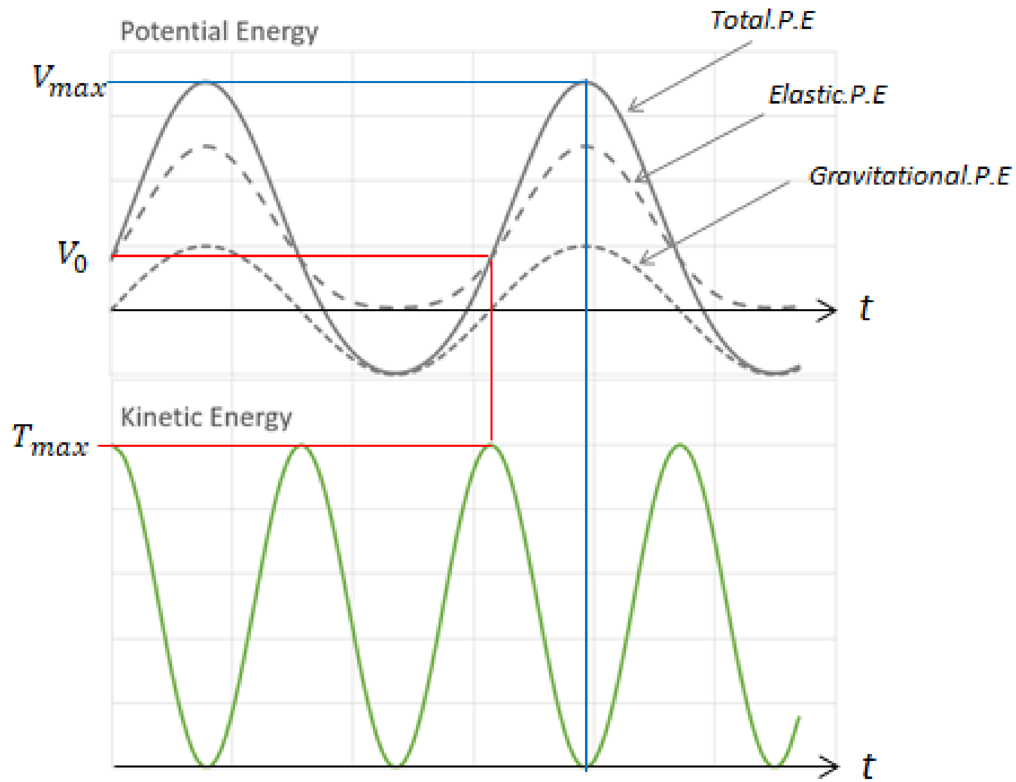
- Finally, taking its time derivative gives the *EOM* of the free vibration,

$$\frac{d}{dt}(T + V) = 0$$

Principle of Conservation of Energy…

- Recall that the displacement and velocity of a simple harmonic motion are





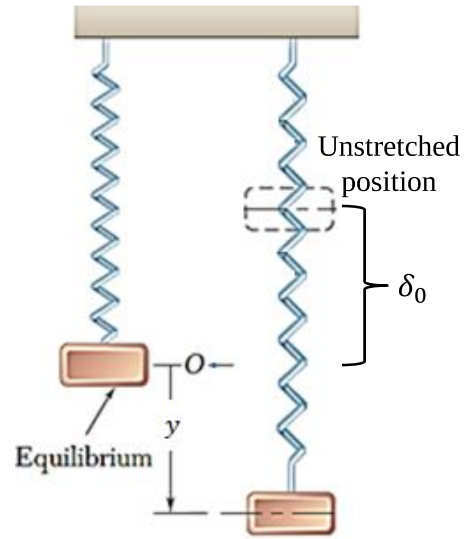
- Applying conservation of energy at these two extreme states, gives

$$T_{max} + V_0 = V_{max} \;\; or \;\; T_{max} = V_{max} - V_0$$

- This relation can be used to compute the *natural frequency* of the system.

EOM of Spring-Mass using Energy method

- Consider the spring-mass system that vibrates in the vertical direction by *y* displacement.



- The kinetic energy is

$$T = \frac{1}{2} m \dot{y}^2$$

- The potential energies of the system are:
  - Elastic potential energy in the spring

$$V_E = \frac{1}{2} k (y + \delta_0)^2 = \frac{1}{2} k y^2 + k y \delta_0 + \frac{1}{2} k {\delta_0}^2$$

Using the static equilibrium condition, i.e. $k\delta_0 = mg$, we have

$$V_E = \frac{1}{2} k y^2 + mgy + \frac{1}{2} k {\delta_0}^2$$

  - Gravitational potential energy $(with\ datum\ at\ equilibrium\ position)$

$$V_G = -mgy$$

Energy Method for Spring-Mass System…

- Hence, the total potential energy is

$$V = V_E + V_G = \tfrac{1}{2}ky^2 + \tfrac{1}{2}k{\delta_0}^2$$

- And the total energy is

$$V + T = \frac{1}{2}ky^2 + \frac{1}{2}k{\delta_0}^2 + \frac{1}{2}m\dot{y}^2$$

- Finally differentiating the expression, gives

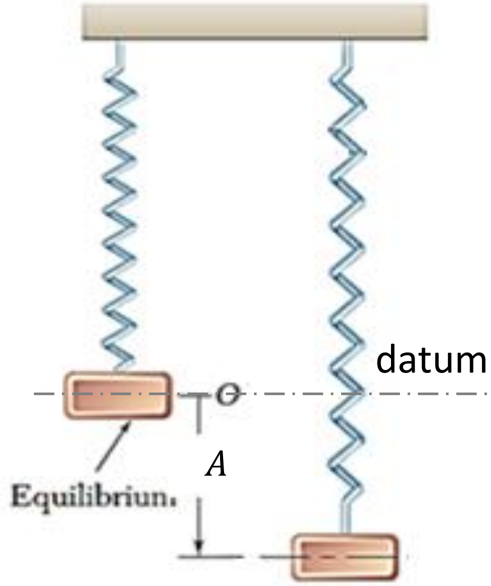
$$\frac{d}{dt}(V + T) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}ky^2 + \frac{1}{2}k{\delta_0}^2 + \frac{1}{2}m\dot{y}^2\right) = 0$$

$$\Rightarrow\ (ky\dot{y}) + 0 + (m\dot{y}\ddot{y}) = 0$$

$$\Rightarrow m\ddot{y} + ky = 0$$

Energy Method for Spring-Mass System…



- Suppose the displacement and velocity are given by

$$y = A\sin(\omega_n t + \phi) \qquad\qquad y_{max} = A$$
$$\Rightarrow$$
$$\dot{y} = A\omega_n \cos(\omega_n t + \phi) \qquad\qquad \dot{y}_{max} = A\omega_n$$

- At the equilibrium zero position,

$$V_0 = \frac{1}{2}k{\delta_0}^{2\dagger}$$

$$T_{max} = \frac{1}{2}m{\dot{y}_{max}}^2 = \frac{1}{2}m(A\omega_n)^2$$

† *The potential is due to the initial elongation of spring from the weigh of the mass. And the datum is taken to be at the equilibrium position, and hence the gravitational potential is zero.*

Energy Method for Spring-Mass System…

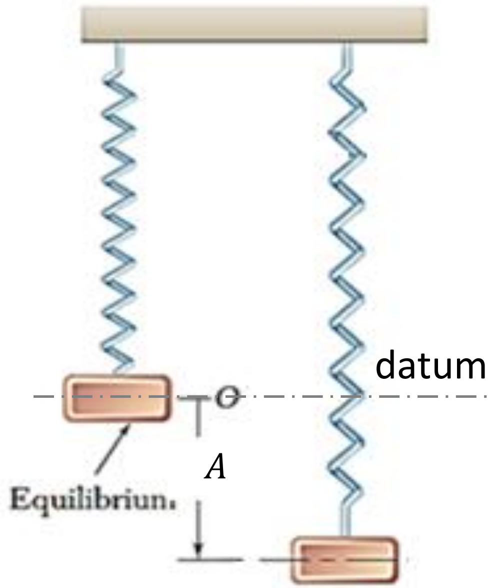
- At maximum displacement position,

$$V_{max} = \frac{1}{2}k(A+\delta_0)^2 - mgA$$

$$= \frac{1}{2}kA^2 + \frac{1}{2}k{\delta_0}^2 + (k\delta_0 - mg)A$$

Equal to zero from static equilibrium

$$\Rightarrow V_{max} = \frac{1}{2}kA^2 + \frac{1}{2}k{\delta_0}^2$$



- Applying the conservation of energy at the two extreme states, gives

$$T_{max} = V_{max} - V_0$$

$$\Rightarrow \frac{1}{2}m(A\omega_n)^2 = \frac{1}{2}kA^2 + \frac{1}{2}k{\delta_0}^2 - \frac{1}{2}k{\delta_0}^2$$

$$\Rightarrow \frac{1}{2}m(A\omega_n)^2 = \frac{1}{2}kA^2 \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

Energy Method for Spring-Mass System…

- For this example, since the *initial spring force balances to weight completely*, we can choose to ignore the potential energies associated to these forces. This implies that the potential energy at equilibrium state *V_0 is zero*.

- And for the potential energy at maximum displacement position, it reduces to

$$V_{max} = \frac{1}{2} k A^2$$

- With this, the conservation of energy at the two extreme states becomes

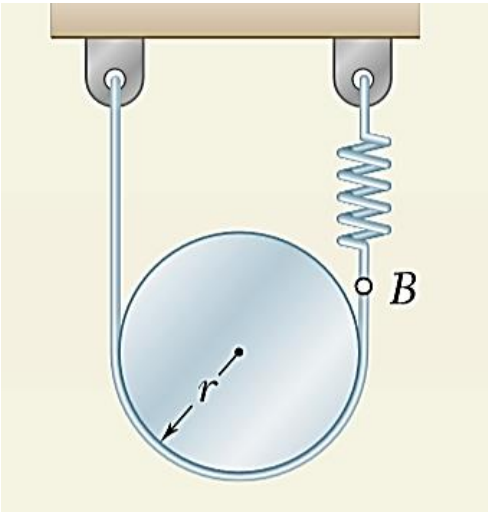
$$T_{max} = V_{max}$$

$$\Rightarrow \frac{1}{2} m (A\omega_n)^2 = \frac{1}{2} k A^2$$

$$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

Example 4

Redo Example 2 using the principle of conservation of energy.



In this case, we determine the kinetic and potential energies at the equilibrium and maximum potential positions.

The same kinematic conditions apply, i.e.

$$x_G = r\theta, \quad v_G = r\dot{\theta}, \quad \text{and} \quad \delta = 2r\theta$$

The angular oscillation can be given by

$$\theta = \Theta sin(\omega_n t + \varphi)$$

Then

$$\theta_{max} = \Theta \Rightarrow \begin{aligned} x_{G,max} &= r\Theta \\ \delta_{max} &= 2r\Theta \end{aligned}$$

And

$$\dot{\theta} = \Theta\omega_n cos(\omega_n t + \varphi) \Rightarrow \begin{aligned} \dot{\theta}_{max} &= \Theta\omega_n \\ v_{G,max} &= r\dot{\theta}_{max} = r\Theta\omega_n \end{aligned}$$

Example 4…

Since the initial spring force balances the weight of the cylinder, hence we can ignore the potential energies due to them.

At maximum displacement position,

$$V_{max} = \frac{1}{2} k {\delta_{max}}^2 = \frac{1}{2} k (2r\Theta)^2$$

At equilibrium position,

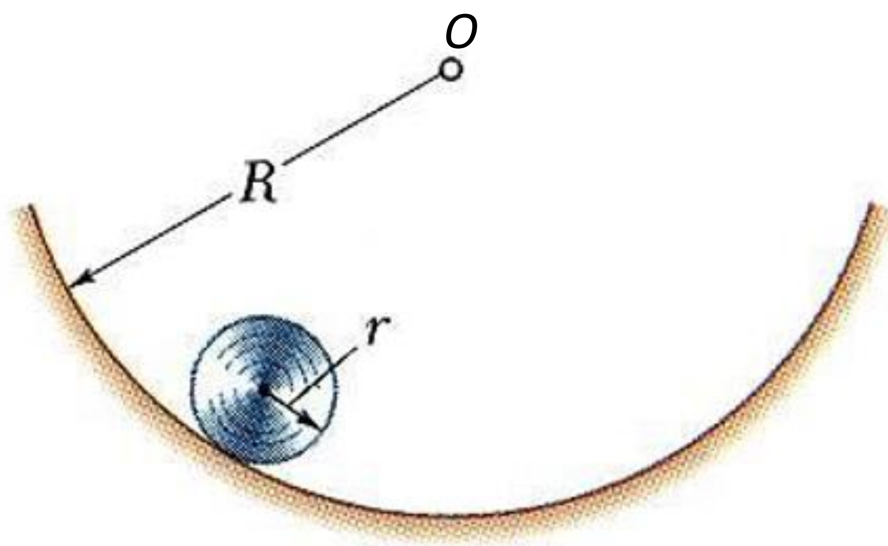
$$T_{max} = \frac{1}{2} m\left(v_{G,max}\right)^2 + \frac{1}{2} I_G\left(\dot{\theta}_{max}\right)^2$$
$$= \frac{1}{2} m\left(r\dot{\theta}_{max}\right)^2 + \frac{1}{2}\left(\frac{1}{2} mr^2\right)\left(\dot{\theta}_{max}\right)^2 = \frac{3}{4} m(r\Theta\omega_n)^2$$

Finally, applying the principle of conservation of energy,

$$\frac{3}{4} m(r\Theta\omega_n)^2 = \frac{1}{2} k(2r\Theta\,)^2 \Rightarrow \omega_n = \sqrt{\frac{\boldsymbol{8k}}{\boldsymbol{3m}}}$$

Example 5

Determine the natural oscillating frequency of small rotating motion of a cylinder which rolls without slipping inside a curved surface as shown.



Example 5…

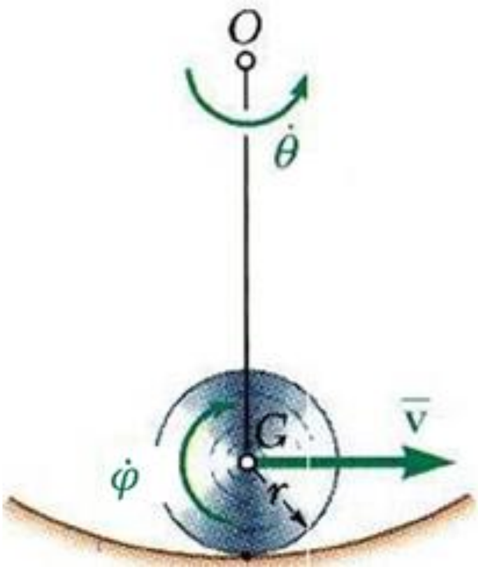
In this example, two rotational motions can be identified, namely:

i) Circular motion of point G about the center of the curved surface, i.e. point O. We let this angular motion be defined as: $\theta = \Theta sin(\omega_n t + \phi)$

ii) The rotation of the cylinder as it rolls on the curved surface, φ.

From the first circular motion, the velocity at G is

$$v_G = (R - r)\dot{\theta}$$



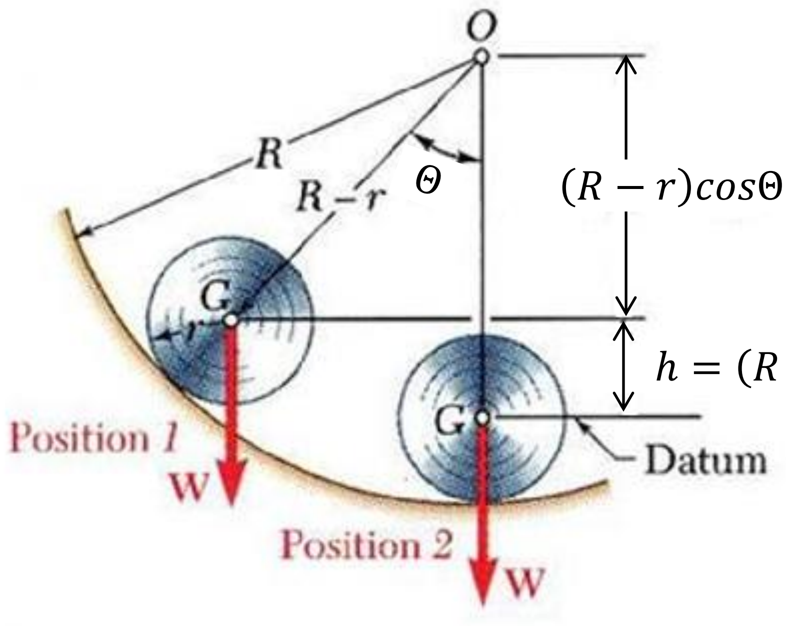
And due to non-slip condition, this velocity also be given by

$$v_G = r\dot{\varphi}$$

Equating the two expressions, we get

$$(R - r)\dot{\theta} = r\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{(R - r)}{r}\dot{\theta}$$

Example 5…



At position 1,

$$cos\Theta = 1 - \frac{\Theta^2}{2!} + \frac{\Theta^4}{4!} + \cdots$$

(Taylor series of $cos\Theta$)

$$V_{max} = Wh = W(R-r)(1-cos\Theta)$$

$$\Rightarrow V_{max} \approx mg(R-r)\frac{\Theta^2}{2}$$

Example 5…

At position 2,

$$V_0 = 0 \text{ (at datum)}$$

And the kinetic energy is,

$$T_{max} = \frac{1}{2} m {v_{max}}^2 + \frac{1}{2} I {\dot{\varphi}_{max}}^2$$

Using the following kinematic relations,

$$v_{max} = (R-r)\dot{\theta}_{max}, \qquad and \qquad \dot{\varphi}_{max} = \frac{(R-r)}{r}\dot{\theta}_{max}$$

The kinetic energy becomes

$$T_{max} = \frac{1}{2} m \left[(R-r)\dot{\theta}_{max}\right]^2 + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\left(\frac{(R-r)}{r}\dot{\theta}_{max}\right)^2$$

$$\Rightarrow T_{max} = \frac{3}{4} m (R-r)^2 {\dot{\theta}_{max}}^2$$

Example 5…

And since $\theta = \Theta sin(\omega_n t + \phi) \Rightarrow \theta = \Theta\omega_n cos(\omega_n t + \phi)$, we have

$$\dot{\theta}_{max} = \Theta\omega_n$$

With this relation, the kinetic energy becomes

$$T_{max} = \frac{3}{4}m(R-r)^2\Theta^2{\omega_n}^2$$

Using the conservation of energy, gives

$$T_{max} = V_{max} - V_0$$

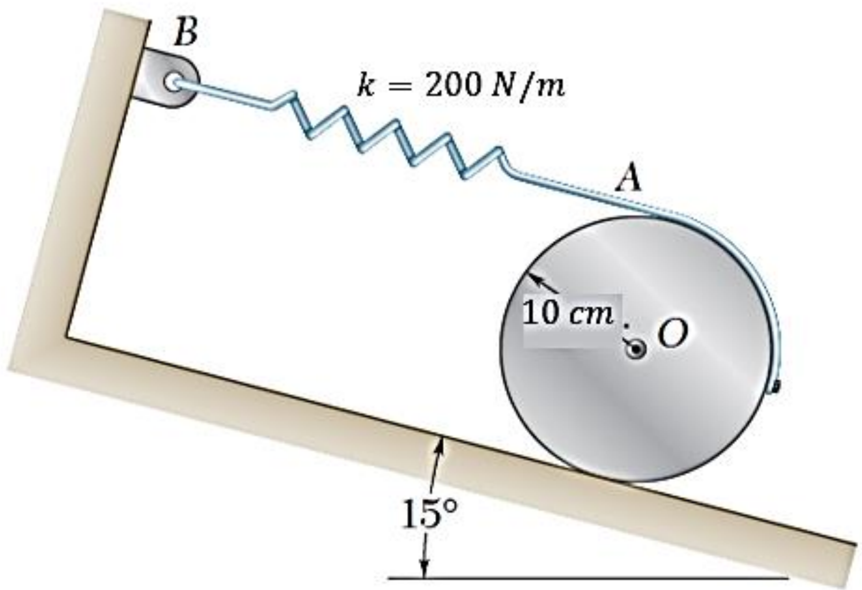
$$\Rightarrow \frac{3}{4}m(R-r)^2\Theta^2{\omega_n}^2 = mg(R-r)\frac{\Theta^2}{2}$$

$$\Rightarrow \omega_n = \sqrt{\frac{2g}{3(R-r)}}$$

Question 1

A 10-*kg* uniform cylinder can roll without sliding on a 15°-incline. A belt is attached to the rim of the cylinder, and a spring holds the cylinder at rest in the position shown. If the center of the cylinder is moved 5 *mm* down the incline and released, determine the period of oscillation

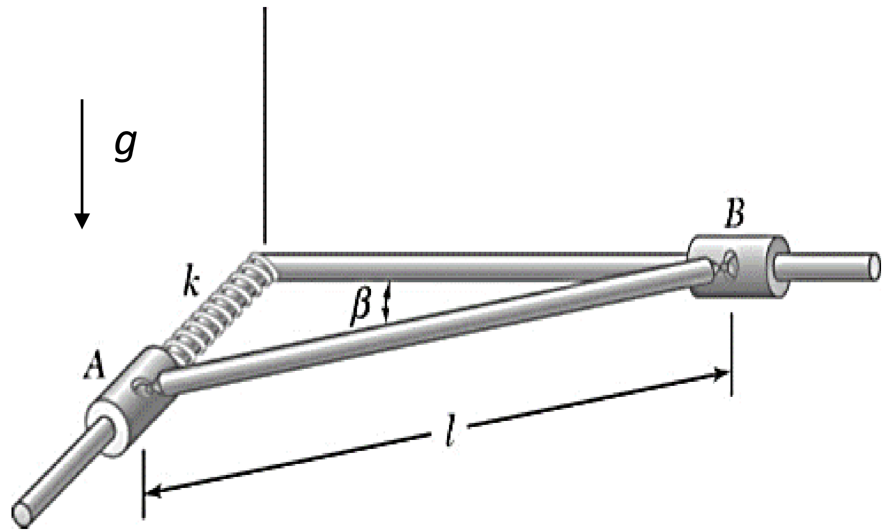
Answers: $\tau_n = 0.860\ s$



Question 2

A slender rod *AB* of mass *m* and length *l* is connected to two collars of mass m_c in the horizontal plane as shown. Collar *A* is also attached to a spring of constant *k*. Assume that the collars can slide freely on their respective rods, and it is in equilibrium at the position shown. Determine the natural angular frequency of this system

Answers: $\omega_n = \sqrt{\dfrac{\boldsymbol{k\cos^2\beta}}{\left(\frac{1}{3}\boldsymbol{m}+\boldsymbol{m_c}\right)}}$



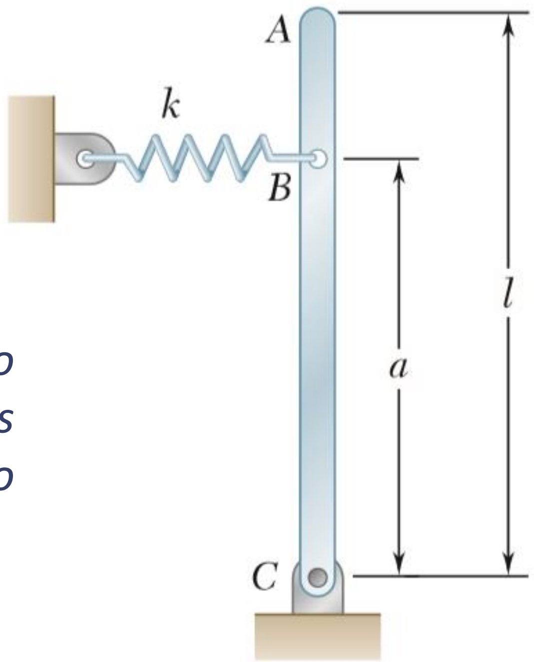
Question 3

An inverted pendulum consisting of a rigid bar *ABC* of length *l* and mass *m* is supported by a pin and bracket at *C*. A spring of constant *k* is attached to the bar at *B* and is undeformed when the bar is in the vertical position shown. Determine:

(a) the frequency of small oscillations,

(b) the smallest value of *a* for which these oscillations will occur.

Answers: (a) $\omega_n = \sqrt{\frac{6ka^2 - 3mgl}{2ml^2}}$, (b) $a_{min} > \sqrt{\frac{mgl}{2k}}$



Note: This system will not oscillate if the mass of the rod is too heavy for the spring to restore it to its equilibrium position. In this case, the system would cease to a steady-stationary state titling to either side, depending on which side it is being displaced.