ME2115/ME2115E/TME2115 -

Mechanics of Machines

Kinematics of rigid bodies

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KINEMATICS OF RIGID BODIES

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Introduction

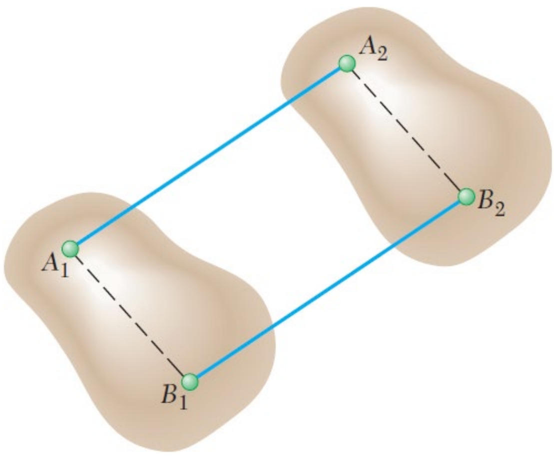
- Recall that we have studied particle dynamics. Particles refer to bodies for which only the motion as an entire unit is considered and any rotation is neglected. However, when rotation is not negligible, the bodies cannot be treated as particles ➔ **Rigid Body Dynamics.** We will learn kinematics of rigid bodies in this Chapter and kinetics in Chapter 6.
- You will learn the relations exiting between time, positions, velocities and accelerations of the various particles forming a rigid body.
- In this Chapter, we mainly focus on **plane motion** of rigid bodies - all particles of the body move in parallel planes.
- Classification of rigid-body plane motion:
  - Translation
    - Rectilinear translation
    - Curvilinear translation
  - Rotation about a fixed axis
  - General plane motion: Translation + Rotation

Translation

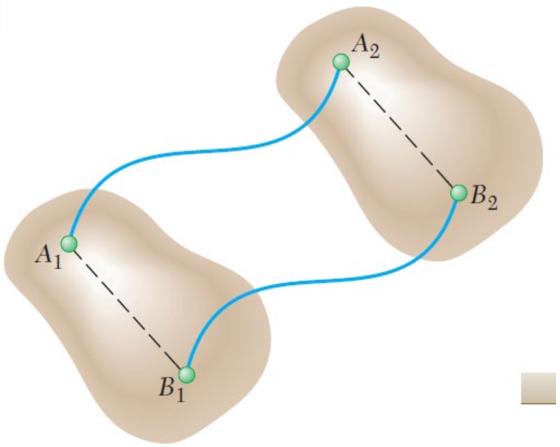
Translation: Any straight line inside the body remains parallel to its original direction or keeps the same direction during the motion, such as $A_1B_1 \parallel A_2B_2$ during the motion.

In a translation all the particles forming the body move along parallel paths. If the paths, are straight lines, such as A_1A_2, B_1B_2, as shown in Fig. a, rectilinear translation; if curved lines (Fig. b), curvilinear translation.

(a) **Rectilinear translation**



(b) **Curvilinear translation**



- Black dashed line: straight line connecting two arbitrary points
- Blue solid line: the path of an arbitrary point

Translation

Velocity and acceleration

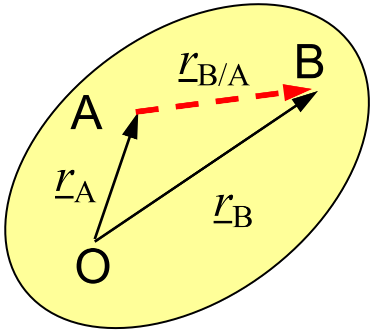
During translation, all the points of the body have the same linear velocity and acceleration at any given instant.

$\underline{v}_A = \underline{v}_B$=const. and $\underline{a}_A = \underline{a}_B$=const.

Proof:

During translation, any vector joining the two points in the body is a constant vector.

i.e., $AB = constant\ (\because rigid\ body)$



$\underline{r}_{B/A}$= constant

$\underline{r}_B = \underline{r}_A + \underline{r}_{B/A} = \underline{r}_A + constant$

by differentiating,

(Recall $v = dr/dt;\ a = dv/dt$)

$\underline{v}_B = \underline{v}_A$ and $\underline{a}_B = \underline{a}_A$

So, during translation, all the points of the body have the same **linear** velocity and acceleration at any given instant.

Rotation

(2) **Rotation about a fixed axis**: All particles move in circular paths about the axis of rotation.

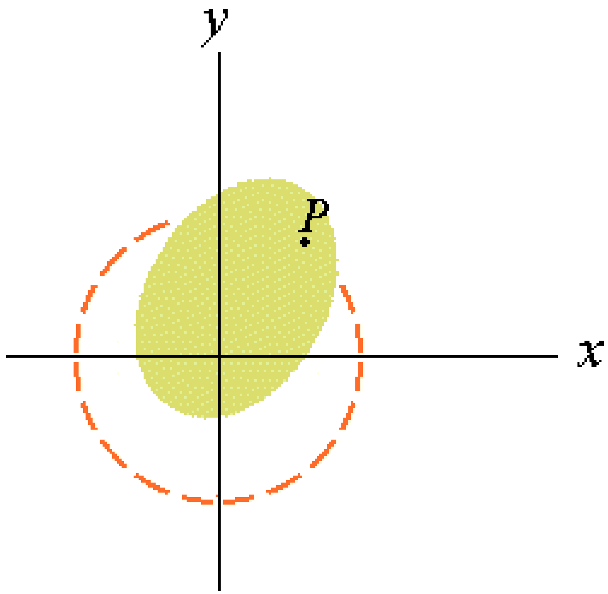
Angular motion of a rigid body in plane motion

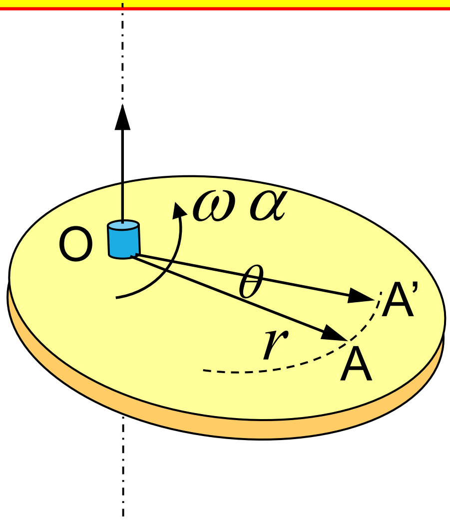
Angular displacement θ in radians

Angular velocity $\omega = d\theta/dt$ rad/sec

Angular acceleration $\alpha = d\omega/dt$ rad/sec^2

$= d^2\theta/dt^2$ rad/sec^2





c.f.

Linear displacement x in meters

Linear velocity $v = dx/dt$ m/sec

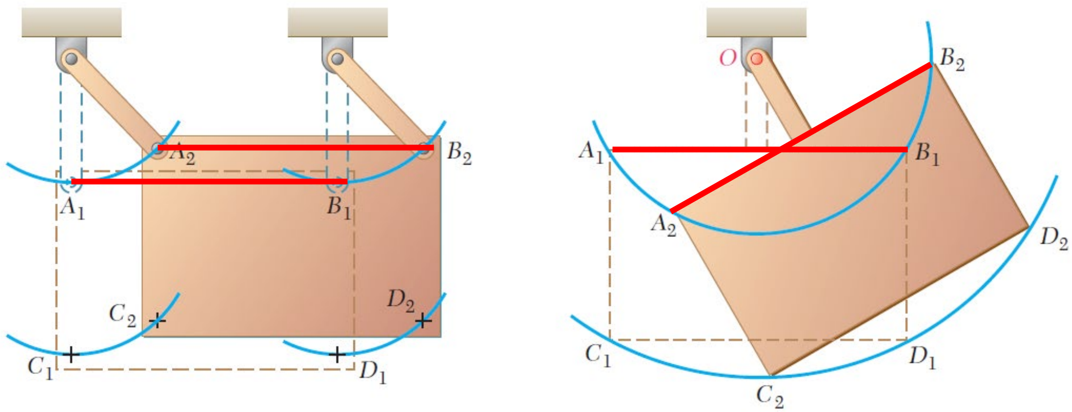
Linear acceleration $a = dv/dt$ m/sec^2

or, $= d^2x/dt^2$

*Rotation

Rotation should not be confused with certain types of curvilinear translation.

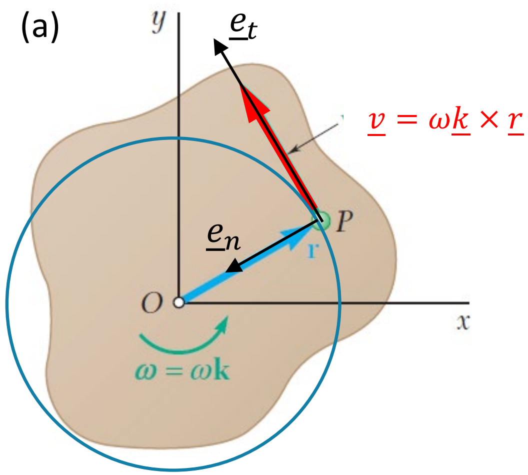
In Fig. a, all particles move along **parallel** circles; while along **concentric** circles in the rotation motion (Fig. b). Or $A_1B_1 \parallel A_2B_2$ in Fig. a; but not in Fig. b.



(*a*) Curvilinear translation (*b*) Rotation

*Rotation – Representative slab

Motion of a slab in the x-y plane perpendicular to the axis of rotation



In vector form:

Let $\underline{k}$ be the unit vector normal to the plane of rotation.

Angular Displacement: $\underline{\theta} = \theta \underline{k}$

Angular Velocity: $\underline{\omega} = \omega \underline{k} = \dot{\theta} \underline{k}$

Angular Acceleration: $\underline{\alpha} = \alpha \underline{k} = \dot{\omega} \underline{k} = \ddot{\theta} \underline{k}$

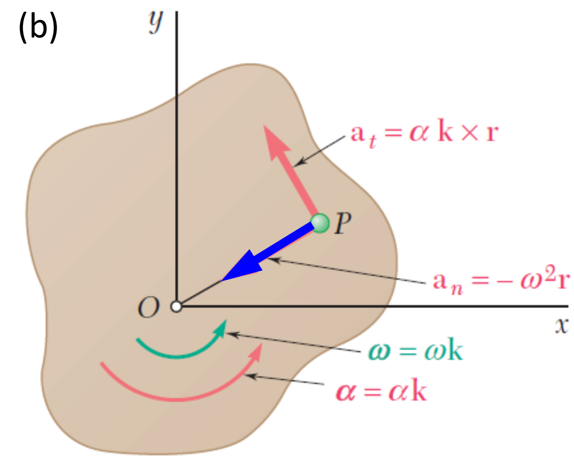
Motion of particle in rigid body rotating about a fixed axis:

Velocity: $\underline{v} = \dot{\underline{r}} = \underline{\omega} \times \underline{r} = \underline{v}_t$

Acceleration: $\underline{a} = \dot{\underline{v}} = \underline{\omega} \times \dot{\underline{r}} + \dot{\underline{\omega}} \times \underline{r}$

$$= \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \underline{\alpha} \times \underline{r}$$

$$= -\omega^2 \underline{r} + \underline{\alpha} \times \underline{r} = \underline{a}_n + \underline{a}_t$$



In scalar form:

$v = v_t = r\omega$ (in tangent only)

$a = a_n + a_t = r\omega^2 + r\alpha$ (normal and tangent)

Note: In tangent means perpendicular to $\underline{r}$; normal is along $-\underline{r}$ as shown in Fig. a.

Conceptual Quiz 1

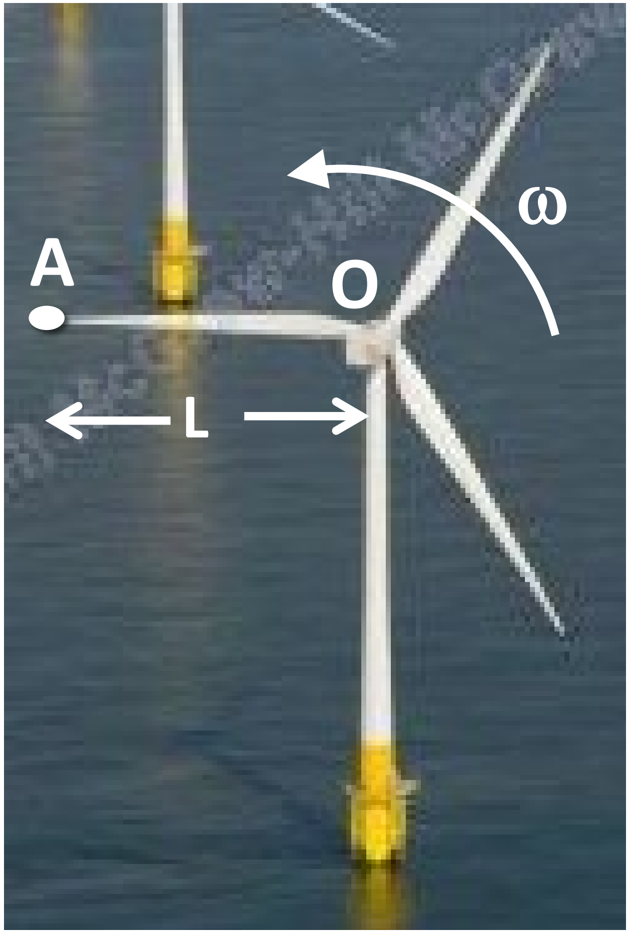
What is the direction of the velocity of point A on the turbine blade?

a) →

b) ←

c) ↑

d) ↓



Conceptual Quiz 2

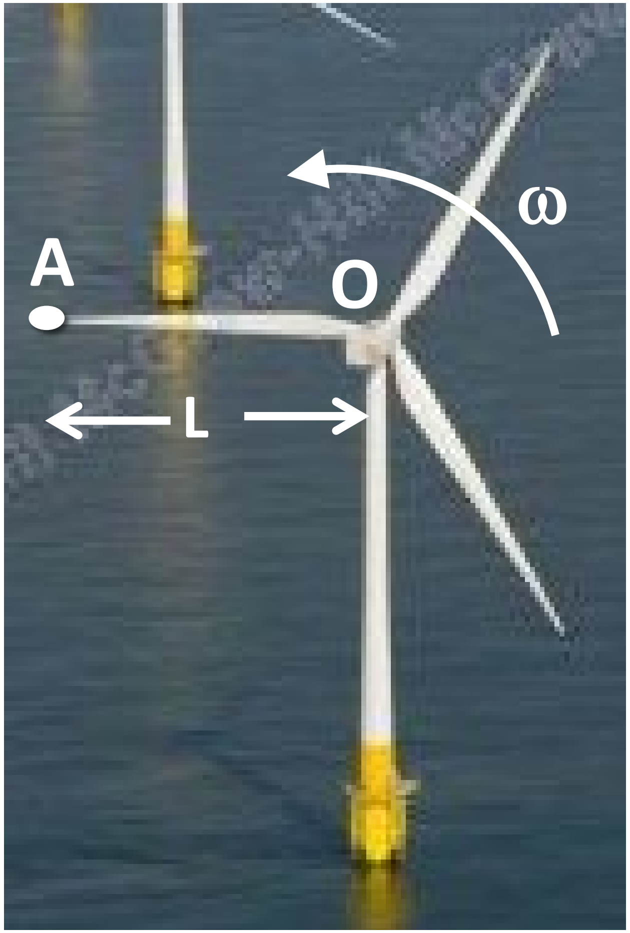
What is the direction of the normal acceleration of point A on the turbine blade?

a) →

b) ←

c) ↑

d) ↓



*Equations defining the rotation

Recall the four types of problems in rectilinear motion of particles **in L3, slide 9**. Not $x(t)$, but $a(t), a(x)$ or $a(v)$ are usually given to determine particle motion.

Similarly, motion of a rigid body rotating around a fixed axis is often specified by the type of angular acceleration $\alpha(t), \alpha(\theta)$ or $\alpha(\omega)$, not $\theta(t)$.

<u>**In dot form**</u>:

$$\boxed{\omega = \frac{d\theta}{dt} = \dot{\theta}}$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega}$$

or $$\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

or $$\alpha = \omega \frac{d\omega}{d\theta}$$

$$\left(\text{c.f.} \quad a = \dot{v} = \ddot{x} = v\frac{dv}{ds} \right)$$

<u>**Special cases**</u>:

1) Uniform rotation ($\omega = const.;\ \alpha = 0$)

Angular displacement: $\boxed{\theta = \theta_0 + \omega t}$

2) Uniformly accelerated rotation ($\alpha = const.$)

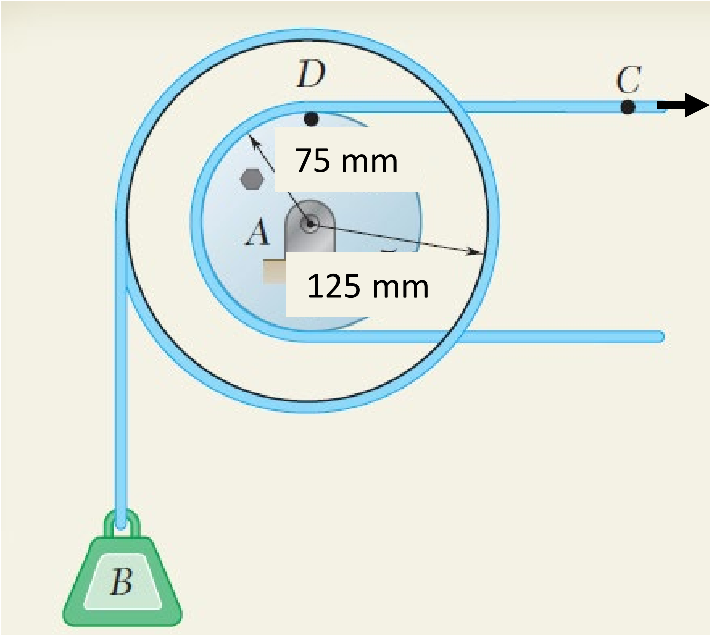
Angular displacement: $\boxed{\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2}$

Angular velocity: $\boxed{\omega = \omega_0 + \alpha t}$

$$\boxed{\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)}$$

(c.f. **L3, slide 12**)

Example 4.1



Load *B* is connected to a double pulley by one of the two inextensible cables shown. The motion of the pulley is controlled by cable *C*, which has a constant acceleration of 225 mm/s^2 and an initial velocity of 300 mm/s, both directed to the right. Determine (*a*) the number of revolutions executed by the pulley in 2 s, (*b*) the velocity and change in position of the load *B* after 2 s, and (*c*) the acceleration of point *D* on the rim of the inner pulley at $t = 0$.

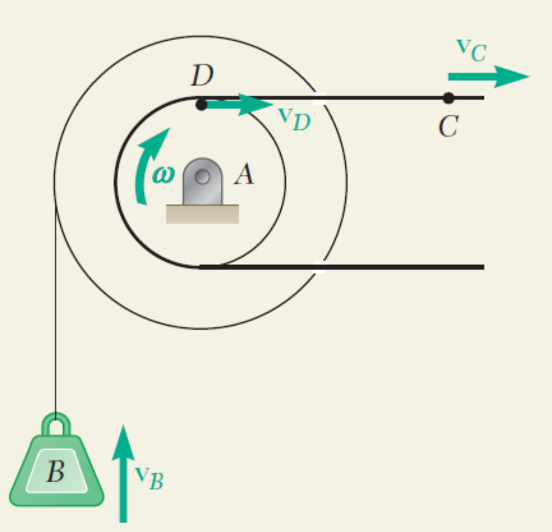
<u>SOLUTION</u>:

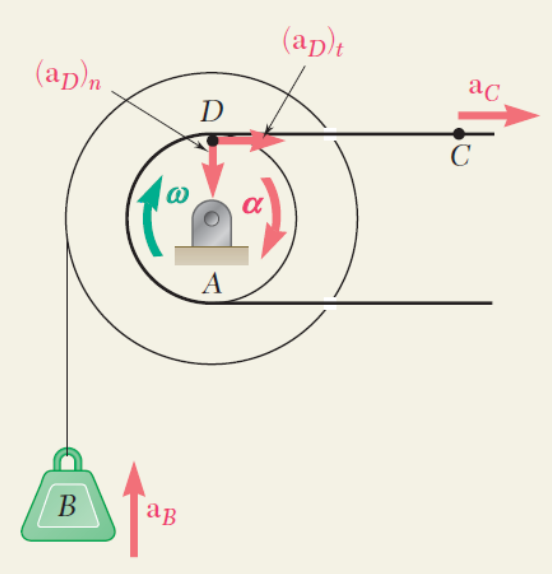
1) Due to the action of the cable, the tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*. Calculate the initial angular velocity and acceleration.

2) Apply the relations for uniformly accelerated rotation to determine the velocity and angular position of the pulley after 2 s.

3) Evaluate the initial tangential and normal acceleration components of *D*.

Solution





a. Motion of Pulley. Since the cable is inextensible, the velocity of point D is equal to the velocity of point C and the tangential component of the acceleration of D is equal to the acceleration of C.

$$(\mathbf{v}_D)_0 = (\mathbf{v}_C)_0 = 300 \text{ mm/s} \rightarrow \qquad (\mathbf{a}_D)_t = \mathbf{a}_C = 225 \text{ mm/s}^2 \rightarrow$$

Noting that the distance from D to the center of the pulley is 75 mm, we write

$$(v_D)_0 = r\omega_0 \qquad 300 \text{ mm/s} = (75 \text{ mm})\omega_0 \qquad \boldsymbol{\omega}_0 = 4 \text{ rad/s} \curvearrowright$$
$$(a_D)_t = r\alpha \qquad 225 \text{ mm/s}^2 = (75 \text{ mm})\alpha \qquad \boldsymbol{\alpha} = 3 \text{ rad/s}^2 \curvearrowright$$

Using the equations of uniformly accelerated motion, we obtain, for $t = 2$ s,

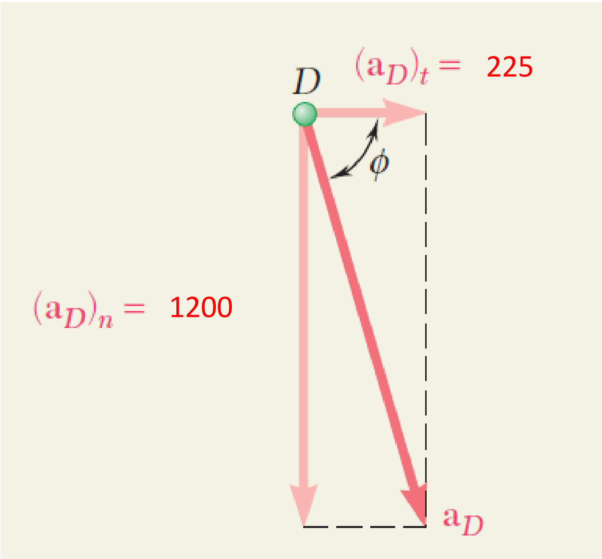
$$\omega = \omega_0 + \alpha t = 4 \text{ rad/s} + (3 \text{ rad/s}^2)(2 \text{ s}) = 10 \text{ rad/s}$$
$$\boldsymbol{\omega} = 10 \text{ rad/s} \curvearrowright$$
$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (4 \text{ rad/s})(2 \text{ s}) + \tfrac{1}{2}(3 \text{ rad/s}^2)(2 \text{ s})^2 = 14 \text{ rad}$$
$$\theta = 14 \text{ rad} \curvearrowright$$

$$\text{Number of revolutions} = (14 \text{ rad})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 2.23 \text{ rev} \quad \blacktriangleleft$$

b. Motion of Load B. Using the following relations between linear and angular motion, with $r = 125$ mm, we write

$$v_B = r\omega = (125 \text{ mm})(10 \text{ rad/s}) = 1250 \text{ mm/s} \qquad \mathbf{v}_B = 1.25 \text{ m/s} \uparrow \quad \blacktriangleleft$$
$$\Delta y_B = r\theta = (125 \text{ mm})(14 \text{ rad}) = 1750 \text{ mm} \quad \Delta y_B = 1.75 \text{ m upward} \quad \blacktriangleleft$$

Solution



c. Acceleration of Point *D* at *t* = 0. The tangential component of the acceleration is

$$(\mathbf{a}_D)_t = \mathbf{a}_C = 225 \text{ mm/s}^2 \rightarrow$$

Since, at $t = 0$, $\omega_0 = 4$ rad/s, the normal component of the acceleration is

$$(a_D)_n = r_D\omega_0^2 = (75 \text{ mm})(4 \text{ rad/s})^2 = 1200 \text{ mm/s}^2 \qquad (\mathbf{a}_D)_n = 1200 \text{ mm/s}^2 \downarrow$$

The magnitude and direction of the total acceleration can be obtained by writing

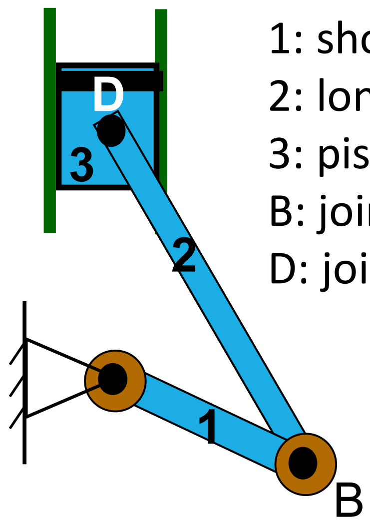
$$\tan\phi = (1200 \text{ mm/s}^2)/(225 \text{ mm/s}^2) \qquad \phi = 79.4°$$

$$a_D \sin 79.4° = 1200 \text{ mm/s}^2 \qquad a_D = 1220 \text{ mm/s}^2$$

$$\mathbf{a}_D = 1.22 \text{ m/s}^2 \measuredangle 79.4° \quad \blacktriangleleft$$

*Mechanism consisting of several moving parts

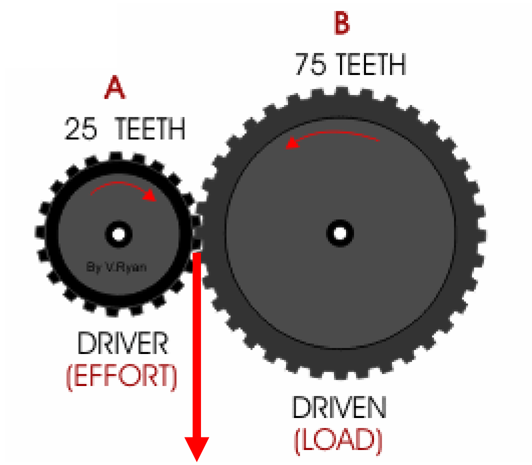
- Treat each part as a rigid body (each has its own ω and a)
- **Pin-connected joints:** bodies have same absolute velocity and acceleration at joints
- **Gear tooth or contact surfaces without slip**: same absolute velocity and tangential acceleration, but may have different normal accelerations



1: short bar
2: long bar
3: piston
B: joint of 1 & 2
D: joint of 2 & 3

$$v_{D2} = v_{D3} \text{ and } a_{D2} = a_{D3}$$

$$v_{B2} = v_{B1} \text{ and } a_{B2} = a_{B1}$$



$v_A = v_B; (a_A)_t = (a_B)_t$
at contact point $\boldsymbol{E}$

$r_A \neq r_B; \omega_A \neq \omega_B$

The relation depends on the number of teeth.

$(a_A)_n \neq (a_A)_n$ **Why?** $\boldsymbol{a_n = \frac{v^2}{r}}$

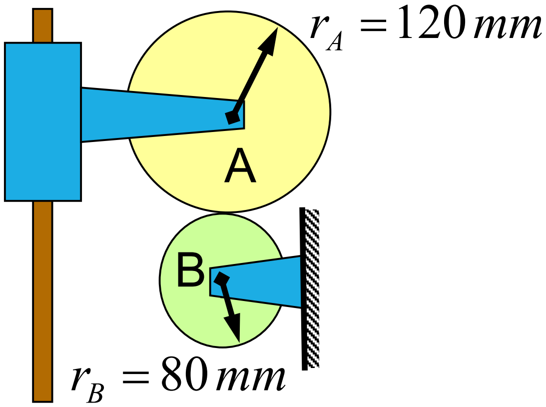
Example 4.2

Initially A and B are separated. $\omega_A = 0, \omega_B = 500 rpm$

B will coast to a rest in 60 s (without contact)

At $t = 0$, disk A is started with $\alpha_A = 3 rad/s^2$.

Find (a) time when disks may be brought together without slipping and (b) ω_A and ω_B as contact is made.



Solution :

Study A & B seperatelly

Disk A : $\alpha_A = 3 \quad (\omega_A)_0 = 0 \quad \therefore \boxed{\omega_A = (\omega_A)_o + \alpha_A t = 3t}$

Disk B : $(\omega_B)_0 = 500 rpm = 52.36 rad/s$

At $t = 60$, $\boxed{(\omega_B)_{60} = 0 = (\omega_B)_0 + \alpha_B t \Rightarrow \alpha_B = -0.873}$

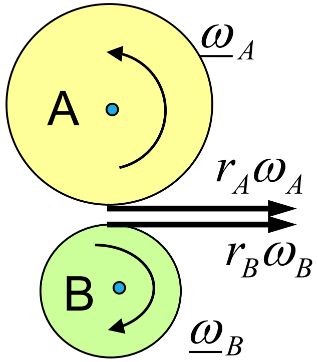
At time t, $\quad \omega_B = 52.36 - 0.873t$

Uniformly accelerated rotation:

$$\boxed{\omega = \omega_0 + \alpha t}$$

c.f. Uniformly accelerated rectilinear motion:

$$v = v_0 + at$$



For no slipping as disks touch, edge speeds must be equal, i.e. $v_A = v_B$ at contact point.

Therefore $r_A\omega_A = r_B\omega_B \qquad .....(1)$

(a) Bring disks together : Eq. (1) $\Rightarrow r_A\omega_A = r_B\omega_B$

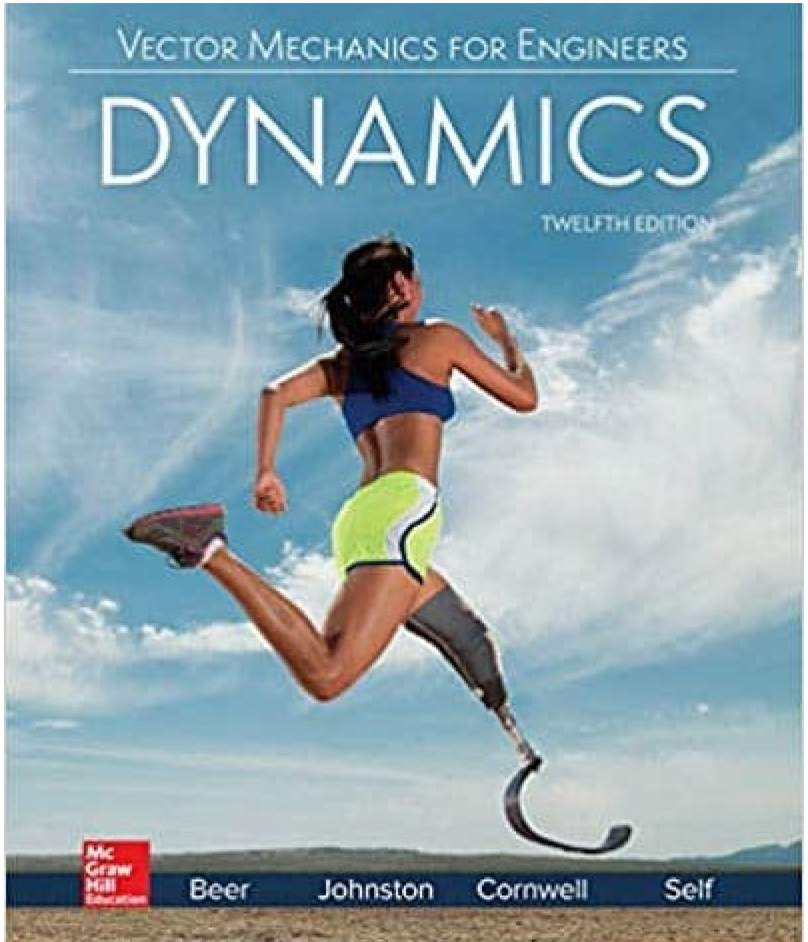
$$\therefore 0.120 \times 3t = 0.080 \times (52.36 - 0.873t) \Rightarrow t = 9.75s$$

(b) At $t = 9.75s$: $\omega_A = 3t = 29.25 rad/s$

$$\omega_B = 52.36 - 0.873t = 43.85 rad/s$$

General Plane Motion=Translation+Rotation

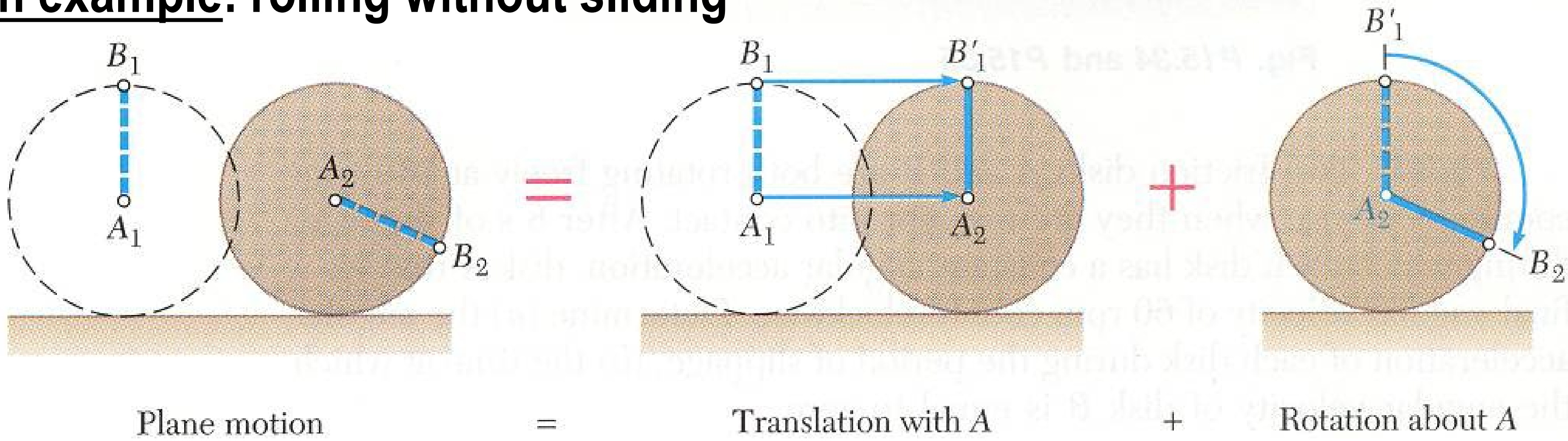
= **Translation** with respect to a reference point + **Rotation** about that reference point



The knee has linear velocity and acceleration from both translation (the runner moving forward) as well as rotation (the leg rotating about the hip).

General Plane Motion

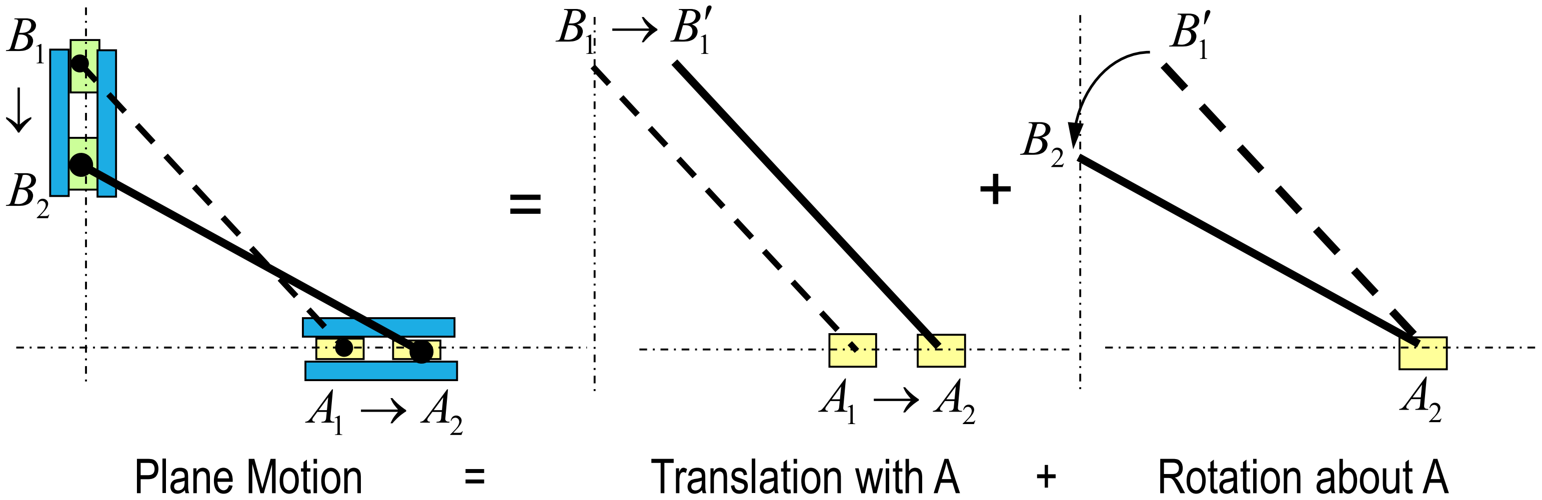
<u>An example</u>: rolling without sliding



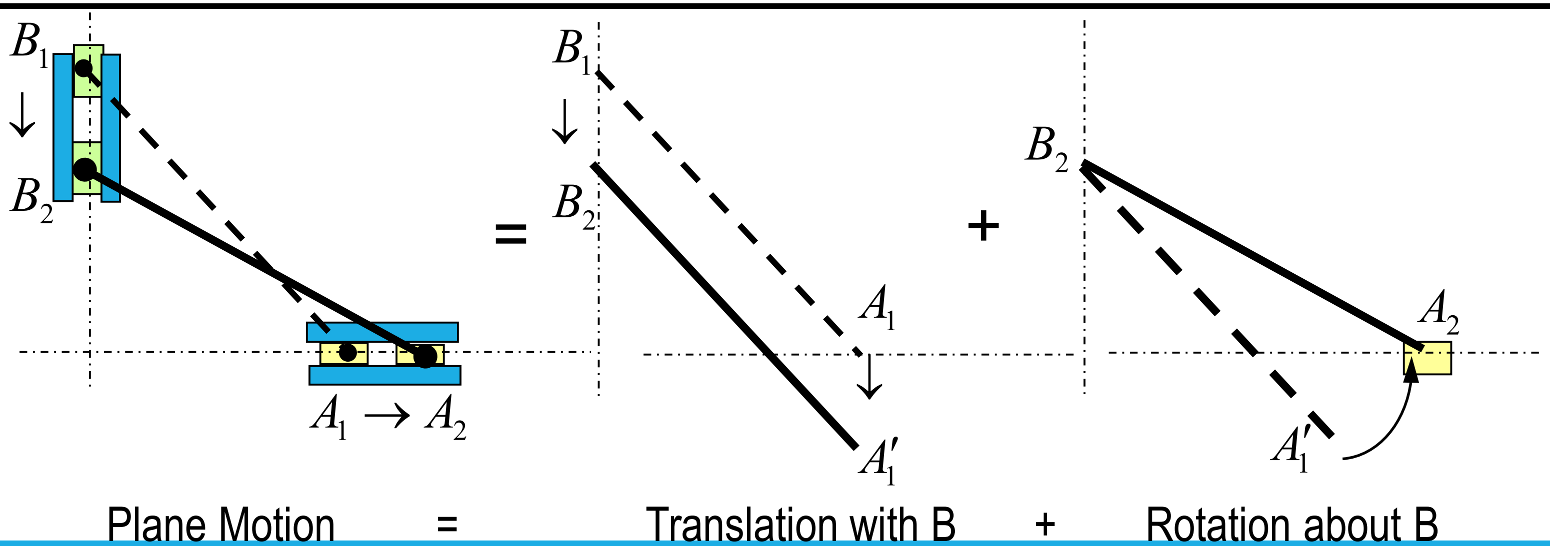
- *General plane motion* is neither a translation nor a rotation.
- General plane motion can be considered as the *sum* of a translation and rotation.
- Displacement of particles A_1 and B_1 into A_2 and B_2 can be divided into two parts:
  - Translation from A_1 to A_2 and B_1 to B'_1
  - Rotation of B'_1 about A_2 into B_2

General Plane Motion

Another example: constrained rod



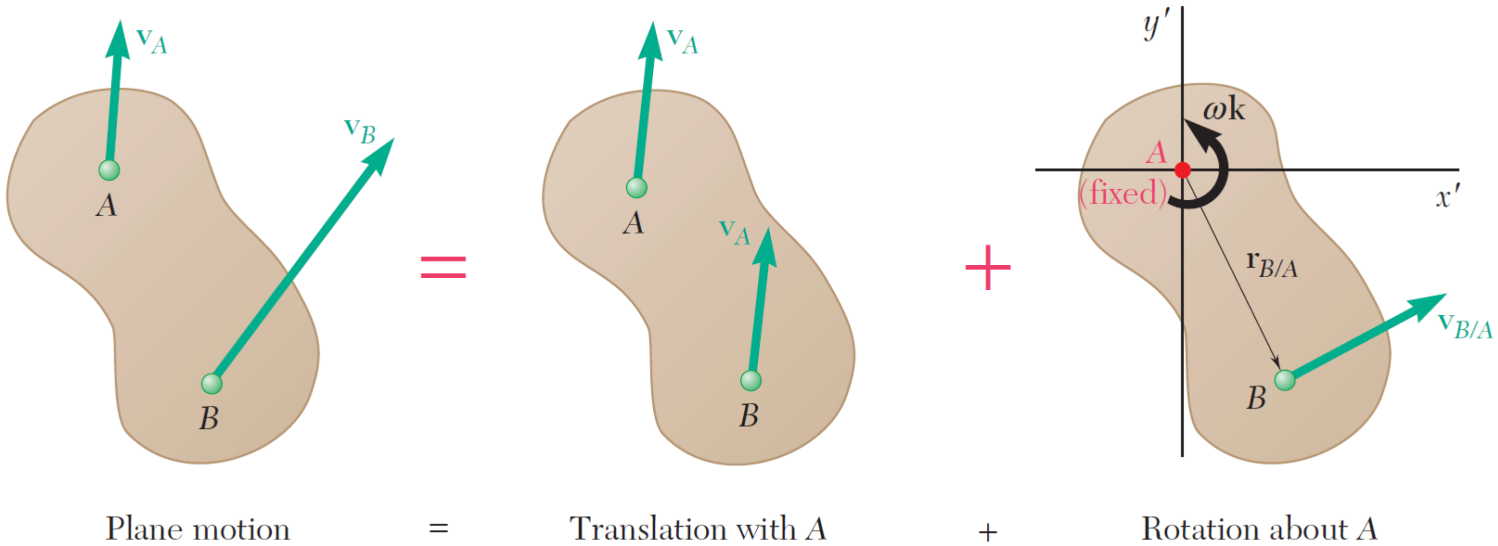
Point ***A*** is taken as the reference point

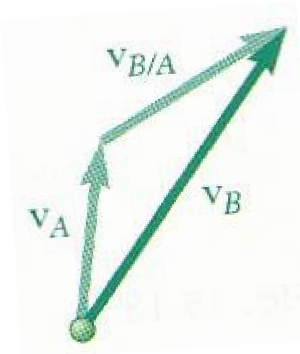


Point ***B*** is taken as the reference point

*Absolute and Relative Velocity

Translation Rotation





Vector triangle

$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}$

- Any plane motion can be replaced by a translation of an arbitrary reference point A and a simultaneous rotation about A.

$$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}$$

where, $\underline{v}_{B/A} = \omega\underline{k} \times \underline{r}_{B/A}$

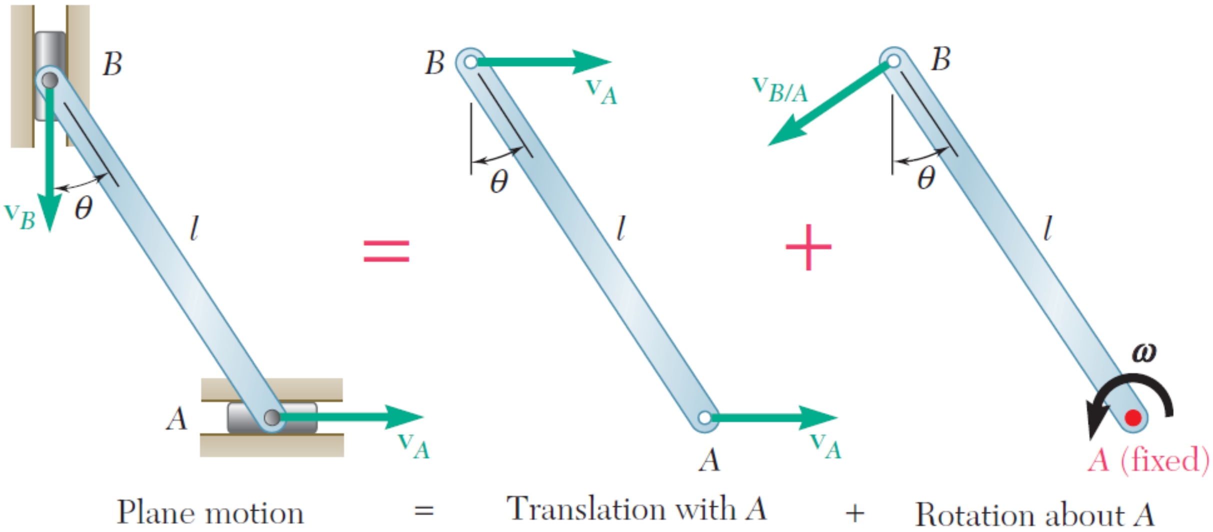
Magnitude: $v_{B/A} = r\omega$

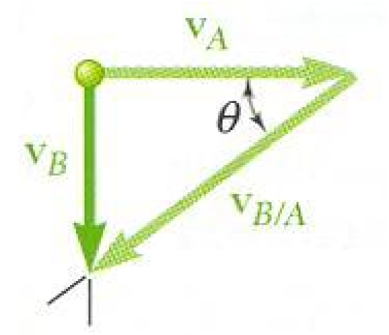
$\underline{v}_A$ is the translational component of $\underline{v}_B$

$\underline{v}_{B/A}$ is the rotational component of $\underline{v}_B$

Example 4.3

A constrained rod
Given v_A, l, and θ
Find v_B and ω_{AB}





Vector triangle
$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}$

- Assuming that the velocity v_A of end *A* is known, wish to determine the velocity v_B of end *B*, and the angular velocity ω in terms of v_A, l, and θ.
- The direction of v_B and $v_{B/A}$ are known. Complete the velocity diagram.

From the vector triangle above:

$$\frac{v_B}{v_A} = \tan\theta \qquad \frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

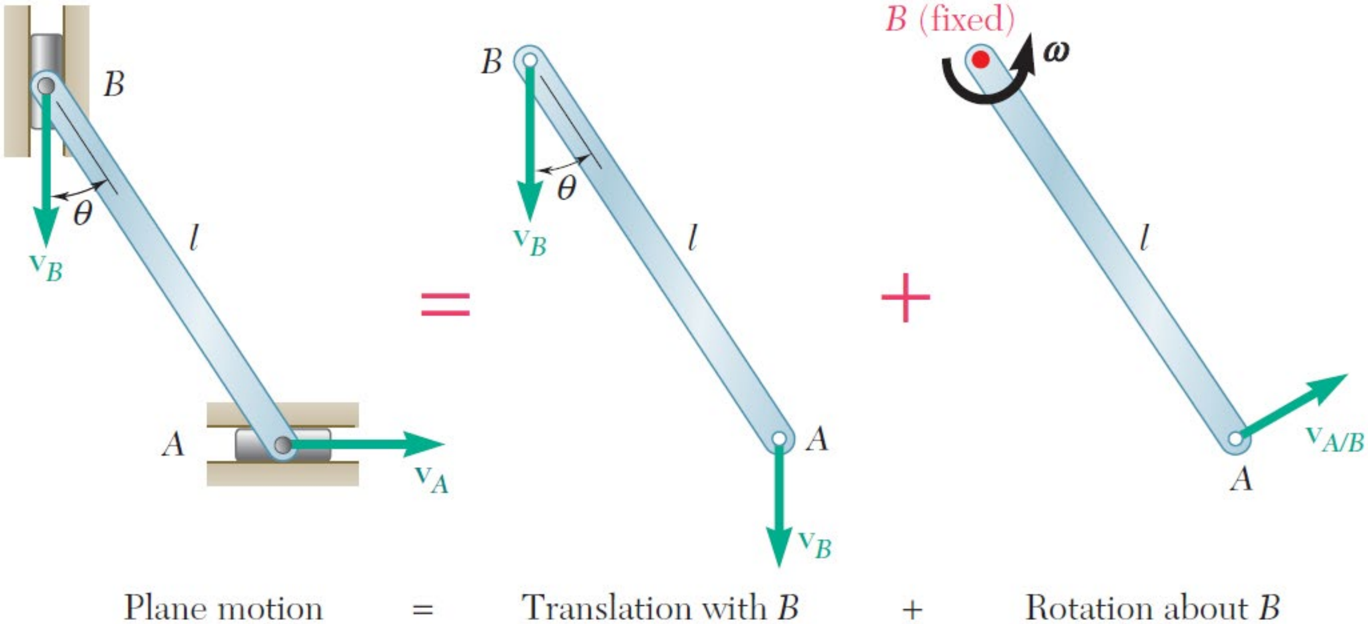
$$v_B = v_A \tan\theta \downarrow$$

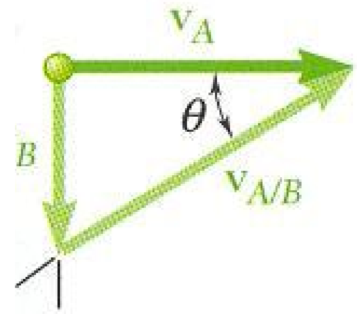
$$\omega = \frac{v_A}{l\cos\theta}$$

Example 4.3

Given v_B
Find v_A

Now, ***B*** becomes the reference point!



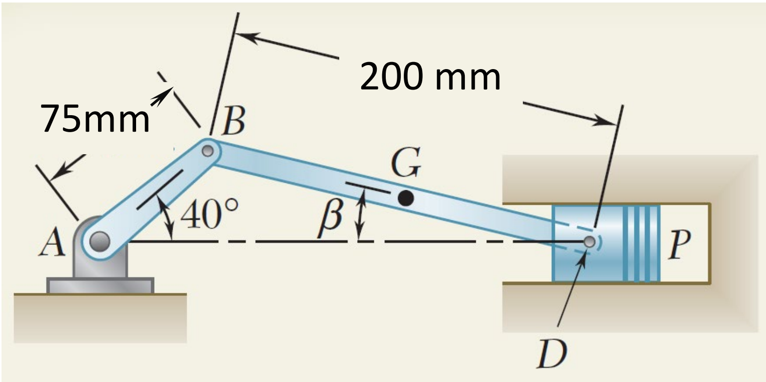


Vector triangle
$\underline{v}_A = \underline{v}_B + \underline{v}_{A/B}$

- Selecting point *B* as the reference point and solving for the velocity v_A of end *A* and the angular velocity ω leads to an equivalent velocity triangle.
- $v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point.

Evaluate: Angular velocity ω of the rod in its rotation about *B* is the *same* as its rotation about *A*. Angular velocity is not dependent on the choice of reference point.

Example 4.4



In the engine system shown, the crank *AB* has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

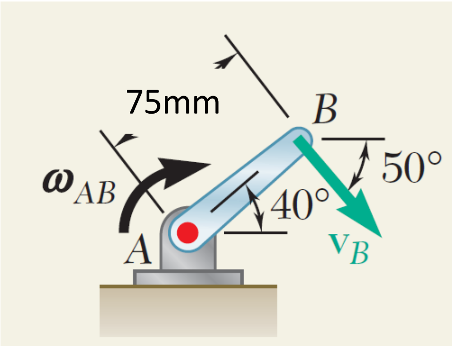
SOLUTION:

- Will determine the absolute velocity of point *D* with

$$\underline{v}_D = \underline{v}_B + \underline{v}_{D/B}$$

- The velocity $\underline{v}_B$ is obtained from the given crank rotation data.
- The directions of the absolute velocity $\underline{v}_D$ and the relative velocity $\underline{v}_{D/B}$ are determined from the problem geometry.
- The unknowns in the vector expression are the velocity magnitudes $\underline{v}_D and\ \underline{v}_{D/B}$ which may be determined from the corresponding vector triangle.
- The angular velocity of the connecting rod is calculated from $\underline{v}_{D/B}$

Solution



Motion of crank AB:

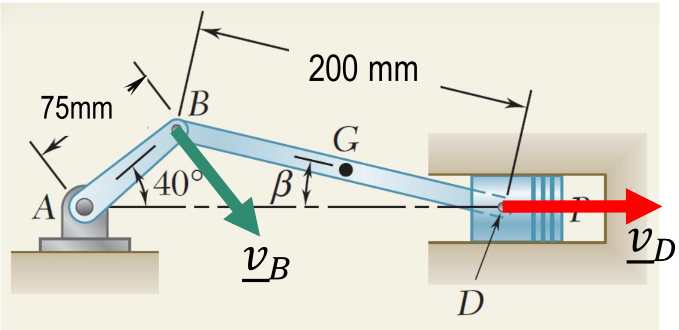
- The crank *AB* purely rotates about point *A*. Will determine the absolute velocity of point *D* with $\underline{v}_D = \underline{v}_B + \underline{v}_{D/B}$

- The velocity $\underline{v}_B$ is obtained from the crank rotation data.

$$\boxed{\begin{aligned}\omega_{AB} &= \left(2000\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 209.4\ \text{rad/s} \\ v_B &= (AB)\omega_{AB} = (0.075\ \text{m})(209.4\ \text{rad/s}) = 15.705 m/s \quad 50^\circ\end{aligned}}$$

The $\underline{v}_B$ direction is in tangent, i.e., perpendicular to *AB* as shown.

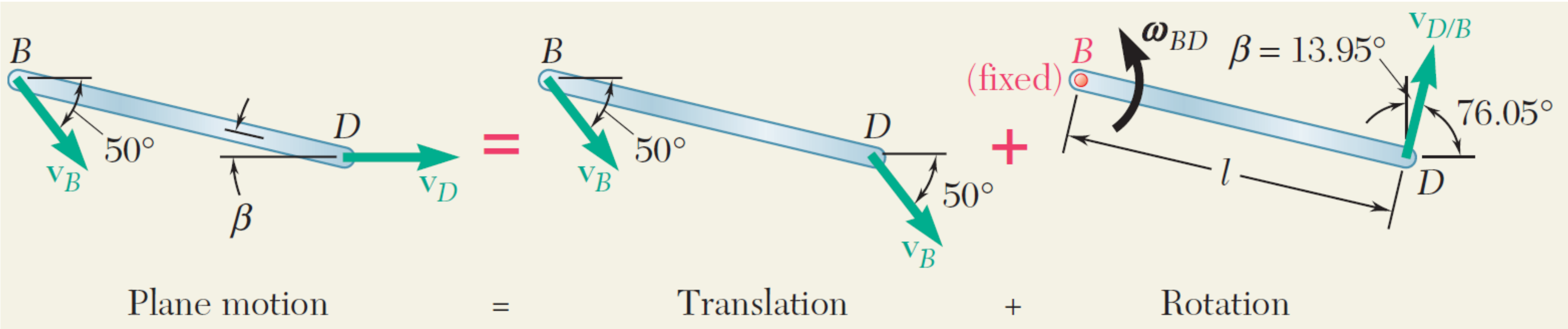
Motion of connecting rod BD:

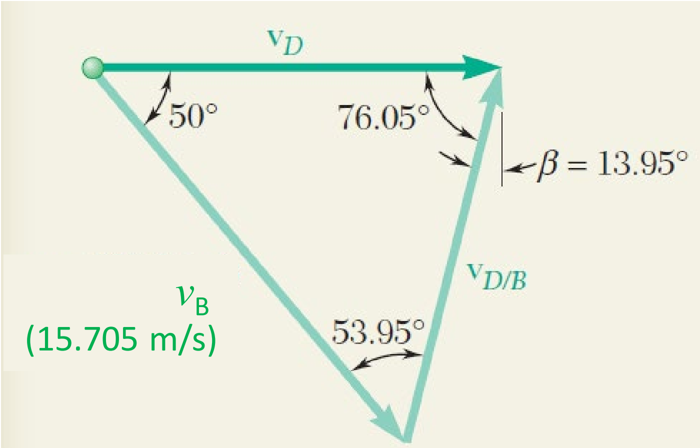


- The direction of the absolute velocity $\underline{v}_D$ is horizontal. The direction of the relative velocity $\underline{v}_{D/B}$ is perpendicular to *BD*. Compute the angle between the horizontal and the connecting rod from the **law of sines** in the triangle ABD.

$$\frac{\sin 40^\circ}{200\ \text{mm}} = \frac{\sin\beta}{75\ \text{mm}} \qquad \boxed{\beta = 13.95^\circ}$$

Solution





Vector triangle

$$\underline{v}_D = \underline{v}_B + \underline{v}_{D/B}$$

- Determine the velocity magnitudes v_D and $v_{D/B}$ from the vector triangle with the law of sines.

$$\frac{v_D}{\sin 53.95°} = \frac{v_{D/B}}{\sin 50°} = \frac{15.705 m/s}{\mathbf{\sin 76.05°}}$$

$$v_D = 13.08\ m/s$$

$$\boxed{v_P = v_D = 13.08\ m/s} \longrightarrow$$

$$v_{D/B} = 12.40\ m/s \measuredangle 76.05°$$

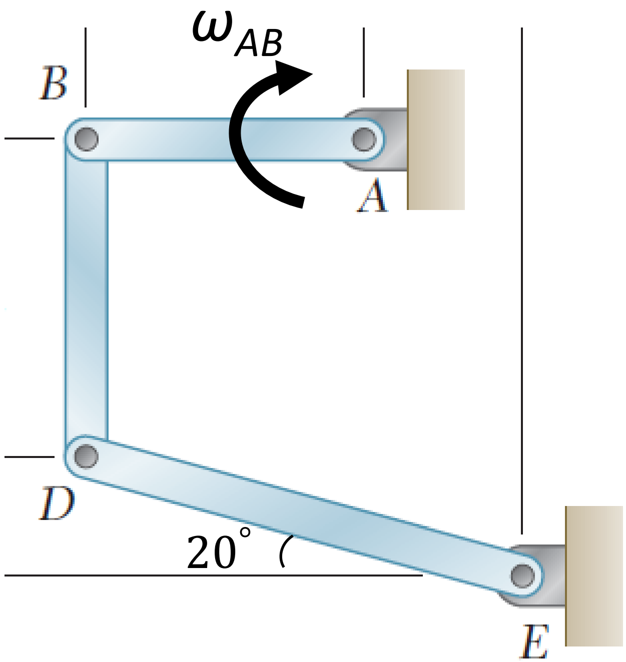
$$v_{D/B} = l\omega_{BD}$$

$$\omega_{BD} = \frac{v_{D/B}}{l} = \frac{12.40\ m/s}{0.20\ m}$$

$$= 62.0\ \mathrm{rad/s}$$

$$\boxed{\omega_{BD} = 62.0\ rad/s}$$

Conceptual quiz 3



In the position shown, bar *AB* has an angular velocity of ω_{AB} clockwise. Determine the direction of $\underline{v}_B, \underline{v}_D, \underline{v}_{D/B}$

Tips:

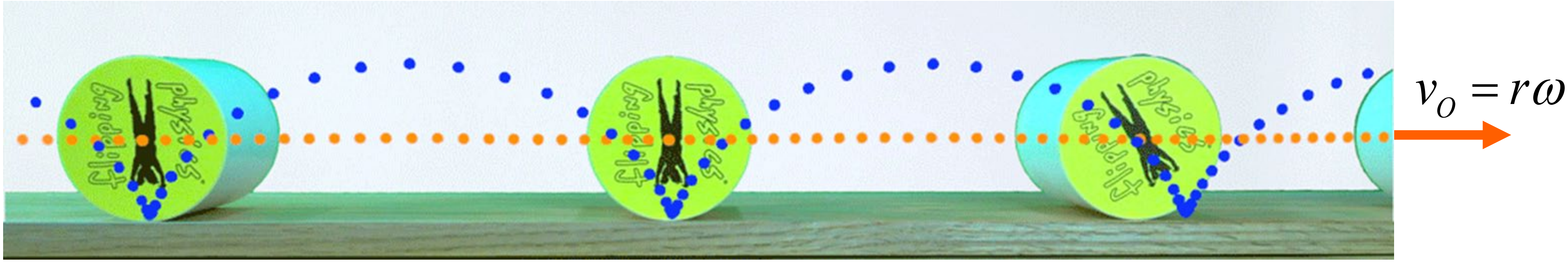
1) Isolate each bar for analysis
2) Direction of velocity in rotational motion is always in tangent.
3) Using vector triangle to find the sign of $\underline{v}_{D/B}$

Answer:

$$\underline{v}_D \;\angle 70^\circ = \underline{v}_B \uparrow + \underline{v}_{D/B} \rightarrow$$

*General Plane Motion

A special case: Rolling without sliding

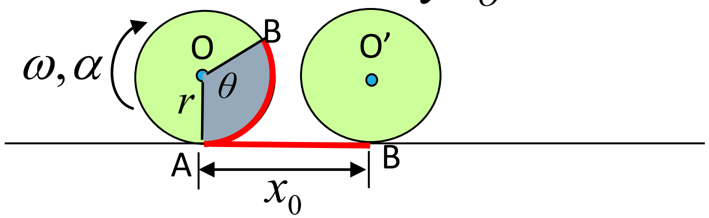


A curve traced the point of centre of geometry is a straight line, and the rectilinear velocity of *O* is

$$\boxed{v_O = r\omega}$$

For rolling without sliding, the geometric centre O moves at a velocity $v_O = \omega r$

Proof :

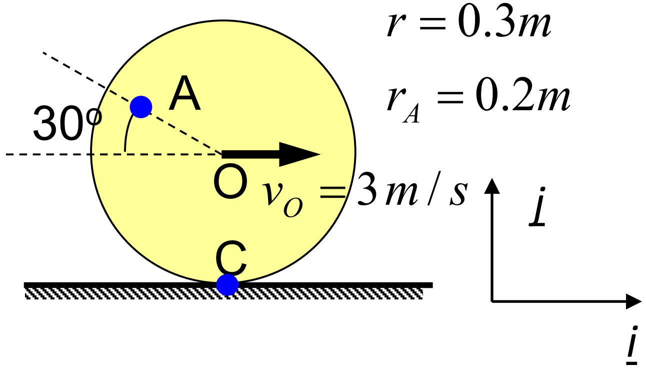


$$x_O = arcAB = \theta r \rightarrow$$

$$\frac{d}{dt}: \quad v_O = \dot{\theta} r = \omega r \rightarrow$$

$$\frac{d}{dt}: \quad a_O = \dot{\omega} r = \alpha r \rightarrow$$

Example 4.5



$r = 0.3m$

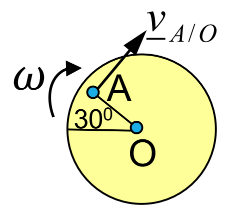
$r_A = 0.2m$

Wheel rolls without sliding to the right.

Find $\underline{v}_A$ and $\underline{v}_C$.

Solution

$v_O = r\omega \quad \Rightarrow \quad \omega = \frac{v_O}{r} = \frac{3}{0.3} = 10\ rad/s$ ↻

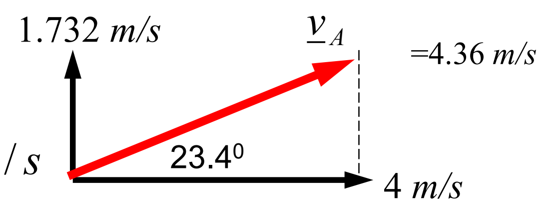


Find $\underline{v}_A$: $\quad \underline{v}_A = \underline{v}_O + \underline{v}_{A/O}$

where $\quad \underline{v}_{A/O} = \omega r_A \quad \angle 60^0$

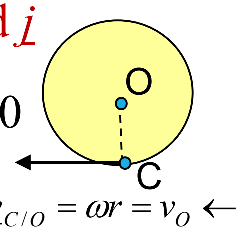
Therefore $\quad \underline{v}_A = 3 \rightarrow + \left(\frac{3}{0.3}\right) \times 0.2 \ \angle 60^0 = 3 \rightarrow + \ (1 \rightarrow + 1.732 \uparrow) = 4 \rightarrow + 1.732 \uparrow m/s$

Resolve into $\underline{i}$ and $\underline{j}$



v_A=4.36 m/s $\angle 23.4^0$

Find $\underline{v}_C$: $\quad \underline{v}_C = \underline{v}_O + \underline{v}_{C/O} = \underline{v}_O \rightarrow + \ \underline{v}_O \leftarrow \ = 0$

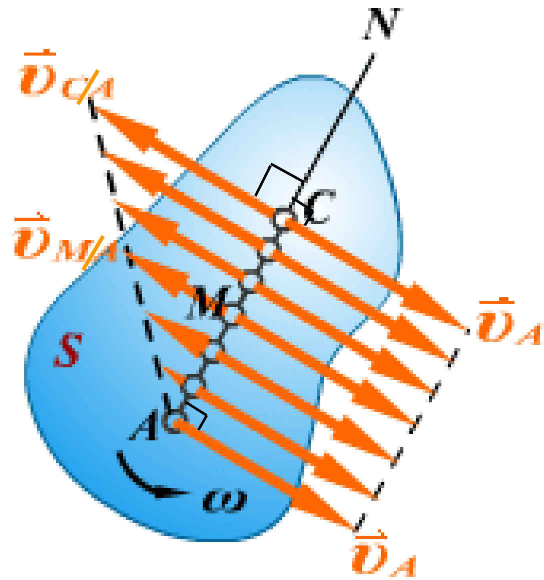


Evaluation: Interesting phenomenon – the absolute velocity of the non-fixed contact point, *C*, is always zero at **any instant** !

*Instantaneous centre of rotation (I.C.R)

Consider the general plane motion of slab shown.
Its angular velocity ω and linear velocity of a point A are known.

AN normal to v_A



- Draw a normal AN to the line of v_A
- M is an arbitrary point on AN.
- v_M can be determined using the reference A as:

$$\vec{v}_M = \vec{v}_A + \vec{v}_{M/A}$$

where, $v_{M/A} = \omega r_{AM}$, the same direction as v_A, but opposite in sense

- By gradually increasing the distance along AN, we can have a particular point C, which fulfils

$$v_{C/A} = \omega r_{AC} = v_A$$

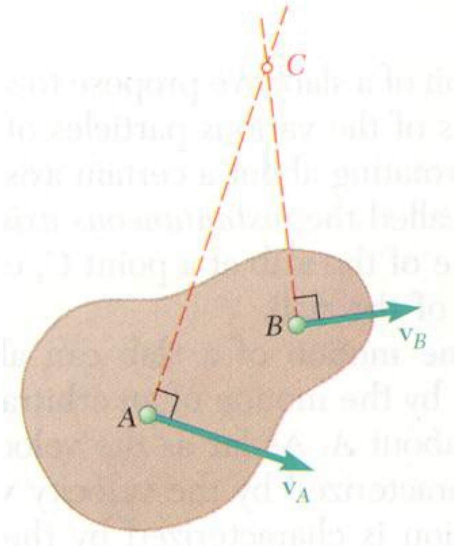
Then, $v_C = v_A - \omega r_{AC} = 0$

- As far as the velocities are concerned, the slab seems to rotate about the *instantaneous center of rotation C*.

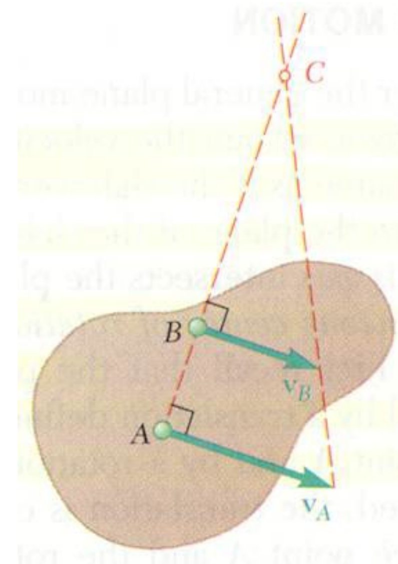
Every points have the same translational velocity, $\vec{v}_A$.

C is called instantaneous centre of rotation. It's velocity is zero just at that instant.

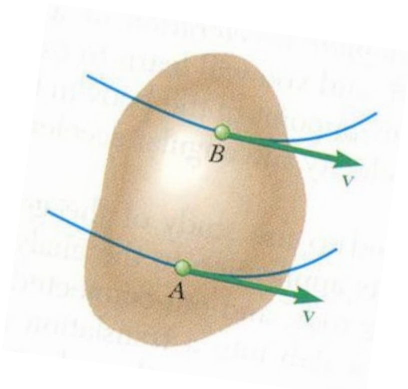
*Instantaneous centre of rotation (I.C.R)



- If the velocity at two points A and B are known, the instantaneous centre of rotation lies at the intersection of the perpendiculars to the velocity vectors through A and B .



- If the velocity vectors at A and B are **parallel**, the I.C.R lies at the intersection of the line AB with the line joining the extremities of the velocity vectors at A and B.

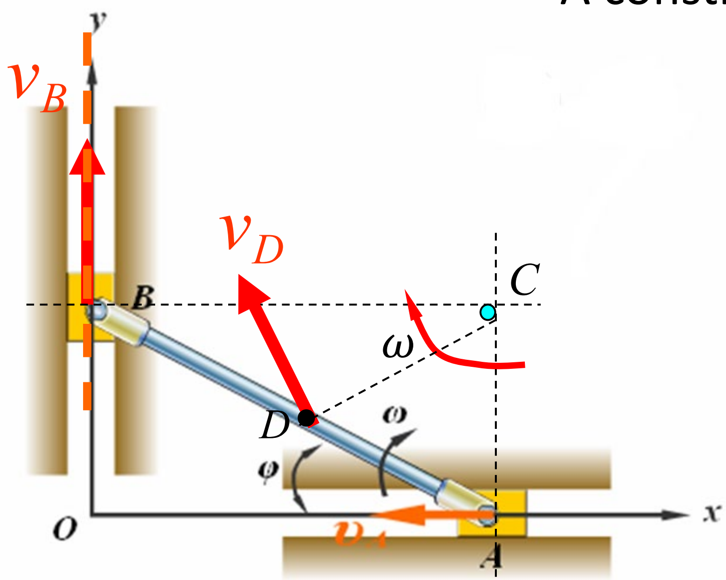


- If the velocity magnitudes are **equal**, the I.C.R is at infinity and the angular velocity is zero. It is actually a translational motion! $\omega = 0$ and $\alpha = 0$

<u>Caution:</u> In general, the acceleration at C may not be zero. The acceleration of any point cannot be determined as if the slab was rotating about C.

Example 4.6

A constrained bar AB. Given v_A, φ and l_{AB} , find I.C.R, ω, and v_B



- The instantaneous centre of rotation lies at the intersection of the perpendiculars to the velocity vectors through *A* and *B* as shown at point *C*.
- The velocity of *C* is zero.
- The velocities of all particles on the rod are as if they rotate about *C* with angular velocity *ω* .

As *A* rotates about *C* which is "fixed",

$$v_A = |AC|\omega_{AB}$$

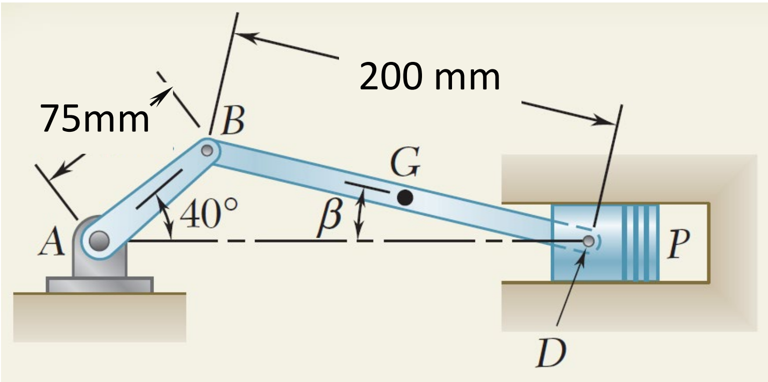
$$\omega_{AB} = \frac{v_A}{AC} = \frac{v_A}{l\sin\varphi}$$

$$v_B = (BC)\omega_{AB} = (l\cos\varphi)\frac{v_A}{l\sin\varphi}$$

$$= v_A \cot\varphi$$

Example 4.7

Solve the same **Example 4.4**, but using the method of I.C.R.

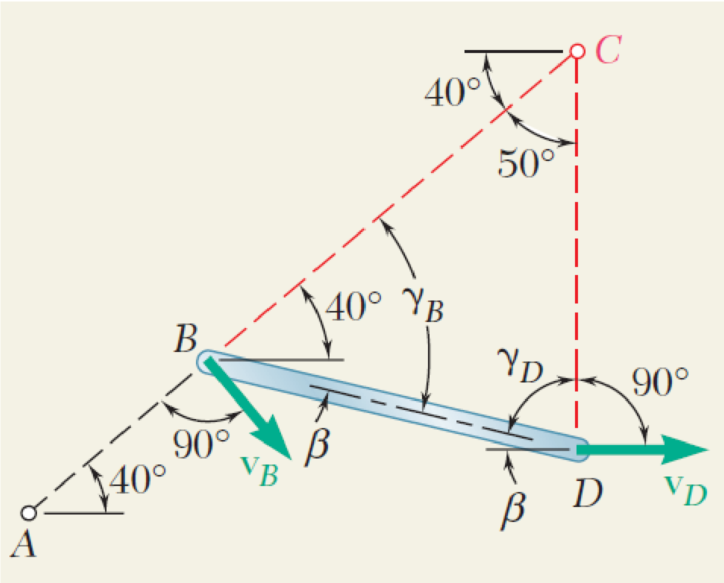


In the engine system shown, the crank *AB* has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

<u>SOLUTION</u>:

- The velocity $\vec{v}_B$ is obtained from the given crank rotation data.
- The direction of the velocity vectors at *B* and *D* are known. The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D.*
- Determine the angular velocity about the center of rotation based on the velocity at *B.*
- Calculate the velocity at *D* based on its rotation about the instantaneous center of rotation.

*Solution



$$\gamma_B = 40° + \beta = 53.95°$$

$$\gamma_D = 90° - \beta = 76.05°$$

$$v_B = (AB)\omega_{AB} = 15.705 m/s$$

$$BC = 260 \text{ mm} \quad CD = 211 \text{ mm}$$

Motion of crank AB:

- The velocity $\underline{v}_B$ is obtained from previous example.

$$v_B = (AB)\omega_{AB} = 15.705 m/s \quad 50^\circ$$

Motion of the connecting rod *BD*:

- We first locate the I.C.R, *C*, by drawing lines perpendicular to the absolute velocities $\underline{v}_B$ and $\underline{v}_D$. Recalling from previous example that $\beta = 13.95^\circ$ *and* $BD = 200\ mm$, we solve the triangle *BCD*.

- Since the connecting rod *BD* seems to rotate about point *C*, we write

$$v_B = (BC)\omega_{BD}$$

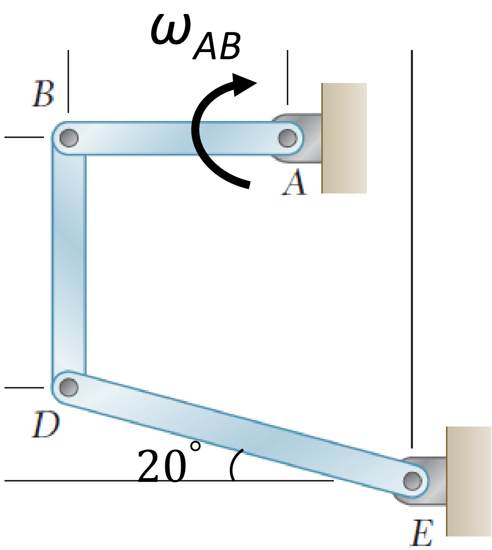
$$15.705 m/s = (0.260 m)\omega_{BD} \qquad \boxed{\omega_{BD} = 62.0\ rad/s}$$

$$v_D = (CD)\omega_{BD} = (0.211\ m)(62.0\ rad/s)$$

$$\boxed{v_P = v_D = 13.08\ m/s} \rightarrow$$

Evaluate: The same results, but simpler and easier because I.C.R is a special case of “reference-point method”. The velocity of the reference point, *C*, is zero.

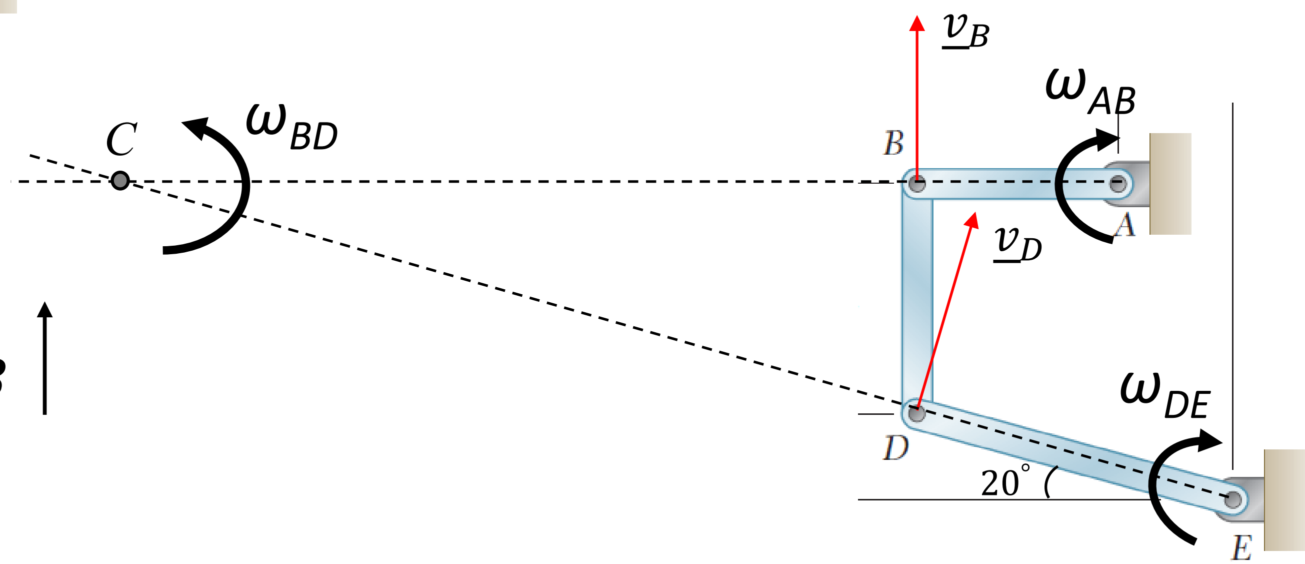
Conceptual quiz 4



In the position shown, bar *AB* has an angular velocity of ω_{AB} clockwise. Locate I.C.R of bar *BD*.

Tip:

I.C.R is at intersection of two lines drawn perpendicular to $\underline{v}_B$ and $\underline{v}_D$



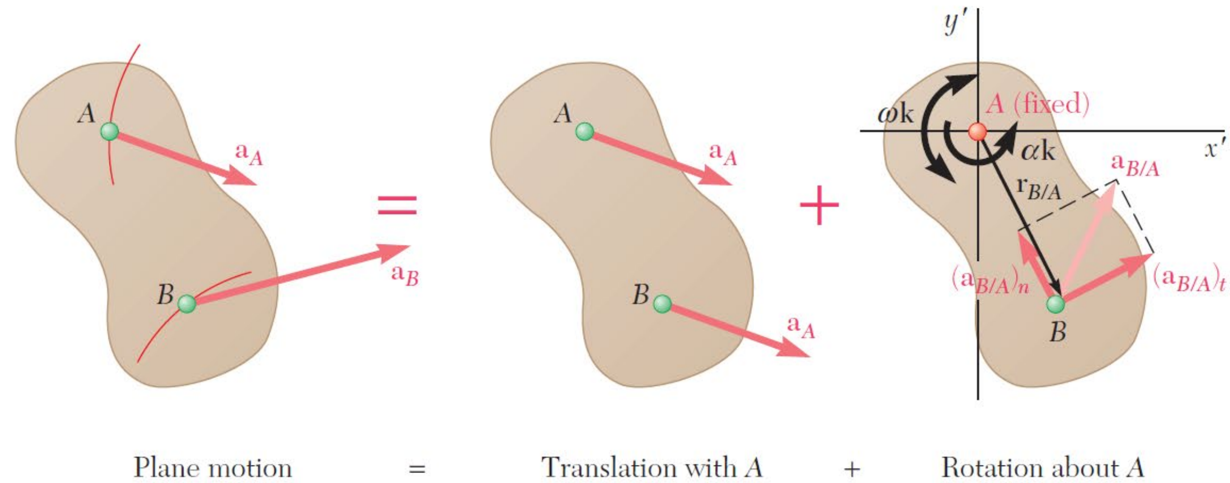
Recall

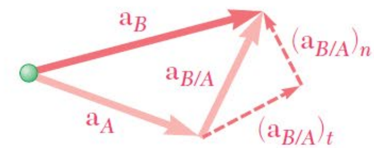
$\underline{v}_D$ ∠ 70° ; $\underline{v}_B \uparrow$

Acceleration in general motion

As the bicycle accelerates, a point on the top of the wheel will have acceleration due to the acceleration from the axle (the overall linear acceleration of the bike), the tangential acceleration of the wheel from the angular acceleration, and the normal acceleration due to the angular velocity.

*Absolute and Relative Acceleration





2 vector triangles

$\underline{a}_B = \underline{a}_A + \underline{a}_{B/A}$

$\underline{a}_{B/A} = (\underline{a}_{B/A})_t + (\underline{a}_{B/A})_n$

- Absolute acceleration of a particle of the slab,

$$\boxed{\underline{a}_B = \underline{a}_A + \underline{a}_{B/A}}$$

$\underline{a}_A$ is the translational component of $\underline{a}_B$

$\underline{a}_{B/A}$ is the rotational component of $\underline{a}_B$

- Relative acceleration $\underline{a}_{B/A}$ associated with rotation about A includes t & n components,

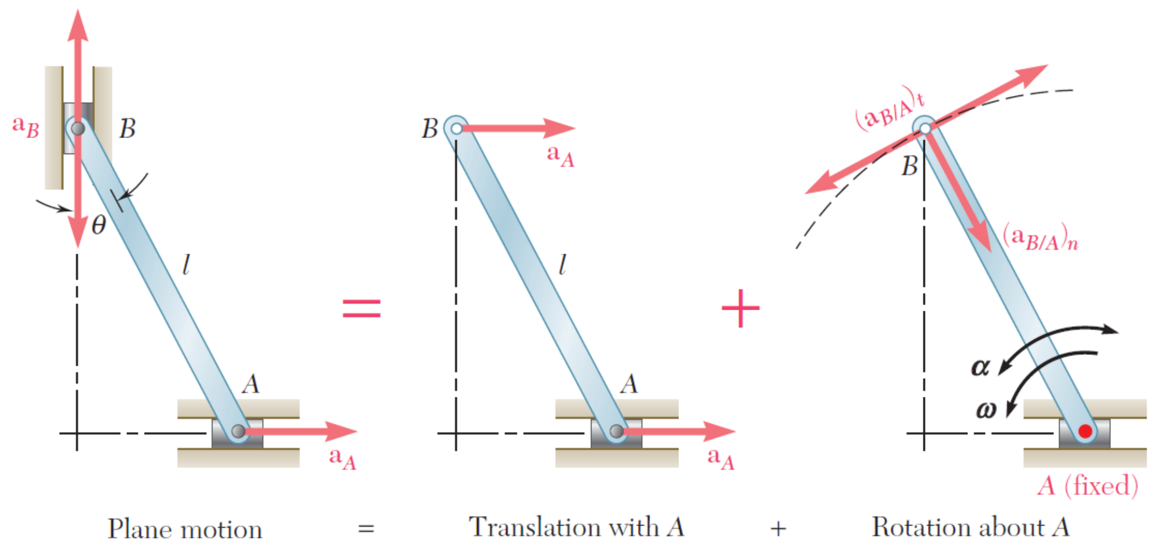
$$\boxed{\underline{a}_{B/A} = (\underline{a}_{B/A})_t + (\underline{a}_{B/A})_n}$$

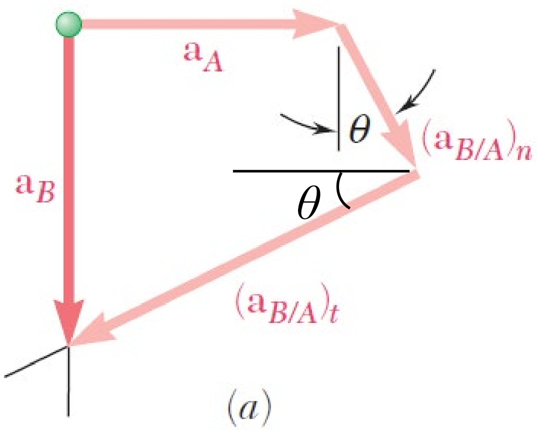
Vector form	Scalar form
$(\underline{a}_{B/A})_t = \alpha \underline{k} \times \underline{r}_{B/A}$	$(a_{B/A})_t = r\alpha$
$(\underline{a}_{B/A})_n = -\omega^2 \underline{r}_{B/A}$	$(a_{B/A})_n = r\omega^2$

Example 4.8

Let us again consider the structure in **Example 4.3**

Given $\underline{a}_A, \underline{v}_A$, determine $\underline{a}_B, \underline{\alpha}$.



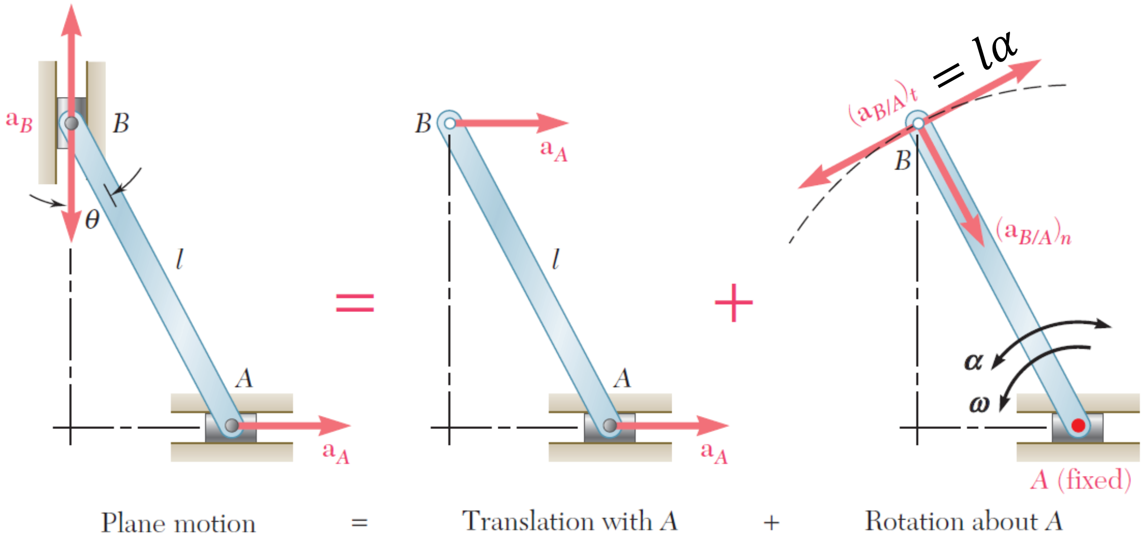


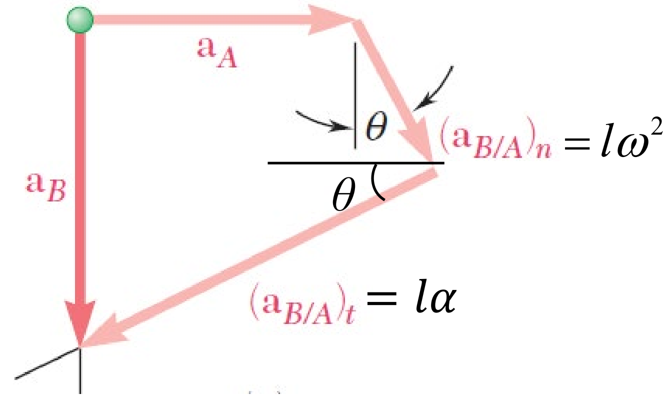
$$\boxed{\begin{aligned} \underline{a}_B &= \underline{a}_A + \underline{a}_{B/A} \\ &= \underline{a}_A + (\underline{a}_{B/A})_t + (\underline{a}_{B/A})_n \end{aligned}}$$

Recall the addition of 3 vectors

- Vector result depends on sense of $\underline{a}_A$ and the relative magnitudes of $\underline{a}_A$ and $\underline{a}_{B/A}$
- Must also know angular velocity ω.

Solution





$$\underline{a}_B = \underline{a}_A + \underline{a}_{B/A}$$
$$= \underline{a}_A + (\underline{a}_{B/A})_t + (\underline{a}_{B/A})_n$$

- Write $\underline{a}_B = \underline{a}_A + (\underline{a}_{B/A})_n + (\underline{a}_{B/A})_t$ in terms of the two component equations,

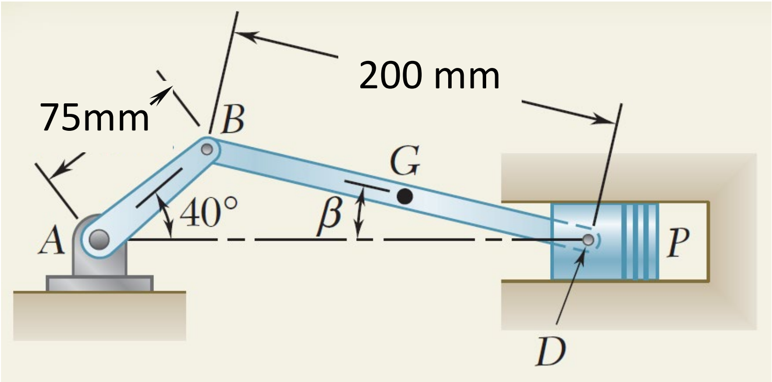
$\overset{+}{\rightarrow}$ x components: $0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$

$+\uparrow$ y components: $-a_B = -l\omega^2\cos\theta - l\alpha\sin\theta$

- Solve for a_B and α.

Example 4.9

Again use the same structure in **Example 4.4**, but solving the angular/linear accelerations.



In the engine system shown, the crank *AB* has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine the angular acceleration of the connecting rod *BD* and the linear acceleration of point *D*.

SOLUTION:

- The angular acceleration of the connecting rod *BD* and the acceleration of point *D* will be determined from

$$\boxed{\underline{a}_D = \underline{a}_B + (\underline{a}_{D/B})_n + (\underline{a}_{D/B})_t}$$

- The acceleration of *B* is determined from the given rotation speed of *AB*.
- The directions of the accelerations $\underline{a}_D, \underline{a}_B, (\underline{a}_{D/B})_n, (\underline{a}_{D/B})_t$ are determined from the geometry.
- Component equations for acceleration of point *D* are solved simultaneously for acceleration of *D* and angular acceleration of the connecting rod.

Solution

SOLUTION:

- Recall the velocities from the previous example.

$$\omega_{AB} = 209.4\,\text{rad/s} = const. \quad so, \ \alpha_{AB} = 0$$

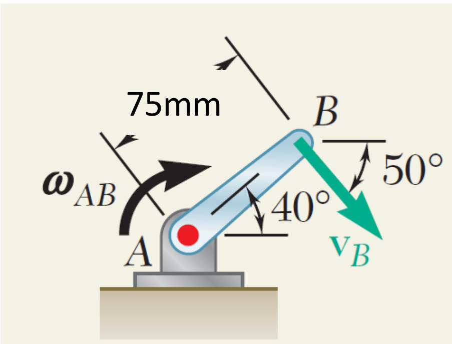
$$v_B = 15.705\,m/s \quad 50^\circ$$

$$v_D = 13.08\,m/s \ \longrightarrow$$

$$\omega_{BD}|_{t=0} = 62.0\,rad/s$$

Note: $\omega_{BD} \neq const.$

$$\beta = 13.95^\circ$$

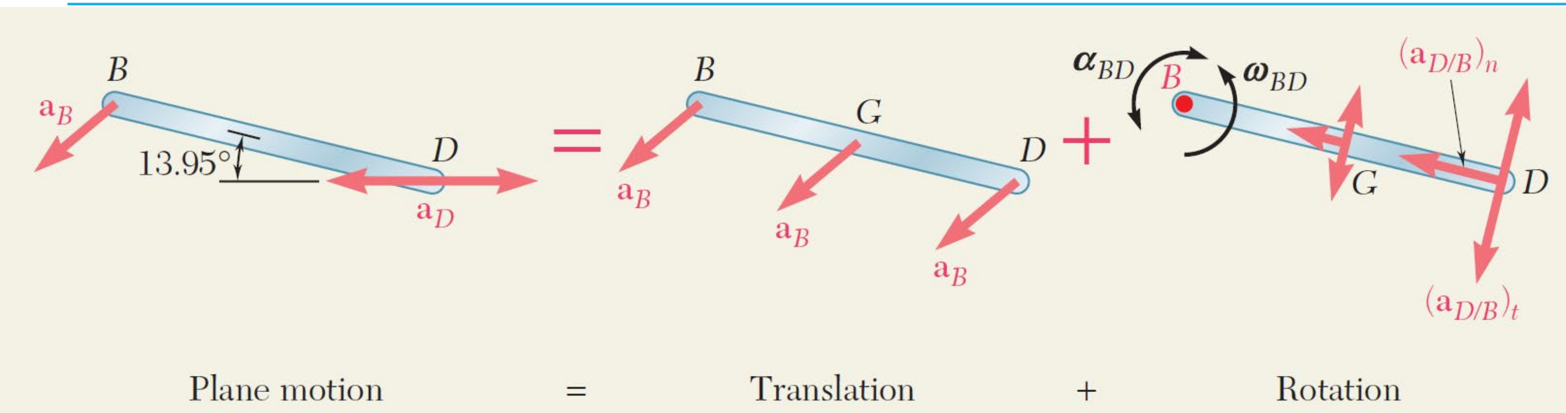


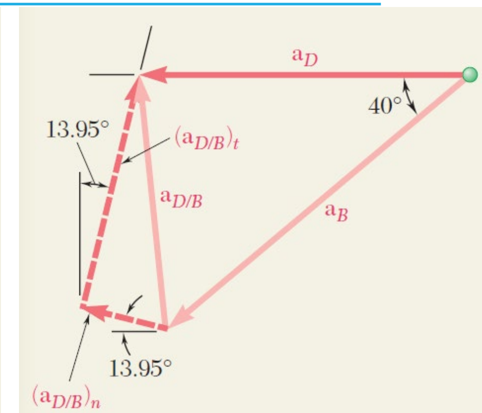
Motion of Crank AB. Since the crank rotates about A with constant ω_{AB} we have $\alpha_{AB} = 0$. The acceleration of B is therefore directed toward A and has a magnitude

$$a_B = r\omega_{AB}^2 = \left(\frac{75}{1000}\,\text{m}\right)(209.4\ \text{rad/s})^2 = 3289\ \text{m/s}^2$$

$$\mathbf{a}_B = 3289\ \text{m/s}^2 \ \measuredangle\ 40^\circ$$

Solution





Motion of the connecting rod *BD*:

The motion of BD is resolved into a translation with B and a rotation about B. The relative acceleration $\mathbf{a}_{D/B}$ is resolved into normal and tangential components:

$$(a_{D/B})_n = (BD)\omega_{BD}^2 = \left(\frac{200}{1000}\text{m}\right)(62.0\ \text{rad/s})^2 = 768.8\text{m/s}^2 \quad (\mathbf{a}_{D/B})_n = 768.8\text{m/s}^2 \measuredangle\ 13.95°$$

$$(a_{D/B})_t = (BD)\alpha_{BD} = \left(\frac{200}{1000}\text{m}\right)\ \alpha_{BD} = 0.2\,\alpha_{BD} \qquad (\mathbf{a}_{D/B})_t = 0.2\,\alpha_{BD} \measuredangle\ 76.05°$$

While $(\mathbf{a}_{D/B})_t$ must be perpendicular to BD, its sense is not known.

Noting that the acceleration $\mathbf{a}_D$ must be horizontal, we write

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = \mathbf{a}_B + (\mathbf{a}_{D/B})_n + (\mathbf{a}_{D/B})_t$$

$$[a_D \leftrightarrow] = [3289 \measuredangle\ 40°] + [768.8 \measuredangle\ 13.95°] + [0.2\alpha_{BD} \measuredangle\ 76.05°]$$

Equating x and y components, we obtain the following scalar equations:

$\xrightarrow{+} x$ components: $\quad -a_D = -3289 \cos 40° - 768.8 \cos 13.95° + 0.2\alpha_{BD} \sin 13.95°$

$+\uparrow y$ components: $\quad 0 = -3289 \sin 40° + 768.8 \sin 13.95° + 0.2\alpha_{BD} \cos 13.95°$

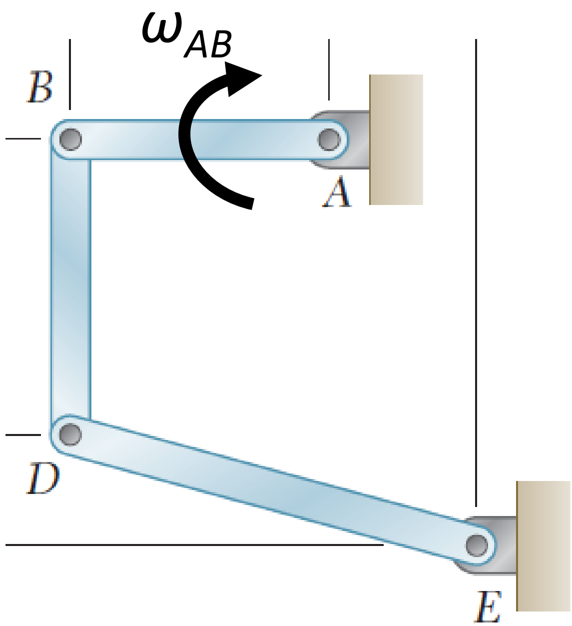
Solving the equations simultaneously, we obtain $\alpha_{BD} = +9940\ \text{rad/s}^2$ and $a_D = +2790$ m. The positive signs indicate that the senses shown on the vector polygon are correct; we write

$$\alpha_{BD} = 9940\ \text{rad/s}^2 \circlearrowleft$$

$$\mathbf{a}_D = 2790\ \text{m/s}^2 \leftarrow$$

Conceptual quiz 5

NUS National University of Singapore

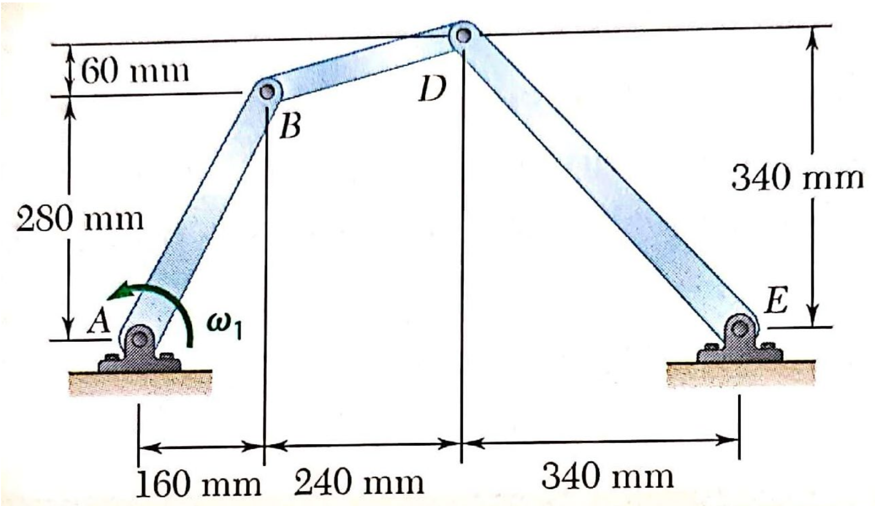


Knowing that at the instant shown bar *AB* has a constant angular velocity of ω clockwise.

Which of the following is true?

a) The direction of a_D is $\nearrow$
b) The angular acceleration of *BD* must also be constant
c) The direction of the linear acceleration of *B* is $\rightarrow$

Example 4.10 (for practice)

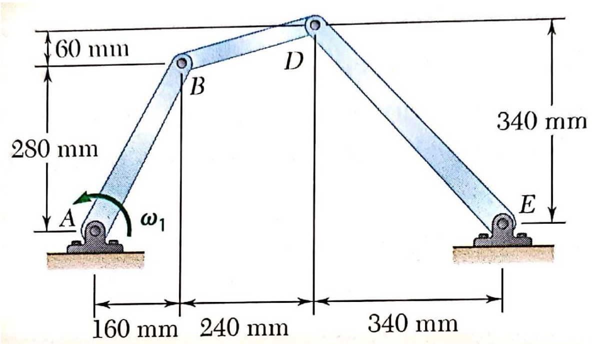


The linkage *ABDE* moves in the vertical plane. Knowing that in the position shown crank *AB* has a constant angular velocity $\underline{\omega}_1$ of 20 rad/s counterclockwise, determine the **angular velocities** and **angular accelerations** of the connecting rod *BD* and of the crank *DE*.

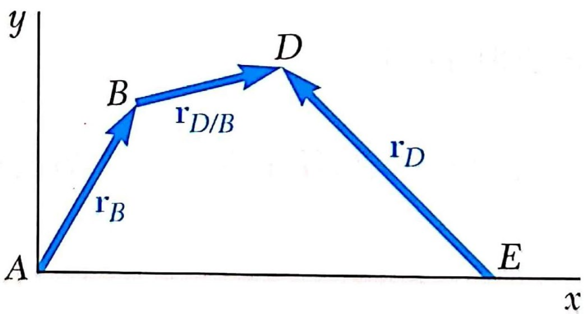
<u>NOTICE</u>:

- This problem could be solved by the method used in the previous example, however, the vector approach will be taught here.
- Students can choose either approach for their convenience.
- For **practice**, solve this problem using the "magnitude + arrow" approach in the previous example

Solution



Position vectors:



$$\mathbf{r}_B = 160\mathbf{i} + 280\mathbf{j} \text{ mm}$$
$$\mathbf{r}_D = -340\mathbf{i} + 340\mathbf{j} \text{ mm}$$
$$\mathbf{r}_{D/B} = 240\mathbf{i} + 60\mathbf{j} \text{ mm}$$

SOLUTION:

The position vectors are chosen as shown in the sketch, and the values are obtained from the given dimensions.

Velocities. Since the motion of each element of the linkage is contained in the plane of the figure, we have

$$\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} = (20 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$

where $\mathbf{k}$ is a unit vector pointing out of the paper. We now write

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$
$$\omega_{DE}\mathbf{k} \times \mathbf{r}_D = \omega_{AB}\mathbf{k} \times \mathbf{r}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B}$$
$$\omega_{DE}\mathbf{k} \times (-340\mathbf{i} + 340\mathbf{j}) = 20\mathbf{k} \times (160\mathbf{i} + 280\mathbf{j}) + \omega_{BD}\mathbf{k} \times (240\mathbf{i} + 60\mathbf{j})$$

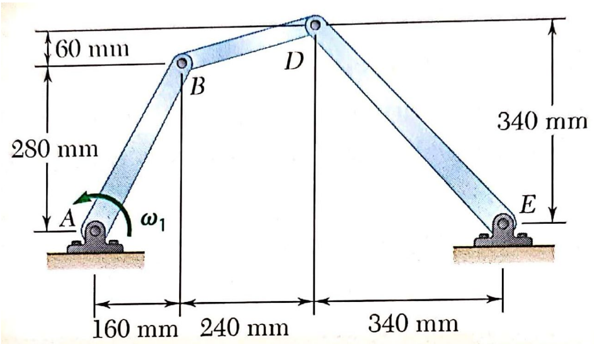
Dividing each term by 20 we get

$$-17\omega_{DE}\mathbf{j} - 17\omega_{DE}\mathbf{i} = 160\mathbf{j} - 280\mathbf{i} + 12\omega_{BD}\mathbf{j} - 3\omega_{BD}\mathbf{i}$$

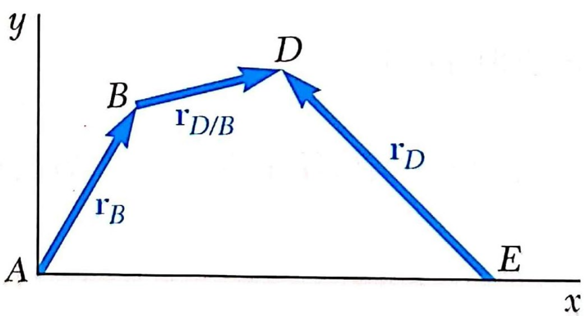
Equating the coefficients of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we obtain the following two scalar equations:

$$-17\omega_{DE} = -280 - 3\omega_{BD}$$
$$-17\omega_{DE} = +160 + 12\omega_{BD}$$
$$\boldsymbol{\omega}_{BD} = -(29.33 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = (11.29 \text{ rad/s})\mathbf{k}$$

Solution



Position vectors:



$$\mathbf{r}_B = 160\mathbf{i} + 280\mathbf{j} \text{ mm}$$
$$\mathbf{r}_D = -340\mathbf{i} + 340\mathbf{j} \text{ mm}$$
$$\mathbf{r}_{D/B} = 240\mathbf{i} + 60\mathbf{j} \text{ mm}$$

SOLUTION:

Accelerations: Noting that at the instant considered crank *AB* has a constant angular velocity

$\underline{\omega}_{AB} = \underline{\omega}_1 = 20\underline{k}\ rad/s$ = const. So, $\alpha_{AB} = 0$.

$$\boldsymbol{\alpha}_{AB} = 0 \qquad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\mathbf{k}$$

Note: take +**k** first

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} \quad (1)$$

Expressing **r** in *m* we have

$$\mathbf{r}_B = 0.16\mathbf{i} + 0.28\mathbf{j}$$
$$\mathbf{r}_D = -0.34\mathbf{i} + 0.34\mathbf{j}$$
$$\mathbf{r}_{D/B} = 0.24\mathbf{i} + 0.06\mathbf{j}$$

Each term of Eq. (1) is evaluated separately:

$$\begin{aligned}
\mathbf{a}_D &= \alpha_{DE}\mathbf{k} \times \mathbf{r}_D - \omega_{DE}^2\mathbf{r}_D \\
&= \alpha_{DE}\mathbf{k} \times (-0.34\mathbf{i} + 0.34\mathbf{j}) - (11.29)^2(-0.34\mathbf{i} + 0.34\mathbf{j}) \\
&= -0.34\alpha_{DE}\mathbf{j} - 0.34\alpha_{DE}\mathbf{i} + 43.33\mathbf{i} - 43.33\mathbf{j} \\
\mathbf{a}_B &= \alpha_{AB}\mathbf{k} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B = 0 - (20)^2(\underset{0.16\mathbf{i}}{\cancel{16\mathbf{i}}} + 0.28\mathbf{j}) \\
&= -64\mathbf{i} - 112\mathbf{j} \\
\mathbf{a}_{D/B} &= \alpha_{BD}\mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} \\
&= \alpha_{BD}\mathbf{k} \times (0.24\mathbf{i} + 0.06\mathbf{j}) - (29.33)^2(0.24\mathbf{i} + 0.06\mathbf{j}) \\
&= 0.24\alpha_{BD}\mathbf{j} - 0.06\alpha_{BD}\mathbf{i} - 206.4\mathbf{i} - 51.61\mathbf{j}
\end{aligned}$$

Substituting into Eq. (1) and equating the coefficients of **i** and **j**, we obtain

$$\mathbf{i} \overset{+}{\rightarrow}: \quad -0.34\alpha_{DE} + 0.06\alpha_{BD} = -313.7$$
$$\mathbf{j} + \uparrow: \quad -0.34\alpha_{DE} - 0.24\alpha_{BD} = -120.28$$
$$\boldsymbol{\alpha}_{BD} = -(645 \text{ rad/s}^2)\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = (809 \text{ rad/s}^2)\mathbf{k}$$

Evaluate: assumption of +**k** for α_{BD} is wrong; while +**k** for α_{DE} is correct

The END