National University of Singapore

ME2115/TME2115
Mechanics of Machines

FREE VIBRATION WITH DAMPING

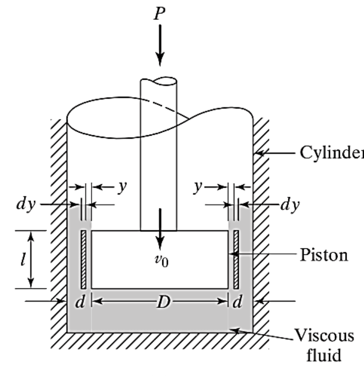
Learning Outcomes

- Able to understand the effects of *viscous damping* on the free vibration of a spring-mass-damper system, which can be classified as:
  - *Under-damped*,
  - *Critical damped*, and
  - *Over-damped*.
- Able to recognize the *free vibration* of an *under-damped* system, which is *oscillatory* but with *decaying amplitude*.
- Able to derive the *equation of motion* of a vibrating system *with damping*, and recognize relations between the *system parameters* (mass, spring stiffness and damping coefficient) and the system *vibration characteristic* (natural frequency, critical damping constant, exponential decaying constant, damping ratio).

Damping Mechanisms

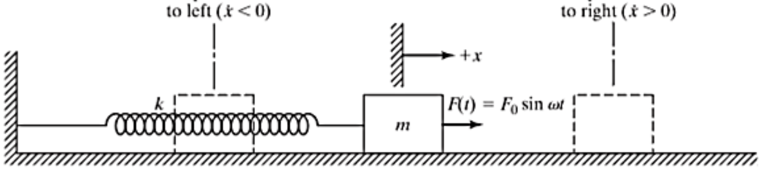
- All mechanical systems will experience damping to some degrees. They may be due to:

  - *Fluid friction* – when the body is moving in a viscous fluid. It is also known as viscous damping in vibration study.



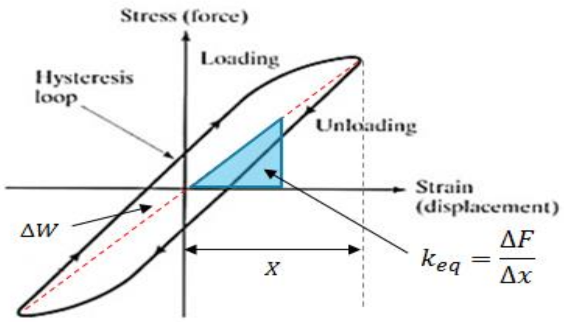
$$c = \mu\left[\frac{3\pi D^3 l}{4d^3}\left(1 + \frac{2d}{D}\right)\right]$$

$$F_D = sgn(\dot{x})\mu N$$



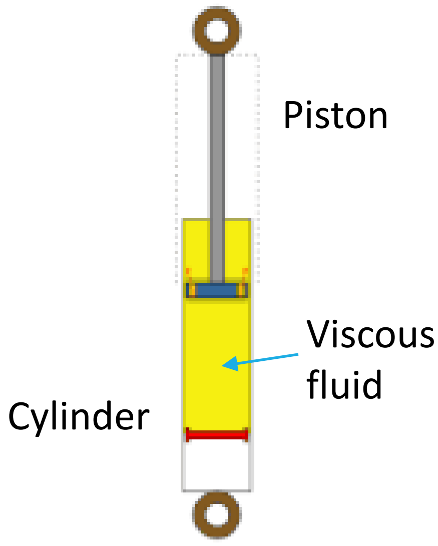
  - *Coulomb friction* – at the interfaces between rigid bodies. It is also known as dry friction.

  - *Structural or hysteresis friction* – due to the internal structure of the material, e.g. friction between grain boundaries of the material.



Dashpot Damper

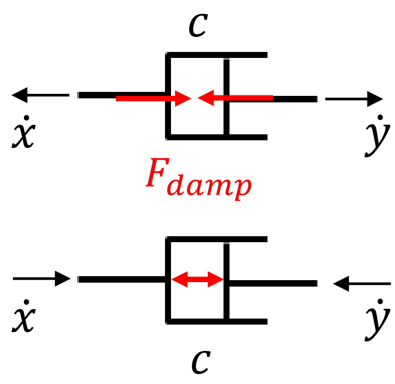
- The working principle of a viscous damper (dashpot damper):
  - It comprises three basic components, a cylinder, a piston and viscous fluid.
  - A perforated disk separates the cylinder into two chambers.
  - As the disk moves in the cylinder, fluid is forced to move from one chamber to another.
  - Due to limited flow path, a pressure difference is created which leads to the damping force.



- The resistance force arises is *directly proportional and opposite to the relative velocity of the piston and cylinder*,

$$F_{damp} = -c(\dot{x} - \dot{y})$$

where c is known as the *damping coefficient or constant*.



Free Vibration of Spring-Mass-Damper System

- The equation of motion ($+ve \downarrow$) is

$$W - c\dot{x} - k(x + \delta_0) = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = 0$$

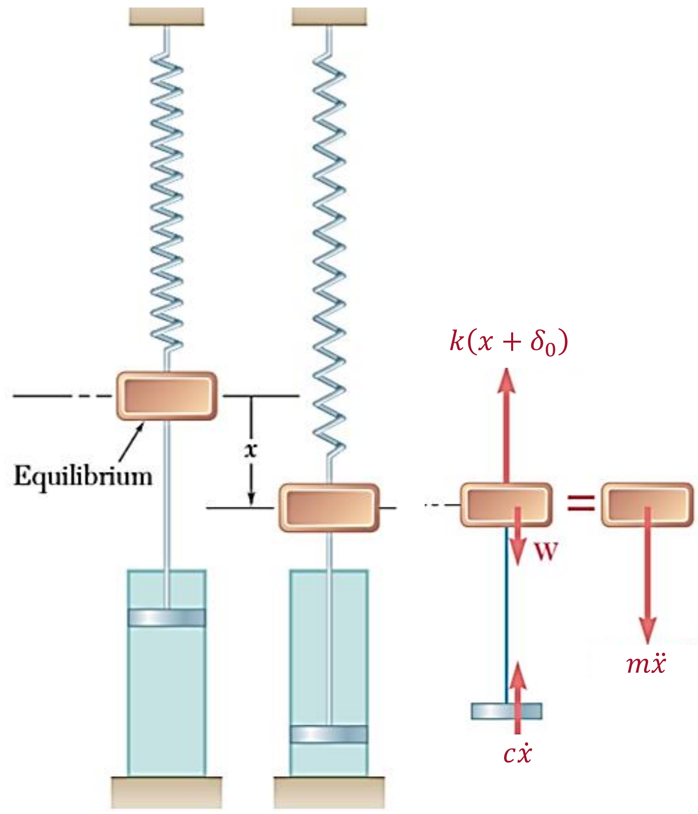
- Suppose the solution for this differential equation is given by

$$x = Ae^{\lambda t}$$

where A is the amplitude of motion, and λ gives the time dependent characteristic of the motion.

- Hence, the velocity and acceleration are

$$\dot{x} = A\lambda e^{\lambda t}, and \;\; \ddot{x} = A\lambda^2 e^{\lambda t}$$



Free Damped Vibration…

- Substituting this expression into the *EOM*, gives

$$m\left(\lambda^2 A e^{\lambda t}\right) + c\left(\lambda A e^{\lambda t}\right) + k\left(A e^{\lambda t}\right) = 0$$

$$\Rightarrow (m\lambda^2 + c\lambda + k) A e^{\lambda t} = 0$$

$$\Rightarrow m\lambda^2 + c\lambda + k = 0 \qquad (\text{characteristic equation})$$

- The roots of this quadratic equation is

$$\boxed{\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$$

- Depending on the value of the discriminant (square root term), there are three different cases for λ. They give rise to different damping motions, namely:
  - Over-damped vibration,
  - Critically damped vibration, and
  - Under-damped vibration.

Over-Damped Vibration

- Over-damping occurs when

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0$$

- In this case, roots λ_1 and λ_2 are real and distinct, given by

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}, \qquad \text{and} \quad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

  which are both negative values, and λ_1 is always less negative than λ_2.

- The general solution is given by

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

- This is a *non-oscillatory* motion, since both the terms decay exponentially with time. In general, the motion is dominated by the first term because it decays slower than the second term.

Critically Damped Vibration

- Critical-damping occurs when

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$$

- This is a special case where the damping coefficient is related to the system mass and spring, i.e.

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\Rightarrow c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{mk} = 2m\omega_n$$

- Due to its uniqueness, this damping coefficient is called the ***critical damping coefficient***, that is

$$\boxed{c_{cr} = 2\sqrt{mk} = 2m\omega_n}$$

Critically Damped Vibration…

- In this case, there is only one lambda, that is

$$\lambda_c = -\frac{c_{cr}}{2m} = -\omega_n$$

- And the general solution is given by

$$x = (A_1 + A_2 t)e^{-\omega_n t}$$

- This is also a *non-oscillatory* motion. It corresponds to the case in which the system can *regain its equilibrium in the shortest time*. This is a desirable feature in many applications where vibration control is needed, such as in shock damper designs in vehicle system.

Under-Damped Vibration

- Under-damping occurs when

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0$$

- In this case, the second term in the root expression becomes *imaginary*, and the roots are *complex conjugates*†, i.e.

$$\lambda_1 = -\left(\frac{c}{2m}\right) + i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,, and$$

$$\lambda_2 = -\left(\frac{c}{2m}\right) - i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

† *Given a complex number,* $Z = a + ib$*, then its conjugate is* $\bar{Z} = a - ib$.

Under-Damped Vibration – Exponential form

- Suppose if we define

$$\boxed{\alpha = \frac{c}{2m},} \qquad \text{and} \qquad \boxed{\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}$$

where α corresponds to the exponential decaying coefficient, and ω_d is the damped oscillation frequency.

- The roots then be rewritten as

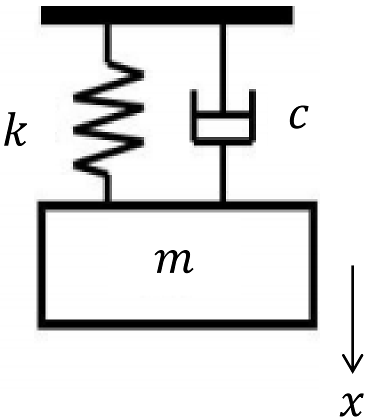
$$\lambda_1 = -\alpha + i\omega_d, \qquad and \quad \lambda_2 = -\alpha - i\omega_d$$

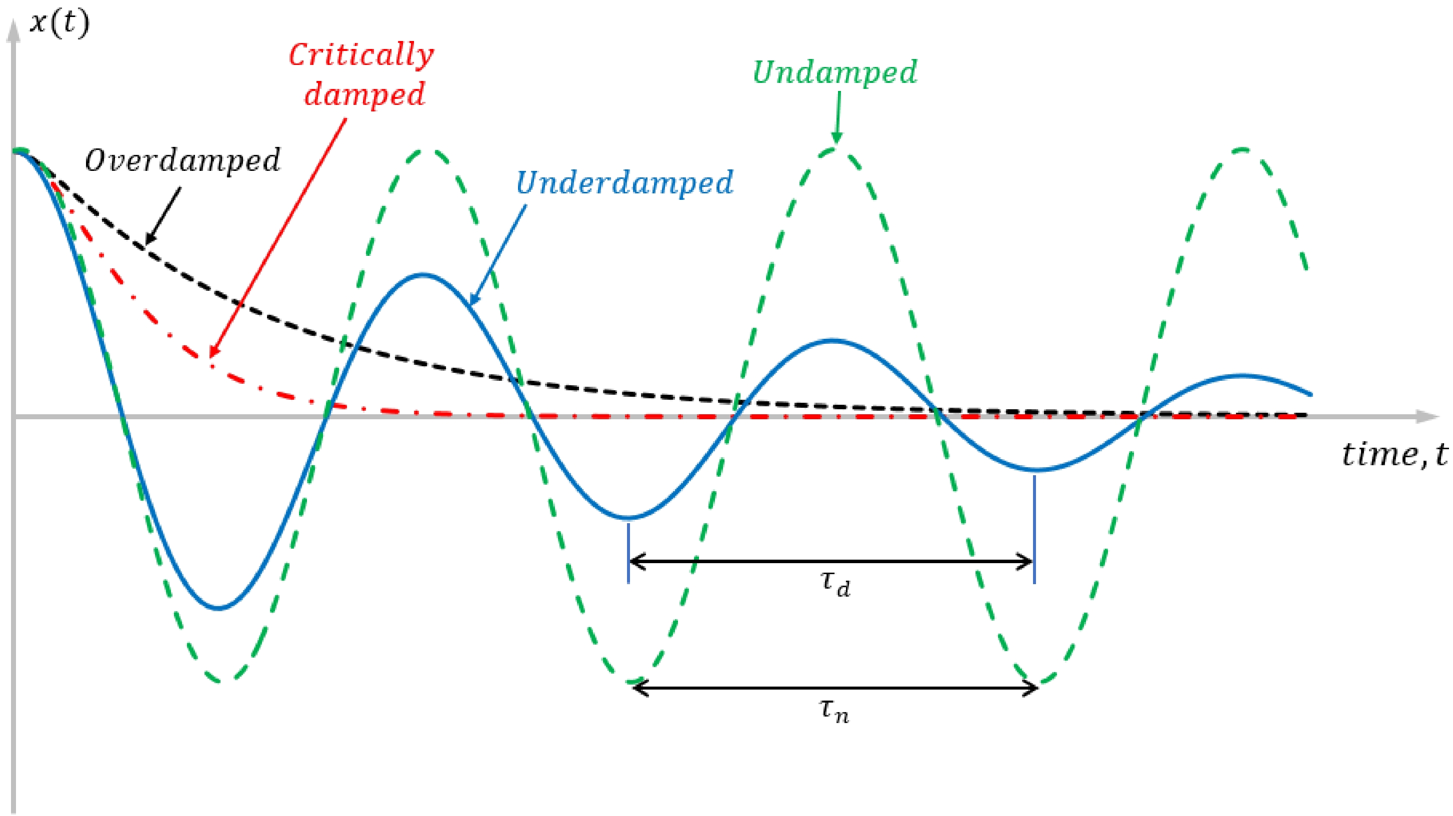
- Hence, the general solution for a free under-damped vibration is

$$x = A_1 e^{(-\alpha + i\omega_d)t} + A_2 e^{(-\alpha - i\omega_d)t} = e^{-\alpha t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$

where the constants A_1 and A_2 are complex numbers, and they are complex conjugate pairs.

Free Damped Vibration...





Under-Damped Vibration – Sinusoidal form

- From trigonometric identities, exponential function with imaginary exponent can be expressed in terms of sine and cosine functions, i.e.

$$e^{i\theta} = cos\theta + isin\theta, \quad \text{and} \quad e^{-i\theta} = cos\theta - isin\theta$$

- Replacing the $e^{i\omega_d t}$ and $e^{-i\omega_d t}$ in the under-damped solution, gives

$$x = e^{-\alpha t}\left[A_1\left(cos(\omega_d t) + isin(\omega_d t)\right) + A_2\left(cos(\omega_d t) - isin(\omega_d t)\right)\right]$$

$$= e^{-\alpha t}[\underbrace{(A_1 + A_2)}_{C_1}cos(\omega_d t) + \underbrace{i(A_1 - A_2)}_{C_2}sin(\omega_d t)]$$

- Now, C_1 and C_2 must be real numbers simply because x is a real function of time. With this, it can be shown that

$$\begin{cases} C_1 = A_1 + A_2 \\ C_2 = i(A_1 - A_2) \end{cases} \Rightarrow \begin{cases} A_1 = \dfrac{1}{2}(C_1 - iC_2) \\ A_2 = \dfrac{1}{2}(C_1 + iC_2) \end{cases}$$

Under-Damped Vibration – Sinusoidal form…

- In summary, the free under-damped vibration motion is

$$\boxed{x = e^{-\alpha t}[C_1 cos(\omega_d t) + C_2 sin(\omega_d t)]}$$

$$\boxed{x = Xe^{-\alpha t} sin(\omega_d t + \phi)}$$

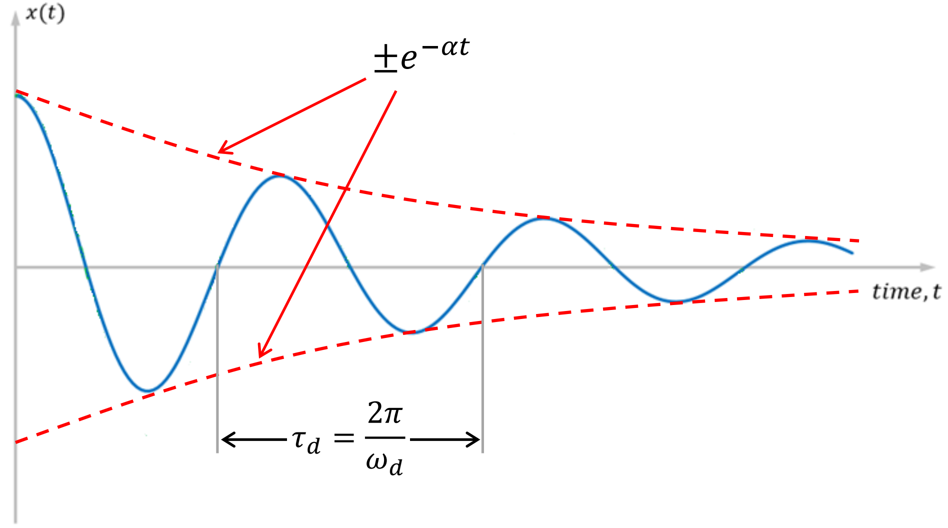
where

$$X = \sqrt{C_1^2 + C_2^2}, \qquad \text{and} \qquad \phi = tan^{-1}\left(\frac{C_1}{C_2}\right)$$

This expression contains:

i) an exponentially decaying function $e^{-\alpha t}$, which encapsulates the damped oscillation profile, and

ii) a sinusoidal function $sin(\omega_d t + \phi)$ that is oscillating at $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$, which is lower than the natural frequency ω_n.



Under-Damped Vibration – Damping ratio form

- Recall that, $\omega_n = \sqrt{\frac{k}{m}}$, and $c_{cr} = 2m\omega_n$. And let us define the damping ratio

$$\boxed{\zeta = \frac{c}{c_{cr}}}$$

where the types of damping cases can be easily quantify by

- $\zeta > 1$ (over-damped)
- $\zeta = 1$ (critical damped)
- $\zeta < 1$ (under-damped)

- With these three parameters, we can express can then express

$$\alpha = \frac{c}{2m} = \frac{c\omega_n}{2m\omega_n} = \left(\frac{c}{c_{cr}}\right)\omega_n \Rightarrow \boxed{\alpha = \zeta\omega_n}$$

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{\omega_n{}^2 - (\zeta\omega_n)^2} \Rightarrow \boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

Under-Damped Vibration – Damping ratio form…

- Using the definitions for ζ and ω_n, the generic form of the *EOM* for under-damped vibration can be rewritten as

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \left(\frac{c}{2\omega_n m}\right)(2\omega_n)\dot{x} + \frac{k}{m}x = 0$$

c_{cr}

$$\Rightarrow \boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = 0}$$

- And the general solution becomes

$$\boxed{x = Xe^{-\zeta\omega_n t} sin(\omega_d t + \phi)}$$

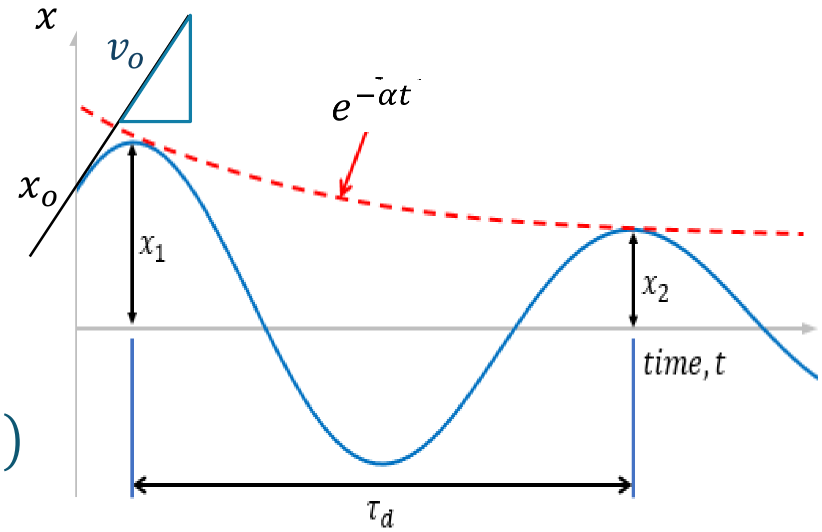
where

$$\omega_n = \sqrt{\frac{k}{m}}, \qquad \zeta = \frac{c}{2\sqrt{km}}, \qquad and \qquad \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Under-Damped Vibration – Initial conditions

- Suppose the initial conditions are given by

$$x(0) = x_o, \quad \text{and} \quad \dot{x}(0) = v_o$$



- Consider the general solutions, i.e.

$$x = Xe^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$

$$\dot{x} = Xe^{-\zeta\omega_n t}\left(-\zeta\omega_n sin(\omega_d t + \phi) + \omega_d cos(\omega_d t + \phi)\right)$$

- Next, applying the above initial conditions, give

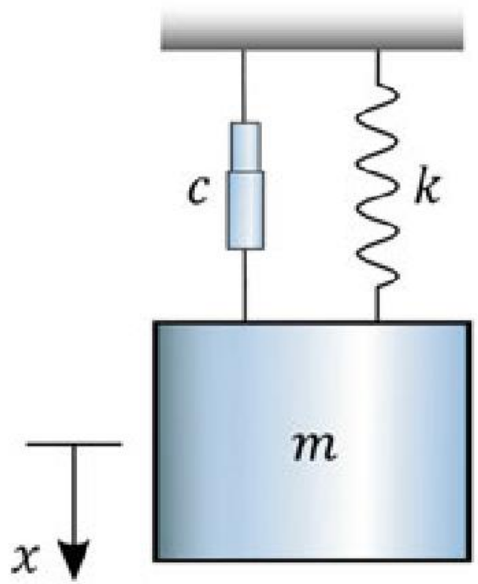
$$x_o = Xsin\phi;\ and\ \ v_o = X(-\zeta\omega_n sin\phi + \omega_d cos\phi)$$

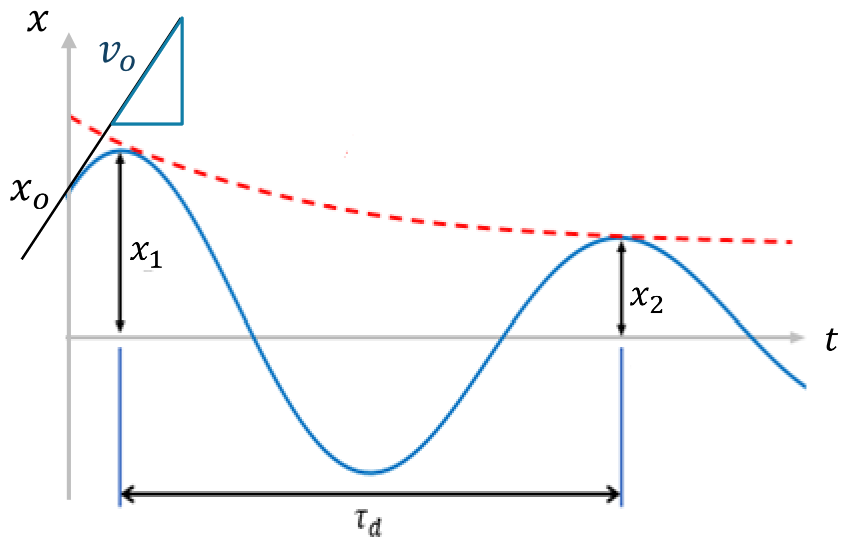
which can be solved to give

$$\boxed{X = \sqrt{\left(\frac{v_o + \zeta\omega_n x_o}{\omega_d}\right)^2 + {x_o}^2}}\ ;\ and\ \boxed{\phi = tan^{-1}\left(\frac{\omega_d x_o}{v_o + \zeta\omega_n x_o}\right)}$$

Example 1

The damped mass–spring system shown in the figure has mass *m* = 10 *kg*, stiffness coefficient *k* = 1000 *N/m*, and damping coefficient *c* = 10 *N·s/m*. Determine the displacement of the mass as a function of time when the initial conditions of the system are given by: $x_0 = 1\ mm$ and $v_0 = 10\ mm/s$. And determine the amplitudes of peaks x_1 and x_2, as shown in the response plot.





Example 1…

Using the general solution of the under-damped system, i.e.

$$x = Xe^{-\alpha t} sin(\omega_d t + \phi)$$

The natural angular frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\, rad/s$$

The decaying coefficient is

$$\alpha = \frac{c}{2m} = \frac{10}{2(10)} = 0.5\, rad/s$$

And the under-damped oscillating frequency is

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{(10)^2 - (0.5)^2} = 9.9875\, rad/s$$

Example 1...

Or we can also use the under-damped solution in the damping ratio form, i.e.

$$x = Xe^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$

The damping ratio is given by

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{10}{2\sqrt{(1000)(10)}} = 0.05$$

Together with the natural frequency $\omega_n = 10\, rad/s$, the under-damped oscillating frequency can be calculated as

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = (10)\sqrt{1-(0.05)^2} = 9.9875\, rad/s$$

Using either approach, the general solution of the under-damped system becomes

$$x = Xe^{-0.5t} sin(9.9875t + \phi)$$

Example 1…

Finally applying the initial conditions into the amplitude and phase formulas, we get

$$X = \sqrt{\left(\frac{v_o + \zeta\omega_n x_o}{\omega_d}\right)^2 + {x_o}^2} = \sqrt{\left(\frac{10 + (0.05)(10)(1)}{9.9875}\right)^2 + (1)^2} = 1.451\, mm$$

$$\phi = tan^{-1}\left(\frac{\omega_d x_o}{v_o + \zeta\omega_n x_o}\right) = tan^{-1}\left(\frac{(9.9875)(1)}{10 + (0.05)(10)(1)}\right) = 0.7604\, rad$$

With this, the exact response of the system is

$$x = 1.451e^{-0.5t} sin(9.9875t + 0.7604)$$

Example 1…

As observed from the response plot, the amplitudes x_1 and x_2 occur when the gradients (velocities) are zeros.

Hence, differentiating the displacement function,

$$\dot{x} = 1.451e^{-0.5t}\left(-0.5sin(9.9875t + 0.7604) + 9.9875cos(9.9875t + 0.7604)\right)$$

And then solving for the time at which the velocity is zero, gives

$$-0.5sin(9.9875t + 0.7604) + 9.9875cos(9.9875t + 0.7604) = 0$$

$$\Rightarrow tan(9.9875t + 0.7604) = \frac{9.9875}{0.5}$$

$$\Rightarrow 9.9875t + 0.7604 = tan^{-1}\left(\frac{9.9875}{0.5}\right) + n\pi$$

$$\Rightarrow t = \frac{1}{9.9875}(0.7604 + n\pi)$$

Example 1…

Hence, peak x_1 occurs at

$$t = \frac{1}{9.9875}(0.7604) = 0.076135\ s$$

And the corresponding amplitude is

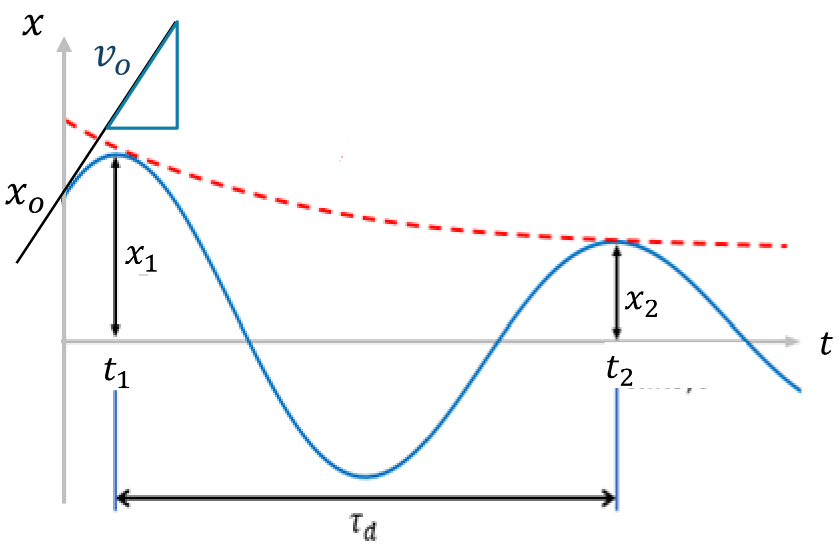
$$x_1 = 1.451e^{-0.5(0.076135)}sin(9.9875(0.076135) + 0.7604) = 1.395\ mm$$

For peak x_2, one should be aware that it is actually the third zero velocity point. Hence,

$$t = \frac{1}{9.9875}(0.7604 + 2\pi) = 0.70524\ s$$

And with this,

$$x_2 = 1.019\ mm$$

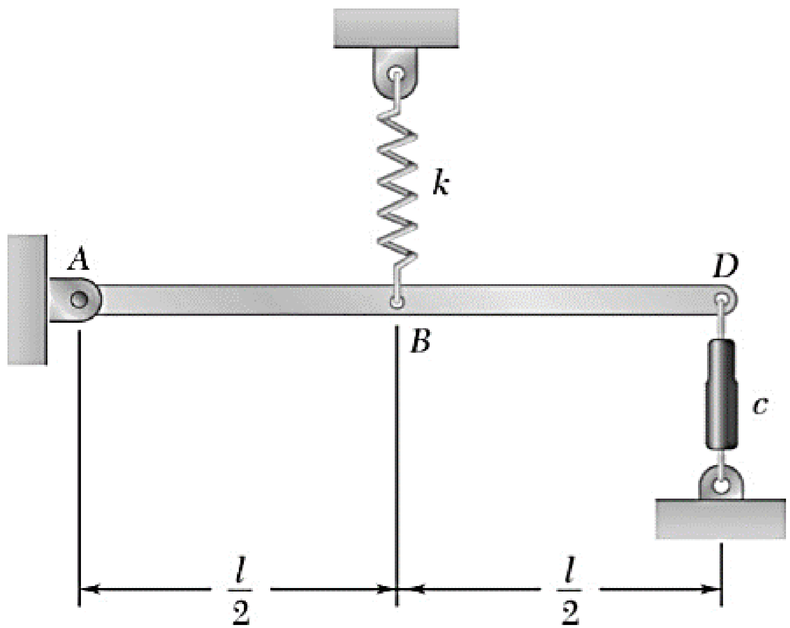


Example 2

A uniform rod of mass m is supported by a pin at A, a spring at B, and a dashpot at D, as shown in the figure. Determine in terms of m k, and c, for small oscillations,

(i) the critical damping coefficient of this system, and

(ii) its damping ratio, if it is an under-damped system.

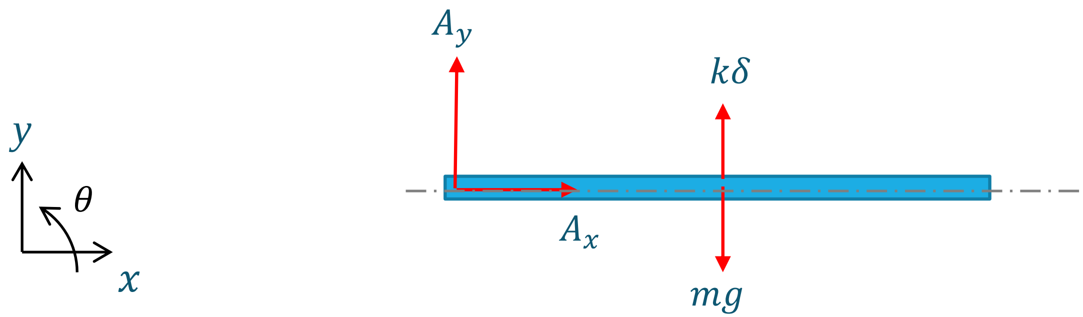


Example 2...

From kinematic constraint, and for small angle rotation,

$$y_B = \frac{l}{2}\theta, \qquad y_D = l\theta \Rightarrow \dot{y}_D = l\dot{\theta}$$

At static equilibrium,



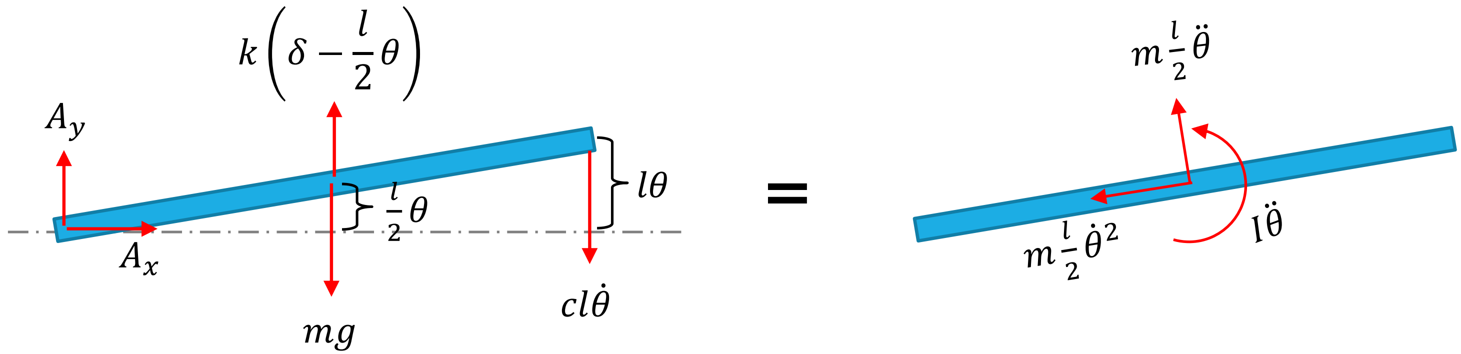
Consider the moment equation about *A*,

$$(k\delta)\frac{l}{2} - mg\frac{l}{2} = 0 \Rightarrow k\delta - mg = 0$$

where δ is the initial elongation of the spring.

Example 2…

The FBD of the rod is



Consider the angular *EOM* about *A*,

$$k\left(\delta - \frac{l}{2}\theta\right)\frac{l}{2} - mg\frac{l}{2} - (cl\dot{\theta})l = I\ddot{\theta} + \left(m\ddot{\theta}\frac{l}{2}\right)\frac{l}{2}$$

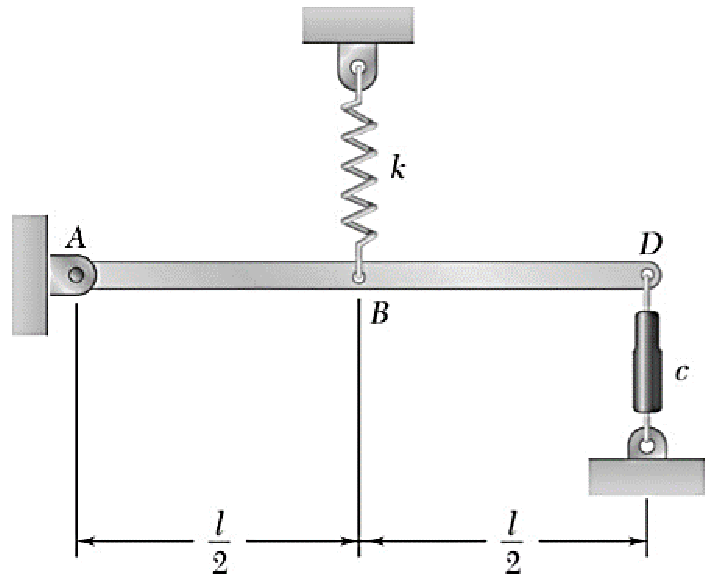
$$\Rightarrow \left[\frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2\right]\ddot{\theta} + cl^2\dot{\theta} + k\left(\frac{l}{2}\right)^2\theta = 0$$

Example 2…

After some manipulations, the *EOM* becomes

$$\left[\frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2\right]\ddot{\theta} + cl^2\dot{\theta} + k\left(\frac{l}{2}\right)^2\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left(\frac{3c}{m}\right)\dot{\theta} + \left(\frac{3k}{4m}\right)\theta = 0$$

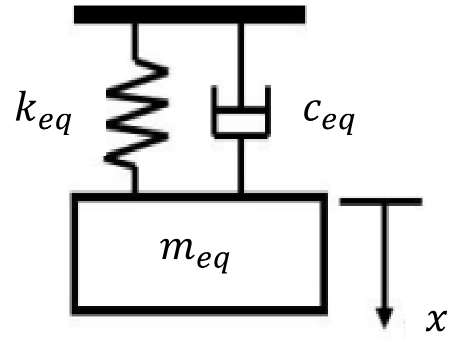


This rotating rod system can be mapped into a simple spring-mass-damper system as shown

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = 0$$

where

$$m_{eq} = 1, \qquad c_{eq} = \frac{3c}{m}, \qquad \text{and} \quad k_{eq} = \frac{3k}{4m}$$



Example 2...

(i) Based on the equivalent spring-mass-damper equation, the critical damping coefficient is

$$\left(c_{eq}\right)_{cr} = 2\sqrt{k_{eq}m_{eq}} = 2\sqrt{\left(\frac{3k}{4m}\right)(1)} = \sqrt{\frac{3k}{4m}}$$

And mapping it back to the original system,

$$\frac{3c_{cr}}{m} = \sqrt{\frac{3k}{4m}} \Rightarrow c_{cr} = \sqrt{\frac{km}{3}}$$

(ii) The damping ratio is

$$\zeta = \frac{c}{c_{cr}} = c\sqrt{\frac{3}{km}}; \quad or \quad = \frac{c_{eq}}{\left(c_{eq}\right)_{cr}} = \frac{(3c/m)}{2\sqrt{\left(\frac{3k}{4m}\right)(1)}} = c\sqrt{\frac{3}{km}}$$

Example 2...

Alternatively, we can relate the coefficients from the *EOM* to the damping ratio form, which gives

$$2\zeta\omega_n = \frac{3c}{m}; \qquad and \qquad {\omega_n}^2 = \frac{3k}{4m} \Rightarrow \omega_n = \sqrt{\frac{3k}{4m}}$$

Substituting ω_n into the damping coefficient relation, we have

$$2\zeta\sqrt{\frac{3k}{4m}} = \frac{3c}{m} \Rightarrow \zeta = c\sqrt{\frac{3}{km}}$$

And the critical damping coefficient is

$$\Rightarrow \zeta = \frac{c}{c_{cr}} \Rightarrow c\sqrt{\frac{3}{km}} = \frac{c}{c_{cr}} \Rightarrow c_{cr} = \sqrt{\frac{km}{3}}$$

Example 3

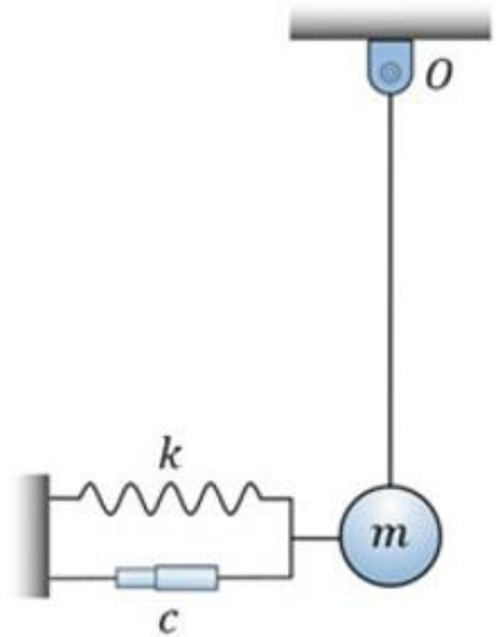
A pendulum is attached to a spring and damper device as shown. Assuming small oscillations and that the rod is massless, obtain the equation of motion the free vibration of assembly. And determine also:

(i) the natural angular frequency,

(ii) critical damping constant,

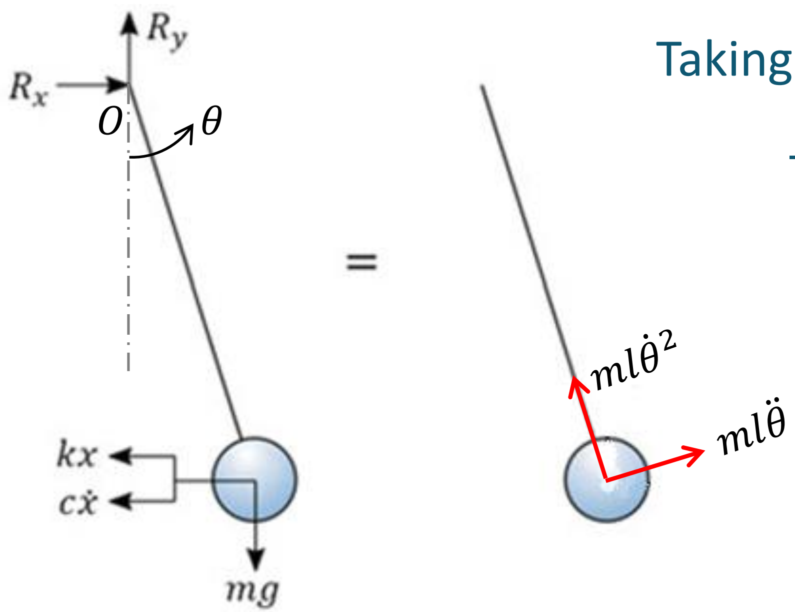
(iii) damping ratio of this system, and

(iv) damped oscillation frequency.



Example 3...

The FBD of this system



Taking moment about *O*, we have

$$-(kx)lcos\theta - (c\dot{x})lcos\theta - mglsin\theta = (ml\ddot{\theta})l$$

And from the kinematic observation,

$$x = lsin\theta \Rightarrow \dot{x} = lcos\theta\dot{\theta}$$

Substituting these relations into the moment equation, gives

$$-(klsin\theta)lcos\theta - (cl\dot{\theta}cos\theta)lcos\theta - mglsin\theta = (ml\ddot{\theta})l$$

$$\Rightarrow ml^2\ddot{\theta} + c\dot{\theta}l^2(cos\theta)^2 + kl^2sin\theta cos\theta + mglsin\theta = 0$$

Example 3…

For small angle oscillations, $sin\theta \approx \theta$ and $cos\theta \approx 1$. Hence, the equation becomes

$$ml^2\ddot{\theta} + cl^2\dot{\theta} + kl^2\theta + mgl\theta = 0$$

$$\Rightarrow m\ddot{\theta} + c\dot{\theta} + \left(k + \frac{mg}{l}\right)\theta = 0$$

(i) Based on this *EOM*, the natural frequency is

$$\omega_n = \sqrt{\frac{\left(k + \frac{mg}{l}\right)}{m}} = \sqrt{\frac{kl + mg}{ml}}$$

(ii) The critical damping coefficient is

$$c_{cr} = 2m\omega_n = 2m\sqrt{\frac{kl + mg}{ml}}$$

Example 3…

(iii) The damping ratio is

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\sqrt{\frac{kl+mg}{ml}}} = \frac{c}{2m}\sqrt{\frac{ml}{kl+mg}}$$

(iv) And finally, the damped oscillation frequency is

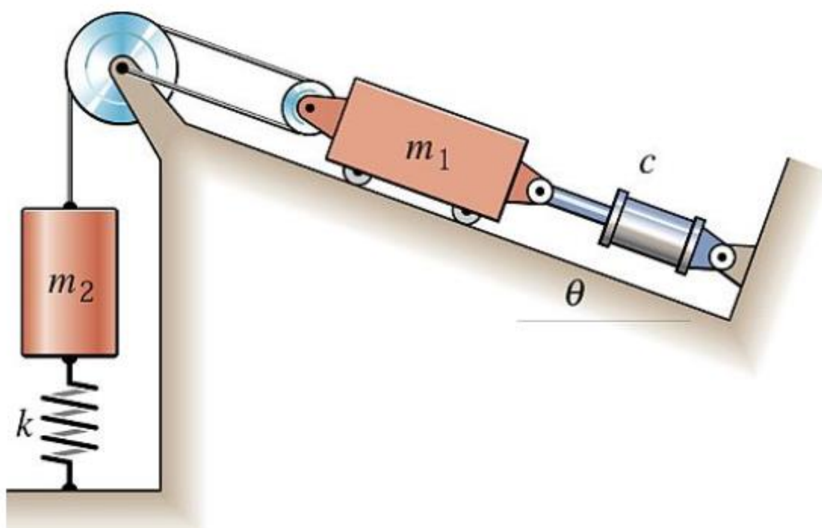
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{kl+mg}{ml}}\sqrt{1-\left(\frac{c}{2m}\sqrt{\frac{ml}{kl+mg}}\right)^2}$$

$$\Rightarrow \omega_d = \sqrt{\frac{4m(kl+mg)-c^2 l}{4m^2 l}}$$

Example 4

The spring-mass-damper system is connected as shown, where the massless rollers roll frictionless about their pivot, and they also roll without slipping on an inextensible cable. It is given also that $m_1 = m_2 = 10\ kg$, and $k = 200\ N/m$.

Suppose it is required that the system to oscillate at $0.6\ Hz$, determine the damping coefficient *c* that is needed to achieve this performance.



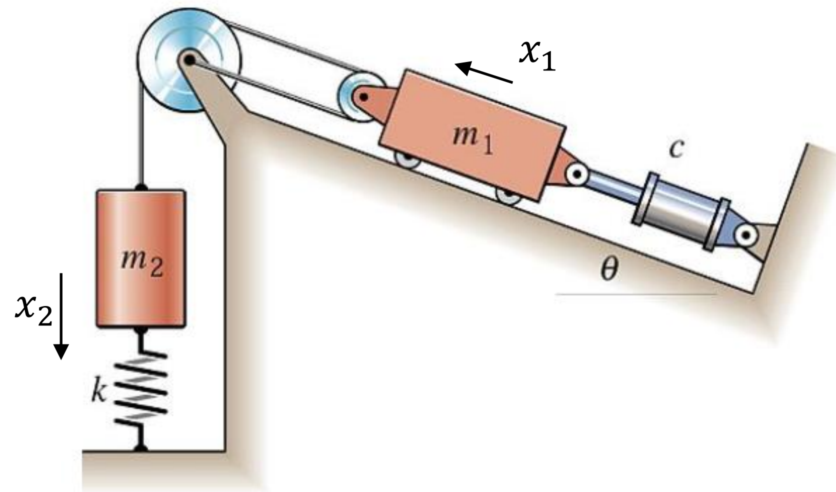
Example 4…

Although this system comprises two masses, they are connected rigidly via the cable. Hence, it is a 1 DOF system.

And the displacements of the masses are related by

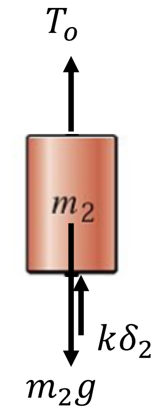
$$x_2 = 2x_1$$

$$\Rightarrow \dot{x}_2 = 2\dot{x}_1, \qquad and \quad \ddot{x}_2 = 2\ddot{x}_1$$



Assume for small amplitude vibration, the cable is always in tension during the motion. And since the rollers are assumed massless, hence one can expect the tension in the cable is uniform throughout the length.

Example 4…



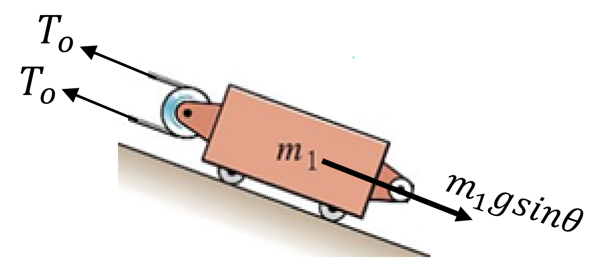
Consider force equation at static equilibrium for m_2,

$$T_o + k\delta_2 - m_2g = 0$$

where T_o is the tension in the cable, and δ_2 is the initial compression of the spring.

Similarly for m_1 along the slope,

$$2T_o - m_1gsin\theta = 0$$



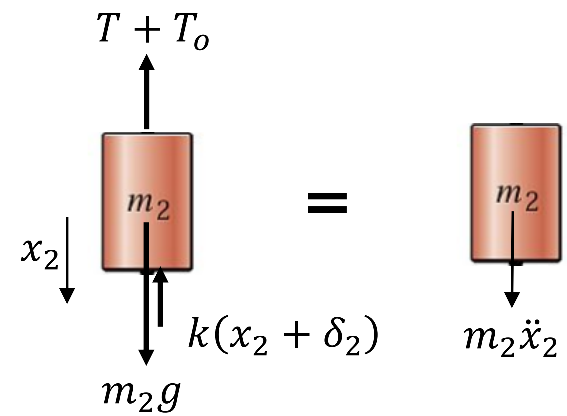
Example 4…

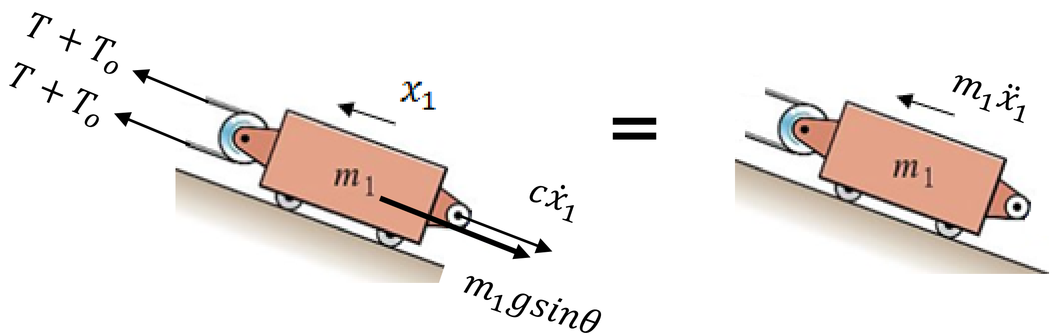
Consider the *FBD* of m_2, the *EOM* in vertical direction is

$$-(T+T_o)-k(x_2+\delta_2)+m_2g=m_2\ddot{x}_2$$

$$\Rightarrow m_2\ddot{x}_2+kx_2=-T-(T_o+k\delta_2-m_2g)$$

where the T is the variation in the cable tension in addition to the initial tension at equilibrium.





Consider the FBD of m_1, the EOM along the slope is

$$2(T+T_o)-m_1gsin\theta-c\dot{x}_1=m_1\ddot{x}_1$$

$$\Rightarrow m_1\ddot{x}_1+c\dot{x}_1=2T+(2T_o-m_1gsin\theta)$$

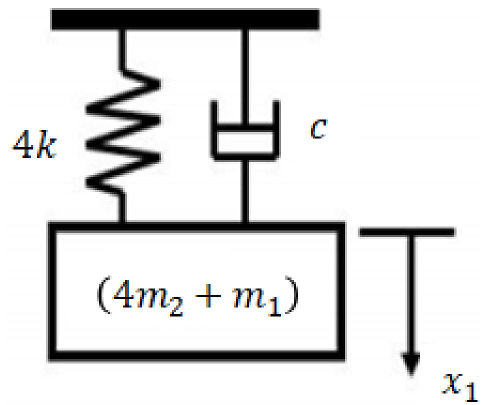
Example 4...

Eliminating T from two equations, gives

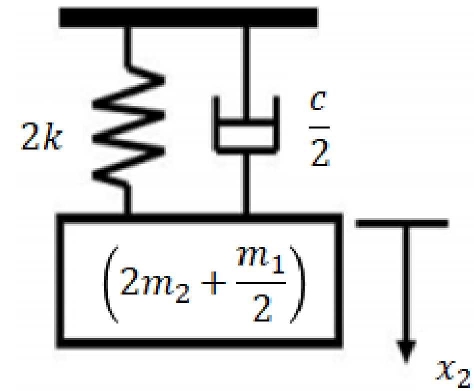
$$2m_2\ddot{x}_2 + 2kx_2 + m_1\ddot{x}_1 + c\dot{x}_1 = 0$$

Now, using the kinematics relation, i.e. $x_2 = 2x_1$, the *EOM* can be expressed in terms of either x_1 or x_2.

$$(4m_2 + m_1)\ddot{x}_1 + c\dot{x}_1 + 4kx_1 = 0$$



$$\left(2m_2 + \frac{m_1}{2}\right)\ddot{x}_2 + \frac{c}{2}\dot{x}_2 + 2kx_2 = 0$$



Example 4…

(a) Substituting the system parameters, the first *EOM* becomes

$$(4(10) + (10))\ddot{x}_1 + c\dot{x}_1 + 4(200)x_1 = 0$$

$$\Rightarrow 50\ddot{x}_1 + c\dot{x}_1 + 800x_1 = 0$$

Given that the system is to oscillate at 0.6 *Hz*, that is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow 2\pi(0.6) = \sqrt{\frac{800}{50}}\sqrt{1-\zeta^2}$$

$$\Rightarrow \zeta = 0.3343$$

Finally, using the relation for damping ratio, gives

$$\zeta = \frac{c_{eq}}{2\sqrt{k_{eq}m_{eq}}} \Rightarrow 0.3343 = \frac{c}{2\sqrt{800(50)}}$$

$$\Rightarrow c = \mathbf{133.7}\ Ns/m$$

Logarithmic Decrement

- Given the motion of a damped free vibration is

$$x = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$

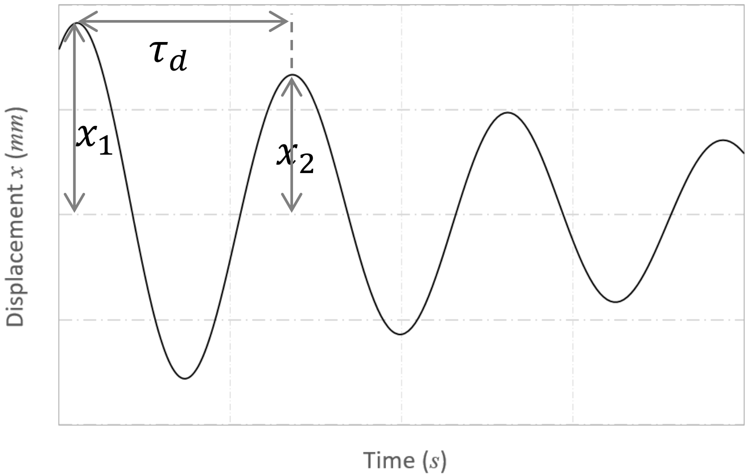
Consider two instants that are separated by a period of the oscillation, i.e. τ_d.

Their displacements are,

$$x_1 = Ae^{-\zeta\omega_n t_1} sin(\omega_d t_1 + \phi)$$

$$x_2 = Ae^{-\zeta\omega_n (t_1+\tau_d)} sin(\omega_d (t_1 + \tau_d) + \phi)$$

$$= Ae^{-\zeta\omega_n t_1} e^{-\zeta\omega_n \tau_d} sin(\omega_d t_1 + \phi)$$



Taking the ratio of the two displacements,

$$\frac{x_1}{x_2} = \frac{Ae^{-\zeta\omega_n t_1} sin(\omega_d t_1 + \phi)}{Ae^{-\zeta\omega_n t_1} . e^{-\zeta\omega_n \tau_d} sin(\omega_d t_1 + \phi)} = e^{\zeta\omega_n \tau_d} \Rightarrow \boxed{ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n \tau_d}$$

Logarithmic Decrement...

- Let $\delta = ln\left(\frac{x_1}{x_2}\right)$, which is called the *logarithmic decrement*.
- And using the definition of the damped period of oscillation, i.e.

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

We have

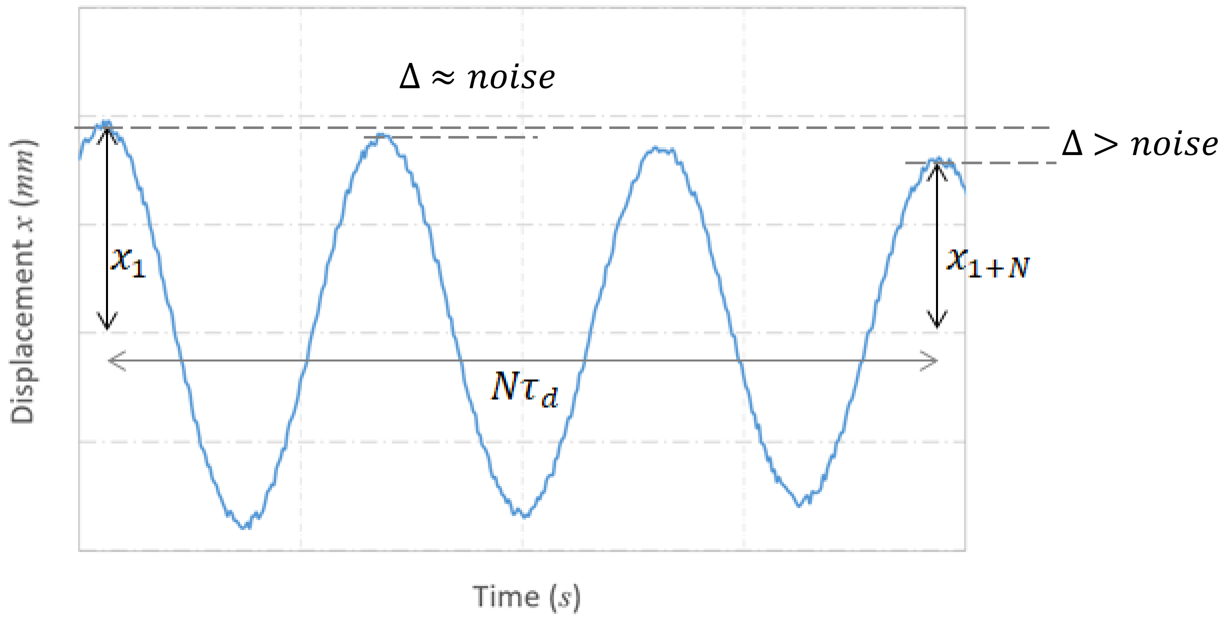
$$\delta = \zeta\omega_n\left(\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}\right)$$

$$\Rightarrow \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

- Inversing the above expression gives

$$\boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}}$$

Logarithmic Decrement…



- Using the same concept, one can also show that

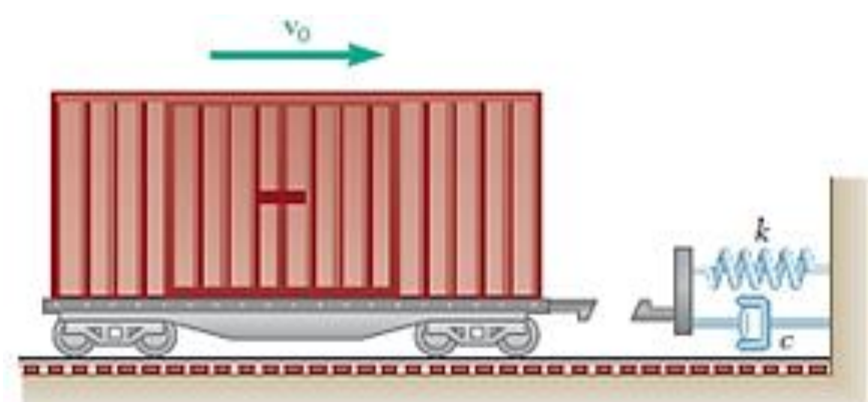
$$\boxed{\delta = \frac{1}{N} ln\left(\frac{x_1}{x_{1+N}}\right)}$$

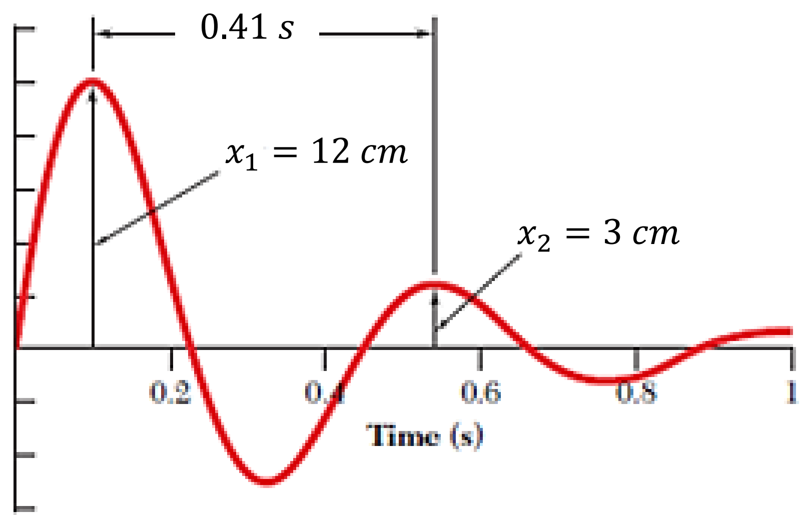
where x_1 and x_{1+N} are the vibration amplitudes at two instants that are separated by $N\tau_d$ interval.

Example 5

A loaded railroad car weighing 1,500 *kg* is rolling at a constant velocity v_0 when it couples with a spring and dashpot bumper system. The recorded displacement-time curve of the loaded railroad car after coupling is as shown. Determine the damping and spring constants used in the system.

Suppose it is required that amplitude should be reduced by more than 90 *%* after one cycle, and the time taken cannot be longer than 0.45 *second*. What is the necessary damping constant for these requirements to be satisfied?





Example 5...

Using the amplitudes $x_1 = 12\ cm$, and $x_2 = 3\ cm$, the logarithm decrement is given by

$$\delta = ln\left(\frac{12}{3}\right) = 1.386$$

With this value, the damping ratio of the system is

$$\zeta = \frac{1.386}{\sqrt{(2\pi)^2 + (1.386)^2}} = 0.2155$$

From the period of oscillation, we get

$$\tau_d = 0.41s \Rightarrow \omega_d = \frac{2\pi}{\tau_d} = 15.325\, rad/s$$

Example 5...

Next, using the damped oscillation frequency to determine the natural frequency, i.e.

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\Rightarrow \omega_n = \frac{15.325}{\sqrt{1-(0.2155)^2}} = 15.694$$

Finally, the spring constant are calculated is as

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m{\omega_n}^2$$

$$\Rightarrow k = (1500)(15.694)^2 = \mathbf{369.4}\ kN/m$$

And the damping constant is

$$\zeta = \frac{c}{2m\omega_n} \Rightarrow c = 2m\omega_n\zeta$$

$$\Rightarrow c = 2(1500)(15.694)(0.2155) = \mathbf{10,146}\ N.m/s$$

Example 5...

If the amplitude is to be reduced by 80 % in the second cycle, this implies that $x_2 = 0.1x_1$. And the corresponding logarithmic decrement is

$$\delta = ln\left(\frac{x_1}{0.1x_1}\right) = 2.303$$

which in turn gives the damping ratio to be

$$\zeta = \frac{2.303}{\sqrt{(2\pi)^2 + (2.303)^2}} = 0.3442$$

Hence to satisfy this displacement requirement, the damping constant must be greater than 0.3442, which translates to the damping constant to be

$$c > 2m\omega_n\zeta = 2(1500)(15.694)(0.3442) = 16{,}200\ N.m/s$$

Example 5...

Next for the time taken requirement, the damped oscillating frequency becomes

$$\omega_d = \frac{2\pi}{(0.45)} = 13.963\, rad/s$$

And using the damped oscillating frequency relation, we get the damping ratio for this requirement to be

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\Rightarrow \zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{13.962}{15.694}\right)^2} = 0.4567$$

which in turn gives

$$c < 2m\omega_n\zeta = 2(1500)(15.694)(0.4567) = 21{,}500\, N.m/s$$

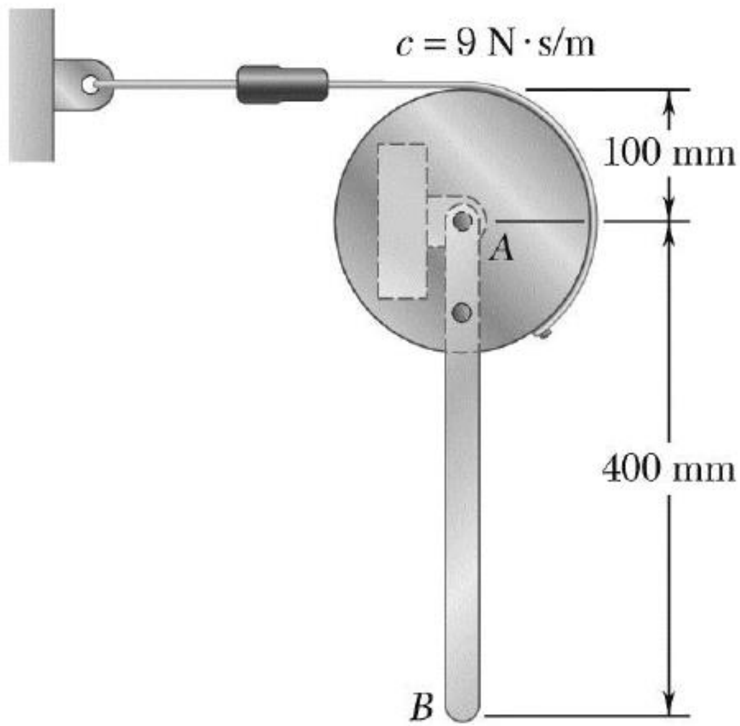
Hence, the damping constant should be chosen to be

$$16{,}200\, N.m/s < c < 21{,}500\, N.m/s$$

Question 1

A 3-*kg* slender rod *AB* is bolted to a 5-*kg* uniform disk. A dashpot of damping coefficient *c* = 9 *Ns/m* is attached to the disk as shown. Determine the damping ratio ζ of this given system.

Answers: $\zeta = 0.0431, \omega_d = 5.635\ rad/s$

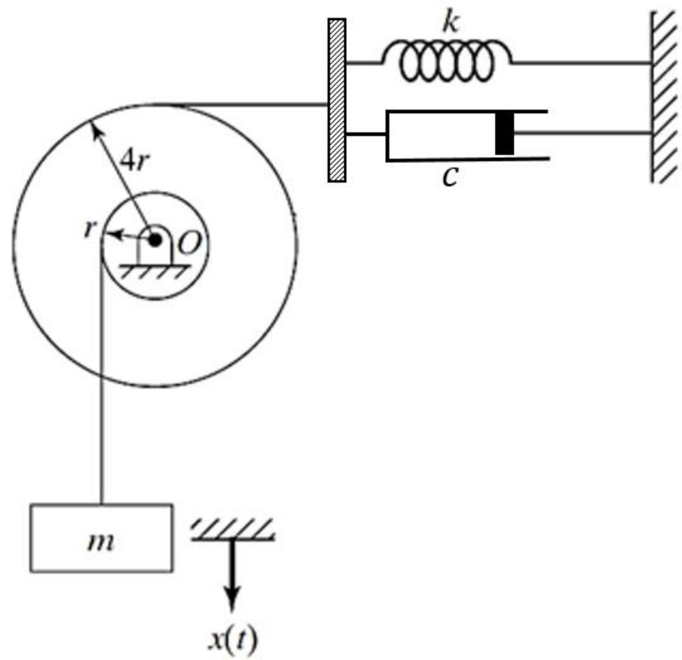


Question 2

The figure shows a double-pulley which has an inner spool of radius r, and an outer spool of radius $4r$. Its moment of inertia about O is given by J_O. It is free to rotate about a hinge at O, and cable does not slip during motion.

(a) Determine the equation of motion for the free vibration of this system for small oscillation amplitude.

(b) Determine the stiffness of the spring that can make the system a critically damped system.



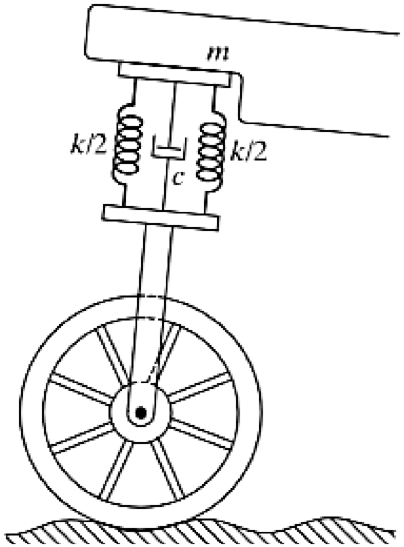
Answers:

(a) $\left(m + \frac{J_O}{r^2}\right)\ddot{\theta} + 16c\dot{\theta} + 16k\theta = 0$

(b) $k = \dfrac{4c^2}{\left(m + \frac{J_O}{r^2}\right)}$

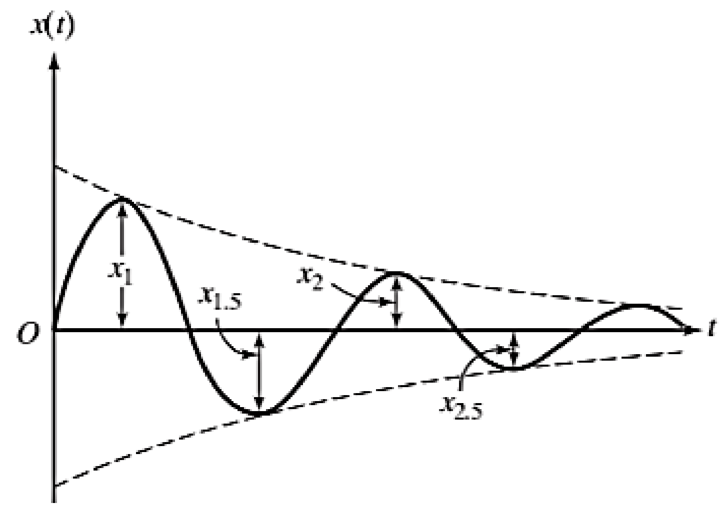
Question 3

An under-damped shock absorber is to be designed for a motorbike of mass 200 *kg*, which can be approximated as a 1 DOF spring-mass-damper system.



(a) Find the necessary stiffness of the springs, and the damping coefficient of the shock absorber if the damped period of vibration is to be 2 *s*, and the amplitude is to be reduced to 0.25 in one half cycle (i.e. $x_{1.5} = x_1/4$)

(b) Find the minimum initial velocity that will displaced the mass by *250 mm*.



Answers:

(a) $k = 2358\ N/m$, and $c = 554.5\ Ns/m$

(b) $v_o = 1.43\ m/s$