4. If the boundary conditions were instead *homogeneous Neumann boundary conditions* at $x = 0$ and $x = \pi$,

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=\pi} = 0,$$

and the initial condition $u(x, 0) = \cos(x)$, what is the solution to the heat equation, $\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$?
Solution: The first step is separation of variables: let $u(x,t) = \sum_\alpha T_\alpha(t) X_\alpha(x)$. We take a single mode, $T_\alpha X_\alpha$, and put it into the PDE,

$$\frac{\partial T_\alpha X_\alpha}{\partial t} = \beta \frac{\partial^2 T_\alpha X_\alpha}{\partial x^2} \quad \Rightarrow \quad \frac{1}{\beta}\frac{T'_\alpha}{T_\alpha} = \frac{X''_\alpha}{X_\alpha} = \alpha$$

Solving for T_α we get $T_\alpha(t) = T_\alpha(0)e^{\alpha\beta t}$. Solving for X_α is more complicated, as there are three cases:

(a) $\alpha = \lambda^2 > 0$: In this case we get $X_\alpha = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x)$. To match the boundary conditions, we need $X'_\alpha = \lambda C_1 \sinh(\lambda x) + \lambda C_2 \cosh(\lambda x)$. Applying the first boundary condition,

$$X'_\alpha(0) = \lambda C_2 = 0 \quad \Rightarrow \quad C_2 = 0,$$

so $X_\alpha = \lambda C_1 \cosh(\lambda x)$. Applying the second boundary condition gives

$$X'_\alpha(\pi) = \lambda C_1 \sinh(\lambda\pi) = 0.$$

Since $\lambda \sinh(\lambda\pi) \neq 0$, we conclude that $C_1 = 0$, and that this mode is trivial.

(b) $\alpha = 0$: In this case $X_\alpha = C_1 + C_2 x$, and $X'_\alpha = C_2$. Then the left boundary condition gives

$$X'_\alpha(0) = C_2 = 0$$

and the right boundary condition gives us the same thing:

$$X'_\alpha(\pi) = C_2 = 0.$$

Let's use $a_0/2$ instead of C_1, and we're left with $X_\alpha = a_0/2$, a constant.

(c) $\alpha = -\lambda^2 < 0$: In this case $X_\alpha = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$, and $X'_\alpha = -C_1 \lambda \sin(\lambda x) + C_2 \lambda \cos(\lambda x)$. The left boundary condition implies that

$$X'_\alpha(0) = C_2 \lambda = 0$$

and, since $\lambda \neq 0$ in this case, we get $C_2 = 0$, so that $X_\alpha = C_1 \cos(\lambda x)$. the right boundary condition implies that

$$X'_\alpha(\pi) = -C_1 \lambda \sin(\lambda\pi) = 0$$

which implies that either $C_1 = 0$ or $\sin(\lambda\pi) = 0$. This second case happen when $\lambda\pi = n\pi$, i.e. $\lambda = n$. Thus, the non-trivial modes for this case are

$$X_\alpha = C_2 \cos(nx).$$

Having determined all the modes $T_\alpha X_\alpha$, we can put them into the series solution for u, which is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta n^2 t} \cos(nx).$$

The initial conditions are $u(x,0) = \cos(x)$, so

$$u(0,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \cos(x),$$

which is a Fourier cosine series for $\cos(x)$. We could use the formula for cosine series to calculate a_n, but, since $\cos(x)$ is already a Fourier cosine series, we can just match term-by-term, getting $a_1 = 1$, and $a_n = 0$ if $n \neq 1$. The solution is then

$$\begin{aligned} u(x,t) &= e^{-\beta 1^2 t} \cos(1x) \\ &= e^{-\beta t} \cos(x). \end{aligned}$$