ME2115/ME2115E/TME2115 -

Mechanics of Machines

Review of particle kinematics

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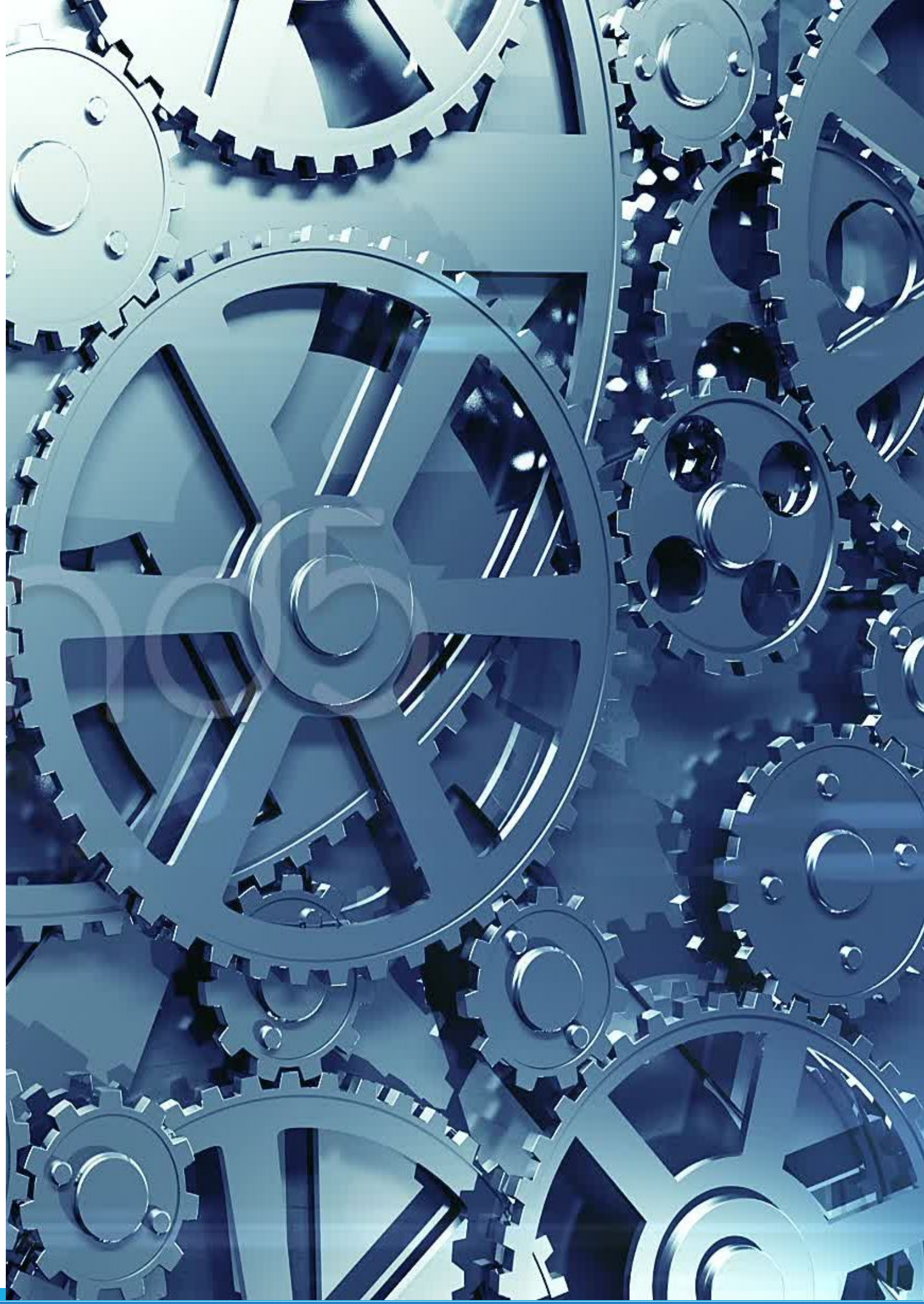

Contents

KINEMATICS OF PARTICLES

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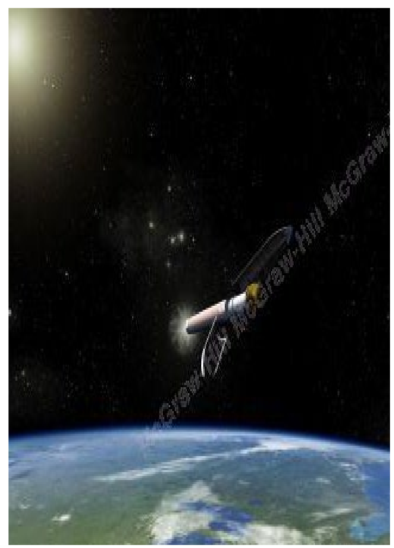
Introduction

Kinematic relationships are used to help us determine the trajectory of a golf ball, the orbital speed of a satellite, and the accelerations during acrobatic flying.

To establish the **motion** of a particle $f(\underline{x}, \underline{v}, \underline{a}, t) = 0$

- **Kinematics**: to study the geometry of motion with no reference to forces.
- **Particle**: A body with relatively negligible dimensions, but has finite mass.



Particle motions

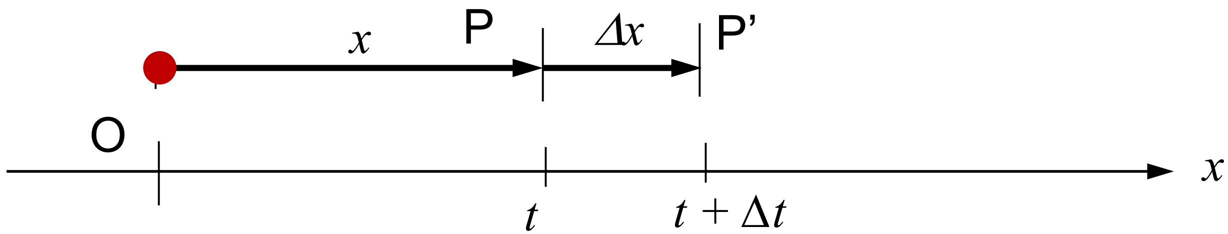
- **Particle motions include:**

  - ***<u>Rectilinear motion</u>***: position, velocity, and acceleration of a particle as it moves along a **straight line**.

- ***<u>Curvilinear motion</u>***: position, velocity, and acceleration of a particle as it moves along a **curved line** in two or three dimensions.

Position

Rectilinear Motion of Particles – motion is along a straight path, i.e., one-dimension



Some important concepts:

Motion – the motion of a particle is known if the position coordinate for particle is known for every value of time *t*. May be expressed in the form of a function of x(t).

$$e.g.\ x(t) = 6t^2 - t^3$$

Position, Displacement, Velocity and Acceleration

Position of P with respect to a chosen fixed origin O = x m; **a vector**

Displacement of P is the change of position of P = Δx m; **a vector**

Distance of P and P' is the length measured from P to P', independent on the direction; **a scalar**

Note: For 1D motion, we usually use sign +/- to indicate the direction, for example, +5 m represents the position vector P, instead of $5\underline{i}$ m if x=5 m in the figure above.

Velocity and acceleration

Velocity is the rate of change of position:

Average velocity, v from A to B

$$v_{av} = \frac{\Delta x}{\Delta t} \quad m/s$$

Instantaneous velocity, v at A or B

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad m/s \qquad (4.1)$$

e.g.,

$$x = 6t^2 - t^3 m$$

$$v = \frac{dx}{dt} = 12t - 3t^2 m/s$$

Acceleration is the rate of change of velocity:

Average acceleration,

$$a_{av} = \frac{\Delta v}{\Delta t} \quad m/s^2$$

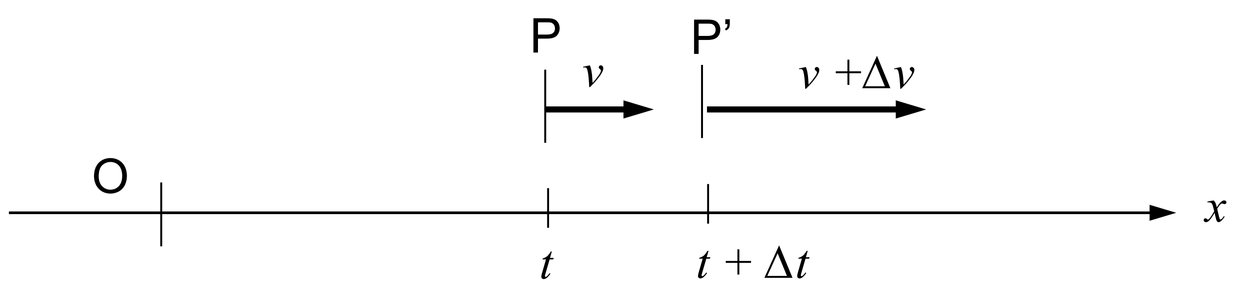
Instantaneous acceleration,
From the definition of a derivative,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2} \quad m/s^2 \qquad (4.2)$$

e.g.,

$$v = 12t - 3t^2 m/s$$

$$a = \frac{dv}{dt} = (12 - 6t) m/s^2$$



Acceleration (cont'd)

In view of Eqs. (4.1) and (4.2 by eliminating ; dt gives

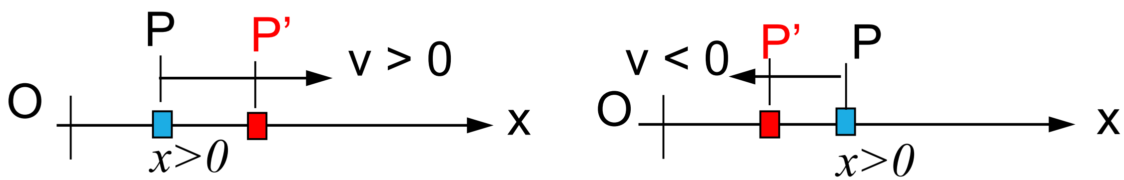
$$\frac{dx}{v} = \frac{dv}{a}$$

or $$a\,dx = v\,dv$$

or $$\boxed{a = v\frac{dv}{dx}} \quad (4.3)$$

Remarks The direction of *v* depends on displacement, not position.

When x is + ve, v can be either + ve or - ve



When Δx is + ve, v is + ve

When Δv is - ve, v is – ve

$$v_{av} = \frac{\Delta x}{\Delta t} \quad m/s$$

Δx is a vector

Δt is a scalar

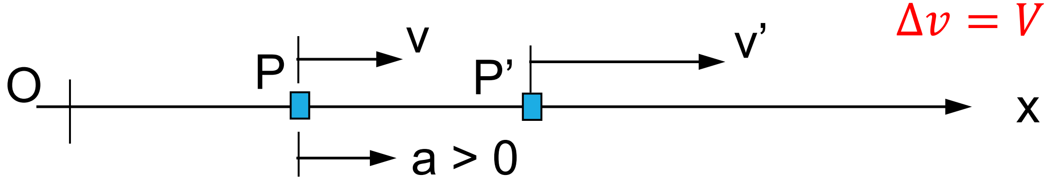
Thus, sense of v depents on displacement sense

Direction and sign

(1) Particle is moving faster in the +ve direction

$\Delta x = P' - P$

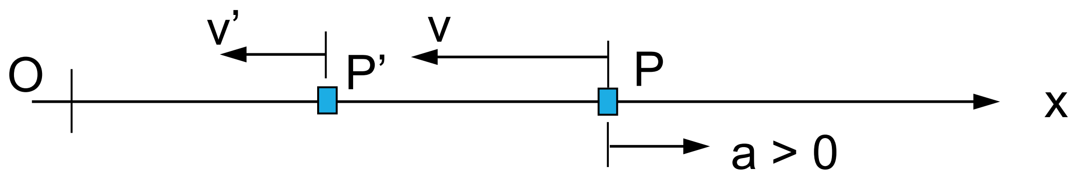
$\Delta v = V' - V$



Δx is + ve, v is + ve

Δv is + ve, a is + ve

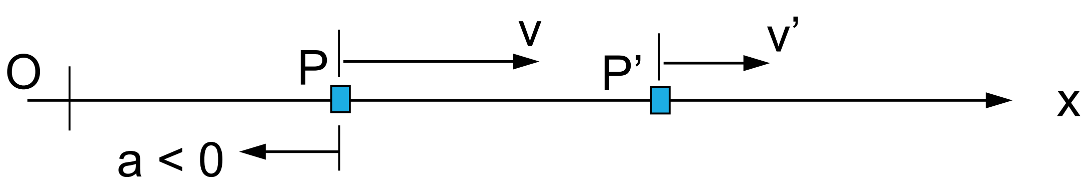
(2) Particle is moving more slowly in -ve direction



Δx is – ve, v is – ve

Δv is + ve, a is + ve

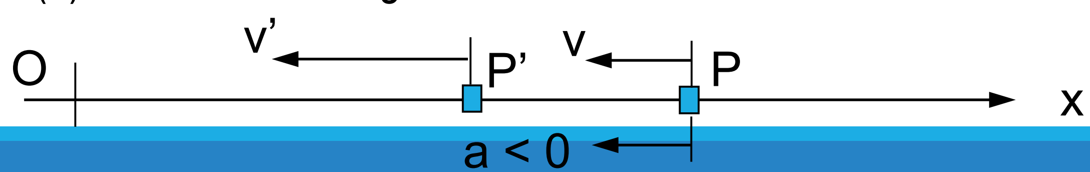
(3) Particle is moving more slowly in +ve direction



Δx is + ve, v is + ve

Δv is – ve, a is – ve

(4) Particle is moving faster in -ve direction



Δx is – ve, v is – ve

Δv is – ve, a is – ve

*Determination of the motion

Motion of a Particle

$$dx = v\,dt \qquad (4.1)$$

$$dv = a\,dt \qquad (4.2)$$

$$v\,dv = a\,dx \qquad (4.3)$$

Four types of problems in rectilinear motion of particles:

1. Given $x = x(t)$, find $v(t)$ and $a(t)$

   Solution: Use Eqs. (4.1) and (4.2)

2. $\boxed{\text{Given } a = a(t)}$, find $v(t)$ and $x(t)$

   Solution: Use Eq. (4.2) $\quad \int_{v_o}^{v} dv = \int_{t_o}^{t} a(t)dt \qquad (4.4)$

   Use Eq. (4.1) $\quad \int_{x_o}^{x} dx = \int_{t_o}^{t} v(t)dt \qquad (4.5)$

*Determination of the motion

Types of problems (cont'd):

3. $\boxed{\text{Given } a = a(x)}$, find $v(x)$ and $x(t)$

Solution: Use Eq. (4.3) $\quad \int_{v_o}^{v} v\,dv = \int_{x_o}^{x} a(x)\,dx \quad (4.6)$

Use Eq. (4.1) $\quad \int_{t_o}^{t} dt = \int_{x_o}^{x} \frac{1}{v(x)}\,dx \quad (4.7)$

4. $\boxed{\text{Given } a = a(v)}$, find $v(t)$ and $v(x)$

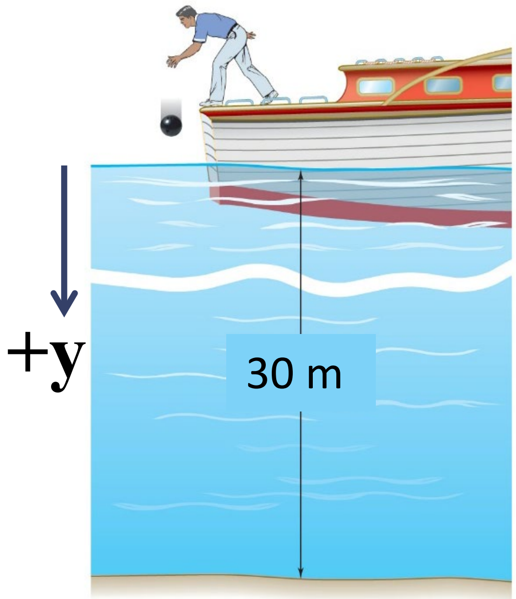
Solution: Use Eq. (4.2) $\quad \int_{t_o}^{t} dt = \int_{v_o}^{v} \frac{1}{a(v)}\,dv \quad (4.8)$

Use Eq. (4.3) $\quad \int_{x_o}^{x} dx = \int_{v_o}^{v} \frac{v}{a(v)}\,dv \quad (4.9)$

*Summary

Given	Kinematic relationship	Integrates
$a = a(t)$	$\dfrac{dv}{dt} = a(t)$	$\int_{v_0}^{v} dv = \int_0^t a(t)\,dt \Rightarrow v(t) \xrightarrow{\int_{x_0}^{x} dx = \int_{t_0}^{t} v(t)dt} x(t)$
$a = a(x)$	$v\,dv = a(x)\,dx$	$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx \Rightarrow v(x) \xrightarrow{\int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{1}{v(x)} dx} x(t)$
$a = a(v)$	$\dfrac{dv}{a(v)} = dt$	$\int_{v_0}^{v} \dfrac{dv}{a(v)} = \int_0^t dt \Rightarrow v(t) \xrightarrow{\int_{x_0}^{x} dx = \int_{t_0}^{t} v(t)dt} x(t)$
	$\dfrac{v\,dv}{a(v)} = dx$	$\int_{v_0}^{v} \dfrac{v\,dv}{a(v)} = \int_{x_0}^{x} dx \Rightarrow v(x) \xrightarrow{\int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{1}{v(x)} dx} x(t)$

Conceptual quiz



A bowling ball is dropped from a boat so that it strikes the surface of a lake with a speed of 15 m/s. Assuming the ball experiences a downward acceleration of $a = 10 - 0.01v^2$ when in the water, determine the velocity of the ball when it strikes the bottom of the lake.

Which integral should you choose?

(a) $$\int_{v_0}^{v} dv = \int_{0}^{t} a(t)\,dt$$

(c) $$\int_{v_0}^{v} v\,dv = \int_{y_0}^{y} a(x)\,dx$$

(b) $$\int_{y_0}^{y} dx = \int_{v_0}^{v} \frac{v\,dv}{a(v)}$$

(d) $$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{0}^{t} dt$$

*Equations defining uniform motion

$$\int_{v_o}^{v} dv = \int_{t_o}^{t} a(t)dt \quad (4.4) \qquad \int_{x_o}^{x} dx = \int_{t_o}^{t} v(t)dt \quad (4.5) \qquad \int_{v_o}^{v} vdv = \int_{x_o}^{x} a(x)dx \quad (4.6)$$

Special Cases:

If $\underline{v \text{ is constant}}$ (uniform rectilinear motion, $\underline{a} = 0$)

Eq. (4.5) yields $\int_{x_o}^{x} dx = v\int_{t_o}^{t} dt$

$$\Rightarrow \boxed{x = x_o + v(t - t_o) \qquad (4.10)}$$

If $\underline{a \text{ is constant}}$ (uniformly accelerated rectilinear motion)

(i) From Eq. (4.4) $\int_{v_o}^{v} dv = a\int_{t_o}^{t} dt$

$$\Rightarrow \boxed{v = v_o + a(t - t_o) \qquad (4.11)}$$

(ii) From Eq. (4.5) $\int_{x_o}^{x} dx = \int_{t_o}^{t} vdt$ and substituting Eq. (4.11) gives

$$\Rightarrow \boxed{x = x_o + v_o(t - t_o) + \frac{1}{2}a(t - t_o)^2 \qquad (4.12)}$$

(iii) From Eq. (4.6) $\int_{v_o}^{v} vdv = a\int_{x_o}^{x} dx$

$$\Rightarrow \boxed{v^2 - v_o^2 = 2a(x - x_o) \qquad (4.13)}$$

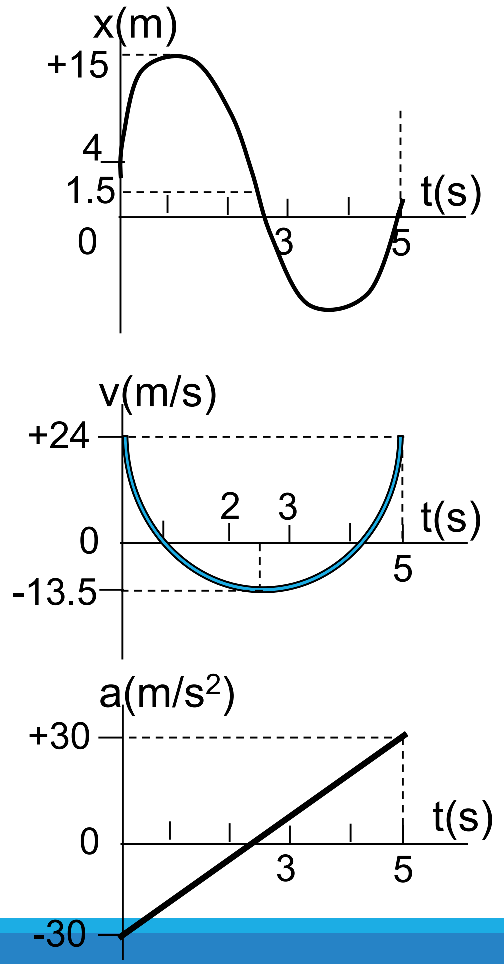
Careful – these only apply to uniform rectilinear motion!

Scalar formulas (because of one-dimensional rectilinear motion)

Example 3.1

The motion of a particle is defined by $x = 2t^3 - 15t^2 + 24t + 4$ where x is in meters and t is in seconds. Determine (a) when $v = 0$, and (b) the position and total distance traveled when the acceleration is zero.

For reference ONLY



The equations of motion are:

$$x = 2t^3 - 15t^2 + 24t + 4$$

$$v = \frac{dx}{dt} = 6t^2 - 30t + 24$$

$$a = \frac{d^2x}{dt^2} = 12t - 30$$

(a) $v = 0$ when $6t^2 - 30t + 24 = 0 \Rightarrow t = 1s$ or $4s$

(b) $a = 0$ when $12t - 30 = 0 \Rightarrow t = 2.5s$

Position when $a = 0$:

$$x_{2.5} = 2(2.5)^3 - 15(2.5)^2 + 24(2.5) + 4 = 1.5\,m$$

Note: $v = 0$ when $t = 1s$:

$$x_1 = 2(1)^3 - 15(1)^2 + 24(1) + 4 = 15\,m$$

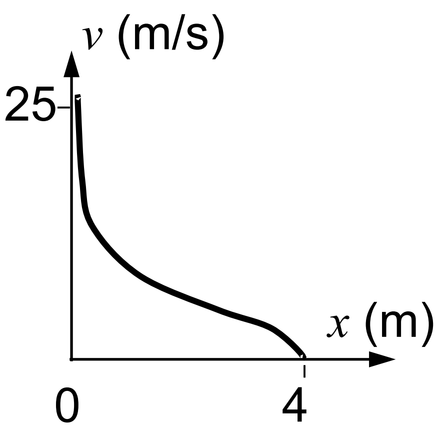
From $t = 0$ to $t = 1s$: $x_1 - x_0 = 15 - 4 = 11\,m$

From $t = 1$ to $t = 2.5s$: $x_{2.5} - x_1 = 1.5 - 15 = -13.5\,m$

Total distance $= 11 + 13.5 = 24.5\,m$

Example 3.2

The acceleration of a particle is $a = -60x^{-1.5}$ where a is expressed in m/s^2 and x is in meters. If $v = 0$ at $x = 4$ m, determine the velocity when (a) $x = 2$ m, and (b) $x = 0.1$ m



$$a = v\frac{dv}{dx} \Rightarrow \quad v\frac{dv}{dx} = -60x^{-1.5}$$

$$\int_0^v v\,dv = -\int_4^x 60x^{-1.5}dx$$

$$\frac{1}{2}v^2 = 120\left[\frac{1}{\sqrt{x}}\right]_4^x \Rightarrow v^2 = 240\left(\frac{1}{\sqrt{x}} - \frac{1}{2}\right)$$

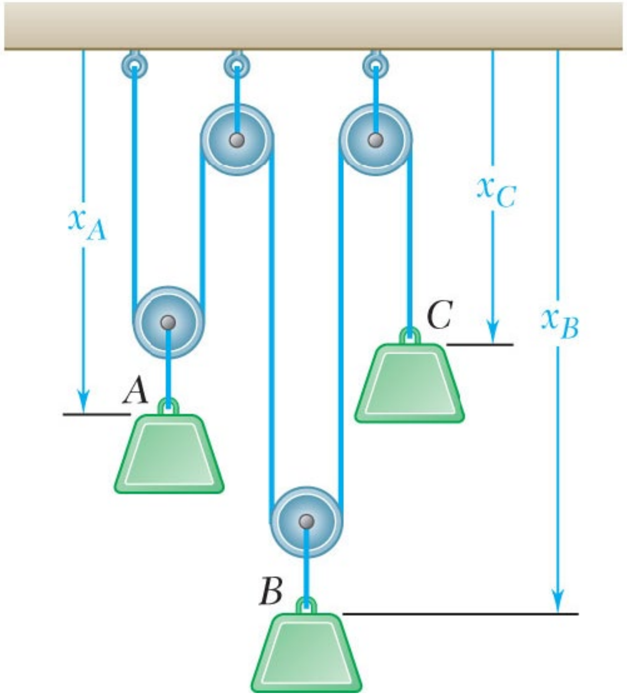
$$(a)\ x = 2m: \quad v^2 = 240\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = 49.706 \Rightarrow v = 7.05\,m/s$$

$$(b)\ x = 0.1m: \quad v^2 = 240\left(\frac{1}{\sqrt{0.1}} - \frac{1}{2}\right) = 638.9 \Rightarrow v = 25.3\,m/s$$

Note: velocity becomes infinitely large as particle approaches $x = 0$
(see in the figure above)

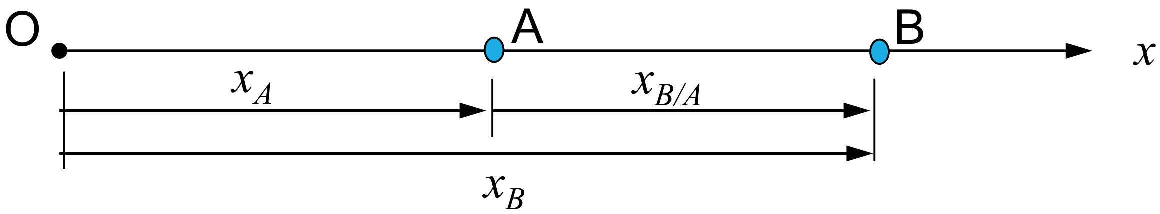
Motion of Several Particles

We may be interested in the motion of several different particles, whose motion may be independent or linked together (dependent).



*Relative motion of particles

Consider two particles A and B



For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

Relative position of B with respect to A

$$x_{B/A} = x_B - x_A \qquad (4.14a)$$

or

$$x_B = x_A + x_{B/A} \qquad (4.14b)$$

Relative velocity of B with respect to A

$$v_{B/A} = v_B - v_A \qquad (4.15a)$$

or

$$v_B = v_A + v_{B/A} \qquad (4.15b)$$

Relative acceleration of B with respect to A

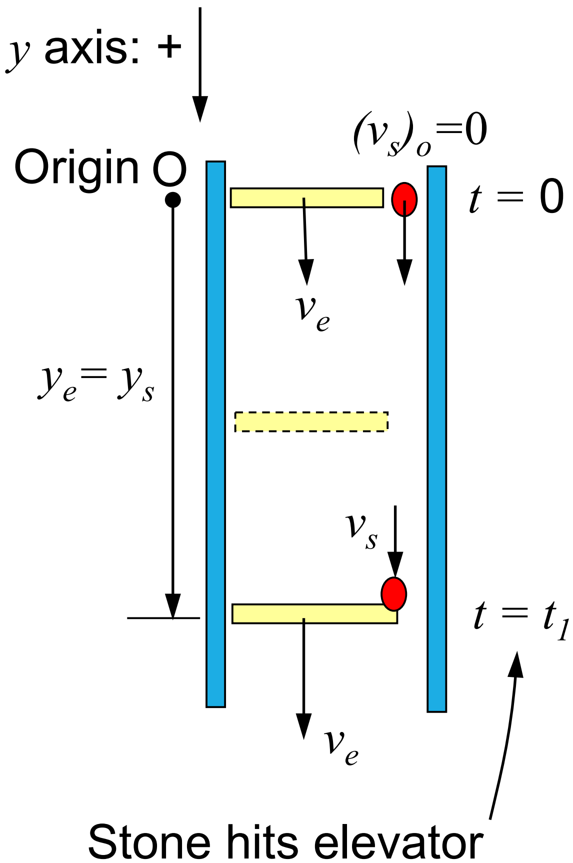
$$a_{B/A} = a_B - a_A \qquad (4.16a)$$

or

$$a_B = a_A + a_{B/A} \qquad (4.16b)$$

Example 3.3

Elevator moving downward at v_e = constant dislodges stone at t = 0. (a) If the stone starts falling with v_s = 0, show that the stone will hit the elevator with $v_{s/e} = v_e$. (b) If v_e = 7.5 m/s, when and where will the stone hit the elevator?



$$\text{Elevator: Uniform rectilinear motion}$$

$$v_e = \text{constant, } so, a_e = 0,\ \ y_e = v_e t$$

$$\text{Stone: Uniformly accelerated rectilinear motion}$$

$$a = g = 9.81\, m/s^2,$$

$$v_s = (v_s)_{t=0} + at = gt, \quad y_s = \frac{1}{2} g t^2$$

$$\text{(a) Stone hits elevator: } t = t_1$$

$$y_e = y_s \Rightarrow v_e t_1 = \frac{1}{2} g t_1^2 \Rightarrow t_1 = 2\frac{v_e}{g}$$

$$v_{s/e} = v_s - v_e = g t_1 - v_e = g\left(\frac{2v_e}{g}\right) - v_e = v_e \qquad v_s = 2v_e$$

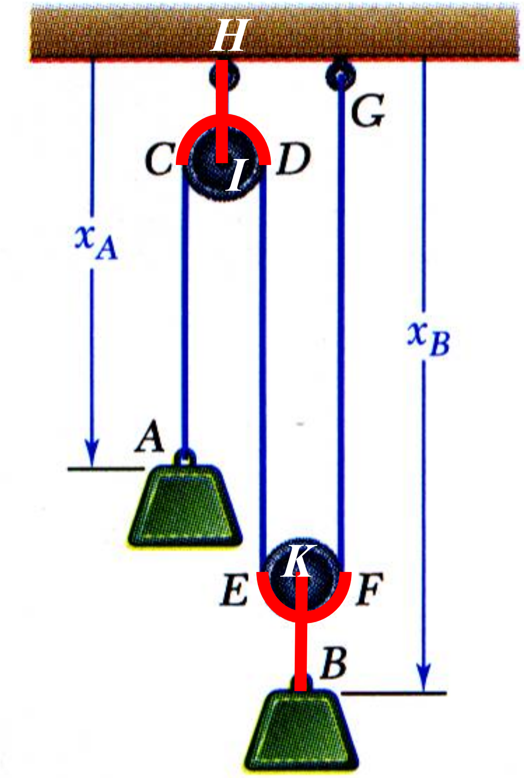
$$\text{(b) For } v_e = 7.5\, m/s: \quad t_1 = 2\frac{v_e}{g} = 2\left(\frac{7.5}{9.81}\right) = 1.529\, \text{sec}$$

$$y_e = v_e t_1 = 7.5 \times 1.529 = 11.47\, m$$

*Dependent motions

Constrained Motion of Connected Particles

When several blocks are connected by inextensible cords/ropes/strings/chains, it is possible to write linear relations between their position coordinates, velocities and accelerations. The number of independent linear relations is equal to the number of inextensible cords.



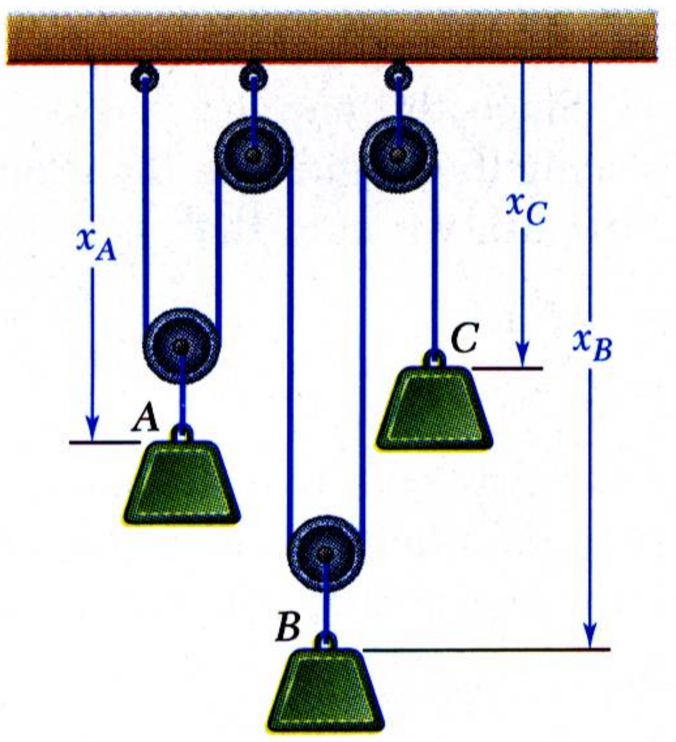
- Position of block *B* depends on position of block *A*. Since rope is of constant length, it follows that sum of lengths of segments must be constant.

The length of the rope is a constant, l = ***AC***+*CD*+***DE***+*EF*+***FG***
The length of red segments (*CD, EF, HI, BK*) keeps a constant during the motion no matter how the blocks move.

$$x_A + 2x_B = (\boldsymbol{AC} + HI) + (HI + \boldsymbol{DE} + BK) + (BK + \boldsymbol{FG})$$
$$= \underbrace{(l - CD - EF) + 2HI + 2BK}_{\text{All are constants}} = Const.$$

Example 3.4

Find the linear relations between position coordinates, velocities and accelerations of three blocks



- Positions of three blocks are dependent.

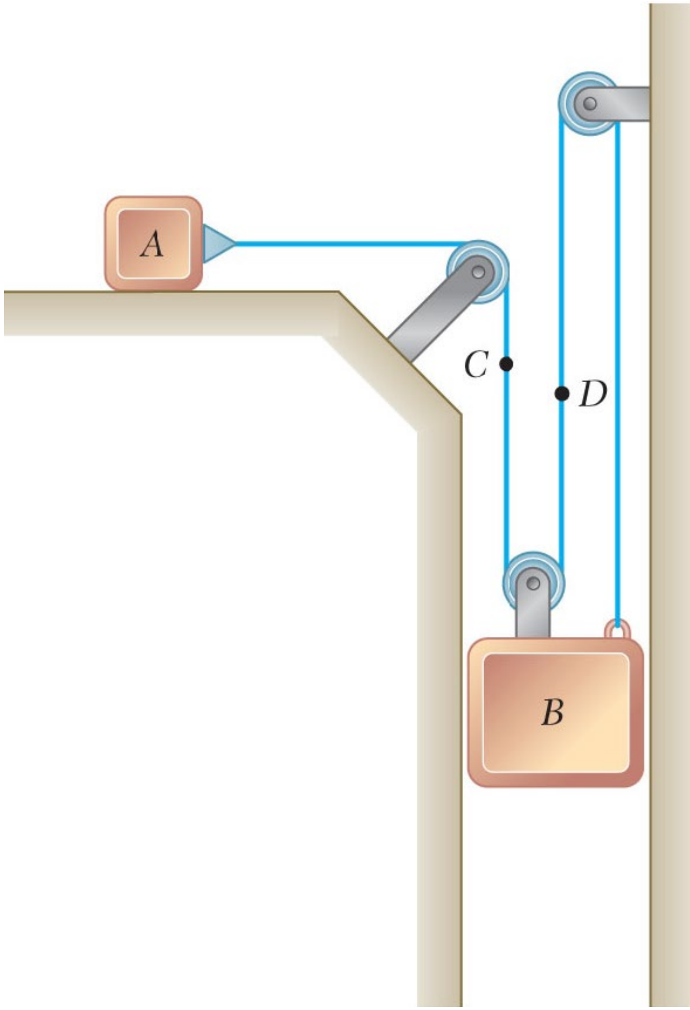
  $2x_A + 2x_B + x_C = \text{constant}$ (1)

- For linearly related positions, similar relations also hold between velocities and accelerations.

$$2\frac{dx_A}{dt} + 2\frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0 \qquad (2)$$

$$2\frac{dv_A}{dt} + 2\frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0 \qquad (3)$$

Example 3.5

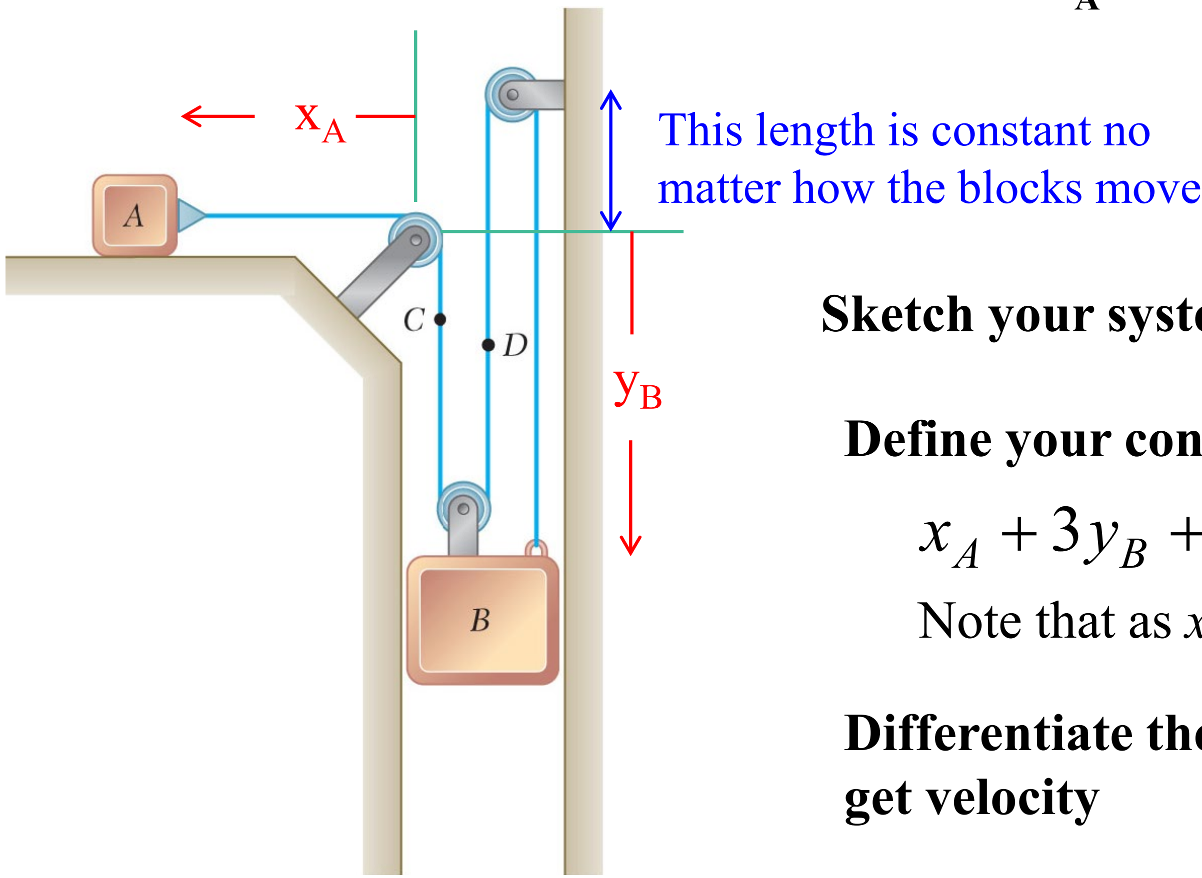


Slider block A moves to the left with a constant velocity of 6 m/s. Determine the velocity of block B.

Suggested steps:

1. **Sketch your system and choose coordinate system**
2. **Write out constraint equation (one rope one equation)**
3. **Differentiate the constraint equation to get velocity**

Solution



Given: v_A = 6 m/s left   Find: v_B

Sketch your system and choose coordinates

Define your constraint equation(s)

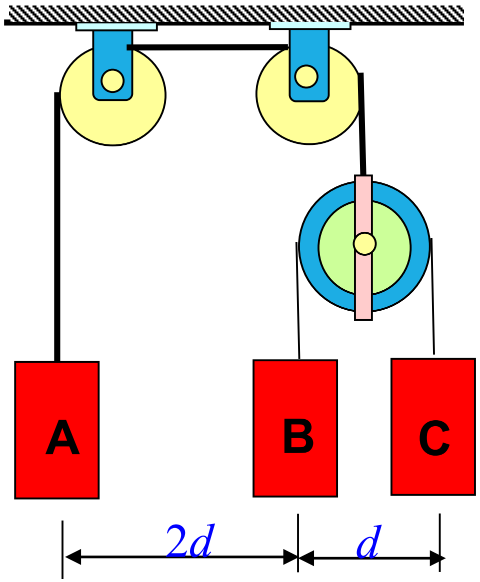
$$x_A + 3y_B + \text{constants} = L$$

Note that as x_A gets bigger, y_B gets smaller.

Differentiate the constraint equation to get velocity

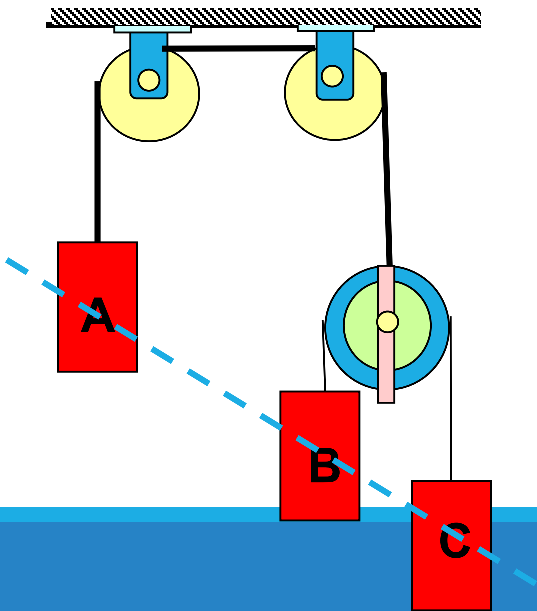
$$6 \text{ m/s} + 3v_B = 0 \qquad \boxed{\mathbf{v}_B = 2 \text{ m/s} \uparrow}$$

Example 3.6



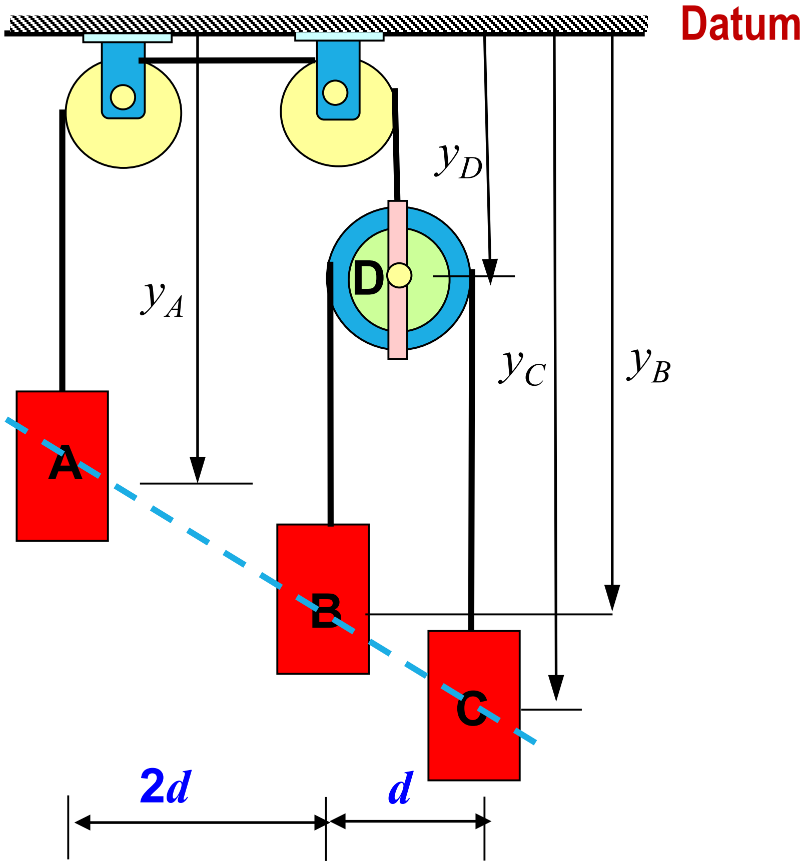
Blocks A, B and C move vertically with constant velocities. At $t = 0$, the three blocks are at the same level and $v_{C/B} = 200$ mm/s (downwards). Determine the velocity of each block so that the three blocks will remain in a straight line alignment during their motion.

We can determine the direction of motion of 3 blocks without knowing their mass. How?



Answer: Using $v_C - v_B > 0$ and ABC in a straight line

Solution



We first define the datum. Then we choose the positive direction, say downwards.

<u>Cord AD</u>: $y_A + y_D = \text{constant}$

By differentiating with respect to time, we get

$$v_A + v_D = 0 \Rightarrow v_D = -v_A \qquad ...(1)$$

<u>Cord BDC</u>: $(y_B - y_D) + (y_C - y_D) = \text{constant}$

By differentiating with respect to time, we get

$$v_B + v_C - 2v_D = 0 \qquad ...(2)$$

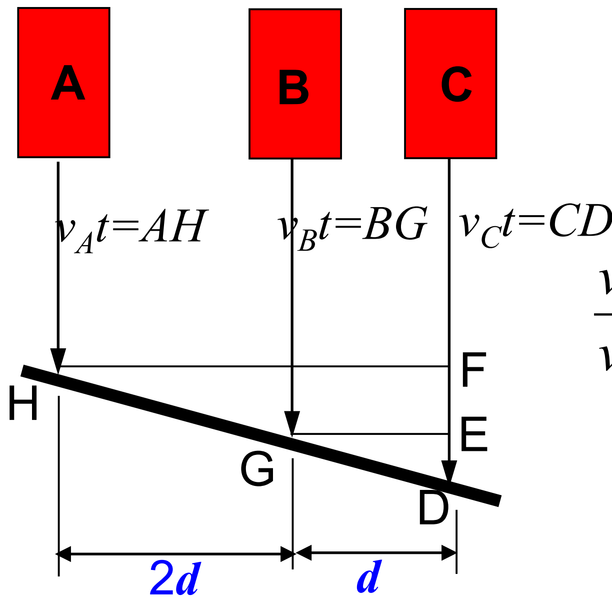
In view of (1), we can write $v_B + v_C + 2v_A = 0 ...(3)$

Given: $v_{C/B} = v_C - v_B = 200 \quad ....(4)$

Note that Eqs. (3) and (4) involves 3 unknowns (v_A, v_B, v_C). Therefore, one more equation is needed.

The number of cords is the number of equations of position (velocity or acceleration)

Solution



The additional equation is obtained from the condition of straight line alignment. From similar triangles, we obtain

$$\frac{v_C t - v_B t}{v_C t - v_A t} = \frac{DE}{DF} = \frac{EG}{FH} = \frac{d}{3d} \Rightarrow 2v_C - 3v_B + v_A = 0 \quad ...(5)$$

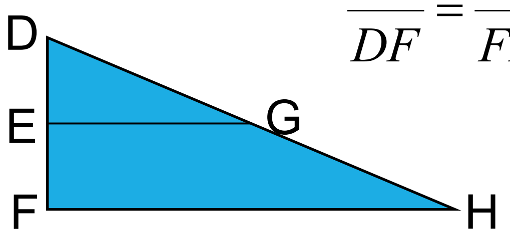
By solving Eqs. (3), (4) and (5) simultaneously, we obtain

$$v_A = -250\, mm/s$$
$$v_B = 150\, mm/s$$
$$v_C = 350\, mm/s$$

Positive means downwards
Negative means upwards

Note: in $\triangle$ DFH:

$$\frac{DE}{DF} = \frac{EG}{FH}$$



Curvilinear motion

Roller-coaster car and flyer capsule both undergo curvilinear motion.

A particle moving along a curve other than a straight line is in *curvilinear motion*.

*Position, velocity and acceleration

Let $\underline{r}$ be a position vector which is a function of time and space.

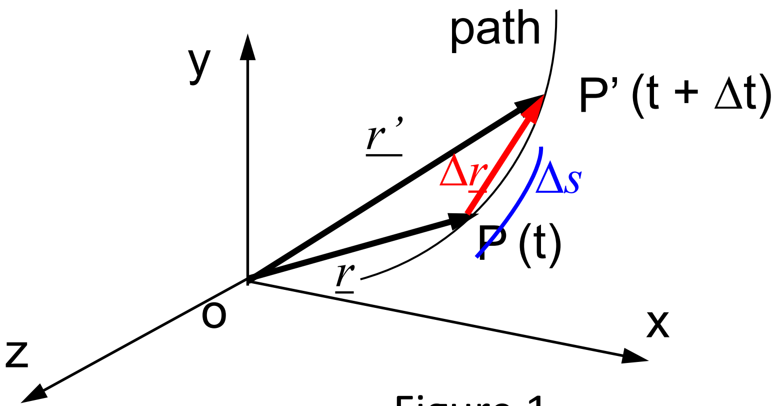


Figure 1

In Figure 1:

$\Delta\underline{r}$ (vector displacement) represents a change in direction and a change in magnitude of $\underline{r}$

Δs : scalar distance

In Figure 2:

$\dfrac{\Delta\underline{r}}{\Delta t}$ = average velocity

(the same direction as $\Delta\underline{r}$)

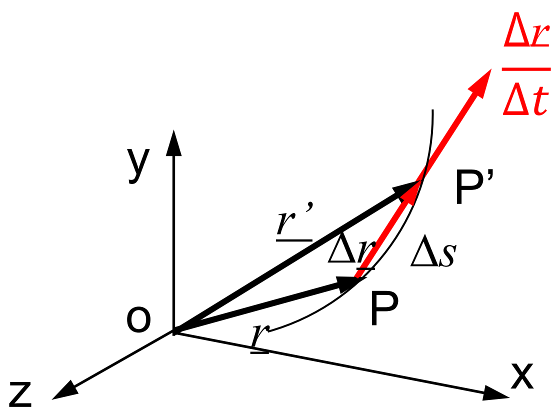


Figure 2

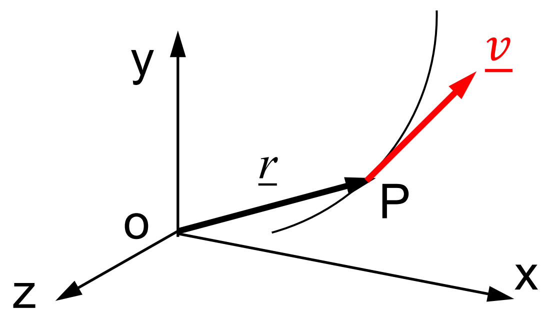


Figure 3

In Figure 3:

$\underline{v}$ = instantaneous velocity

$$\boxed{\underline{v} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} = \frac{d\underline{r}}{dt} \qquad (4.17)}$$

$\boxed{d\underline{r}/dt}$ is tangent to the curve at P.

(as $\Delta\underline{r}$ and Δt are infinitesimal)

*Position, velocity and acceleration

$$\underline{a} = \lim_{\Delta t \to 0} \frac{\Delta \underline{v}}{\Delta t} = \frac{d\underline{v}}{dt} = \frac{d^2 \underline{r}}{dt^2} \qquad (4.18)$$

Direction of $\underline{a}$ is arbitrary but its normal component is always towards the centre of curvature of the path. Do not make the mistake of assuming that $\underline{a}$ is tangent to the path as $\underline{v}$.

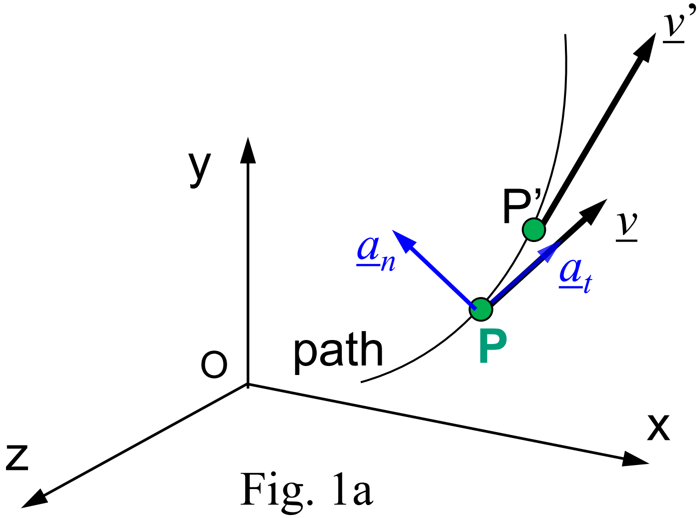


Fig. 1a

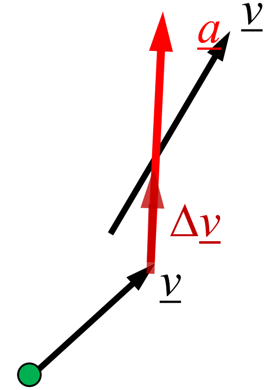


Fig. 1b

Note:

1. In order to find vector $\Delta\underline{v}$ we need to set $\underline{v}$ and $\underline{v}'$ from the same point
2. Then, the direction of $\underline{a}$ is the same as $\Delta\underline{v}$ (see Fig. 1b).
3. $\underline{a}$ can be resolved into normal and tangential components (see Fig. 1a)

Components of velocity & acceleration

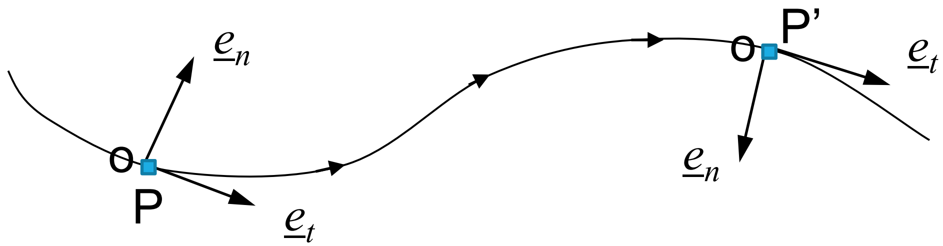
Coordinate Systems for Curvilinear Motion

We focus on **Cartesian coordinates** in the course

- **Rectangular Components**

$$\underline{r} = \underline{r}(x, y, z, t)$$

- **Normal and Tangential Components**



Note:

1) Origin and coordinates are moving with the particle! The particle is always on the origin O. Thus, $\underline{r}$ is always equal to zero.

2) $\underline{e}_t$ and $\underline{e}_n$ are unit vectors (magnitude is 1), representing the tangential and normal direction, respectively. $\underline{e}_n$ always points towards the center of path curvature.

Rectangular components

$$\underline{r} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k} \qquad (4.19)$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k} \qquad (4.20)$$

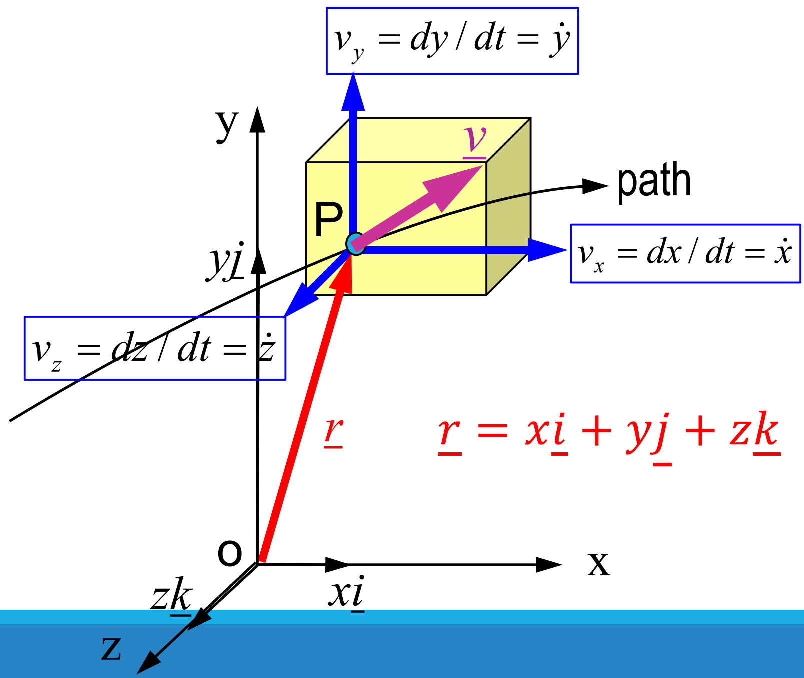
$$\underline{a} = \frac{d^2\underline{r}}{dt^2} = \ddot{x}(t)\underline{i} + \ddot{y}(t)\underline{j} + \ddot{z}(t)\underline{k} \qquad (4.21)$$

New notations:

$$\dot{x} = \frac{dx}{dt} = v_x; \qquad \ddot{x} = \dot{v} = \frac{dv}{dt} = a_x$$

$$\dot{y} = v_y; \qquad \ddot{y} = a_y$$

$$\dot{z} = v_z; \qquad \ddot{z} = a_z$$

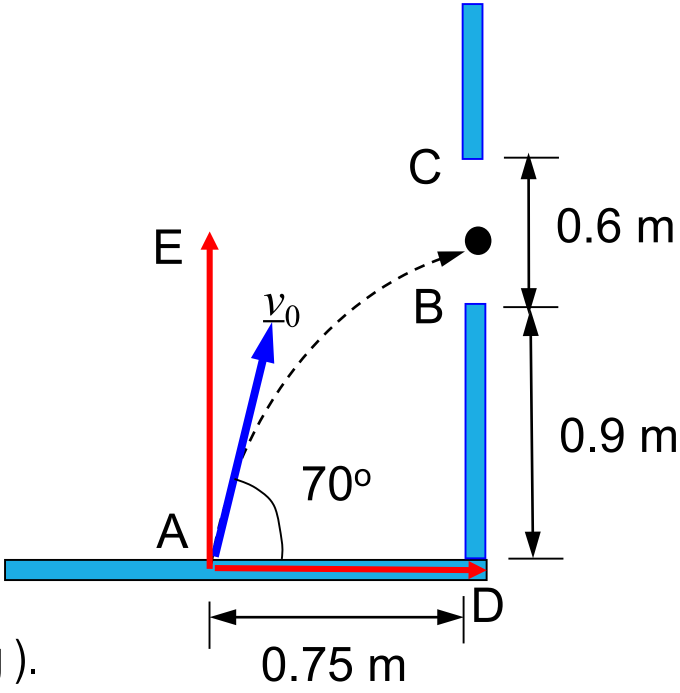


Example 3.7

Curvilinear motion of a particle may be treated as a superposition of rectilinear motions of that particle in three perpendicular directions.

The rectangular coordinates system is the simplest coordinates system. It is particularly effective in the study of motion of projectiles.

Find the range of values of v_0 for which the ball will enter the opening BC, i.e., $h_B < h_{Ball} < h_C$

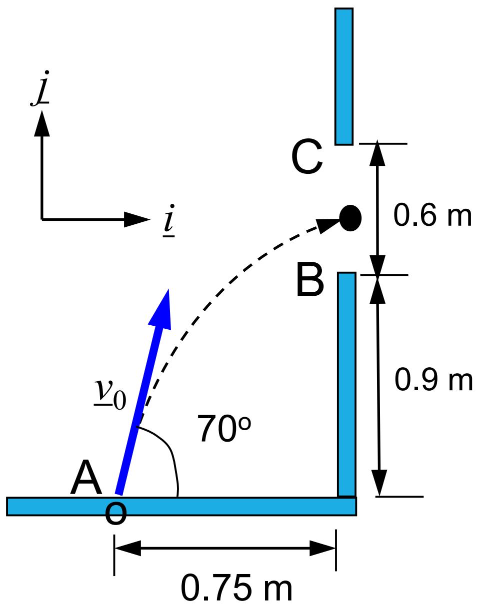


Tips: Motion of projectile (A ➡B) could be replaced by two dependent rectilinear motions, which have the same time.

1. Horizontal motion (A➡D) is uniform with **constant speed.**

2. Vertical motion (A➡E) is uniform with **constant acceleration** (a = -g).

Solution



At Point A : $(v_x)_0 = v_0 \cos 70^o = 0.3420 v_0$; $(v_y)_0 = v_0 \sin 70^o = 0.9397 v_0$

<u>Horizontal Motion</u> : $x = (v_x)_0 t = 0.3420 v_0 t$

At B or C, $x = 0.75\, m$

$$\therefore \quad 0.75 = 0.3420 v_0 t \Rightarrow t = \frac{2.19285}{v_0}$$

<u>Vertical Motion</u> : $y = (v_y)_0 t - \frac{1}{2} g t^2 = 0.9397 v_0 t - \frac{1}{2}(9.81) t^2$

$$= 0.9397(2.19285) - \frac{1}{2}(9.81)\left(\frac{2.19285}{v_0}\right)^2$$

<u>At Point B</u> : $y_B = 0.9\, m$

For ball to enter opening, $y > y_B \Rightarrow 2.0606 - \frac{23.5861}{v_0^2} > 0.9$

$$\Rightarrow v_0 > 4.508\, m/s$$

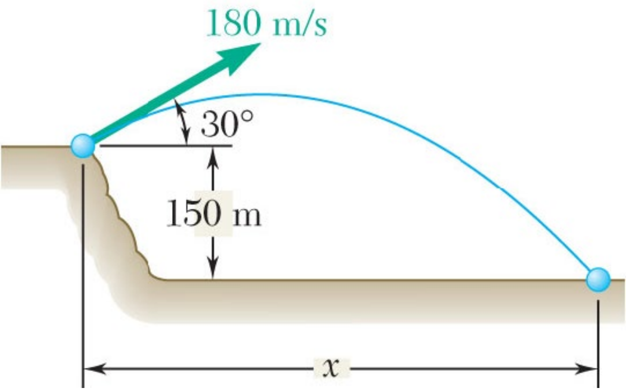
<u>At Point C</u> : $y_C = 1.5\, m$

For ball to enter opening, $y < y_C \Rightarrow 2.0606 - \frac{23.5861}{v_0^2} < 1.5$

$$\Rightarrow v_0 < 6.486\, m/s$$

Therefore, for ball to enter opening BC : $4.508 < v_0 < 6.486\, m/s$

Example 3.8



A projectile is filed from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30° with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.

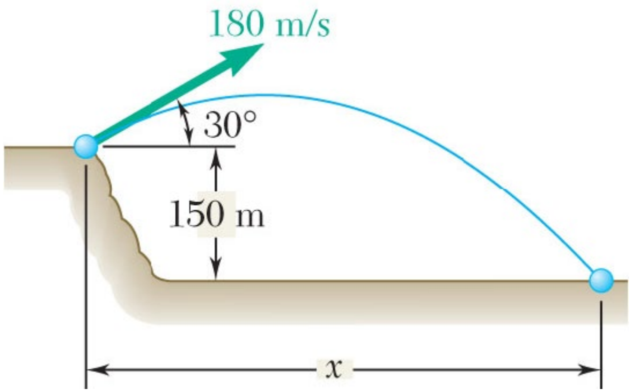
SOLUTION:

Given: $(v)_o = 180$ m/s $\quad (y)_o = 150$ m

$(a)_y = -9.81$ m/s^2 $\quad (a)_x = 0$ m/s^2

The vertical and the horizontal motion will be considered separately.

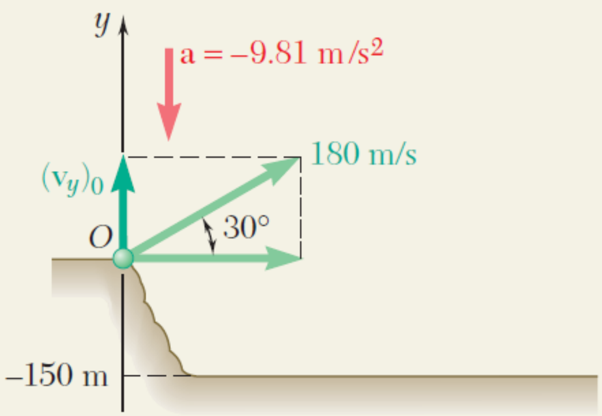
Solution



SOLUTION:

Given: $(v)_o$ =180 m/s $(y)_o$ =150 m

$(a)_y$ = - 9.81 m/s^2 $(a)_x$ = 0 m/s^2



The vertical and the horizontal motion will be considered separately.

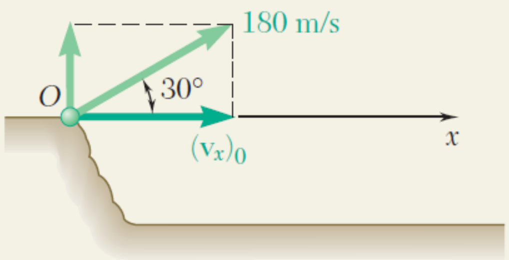
Vertical Motion. ***Uniformly Accelerated Motion.*** Choosing the positive sense of the y axis upward and placing the origin O at the gun, we have

$$(v_y)_0 = (180 \text{ m/s}) \sin 30° = +90 \text{ m/s}$$
$$a = -9.81 \text{ m/s}^2$$

Substituting into the equations of uniformly accelerated motion, we have

$$v_y = (v_y)_0 + at \qquad v_y = 90 - 9.81t \qquad (1)$$
$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = 90t - 4.90t^2 \qquad (2)$$
$$v_y^2 = (v_y)_0^2 + 2ay \qquad v_y^2 = 8100 - 19.62y \qquad (3)$$

Solution



Horizontal Motion. ***Uniform Motion.*** Choosing the positive sense of the x axis to the right, we have

$$(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$$

Substituting into the equation of uniform motion, we obtain

$$x = (v_x)_0 t \qquad x = 155.9t \qquad (4)$$

a. Horizontal Distance. When the projectile strikes the ground, we have

$$y = -150 \text{ m}$$

Carrying this value into Eq. (2) for the vertical motion, we write

$$-150 = 90t - 4.90t^2 \qquad t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

Carrying $t = 19.91$ s into Eq. (4) for the horizontal motion, we obtain

$$x = 155.9(19.91) \qquad x = 3100 \text{ m} \blacktriangleleft$$

b. Greatest Elevation. When the projectile reaches its greatest elevation, we have $v_y = 0$; carrying this value into Eq. (3) for the vertical motion, we write

$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

$$\text{Greatest elevation above ground} = 150 \text{ m} + 413 \text{ m} = 563 \text{ m} \blacktriangleleft$$

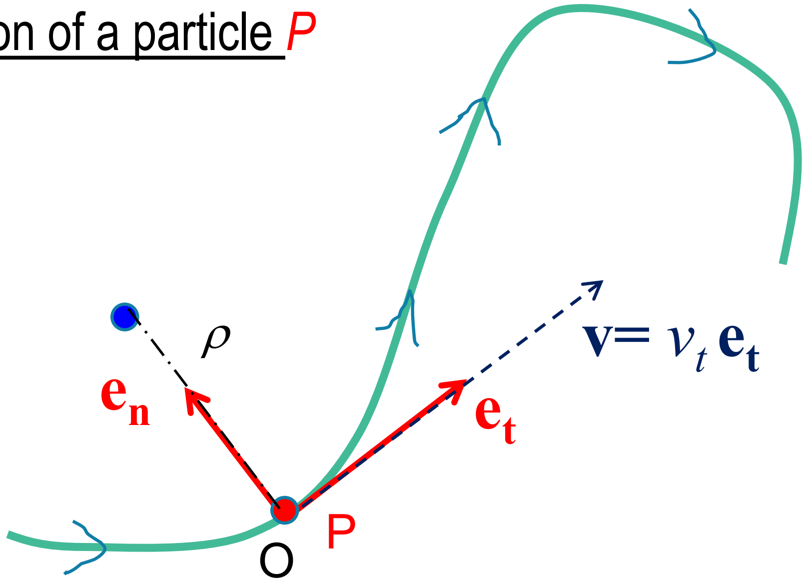
Tangential and Normal Components

If we have an idea of the path of a vehicle, it is often convenient to analyze the motion using tangential and normal components (sometimes called *path* coordinates).

Tangential and Normal Components

The acceleration, in general, is not tangent to the path. It is thus sometimes more convenient to resolve the acceleration into components directed, respectively, along the tangent and the normal to the path of the particle.

Plane motion of a particle *P*



$$\mathbf{v} = v\,\mathbf{e_t}$$

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e_t} + \frac{v^2}{\rho}\mathbf{e_n}$$

ρ = the instantaneous radius of curvature

The origin *o* and *t-n* coordinates move along the path with the particle!

1. Positive $\underline{e}_t$ is tangential to the path and pointing toward the direction of motion. Velocity vector of the particle is in this direction.

2. Positive $\underline{e}_n$ is towards the centre of curvature C of the path.

3. The acceleration can have components in both $\underline{e}_t$ and $\underline{e}_n$ directions.

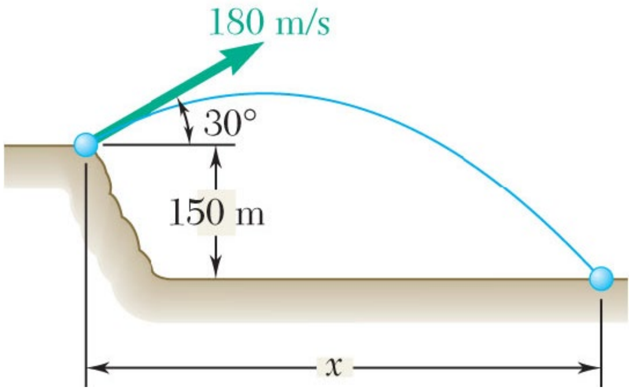
Conceptual quiz

In 2001, a race scheduled at the Texas Motor Speedway was cancelled because the normal accelerations ($\boldsymbol{a}_n$) were too high and caused some drivers to experience excessive g-loads (similar to fighter pilots) and possibly pass out. As a mechanical engineer, can you propose some solutions to solve this problem?

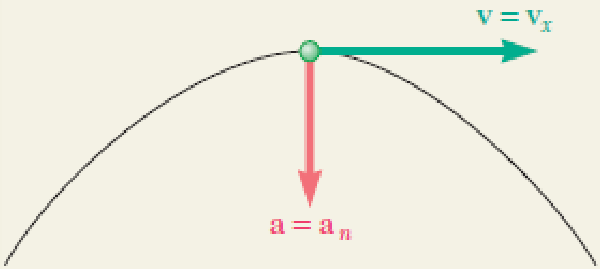
Some possibilities:

1. **Reduce the allowed speed**
2. **Increase the turn radius (difficult and costly)**
3. **Have the racers wear anti-g suits**

Example 3.9



A projectile is filed from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30º with the horizontal. Neglecting air resistance, determine the minimum radius of curvature of the trajectory described by the projectile.

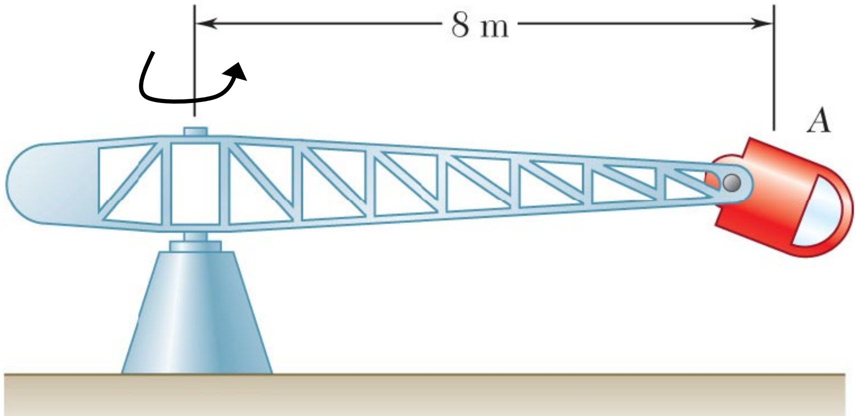


Since $a_n = v^2/\rho$, we have $\rho = v^2/a_n$. The radius will be small when v is small or when a_n is large. The speed v is minimum at the top of the trajectory since $v_y = 0$ at that point; a_n is maximum at that same point, since the direction of the vertical coincides with the direction of the normal. Therefore, the minimum radius of curvature occurs at the top of the trajectory. At this point, we have

$$v = v_x = 155.9 \text{ m/s} \qquad a_n = a = 9.81 \text{ m/s}^2$$

$$\rho = \frac{v^2}{a_n} = \frac{(155.9 \text{ m/s})^2}{9.81 \text{ m/s}^2} \qquad \rho = 2480 \text{ m}$$

Example 3.10



The tangential acceleration of the centrifuge cab is given by

$$a_t = 0.5t \quad (\text{m/s}^2)$$

where t is in seconds and a_t is in m/s^2. If the centrifuge starts from fest, determine the total acceleration magnitude of the cab after 10 seconds.

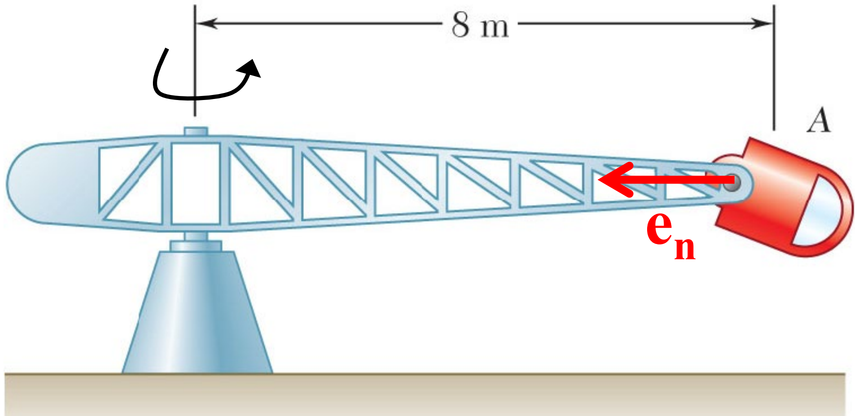
Suggested steps:

- Define your coordinate system
- Calculate the tangential velocity and tangential acceleration
- Calculate the normal acceleration
- Determine overall acceleration magnitude

Solution

Define your coordinate system

In the side view, the tangential direction points into the "page"



Determine the tangential velocity

$$a_t = 0.5t$$

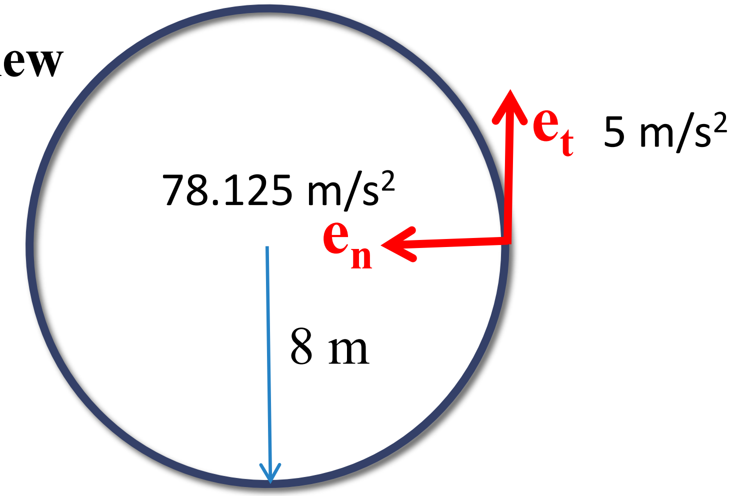
$$v_t = \int_0^t 0.5\, t\, dt = 0.25t^2 \Big|_0^t = 0.25t^2$$

$$v_t = 0.25(10)^2 = 25 \text{ m/s}$$

Determine the normal acceleration

$$a_n = \frac{(v_t)^2}{r} = \frac{25^2}{8} = 78.125 \text{ m/s}^2$$

Top View



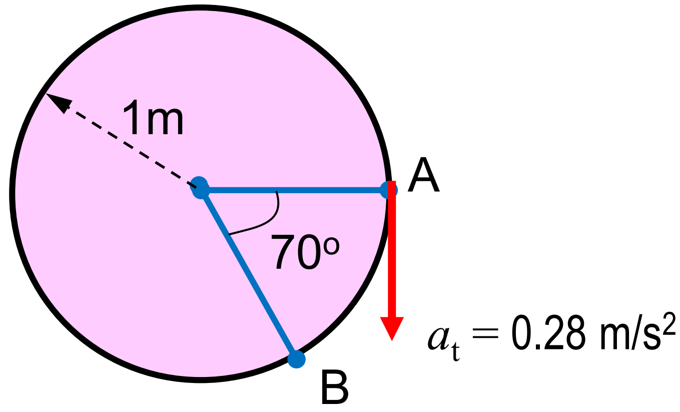
Determine the total acceleration magnitude

$$a_{mag} = \sqrt{a_n^2 + a_t^2} = \sqrt{78.125^2 + [(0.5)(10)]^2}$$

$$\boxed{a_{mag} = 78.285 \text{ m/s}^2}$$

Example 3.11

A particle moves along a circular path with a constant tangential acceleration equals to 0.28 m/s^2. The particle starts from **rest** at point A. Determine the velocity and acceleration when the particle reaches point *B*.



Given: r = 1m; v_0 = 0 m/s; θ = 70°; a_t = 0.28 m/s^2

$$s = r\theta = 1 \times \frac{70}{360} \times 2\pi = 1.2217\, m$$

$$v_B^2 = 2a_t(s - s_o) = 2(0.28)(1.2217) \Rightarrow v_B = 0.827\, m/s$$

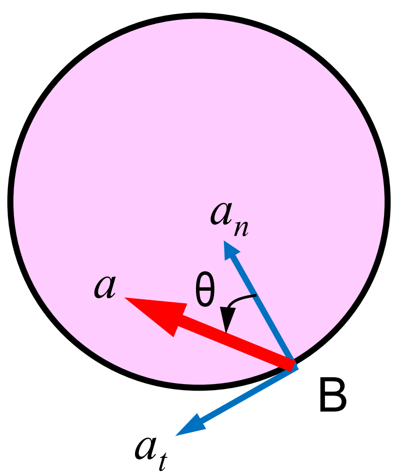
At point *B*:

$$a_n = \frac{v_B^2}{\rho} = \frac{(0.827)^2}{1} = 0.684\, m/s^2$$

$$a_t = 0.28\, m/s^2$$

$$\therefore\ a = \sqrt{a_t^2 + a_n^2} = 0.739\, m/s^2$$

$$\theta = \tan^{-1}\frac{0.28}{0.684} = 22.3^o$$



*Summary of kinematics of particles

Rectilinear motion (**1D** motion):

$f(\underline{x}, \underline{v}, \underline{a}, t) = 0$ **4 equations, 4 problems, 2 special cases**

Curvilinear motion (**2D** motion):

Resolving vectors $\underline{r}$, $\underline{v}$, $\underline{a}$ into components

- **Rectangular**

$$\underline{r} = x(t)\underline{i} + y(t)\underline{j}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j}$$

$$\underline{a} = \frac{d^2\underline{r}}{dt^2} = \ddot{x}(t)\underline{i} + \ddot{y}(t)\underline{j}$$

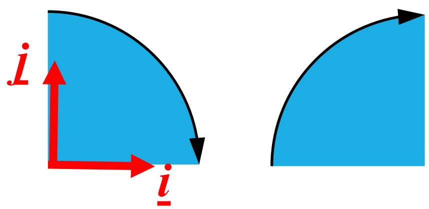
- **Tangential & normal**

$$\underline{v} = v\underline{e}_t$$

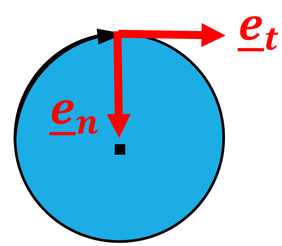
$$\underline{a} = \frac{dv}{dt}\underline{e}_t + \frac{v^2}{\rho}\underline{e}_n$$

2D planar motion of particles only

Tips:



motion of projectiles
(recommend rectangular components)



Circular motion
(Suggest using T&N components)