ME2115 Formulae

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Vector Mechanics

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$$

$$u = |\mathbf{u}| = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = u_x v_x + u_y v_y + u_z v_z$$

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$$

$$\mathbf{u} \times \mathbf{u} = 0$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \text{ (if } \mathbf{u} \perp \mathbf{v})$$

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta$$

$$\mathbf{u} \times \mathbf{v} = uv \sin\theta \cdot \mathbf{n}$$

Particle Kinematics

Rectilinear Motion

$$dx = v\,dt \parallel \int_{x_0}^{x} dx = \int_{t_0}^{t} v(t)\,dt$$

$$dv = a\,dt \parallel \int_{v}^{v_0} dv = \int_{t_0}^{t} a(t)\,dt$$

$$v\,dv = a\,dx \parallel \int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$$

Given $a = a(t)$:

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} a(t)\,dt \parallel \int_{x_0}^{x} dx = \int_{t_0}^{t} v(t)\,dt$$

Given $a = a(x)$:

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx \parallel \int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{1}{v(x)}\,dx$$

Given $a = a(v)$:

$$\int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{1}{a(v)}\,dv \parallel \int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v}{a(v)}\,dv$$

If v is constant:

$$x = x_0 + v(t - t_0)$$

If a is constant:

$$v = v_0 + a(t - t_0)$$

$$x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

Curvilinear Motion

The previous section can apply to angular motion by replacing x with θ, v with ω, and a with α.

$$s = r\theta$$

$$v_t = r\omega$$

$$a_t = r\alpha$$

$$a_n = \frac{v_t^2}{\rho}$$

Rigid Body Mechanics

General Plane Motion

$$v_B = v_A + v_{B/A}$$

$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A} = v_t$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$a_{B/A} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) + \vec{\alpha} \times \vec{r}_{B/A} = \vec{a}_n + \vec{a}_t$$

Rolling without Sliding

* velocity at contact point always 0

$$v_o = r\omega$$

$$a_t = r\alpha$$

$$\vec{a_c} = \vec{\alpha} \times \vec{r}_{c/o}$$

$$\vec{v_c} = \vec{\omega} \times \vec{r}_{c/o}$$

Mass Properties

$$dW = \gamma t\,dA \equiv W = \gamma t A$$

$$\bar{x}A = \int x\,dA = Q_y$$

$$\bar{y}A = \int y\,dA = Q_x$$

Compound Shapes

$$Q_y = \bar{X} \sum A = \sum \bar{x}A$$

$$Q_y = \bar{Y} \sum A = \sum \bar{y}A$$

Mass Moment of Inertia

$$I_O = \int r^2\,dm = mk_o^2$$

$$I = I_O + md^2$$

Rigid Body Kinetics

$$\Sigma F = m\bar{a} = \Sigma F_{eff.}$$

$$\Sigma M_G = \bar{I}\alpha = \Sigma M_{eff.}$$

More generally.

$$\Sigma M_i + \Sigma(\vec{r}_{i/A} \times \vec{F}_i) = M_A = I_G\vec{\alpha} + \vec{r}_{G/A} \times m\vec{a}_G$$

$$M_A = (I_G + mr_{G/A}^2)\vec{\alpha} = I_A\vec{\alpha}$$

$$\bar{a} = a_{ref} + r\omega^2 e_n + r\alpha e_t$$

Principle of Work and Energy

$$U_{1\to2} = \int \vec{F}\,d\vec{r} + \int M\,d\theta = T_2 - T_1$$

$$V_{grav.} = mg$$

$$V_{elas.} = \frac{1}{2}k(\Delta x)^2$$

$$T_i = \frac{1}{2}m|\vec{v_i}|^2 + \frac{1}{2}I_G\omega^2$$

Work of Conservative Forces

$$U_{1\to2} = U_2 - U_1 = V_1 - V_2$$

$$V_1 + T_1 = V_2 + T_2$$

Free Vibration Without Damping

$$\ddot{u} + \omega_n^2 u = 0$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Natural Parameters

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$f_n = \frac{\omega_n}{2\pi} \text{ (in Hz)}$$

General Solution

$$u = A\sin(\omega_n t + \phi)$$

$$\dot{u} = A\omega_n \cos(\omega_n t + \phi)$$

$$\dot{\theta}_{max} = \omega_n\theta$$

$$\dot{x}_{max} = \omega_n x$$

$$A = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_n}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_o\omega_n}{v_o}\right)$$

Pendulum

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{l}}$$

Generic Rigid Body

$$\omega_n = \sqrt{\frac{mgd}{I_O}}$$

Free Vibration with Damping

Spring-Mass-Damper System

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Overdamped $\left(\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0\right)$

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

Critically Damped $\left(\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0\right)$

$$c_{cr} = 2\sqrt{mk} = 2m\omega_n$$

$$x = (A_1 + A_2 t)e^{-\omega_n t}$$

Underdamped $\left(\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0\right)$

Equation of Motion

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$x = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

Exponential Decay Coefficient

$$\alpha = \frac{c}{2m} = \zeta\omega_n$$

Damped Oscillation Frequency

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_n\sqrt{1 - \zeta^2}$$

Damping Ratio

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$$

Stiffness Coefficient

$$k = m\omega_n^2$$

Initial Conditions

$$X = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{v_o + \zeta\omega_n x_o}{\omega_d}\right)^2 + x_o^2}$$

$$\phi = \tan^{-1}\left(\frac{C_1}{C_2}\right) = \tan^{-1}\left(\frac{\omega_d x_o}{v_o + \zeta\omega_n x_o}\right)$$

Logarithmic Decrement

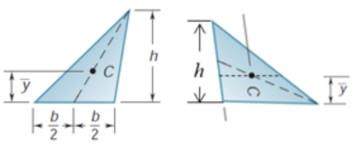
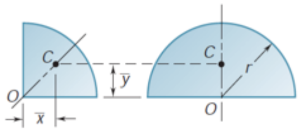
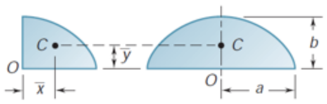
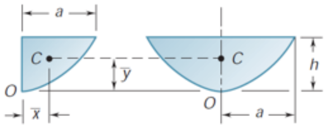
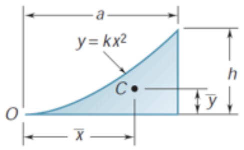
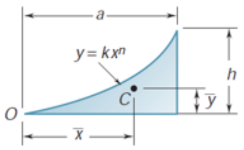
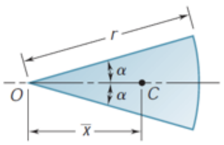
$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n\tau_d$$

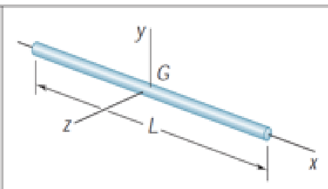
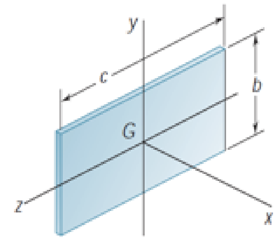
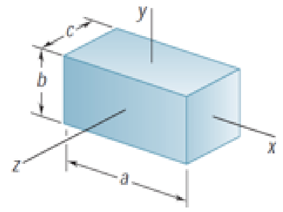
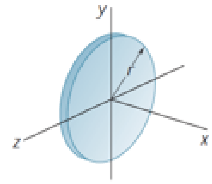
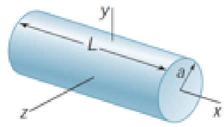
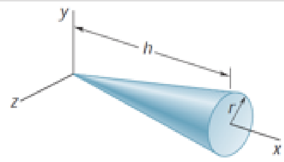
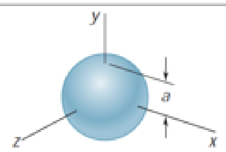
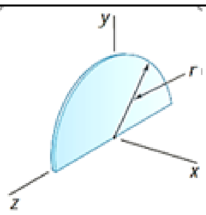
$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

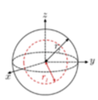
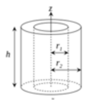
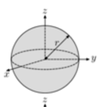
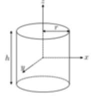
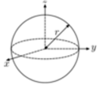
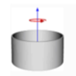
$$\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right)$$

Tips

- Damping/stiffness formula order: find $\delta \to \zeta \to \omega_n \to c_{cr} \to c \wedge k$
- Funky approximations: $\sin\theta \approx \theta \parallel \cos\theta - 1 \approx -\frac{\theta^2}{2}$
- For a small enough slice of theta in a circle, a triangle can have 2 90° angles (lol)
- **Law of Sines**: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ for any triangle

Shape	$\bar{x}$	$\bar{y}$	Area
		$\frac{h}{3}$	$\frac{bh}{2}$
	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
	0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
	$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
	0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
	$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
	0	$\frac{3h}{5}$	$\frac{4ah}{3}$
	$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
	$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
	$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Slender rod		$I_y = I_z = \frac{1}{12}mL^2$
Thin rectangular plate		$I_x = \frac{1}{12}m(b^2+c^2)$ $I_y = \frac{1}{12}mc^2$ $I_z = \frac{1}{12}mb^2$
Rectangular prism		$I_x = \frac{1}{12}m(b^2+c^2)$ $I_y = \frac{1}{12}m(c^2+a^2)$ $I_z = \frac{1}{12}m(a^2+b^2)$
Thin disk		$I_x = \frac{1}{2}mr^2$ $I_y = I_z = \frac{1}{4}mr^2$
Circular cylinder		$I_x = \frac{1}{2}ma^2$ $I_y = I_z = \frac{1}{12}m(3a^2+L^2)$
Circular cone		$I_x = \frac{3}{10}ma^2$ $I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2+h^2)$
Sphere		$I_x = I_y = I_z = \frac{2}{5}ma^2$
Semicircular disk		$I_x = \frac{1}{2}mr^2$ $I_y = I_z = \frac{1}{4}mr^2$

Thick-walled hollow sphere		$I = \frac{2m}{5}\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right)$	Thick-walled hollow cylinder		$I_z = \frac{1}{2}m(r_1^2+r_2^2)$ $I_x = I_y = \frac{1}{12}m[3(r_1^2+r_2^2)+h^2]$
Solid sphere		$I = \frac{2}{5}mr^2$	Solid cylinder		$I_z = \frac{1}{2}mr^2$
Hollow sphere		$I = \frac{2}{3}mr^2$	Thin-walled hollow cylinder		$I = mr^2$