# Heat Equation Flowchart

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## The Heat Equation

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

## Separation of Variables

$$u(x,t) = \sum_\alpha T_\alpha(t) X_\alpha(x)$$

$$\frac{X''_\alpha}{X_\alpha} = \frac{T'_\alpha}{\beta T_\alpha} = \alpha$$

## Modes

$$T_\alpha(t)X_\alpha(x) = \begin{cases} e^{\alpha\beta t}\left[C_1(\alpha)\cos(\sqrt{-\alpha}x) + C_2(\alpha)\sin(\sqrt{-\alpha}x)\right], & \alpha < 0 \\ C_1(0) + C_2(0)x, & \alpha = 0 \\ e^{\alpha\beta t}\left[C_1(\alpha)\cosh(\sqrt{\alpha}x) + C_2(\alpha)\sinh(\sqrt{\alpha}x)\right], & \alpha > 0 \end{cases}$$

## Boundary Conditions?

### Homogeneous Dirichlet

$u(0,t) = 0$, $u(L,t) = 0$

$$T_\alpha(t)X_\alpha(0) = 0$$
$$T_\alpha(t)X_\alpha(L) = 0$$

$$u(x,t) = \sum_{n=1}^\infty b_n e^{-\beta \frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)$$

$u(x,0) = f(x)$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(nx)\,dx$$

### Non-Homogeneous Dirichlet

$u(0,t) = u_0$, $u(L,t) = u_L$

$\alpha \neq 0$:
$$T_\alpha(t)X_\alpha(0) = 0$$
$$T_\alpha(t)X_\alpha(L) = 0$$

$\alpha = 0$:
$$T_0(t)X_0(0) = u_0$$
$$T_0(t)X_0(L) = u_L$$

$$u(x,t) = u_0 + \frac{u_L - u_0}{L}x + \sum_{n=1}^\infty b_n e^{-\beta \frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)$$

$u(x,0) = f(x)$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - u_0 - \frac{u_L - u_0}{L}x\right)\sin(nx)\,dx$$

### Homogeneous Neumann

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L} = 0$$

$$T_\alpha(t)X'_\alpha(0) = 0$$
$$T_\alpha(t)X'_\alpha(L) = 0$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n e^{-\beta \frac{n^2\pi^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right)$$

$u(x,0) = f(x)$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(nx)\,dx$$

## Solution

$u(x,t)$