Modality and Alternative Functional Harmony in Diatonic and Non-diatonic Scales

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Abstract

The topic of this thesis is the properties of scales and modes other than the familiar major and minor, and the possibilities these afford for extended tonality and alternative functional harmony. In chapter one, I develop a systematic account of how to describe a large set of musically useful scales by using a generated line of intervals. This account is based on Hook's (2011) use of the line of fifths, but generalized to any eligible generating interval. In chapter two, I review the theoretical literature on various relevant scale properties, such as transpositional asymmetry and intervallic content. I discuss the implications of these properties when modes of these scales are realized in actual music. In chapter three, I discuss the placement of "tendency tones"—minor seconds that resolve to a member of the tonic triad—in various scales and modes, and how this placement affects the tonal implications of a given mode. In connection with this argument I introduce the idea of modal pitch-class sums and show that a mapping exists between these sums and a scale's structure upon the line of fifths. Finally, chapter four discusses how different structures of harmonic functions (analogous to the familiar tonic, subdominant, and dominant functions) can be realized within a mode. Building on the account of tendency tones developed previously, I discuss the contribution of these functions to the tonal dynamics of a given mode.

Abrégé

Ce mémoire porte sur les propriétés des gammes et des modes autres que le majeur et le mineur, et sur les possibilités qu'ils offrent pour la tonalité élargie et l'harmonie fonctionnelle non conventionnelle. Dans le premier chapitre, je développe une approche systématique permettant de décrire un large ensemble de gammes utiles musicalement en utilisant une chaîne générée d'intervalles. Cette approche est fondée sur Hook 2011 qui utilise une chaîne de quintes, mais applique sa méthode à tous les intervalles générateurs éligibles.

Dans le deuxième chapitre, je résume la littérature théorique sur les différentes propriétés pertinentes des gammes, telles que l'asymétrie transpositionnelle et le contenu intervallique. Je traite des implications de ces propriétés quand les modes de ces gammes sont réalisés dans la musique. Dans le troisième chapitre, je traite de l'emplacement des "tendency tones"–expression qui fait référence aux secondes mineures qui se résolvent à un membre de l'accord de tonique–dans différentes gammes et modes, et comment cet emplacement affecte les implications tonales d'un mode donné. En lien avec cet argument, j'introduis l'idée d'une somme des classes de hauteur et montre qu'une correspondance existe entre ces sommes et la structure d'une gamme sur la chaîne de quintes. Finalement, le chapitre 4 traite de la façon dont différentes structures des fonctions harmoniques (analogues aux familières fonctions de tonique, sous-dominante et dominante) peuvent être réalisées à l'intérieur d'un mode. En m'appuyant sur l'approche des "tendency tones" développée dans le chapitre précédent, je montre comment ces fonctions contribuent à la dynamique tonale d'un mode donné.

Acknowledgements

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If not for my thesis advisor, Jon Wild, I would certainly never have developed the thoughts outlined in this thesis. Complaints, therefore, will be most efficient if addressed directly to him. I must also thank Jon for his manifold assistance with matters intellectual, academic, and personal throughout my time at McGill, and—above all—for putting up with my daily barrage of emails without protest.

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As in all my endeavours, I am indebted to my parents, Ken Sailor and Jan Norris. Without their unwavering moral support, I would undoubtedly have felt obliged to do something responsible with my life.

Thanks, finally, to Marie-Ève Piché. Sailing the rarefied seas of music theory can be a lonely pursuit. I have been lucky to have such a close companion in my voyages across these strange waters. Thanks, Marie-Ève, for all the nifty chord progressions, heart-rending melodies, and moments great and small.

Contribution of Author

This Master's thesis is exclusively the work of the student, Malcolm Sailor.

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Web apps

As a companion project to this thesis, I built a suite of web applications that allow the user to experiment with many of the musical materials that I treat in the ensuing text. I will introduce each of these below when they are relevant to the discussion, but I have listed them here as well for easy reference. Although I presume that many of these apps were written in a spirit of play, and their output can be amusing, I hope that the reader will nevertheless feel that they serve a useful purpose in making concrete the sometimes abstract discussion of musical materials.

The technical knowledge required to build these apps was largely new to me, and so this project was an immense learning experience. This was alternately rewarding and frustrating, but in any case, the scale of the challenge means that the site is, at time of the initial thesis submission, still in something of a "beta" version. In particular, the midi playback is unreliable. For the best experience, Chrome or Safari browsers are recommended (Firefox does not seem to work as well). I would greatly appreciate feedback and bug reports!

The web apps are numbered according to the chapter of the thesis to which they are most relevant.

1.1 Generating scales (see section 1.5, page 46)

http://sailor.music.mcgill.ca/generatescale

2.1 Scale patterns (see section 2.5, page 75)

http://sailor.music.mcgill.ca/scalepatterns

2.2 Change mode of Bach keyboard work (see section 2.5, page 75)

http://sailor.music.mcgill.ca/bachchangemode

3.1 Combine tendency tones (see section 3.2, page 92)

http://sailor.music.mcgill.ca/tendencytones

4.1 Function structures (see 4.4, page 135)

http://sailor.music.mcgill.ca/funcstructs

4.2 Cycle-of-fifths function structures (see 4.4, page 135)

http://sailor.music.mcgill.ca/cycleoffifthsmodes

4.3 Cycle-of-seconds function structures (see 4.4, page 135)

http://sailor.music.mcgill.ca/cycleofsecondsmodes

Introduction

The topic of this thesis is the properties of scales and modes other than the familiar major and minor. These modes include not only the other modes of the diatonic scale, but also modes of other closely related scales such as the ascending melodic minor, as well as more unusual scales. I examine these questions from both an abstract perspective, inquiring about the scale-theoretic properties of different scale formations, as well as from a more concrete musical perspective, inspired by my first approach to this topic through my own work as a composer. I therefore ask questions of the form "How, and to what extent could such a scale or mode be realized compositionally? What would it *sound* like?" Above all, I ask about the possibilities allowed by the use of scales and modes other than major and minor for what I call "alternative functional harmony." Throughout, I largely constrain my investigations to heptatonic scales (which I refer to as 7-scales, for the ease of author and reader alike).

In the remainder of this introduction, I outline the plan of this thesis, and then define some of the terms that I will be using throughout.

In chapter one, I first explore some foundational questions about what scales are and what they are good for. I then develop a systematic account of how to build and describe scales. This account begins from Julian Hook's (2011) exposition of "spelled heptachords," where scales are constructed on the line of fifths. Whereas Hook only considers 12-tone equal temperament, I demonstrate how to extend this method of scale construction/description to generated lines for any size of generating interval that can construct a well-formed 7-scale (Carey and Clampitt 1989). Finally, circling back to 12-tone equal temperament, I show that this style of thinking reveals several perhaps advantageous features of our predominant tuning system.

Chapter two reviews the literature on scale properties, comparing the diatonic to non-diatonic scales. Features discussed include uneven step sizes, transpositional asymmetry, cardinality, intervallic features such as "diatonic seconds" and "diatonic thirds" (Tymockzo 2004), and consecutive semitones. I expound a "fuzzy" conception of these properties that orders the scales of equal temperament according to their "quasi-diatonicity" and allows a nuanced consideration of how different scales might be more or less suited to the composition of certain types of music. Finally, a brief appendix considers some alternative scale properties related to "diatonic thirds."

Building on the framework for constructing scales of the previous chapters, chapter three introduces the subject of mode and, in particular, considers how the placement of minor seconds within a mode affects the tonal implications—the tendencies towards tension and resolution—of that mode. Extrapolating from the familiar leading tone, I develop the idea of a "tendency tone" as a scale degree that can resolve by minor second to a member of a consonant tonic triad. The chapter introduces this subject by considering a combinatoric approach to tendency tones, and examining the resulting scales and modes. I then discuss how the arrangement of tendency tones in the diatonic modes is structured. This allows the construction of a "line of minor seconds" and a "circle of minor seconds" analogous to the line and circle of fifths, and this device allows us to expand our discussion of tendency tones to non-diatonic modes. In the course of the chapter, I introduce the idea of *modal pitch-class sums* as a metric for measuring the relative bright- or darkness of a mode. An appendix discusses a surprising mapping between the modal pitch-class sums of a scale and the same scale's structure upon the line of fifths.

Finally, chapter four considers how different modes permit different harmonic *functions*, analogous to the familiar tonal functions of tonic, dominant, and subdominant. I begin by exploring *complete function structures*—sets of triads that can account for all the pitches of a 7-scale. I show how minimal complete function structures—like tonic, dominant, and subdominant—are always segments of scalar interval cycles. I also show how the operation of functional substitution can operate to curl these segments themselves into cycles—an idea whose antecedents go back to Rameau (1737). Next, I consider the actual modes and scales that are formed by realizing function structures with consonant triads, and the tonal implications thereof. Finally, I explore how these various modes interact with Daniel Harrison's (1994) conception of functions as assemblies of scale degrees.

In general, tonally centric Western music has used a relatively circumscribed set of background scales, especially the diatonic scale, and a similarly circumscribed set of tonal centers within these scales (e.g., the major and minor modes). Some readers may find themselves wondering if many of the more unusual modes discussed herein really exist, in the sense that it is possible for a listener to infer the intended tonal center within the intended scale. This is an interesting (and non-trivial) question. Personally, my musical experience leads me to suspect that the answer is often yes but may sometimes be no. Nevertheless, this is not the question with which I am concerned in this thesis. Instead, this thesis simply assumes the possibility and perceptual validity of scales and modes. I will attempt to explain this choice in metaphorical terms: imagine a cross-country ski trail with a varied topography of hills, valleys, and plateaus. There are a number of starting points along the trail. At any of these, one can park, put on one's skis, and begin the trail, but the trail forms a loop, so that no matter where you began, you will ski through the whole trail before getting back to your car. This trail is a metaphor for the scales and modes discussed in this thesis. The whole trail is a scale, and beginning from any of the different starting points provides a different mode of this scale. This thesis treats two types of questions about the ski trail: 1) what is the topography of the trail? (that is, what are the properties of the scale?) and, 2) how do our different starting points affect the trajectory of our ski? (that is, what are the properties of the different modes of this scale?) There is, however, a third type of question that one might wish to ask about our ski trail: how does one get to the different starting-points along the trail? (that is, how does a piece of music establish a particular scale degree as the central tonic? How do we get from a *scale* to a *mode*?) And can we even get to the ski trail at all? (Perhaps it is located on another continent, and we don't have time or money to travel there.) I do not treat this third type of question here. In the present work, I simply assume that it is possible to treat various pitches within a scale as a central tonic, and therefore to establish a mode. In other words, this thesis will tell you about the sights you will see along the ski trail, and it will tell you how beginning at one or another starting-point will affect your ski, but it will not give you directions for how to drive to the different starting-points.

Some preliminary definitions

For the purposes of this thesis:

SCALES: *Scale* refers to a circularly-ordered collection of pitch-classes. In principle, a scale could have any number of pitch-classes, but in practice, collections with too few or too many pitch-classes tend not to be heard as scale-like (or used in a scalar manner by composers). Ordinarily, scales are ordered by ascending pitch-class[1] (e.g., (0, 2, 3, …))—these are what I will later call *natural* scales—but it is logically possible for a scale to have any other arbitrary order (e.g., (2, 0, 3, …)) and the spelling system of Western music (which we shall examine in greater detail in Chapter 1) is capable of accommodating this (e.g., C♯♯, D♭♭, E♭, …).

MODES: *Mode* refers to a scale, together with a centric pitch-class or *tonic*. For example, what we might usually call "the major scale" is more properly called "the major mode of the diatonic scale"—major[2] being just one of the seven modes of the diatonic scale. The number of modes of a scale is equal to the cardinality of that scale, divided by its number of degrees of transpositional symmetry. A transpositionally asymmetrical scale, such as all the 7-scales with which we shall be primarily concerned in this thesis,

[1] This statement is somewhat informal because, strictly speaking, there is no such thing as "ascending pitch-class." We can formalize it by saying that scales are ordered by clockwise procession on the circle of pitch-class space.

[2] In a modal context, the major mode is often referred to as "Ionian," but in general I will favour the more familiar "major" and "minor" over "Ionian" and "Aeolian."

has as many modes as it has pitch-classes, although some of these modes may be more musically useful than others.

INTERVALS: I use the traditional terms for intervals (e.g., "minor second," "diminished fourth," etc.). The important thing to note about these terms is that they measure distance within a scale, and I therefore call them *scalar intervals*. Their description is necessarily rich, consisting of two layers of description, *generic* and *specific*. Generic scalar intervals (e.g., "third") refer only to the number of scale steps between two pitches. Specific scalar intervals (e.g., "major third") refer both to the number of scale steps and the literal frequency ratio between the pitches. (In Chapter 1, we will see how to define scalar intervals in terms of directed spans on the generated line of fifths, and it is this sense that interval names will usually be intended in this thesis.)

As a scalar interval, the major third C–E (e.g., in C major) differs from the diminished fourth B♯–E (e.g., in C-sharp minor), just as we teach in rudiments class. Scalar intervals therefore differ from atonal intervals, which can simply be represented by integers (such as "4 semitones").[3] When music theorists speak about intervals in less traditional contexts, however, we frequently become somewhat less precise, using specific scalar interval names where they do not strictly speaking apply. For instance, we might speak of "a major seventh" in the context of the octatonic scale. If it is understood that, by "major seventh," we are simply referring to the interval of 11 semitones, then no harm is done by speaking this way. But in this thesis, because of the priority placed upon scalar thinking, I will maintain a stricter intervallic vocabulary, where, if we were to speak about the interval of 11 semitones in the octatonic scale as a scalar interval, we would have to call it an eighth. (Although "octave" is really another term for an "eighth,"

[3] My use of "specific scalar intervals" thus differs somewhat from Clough and Myerson's (1985) "specific intervals," which refers simply to what I call atonal intervals. When using "atonal" in contexts like this, I mean only that the description contains no information about tonal context, and instead merely measures the pitches or intervals absolutely. I don't mean that the music being described is necessarily atonal.

it is probably best to reserve this special term for the ratio 2:1.) From this rigorously scalar vocabulary, some perhaps counterintuitive consequences follow. For example, an atonal interval such as "4 semitones," in the absence of any implicit scale, neither is, nor is not, a major third,[4] and an octatonic scale (for instance) contains no major thirds, although it contains many intervals of 4 semitones. Moreover, in scales that differ radically from the diatonic, the familiar generic interval names like "third" will correspond to unusual atonal intervals (e.g., a third within the chromatic 7-chord {0, 1, 2, 3, 4, 5, 6} is either 2 semitones [i.e., a diminished third], as from 0 to 2, or 7 semitones [i.e., a triply-augmented third], as from 5 to 0).

TRIADS AND OTHER CHORDS: *triad* is understood to refer to the three-note chords that can be formed out of combinations of major and minor thirds. (Assuming a reasonable range of tuning systems, these scalar intervals will approximate the just intervals 5:4 and 6:5.) In other words, what we ordinarily call the minor, major, diminished, and augmented triads; the former two are the *consonant triads*. In tonal music, triads and other chords are most often formed out of pitches spaced a third apart from one another within the scale. This is an example of what I call a *regular chord*. A regular chord of cardinality k is formed by taking k notes from the scale, spaced at some regular interval n. For example, the familiar triads of the diatonic scale are formed by taking three notes from the scale, spaced at the regular interval of a third. I will refer to this as a "tertian 3-chord": "tertian," because the regular interval is the third, and "3-chord," because there are three notes in the chord.

Because in tonal music, triads are more-or-less invariably formed out of thirds, distinguishing between triads and regular chords in this way has some counterintuitive implications. For instance, although the adjective "quartal" is more typically associated

[4] An interval such as "major third" would often also be defined as the harmonic ratio 5:4. I don't mean to enter the intractable debate about whether scales preceded harmonic intervals or vice versa with my assertions above. My simple point is that our nomenclature ("major third") is based upon a scale—otherwise, what is "third"-like about the ratio 5:4?

with chords formed out of perfect or augmented fourths, a "quartal 3-chord" in the octatonic scale (e.g., {0, 4, 9} from the scale {0, 1, 3, 4, 6, 7, 9, 10}) is actually a consonant triad. Nevertheless, I believe any extra mental effort this requires is worthwhile. The practice of tonal music suggests that both triads and regular chords are important phenomena. If this is so, then by considering each independently we can more fully evaluate the possible utility of unfamiliar scales. (I trust that the reader does not need convincing that triads are important phenomena in tonal music, but she may well be more suspicious of the novel category of regular chords. Why should we bother about regular chords? I will defer my response to this question until section 1.1 below, when we shall be better equipped to address it.)

EQUIVALENCE-CLASSES: I consider scales and modes to be equivalent under transposition, but not under inversion. I believe that these equivalences are easily justified by our intuitions about (extended) tonal music.

ENHARMONIC EQUIVALENCE: Two competing, but equally important, senses of enharmonic equivalence will be employed in this thesis. The difference between them is most readily illustrated with an example: the pitch-class set {G♭, B♭, D♭} is *locally enharmonically equivalent* to {G♭, B♭, C♯} and {F♯, A♯, D♭}; it is *globally enharmonically equivalent* to {F♯, A♯, C♯} (or to {E♯♯, G♯♯♯, B♯♯}).[5] These terms will be defined more explicitly using the line of fifths developed in 1.2 below.

SCALE AND MODE NOMENCLATURE: Many of the scales and modes discussed in this thesis have been given a host of different names by various theorists over the years. I give no attempt to list these. My goal with scale and mode nomenclature is simply to name them in such a way that the reader can quickly understand the pitches involved.

Common scales are named after their best-known modes. For instance, I will speak of the "harmonic minor" and "harmonic major" scales. I follow Hook 2011 and Meredith

[5] These enharmonic equivalences all assume 12-tone equal temperament.

2011 in speaking of the "(ascending) melodic minor" scale. (Tymoczko 2004, 2011 calls this the "acoustic scale.") Less common scales may be given mnemonic names that should help both author and reader rapidly understand their pitch-class structure (e.g., "Major ♯2"). Scales that are too distant from the diatonic for any such name to be helpful will simply be referred to by their prime form under T_n equivalence (e.g., {0, 2, 3, 4, 5, 6, 7}).[6]

I refer to diatonic modes by their familiar Greek names, and non-diatonic modes are often referred to by a Greek name together with one or more alterations (e.g., Phrygian ♯6). I also number modes in scale order. The choice of first mode is arbitrary, but if the scale is named, then the first mode is the mode after which the scale is named (e.g., the harmonic minor mode is the first mode of the harmonic minor scale); in the case of the diatonic scale, the first mode is the major mode. I use a scale name followed by a colon and a number to rapidly communicate scale and mode, so "diatonic:1" is the major mode. The four "quasi-diatonic" scales (introduced below) will each be referred to by a single letter: M: diatonic; m: melodic minor; h: harmonic minor; H: harmonic major. So I will more often refer to the major mode as "M:1." (The diatonic scale is abbreviated by "M" [for "major"] because of the strong association of the letter D with a pitch-class.)

One final note: throughout the thesis, I will sometimes use "composition" as a catch-all term meant to refer equally well to improvised music as well as music conveyed through unwritten tradition. The pitches and rhythms of the music were put together, by someone or some group of people (or even some machine), at some point in time. This process, whatever it may have been, was the composition, broadly speaking, of the music.

[6] I use John Rahn's (1987) "least right-packed" algorithm for calculating prime forms. When using Forte numbers to refer to inversionally asymmetrical T_n classes, I follow Larry Solomon in affixing the letter A to the prime form under inversional equivalence, and B to its inversion (Monzo [n.d.]).

Chapter 1 Scales and generated lines

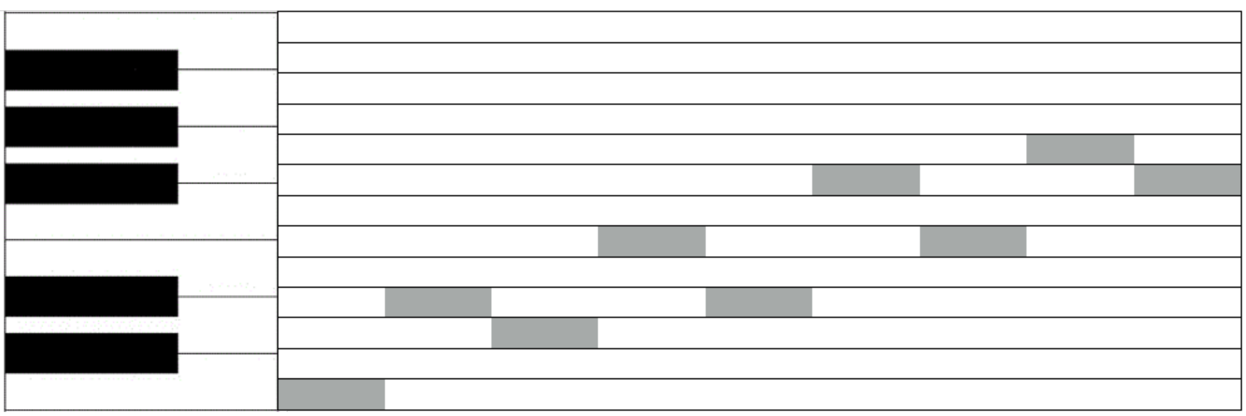
In this chapter, I first make a brief attempt to answer the questions "what are scales?" and "why should we bother with them?" After (hopefully) establishing that scales are important objects worthy of study, I then explain one systematic way of building scales, using a generated line of intervals. This explanation builds upon Julian Hook's "spelled heptachords" (2011).

1.1 Why scales?

Consider the sequence of equal-tempered pitches (60, 63, 62, 65, 63, 67, 65, 68, 67). In specifying these pitches, I wished to avoid specifying a spelling, and to achieve this I have notated these pitches with midi numbers (where C4 or "middle C" is enharmonically equivalent to midi number 60). If I wished to notate these pitches graphically, but still refrain from specifying a spelling, I would be obliged to do so with something like a piano roll, as in Figure 1. This is not, however, the usual means of notating music. Pitches are more often indicated with staff notation. But to notate on a staff obliges us to decide upon a spelling. In the case of the given pitches, even assuming that all the "white keys" will be spelled without accidentals, we are nevertheless confronted with the question of whether to spell the "black-key" pitches 63 and 68 with sharps (as in Figure 2), with flats (as in Figure 3), or with some combination of the two. How might we decide between these possibilities? And is our decision of any practical or musical significance?

I propose that we can answer these questions by first asking ourselves "what do we *hear*, when we hear these pitches?" I would submit that anyone familiar with Western music (and many other varieties of music as well) will hear such a pattern in terms of regular motion—a certain larger distance up, a certain smaller distance down—within some space. (Of course, for most listeners, the knowledge of this structure will be implicit, rather than explicit, in the same manner as native language speakers have, say, implicit knowledge about how to produce the phonemes of their language.) The space within which this regular motion takes place is the *scale*, and Western music's conventions of pitch-naming and music notation are optimized to display motion within scales (and, more specifically, within scales of seven notes). The notation adopted in Figure 3 conveys this regular motion, whereas that in Figure 2 obscures it. There is no acoustic difference between these two notations (at least in 12-tone equal temperament), but there is a real *psychological* difference in the mind.[7] (Note, moreover, that while in the absence of any other context the notation of Figure 2 is an improbable interpretation of the tonal meaning of the given pitches, it is possible to devise a context that *will* convincingly imply the notated tonal meaning, as I have attempted to do in Figure 4. [I have added the melody pitch B3 before the first C4 in order to permit me to establish an E minor context.])

Figure 1. Piano roll representation of pitches (60, 63, 62, 65, 63, 67, 65, 68, 67).



[7] One way of formalizing the intuitions discussed here would be in terms of the "pitch alphabets" of Deutsch and Feroe 1981. Much as these authors do for certain other pitch patterns, we could verify that listeners really do hear in terms of this regular motion by perturbing the pattern rhythmically in ways that do or do not accord with the presumed scalar structure and observing how listeners ability to recall and or reproduce the melody is affected.

Figure 2. Pitches (60, 63, 62, 65, 63, 67, 65, 68, 67) in staff notation, spelled with sharps.

Figure 3. Pitches (60, 63, 62, 65, 63, 67, 65, 68, 67) in staff notation, spelled with flats.

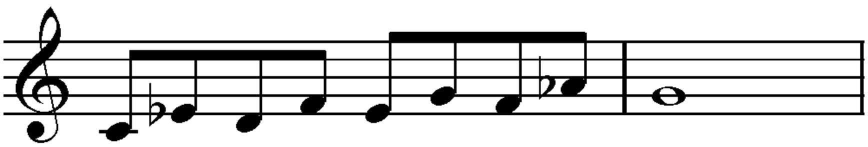


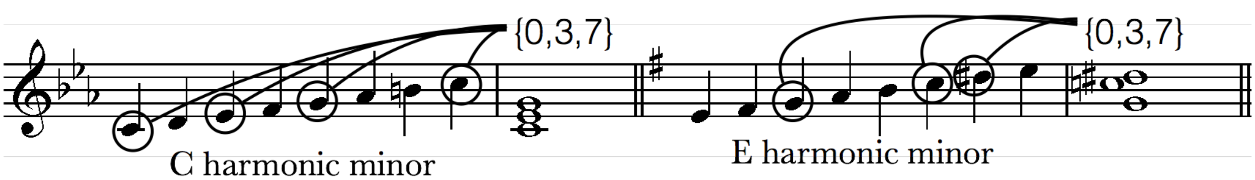
Figure 4. A possible harmonization of Figure 2. (The initial B has been added in order to establish an E minor context.)

In sum, scales are, geometrically speaking, the space within which tonal music takes place (in Tymoczko's formulation, "a scale is a ruler" [2011, 116]); in terms of combinations, scales are the "pitch alphabet" (Deutsch and Feroe 1981) out of which the words and phrases of a passage of tonal music is formed.

In order to work well as a ruler and alphabet, a collection of pitch-classes should possess a number of features. This chapter and the next are largely concerned with the examination of these features. Loosely speaking, we might call the extent to which a collection possesses such features its "scale-ness." The diatonic scale, as we will see below, has the highest scale-ness, while all other 7-collections are, in some way, less useful as scales. There is, however, no firm boundary between those pitch-class collections that can form useful scales—those with high "scale-ness"—and those that cannot. We might think of the composition of scalar music in a scale of low scale-ness as being much like the lifting of a heavy weight. Just as the heavier the weight, the more energy we must expend to lift it, the lower the scale-ness of a collection, the more energy the music must expend in order that the collection be heard as a scale.[8]

If it is true, as I hope I have shown, that the pitches of tonal music assume their meaning at least in part due to their position within a scale, then it follows that the scale also helps determine how groups of pitches assume their meaning in relation to one another. For instance, the pitch-class set {0, 3, 7} takes on rather different meaning when embedded in one or the other of the scales in Figure 5—or as realized in the works excerpted in Figure 6 and Figure 7.[9]

Figure 5. Pitch-classes {0, 3, 7}, embedded in the C harmonic minor and E harmonic minor scales.



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[8] This idea of scale-ness, and of the compositional work that must be performed to establish a scale being in inverse proportion to its scale-ness, can be developed somewhat further by a comparison to certain works of art. For instance, compare the Mozart *Requiem* to Cage's *4'33"*: regardless of setting, no one needs to be told that the former is music (i.e., it has high "music-ness") whereas even when the latter is performed, at a concert, in a concert hall, many (or perhaps most) people still need to be told that what they are witnessing is music. A similar contrast could be drawn between Duchamp's *Fountain*, whose status as art remained controversial even after being placed in a gallery, and, say, Da Vinci's *Last Supper*. This argument should in no way be taken as implying a difference in aesthetic value between cultural productions of relatively high or low "music-" or "art-ness."

[9] C.P.E. Bach may not have assumed equal temperament, and may therefore not have considered E-flat and D-sharp to be equivalent. But he would certainly have played both with the same key on his Klavier.

Figure 6. Pitch-classes {0, 3, 7} in C.P.E. Bach, Sonata Wq 48/4, i.

Figure 7. Pitch-classes {0, 3, 7} in C.P.E. Bach, Sonata Wq 49/3, iii.

The converse of this situation occurs when two pitch-class sets belong to different T_n-classes—that is, they have differing atonal intervallic structures—but have the same generic scalar intervals, and are thus equivalent under scalar transposition. Such "scalar T_n-classes," which share a scalar structure, but have differing specific interval structure, underlie such important techniques of tonal music as imitation and inversion. Even more fundamentally, they underlie the very nature of tonal harmony, because chord types in tonal music tend to be formed from such scalar T_n-classes (e.g., triad, seventh chord), which then take on a specific quality depending on their transposition level within the scale. Moreover, while it might in principle be possible to use scalar T_n-classes that possess any arbitrary structure of scalar intervals, in practice, the harmonies used by common-practice tonal music are *regular chords*, as defined above.

Many questions of a more perceptual nature will of necessity go unaddressed in this thesis. For instance, how does a listener infer a scale? And, relatedly, how does a composer establish a scale in such a way that a listener might infer it?

1.2 Describing scales: note-names, fifth-strings, letter-distinct and proper scales

The preceding discussion has, I hope, established that atonal pitch-classes are not sufficient for describing scales, and traditional "note names" are necessary. But what exactly is a note name? A note name, such as "C-sharp" or "D-natural," is made of two elements: a letter, and an accidental.

The letter is a means of indicating the ordering of pitches within a scale: B is the pitch that comes after A and before C. The order is circular, so that A comes after G; since B comes after A, B therefore comes two pitches after G. By performing such "modular letter arithmetic," we can calculate all scalar intervals. G to B is a third, regardless of whether G is sharp and B is flat, or G is double-flat and B is natural, or any other accidentals that may be associated with each of the pitches. The use of letters is arbitrary: we could also use numbers (as we do when we speak of scale degrees like $\hat{2}$ and $\hat{7}$ or roman numerals such as III and vi) or any other type of symbol (such as the syllables of solfège). But we must make an important qualification: letter arithmetic can only be depended upon when the number of letters matches the cardinality of the scale. We use seven letters (or seven scale degrees, roman numerals, or solfège syllables) because our scales traditionally have seven pitch-classes. For this reason, it is not only that traditional note names (or something like them) are necessary for adequately describing scales, but also that scales, and in particular 7-scales, are implied by our very use of these note names (as well as by the very design of many of our instruments, such as keyboards). Moreover, these note names permeate our means of notating music, because each location on the staff corresponds to one of seven letters, and not to (say) a chromatic pitch-class or to any other "scale-less" notion. The result is that our notation is optimized to convey scalar information, such as the interval of a third, by obscuring chromatic (or "specific") information, such as the interval of three or four semitones[10]—but only in 7-scales. As

[10] The fact that notation conveys scalar, rather than specific, information poses a barrier to musical beginners, who find it easier to read representations of the specific notes that they should play. (This fact is attested to by the huge number of keyboard tutorials on the internet which illustrate directly upon a

long as we use our usual note-naming apparatus, scales of fewer than seven pitch-classes will always be missing letters, while scales of more will always have doubled letters. In these other cardinalities, we will not be able to use letter arithmetic to measure scalar intervals, and when music in these scales is notated upon the staff, scalar intervals will often be obscured. As an illustration, Figure 8, notates thirds upon all scale degrees in three scales of differing cardinalities. In the 7-scale, all thirds are separated by a constant number of staff-locations, but in the 6- and 8-scales, some thirds (those placed in boxes) appear to span either a greater or smaller number of staff locations. It would be possible to solve this problem by notating such scales on staves where the number of letter-positions corresponds to the cardinality of the scale, but the modest gains of theoretical consistency would presumably not be worth the immense effort of learning an entirely new notational convention. (Moreover, seven is not some arbitrarily chosen number, but happens to be an especially useful scale cardinality, as we shall see in Chapter 2.)

Figure 8. Thirds notated on the staff in A) a 7-scale, B) a 6-scale, and C) an 8-scale.

We have now explored the role of the first element of a note-name, the letter, but what of the second, the accidental? Clearly, sharps and flats inflect pitches upwards and downwards by chromatic semitone, respectively. But what exactly is it they inflect? We might simply say that the uninflected note-names (i.e., those in which the accidental, although often omitted, is a natural) are arbitrarily assigned to their values in the C diatonic scale, and indeed, this is probably how most musicians (including myself) think

keyboard the notes to be played.) Since the beginner does not yet know any scales that are implicit in the notation, these scales are an obstacle to her reading. The expert, on the other hand, knows these scales so well that, at least in the case of tonal music, they greatly facilitate her ability to read musical notation. (No concert pianist prepares Rachmaninoff concertos from Youtube tutorials; no professional orchestra performs from scores that illustrate the position of the notes to be performed upon their instruments.)

of note names in practice. But it is also possible to give another account of note-names, where the natural note values need not be fixed arbitrarily. Doing so will, moreover, give us the apparatus to provide a scalar spelling for *all* equal-tempered 7-scales—and many other scales, besides.

To begin, we array note names along an infinite "line of fifths" as in Figure 9. I believe Temperley [2001] was the first to explicitly write about the line of fifths. While Temperley refers to combinations of letter and accidental [e.g., C-flat, F-natural] as *tonal pitch-classes*, I prefer to simply call these "note names," following Hook [2011, 85]. Given the frequency of any note name on the line of fifths (e.g., A4=440),[11] and the size of the fifth (e.g., an equal-tempered fifth of 700 cents), the frequencies of all note names upon the line are determined. The line of fifths is very similar to the familiar circle of fifths, and the operation of enharmonic equivalence winds the line into the circle.[12] In fact, the circle of fifths is so similar and familiar to the line that it is worth re-emphasizing that the line does *not* recognize enharmonic equivalence and continues (in principle) indefinitely, unbounded in both the flat and sharp directions. Each letter repeats every seven positions upon the line, inflected sharpwards (… G double-flat, G-flat, G-natural, G-sharp, G double-sharp, …).

We can put note names into formal mathematical terms by putting them into correspondence with the integers. I will call such integers "fifth-classes," but will more often refer to the more intuitive but equivalent category of "note names." The exact choice of mapping from note-name to fifth-classes is arbitrary, but I favour F-natural = 0.[13] Then:

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[11] Since a pitch-class has no frequency, this should, strictly speaking, read "frequency class"—i.e., the set of octave related frequencies of the constituent pitches of a given pitch-class—rather than "frequency".

[12] The number of enharmonically equivalent nodes on the circle depends on the cardinality of the temperament, which in turn is determined by the generating interval.

[13] The advantage of assigning F to 0 is that it allows us to most readily calculate the accidental of a fifth-class. Hook (2011) prefers to assign fifth-class class 0 to D-natural. The advantage of this choice of assignment is that it allows him to most readily develop his ideas about "balanced" forms (88), i.e., those that contain equal numbers of both sharps and flats. Another useful convention would be to assign 0 to C; then a conventional 12-tone equal temperament pitch-class could be extracted from a fifth-class by

- the letter represents the note name's residue class, mod 7. So, for example, because the fifth-class of A♭ is -3, and -3 mod 7 = 4, 4 is the residue class of all note names containing the letter A. (Hook [2011] calls a pitch's residue class its "diatonic pitch class"; since letters are an eminently familiar means of representing the same information, I shall simply use "letter" instead.)
- the accidental represents the quotient of the fifth-class, divided by 7, rounded downwards to the next lowest integer, where negative numbers correspond to flats, positive numbers to sharps, and 0 corresponds to a natural. (This rounding downwards is often called the floor function, represented symbolically by $\lfloor x \rfloor$.) So, for example, the fifth-class of A♭ is -3, and $\lfloor -3/7 \rfloor = -1$. Therefore, the accidental part of the note name "A♭" corresponds to the integer -1, or one flat.

Any set of note names can be represented on the line of fifths as a string of 1's and 0's, where 1 indicates the presence of a note-name and 0 indicates its absence. Following Hook (2011, 88) I will call this a *fifth-string*.[14] Since the line of fifths is infinite, every fifth-string contains an infinite number of leading and trailing zeroes, but due to space constraints I will omit these. As examples, beneath the line of fifths in Figure 9, I have indicated several fifth-strings. Fifth-strings can be translated left or right upon the line of fifths. Any two note-name sets that have the same fifth-string under such translation, such as the C major and E-flat major triads in Figure 9, are transpositionally equivalent. A fifth-string is therefore equivalent to a T_n-class of note-names. However, the fifth-strings will differ for two non-enharmonically equivalent spellings of the same pitch-class set, such as "E-flat major triad" and "Not an E-flat major triad" in Figure 9. In other words, a fifth-string is not equivalent to a T_n-class of pitch-class integers, because any

multiplication by 7, mod 12. With F = 0, we can calculate a pitch-class from a fifth-class by first subtracting 1, then multiplying by 7 mod 12. (E.g., F = ([0 – 1] * 7) mod 12 = 5.)

[14] My representation of fifth-strings differs from Hook's, however. What I notate as (1, 1, 0, 0, 1), he would notate as (1, 3)—i.e., he notates only the intervals on the line between adjacent fifth-classes.

such T_n-class has many different possible fifth-strings. The *span* of a fifth-string (Hook 2011, 86) is the distance from its flattest element (i.e., its leftmost "1") to its sharpest (its rightmost "1"). In the case of either major triad in Figure 9, the span is 4, but in the case of "Not an E-flat major triad," the span is 11.

Figure 9. A segment of the line of fifths, with examples of several fifth-strings.

	…	Db	Ab	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#	D#	…
	…	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	…
Fifth-strings:																	
C major triad:							1	1	0	0	1						
E-flat major triad:				1	1	0	0	1									
Not an E-flat major triad:					1	0	0	1	0	0	0	0	0	0	0	1	

We are now in a position to define *local* and *global* enharmonic equivalence more concretely. Two sets are enharmonically equivalent wherever their atonal pitch-class content is the same. But they are only *globally* enharmonically equivalent if there is a single operation that, by translating the relevant fifth-string some multiple of twelve along the line of fifths, can transform one into the other. For instance, {G♭, B♭, D♭} can be transformed into {F♯, A♯, C♯} by translating the whole fifth-string twelve spaces right along the line of fifths. In contrast, no such operation exists to transform {G♭, B♭, D♭} into {G♭, B♭, C♯}: here, only the third note name is translated by twelve spaces along the line of fifths, while the first two remain constant (i.e., they are translated by zero spaces along the line of fifths). Where no uniform translation exists, the set is locally, rather than globally, enharmonic.

Hook (2011) defines two more concepts that will be useful to us. The first of these is *letter distinctness*. A collection of tonal pitch-classes is letter distinct if it contains no letter more than once—that is, if it contains no cross relations such as C-natural/C-sharp

(87). In terms of fifth-strings, a fifth-string is letter distinct when no two members of the fifth-string are separated by a span of a multiple of 7 (i.e., no pairs belong to the same residue class mod 7). Hook calls a letter-distinct 7-collection a "spelled heptachord"; I will simply refer, with some loss of precision but with increased generality, to *spellings*, specifying the cardinality of the spelled collection where necessary.

The second of Hook's useful concepts is *proper-ness* (Hook 90).[15] A spelling is proper when it contains no enharmonic doublings (such as B-sharp and C-natural) and no "crossings" (such as B-sharp and C-flat).[16] It is important to note that, in determining whether any scale is proper, we need only consider its seconds. A second is the interval between two adjacent pitches in a scale, and it is not possible for non-adjacent pitches to cross without having adjacent pitches cross as well. To see that this is so, suppose that two pitches C and E a third apart cross. (These pitches have letters C and E and arbitrary accidentals.) Either 1) C crosses D, or 2) C does not cross D. 1) If C crosses D, then adjacent pitches cross. 2) If C does not cross D, then D is above C, but C is above E, so D is above E, and adjacent pitches cross. Similar reasoning can be repeated for all larger scalar intervals.

A letter-distinct, proper spelling of a given pitch-class set (or its globally enharmonic equivalents) can be seen as describing the *natural* scale associated with the set. This is because this spelling of a scale can be recovered from the minimal representation of a scale—that is, its sounding frequencies (or frequency-classes)—while

[15] I believe Hook only uses "proper" as an adjective, and never uses a noun form. I have used "proper-ness" and not "propriety" to avoid confusion with Rothenberg propriety (Carey 2007, 83).

[16] Hook defines this condition as follows: "as the seven letters proceed around the circle of [diatonic pitch-class] space [i.e., through the note-names in letter order], the corresponding projected pitch classes of the heptachord proceed similarly around the circle of mod-12 pc space, without pausing (for an enharmonic doubling such as B-sharp–C […]) or reversing direction (for a crossing)" (Hook 2011 90). Although this definition invokes both 7-scales and 12-tone equal temperament, it can be readily expanded to other scale cardinalities and tunings.

no other description can be.[17] If we hear some sequence of frequencies (which, for present purposes, are most readily represented as pitch-classes) such as (0, 11, 0, 7, 8, 0, 11, 0, 2, 7, 0, 11, 0, 2, 5, 7, 8, 7, 5, 3), we can always reconstruct a proper, letter-distinct description of a scale from these frequencies,[18] such as {C, D, E♭, F, G, A♭, B}. Perhaps the composer might have arbitrarily intended the scale to have a crossing (e.g., {C, D♯, E♭♭, …}), or an enharmonic doubling, (e.g., {B♯, C, D, …}), or to be non-letter-distinct, (e.g., {C, E♭, E, …}), but all such spellings imply an *artificial* scale—artificial in the sense that there is no way to reconstruct it purely from auditory experience. It is not possible to hear two different pitches and reliably infer that the higher pitch is "lower," according to some arbitrarily decreed scale order, nor is it possible to hear one pitch repeated twice and know that the first occurrence of the pitch is a "different" scale degree from the second. We can put the distinction between natural and artificial scales in terms of *scale-ness*, as defined above. While various other factors might affect scale-ness, it is plain to see that an artificial scale, that is, an arbitrarily defined ordering containing pitch-class doublings and/or crossings, is so unlikely to be heard as a scale (at least in the prescribed ordering) that its scale-ness is effectively zero.

An important fact about proper spellings is that they exist for every 7-chord in 12-tone equal temperament. (An explanation of why this is so will have to wait for Section 1.6.) According to Hook, the proper spelling of any 7-chord is unique, under global enharmonic equivalence, but this is not quite true. For example, we might reasonably be inclined to spell {0, 2, 4, 5, 7, 9, 11} as {C, D, E, F, G, A, B}, or even as {B♯, C♯♯, D♯♯, E♯, F♯♯, G♯♯, A♯♯}. But it is also possible, logically speaking, for some perverse individual to elect to spell these pitch-classes as {C, D-duodecuple-flat [i.e., D♭♭♭♭♭♭♭♭♭♭♭♭], E, F, G, A, B}—

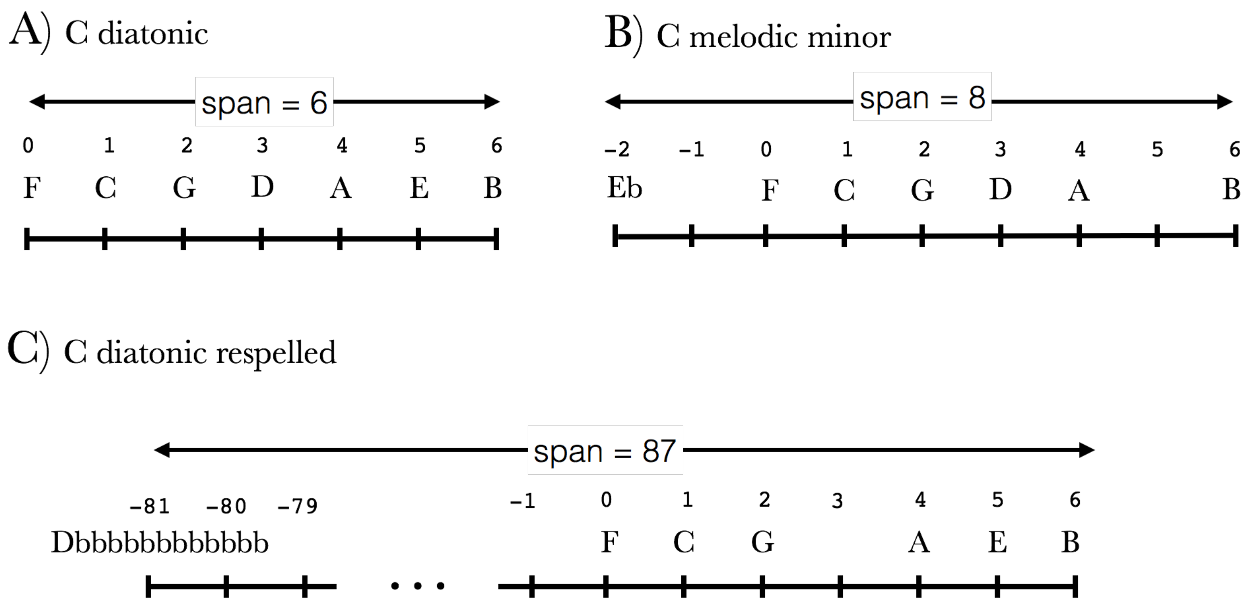
[17] Note that conventional music notation is not minimal in this sense. Because it always contains information about spelling, it implicitly contains a scalar interpretation of whatever it notates, even if this scalar interpretation is arbitrary and even nonsensical, as it might be in the case of an atonal work.

[18] Provided the sequence has the same number of pitch-classes as the scale we are trying to construct.

or in an infinity of other (similarly absurd) ways that are not globally enharmonically equivalent. Although, the existence of such spellings is unlikely to be of musical relevance, they are nevertheless logically consistent with Hook's definition of properness. For the time being, we can usefully exclude them by reference to the span of each spelling. As shown in Figure 10, the span of C diatonic, spelled ordinarily, is six—this is the shortest possible span for a 7-chord. For comparison, the span of C melodic minor, spelled ordinarily, is eight. On the other hand, the span of C-diatonic, respelled as above with a D-duodecuple-flat, is much longer: 87. The proper spellings that Hook treats as unique are in fact simply the proper spellings of least span: these are indeed unique (i.e., there is never a tie for spelling of shortest span, under global enharmonic equivalence). Since there is never any reason to use proper spellings other than those of least span, I will henceforward use "proper spelling" to refer simply to these latter. (Below, we will see that there are other, perhaps more parsimonious ways, of delimiting the set of proper spellings, without appealing to least span.) These proper spellings can be sorted into proper-spelling equivalence-classes that can be usefully ordered by the span of the associated fifth-string, from the diatonic scale {0, 2, 4, 5, 7, 9, 11} (which, at this transposition level, can be spelled {C, D, E, F, G, A, B}, and has a span of 6) to the chromatic 7-chord {0, 1, 2, 3, 4, 5, 6} (which can be spelled {B♯, C♯, D, E♭, F♭, G♭♭, A♭♭♭}, and has a span of 30).[19] This ordering has been done a number of times (Audétat and Junod 2017, Hook 2011, Sailor 2016); it provides a rough index of the "quasi-diatonicity" of each scale, as we will discuss further in Section 2.3.

[19] This ordering is only partial, because not every proper spelled 7-chord has a unique span. For instance, both the Hungarian minor and the "whole-tone plus one" scales have a span of 10.

Figure 10. Possible spellings of several 7-scales.



PROPER SPELLINGS ARE SHORT: A further fact about proper spellings, unnoted by Hook, is that the proper spelling of any 7-chord is also its shortest spelling (i.e., its spelling of least span).[20] To show why, let us first note the fact that, for any two pitch-classes, the proper spelling of these pitch-classes is always shorter than any other spelling. For example, consider the pitch-classes 0 and 1. A spelling as C to D-flat has a span of five fifths,[21] whereas a spelling as C-sharp to D-double-flat (or B double-sharp to C-natural, or any other doubly-diminished second) has a span of nineteen fifths.[22] It is not hard to see that any other possible spellings, such as quadruply-diminished fourths, will have even greater spans.[23] Now consider that any spelling of a larger pitch-class set will always have a flattest note name and a sharpest note name, and that the span of the pitch-class set as a whole will always be the span of these two extreme pitch-classes. Since, as we

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[20] Since, as noted above, we are using "proper spellings" as a shorthand for "proper spelling of least span," it would be more strictly accurate to say that the shortest spelling of any 7-chord is letter-distinct and crossing- and doubling-free.

[21] I.e., D♭–A♭–E♭–B♭–F–C

[22] I.e., D♭♭–A♭♭–E♭♭–B♭♭–F♭–C♭–G♭–D♭–A♭–E♭–B♭–F–C–G–D–A–E–B–F♯–C♯.

[23] It is possible to verify by exhaustion that similar reasoning holds for all other atonal pitch-class intervals.

have just seen, the proper spelling of any two pitch-classes is always shorter than any non-proper spelling, the shortest possible span for a spelling of the whole pitch-class set will always be the proper spelling with the shortest span between its two extreme pitch-classes.

Proper spellings are not necessarily uniquely short, however. If the span of the extreme pitch-classes is sufficiently great, it may contain non-proper spellings of other pitch-class pairs. In this case there may be more than one spelling of equally short span, only one of which will be proper. For instance, the pitch-class set {0, 1, 3, 4, 5, 6, 7} might be given a proper spelling as {B♯, C♯, D♯, E, F, G♭, A♭♭}. The extreme flat and sharp note-names of this spelling are A♭♭ and B♯, respectively, and they have a span of 23, which is therefore the span of the entire spelling. However, this span is longer than the span of a non-proper semitone (i.e., a doubly-diminished second), which is 19. It is therefore also possible to give the pitch-class set the non-proper spelling {B♯, C♯, D♯, **E♯**, **F♭**, G♭, A♭♭}, because the span of 19 between the crossing note-names F♭ and E♯ is entirely contained within the span of 23 from A♭♭ to B♯.

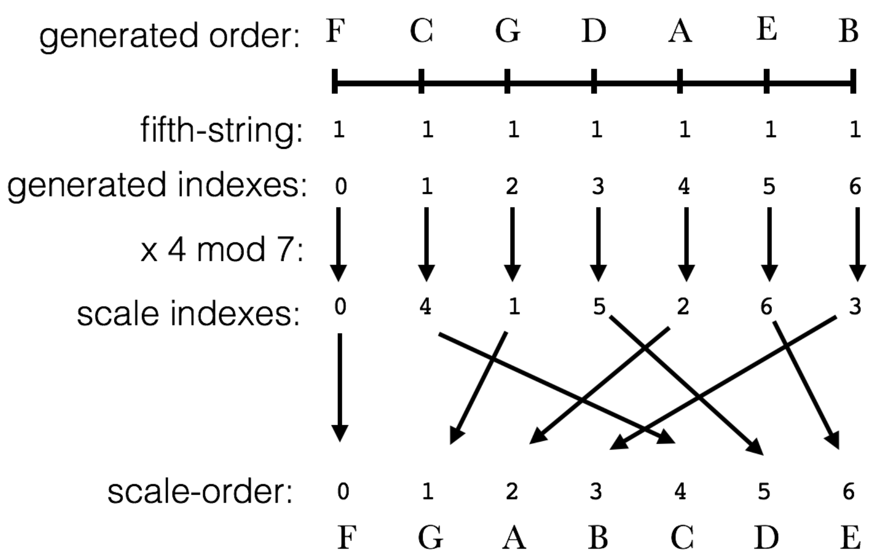
1.3 Near well-formedness and scalar intervals as directed spans

Carey and Clampitt (1989, 196) state that a scale generated by consecutive fifths is *well-formed* when there exists an automorphism that rearranges the scale from generated order to scale order. I will formulate this in terms of *generated indexes* and *scale indexes*, illustrated in Figure 11. The *generated index* of each pitch-class in a scale is the span on the line of fifths between this pitch-class and the origin pitch-class of the generation (i.e., the flattest pitch-class on the line of fifths). The *scale index* of each pitch-class defines its order relative to the other pitch-classes of the scale, when the pitch-classes are placed in natural scale order (where the origin pitch-class of the generation is numbered 0).[24] In the case of

[24] "Generated indexes" and "scale indexes" are my terms, which allow me to introduce "near well-formedness" below. These terms are not explicitly used by Carey and Clampitt, who simply represent scale order by literal position on the page, but the ideas are implicit in their argument.

the diatonic scale, the automorphism that maps the generated indexes to the scale indexes is then multiplication by 4, mod 7 (consider Figure 11).

Figure 11. The well-formedness automorphism of the diatonic scale.

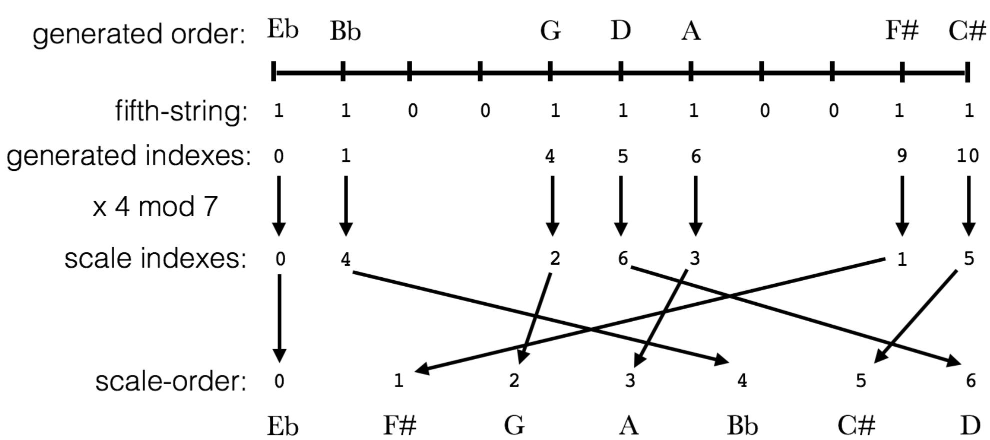


The virtue of formulating well-formedness in terms of generated and scale indexes is that it allows us to define *near well-formedness* for all scales that are constructed upon the line of fifths, and not only scales generated by consecutive fifths. Since the line of fifths is itself generated by consecutive fifths, all scales constructed upon it can be seen as generated from a set of consecutive fifths whose members either are present (in which case they are represented by a "1" in the corresponding fifth-string) or are not present (in which case they are represented by a "0"). To contrast such scales with the "generated scale" formed by consecutive fifths, in this thesis I will simply call them "non-generated scales" (although in a broader context this appellation is clearly sufficiently vague to apply to a wide range of other scales). A generated scale, as we have just now seen above, is well-formed when there is an automorphism that maps its generated indexes to its scale indexes. A non-generated scale is then *nearly well-formed* when

1) the generated scale of the same cardinality and generating interval is well-formed, and
2) the same automorphism that maps the generated scale from its generated indexes to its scale indexes also maps the non-generated scale from its generated indexes to its scale indexes.

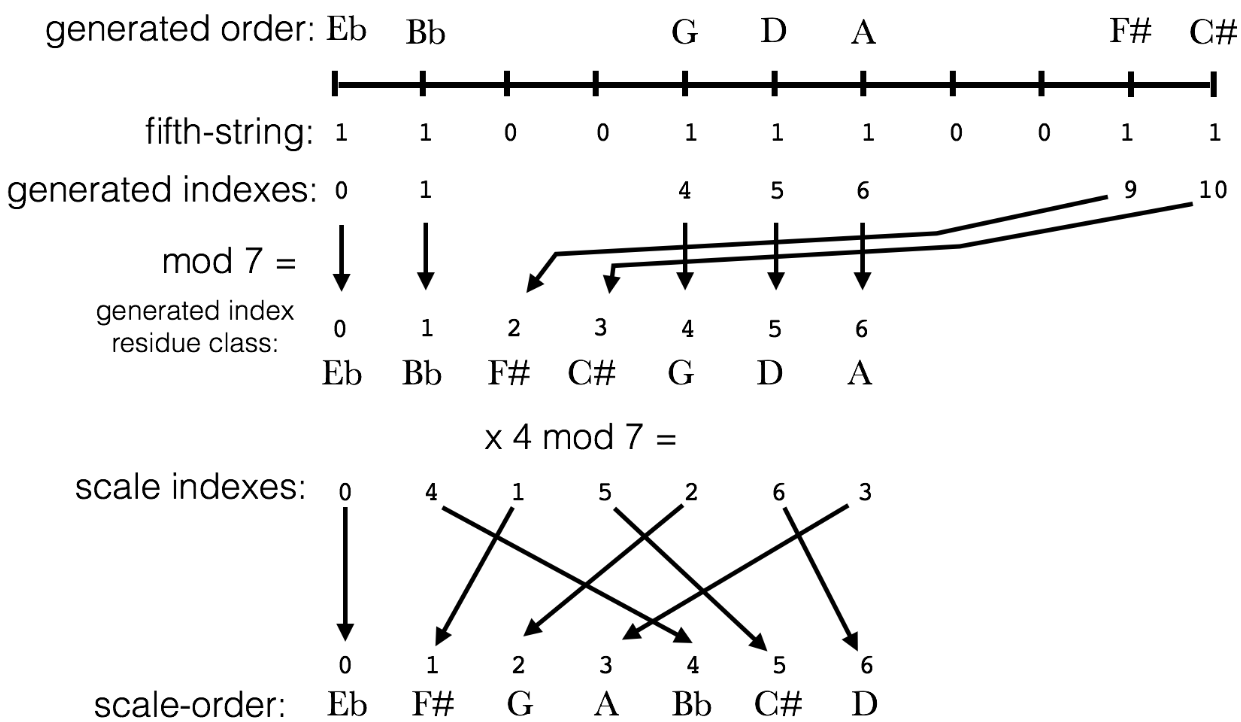
As we have just seen, in the case of 7-scales generated by consecutive fifths, the generated scale is well-formed and the automorphism is multiplication by 4, mod 7. For example, consider the Hungarian minor scale in Figure 12. Because the automorphism maps its generated indexes to its scale indexes, it is nearly well-formed. In fact, as we shall see below, any proper spelling is a nearly well-formed scale and vice versa, as long as the scale of the same cardinality generated by consecutive fifths is well-formed.

Figure 12. The near well-formedness automorphism of the Hungarian minor scale 1.



Rather than proceeding as in Figure 12, however, the reader may find the procedure illustrated in Figure 13 somewhat more intuitive. Here, a mathematically superfluous step has been inserted into the calculation. Before we multiply by 4, mod 7, the generated indexes have first been ordered according to their residue class, mod 7. The effect is to sort the pitch-classes into "cycle-of-fifths order"—the familiar order (F, C, G, D, A…) by fifths within a 7-scale. The reader may now be wondering "what is the point of all this mathematical formalism, if its only result is to put the notes into the familiar order of the cycle of fifths?" Answering this question permits us to observe a previously unstated feature of the line of fifths: the assumption of scale order as multiplication by 4, mod 7, is already built into our means of labelling the line with letter-names. On the line of fifths, we do not label the node next to "F" as "G," as we should do if we were simply labelling the nodes in alphabetical order. This is because the line is a line of *fifths*, and so we must label this node as "C," a fifth above F. It just so happens that ascending a fifth in a 7-scale, mathematically speaking, is the interval of 4, mod 7. In other words, the "fifths" of the line of fifths really are fifths [i.e., scalar intervals], not just acoustic fifths [i.e., approximations to the ratio 3:2].

Figure 13. The near well-formedness automorphism of the Hungarian minor scale 2.



The automorphism of any nearly well-formed scale means that directed spans upon the line of fifths correspond to generic and specific scalar intervals in a similar way to how letters and accidentals correspond to integers upon the line of fifths. (When measuring intervals on the line of fifths, we must measure them as *directed* spans [i.e., a span with a sign, either positive or negative]. For example, the directed span from F to C is 1, but the span from C to F is -1. It is arbitrary whether we choose to consider a span such as either of these as one interval [e.g., a perfect fifth] or its complement [a perfect fourth]. What is important is that we be consistent. In this thesis, I always calculate the span beginning from the first listed pitch-class, with the understanding that, if the interval were realized in pitch-space, this pitch-class would be lower in pitch. So (F, C) is understood to be a perfect fifth [or one of its compounds] with a span of 1, while (C, F) is understood to be a perfect fourth [or compound] with a span of -1.)

To get the generic scalar interval associated with any directed span on the line of fifths, we multiply the span by 4, mod 7. For instance, the three intervals (C, E), (C, E♭), and (C, E♭♭), shown in Figure 14, have directed spans of 4, -3, and -10, respectively. Multiplied by 4, mod 7, these directed spans are all equal to 2; in other words, each of these intervals is a third.[25] If we prefer to proceed in the opposing direction, from scalar interval to directed span upon the line of fifths, we multiply the scalar interval by 2, mod 7 (because 2 is the multiplicative inverse of 4, mod 7). Therefore, the residue class of the directed spans of all thirds (such as 4, -3, -10) is 4 (because 2 * 2 mod 7 = 4); for fourths, it is 6 (because 3 * 2 mod 7 = 6); and so on.

We can get the species of a scalar interval (i.e., whether it is minor, major, augmented, doubly-augmented, etc.) by dividing the directed span by 7, and then taking the floor of the result (i.e., $\lfloor s/7 \rfloor$, where s is the directed span). In general,

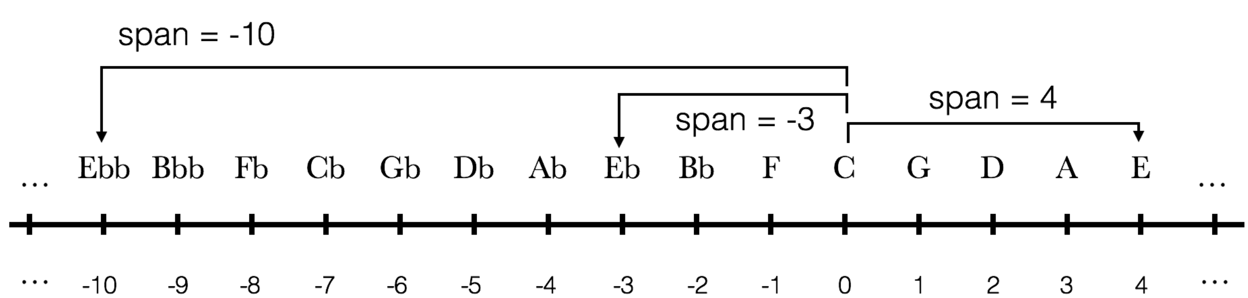
- if the result is 0, then the interval is major,
- if the result is greater than 0, it is augmented the corresponding number of times (e.g., if the result is 2, then the interval is doubly-augmented),
- if the result is -1, then the interval is minor, and

[25] To avoid any potential confusion, it is worth reiterating that when converting generic scalar intervals from integers (e.g., 2) to their traditional ordinal names (e.g., third), one must always add 1.

- if the result is less than -1, then the interval is correspondingly diminished (e.g., if the result is -2, then the interval is singly-diminished)

However, this neat picture is complicated by the unique status of the fourth and fifth.[26] Unlike the other intervals, which can be major or minor, in the traditional nomenclature, fourths and fifths each come in only one non-augmented or diminished (i.e., perfect) quality. If the two different diatonic qualities of fourth and fifth were considered major and minor (i.e., if perfect and augmented fourth were considered the “minor” and “major” fourth, respectively, and if the diminished and perfect fifth were considered the “minor” and “major” fifth, respectively), then the above procedure would give the right interval species in these cases as well. Instead, a fifth is diminished (rather than minor) when the above result is -1, and it is doubly-diminished at -2, etc., while a fourth is augmented (rather than major) when the above result is 1, and it is doubly-augmented when it is 2, etc. In any case, the important point is that both a scalar interval’s generic and specific components are determined by its span on the line of fifths.

Figure 14. Three directed spans upon the line of fifths.



PROPER SPELLINGS AND NEAR WELL-FORMEDNESS: Among 7-scales, all and only proper spellings are nearly well-formed. To show this, we can reason as follows. Nearly well-formed scales must include every residue class (i.e., letter name) once and only once among their generated indexes. If they omitted any residue class, then either they would

[26] The octave/unison could be added to this list, but because we are only considering intervals between different pitch-classes within a scale, these intervals do not come into play.

be of the wrong cardinality, or else they would have to contain another residue class twice, and if they contained any residue class twice, then all multiplications would map the two generated indexes of same residue class to the same scale order index, and no scale-order automorphism would exist. Therefore, they contain every residue class once, and since letters represent residue classes, this means they are letter-distinct. Moreover, because by definition, the automorphism for a nearly well-formed scale places it in natural scale order—that is, ordered without crossings or enharmonic doublings—a nearly well-formed scale must be proper. (To show that a proper scale must be nearly well-formed, we can simply follow the reasoning of this paragraph in reverse.)

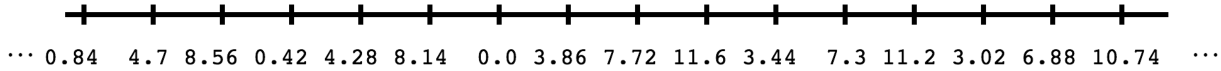
1.4 Generalizing proper spellings to other temperaments and cardinalities

The discussion so far has restricted itself to 12-tone equal temperament and 7-scales, but in fact nothing about the line of fifths and the other associated ideas requires this restriction. The following section will generalize these ideas to other tunings and cardinalities.

Broadly speaking, the apparatus developed so far requires two features: a generating interval, and a scale cardinality. We have so far generated the sounding pitch-classes of the line of fifths by consecutive 12-tone equal-tempered fifths, but if we allow the generating interval to vary, we can see the line of fifths as an instance of the more general idea of a *generated line*. As an example, Figure 15 illustrates the "line of just thirds" (a just third is approximately 3.86 semitones).[27] Beneath the line is listed the pitch-class of each node in 12-tone equal-tempered semitones.

[27] "Just third" here refers only to the acoustic interval 5:4, and not to any scalar interval. Because no associated scale cardinality has yet been defined, the scalar interval of any span on this line remains undefined.

Figure 15. A segment of the "line of just thirds."



If the generating interval of a generated line is a rational proportion of the octave, then the interval will define an equal temperament. For instance, if the interval is 7/12 of an octave, then after repeating it twelve times, we will arrive at a pitch-class enharmonically equivalent to our starting pitch-class. It will thus define 12-tone equal temperament: all and only the pitch-classes of the 12-tone equal tempered scale will be available upon the generated line.[28] Similarly, generating intervals of 5/7 octave, 10/19 octave, and 31/1200 octave will generate 7-, 19-, and 1200-tone equal temperaments, respectively. (Of course, I don't mean to suggest that the latter of these is a practicable temperament!) On the other hand, if the generating interval is not a rational proportion of the octave, then there are no enharmonic equivalents on the line, and no temperament is defined. In this case we can speak of an "untempered tuning system." Just intervals, such as the fifth, generate such untempered tuning systems, which is one of the motivations for equal temperament.

Up to this point, we have always labelled the line of fifths with seven letters, because we have been concerned with 7-scales. But if we were concerned with some other cardinality of scale, then we could label the line with the corresponding number of letters. In Figure 16, for example, the line of just thirds has been labelled for 5-scales. (Because I am labeling this generated line for a scale of cardinality other than 7, I begin the alphabet from H, rather than A, so as not to confuse these letter names with the familiar note-names of our 7-scale-centric system of notation.) The combination of generating interval

[28] The generating intervals of 1/12, 5/12, or 11/12 of an octave would also define 12-tone equal temperament. In general, the cardinality of the temperament generated by a given rational interval can be inferred from the denominator of the interval, when the interval is expressed as a fraction of an octave in simplest form. 9/12 of an octave would generate 4-tone equal temperament, since this fraction expressed in simplest form is 3/4.

and scale cardinality defines a generated scale, corresponding to the region of the line of just thirds where the note names have no accidentals.

Figure 16. The "line of just thirds" labelled for 5-scales.

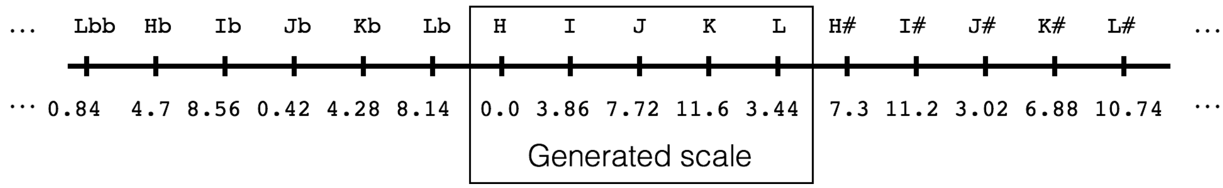
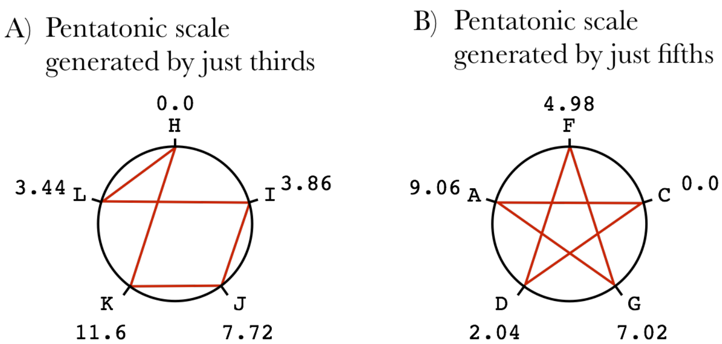


Figure 17. Visual illustration of well-formedness (or lack thereof) in two generated scales.



Depending on the combination of generating interval and cardinality, the generated scale may or may not be well-formed. With the familiar combinations of a generating interval of a just or equal-tempered fifth and a scale cardinality of 5 or 7, the generated scale is well-formed. The generated scale in Figure 16, on the other hand, is not well-formed. As Carey and Clampitt [1989, 188–9] show, one intuitive way of showing this is by placing the generated pitch-classes around a circle according to their generated order, and then tracing line segments connecting the pitch-classes by scale order, as has been done in red in Figure 17. If the numbers of degrees of symmetry of the resulting shape matches the cardinality of the scale, then the scale is well-formed. It can be seen that the scale in Figure 17A is not well-formed, while that in Figure 17B is. In prose, steps in the generation of the just-thirds scale, such as H to I and I to J, do not span a constant number of scale degrees when the pitch-classes are placed into scale order. For example, H to I spans one other scale degree, L (i.e., H to I is a third), while I to J spans no other scale degrees (I to J is a second).

Note that the placement of letters upon the generated line in Figure 16 differs from the way in which we ordinarily place letters upon the line of fifths. The line of fifths is customarily labelled with letters in order to indicate their scale order, and not their generated order. We can do this because the 7-scale generated by the line of fifths is well-formed, and we can therefore use the automorphism that maps from scale-order to generated-order to define the appropriate letter-order on the generated line. If there is no such automorphism, then there are no proper spellings, because the letter order is undefined.[29]

If the generated scale is well-formed, making proper spellings possible, then the number of proper-spelling classes that can be constructed on the generated line is limited by the size of the smallest proper interval. To see what I mean by this, let us return to thinking about 12-tone equal temperament. In 12-tone equal temperament, the smallest proper interval is the minor second. If we diminish a minor second further, we get a diminished second, which is not a proper interval, because it is an enharmonic doubling, and if we diminish any interval beyond an enharmonic doubling, we will get crossings. Moreover, there cannot be any proper non-second interval as small as, or smaller than, the smallest second. If there were, it would impose a crossing or enharmonic doubling upon the pitch-classes that, by definition, should lie between the extreme pitch-classes of any larger scalar interval (for example, a doubly-diminished third, such as (C♯, E♭♭), would impose a crossing or enharmonic doubling upon D). The minor second is therefore the smallest proper interval, and upon the line of fifths, it has a span of -5. All further

[29] In fact, we could label the line in Figure 16 according to the scale order of the generated scale, beginning from "H." This order would not be generalizable to other pitch-classes, however. If we instead generated the scale from the nodes labeled "I" or "J", etc., we would have to relabel the line. Only the line associated with a well-formed scale can be labelled in a way that holds for all transpositions along the line.

diminished seconds have a span smaller than -5 (e.g., -12 [a diminished second], -19 [a doubly-diminished second], etc.). Therefore, in 12-tone equal temperament, a 7-scale spelling is proper if and only if the span of all the seconds that it contains are >= -5.[30] Note that this definition (unlike Hook's [2011]) eliminates the perverse spellings that we saw above such as {C, D♭♭♭♭♭♭♭♭♭♭♭♭, E, F, G, A, B}; therefore, under this definition of a proper spelling, proper spellings are unique under global enharmonic equivalence.[31]

We are now in a position to expand this definition beyond 12-tone equal temperament, although we will remain with 7-scales for the moment. Consider Figure 18. This figure shows the size, in semitones, of different values of specific seconds on the line of fifths, for several different values of fifth. Non-proper seconds—that is, those that involve enharmonic doublings or crossings—are highlighted in green, while the smallest proper second on each line of fifths is highlighted in blue. With a just fifth of 7.02 semitones, the smallest proper second has a span of -5, exactly as in 12-tone equal temperament. Exactly as in 12-tone equal temperament, therefore, a 7-scale spelling on the line of just fifths is proper if and only if the span of all the seconds that it contains have a span of >= -5. Therefore, even though the line of just fifths has an infinite number of enharmonically distinct pitch-classes, whereas the line of fifths in 12-tone equal temperament has only 12, the former line has exactly the same 66 proper-spelling classes as the latter. With a fifth smaller than 7 semitones, however, such as the fifth of 6 30/31 semitones that generates 31-tone equal temperament, the span of -12 (i.e., the diminished second) is also proper. Therefore, in 31-tone equal temperament, a letter-distinct 7-scale

[30] We could also look at how the possible spellings are constrained by the "large second," the largest possible scale step. However, because the largest second only occurs when all the other seconds are as closely packed (i.e., as small) as possible, it is fully determined by the size of small second together with the scale cardinality. (To be precise, the size of the large second in cents is always equal to 1200 – (n – 1) * s, where n is the cardinality of the scale and s is the size of the small semitone in cents.)

[31] Some might object that the idea of a "smallest second" seems to invoke pitch, rather than pitch-class, since every "small second" is also a "large seventh," depending only on the arbitrary choice of in which direction we measure the interval. We can get around this objection by formulating a more rigorous definition of the "smallest second," defined in terms of the linear equation for the different species of a generic interval, discussed below.

spelling is proper if and only if the spans of all the seconds that it contains are >= -12, rather than >= -5, as is the case for 12-tone equal temperament. This fact means that there are many more proper-spelling classes in 31-tone equal temperament (2652 to be exact).[32]

Figure 18. Seconds in 7-scales generated by various sizes of fifth.[33]

		Specific interval									
		dddd2	ddd2	dd2	d2	m2	M2	A2	AA2	AAA2	AAAA2
		Span on line of fifths									
	Size of fifth in 12-tet semitones	-33	-26	-19	-12	-5	2	9	16	23	30
Fifth in 16-tet	6.75	5.25	4.5	3.75	3	2.25	1.5	0.75	0	-0.75	-1.5
Fifth in 7-tet	6 6/7	1.71	1.71	1.71	1.71	1.71	1.71	1.71	1.71	1.71	1.71
Fifth 31-tet	6 30/31	-1.96	-1.18	-0.4	0.38	1.16	1.94	2.72	3.5	4.28	5.06
Fifth in 12-tet	7	-3	-2	-1	0	1	2	3	4	5	6
Just fifth	7.02	-3.66	-2.52	-1.38	-0.24	0.9	2.04	3.18	4.32	5.46	6.6
Fifth in 5-tet	7.2	-9.6	-7.2	-4.8	-2.4	0	2.4	4.8	7.2	9.6	12

Smallest second | Non-proper seconds (<= 0 semitones)

Notice also in Figure 18 that, at the top of the table, with a fifth of 6.75 semitones, the small second (in blue) and the non-proper seconds (in green) are found at the right of the table, rather than at the left. In the second row of Figure 18, on the other hand, with a fifth of 6 6/7 semitones (or 4/7 of an octave, generating 7-tone equal temperament), all seconds are equal. These and other features are better appreciated when we plot the numbers from Figure 18 as a line graph, as has been done in Figure 19. When the slope of the extrapolated lines in Figure 19 is positive, as it is for values of fifth greater than 6 6/7, then spans of -5 are smaller than spans of 2, and the usual nomenclature of minor and major seconds, respectively, makes sense. When the value of the fifth is 6 6/7, the slope of the extrapolated line is flat, and the value of the second is always 1/7 octave (i.e.,

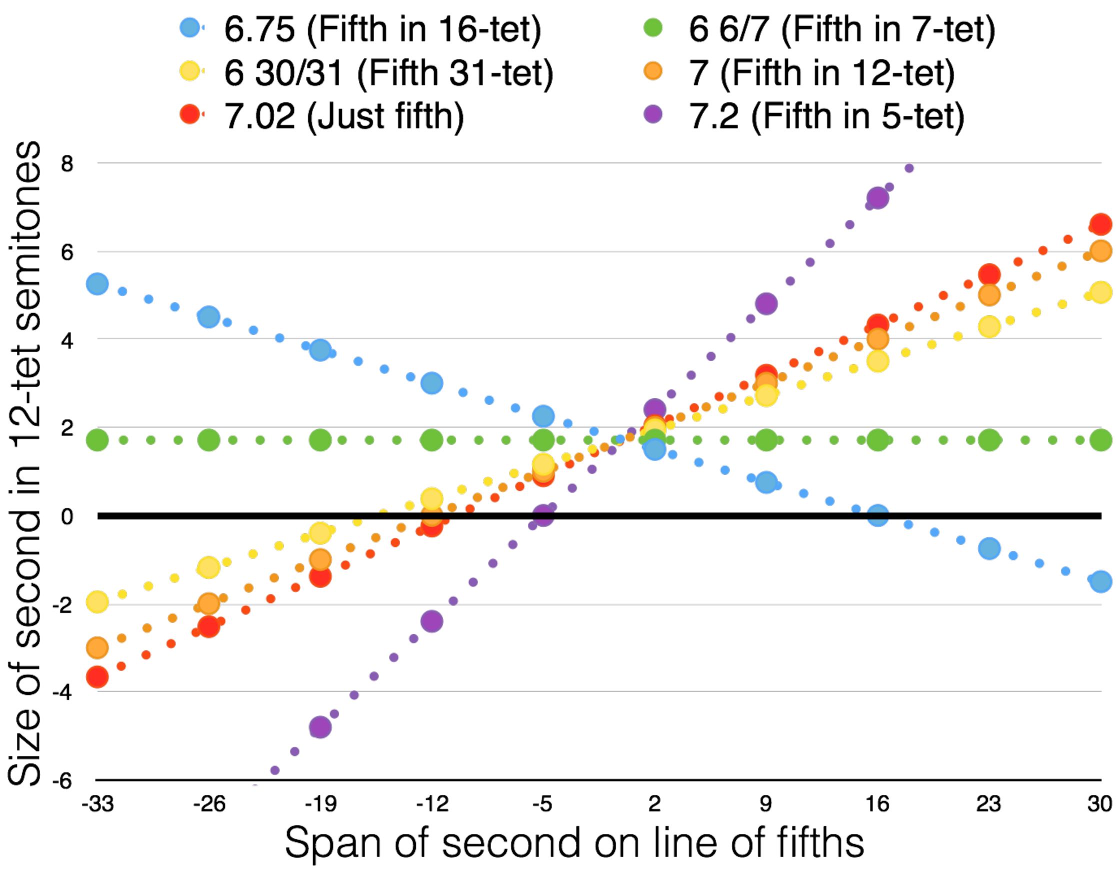
[32] These numbers were calculated with a computer program that, given a cardinality and generating interval, simply produced every possible fifth-string, and then counted them. I have not yet worked out how these numbers might be calculated more directly. As we will see, however, it would be necessary to do so for values of fifth close to 6 6/7, because as the fifth approaches 6 6/7, the span of the smallest second approaches infinity, meaning that the number of proper fifth-strings also approaches infinity.

[33] The use of negative numbers in this chart will be explained below.

Also, there are no proper 7-scales in 5-tone equal temperament. The “5-tet fifth” of 7.2 semitones is given in this chart because it is the upper bound for well-formed 7-scales generated by scalar fifths.

12/7, or about 1.71, semitones). For smaller values of fifth such as 6.75, the slope of the extrapolated line is negative, and "minor seconds" (i.e., spans of -5) are larger than "major seconds" (i.e., spans of 2).

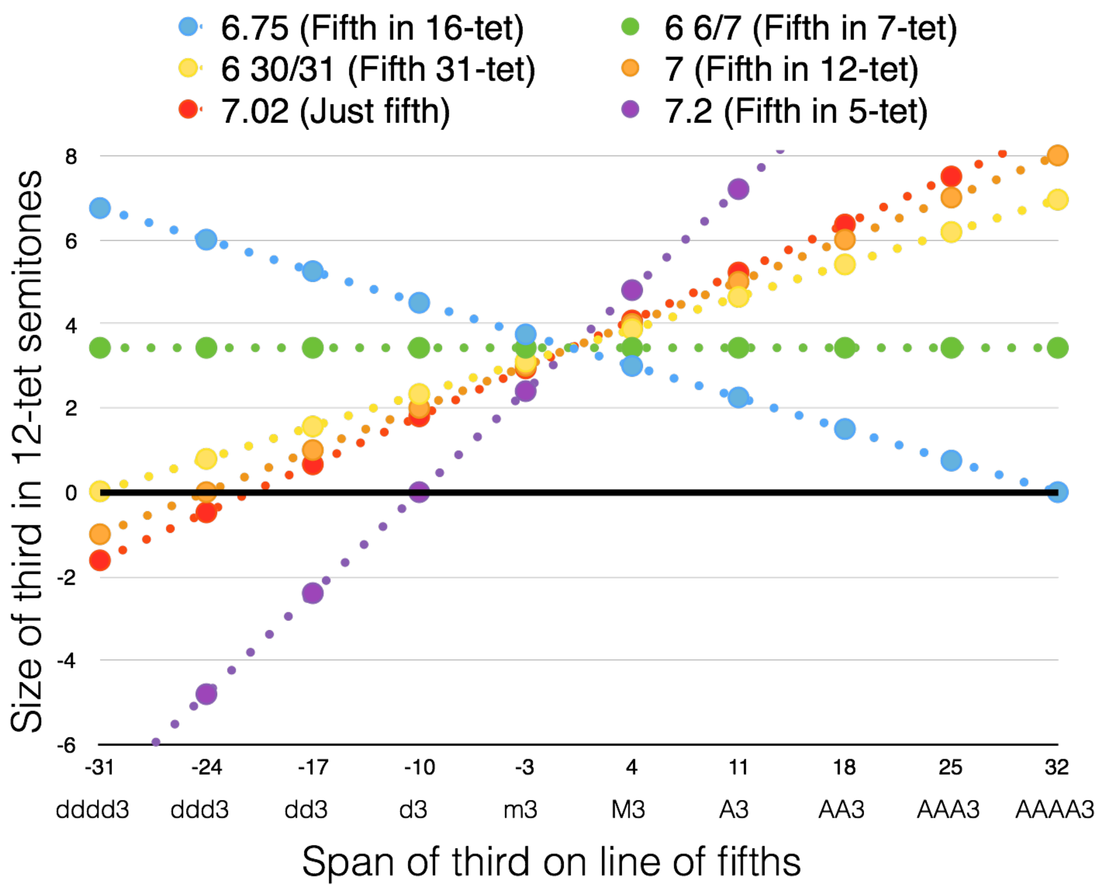
Figure 19. Extrapolated line graph for seconds in 7-scales generated by various sizes of fifth.



The seconds generated by each size of fifth lie upon the dotted lines depicted in this chart. The lines themselves, however, are no more than extrapolations from these points. The line of fifths is discrete, not continuous (e.g., there is no pitch between F and C on the line of fifths) and, moreover, even at intermediate whole number values (e.g., 4, a major third) the spans of the the line of fifths do not have the values depicted by the dotted lines.

It is possible to plot a similar chart for any other size of generic interval; for example, this has been done for thirds in Figure 20. The slope of the extrapolated line for each generic interval is always 1/7 * the size of the augmented unison on the relevant line of fifths. This follows from the fact that augmented unisons are always found seven steps apart upon the line of fifths (in the case of 7-scales) together with the fact that an augmented unison is always the absolute difference between the size of any two "adjacent" qualities of the same interval (such as a major interval [e.g., C to D] and a minor interval [e.g., C to D♭], or a major and an augmented interval, or a diminished and a doubly diminished interval, etc.). (The diminished unison is always equal to -1 * the augmented unison, so the absolute value of the diminished and augmented unisons is the same).

Figure 20. Extrapolated line graph for thirds in 7-scales generated by various sizes of fifth.



All lines for a scalar interval *n* in any scale cardinality *c*, such as the lines in Figure 19 and Figure 20, will always intersect at a span of 0 (i.e., at the *y*-intercept)—although, because these are extrapolated lines, this span will only correspond to a pitch-class in the case of the unison. The point of intersection is always (*n*/*c*) * the octave, regardless of the size of the generating interval. For instance, the y-intercept of the lines for scalar seconds ($n = 1$) in 7-scales ($c = 7$) is $n / c = 1/7$ octave (or 12/7 semitones, about 1.714), and it can be verified visually that this is the point of intersection of the lines in Figure 19. The y-intercept of the lines for scalar thirds ($n = 2$) is 2/7 octave (or 24/7 semitones, about 3.429), and this can be seen to be the point of intersection in Figure 20. It follows that, for any well-formed combination of generating interval and scale cardinality, the only other number we require to calculate the size of any scalar interval of directed span s is the size of the augmented unison, u. With a little algebra, we can show that the size of the interval in semitones will then be $(12n + su) / c$ (assuming the augmented unison is itself expressed in semitones). Moreover, if we derive the appropriate multiplicative automorphism a from generic scalar intervals to generated order residue classes, we can eliminate the generic interval n from this calculation, giving $(12(sa \bmod c) + su)/c$ (again assuming that u is expressed in semitones). So, for instance, for 7-scales in 12-tone equal temperament $c = 7$, $a = 4$, and $u = 1$ semitone. Therefore,

- the size of the span 2 is (12(2 * 4 mod 7) + 2 * 1/7) = (12 + 2)/7 = 14/7 = 2, (a major second is two semitones),
- the size of the span -5 is (12(-5 * 4 mod 7) + -5 * 1)/7 = (12 – 5)/7 = 7/7 = 1, (a minor second is one semitone),
- the size of the span 4 is (12(4 * 4 mod 7) + 4 * 1)/7 = (24 + 4)/7 = 28/7 = 4, (a major third is four semitones),

and so on.

The virtue of using these linear equations to calculate intervals, rather than simply multiplying the span by the generating interval, mod 12, is that the linear equations calculate intervals in pitch space, rather than pitch-class space. (Note the absence of modular arithmetic in the above calculations.) In pitch-class space, an 11-times diminished second with a span of -82 (e.g., C to D-duodecuple flat) is equivalent to a major second (-82 * 7 mod 12 = 2), but in pitch space, it is -10 semitones, as given by 12(-

82 * 4 mod 7)/7 + -82 * 1/7 = 12/7 + -82/7 = -70/7 = -10. This difference allows us to rigorously define the idea of a "smallest second" as the least second > 0, and provides a parsimonious formalization of the intuition that C and D-duodecuple flat should cross.

1.5 Web app 1.1

Web app 1.1, "Generating scales," allows the reader to experiment with some of the ideas in the preceding sections.[34] The app first asks for a generating interval. If this generating interval produces a line of fifths with a well-formed 7-scale (for now, only 7-scales are supported), then the user is asked to choose a fifth-string in order to construct a scale. (In fact, the user may be asked to choose a second-string, or a third-string, etc., depending on the choice of generator.) The user will next be shown the step sizes of the resulting scale, and selects a mode of this scale for playback. The scale is then played back in melodic form. At this point, the user can also apply the scale to a keyboard work of Bach, or hear the effect of a sequential pattern repeatedly transposed through the scale. This latter functionality is the same as that for non-generated scales provided by web apps 2.1 and 2.2, respectively, and will be described in greater detail in section 2.5.

1.6 Some special features of 12-tone equal temperament

In this section, we will see that the apparatus developed above for describing nearly well-formed scales can help make apparent some special properties of 12-tone equal temperament. These are as follows:

1) 12-tone equal temperament is the only temperament where the augmented unison and minor second are enharmonically equivalent. This may have advantages for chromatic tonal music.
2) 12-tone equal temperament is *diatonically closed*. This helps make enharmonicism more practicable.

[34] http://sailor.music.mcgill.ca/generatescale

3) In 12-tone equal temperament, the augmented unison is equal to one chromatic step. From this it can be shown to follow that every 7-scale has a proper spelling.

When I speak of an "equal temperament," I'm referring to the set of pitch-classes available on the generated line for some choice of generating interval, where the generating interval is a rational proportion of the octave. (If the generating interval is not a rational proportion of the octave, then we have an untempered tuning system.) Given a generating interval of 7/12 octave (or 7 semitones), all and only the pitch-classes of 12-tone equal temperament are available. These pitch-classes are also available in 24- or 36-tone equal temperament, but this generating interval can only ever generate a subset of their pitch classes—the 12 pitch-classes of 12-tone equal temperament. (It is possible to generate 24-tone equal temperament with a fifth of 13/24 octave. The resulting "diatonic" scale has five "major seconds" of 1 semitone and two "minor seconds" of 3.5 semitones.)

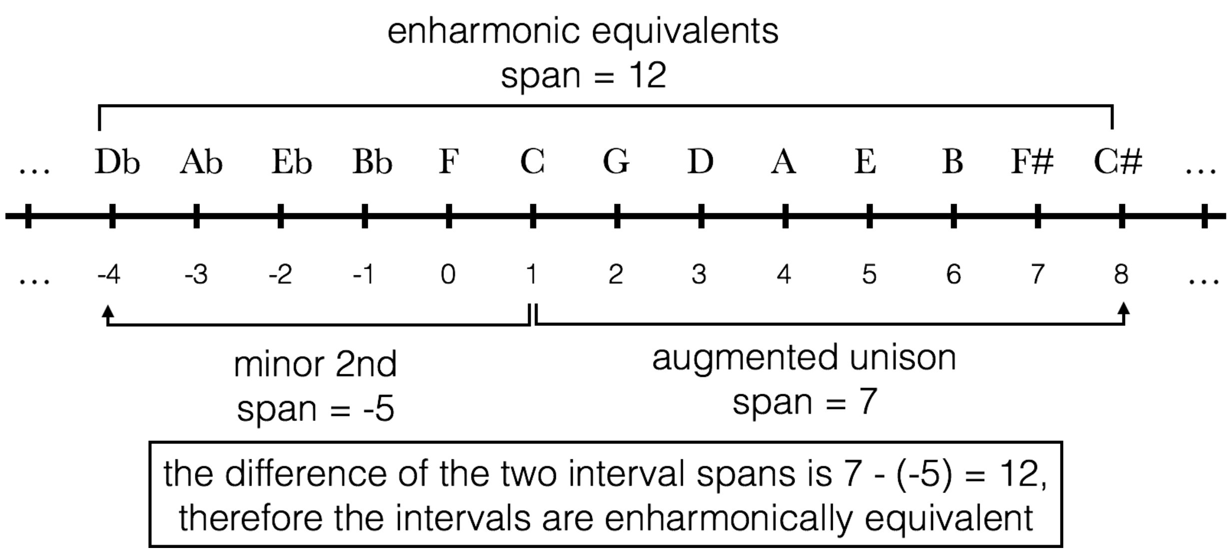
In the following, I will refer to what I call the *simplicity* of an interval. I believe that an example makes it easy to see what I mean by this term: a major second is a relatively simple interval, a diminished seventh is a somewhat less simple interval, and a triply-augmented fourth is a much less simple interval. In general, simpler intervals have shorter spans. Any reader who would prefer a more rigorous definition can see the footnote to this sentence.[35]

IN 12-TONE EQUAL TEMPERAMENT, THE AUGMENTED UNISON AND MINOR SECOND ARE ENHARMONICALLY EQUIVALENT: Let us begin by discussing the conditions under which pitches and intervals on a generated line are enharmonically equivalent. If a generating interval generates an equal temperament of cardinality *n*, then any two pitch-classes

[35] Simplicity can be defined in terms of the scale of least span (i.e., of highest near well-formedness) that contains a given interval. So, according to this definition, the intervals of the diatonic scale (span = 6) have the highest simplicity, followed by the augmented fifth and diminished fourth of the melodic-minor scale (span = 8), followed by the augmented second and diminished seventh of the harmonic minor and major scales (span = 9), and so on.

separated by a span of n on the generating line—or a whole multiple thereof—must be enharmonic equivalents. For instance, in 12-tone equal temperament, the span from D-flat to C-sharp is 12, and these pitches are enharmonic equivalents (see Figure 21), as are D-flat and B-double-sharp (with a span of 24 between them). In 19-tone equal temperament, in contrast, neither of these pairs of note names are enharmonic equivalents, but all pairs separated by a span of 19 are. A logical corollary is that any two intervals whose spans differ by n—or a whole multiple thereof—are of equal size in pitch or pitch-class space (i.e., the intervals are locally enharmonically equivalent). This follows from the fact that, if we construct both intervals on the line of fifths starting from the same pitch-class, the pitch-classes attained will differ by n or some multiple thereof, and will therefore be enharmonic equivalents (again, consider Figure 21). Plainly, two intervals joining a single pitch-class to two enharmonically equivalent pitch-classes will necessarily be enharmonically equivalent.

Figure 21. Enharmonically equivalent intervals upon the line of fifths.



The conditions above can be usefully restated as follows. Any two pitch-classes belonging to the same residue class, modulo n (the cardinality of the temperament), are enharmonic equivalents (e.g., C♯ [8 on the line in Figure 21] mod 12 and D♭ [-4] mod 12 are both 8; this is simply a restatement of the definition of atonal pitch-class). Any two intervals whose spans belong to the same residue class, modulo n, are likewise enharmonic equivalent (e.g., a minor second, with a span of -5, and an augmented unison, with a span of 7, both have residue class 7, mod 12).

From the above, it follows that, in any equal temperament, we can always find the enharmonic equivalent intervals of the augmented unison by adding or subtracting the cardinality of the temperament n from 7 (assuming we are investigating 7-scales) a whole number of times. [36] Moreover, because simpler intervals have smaller absolute spans, the simplest enharmonic equivalent interval will always be found by subtracting n once from 7.[37] Therefore, in 12-tone equal temperament, the simplest enharmonic equivalent to the augmented unison has span 7 – 12 = -5; it is the minor second. In 19-tone equal temperament, on the other hand, it is 7 – 19 = - 12, a diminished second, while in 31-tone equal temperament, it is 7 – 31 = -24, a doubly-diminished third. In 7-scales generated on a line of fifths, the minor second always has a span of -5, and because 12 – 7 = -5, 12-tone equal temperament is plainly the only temperament where the minor second is equivalent to the augmented unison. Moreover, these intervals are also equal to the chromatic step size in the temperament. (We will see below that the equality of the augmented second and chromatic step size implies that all pitch sets are nearly well-formed.)

These two intervals, the minor second (as the smallest interval of the diatonic [and quasi-diatonic] scales), and the augmented unison (as the interval of alteration between one diatonic scale and another), are the smallest intervals likely to be found in quasi-diatonic music, regardless of the tuning system adopted. The fact that, in 12-tone equal temperament, these two intervals are equal to each other and, moreover, to the "lower

[36] Untempered tuning systems do not have enharmonic equivalents.

[37] Because the result of 7 - n will always be closer to 0 than any other member of the residue class of 7, except for possibly 7 itself. For instance, 7 – 12 = -5, which is closer to 0 than 7 + 12 or 7 – (2 * 12), etc.

bound" chromatic step size, may provide advantages in the composition of chromatic tonal music: a composer may be somewhat indiscriminate in his application of chromaticism, and yet be confident that her alterations can be heard as at least one of these two intervals. On the other hand, there may also be a disadvantage to this enharmonicism as well, insofar as a potentially meaningful difference of colour between the augmented unison and the minor second is lost. These questions may be a worthwhile avenue of future study.

12-TONE EQUAL TEMPERAMENT IS DIATONICALLY CLOSED: I call a given temperament *diatonically closed* when all atonal intervals within the temperament are enharmonically equivalent to intervals of the diatonic scale in that temperament. That 12-tone equal-temperament is diatonically closed can be verified from Figure 22A, where all 12 atonal intervals appear. There are thirteen different interval sizes in the diatonic scale, including the unison/octave. It follows that any diatonically closed tuning system can have at most 13 pitch-classes, so 12-tone equal temperament is very nearly the largest diatonically closed temperament; a larger temperament than 13-tone equal temperament will necessarily have more than twelve different atonal interval classes. (Consider Figure 22B, where several atonal intervals are missing).

Figure 22. Diatonic interval species in two different temperaments.

A) 12-tone equal temperament

	Atonal intervals (i.e., size in 12-tone equal-tempered chromatic steps)	
Generic scalar interval class	**Minor***	**Major†**
Unison	0	
Second	1	2
Third	3	4
Fourth	5	*6*
Fifth	*6*	7
Sixth	8	9
Seventh	10	11

All atonal intervals present

B) 19-tone equal temperament

	Atonal intervals (i.e., size in 19-tone equal-tempered chromatic steps)	
Generic scalar interval class	**Minor***	**Major†**
Unison	0	
Second	2	3
Third	5	6
Fourth	8	9
Fifth	10	11
Sixth	13	14
Seventh	16	17

Atonal intervals 1, 4, 7, 12, 15, and 18 missing

*includes the "minor" (i.e., smaller) perfect fourth and diminished fifth
†includes the "major" (i.e., bigger) augmented fourth and perfect fifth

A consequence of diatonic closure is that all non-diatonic scalar intervals (i.e., diminished and augmented intervals on the generated line) will be enharmonically equivalent to diatonic intervals. For example, in 12-tone equal temperament, an augmented second is enharmonically equivalent to a minor third, an augmented sixth is enharmonically equivalent to a minor seventh, etc. A diatonically closed temperament is therefore richest in possibilities for enharmonicisms (at least in music that takes diatonic structures as its starting point). Beginning from any diatonic scale, any alteration that creates a diminished or augmented interval will be enharmonically equivalent to a non-altered interval in some other diatonic scale. All the familiar enharmonic devices of tonal music (e.g., those involving augmented sixths or diminished sevenths) rely on this fact. In contrast, in 19-tone equal temperament, we must extend as far as doubly augmented or diminished intervals before finding any enharmonic equivalents of the diatonic scale, and in 31-tone equal temperament, enharmonic equivalents of the diatonic scale are found only among triply and quadruply augmented and diminished intervals. (These equivalences are illustrated in Figure 23, where intervals in all three-temperaments are listed beside their enharmonic equivalents.) Assuming (as seems reasonable) that it is relatively more difficult to perceive doubly-, triply-, etc., augmented and diminished intervals, enharmonicism in these temperaments is likely to be a correspondingly more difficult compositional device to achieve.

It is a remarkable circumstance that 12-tone equal-temperament is not only diatonically closed, but also provides reasonably good approximations to acoustically just intervals. It did not "have" to be this way, in the sense that one can imagine an alternate universe where the properties of acoustics and musical culture combine to produce a diatonically closed scale that imposes intolerable approximations of the diatonic intervals.

Figure 23. Enharmonically equivalent intervals in various temperaments.

Diatonic interval

12-tone equal temperament

Enharmonically Equivalent Spans				Enharmonically Equivalent Spans			
Span	Interval	Span	Interval	Span	Interval	Span	Interval
0	P1	-12	d2	0	P1	12	A7
1	P5	-11	d6	-1	P4	11	A3
2	M2	-10	d3	-2	m7	10	A6
3	M6	-9	d7	-3	m3	9	A2
4	M3	-8	d4	-4	m6	8	A5
5	M7	-7	d1	-5	m2	7	A1
6	A4	-6	d5	-6	d5	6	A4

19-tone equal temperament

Enharmonically Equivalent Spans				Enharmonically Equivalent Spans			
Span	Interval	Span	Interval	Span	Interval	Span	Interval
0	P1	-19	dd2	0	P1	19	AA7
1	P5	-18	dd6	-1	P4	18	AA3
2	M2	-17	dd3	-2	m7	17	AA6
3	M6	-16	dd7	-3	m3	16	AA2
4	M3	-15	dd4	-4	m6	15	AA5
5	M7	-14	dd1	-5	m2	14	AA1
6	A4	-13	dd5	-6	d5	13	AA4
7	A1	-12	d2	-7	d1	12	A7
8	A5	-11	d6	-8	d4	11	A3
9	A2	-10	d3	-9	d7	10	A6

31-tone equal temperament

Enharmonically Equivalent Spans				Enharmonically Equivalent Spans			
Span	Interval	Span	Interval	Span	Interval	Span	Interval
0	P1	-31	dddd3	0	P1	31	AAAA6
1	P5	-30	dddd7	-1	P4	30	AAAA2
2	M2	-29	dddd4	-2	m7	29	AAAA5
3	M6	-28	dddd1	-3	m3	28	AAAA1
4	M3	-27	dddd5	-4	m6	27	AAAA4
5	M7	-26	ddd2	-5	m2	26	AAA7
6	A4	-25	ddd6	-6	d5	25	AAA3
7	A1	-24	ddd3	-7	d1	24	AAA6
8	A5	-23	ddd7	-8	d4	23	AAA2
9	A2	-22	ddd4	-9	d7	22	AAA5
10	A6	-21	ddd1	-10	d3	21	AAA1
11	A3	-20	ddd5	-11	d6	20	AAA4
12	A7	-19	dd2	-12	d2	19	AA7
13	AA4	-18	dd6	-13	dd5	18	AA3
14	AA1	-17	dd3	-14	dd1	17	AA6
15	AA5	-16	dd7	-15	dd4	16	AA2

EVERY 7-SCALE HAS A PROPER SPELLING IN 12-TONE EQUAL TEMPERAMENT: In 12-tone equal temperament, an augmented unison is equal to one step in the chromatic scale. (I assume 7-scales generated by fifths.) Because an augmented unison is always the difference between adjacent qualities of any interval (e.g., between a minor second and a major second, or between a major second and an augmented second, etc.), and because every interval in an equal temperament will necessarily be equal to a whole number multiple of a single step in the chromatic scale of that temperament, it follows that every generic scalar interval (e.g., second) has a proper spelling corresponding to every atonal pitch-class interval in the temperament. We need only begin from the smallest species of the interval, which will always be 1 chromatic step, and add augmented unisons. So the smallest second in 12-tone equal temperament is the minor second, measuring 1 chromatic step. We can then successively raise it, first to a major second (2 chromatic steps), through an augmented second (3 chromatic steps), all the way to, in principle, a nine-times augmented second (11 chromatic steps). Similar reasoning applies to all other sizes of generic scalar interval— any number of steps in the chromatic scale can be given a proper spelling as any required generic scalar interval.[38] Since all intervals have proper spellings, no matter what the intervals in a scale are, they can be spelled properly. Therefore, any possible 7-scale in 12-tone equal temperament has a proper spelling.[39]

Similar reasoning applies to 19-tone equal temperament, where the augmented unison is one step in the 19-tone chromatic scale (see Figure 24). In 31-tone equal temperament, however, the augmented unison is *two* steps in the 31-tone chromatic scale. In any case such as this, where the augmented unison is greater than one step in the chromatic scale, the possible species of each generic scalar interval will only correspond to a subset of the possible atonal pitch-class intervals. For instance, in 31-tone equal temperament, the major second (e.g., C to D, with a span of 2) is 5 chromatic steps. If we diminish it to a minor second (e.g., C to D♭, span of -5), its size will decrease by an

[38] The intervals that result are all themselves proper, but it may be impossible to build a proper scale that contains them. Such intervals as a nine-times augmented second are incompatible with proper scales (at least in 12-tone equal temperament, and probably any other practicable tuning system).

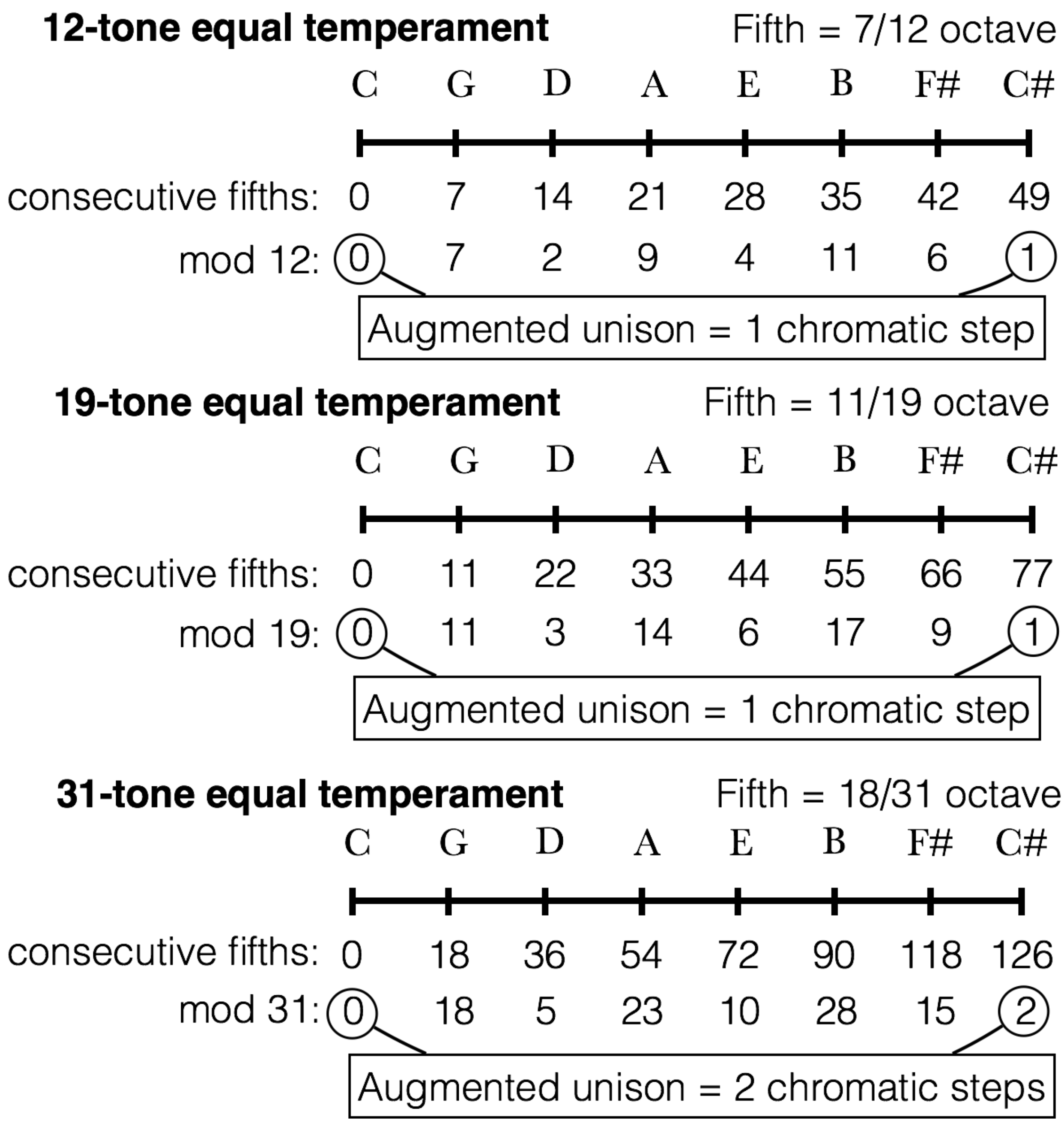
[39] Also, all sets of cardinality less than seven can be given at least one proper spelling as subsets of some 7-scale.

augmented unison of 2 chromatic steps, to 3 chromatic steps; if we decrease it again to a diminished second (e.g., C to D♭♭, span of -12), it will decrease to 1 chromatic step. If we decrease the second further, it will no longer be positive, so this is the smallest proper second. This means there are no proper seconds of 2 or 4 chromatic steps; therefore, any pitch-class set that contains these intervals between adjacent members will be impossible to give a proper spelling. In other words, whereas in 12- or 19-tone equal temperament, all 7-chords can be spelled as proper scales, in 31-tone equal temperament, many 7-chords do not have proper spellings. In fact, since there are 2652 proper-spelling classes in 31-tone equal temperament, but 84825 T_n-classes of 7-chord, only about 3% of 7-chords have proper spellings in this temperament.[40]

[40] The number of T_n-classes of 7-chord was calculated using the Polyà enumeration theorem. Thanks to Jon Wild for showing me this theorem and how it works.

It is easy to find temperaments where even fewer T_n-classes will have proper spellings. For instance, in any temperament where the diminished second is the smallest second, there will be 2652 proper spelling classes. In any temperament of higher cardinality than 31 where this is true of the diminished second, therefore, there will be an even smaller proportion of Tn-classes with proper spellings. Two examples are 43-tet, where ca. 0.35 % of 7-chords have proper spellings, and 79-tet, where ca. 0.007 % of 7-chords do.

Figure 24. Size of the augmented unison in several different temperaments.



Chapter 2 Aspects of quasi-diatonicity and scale-ness

It seems safe to assume that the diatonic scale has high "scale-ness." Its persistent ubiquity in Western music, and the music of many other parts of the world as well, suggests that, whatever the features are that make a given T_n-class suited to use as a scale, the diatonic scale is likely to possess them. Taking this observation as a starting point, in this chapter I begin by considering some basic features of the diatonic scale, such as its cardinality, that will delimit the following discussion of scales. (I take certain basic features as needing no explanation—for instance, the fact that the diatonic scale is octave repeating. This is not to say that the human proclivity to perceive octave equivalence is not remarkable, but it belongs in a study of psychoacoustics, rather than in the present thesis.) In the following section, I review the literature on some of the diatonic scale's special properties and consider whether, and to what extent, they are shared by other scales. In a third section, I give a brief account of how these scale properties might be used to sort scales according to their "quasi-diatonicity," and in the fourth section, I then discuss the idea that many of these same properties can be thought of as applying to actual music, and not simply abstract scales. Finally, an appendix discusses some alternatives to the "diatonic thirds" scale property.

Before continuing further, we should define exactly what is meant in this thesis by a "diatonic" scale. Intuitively speaking, we can associate the diatonic scale with the "white keys" of a piano keyboard (or transpositions thereof), but this begs the question of how these keys are tuned. Using the framework of generated lines developed in the previous chapter, we can give the following definition of a diatonic scale: a diatonic scale is a 7-scale generated by consecutive scalar fifths. The tuning of the fifths can vary—some common values are the just fifth of ca. 702 cents, the 12-tone equal tempered fifth of 700 cents, […]—as long as it remains in the interval (600 cents, 720 cents). If any interval in this range is used to generate a 7-scale, the interval will be a scalar fifth within the resulting scale. (In other words, the resulting scale will be a well-formed scale, as it has to be if every step in its generation corresponds to the same scalar interval, and the automorphism from generated order to scale order will be multiplication by 4, mod 7, the mathematical operation corresponding to a scalar fifth.) As a consequence, all the abstract features we have discussed and will discuss—from the correspondence of directed spans with intervals to "diatonic thirds" (see below)—hold for any scale generated by an interval within this range. It is therefore useful for us to speak of these all as "diatonic scales" (or as specific manifestations of "the diatonic scale"). Nevertheless, it should be noted that only when the generating fifth is reasonably close to a just fifth do we get a scale that is, in terms of its actual acoustic content, recognizably close to the diatonic scale of any familiar tuning system. In other words, while the scales generated by fifths in the interval (600 cents, 720 cents) share abstract properties, in actual music, these properties are best realized by scales generated by a much narrower interval of fifths focused around 702 cents—at least if we wish to produce music that is recognizably diatonic.

In this context, the reader is invited to experiment with generating intervals in the interval (600 cents, 720 cents) with an associated web app.[41]

I will speak of the line of fifths generated by a fifth within the range (600 cents, 720 cents) as a "well-formed line of fifths," because the generated scale on this line will be well-formed. In order for the theoretical framework of note spellings and scalar intervals developed in the previous chapter to apply, scales must be constructed upon a well-formed line of fifths. Even when extending our investigations beyond the diatonic scale, therefore, I will assume a well-formed line of fifths, for instance when I speak of scalar intervals in terms of directed spans. However, some scale cardinalities, such as 6 and 8, cannot be associated with well-formed lines of fifths. Scales of these cardinalities appear to require a different sort of theoretical framework, as is perhaps appropriate since these

[41] http://sailor.music.mcgill.ca/generatescale

cardinalities are most associated with transpositionally symmetrical scales like the whole-tone and octatonic. These sorts of scales have very different properties and idiomatic uses than the diatonic or quasi-diatonic scales with which this thesis is most concerned.

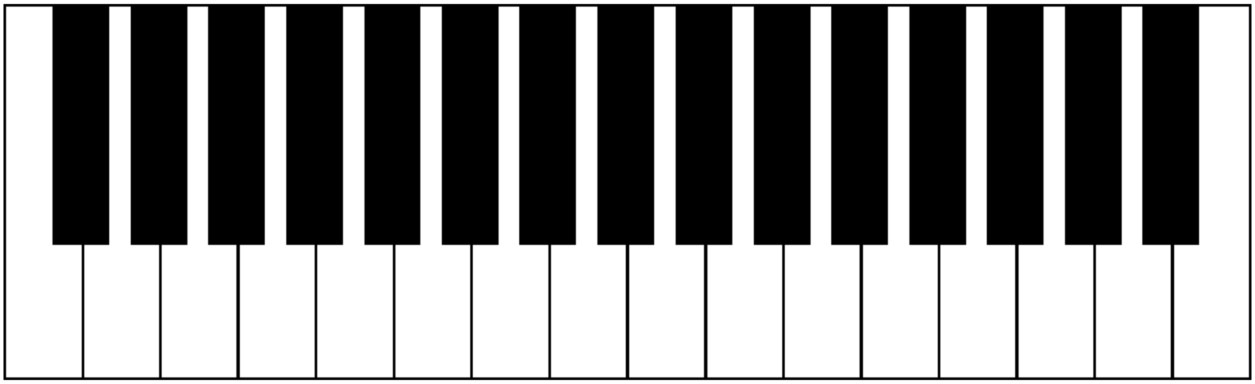
2.1 Basic features of the diatonic scale

THE DIATONIC SCALE HAS UNEVEN STEP SIZES: The uneven step sizes of the diatonic scale permit us to orient our listening in much the same way as an uneven topography permits us to situate ourselves in a landscape. The difficulty of "tonal orientation" in a scale with even step sizes can be metaphorically compared to the difficulty of playing piano on a keyboard such as that in Figure 25. There is empirical evidence in support of this idea: for instance, in Trehub et al. 1999 (cited in Patel 2008), infants were better at discriminating pitch changes in scales with uneven step sizes (both the diatonic scale as well as a presumptively unfamiliar novel scale were tested) than in scales with even step sizes.[42] In Savage et al.'s (2015) study of "statistical universals" of human music, the use of "nonequidistant scales" was the most universal pitch feature of human music (8988–9).[43]

[42] Adults were also tested, and they performed well on the diatonic scale, but equally poorly on the unfamiliar uneven scale as on the even scale. The difference in adult and infant performance, as illustrated in the accompanying graphs, is stark, and tells a powerful parable about the role of both learning and intrinsic and/or innate constraints in the understanding of music.

[43] "Statistical universals" might also be termed "important generalizations." I.e., it is not that they are not without exceptions, but that they are nevertheless so pervasive that their pervasiveness requires explanation.

Figure 25. A redesigned piano keyboard.

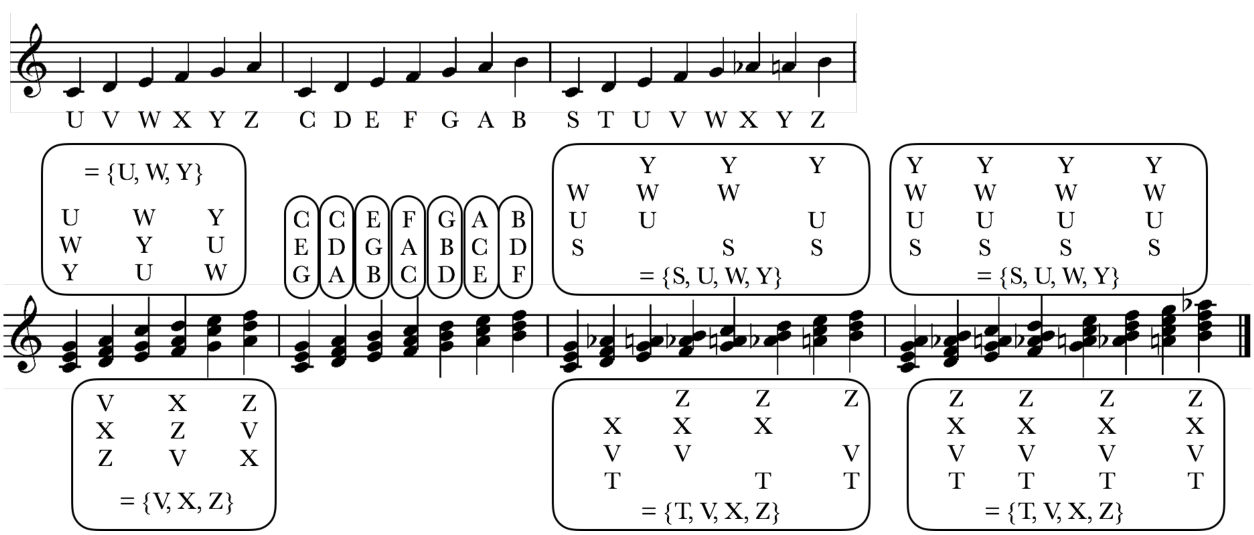


THE DIATONIC SCALE IS TRANSPOSITIONALLY ASYMMETRICAL: This can be seen as the more general case of the previous property. In a transpositionally asymmetrical scale, every generic interval will come in uneven sizes. The diatonic scale not only has uneven steps (i.e., major and minor seconds), but it also has uneven thirds, fourths, etc. In a transpositionally symmetrical scale, on the other hand, every generic interval that comes in a specific size that is a whole multiple of the interval of transpositional symmetry will only come in that specific size. In the octatonic scale, for instance, thirds are uniquely 3 semitones, and fifths are uniquely 6 semitones. The most important musical result is that the harmonies of an asymmetrical scale will form an asymmetrical pattern, such as the familiar pattern of triad qualities in the major mode, and this asymmetrical pattern permits and indeed facilitates the emergence of a tonal center.[44] (In their study of statistical universals, Savage et al. 2011 only considered whether scales had equidistant step sizes or not. Since a scale with a single step size [such as the whole-tone scale] is only a special case of a transpositionally symmetrical scale, it would be interesting to know how widespread other transpositionally symmetrical scales are.)

[44] This discussion should not be taken to suggest that it is impossible to establish a centric pitch in a transpositionally symmetrical scale. We can frame the question in terms of the "amount of tonal weight" the composer has to lift in order to do so. To ensure that we hear a given tonic in a whole tone scale, for instance, a composer might be obliged to hammer out a repeated pedal point, whereas in the diatonic scale, she might well find it comparatively effortless to establish a central tonic.

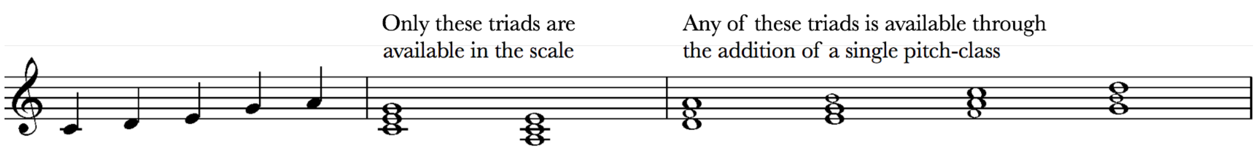
THE CARDINALITY OF THE DIATONIC SCALE IS ODD: If the cardinality of a scale is even, proceeding through that scale by thirds (i.e., by two scale steps at a time) will always cycle through a subset of half of the pitches in the scale. (One can think of this as dividing the original scale into two equal subsets by taking every second pitch.) In a 6-scale, for instance, because six divided by two is three, taking successive tertian 3-chords will alternate through two 3-chords in successive inversions. This means that there are at most two distinct consonant triads in a six-note scale. (Each of the eleven ways of combining two interlocking triads will give rise to such a "scale"; the diatonic hexachord [which can be seen as the combination of C major and D minor triads] is given in Figure 26].) In an eight-note scale, the situation is somewhat different. Here, taking successive tertian 4-chords from the scale will alternate through two 4-chords in successive inversions. If we take 3-chords instead, the 3-chords will alternate between subsets of the two 4-chordal "halves" of the scale. The effect of two alternating background collections is palpable, and very different from the effect of seven audibly distinct 3-chords obtainable in 7-scales.

Figure 26. Tertian 3-chords in 6-scales, 7-scales, and 8-scales.



THE DIATONIC SCALE HAS SEVEN PITCH-CLASSES: Each of the basic features seen so far—uneven step sizes, transpositional asymmetry, and odd cardinality—follow as a necessary consequence of the fact that the diatonic scale has seven pitch-classes (i.e., in the jargon of this thesis, it is a 7-scale). In Savage et al. (2011), the use of scales of seven or fewer pitch-classes is listed as the second most universal pitch feature of human music. Although this clearly leaves the door open to scales with fewer than seven pitch-classes, in music where harmony is based on motion between consonant triads, it seems likely that there may be a sort of natural momentum for scales with fewer pitch-classes to accumulate up to seven pitch-classes. This is simply because there will necessarily be "gaps" in such a scale where triads are unavailable, and these gaps can readily be filled through the addition of a single pitch-class. Such a procedure is arguably evident in a great deal of recent pop music, where melodies are heavily pentatonic, but harmonizations draw from the consonant triads of the diatonic scale. This sort of thinking, while hardly unassailable, suggests that 7-scales may be in a sort of "Goldilocks' zone" for scale cardinality in triadic music.[45] Whether or not this is true, this thesis largely constrains its remaining investigations to the domain of 7-scales.

Figure 27. Consonant triads available in the pentatonic scale.



2.2 Some important properties of the diatonic scale and its friends and neighbours

DIATONIC THIRDS: A scale with "diatonic thirds" is a scale in which all thirds are either major or minor. Tymoczko [2004, 225] defines major or minor thirds as those that are either 3 or 4 semitones, but we could also define them as intervals with a directed span of either -3 or 4 on the line of fifths. In equal temperament, these two formulations

[45] Within 12-tone equal temperament (and any other temperament not divisible by seven), the cardinality of the diatonic scale also enforces its transpositional asymmetry.

are equivalent,[46] but the latter formulation is of broader relevance for 7-scales because it allows us to observe that diatonic thirds can be a property of non-12-tone equal tempered scales as well. Four important 7-scales have diatonic thirds: the diatonic scale, the melodic-minor scale, the harmonic minor scale, and the harmonic major scale.[47] When it comes to scales of other cardinalities, however, the definition in terms of directed spans upon the line of fifths is more problematic because the cardinality may not have a well-formed line of fifths associated.

As Tymoczko shows, under his definition of diatonic thirds in terms of 3- or 4-semitone intervals, among 12-tone equal tempered T_n-classes of cardinality other than 7, there are three with diatonic thirds. Each of these are transpositionally symmetrical: the hexatonic, whole-tone, and octatonic scales. Because these are not 7-scales, however, there is no well-formed line of fifths associated with them, and scalar intervals within these scales cannot be put into correspondence with directed spans, and a definition of diatonic thirds in terms of spans on the line of fifths cannot be made to apply.

On the other hand, compensating for this loss somewhat is that defining diatonic thirds in terms of spans allows us to admit a new scale to the ranks of those having diatonic thirds, although this scale is only available in temperaments where the diminished second is proper (i.e., not including 12-tone equal temperament). This scale is the "harmonic major ♯5 scale," which might be spelled (C, D, E, F, G♯, A♭, B). It can be considered the union of a diminished seventh chord (G♯, B, D, F) with an augmented triad (A♭, C, E) a diminished second distant.[48] (One easy way to conceive of this scale is

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[46] This follows from the fact that (3 * 7) mod 12 = 9 [i.c. 3] and (4 * 7) mod 12 = 4.

[47] This is true for all usual values of fifth. If the fifth is such that the augmented second is not proper, however, then the harmonic minor and harmonic major scales are not available—but this only occurs with a fifth such as 7/13 octave (which generates 13-tone equal temperament). Because this fifth is smaller than 4/7 octave, the augmented unison is negative and the "major second" (i.e., span of 2) is the smallest second (see Chapter 1). This is very far removed from any practicable tuning system! (Note that 7/13 is not the best approximation to a just fifth in 13-tone equal temperament—this is instead 8/13. 8/13 does not give a well-formed scale, being larger than 3/5, the upper bound for well-formed lines of fifths.)

[48] After finding this scale, I discovered that Tymoczko (2004, 285) writes that "the computer scientist David Rappaport (2006) has observed that the multiset C-C-D♯-E-F♯-G♯-A also possesses diatonic

as a diatonic scale that has had *both* the alteration that would transform it into the harmonic minor scale [♯5] *and* the alteration that would transform it into the harmonic major scale [b6].)[49]

Note that either of the definitions of diatonic thirds discussed above require only that all thirds be either major *or* minor, not that both major *and* minor thirds be present in the scale. In fact, all the transpositionally symmetrical scales above have exclusively minor thirds (in the case of the octatonic) or exclusively major thirds (in the case of the hexatonic and whole-tone). In such a scale, scalar triads will be either uniformly diminished or uniformly augmented. A stronger constraint, which I call *strong diatonic thirds*, is that all thirds in the scale be either major or minor, *and* both major and minor thirds are found. Among twelve-tone equal-tempered pitch class sets, only the four transpositionally asymmetrical 7-chords fulfill this criterion, regardless of how we define major and minor thirds.[50] (In other temperaments, we can add the "harmonic major ♯5 scale to this list as well.) These four scales form an important group that we shall return to many times. I therefore give them a name—the "quasi-diatonic scales." Below, we shall discuss a few of their other properties, but for now, I would merely like to note that strong diatonic thirds suggests a reason why these scales have been so useful to composers of tonal music: tertian 3-chords within the scale will always be triads, and they will

thirds." If spelled beginning with B♯ (rather than C), the resulting scale is a transposition of the scale I discuss.

[49] In 31-tone equal temperament the scale can be heard here: http://sailor.music.mcgill.ca/generatescale/18_31/octaves/19/2 .

[50] I believe that Tymoczko does not discuss "strong diatonic thirds" or any analogy to it. He does, however, discuss something similar in the case of what we might call *pentatonic thirds*, where all thirds are 4 or 5 semitones (2004, 229; he calls this property "DT+"). Pentatonic thirds is possessed by only three collections: the hexatonic, whole-tone, and pentatonic scales, but only the latter possesses *strong pentatonic thirds* (my term) since, as we have seen above, the hexatonic and whole-tone scales contain exclusively major thirds. Tymoczko writes that, because of their single size of third, the hexatonic and whole-tone collections satisfy pentatonic thirds "in a less interesting way." I agree (although I might propose replacing "interesting" with "useful for the composition of music that uses consonant triads"), but I do not see why Tymoczko did not extend similar reasoning to the case of diatonic thirds.

sometimes be consonant triads. (If strong diatonic thirds is satisfied, and thirds circulate,[51] there will be at least one major and one minor triad in the scale.[52]) Since the consonant triad is the central harmonic object of tonal music, and division into two scale steps seems to be a more-or-less "natural" division of the acoustic third,[53] strong diatonic thirds is arguably the ideal condition for a scale in tonal music. (In the appendix to this chapter in Section 2.6, I discuss some possible extensions and/or alternatives to diatonic thirds.)

DIATONIC SECONDS: A scale with *diatonic seconds* is a scale in which all seconds are either major or minor. Much like with diatonic thirds, diatonic seconds can be applied to 7-scales in many temperaments if we define major and minor seconds in terms of their spans upon the line of fifths (of 2 and -5, respectively). On the other hand, diatonic seconds is applicable to a wider range of cardinalities if we define it, as Tymoczko [2004, 223] does, in terms of intervals of one or two 12-tone equal tempered semitones. In the following paragraph, I will adopt this latter formulation and restrict my purview to 12-tone equal temperament.

Many more T_n-classes have diatonic seconds than have diatonic thirds—31, rather than just 7, over four times as many. If we define a *strong diatonic seconds* property analogous to strong diatonic thirds (i.e., the scale has only *and* both major and minor seconds), 29 T_n-classes possess it, over seven times as many as the merely 4 that possess

[51] By "circulate," I mean that, if we begin on any pitch-class of the scale and ascend by successive thirds (or descend by successive thirds), we will return to the pitch-class we started upon after *circulating* through every third of the scale at least once. Thirds in the diatonic scale circulate, as they music in any 7-scale because 2 (the integer associated with a scalar third) and 7 are relatively prime. Thirds do not circulate in the octatonic scale or any other 8-scale, because 8 is divisible by 2.

[52] To show that this is so, reason as follows: if there are major thirds in the scale, there is at least one run of at least one consecutive major thirds in the scale. Since there are minor thirds in the scale as well, then this run of major thirds must be bordered by minor thirds both above and below. Therefore, at least one major third will be found beneath a minor third, forming at least one major triad, and at least one major third will be found above a minor third, forming at least one minor triad.

[53] In favour of this somewhat broad statement, I can appeal to the fact that, as seen above, human music tends to use scales of about seven pitches per octave; if these pitches are distributed anywhere near evenly, then acoustic thirds will tend to correspond to two scale steps. Nevertheless, since five is also a common scale cardinality, it could be argued that acoustic minor thirds are also likely to correspond to a single step (as in what we might call "pentatonic seconds," where each scale step is either two or three equal-tempered semitones).

strong diatonic thirds. (The only T_n-classes with "weak diatonic seconds" [i.e., with step sizes of only a minor second or only a major second] are, evidently, the chromatic and whole-tone scales, respectively.) It is therefore evident that the property of diatonic seconds alone, whether in its weak or strong form, is insufficient to generate a musically relevant set of scales in the way that diatonic thirds—especially strong diatonic thirds—does. We might, therefore, wonder whether diatonic seconds is really an important constraint on scale formation, especially given the important role of the harmonic minor scale in musical pedagogy and of prominent augmented seconds in many musical styles (e.g., Klezmer and other Eastern European music, Indian classical music, etc.). It is true that a diatonic seconds constraint, or something like it, is relevant to musical practice within some idioms, since—as anyone who has taken an introductory music theory class knows—common-practice music avoids melodic augmented seconds. But even here, this prohibition applies only to *melodic* augmented 2nds. If harmonic augmented 2nds were not permitted, many common voicings of the diminished seventh chord, including all closed spacings except for root position, would be prohibited, as well as a great many common melodic formulas, such as $\hat{5}$–$\hat{6}$–$\hat{5}$ upon any minor-key dominant.

NO CONSECUTIVE MINOR SECONDS: A third scale property discussed by Tymoczko (indeed, it was the title of an earlier article [Tymoczko 1997]) is the *no consecutive minor seconds* constraint. [54] Since we are speaking about scales, what Tymoczko calls "semitones" are, in the terminology of this thesis, minor seconds, so I will instead speak of the "no consecutive minor seconds" constraint. However, because it is impossible for a scale with diatonic thirds to have consecutive minor seconds (since two consecutive minor seconds produces a diminished third), we might ask whether diatonic thirds renders the prohibition of consecutive minor seconds unnecessary to an account of extended-tonal scale formation. To answer this, we must consider those scales that obey

[54] In fact, Tymoczko calls this the "no consecutive semitones" constraint, but since we are speaking about scales, in the terminology of this thesis this is really the "no consecutive minor seconds" constraint.

the no consecutive semitones constraint, yet do not have diatonic thirds. (Again, the following discussion assumes 12-tone equal temperament.) In fact, because there are 124 T_n-classes (of any cardinality) that do not have consecutive minor seconds, but only seven that have diatonic thirds, the vast majority of collections without consecutive minor seconds do not have the diatonic thirds property. Most of these are not conventionally scale-like. (It is not difficult to think of examples, such as {0, 1, 3, 4} or {0, 4, 8}.) To get around this profusion of no-consecutive-minor-second scale-types, Tymoczko appeals to *maximal* no-consecutive-minor-second scales: those scales to which it would be impossible to add another pitch-class without creating consecutive minor seconds. He then proves that these are precisely equivalent to the scales with diatonic thirds [285]. This is a remarkable result, but—whereas there are evident musical reasons why one might prefer a scale with only major and minor thirds, or only major and minor seconds—it is not obvious why a musician might prefer to make music with a "maximal" scale. For this reason, it seems to me that it is more musically relevant to emphasize diatonic thirds, and from there show the startling result that it is impossible to add a pitch-class to any scale with diatonic thirds without producing consecutive semitones.[55]

Moreover, I am not convinced by the reason that Tymoczko (2004, 224–5) gives why composers would have avoided scales with consecutive semitones—namely, the dissonance of a simultaneously sounded {0, 1, 2}, which prevents composers from treating scales with consecutive semitones "pandiatonically."[56] In fact, simultaneously

[55] Diatonic thirds can be conceptualized as two constraints: *no diminished thirds* (i.e., no consecutive minor seconds) and *no augmented thirds*. The intersection of the scales without diminished thirds and those without augmented thirds is the scales with diatonic thirds. Since scales with diatonic thirds are the maximal scales without consecutive minor seconds, one might wonder if they are, in a similar fashion, also the minimal scales without augmented thirds. They are not, however: although all scales with diatonic thirds are minimal no-augmented-thirds scales, not all minimal no-augmented-thirds scales have diatonic thirds. Minimal no-augmented-thirds scales with diminished thirds include T_n-classes 7-30A {0,1,2,4,6,8,9} and 7-30B {0,1,3,5,7,8,9}, 7-22 {0,1,2,5,6,8,9} and 8-9 {0,1,2,3,6,7,8,9}.

[56] In the more limited context that this was constraint was originally proposed by Jeff Pressing (1977, 30)—in the "scale forms" of jazz voicings—it is more justifiable, since it is true that jazz musicians 1) often associate a scale with each voicing and 2) avoid consecutive semitones in both scales and voicings.

sounded {0, 1, 2}'s regularly occur in the repertoire Tymoczko is treating, for instance the pitches F, G-flat, and E-natural throughout Figure 28 (which is as long or longer than many of the passages Tymoczko and other analysts point to as belonging to/defining one or another mode or scale) and, indeed, they occur even in various common-practice contexts, such as when an augmented sixth takes place over a dominant pedal, as in Figure 29. While "pandiatonic" styles of post-tonal music obviously exist, I do not believe that pandiatonicism has historically been an important constraint on scale formation as Tymoczko suggests. Even the diatonic scale contains many potential simultaneities that common-practice composers would certainly have avoided (e.g., E-F-B-C). Indeed, since most music, even after the emergence of pandiatonicism as a compositional device, does not show signs of having been composed pandiatonically, it seems strange to suppose that even those composers who sometimes wrote pandiatonically would have constrained their compositional practice in their non-pandiatonic passages according to which scales might have functioned in a pandiatonic context. I believe, instead, that the reason for avoiding scales with consecutive semitones most likely has less to do with difficulties with {0, 1, 2} and more to do with "{0, 2} of {0, 1, 2}"—that is, more to do with the difficulty of absorbing a diminished third into a triadic style.

Figure 28. Consecutive minor seconds E, F, and G-flat in Debussy, *Préludes*, Livre 1, ix, "La sérénade interrompue," mm. 32–42.

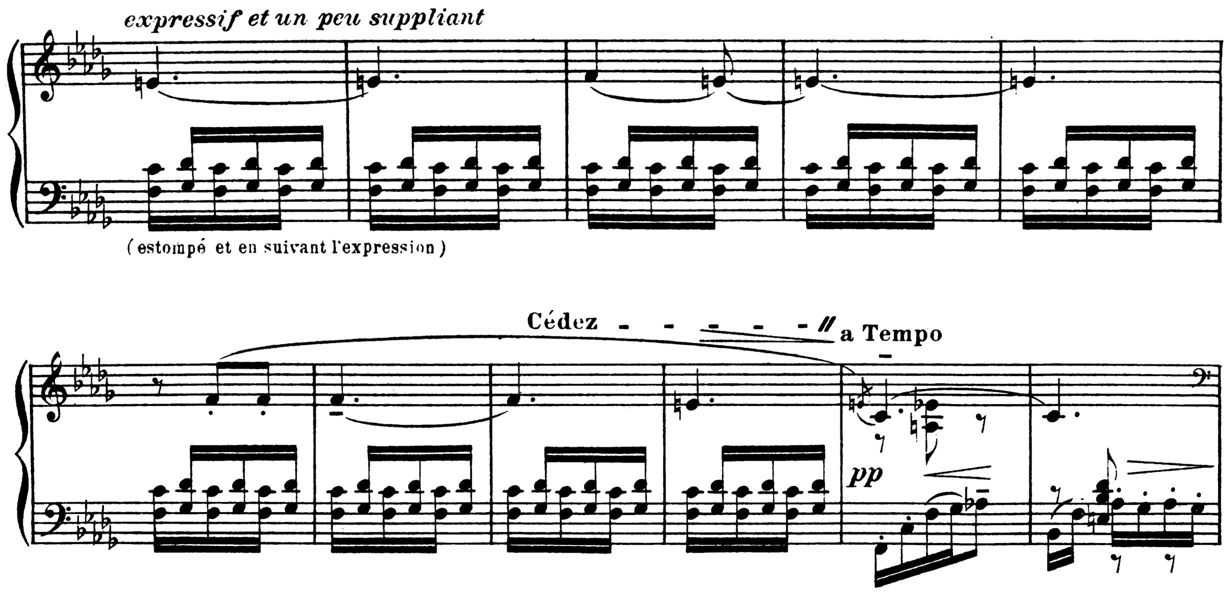


Figure 29. Consecutive minor seconds A-natural, B-flat, and C-flat in Joseph Haydn, String Quartet in C minor, op. 17 no. 4, i, mm. 32–37.

2.3 Fuzzy set membership and near well-formedness as a measure of quasi-diatonicity

The scale properties we have considered so far are categorical, all-or-nothing. A scale is either well-formed, or it is not. A scale either has diatonic seconds, or it does not.

But consider a melodic-minor scale—since it is not generated by consecutive fifths, it is not well-formed.[57] And neither is a chromatic 7-chord, such as {0, 1, 2, 3, 4, 5, 6}. But surely one of these two sets is, in some sense, *more* well-formed than the other? Or consider a harmonic minor scale—since it contains an augmented second, it does not have diatonic seconds. But again, surely it more *nearly* has diatonic seconds than a chromatic 7-chord?

We can answer both of these questions affirmatively. For instance, in the case of well-formedness, we can appeal to the span of a scale upon the line of fifths. Since a well-formed 7-scale has a span of 6,[58] any scale of greater span can be seen as correspondingly more *poorly-formed*. A melodic-minor scale has a span of 8. It is correspondingly much closer to being well-formed than a chromatic 7-chord, which has a span of 30, and is therefore much more poorly-formed.

In the case of diatonic seconds, consider the fact that all and only those 7-scales with diatonic seconds contain exactly two minor seconds. Any 7-scale that contains more than two minor seconds will necessarily contain augmented seconds. For example, if a scale contains three minor seconds (such as the harmonic minor scale), it will necessarily contain an augmented second; if a scale contains four minor seconds (such as the Hungarian minor scale), it will necessarily contain either two augmented seconds or one doubly-augmented second. It is therefore possible to measure a scale's deviation from diatonic seconds by simply summing the number of minor seconds it contains in excess of two. (This assumes that the minor second is the smallest second.)[59]

[57] Although it is *nearly* well-formed, in the sense developed in Chapter 1.

[58] This reasoning assumes that the choice of generating interval is such that the 7-scale of span 6 *is* well-formed. Certain choices of generating interval, such as 600 cents or 720 cents, will not generate a well-formed 7-scale.

[59] If, on the other hand, the tuning system permits diminished seconds, then this calculation is made more complicated. We might attempt to proceed by noting that any 7-scale that contains more than five major seconds will necessarily contain diminished seconds, and therefore adding the number of major seconds the scale contains in excess of five to the number of minor seconds in excess of two. (At least one of these two numbers will always be zero.) However, if diminished seconds are allowed, then there exist scales that contain no minor *or* major seconds. It would therefore be necessary to rethink the diatonic seconds metric more extensively.

Proceeding in a similar fashion, it is possible to define similar metrics for other diatonic scale qualities, such as diatonic thirds, no consecutive semitones, maximal evenness (Clough and Douthett 1991), and Myhill's property (Clough and Myerson 1985). In Sailor (2016), I did this for the case of 7-scales in 12-tone equal temperament. Each metric can then be seen as defining fuzzy membership values for the set of scales with the given property.[60] Figure 30, taken from my earlier paper, lists the results sorted by near well-formedness. (In this table, unlike the rest of this thesis, inversionally equivalent scales are not listed separately. The intended property being measured should be apparent from the column headings; details of how the metrics were constructed are found in the earlier paper. Most numerical values have been normalized so that the value for the diatonic scale is 1, indicating crisp set membership, and the most distant scale among 12-tone equal tempered scales has value 0.)[61] It can be seen that the various properties largely correlate with one another. In the earlier paper, I calculate the correlation of all these properties. Near well-formedness is the scale property that correlates most highly with the others (r = .80),[62] although the margin is slim (evenness and scalar diatonic triads correlate nearly as highly). Near well-formedness can therefore be taken as an especially good measure of "quasi-diatonicity." (Other authors have sorted

[60] An excellent "tutorial" on fuzzy sets, written especially for music theorists, can be found in Quinn (1997, 240–248).

[61] Most of these normalizations are valid wherever the minor second is the smallest second, namely within any temperament where the generating fifth is smaller than 720 cents and greater than or equal to 700 cents. In any such system, the chromatic 7-chord is the most poorly-formed possible scale (i.e., the scale of greatest span). Normalization in a more general case would be more complicated. As we saw in Chapter 1, for instance, for other choices of fifth (such as the mean-tone fifth or the 31-tone equal tempered fifth) there are many more scales possible, all with greater span, with a limit at infinity for a fifth of 6 6/7 semitones. It is not possible, therefore, to normalize well-formedness such that 0 is assigned to the maximally poorly-formed scale in all possible temperaments, since there is no such scale. Perhaps the best choice for a generalizable normalization would be to apply an inverse function like $1/((x-6)+1)$, where x is the span. This function as a value of 1 where x is 6 (the minimum value), and has a limit of 0 as x goes to infinity.

[62] This rises to r = .88 if Myhill's property is removed. Because Myhill's property is also possessed by the chromatic 7-chord {0, 1, 2, 3, 4, 5, 6}, it correlates poorly with the other scale properties.

the scales of 12-tone equal temperament in a similar manner, including Audétat and Junod [2017] and Hook [2011].)

Figure 30. Fuzzy membership values for 7-scales, sorted by near well-formedness (from Sailor [2016]).

ForteNum	Prime form	Mnemonic name	Well-formedness	No consecutive semitones	Diatonic seconds	Diatonic thirds	Diatonic intersection	Scalar consonant triads	Scalar diatonic triads	Myhill's property	Evenness
7-35	{0, 1, 3, 5, 6, 8, 10}	Diatonic	1	1	1	1	7	6	7	1	1
7-34	{0, 1, 3, 4, 6, 8, 10}	Melodic minor	0.92	1	1	1	6	4	6	0.88	0.89
7-32	{0, 1, 3, 4, 6, 8, 9}	Harmonic minor/major	0.88	1	0.75	1	6	4	6	0.75	0.87
7-22	{0, 1, 2, 5, 6, 8, 9}	Hungarian minor	0.83	0.8	0.5	0.91	5	4	4	0.62	0.74
7-30	{0, 1, 2, 4, 6, 8, 9}	Major #2 (aug 6th)	0.83	0.8	0.75	0.91	6	4	4	0.62	0.87
7-33	{0, 1, 2, 4, 6, 8, 10}	WT plus one	0.83	0.8	1	0.91	5	2	3	0.75	0.78
7-20	{0, 1, 2, 5, 6, 7, 9}	Phrygian hungarian mino	0.79	0.6	0.5	0.73	5	2	2	0.5	0.74
7-28	{0, 1, 3, 5, 6, 7, 9}		0.79	0.8	0.75	0.82	5	1	3	0.5	0.76
7-29	{0, 1, 2, 4, 6, 7, 9}	Major #6	0.79	0.8	0.75	0.82	6	3	4	0.5	0.87
7-31	{0, 1, 3, 4, 6, 7, 9}	Octatonic minus one	0.79	1	0.75	0.91	5	2	5	0.62	0.76
7-21	{0, 1, 2, 4, 5, 8, 9}	Hexatonic plus one	0.71	0.8	0.5	0.91	5	3	3	0.5	0.74
7-25	{0, 2, 3, 4, 6, 7, 9}		0.71	0.8	0.75	0.73	4	0	2	0.5	0.65
7-26	{0, 1, 3, 4, 5, 7, 9}		0.71	0.8	0.75	0.82	4	1	2	0.38	0.65
7-27	{0, 1, 2, 4, 5, 7, 9}		0.71	0.8	0.75	0.82	5	2	3	0.38	0.76
7-19	{0, 1, 2, 3, 6, 7, 9}		0.62	0.6	0.5	0.73	5	2	2	0.5	0.74
7-23	{0, 2, 3, 4, 5, 7, 9}		0.62	0.6	0.75	0.64	4	0	0	0.25	0.54
7-24	{0, 1, 2, 3, 5, 7, 9}		0.62	0.6	0.75	0.73	4	1	1	0.25	0.65
7-Z18	{0, 1, 4, 5, 6, 7, 9}		0.62	0.6	0.5	0.73	4	1	1	0.25	0.63
7-14	{0, 1, 2, 3, 5, 7, 8}		0.58	0.6	0.5	0.64	5	0	1	0.25	0.62
7-15	{0, 1, 2, 4, 6, 7, 8}		0.58	0.6	0.5	0.64	5	0	1	0.5	0.72
7-7	{0, 1, 2, 3, 6, 7, 8}		0.58	0.4	0.25	0.55	4	0	0	0.62	0.6
7-Z37	{0, 1, 3, 4, 5, 7, 8}		0.58	0.8	0.5	0.73	3	0	1	0.62	0.51
7-Z38	{0, 1, 2, 4, 5, 7, 8}		0.58	0.8	0.5	0.73	4	0	2	0.5	0.62
7-11	{0, 1, 3, 4, 5, 6, 8}		0.5	0.6	0.5	0.55	3	0	0	0.5	0.4
7-13	{0, 1, 2, 4, 5, 6, 8}		0.5	0.6	0.5	0.55	3	0	1	0.5	0.51
7-16	{0, 1, 2, 3, 5, 6, 9}		0.5	0.6	0.5	0.64	3	1	1	0.25	0.42
7-Z17	{0, 1, 2, 4, 5, 6, 9}		0.5	0.6	0.5	0.64	4	2	2	0.5	0.63
7-Z36	{0, 1, 2, 3, 5, 6, 8}		0.5	0.6	0.5	0.55	4	0	1	0.25	0.51
7-10	{0, 1, 2, 3, 4, 6, 9}		0.42	0.4	0.5	0.45	3	0	0	0	0.52
7-6	{0, 1, 2, 3, 4, 7, 8}		0.42	0.4	0.25	0.55	4	0	0	0.5	0.49
7-8	{0, 2, 3, 4, 5, 6, 8}		0.42	0.4	0.5	0.36	3	0	0	0.5	0.29
7-9	{0, 1, 2, 3, 4, 6, 8}		0.42	0.4	0.5	0.45	4	0	0	0	0.4
7-Z12	{0, 1, 2, 3, 4, 7, 9}		0.42	0.4	0.5	0.55	4	0	0	0.5	0.52
7-4	{0, 1, 2, 3, 4, 6, 7}		0.29	0.4	0.25	0.36	3	0	0	0.5	0.25
7-5	{0, 1, 2, 3, 5, 6, 7}		0.29	0.4	0.25	0.36	3	0	0	0.62	0.36
7-2	{0, 1, 2, 3, 4, 5, 7}		0.21	0.2	0.25	0.18	3	0	0	0.38	0.15
7-3	{0, 1, 2, 3, 4, 5, 8}		0.21	0.2	0.25	0.27	3	0	0	0.38	0.28
7-1	{0, 1, 2, 3, 4, 5, 6}		0	0	0	0	2	0	0	1	0

2.4 Scale properties in actual music

In this thesis I have spoken of properties like "diatonic seconds" as applicable to abstract musical structures like scales, as theorists typically do. In this brief section, I would like to make the observation that we can think of many such properties as inherent not only in abstract scales, but also in specific passages of music. That is to say, if all the seconds in a passage of music are major or minor, then we could say that the *passage* possesses diatonic seconds, regardless of whether the background collection formed by the notes of the passage does. It is possible, for example, to write music in the harmonic minor scale that uses only the major and minor seconds of the scale and avoids the augmented second, and it would not be difficult to find common-practice melodies that fulfill this condition. In fact, although we don't call it "diatonic seconds," this sort of constraint (e.g., "avoid augmented seconds") has in fact been a staple of practical musical theory for hundreds of years—consider Sechter's version of the harmonic minor scale (Figure 31). This sort of melodic constraint can also be relevant when writing non- or quasi-diatonic music. When I wrote the melody in Figure 32, I was consciously employing the h:5, Phrygian #3 mode (the mode itself is shown in Figure 33), but I was unconsciously avoiding the augmented second between $\hat{2}$ and $\hat{3}$ of the mode. In fact, I realized afterwards that I had the melody I had written featured every adjacent scale step of the mode, except for the augmented second—much like Sechter's harmonic minor scale. (Of course, it is also possible to use this type of scale in a manner that *emphasizes* the augmented second, as in a great deal of Klezmer music, for example.)

Figure 31. Harmonic minor according to Simon Sechter (from Harrison [1994, 29]).

EXAMPLE 1.3 Harmonic minor according to Sechter

Figure 32. The opening melody in F♯ Phrygian ♯3 from a piece of mine for violin and piano (Fall, 2013).[63]

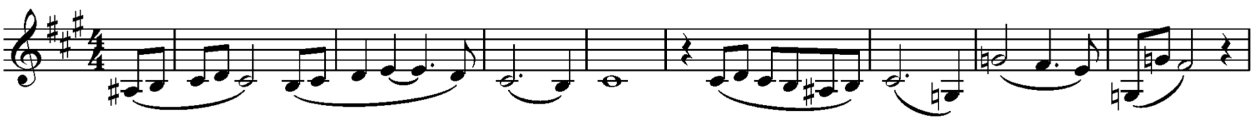
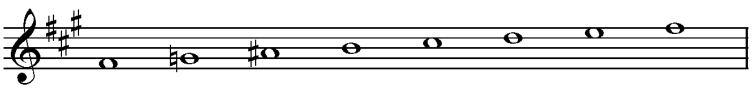


Figure 33. F-sharp Phrygian ♯3.



This observation suggests another way that we might think of some of the fuzzy metrics introduced above: as approximate measures of how constrained a composer is within a given scale, if she wishes to write music that exhibits a given feature. In the diatonic scale, because it possesses the "crisp" diatonic seconds property, one can *only* write music with diatonic seconds. On the other hand, a passage in the harmonic minor scale, if composed judiciously, may have diatonic seconds, but it may not as well—especially if composed injudiciously. As scales become less and less diatonic—however we choose to measure quasi-diatonicity—then it becomes more and more difficult to write music that exhibits "diatonicisms" such as diatonic seconds; the possibilities for doing so become more and more constrained. In the limiting case, the only possibility for realizing music with diatonic seconds is oscillating between two pitches separated by a whole- or semitone.[64]

The "unconstrained diatonicism" of the diatonic scale becomes particularly important wherever a composer wishes to employ techniques such as sequence or

[63] A recording is available at http://malcolmsailor.com/blog/for-violin-and-piano/

[64] However, this limiting case is not possible for a 7-scale in 12-tone equal temperament. Since minor seconds are permitted in diatonic seconds, and a minor second is the smallest scale possible scale step in this temperament, the only way to reduce the number of diatonic seconds is by increasing the number of seconds larger than a major second. But the greatest number of such larger seconds a 12-tone equal tempered 7-scale can contain is 2 (e.g., {Cb, D♯, E♯♯, F♯♯, G♯, A, B♭}, leaving five steps of either a minor second or major second. In 12-tone equal temperament, the greatest cardinality of pitch-class set that can have only a single adjacent major or minor second is four. Many such sets are familiar as seventh chords.

imitation that require the transposition of material more-or-less freely throughout the scale. Suppose that parallel-universe Bach insisted upon sticking exclusively to the pitches of a harmonic major scale for his first two part invention. As the motive of this piece is four adjacent scale degrees long, four out of seven possible scalar transpositions of the motive would be available to him, which might not seem so bad—but, if he wished to avoid augmented seconds, it would entirely prevent him from realizing the imitation or sequence around which the piece is constructed (consider Figure 34). In the diatonic scale, on the other hand, seven out of seven scalar transpositions are always possible. The point is not that it is impossible to write entirely convincing imitation or sequence in such a scale (clearly, parallel universe Bach, if he could hold a candle to our own Bach, would ingeniously adjust his music to its materials), but that doing so is highly constrained, whereas the diatonic scale works for such purposes "out of the box." For this reason, we might expect a style that favours imitation and sequence to flow "downhill" towards diatonic (or at least, near-diatonic) scalar materials.

Figure 34. Hypothetical recomposition in the harmonic major mode of J.S. Bach's Invention no. 1 in C major, BWV 772.

2.5 Web apps 2.1 and 2.2

The following two web apps are relevant to this discussion.

[Web app 2.1, "Scale Patterns"](http://sailor.music.mcgill.ca/scale_patterns) allows the user to construct a scale and then listen to the aural effect of various patterns in that scale. [65] The user can select any cardinality

[65] http://sailor.music.mcgill.ca/scale_patterns

of equal temperament between 1 and 100, and then select a scale within that temperament. She can then choose an intervallic pattern which will be repeated throughout the scale according to the chosen pattern of intervals of transposition. The functioning of this app is similar to that of web app 1.1 (see section 1.5), but rather than generating scales, the user can simply select any pattern of equal-tempered pitches. This means that the resulting scales are less relevant to the theoretical framework developed in this thesis. On the other hand, the app is somewhat more flexible—in particular, the user can select scales of any cardinality from 1, to the full chromatic scale.

Web app 2.2, "Change Mode of Bach Keyboard Work," allows the user to replace the pitches of a Bach keyboard work with those of a mode of her choice.[66] The user selects 7 twelve-tone equal tempered pitch-classes, and the program finds the best spelling for this mode and then replaces the pitches of the original piece with those of the same letter-name from the new mode.[67] The example given in Figure 34 can be heard here.[68] Since I would hardly argue that the results hardly resemble the music that a hypothetical equal to Bach would have composed with the given tonal materials, the app is probably more amusing than instructive. Nevertheless, the results can be interesting.[69]

2.6 Appendix: Alternatives to "diatonic thirds"

Recall the definition of "diatonic thirds" introduced above: every scalar third is either minor or major. In this definition, a generic interval, the third, is associated with two specific intervals, the major and minor thirds. When the generating interval is reasonably close to an acoustic fifth, these specific third qualities approximate acoustically just intervals, and are the component intervals of an important musical

[66] http://sailor.music.mcgill.ca/bachchangemode

[67] In fact, this description glosses over some technical details. Sometimes it is necessary to remap the letter-names, for instance if the algorithm determines that a piece whose tonic was originally C is best spelled in the new mode with a tonic of B-sharp.

[68] http://sailor.music.mcgill.ca/bachchangemode/0_2_4_5_7_8_11/inven01/0

[69] The functioning of the algorithm is somewhat crude, in that it is insensitive to the fact that Bach's music modulates.

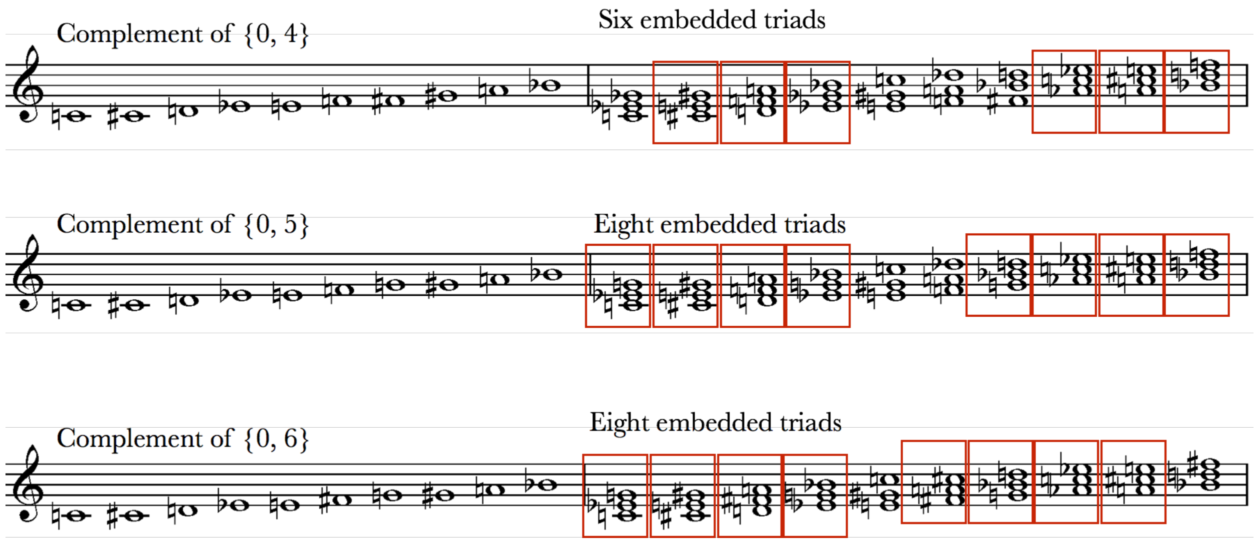
object, the consonant triad. It would be reasonable to suppose, therefore, that if diatonic thirds is a desirable or useful musical object, it has relatively more to do with these approximately just intervals, and relatively less to do with anything important about the scalar interval of the third. Following this line of thought, we might then inquire whether there are scales in which other generic scalar intervals—seconds, fourths, etc.—form near approximations to acoustic major and minor thirds, similar to the manner that scalar thirds do in the case of the quasi-diatonic scales. In this appendix, I briefly consider this question, restricting my investigations to 12-tone equal temperament, where the just minor third is approximated by the interval of 3 semitones and the just major third by the interval of 4.

In 12-tone equal temperament, there are two generic intervals that have an associated set of scales where the generic interval in question comes in species of only and both 3 and 4 semitones.[70] The first of these generic intervals is the third, and the set of associated scales is the quasi-diatonic scales—this is the diatonic thirds property that we have been discussing. The second of these intervals is the fourth, which is associated with a set of scales I have not seen discussed elsewhere. These scales are three 10-scales—the complements of {0, 4}, {0, 5}, and {0, 6}—as well as the unique T_n-class of 11-chord. I have notated the three 10-scales in Figure 35, as well as their embedded consonant triads. (I have notated them with our usual 7-scale-centric notational system, and not made any attempt to show that the triads are constructed out of scalar fourths.) It is doubtful whether these collections could really be of much use as scales in quasi-tonal music, due to the related facts that they 1) contain such a profusion of minor seconds and 2) greatly exceed scale cardinalities typical of scalar music seen above. Nevertheless, they could be of use as a compositional device in triadic music, especially in the case of the

[70] The results in this section were determined by a Python script which simply examined all 12-tone equal-tempered T_n-classes.

complements of {0, 5} and {0, 6}, where—in either case—eight out of the ten quartal 3-chords are consonant triads. (These consonant triads have been boxed in Figure 35.)

Figure 35. 10-scales with scalar fourths of only and both 3 or 4 semitones.



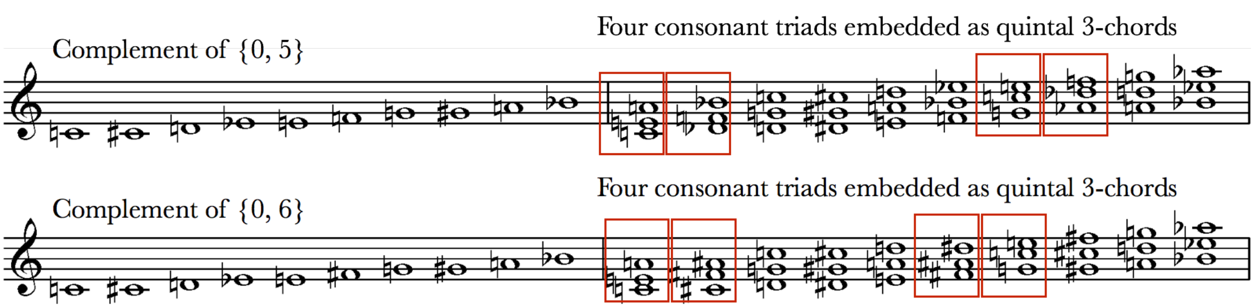
It is not only the intervals of 3 and 4 semitones that can combine to form consonant triads. The intervals of 4 and 5 semitones can also form consonant triads, with the proviso that the triads thereby formed will not be in root position—or not in what we ordinarily think of as root position, at least. (4 semitones below 5 semitones forms [what we would ordinarily consider] a first-inversion minor triad, while 5 semitones below 4 semitones forms a second-inversion major triad.) Observing this, we might wish to consider not only the case where a given generic interval comes in species of only and both 3 and 4 semitones, but also the case where the species are only and both 4 and 5 semitones. 3-chords formed out of the relevant generic interval in the latter sort of scale will always form consonant triads, augmented triads, or "quartal" chords.[71]

[71] By "quartal" chords, I mean the T_n-class {0, 2, 7}. It is conventional to call this T_n-class "quartal" because it is formed out of two perfect fourths within the diatonic scale. But it should be noted that this usage is contrary to the otherwise strictly scalar use of intervallic labels in this thesis, according to which "quartal" {0, 2, 7} chords may or may not be "truly" quartal—that is, formed out of scalar fourths. Hence the use of quotation marks around "quartal."

The only scale with thirds of only and both 4 and 5 semitones is the pentatonic scale. (In jazz harmony, this property is taken advantage of in so-called "So What voicings" and related chord-voicings built out of the pentatonic scale.) There also exist:

- six 8-scales with fourths of 4 and 5 semitones; the best known of these is the octatonic scale.[72]
- two 10-scales and one 11-scale with fifths of 4 and 5 semitones. The 10-scales, together with their quintal 3-chords, are shown in Figure 36; these scales are the complements of {0, 5} and {0, 6}, already seen above. The two scales therefore have the remarkable property that they have consonant triads among both their quartal 3-chords and their quintal 3-chords. This "ambiguity"—a consonant triad could be mistaken as either a quartal or quintal slice of the scale—is evocative of novel compositional possibilities.

Figure 36. 10-scales with scalar fifths of only and both 4 and 5 semitones.



With a single exception, the relevant generic 3-chord types from all of the above scales include not only consonant triads, but also dissonant triads (diminished, augmented, and/or "quartal" chords). The exception is quartal 3-chords in the octatonic scale: as we transpose a quartal 3-chord up or down the octatonic scale, we will *only* find consonant triads. (In this sense, the octatonic scale is more consonant than the diatonic scale!) The octatonic scale is not the only T_n-class through which scalar transposition of a

[72] The others are {0, 1, 2, 4, 5, 7, 8, 9}, Forte number 8-20; {0, 1, 2, 4, 5, 7, 8, 10}, Forte number 8-27A; {0, 1, 3, 4, 5, 7, 8, 10}, Forte number 8-26; {0, 1, 2, 4, 6, 7, 8, 10}, Forte number 8-25; {0, 1, 3, 4, 6, 7, 8, 10}, Forte number 8-27B.

given regular 3-chord will produce *only* consonant triads. Trivially, this is also true of both consonant triads themselves, as well as of eleven of the twelve ways of combining two consonant triads with no common tones,[73] and all six ways of combining three consonant triads with no common tones. However, in every case except the octatonic, the relevant generic interval has three, rather than two, sizes (3, 4, and 5) and the "scale" moves progressively through each inversion of the triads that make it up (consider the 6-scale in Figure 26). The octatonic scale is the unique 12-tone equal-tempered T_n-class where scalar transposition of a given regular 3-chord produces only consonant triads, and every one of these triads is distinct from the others in its pitch-class content.

[73] The exception is two "hexatonic poles" (Cohn 1996, 19), such as C major and A-flat minor.

Chapter 3 Modes and Tendency Tones

This chapter looks at the choice of mode and scale as it affects one important question: namely, the placement of minor seconds. We restrict our purview to those 7-scales in which the minor second is the smallest scale step. These scales are all and only those available in 12-tone equal temperament, but the use of these scales need not be taken as necessarily implying this temperament. As seen in Chapter 1, these are the nearly well-formed scales available on a generated line of fifths where the minor second is the smallest second, and they are available in a variety of other temperaments as well.[74]

The argument in this chapter proceeds from a simple axiom: minor second scale steps have a greater "desire" to resolve than do larger second sizes. In stating this axiom, I certainly don't mean to wade into any fraught debates about, say, whether such a desire for resolution somehow inheres in facts about human audition. I will instead merely observe that something like this axiom is implicit in the practice of a great deal of Western music. This is most conspicuous when we raise the seventh scale degree of minor-mode music, so that the leading-tone will yearn towards its tonic resolution, a minor second away. Larger step sizes (e.g., the major second) are understood to provoke less of a need for resolution. (On the other hand, very much *smaller* step sizes, such as the diminished second in mean-tone temperament [about 41 cents], may be *too* close to be heard as provoking this same need for resolution. This is rather speculative, however, since to my knowledge very little music has been written that both makes much use of such small step sizes and also has something resembling a tonal syntax of resolution.) In any case, the truth or falsity of any greater need for resolution possessed by minor seconds as a statement about *all* music may be highly debatable, but there is nevertheless a broad

[74] To be precise, these scales are available on the line of fifths where the generating fifth is greater than 6 6/7 semitones and smaller than 7.2 semitones.

enough swath of music for which the statement holds true that I propose that it will be productive to simply assume the axiom and see where it leads us.[75]

Based on this axiom, we can give the following definition of a "tendency tone": a tendency tone is a scale degree adjacent by a minor second to a member of the tonic triad. All else being equal, it therefore "tends" towards the resolution into tonic harmony with greater intensity than other scale degrees, which are adjacent by larger seconds. This definition captures the most archetypal examples of tendency tones, namely, $\hat{7}$ of the major mode (the leading tone), $\hat{4}$ of the major mode (most often as seventh of V7), and perhaps the "dual" leading-tone $\hat{6}$ of the minor mode. It does not, however, capture some other scale degrees that may often be described as "tendency tones." Note in particular that, in minor, $\hat{4}$ resolves by descending *major* second (as in the seventh of V7 in a minor key). On the other hand, this definition does include scale degrees—such as $\hat{2}$ in minor, resolving upwards to $\hat{3}$—that are not usually considered tendency tones. To explain such exceptions, we might appeal to the strength of the analogy to common progressions (in the case of $\hat{4}$ in minor), or to the fact that competing melodic considerations may override the tendency of the minor second (in the case of $\hat{2}$ in minor, which falls due to the tendency of cadences to resolve downwards to the tonic). It is beyond question that there are other factors informing a listener's perception of "tendencies" in tonal music beyond the simple arrangement of minor seconds in the scale. Moreover, it is undoubtedly the case that the melodic tendencies perceived in a given style of music by a habituated listener are largely the product of the statistically inferred regularities of melodic motion in that particular style, rather than of the intrinsic structure of the scale employed. Nevertheless, I believe that the minor second resolutions available in a given mode are

[75] To briefly wade into the muddy waters of this type of debate: it seems highly unlikely that the "yearning for resolution quotient" of various melodic intervals is somehow fixed. But at the same time it seems highly likely that this quotient is nevertheless constrained. For instance, it seems unlikely to be arbitrary happenstance that it is the minor second, rather than a much larger interval such as a minor or major third, that is heard as yearning for resolution in common-practice tonal music.

worthy of investigation, that it is useful to have a label for these resolutions, and that "tendency tone" is as useful a label as any.

In this chapter, I begin by taking a combinatoric approach to tendency tones, considering the scales that may be constructed from combinations of from one to four tendency tones (the minimum and maximum numbers within a 7-scale, respectively). I then explore the structure of the placement of minor seconds and tendency tones in the diatonic modes. I use *modal pitch-class sums* to sort these modes from brightest to darkest, and introduce a concept of *characteristic tendency tones*, which suggest a special new feature of the major and minor modes, as well as *counter-tendency tones*. I then examine minor seconds and tendency tones in the modes of non-diatonic scales. To do so, I introduce the "line of minor seconds," as well as the "circle of minor seconds." Each of these constructions is based on the line/circle of fifths. I discuss some of the implications for efficient voice-leadings between different transpositions of scales. The chapter closes with an appendix that discusses a surprising mapping between the modal pitch-class sums of a scale and that scale's structure upon the line of fifths.

3.1 Prelude: a combinatoric approach to tendency tones

A consonant tonic triad has four possible chord factors: root, perfect fifth, minor third and major third (the first two are obligatory, while the latter two are mutually exclusive). Within a mode containing this tonic triad, each of these chord factors can, in principle at least, have a neighbouring pitch-class a minor second above and below. There are therefore eight types of tendency tones, listed in table form in Figure 37. Among these eight types, two are not found in any of the diatonic modes, because they involve intervals not found in the diatonic scale: the diminished fourth and the augmented second.[76]

[76] These tendency tones also cannot be found in the melodic-minor scale, or the whole-tone plus one scale (the other scales that contain only major and minor seconds). This is obvious in the case of the augmented second, but less so in the case of the diminished fourth, since this interval is found in both of

Figure 37. The eight possible tendency tones.

		Minor Triad	Major Triad
Root	**Below (^7 ^8)**	Major ^7	
	Above (^2 ^1)	Minor ^2	
Third	**Below (^2 ^3)**	Major ^2	**Aug ^2**
	Above (^4 ^3)	**Dim ^4**	Perf ^4
Fifth	**Below (^4 ^5)**	Aug ^4	
	Above (^6 ^5)	Min ^6	

Bold: not available in diatonic modes

We shall begin our examination of tendency tones by inquiring as to how these eight tendency tones might be combined. To start, observe that the possible combinations are constrained. Scale degree $\hat{2}$ can be a tendency tone either above $\hat{1}$ or below $\hat{3}$, but it cannot be both at once. The two possibilities are mutually exclusive. The situation is the same for $\hat{4}$.[77] Observe as well that a minor third can only be formed out of the combination of a minor second and a major second, in either order.[78] Any minor third within the tonic triad therefore necessarily imposes a tendency tone, regardless of whether it is found between the root and third of a minor chord (which imposes either $\hat{2}$–$\hat{1}$ or $\hat{2}$–$\hat{3}$), or between the third and fifth of a major chord (which imposes either $\hat{4}$–$\hat{3}$ or $\hat{4}$–$\hat{5}$).

We might begin by considering *minimal* tendency tone modes: those that contain as few tendency tones as possible. Since every consonant triad contains a minor third and, as we have just seen, the interval of a minor third imposes at least one tendency tone,

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these scales. By definition, however, tendency tones resolve to a consonant tonic triad, meaning that the $\hat{5}$ is a perfect fifth above the tonic. If $\hat{4}$ is diminished, the interval from $\hat{4}$ to $\hat{5}$ will therefore be an augmented second, which is not found in either of these scales.

[77] The symmetrical nature of $\hat{2}$ and $\hat{4}$ here, as well as many other such symmetries observed in this chapter, follow from the equivalence under inversion of the major and minor triad.

[78] If smaller seconds were allowed, then other possibilities would exist for constructing a minor third from two seconds (e.g., a minor third can be formed from an augmented second and a diminished second).

there are no scales with zero tendency tones. There are, however, scales with only one tendency tone. If the tonic triad is minor, then this tendency tone must consequently be in either of the two positions between scale degrees $\hat{1}$ and $\hat{3}$ (i.e., minor $\hat{2}$ or major $\hat{2}$);[79] if it is major, then it must be in either of the two positions between $\hat{3}$ and $\hat{5}$ (i.e., perfect $\hat{4}$ or augmented $\hat{4}$). The rest of the mode is then fully determined by the stipulation that we must avoid introducing other tendency tones: the major third of the tonic triad must be divided into two major seconds, while the perfect fourth from $\hat{5}$ to $\hat{8}$ must have its minor second between $\hat{6}$ and $\hat{7}$. The resulting modes are shown in Figure 38.

Figure 38. Minimal-tendency-tone-modes.

	Minor tonic triad		Major tonic triad	
Tendency tone	**Spelling with C as tonic**	**Scale and mode**	**Spelling with C as tonic**	**Scale and mode**
^2 ^1	C Db Eb F G A Bb C	m:2, Phrygian #6		
^2 ^3	C D Eb F G A Bb C	M:2, Dorian		
^4 ^3			C D E F G A Bb C	M:5, Mixolydian
^4 ^5			C D E F# G A Bb C	m:4, Lydian b7

I have argued elsewhere (Sailor 2017) that the sonic aesthetic of Fauré and some of his French contemporaries was, in part, a matter of de-emphasizing the use of tendency-tone resolutions. In this connection, it is certainly suggestive that the diatonic modes of Figure 38—namely, the Dorian and Mixolydian modes—are those most prominently employed in the French music of the period. Similarly suggestive is the fact that also present in the table is the non-diatonic 7-scale mode most favoured by many of these composers—the Lydian b7 or "Acoustic" mode of the melodic-minor scale.

Next, let us consider dual-tendency-tone modes, i.e., those with two and only two tendency tones. All non-cross-relational pairs of tendency tones are listed in Figure 39

[79] By "minor $\hat{2}$," I mean "scale degree $\hat{2}$ a minor second above the tonic." I prefer this means for referring to scale degree quality to the more common "flat $\hat{2}$" because it does not presume the major mode as a reference. (It also avoids the awkward situation in, say, E-sharp Locrian where "flat $\hat{2}$" is F-sharp.)

below, together with the modes thereby determined. Inversionally related pairs of tendency tones are displayed side-by-side in the table to facilitate their comparison. Here, the stipulation to avoid introducing other tendency tones largely but not entirely determines the resulting mode. When a $\hat{6}$–$\hat{5}$ tendency tone is present, (e.g., A♭–G, in the key of C), then the seventh degree can be minor (e.g., B♭), and no new tendency tones will be introduced. It's also the case that no new tendency tones will be introduced, however, if the seventh is diminished (e.g., B♭♭). The situation is similar with a $\hat{7}$–$\hat{8}$ tendency tone (e.g., B–C), where $\hat{6}$ could then be either major (A) or augmented (A♯). In either case, the latter possibility (i.e., the diminished seventh or augmented sixth) produces "gratuitous" (i.e., unnecessary in the construction of the stipulated tendency tones) consecutive minor seconds. In Figure 39, I have only indicated the "non-gratuitous" modes, but the reader should recognize the possibility (albeit, relative implausibility) of the omitted "gratuitous" modes.

Figure 39. Dual-tendency-tone modes.

	Minor tonic triad		Major tonic triad			Minor tonic triad		Major tonic triad	
Tendency tones	**Spelling with C as tonic**	**Scale and mode**	**Spelling with C as tonic**	**Scale and mode**	**Tendency tones**	**Spelling with C as tonic**	**Scale and mode**	**Spelling with C as tonic**	**Scale and mode**
^2 ^1, ^4 ^3	C Db Eb Fb G A Bb C	OCT-1	C Db E F G A Bb C	H:5, Mixolydian b2	**^2 ^3, ^4 ^5**	C D Eb F# G A Bb C	h:4, Dorian #4	C D# E F# G A Bb C	OCT-1
^2 ^1, ^6 ^5	C Db Eb F G Ab Bb C	M:3, Phrygian	*none (minor second required in minor third between ^3 and ^5)*		**^4 ^5, ^7 ^8**	*none (minor second required in minor third between ^1 and ^3)*		C D E F# G A B C	M: 4, Lydian
^2 ^1, ^7 ^8	C Db Eb F G A B C	WT+1	*none (minor second required in minor third between ^3 and ^5)*		**^4 ^5, ^6 ^5**	*none (minor second required in minor third between ^1 and ^3)*		C D E F# G Ab Bb C	WT+1
^2 ^3, ^6 ^5	C D Eb F G Ab Bb C	M: 6, natural minor	*none (minor second required in minor third between ^3 and ^5)*		**^4 ^3, ^7 ^8**	*none (minor second required in minor third between ^1 and ^3)*		C D E F G A B C	M: 1, major
^2 ^3, ^7 ^8	C D Eb F G A B C	m:1, melodic minor	*none (minor second required in minor third between ^3 and ^5)*		**^4 ^3, ^6 ^5**	*none (minor second required in minor third between ^1 and ^3)*		C D E F G Ab Bb C	m: 5, Mixolydian b6
^2 ^1, ^4 ^5	C Db Eb F# G A Bb C	OCT-1	C Db E F# G A Bb C	OCT-1					
^2 ^3, ^4 ^3	C D Eb Fb G A Bb C		C D# E F G A Bb C						
^6 ^5, ^7 ^8	*none (minor second required in minor third between ^1 and ^3)*		*none (minor second required in minor third between ^3 and ^5)*						

Consecutive minor seconds

Other contrary motion tendency tones

Since all tendency tones are minor seconds, but not all minor seconds are tendency tones, it is impossible for a mode to have more tendency tones than it has minor seconds. Therefore, of the three 7-scales with two minor seconds (i.e., the diatonic, melodic minor, and whole tone plus one scales), all of their modes with consonant tonic triad are listed in the two preceding tables. (There are no 7-scales with fewer than two minor seconds.)

An interesting feature of Figure 39 is how few modes it contains where the two tendency tones resolve in contrary motion, especially if we exclude the scales with consecutive minor seconds. The tendency tone-pairs $\hat{2}$–$\hat{1}$ and $\hat{2}$–$\hat{3}$ or $\hat{4}$–$\hat{3}$ and $\hat{4}$–$\hat{5}$ would involve contrary motion, but such pairs involve cross-relations and are therefore excluded by definition from existing in a scale (i.e., a scale only contains one second or fourth degree).[80] The only remaining modes with contrary motion tendency tones are the familiar major and minor modes of the diatonic scale, as well as two modes of the "octatonic minus one" scales with an augmented third between minor $\hat{2}$ and augmented $\hat{4}$. The presence of the major and minor modes as the only quasi-diatonic members of this short list is certainly suggestive. We will revisit this observation below.

Triple-tendency modes are listed in Figure 40. (Again, inversionally related groups of tendency tones are displayed side-by-side). All stable modes of all scales with three or fewer minor seconds (including the harmonic minor and major scales) are either in Figure 40, or one of the two preceding tables. The considerable majority of triple-tendency-tone modes impose consecutive minor seconds and thus lie outside of the quasi-diatonic scales. All of those that do not impose consecutive minor seconds are modes of the harmonic minor or major scales. Four of these harmonic minor/major modes are what we might call the "all-similar" tendency-tone modes. In the all-similar modes, every member of the tonic triad is approached by tendency tone in similar motion. The result

[80] This is most certainly not to say that such cross-relational tendency tones are not interesting and even musically useful, but simply that they necessarily involve extra-scalar alterations and therefore do not concern us here.

is that the scale contains two triads of the same quality, a minor second apart. (It follows that the "all-ascending" mode of a given triad quality will be found one degree higher in the scale from the "all-descending" mode of that same triad quality.) Besides these all-similar modes, the remaining harmonic minor/major modes are the "canonical" harmonic minor and harmonic major modes themselves. It seems worthy of comment that, among quasi-diatonic scales, both of these commonly used modes are maximally embedded with tendency tones.

Figure 40. Triple-tendency-tone modes.

	Minor tonic triad		Major tonic triad			Minor tonic triad		Major tonic triad	
Tendency tones	**Spelling with C as tonic**	**Scale and mode**	**Spelling with C as tonic**	**Scale and mode**	**Tendency tones**	**Spelling with C as tonic**	**Scale and mode**	**Spelling with C as tonic**	**Scale and mode**
^2 ^1, ^4 ^3, ^6 ^5	C Db Eb Fb G Ab Bb C	H:3, Phrygian b4	C Db E F G Ab Bb C	h:5, Phrygian #3	**^2 ^3, ^4 ^5, ^7 ^8**	C D Eb F# G A B C	H:4, Lydian b3	C D# E F# G A B C	h:6, Lydian #2
^2 ^1, ^4 ^3, ^7 ^8	C Db Eb Fb G A B C		C Db E F G A B C		**^2 ^3, ^4 ^5, ^6 ^5**	C D Eb F# G Ab Bb C		C D# E F# G Ab Bb C	
^2 ^1, ^4 ^5, ^6 ^5	C Db Eb F# G Ab Bb C		C Db E F# G Ab Bb C		**^2 ^1, ^4 ^5, ^7 ^8**	C Db Eb F# G A B C		C Db E F# G A B C	
^2 ^1, ^6 ^5, ^7 ^8	C Db Eb F G Ab B C		*none (minor second required in minor third between ^3 and ^5)*		**^4 ^5, ^6 ^5, ^7 ^8**	*none (minor second required in minor third between ^1 and ^3)*		C D E F# G Ab B C	
^2 ^3, ^4 ^3, ^6 ^5	C D Eb Fb G Ab Bb C		C D# E F G Ab Bb C		**^2 ^3, ^4 ^3, ^7 ^8**	C D Eb Fb G A B C		C D# E F G A B C	
^2 ^3, ^6 ^5, ^7 ^8	C D Eb F G Ab B C	h:1, harmonic minor	*none (minor second required in minor third between ^3 and ^5)*		**^4 ^3, ^6 ^5, ^7 ^8**	*none (minor second required in minor third between ^1 and ^3)*		C D E F G Ab B C	H:1, harmonic major

Consecutive minor seconds

All-similar tendency tones

Finally, Figure 41 lists the modes with four tendency tones. (The first two rows of this table are inversionally related to one another. Due to the prohibitive width of the table they have not been displayed next to each other, unlike the preceding tables.) These are the maximal-tendency-tone modes, because a mode can have at most four tendency tones: one each of the tendency tones involving $\hat{2}$ and $\hat{4}$, respectively, and the two remaining possible tendency tones, $\hat{6}$–$\hat{5}$ and $\hat{7}$–$\hat{8}$. Since we are examining 7-scales, and the tonic triad has three pitch-classes, in a maximal-tendency-tone mode all four non-tonic notes will be tendency tones. It is interesting to consider the harmonies formed when all four of these tendency tones are combined. We might call these "maximal-tendency-tone chords."[81] In order for a mode to have four tendency tones, the mode must contain a minor second both above and below at least one of the three members of the tonic triad. This combination of tendency tones both above and below a given pitch-class forms a diminished third, or as we more often think of it, an augmented sixth. All maximal-tendency-tone-chords are therefore augmented-sixth chords. The chord- and resolution-types of the maximal tendency tone chord associated with each mode have been listed in a new column in Figure 41. One of these chord- and resolution-types is familiar from the common practice: the German sixth that resolves to the root of the following major triad. (In common-practice procedure, however, the following major triad is usually a dominant, rather than a tonic, and the resolution usually proceeds through an intermediary 6/4 chord in order to avoid parallels.) The other augmented sixth-types are less well-known, but many can be found in the extended-tonal repertoire. Some of these augmented-sixth types have minor thirds: these belong to the "Swedish sixth" family proposed by Piché (2018).

[81] The "maximal tendency tone chords" do not constitute, however, all harmonies that can resolve to a consonant triad by minor second motion in all four voices, because they exclude all such harmonies (the majority) that involve cross relations (e.g., D♭ resolving to C together with D resolving to E♭). As a highly tentative hypothesis that I will not consider further, I would suggest that extended tonal composers have generally favoured the maximal tendency tone chords over these other cross-relational harmonies.

Figure 41. Maximal-tendency-tone modes.

	Minor tonic triad			Major tonic triad		
Tendency tones	**Spelling with C as tonic**	**Scale and mode**	**Augmented sixth/ tonic pair**	**Spelling with C as tonic**	**Scale and mode**	**Augmented sixth/ tonic pair**
^2 ^1, ^4 ^3, ^6 ^5, ^7 ^8	C Db Eb Fb G Ab B C	7-21A	Germano-Swedish sixth to root	C Db E F G Ab B C	Hungarian minor:5	German sixth to root
^2 ^3, ^4 ^5, ^6 ^5, ^7 ^8	C D Eb F# G Ab B C	Hungarian minor:1	Franco-Swedish sixth to fifth	C D# E F# G Ab B C	7-21B	Germano-Swedish sixth to fifth
^2 ^1, ^4 ^5, ^6 ^5, ^7 ^8	C Db Eb F# G Ab B C	7-20A	Double augmented sixth to root and fifth	C Db E F# G Ab B C	7-20B	Double augmented sixth to root and fifth
^2 ^3, ^4 ^3, ^6 ^5, ^7 ^8	C D Eb Fb G Ab B C	7-21B	German sixth to third	C D# E F G Ab B C	7-21A	Franco-Swedish sixth to third

We should pause to note that, in most music that exhibits augmented sixth resolutions such as those in Figure 41, it is doubtful whether we are intended to infer the scales listed in the figure as providing the background structure of the music. For instance, the combination of German augmented-sixth chord with dominant triad is not ordinarily thought of as forming a scale. Indeed, it seems probable that, without a special effort on the part of the composer to establish a maximal tendency tone mode, any such maximal agglomeration of tendency tones is more likely to be heard as chromatic alterations to some other scale, or perhaps simply as a chromatic structure without any particular scalar scaffolding in the background. I'm not aware of any repertoire that makes such an effort. Music that employs one of the scales from Figure 41 does not necessarily use the harmonic resources in this manner. For instance, the Liszt excerpt in Figure 42 is entirely in G Hungarian minor,[82] but it simply relies on an alteration of tonic and dominant harmonies. On the other hand, the dominant chord *is* approached by a "maximal-tendency-tone" German-sixth chord. (This raises an interesting question: is it the case that in common-practice music, it is *dominant* chords, rather than tonic chords,

[82] Every pitch in the excerpted passage belongs to G Hungarian minor, with the exception of the C's of the D7 harmonies in mm. 23–26 and the Picardy third in mm. 26–27.

that tend to be approached by relatively large conglomerations of tendency tones? Certainly, this is true in the case of common-practice augmented-sixth chords.[83])

Figure 42. Liszt, Hungarian Rhapsody no. 3, mm. 17–27.

[83] Another famous example, arguably an augmented sixth chord, would be the Tristan chord. This famous chord is a *dominant*, and not a tonic, preparation.

3.2 Web app 3.1

In light of the previous discussion, the reader is invited to experiment with web app 3.1, "Combine Tendency Tones."[84] There, she can try out a combinatoric approach to tendency tones for herself, and listen to the music that results.

3.3 Tendency tones in the diatonic modes

Having introduced the subject of tendency tones, in the present section we will restrict our purview to the arrangement of tendency tones in the diatonic modes. The principles of organization which we observe therein will be useful when we later expand our investigations to the modes of other scales.

As is common knowledge, the diatonic modes can be ordered from "brightest" (Lydian) to "darkest" (Locrian), as has been done in Figure 43. A relatively "bright" mode is a mode with larger ascending intervals, as measured from the tonic, while a relatively "dark" mode has smaller intervals, measured from the tonic. In any 7-scale (and, indeed, any transpositionally asymmetrical pitch-class set), there is always a unique ordering of modes from "brightest" to "darkest" which can be defined by sorting the modes according to their *modal pitch-class sums*. A modal pitch-class sum is calculated by simply summing the directed atonal intervals from the modal tonic to the other pitch-classes of the scale.[85] For the diatonic modes, these sums have been listed in the middle column of Figure 43. I will show in the appendix to this chapter that the distribution of modal pitch-class sums for a scale can be mapped to the fifth-string of its near-well-formed spelling. Since each node of the fifth-string corresponds to a unique pitch-class in the scale, each

[84] http://sailor.music.mcgill.ca/tendencytones

[85] As an example, consider the D Dorian mode: measured in 12-tone equal temperament, it contains a major second (2) + a minor third (3) + a perfect fourth (5) + a perfect fifth (7) + a major sixth (9) + a minor seventh (10) = 2 + 3 + 5 + 7 + 9 + 10 = 36.

modal pitch-class sum of the scale is likewise unique. These sums can thus always be used to sort the modes.[86]

In the case of the diatonic scale, moving from bright to dark or vice versa corresponds to motion on the line (or circle) of fifths. (Since we will travel only seven steps [one for each pitch-class of the scale], there is no difference whether we suppose that we are travelling upon the infinite line of fifths or around the 12-node circle of fifths.) This motion can be conceptualized in either of two ways:

1) holding the key signature constant, we move the root on the line of fifths, as from F Lydian through C major, G Mixolydian, etc., to B Locrian. Vincent (1951, 26) calls the modes arranged in this fashion the "ordinal index" of modes; according to the operations defined by Hook (2008), this index results from the repeated application of the *diatonic transposition operator*, specifically t_4.
2) holding the root constant, the key signature can be shifted along the line of fifths, as from C Lydian to C major, C Mixolydian, etc., to C Locrian. Vincent (1951, 27–28) calls this the "lateral index" of modes; for Hook, who gives a very similar illustration (2008, 148), it results from the repeated operation of the signature transformation.[87] (In fact, following the latter procedure, by continuing to inflect the key signature, we can continue past C Locrian to C-flat (or, enharmonically, B) Lydian, and so on. Since twelve [the number of enharmonically-distinct key signatures in 12-tone equal temperament] and seven [the number of diatonic modes] are relatively prime, under enharmonic and octave equivalence, this procedure, repeated enough times—84, to be exact—will bring us all the way back to our starting pitch-class and mode.)

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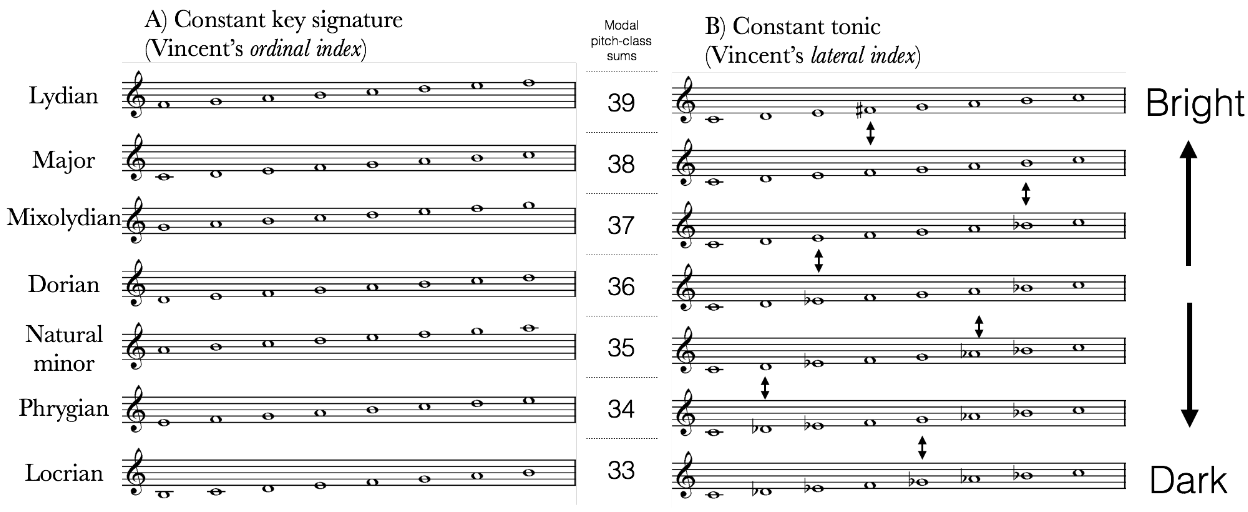
[86] In fact, this can even apply to non-proper multisets, as long as the letter names are taken to specify an ordering. For instance, in the multiset {C D E F♯ G♯ A♯ B♯}, the pitch-class sums for C and B♯ are 42 and 30, respectively.

[87] Each of these procedures are "fixed domain diatonic relationships", according to Bates' terminology [2012, 35–36], where the fixed domain is either the key signature (ordinal index) or tonic pitch-class (lateral index).

Figure 43. The diatonic modes, ordered from bright to dark.



Although the choice between favouring the "ordinal" and "lateral" indexes is basically arbitrary, the latter, where tonic is held constant, is arguably a more useful representation for our purposes, because it makes most immediately visible the differences in interval structure between each mode and its modal "neighbours" (i.e., those modes whose modal pitch-class sums differ by 1). In Figure 43B, I have indicated these differences with arrows indicating the alterations that distinguish each pair of modes.

Now that we have ordered the modes from dark to bright, we can consider the relations among them by comparing the arrangement of their minor seconds. Since a diatonic mode can be defined as a pattern of major and minor seconds (e.g., the major mode is [M2, M2, m2, M2, M2, M2, m2]), and because there are two and only two minor seconds in each diatonic mode, every diatonic mode can be fully specified by specifying the position of its minor seconds (e.g., if there are minor seconds from $\hat{3}$ to $\hat{4}$ and $\hat{7}$ to $\hat{8}$, then there is only one compatible diatonic mode: major). Each mode shares one of its two minor seconds with each of its modal neighbours.[88] The minor second that a mode shares

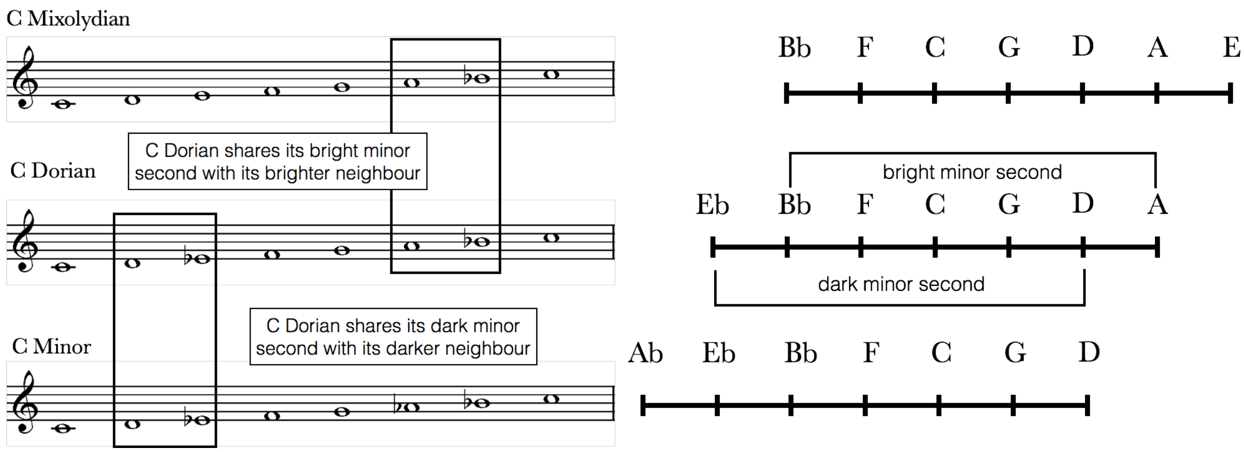
[88] The modes at the extremes—namely, Locrian and Lydian—only have one modal neighbour each, provided we keep the root constant. If we allow the root to drift, then Locrian shares one minor second with the Lydian mode on the root an augmented unison lower (e.g., B Locrian shares the minor second E-

with its brighter neighbour, and not with its darker neighbour, I call its *bright* minor second. The lower pitch of this minor second is the note furthest sharpwards in the segment of the line of fifths that generates the mode. (To make this more concrete, it may be helpful to consider Figure 44.) Similarly, the minor second that a mode shares with its darker neighbour, and not with its brighter neighbour, I call its *dark* minor second. The upper pitch-class of this minor second is the note furthest flatwards in the mode's segment of the line of fifths. As an example, the C Dorian mode in Figure 44 contains the pitches {C, D, E-flat, F, G, A, B-flat}, generated from the segment of the line of fifths {E-flat, B-flat, F, C, G, D, A}. On this segment of the line, A is the pitch-class the furthest sharpwards, and thus the bright minor second of this mode is from A to B-flat (i.e., from $\hat{6}$ to $\hat{7}$). C Dorian shares this minor second with its brighter neighbour C Mixolydian, but not with its darker neighbour C Minor.[89] Conversely, E-flat is the pitch-class furthest flatwards in C Dorian's segment of the line of fifths. Therefore, the dark minor second of this mode is from D to E-flat (i.e., from $\hat{2}$ to $\hat{3}$), and C Dorian shares this minor second with its darker neighbour C Minor, but not with its brighter neighbour C Mixolydian.

F with B♭ Lydian), and, conversely, Lydian shares one minor second with the Locrian mode an augmented unison higher (e.g., B♭ Lydian shares the minor second E-F with B Locrian). To permit these other scales as "neighbours," we must define neighbours in terms of key signatures that differ by one accidental, rather than in terms of modal pitch-class sums.

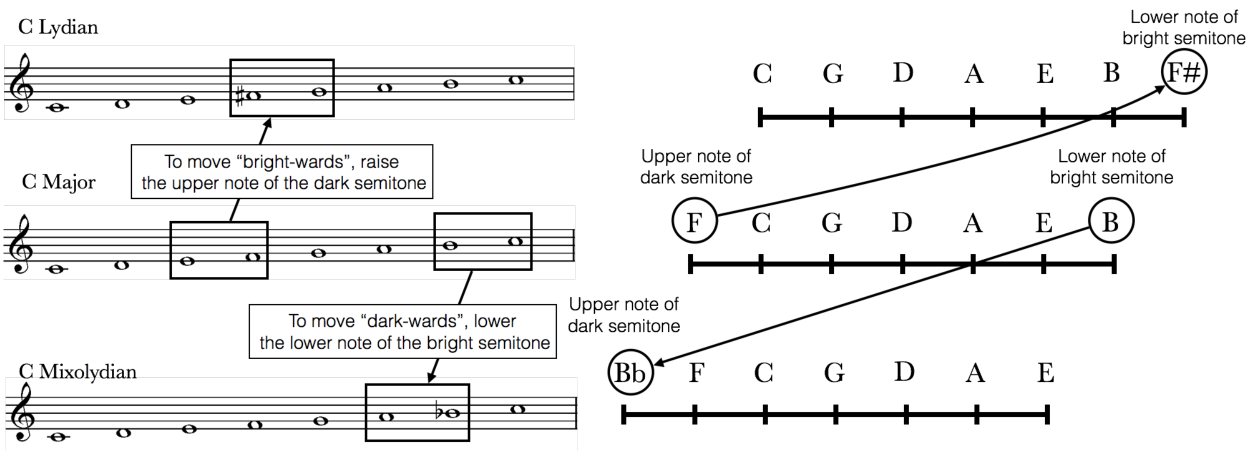
[89] If we hold the key signature constant and change the roots, C Dorian's brighter neighbour is F Mixolydian. In this case, we must reckon the semitones not as pitch-classes but as scale degrees: C Dorian shares the $\hat{6}$–$\hat{7}$ semitone with F Mixolydian. Although the semitone from A to B-flat exists in both scales (as it must, since they share the same key signature and therefore pitch-class content), it is in either case the bright semitone in both modes, because A is the furthest pitch-class sharpwards in the relevant segment of the line of fifths.

Figure 44. C Dorian and its modal neighbours.



It follows from the above that, to move "bright-wards" through the modes, one successively raises the upper pitch-class of each mode's *dark* minor second, which then becomes the lower pitch-class of the resulting mode's bright minor second. For instance, as shown in Figure 45, to move from Major to Lydian one raises the upper pitch-class of Major's dark minor second ($\hat{4}$–$\hat{3}$). This alteration converts $\hat{4}$ into the lower pitch-class of Lydian's bright minor second ($\hat{4}$–$\hat{5}$). On the other hand, to move "dark-wards," one successively lowers the lower pitch-class of each mode's *bright* minor second, which then becomes the upper pitch-class of the resulting mode's dark minor second. For instance, (Figure 45) to move from Major to Mixolydian, one lowers the lower pitch-class of Major's bright minor second ($\hat{7}$–$\hat{8}$), converting $\hat{7}$ into the upper pitch-class of Mixolydian's dark minor second ($\hat{6}$–$\hat{7}$).

Figure 45. Voice-leading between C Major and its modal neighbours.



The above discussion is summarized schematically in Figure 46. In the figure, an additional layer of information is added by highlighting in red those modal minor seconds which are tendency tones. The only pair of adjacent scale degrees that does not contain a member of the tonic triad is $\hat{6}$–$\hat{7}$, and therefore the $\hat{6}$–$\hat{7}$ minor second shared by Dorian and Mixolydian cannot be a tendency tone. Moreover, in Locrian, $\hat{5}$ forms a dissonant diminished fifth in the tonic triad, therefore the $\hat{4}$–$\hat{5}$ minor second in this mode is not a tendency tone, according to the definition adopted in this chapter.

Figure 46. Placement of minor seconds in the diatonic modes. (Tendency tones are highlighted in red.)

To move sharpwards, raise the upper pitch of the dark minor second

<table>
<tr><th colspan="2">Locrian</th><th colspan="2">Phrygian</th><th colspan="2">Minor</th><th colspan="2">Dorian</th><th colspan="2">Mixolydian</th><th colspan="2">Major</th><th colspan="2">Lydian</th></tr>
<tr><td>Dark</td><td>Bright</td><td>Dark</td><td>Bright</td><td>Dark</td><td>Bright</td><td>Dark</td><td>Bright</td><td>Dark</td><td>Bright</td><td>Dark</td><td>Bright</td><td>Dark</td><td>Bright</td></tr>
<tr><td>^4 ^5</td><td colspan="2">^2 ^1</td><td colspan="2">^6 ^5</td><td colspan="2">^2 ^3</td><td colspan="2">^7 ^6</td><td colspan="2">^4^3</td><td colspan="2">^7 ^8</td><td>^4 ^5</td></tr>
</table>

To move flatwards, lower the lower pitch of the bright minor second

Figure 47 divides these tendency tones into two categories: those that resolve upwards, and those that resolve downwards.

Figure 47. Upwards- and downwards-resolving tendency tones in the diatonic modes.

Upwards-resolving tendency tones	Modes	Downwards-resolving tendency tones	Modes
^4 ^5	Lydian	^2 ^1	Phyrgian
^7 ^8	Lydian, major	^6 ^5	Natural minor, Phrygian
^2 ^3	Natural minor, Dorian	^4 ^3	Major, Mixolydian

As we travel sharpwards through the modes, upwards-resolving tendency tones always make their first appearance as the bright minor second of a mode (e.g., $\hat{7}$–$\hat{8}$ appears as we move from Mixolydian to major; $\hat{4}$–$\hat{5}$ as we move from Major to Lydian). Because bright minor seconds are always introduced by raising their lower pitch-class (e.g., B♭ is raised to B-natural when moving from C Mixolydian to C major), the very pitch-class that tends upwards towards resolution (e.g., B-natural tending upwards towards C) is that which has just been introduced, and which therefore distinguishes the mode from its darker neighbour. For this reason, upwards-resolving tendency tones can be heard as especially characteristic of the mode within which they constitute the bright minor second.

Inversely, as we move flatwards through the modes, downwards-resolving tendency tones make their first appearance as the dark minor second of a mode (e.g., $\hat{2}$–$\hat{1}$ appears as we move from natural minor to Phrygian). Dark minor seconds are introduced by lowering their upper pitch-class (e.g., D-natural is lowered to D♭ when moving from C natural minor to C Phrygian), and therefore the very pitch-class that tends downwards towards resolution is that which has just been introduced, and which therefore distinguishes the mode from its brighter neighbour. Downwards-resolving tendency tones can therefore be heard as especially characteristic of the mode within which they constitute the dark minor second.

I therefore speak of tendency tones in either of these two scenarios (bright minor second as upwards-resolving tendency tone; dark minor second as downwards-resolving tendency tone) as *characteristic tendency tones*. Unlike some other questions discussed in this thesis, the extent to which a given tendency tone is truly "characteristic" of a given mode is not amenable to precise quantification. Nevertheless, I hope that the reader will agree that this choice of descriptor has a certain logic. Moreover, as someone who has devoted a great deal of time to the study and composition of modal music, I can attest that calling these tones "characteristic" accords powerfully with my intuitions about which tendencies are most characteristic of which modes.

An information-theoretic perspective might also be invoked in favour of this conception of characteristic tendency tones. The specification of a diatonic mode's characteristic tendency tones is all that is needed to fully specify the entire mode. The modes with a single characteristic tendency tone (Lydian and Phrygian) can both be uniquely specified by the specification of only this tendency tone. (E.g., if a diatonic mode has tendency tone $\hat{4}$–$\hat{5}$, it is Lydian; if it has tendency tone $\hat{2}$–$\hat{1}$, it is Phrygian. Locrian also has minor seconds between these scale degrees, but because it does not have a consonant tonic triad, these are not tendency tones.) Both of the modes with two characteristic tendency tones (major and natural minor) require the specification of both of these tendency tones to be uniquely defined. (E.g., to define major, we need both $\hat{4}$–$\hat{3}$, to differentiate it from Lydian, and $\hat{7}$–$\hat{8}$, to differentiate it from Mixolydian.) The modes without characteristic tendency tones (Mixolydian and Dorian), on the other hand, cannot be uniquely defined by the specification of their (non-characteristic) tendency tone (e.g., Mixolydian shares its tendency tone, $\hat{4}$–$\hat{3}$, with major), and require other information (perhaps the "counter-tendency tones" discussed below) in order to be specified among the diatonic modes. In practice, this means that a composer writing a diatonic modal piece of music signals the mode most efficiently and effectively through the use of the characteristic tendency tone(s) of that mode.

Figure 48. Characteristic tendency tones in the diatonic modes.

Characteristic tendency tones					
Phrygian	Natural minor	Dorian	Mixolydian	Major	Lydian
^2 ^1	^6 ^5, ^2 ^3	none	none	^7 ^1, ^4 ^3	^4 ^5

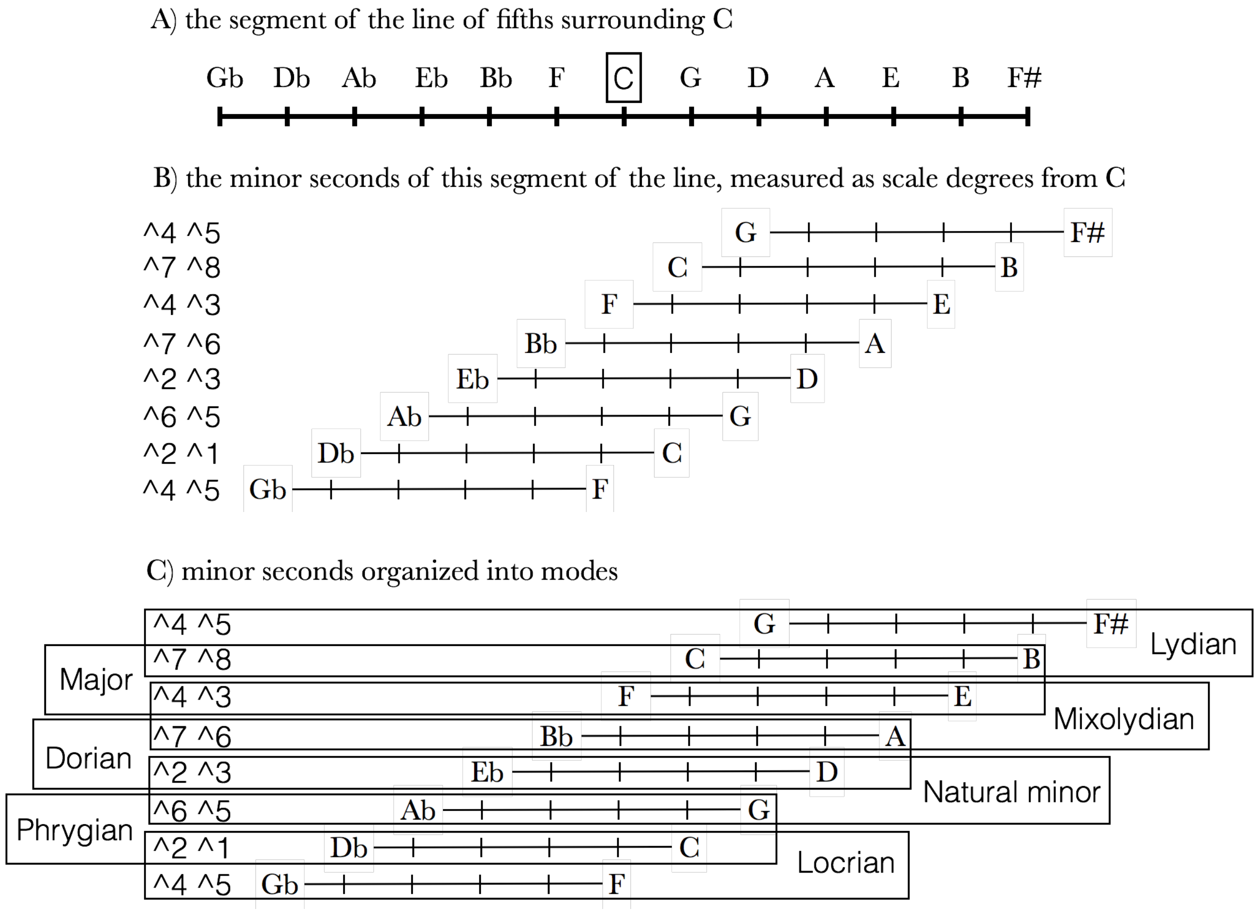
Figure 48 lists the characteristic tendency tones for each of the diatonic modes. The fact that both the major and natural minor modes have two characteristic tendency tones (and are thus maximally endowed with characteristic tendency tones, since a diatonic mode has only two minor seconds) suggests a previously undiscussed reason why tonal music may have "tended" towards these modes. On the other hand, we saw above that Dorian and Mixolydian are both "minimal tendency tone" modes (see Figure 38). Figure 48 demonstrates that they are also "minimal *characteristic* tendency tone" modes, because they contain no characteristic tendency tones at all. The absence of characteristic tendency tones reinforces the suggestion made above as to why French composers such as Fauré, who de-emphasized tendency tone resolutions in their music (Sailor 2017), found these modes so congenial.

In fact, it might be argued that, in place of characteristic tendency tones, the Dorian and Mixolydian modes have characteristic *counter-tendency* tones—i.e., characteristic *major* seconds where minor second tendency tones are expected. To illustrate, consider the fact that both the Mixolydian and major modes possess a $\hat{4}$–$\hat{3}$ tendency tone. For this reason, given a major tonic, we might reasonably expect the mode to possess this tendency tone. However, when in the Lydian mode it is removed by the upwards alteration of $\hat{4}$, a new tendency tone, $\hat{4}$–$\hat{5}$, takes its place. The situation is different with the $\hat{7}$–$\hat{8}$ tendency tone shared by both the major and Lydian modes. Given a major tonic, we might therefore expect the mode to possess this $\hat{7}$–$\hat{8}$ tendency tone. In the Mixolydian mode, however, when this tendency tone is removed by the downwards alteration of $\hat{7}$, no new tendency tone is created. For this reason, we might consider the absence of a minor second from $\hat{7}$–$\hat{8}$ in the Mixolydian mode to be characteristic of this mode—a characteristic counter-tendency tone—very much analogous to how the presence of the minor second from $\hat{4}$ to$\hat{5}$ is characteristic of Lydian. The situation is similar concerning the absence of the minor second from $\hat{6}$ to $\hat{5}$ in the Dorian mode. If this argument holds water, then the fin-de-siècle French affection for these modes can be seen as a motion in precisely the opposite direction from the chromatic thrust of German music. While Wagner and his colleagues and imitators were relying on chromatic alteration to turn nearly every pitch into a tendency tone, French composers (of a slightly later vintage, admittedly) turned to modes with a conspicuous—even "characteristic"—absence of tendency tones.

3.4 Tendency tones in non-diatonic modes

Our next task is to consider the placement of minor seconds, and thus tendency tones, in the modes of non-diatonic scales. To accomplish this, however, we will begin by re-examining the immediately preceding discussion from a new perspective, introducing another way of visualizing the relationships of shared and unshared minor seconds among the diatonic modes depicted in Figure 46. Consider the segment of the line of fifths extending six pitch-classes flatwards and sharpwards from a given tonic pitch-class. This segment will contain, as sub-segments, all the diatonic modes of the given tonic. Such a segment is depicted in Figure 49A, with C as the tonic. The minor seconds present in this segment of the line of fifths can then be measured as scale degrees relative to the tonic, as in Figure 49B, and any two adjacent minor seconds will belong to one or the other of the diatonic modes, as in Figure 49C.

Figure 49. Another look at the diatonic modes.



Notice that the scale degrees of the minor seconds in Figure 49B appear to be circular, beginning where they ended (with $\hat{4}$–$\hat{5}$), although the specific qualities of the scale degrees differ (F and G♭, vs. F♯ and G). Observe, moreover, that if we were to extend the segment beyond that containing all the diatonic modes of C, this circular arrangement would continue: after F♯-G, we would come to D-C♯, and A-G♯, which are $\hat{2}$–$\hat{1}$ and $\hat{6}$–$\hat{5}$, respectively[90]—just as D♭-C ($\hat{2}$–$\hat{1}$) and A♭-G ($\hat{6}$–$\hat{5}$) followed G♭-F. In sum, the logic governing the scale-degree placement of modal minor seconds is the same as for pitch-classes upon the line of fifths: minor seconds whose position is congruent mod 7 share the same generic scale-degree quality (e.g., $\hat{4}$–$\hat{3}$), while being raised or lowered in specific

[90] C♯ is not a legal scale degree within a mode with C as tonic, but nevertheless, we can appreciate that if it were to occur as a passing alteration, we would have to call it ♯$\hat{1}$.

quality according to their particular location[91] (e.g., diminished fourth and minor third; perfect fourth and major third; augmented fourth and augmented third—consider Figure 50). Specific minor seconds can be arranged along a "line of minor seconds" (Figure 51), while the operation of generic equivalence creates a "circle of minor seconds" (Figure 52). (Minor seconds that are not available in a mode with tonic C are italicized. Notice that the non-italicized portion of Figure 51 corresponds precisely to the tendency tones as arranged in Figure 46 above.)

Figure 50. Several different locations of $\hat{4}$–$\hat{3}$ on the line of fifths, with C as tonic.

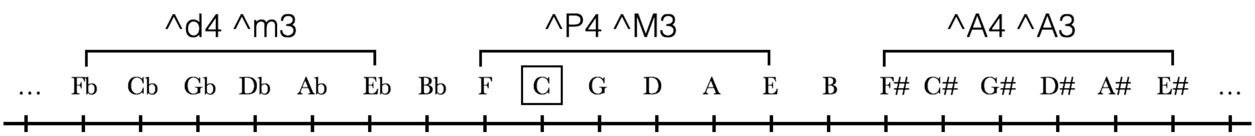
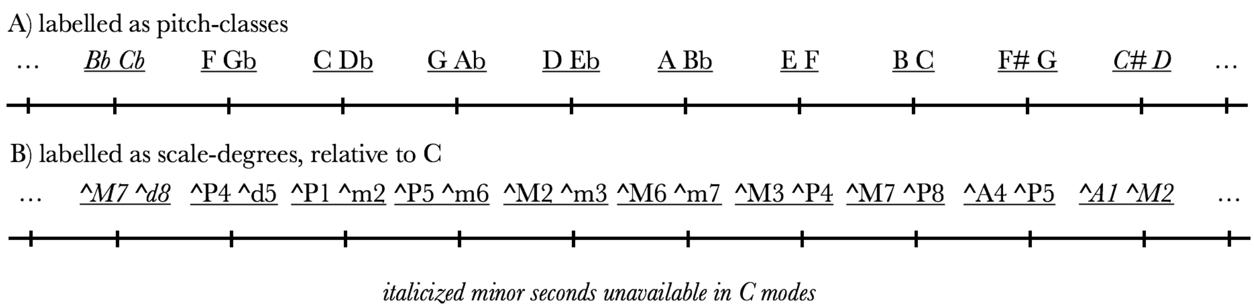
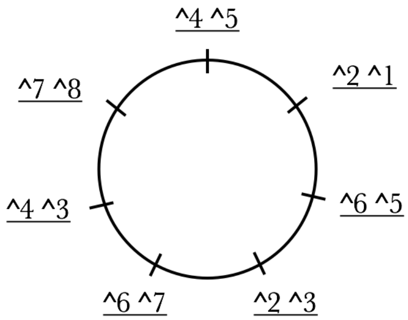


Figure 51. The "line of minor seconds."



[91] A "generic" scalar interval refers to the number of scale degrees the interval spans (e.g., a fourth, a fifth), while a "specific" scalar interval is a generic interval together with that interval's "quality" (e.g., a diminished fourth, a perfect fifth). See the introduction as well as Chapter 1 for more discussion.

Figure 52. The “circle of minor seconds.”



It is also worth appreciating that voice-leadings between neighbouring diatonic modes result from the only operation that, by a single augmented unison (i.e., semitone) alteration, can act to both remove a previously present minor second and add a previously absent one. This follows from the fact that, to remove a minor second, we must alter one of its constituent pitch-classes by inflecting it away from the other.[92] For example, the minor second B–C can be removed by lowering B to B♭. The only way in which this alteration can introduce a new minor second to the scale is if the pitch-class a minor second below the new pitch (e.g., A) was already present. We can think of this “efficient minor-second exchange” in terms of a minor third “frame” (e.g., A–C) that must be present in both scales in order that a single alteration of the middle pitch-class can both introduce one minor second and remove another.[93] Moreover, the new minor second will be a major second higher or lower than the original minor second (e.g., A–B♭ is a major second lower than B–C); in other words, it will be found two steps away on the line of fifths.

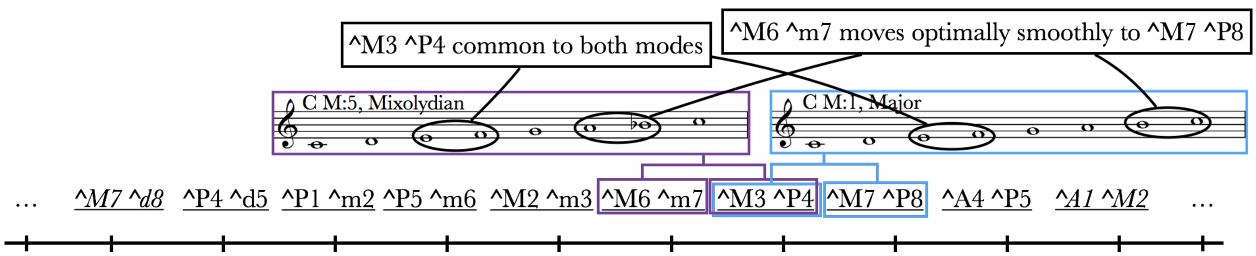
In fact, we have already seen this above in the discussion of “bright” and “dark” minor seconds when moving between neighbouring modes (consider Figure 44 and Figure 45), but the most useful way of representing this for our present purposes is on

[92] Provided we are in a tuning where the minor second is the smallest admissible second, as assumed in this chapter.

[93] B–C could also be removed by raising C to C♯; in this case, D is the pitch that would have to be present for the alteration to introduce a new minor second, and B–D is the minor third “frame.”

the line of minor seconds, as illustrated in Figure 53. A diatonic mode (e.g., C Mixolydian) contains two consecutive minor seconds on the line of minor seconds. When we move to its sharpwards neighbour (e.g., C major), the bright minor second (e.g., M$\hat{3}$–P$\hat{4}$) of the first mode will remain, but the dark minor second (e.g., M$\hat{6}$–m$\hat{7}$) will be removed, being replaced by the minor second two steps sharpwards on the line (e.g., M$\hat{7}$–P$\hat{8}$). Two steps on the line of minor seconds is precisely the interval of "efficient minor-second exchange" which we have just been investigating.

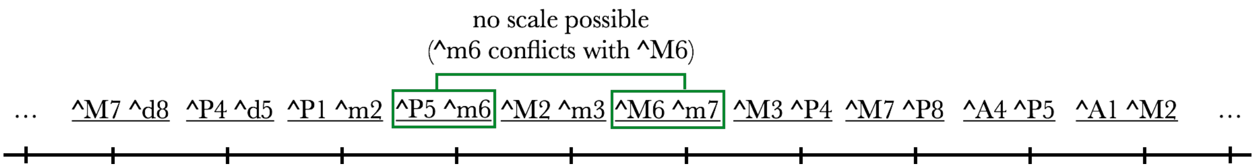
Figure 53. Voice-leading between neighbouring diatonic modes.



With the above-developed apparatus in mind, we are now in a position to consider the modes of non-diatonic scales. The minor seconds of the diatonic scale are separated by one step upon the line of minor seconds, but in non-diatonic scales, the minor seconds will necessarily be separated by other intervals upon the line (or there will be more than two minor seconds). We might begin by wondering about a scale with minor seconds two steps apart upon the line. But no such scale can exist, because the distance between the most distant pitch-classes of the pair of minor seconds will be 7 steps—that is, they will share the same letter name, and therefore are excluded by definition from belonging to the same scale (Figure 54).[94]

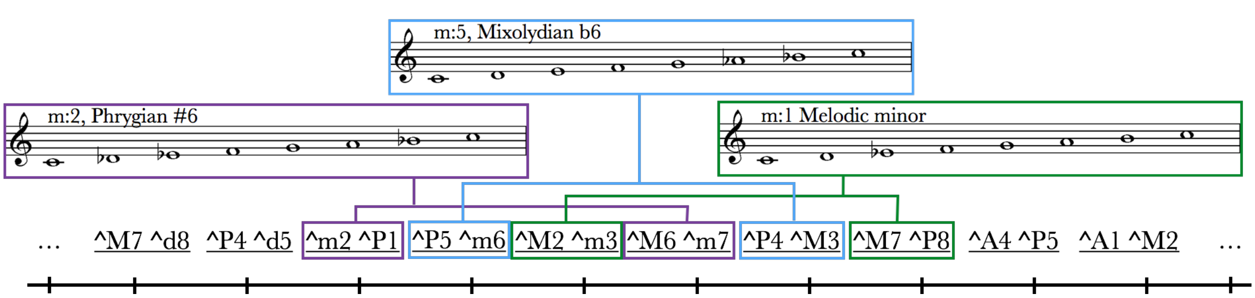
[94] This is also why there is no properly spelled 7-scale with a span of 8.

Figure 54. Impossibility of a scale containing two minor seconds separated by two steps on the "line of minor seconds."



We can, on the other hand, construct a scale with minor seconds *three* steps apart on the line of fifths—this is the melodic-minor scale, the scale of least span after the diatonic. The result is that the modal minor seconds of the melodic minor modes are not adjacent on Figure 46, but instead separated by two spaces (as illustrated in Figure 55). This fact helps to explain why a voice-leading between fifth-related melodic minor modes is less efficient than a voice-leading between fifth-related (i.e., neighbouring) diatonic modes. When moving by sharpwards fifth from one melodic minor mode (e.g., m:5, Mixolydian b6) to another (e.g., m:1, melodic minor), the dark (i.e., flattest) minor second of the second mode (e.g., M$\hat{2}$–m$\hat{3}$) will be two steps *flatwards* on the line of minor seconds from the bright (i.e., sharpest) minor second of the first mode (e.g., P$\hat{4}$–M$\hat{3}$). Therefore, the exchange between one minor second and the other can be achieved by efficient minor-second exchange flatwards, the opposite direction of the overall motion sharpwards on the line of fifths. There is no efficient voice-leading between the remaining minor seconds in each scale (e.g., P$\hat{5}$–m$\hat{6}$ and M$\hat{7}$–P$\hat{8}$), and overall the voice-leading between the two scales requires three augmented unison alterations (e.g., M3 becomes m3, m6 becomes M6, and m7 becomes M7).

Figure 55. Several melodic minor modes on the line of minor seconds.

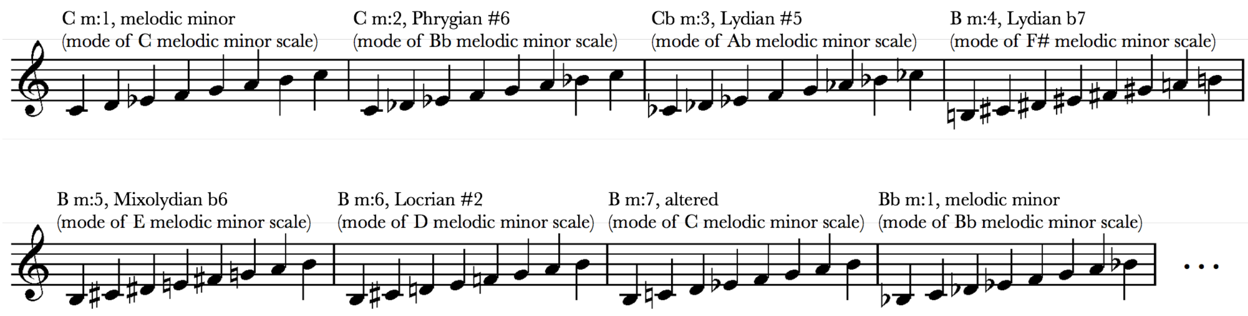


There is, however, a more efficient voice-leading between melodic-minor scales. As we have seen, the efficient minor-second exchange takes place between minor seconds that are two steps apart upon the line of minor seconds. It follows that when voice-leading between modes separated by two steps upon the line, each minor second of the original mode can always move by efficient alteration to its equivalent in the new mode, provided that the minor third "frame" is present in both modes. (This "frame" is guaranteed to be present as long as the scale contains only major and minor seconds, and does not have consecutive minor seconds, because in that case a minor second can only be adjacent to a major second, forming a minor third.) For instance, consider the motion from m:2, Phrygian ♯6 to m:1, melodic minor (Figure 55).[95] The dark minor second of the first mode (m$\hat{2}$–P$\hat{1}$) is two steps on the line from the dark semitone of the second (M$\hat{2}$–m$\hat{3}$), while the bright minor second of the first (M$\hat{6}$–m$\hat{7}$) is also two steps from the bright minor second of the second (M$\hat{7}$–P$\hat{8}$). The repeated application of this type of voice-leading through successive melodic minor modes is notated in Figure 56. (These sorts of patterns can be very useful in practicing non-diatonic modes upon one's instrument.) As we can observe from the figure, this voice-leading passes through all seven modes, because seven is prime, and therefore stepping by two (or any other number not a whole multiple of seven) through the modes will touch on all seven modes. Since 12 (the number of enharmonically equivalent pitch-classes, assuming 12-tone equal-temperament) is divisible by 2, however, this procedure will not proceed through all 12 equal-tempered transpositions of the scale in the manner that it does for the diatonic modes, touching instead on every second scale. We therefore return to the opening scale and mode after repeating the voice-leading 42 (= 6 * 7) times, rather than 84 times as for the diatonic scale. (In 19- or 31-tone equal temperament, on the other hand, the voice-leading would pass

[95] The notation "m:2," meaning "second mode of the melodic-minor scale," is presented in the introduction.

through every transposition of the scale, and return to the opening mode after 133 or 237 iterations, respectively.)

Figure 56. Efficient voice-leading between melodic minor modes.



The remaining quasi-diatonic scales, the harmonic minor and major, combine minor seconds that are separated by four steps on the line of minor seconds. These scales in fact have three minor seconds, as any 7-scale with an augmented second is obliged to (provided the minor second is the smallest second). The structure of harmonic minor modes on the line of minor seconds is given in Figure 57. (The structure of the harmonic major modes on the line is the horizontal reflection of the harmonic minor.) Because in the case of the two minor seconds adjacent to the augmented second, the necessary minor third "frame" is absent, voice-leadings between modes two steps apart on the line (such as Locrian ♮6 and harmonic minor, as illustrated in Figure 57) is not efficient. Instead, there are two, equally efficient, voice-leadings. We can either voice-lead through the harmonic minor/major scales by minor third (as in Figure 58), or by perfect fifth (as in Figure 59).[96] In either case, each voice-leading requires three augmented-second alterations. However, the voice-leadings are different in character. When voice-leading between minor third-related harmonic minor scales, as in Figure 58, all three alterations are in the same direction (i.e., we either raise three pitch-classes or we lower three pitch-classes). The net alteration to the pitch-classes of the scale is therefore ± 3. (Whether this

[96] These figures both assume enharmonic equivalence in 12-tone equal temperament.

net alteration is positive or negative depends on the polarity of the voice-leading.) This means that, once we have returned to a transposition of the same mode as we began with by repeating the voice-leading seven times, the net alteration will be 7 * ± 3 = ± 21. All seven pitch-classes will have been raised or lowered 3 times, meaning the mode will have been transposed by three augmented unisons. (In 12-tone equal temperament, this is enharmonically equivalent to a minor third, but in 19-tone equal temperament it would be equivalent to a major second, and in 31-tone equal temperament to an interval between the major second and the augmented second, 6/31 of an octave.) When voice-leading between fifth-related harmonic minor scales, on the other hand, as in Figure 59, one of the alterations will be in contrary motion to the others, resulting in a net alteration of ± 1. In this case, repeating the voice-leading seven times to return to a transposition of the same mode as we began with will produce a net alteration of 7 * ± 1, meaning the mode will have been transposed by a single augmented unison.

Figure 57. Harmonic minor modes on the line of minor seconds.

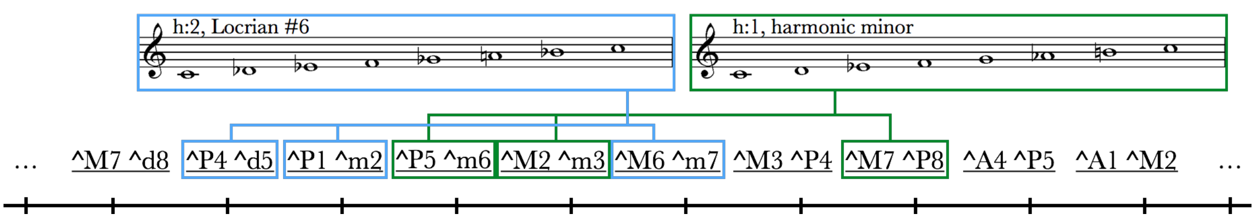


Figure 58. Voice-leading between minor third-related harmonic minor modes.

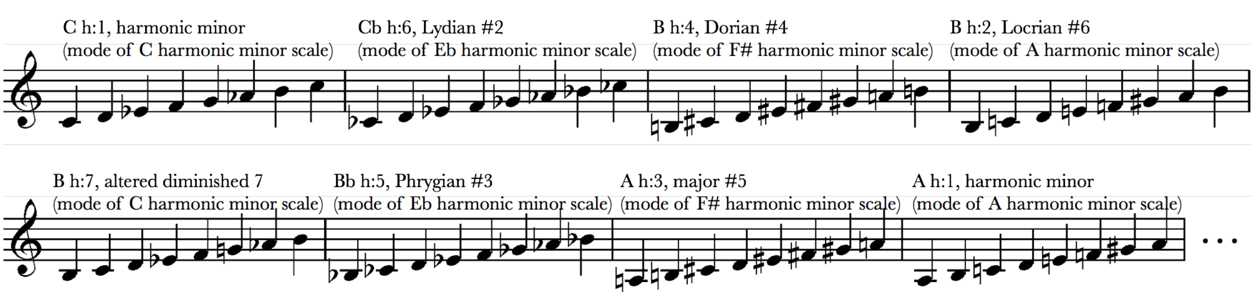
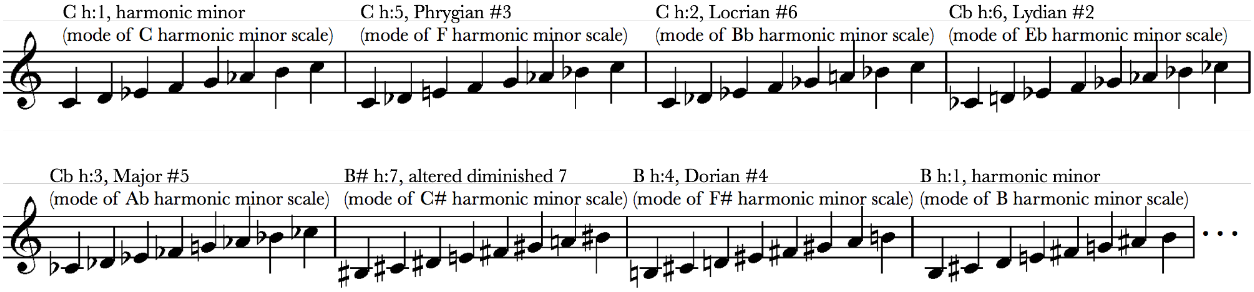


Figure 59. Voice-leading between fifth-related harmonic minor modes.



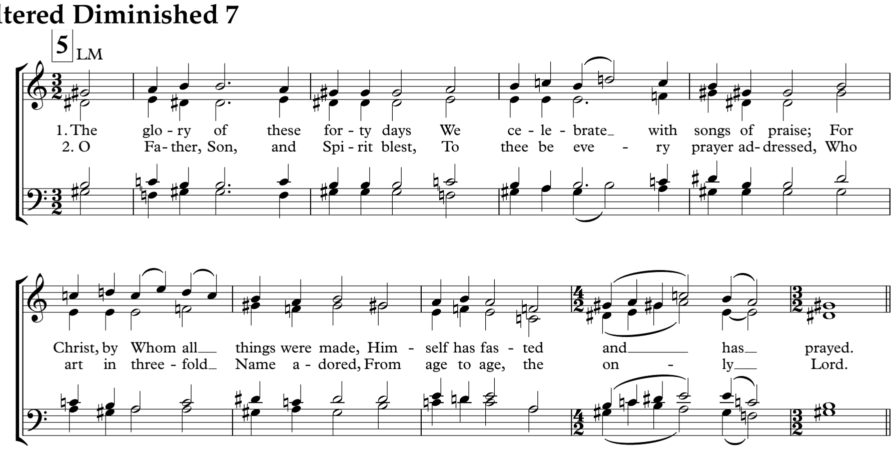
3.5 Appendix: A brief digression concerning modal pitch-class sums

As we saw above, a *modal pitch-class sum* is the sum of the directed intervals from a given modal tonic to each of the other pitch-classes of the scale. Recall from the discussion above that the modal pitch-class sums of the diatonic scale can be arranged from "darkest" to "brightest." Moreover, the sums of adjacent modes in this ordering always differ by an augmented unison (i.e., one semitone in 12-tone equal temperament), as degrees of the scale are raised or lowered to move between neighbouring modes. For example, in the major mode in 12-tone equal temperament, the directed intervals from the tonic to the other pitch-classes are (M2, M3, P4, P5, M6, M7), or (2, 4, 5, 7, 9, 11), which sum to 38; this is the modal pitch-class sum. In the Mixolydian mode, on the other hand, the directed intervals are the same except for the seventh, which is minor rather than major. In other words, the interval of 11 semitones above the tonic is replaced with an interval of 10, and the modal pitch-class sum is correspondingly reduced by one, to 37. In Figure 60, the modal pitch-class sums for all modes of the diatonic scale are given, together with the sums for the other quasi-diatonic modes.[97]

[97] Note that among the modes in this figure, the brightest is H:6, Lydian ♯2 ♯5, and the darkest is h:7, Altered Diminished 7. A remarkable fact about these two modes is that, when they share the same tonic in 12-tone equal temperament, their pitch-class content maximally intersects. For instance, the pitch-classes of C Lydian ♯2 ♯5 are {C, D♯, E, F♯, G♯, A, B} or {0, 3, 4, 6, 8, 9, 11}, and the pitch-classes of C Altered Diminished 7 are {C, D♭, E♭, F♭, G♭, A♭, B♭♭} or {0, 1, 3, 4, 6, 8, 9}. The intersection between these two modes is the six pitch-classes {0, 3, 4, 6, 8, 9}—six being the largest intersection possible between two different 7-chords. If a robust style of music using such scales could be developed, such intersection suggests the possibility of enharmonic "wormholes" between such modes, much like enharmonic dominant

seventh/German sixths in the common-practice repertoire. Moreover, informally speaking, this maximal intersection lends a pleasingly circular nature to modal brightness and darkness. Once we have attained the brightest mode, we are (almost) also at the darkest mode, almost like the usual enharmonic equivalence of the sharpest and flattest scales upon the circle of fifths (although the equivalence here is more approximate). Nevertheless, if we treat these modes in a scalar manner, they are entirely different in affect: not a single scale degree is shared between the two scales (i.e., where one scale has A$\hat{2}$, the other has m$\hat{2}$, etc.). As an illustration of this differing affect, see the two hymn tunes below. (I have had to reach into my own music here because I am not aware of any other music that uses these modes. In either tune, $\hat{5}$ is sometimes altered in the inner parts to create a consonant tonic triad, but the pitch-content otherwise adheres strictly to the given mode. Undoubtedly, these alterations respectively upwards and downwards of $\hat{5}$ magnify the difference between the two modes, but the differing effect persists even when the alterations are removed—for instance, by performing only the outer voices.) We can think of these two pieces as written in subsets of the same 8-chord superset (albeit, at different transpositional levels). The fact that it is possible to write music so different in effect (and affect) with such similar (in pitch-class) materials is suggestive of the psychological reality of 7-scales (a psychological reality most likely inferred from long exposure). I believe that it would be more difficult to achieve the same effect with, say, differing six-note subsets of the diatonic.

Altered Diminished 7



Lydian ♯2 ♯5

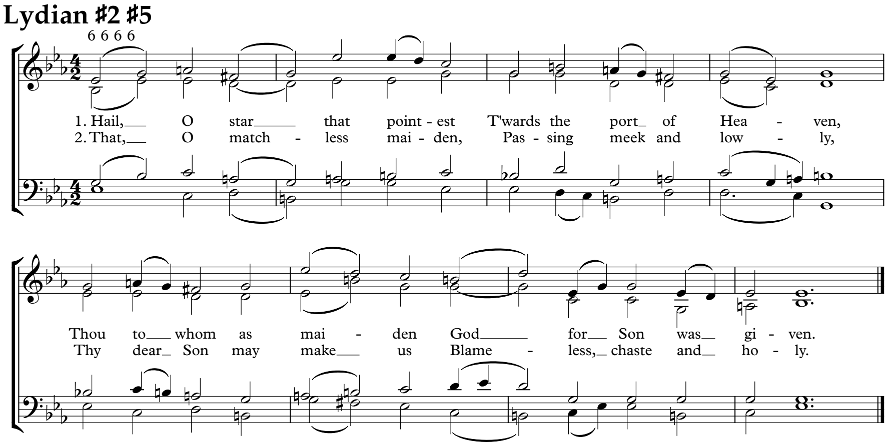
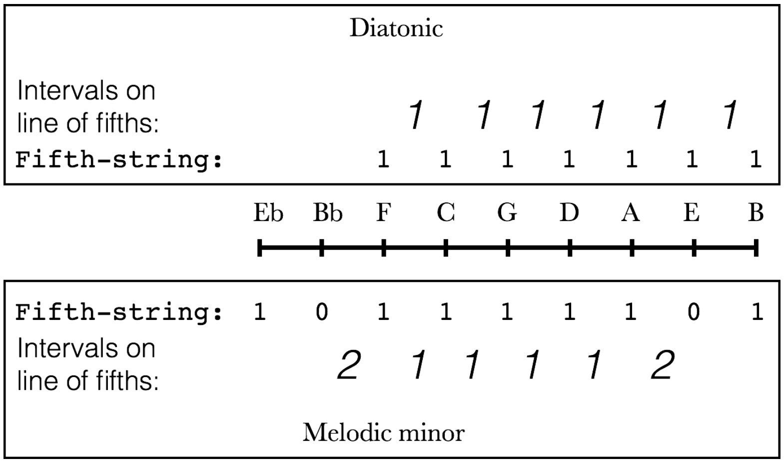


Figure 60. Modal pitch-class sums of the quasi-diatonic scales, in 12-tone equal temperament.

Modal pitch-class sum	Diatonic	Melodic minor	Harmonic major	Harmonic minor
30				
31				h:7, Altered b7
32		m:7, Altered	H:7, Locrian b7	
33	M:7, Locrian		H:3, Phrygian b4	
34	M:3, Phrygian	m:6, Locrian #2		h:2, Locrian #6
35	M:6, Minor	m:3, Phrygian #6	H:2, Dorian b5	h:5, Phrygian #3
36	M:2, Dorian	m:5, Mixolydian b6	H:5, Mixolydian b2	h:1, Harmonic Minor
37	M:5, Mixolydian	m:1, Melodic Minor	H:1, Harmonic Major	h:4, Dorian #4
38	M:1, Major	m:4, Lydian b7	H:4, Lydian b3	
39	M:4, Lydian			h:3, Major #5
40		m:3, Lydian #5		h:6, Lydian #2
41			H:6, Lydian #2 #5	
42				

It can be readily observed from Figure 60 that the sequence of differences between the adjacent sums of the diatonic scale is (1, 1, 1, 1, 1, 1) (i.e., because 34 – 33 = 1, 35 – 34 = 1, and so on). This follows from the just-noted fact that these differences will be equal to the augmented unison, one semitone. The sequence of differences for the melodic-minor scale, on the other hand, is (2, 1, 1, 1, 1, 2) (because 34 – 32 = 2, 35 – 34 = 1, and so on). As illustrated in Figure 61, these two sequences are respectively identical to the intervals between pitch-classes in each scale's fifth-string, when the scale is generated upon the line of fifths.

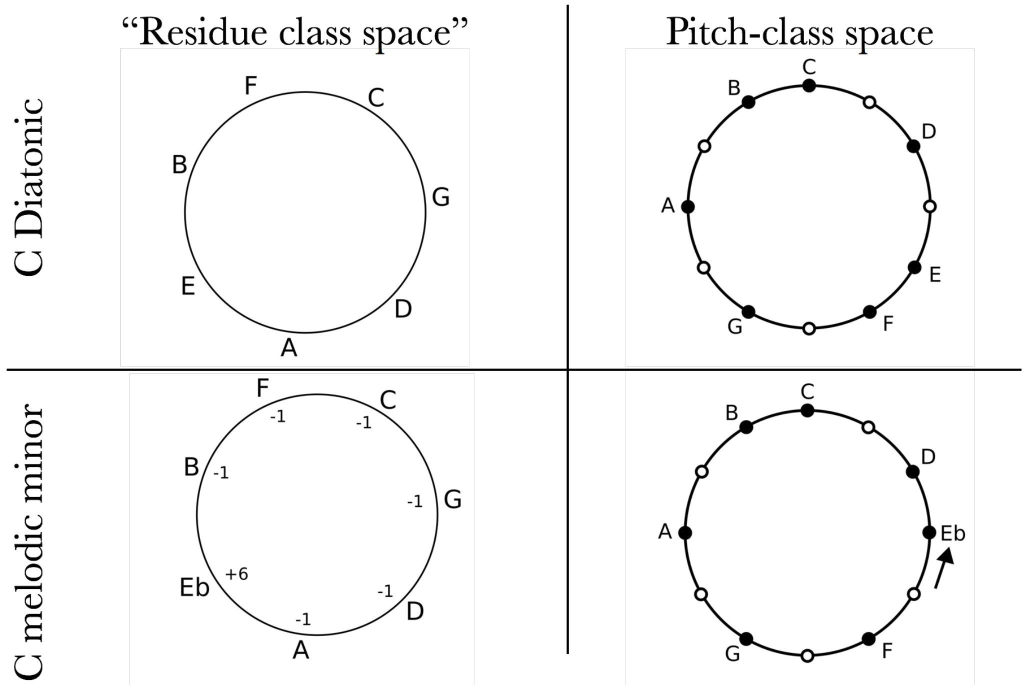
Figure 61. The diatonic and melodic-minor scales on line of fifths.



This might appear to be nothing more than a curious coincidence, but it is in fact a consequence of a deeper connection: the description of *any* scale as intervals upon the line of fifths can always be mapped to the sequence of differences of its modal pitch-class sums. To see why this is so, consider Figure 62. If we start from the pitch-class sums of the modes of C diatonic, when we move to C melodic minor, E will decrease by an augmented unison to E-flat. This means that the directed interval from E(-flat) to each of the other six pitch-classes of the scale will increase by an augmented unison; the effect is to increase the modal pitch-class sum of the E mode by 6 semitones. At the same time, the directed interval from each of the other six pitch-classes to E(-flat) will decrease. This decreases the modal pitch-class sum of each of the other modes by an augmented unison, or 1 semitone. The net effect is to shift the pitch-class sum of the E mode by 7 semitones relative to each of the pitch-class sums of the other six modes. Moreover, *any* alteration we might make to the scale will have a similar effect: for instance, lowering A of the C diatonic scale to A-flat would shift the pitch-class sum of the A mode by 7 semitones relative to the others (consider the harmonic major scale in Figure 60). Since *any* proper spelling of a 7-scale can always be thought of as a diatonic scale where the notes have been raised and lowered by the relevant accidentals, it follows that the differences between the modal pitch-class sums of any such scale can always be reconstructed in this way—that is, by shifting the pitch-class sums of the modes on any inflected scale degrees by 7 semitones relative to the others (positive, if inflected by a flat; negative, if inflected by a sharp). "Shifting by seven relative to the other pitch-classes" is not only the effect upon the pitch-class sums, however: it is also the definition of the action of an accidental on the line of fifths. This is the reason for the equivalence of the description of a scale as a sequence of modal pitch-class sum differences (e.g., (2, 1, 1, 1, 1, 2) for the melodic-minor scale) and the description of that scale as a sequence of intervals upon the line of fifths (e.g., (2, 1, 1, 1, 1, 2)). We must qualify this equivalence somewhat, however: both the diatonic and melodic-minor scales are inversionally symmetrical, but in an inversionally asymmetrical scale such as the harmonic major or minor scales, the relationship between the modal pitch-class sum differences (e.g., (1, 2, 1, 1, 1, 3) for the harmonic major scale in Figure 60) and the line of fifths intervals(e.g., (3, 1, 1, 1, 2, 1)) will be retrograde. This retroversion occurs because flats move modal pitch-class sums in the positive direction, but pitch-classes on the line of fifths in the negative direction. In a very real sense, however, these directions are purely arbitrary: we could equally well use a line of fourths, rather than a line of fifths, or calculate modal pitch-class sums as directed intervals from the other pitch-classes of the scale to the modal tonic (rather than from the modal tonic to the other pitch-classes of the scale).[98] Either of these arbitrary choices would result in an identity between the two sequences, rather than a retroversion.

[98] Calculating the pitch-class sums as descending, rather than ascending, directed intervals would have the same effect.

Figure 62. Voice-leading from C diatonic to C melodic minor.



The directed interval from E(-flat) to all other pitch-classes increases by 1, increasing the pitch-class sum of the E mode by 6
The directed interval from all other pitch-classes to E(-flat) decreases by 1, decreasing the pitch-class sum of all other modes by 1

Strictly speaking, this mapping holds between the fifth-string and the modal pitch-class sum differences measured in *augmented unisons*. In equal temperaments, these augmented unisons will be equal to some number of steps in the chromatic scale of the temperament. Because in 12-tone equal temperament the augmented unison is equal to one chromatic semitone, the fifth intervals and pitch-class sum differences measured in semitones are simply identical, as we saw above. In temperaments where the augmented unison is equal to some other number of chromatic steps, however, the pitch-class sum differences will be scaled by this number of steps. In 31-tone equal temperament, for example, the augmented unison is equal to two chromatic steps, and the pitch-class sum differences of the melodic-minor scale are not (2, 1, 1, 1, 1, 2), but instead (4, 2, 2, 2, 2, 4). (In Figure 63, the modal pitch-class sums of the quasi-diatonic scales in 31-tone equal temperament are shown.) In passing, in 31-tone equal temperament, a curious consequence of the fact that the modal pitch-class sums of any diatonic mode are odd and the augmented unison is two chromatic steps is that it is impossible in this temperament to construct a nearly well-formed mode with an even pitch-class sum. (This is why even numbers are omitted from the first column in Figure 63.)

Figure 63. Modal pitch-class sums of the quasi-diatonic scales, in 31-tone equal temperament.

Modal pitch-class sum	Diatonic	Melodic minor	Harmonic major	Harmonic minor
81				
83				h:7, Altered b7
85		m:7, Altered	H:7, Locrian b7	
87	M:7, Locrian		H:3, Phrygian b4	
89	M:3, Phrygian	m:6, Locrian #2		h:2, Locrian #6
91	M:6, Minor	m:3, Phrygian #6	H:2, Dorian b5	h:5, Phrygian #3
93	M:2, Dorian	m:5, Mixolydian b6	H:5, Mixolydian b2	h:1, Harmonic Minor
95	M:5, Mixolydian	m:1, Melodic Minor	H:1, Harmonic Major	h:4, Dorian #4
97	M:1, Major	m:4, Lydian b7	H:4, Lydian b3	
99	M:4, Lydian			h:3, Major #5
101		m:3, Lydian #5		h:6, Lydian #2
103			H:6, Lydian #2 #5	
105				

Chapter 4 Harmonic Function in Quasi-diatonic Music

The relationship of harmonic functions to scale can be conceived as going from function to scale, or from scale to function, or in both directions at once. For instance, beginning with the pitch-classes of the major mode of the diatonic scale, we can organize them into the standard set of tonal functions—i.e., tonic, dominant, and subdominant. Or, inversely, from the union of these three functions, we can derive the diatonic scale. (In fact, as we saw briefly in Chapter 2, the quasi-diatonic scales can be defined as the set of scales that can be formed out of a combination of three fifth-related functions [i.e., triads] of any combination of major or minor quality—see Meredith [2011, 167].) Since functions and scale mutually imply one another, at least in the case of traditional tonal materials, to which of them we give priority depends largely on our theoretical aims. In this chapter we will proceed in either direction, considering what sort of functions are permitted by various scales, as well as what sort of scales are formed by various combinations of functions, and what musical possibilities these various options suggest.

4.1 Function structures

Interpreted as a means of partitioning a mode into a few fundamental chord types, the standard tonal functions have a few apposite features. For instance, they are *complete*: between the tonic, dominant, and subdominant triads, every pitch-class in the diatonic scale is accounted for. (Other sets of three triads can be incomplete, such as the union of tonic, dominant, and submediant triads, which omits $\hat{4}$.) Moreover, the standard functions are *minimally complete*: they completely account for the seven pitch-classes of the diatonic scale with the minimum number of triadic functions. (In a seven pitch-class scale, three is the minimum number of 3-chords whose union can form the complete collection.)

If we presume that our functions will be tertian 3-chords then, under rotational equivalence, there are two basic classes of structures where three functions can account for all the pitch-classes of a 7-scale. Using zero-origin integers to represent the scale degree at the root of each function, these two classes are {0, 1, 2} and {0, 1, 4}. (An example of the former is {I, II, III}, and an example of the latter is {I, II, V}.) To see why these are the only two function-structure classes, suppose we begin with a tonic triad, I, in an arbitrary scale. (Functions will be indicated with capital roman numerals. This choice of case is not meant to convey anything about the quality of the chords, which is undetermined, since the choice of scale is arbitrary.) This tonic triad accounts for three scale degrees, $\hat{1}$, $\hat{3}$, and $\hat{5}$. For our next function, we might choose any of the remaining triads of the scale:

- If we choose II, we have accounted for scale degrees $\hat{2}$, $\hat{4}$, and $\hat{6}$, together with $\hat{1}$, $\hat{3}$, and $\hat{5}$ already accounted for by I. Only one scale degree now remains unaccounted for. For our third function, we must then choose one of the three triads—namely III, V, and VII—that contain $\hat{7}$:
    - If we choose III, then our three functions are {I, II, III}. These three triads form a {0, 1, 2} function structure.
    - If we choose V, then our three functions are {I, II, V}. These three triads form a {0, 1, 4} function structure.
    - If we choose VII, then our three functions are {I, II, VII}. Like {I, II, III}, these three triads form a {0, 1, 2} function structure, albeit rotated so that the tonic is the second member of the structure.
- If we choose III, we have accounted for scale degrees $\hat{1}$, $\hat{3}$, $\hat{5}$, and $\hat{7}$. Three scale degrees therefore remain unaccounted for: $\hat{2}$, $\hat{4}$, and $\hat{6}$. There is only one triad that contains these three scale degrees, namely II. Our third function must therefore be II, giving the three functions {I, II, III} already seen above, a {0, 1, 2} function structure.

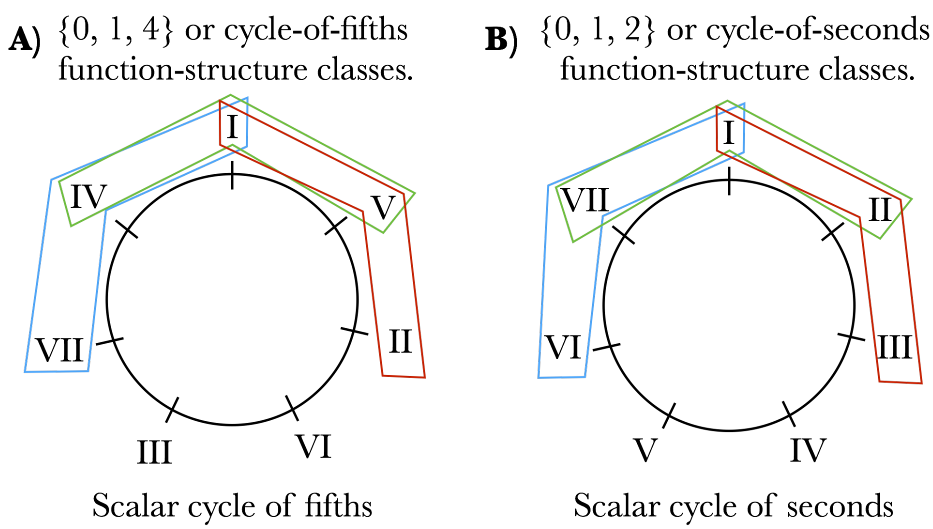
- If we choose IV, we have accounted for scale degrees $\hat{1}$, $\hat{3}$, $\hat{5}$, $\hat{4}$ and $\hat{6}$, and scale degrees $\hat{2}$ and $\hat{7}$ remain unaccounted for. Two triads, V and VII, contain both of these degrees. As our third function, we must choose one of these two triads:
    - If we choose V, yielding {I, IV, V}, we have constructed a rotation of a {0, 1, 4} function structure.
    - If we choose VII, yielding {I, IV, VII}, we have again constructed a rotation of a {0, 1, 4} function structure.
- If we choose any of the remaining three triads V, VI, and VII, then similar reasoning will apply, in inversion, as for the preceding three triads. Because the two function structures {0, 1, 2} and {0, 1, 4} are each inversionally symmetrical, they are therefore the only possible complete function structures in a 7-scale.

Both the {0, 1, 2} and {0, 1, 4} function-structure classes are segments of diatonic interval cycles. In the case of {0, 1, 2}, the cycle is a cycle of seconds, while in the case of {0, 1, 4}, it is a cycle of fifths. (This may be somewhat more obvious if we re-order the latter structure as {0, 4, 1}.) For this reason, I will call these structures by the more mnemonic names *cycle-of-seconds function-structure class* and *cycle-of-fifths function-structure class*, respectively. For example, as illustrated in Figure 64A, the three members of the cycle-of-fifths function-structure class correspond to the three successive slices of the scalar cycle of fifths that include the tonic triad. The root of the tonic triad can be:

- The first pitch-class in the cycle of fifths, giving {I, II, V}.
- The second pitch-class in the cycle of fifths, giving {I, IV, V}.
- The third pitch-class in the cycle of fifths, giving {I, IV, VII}.

It is interesting to note that, if we posit three fifth-related “primary triads” of similar quality (i.e., all major or all minor), the three function structures in Figure 64A allow us to construct all and only the stable diatonic modes. Fifth-related major triads on {I, IV, VII} construct the Mixolydian mode, on {I, IV, V} the major mode, and on {I, V, II} the Lydian mode. Minor triads, on the other hand, would construct the Phrygian, natural minor, and Dorian modes, respectively.[99] Because Mixolydian and Phrygian are the darkest (or flattest) of these three modes, Lydian and Dorian the brightest, and major and natural minor neutral, we might consider {I, IV, VII} to be the "flat" member of the function-structure class, {I, IV, V} to be the "neutral" member, and {I, II, V} to be the "sharp" member.

Figure 64. Realizations of the two complete function-structure classes.



In sum, three positions for the tonic triad in the two possible function-structure classes give six possible minimally complete function structures, as shown in musical notation in Figure 65. (These function structures are abstract, meant to apply to an arbitrary mode, hence the absence of clef from this figure.) One virtue of the completeness of these function structures is that it permits every scale-degree of the mode to be the

[99] We might like to follow a similar procedure for the seconds-cycle function-structure class, but in fact it is not possible to construct a scale where three consonant triads in a {0, 1, 2} configuration all share the same quality. Indeed, this is not possible for *any* set of consonant triads where at least two of the triads are related by scalar third. This follows from the fact that, in either type of consonant triad, the lower third differs in quality from the upper third (e.g., a major triad is constructed from a major third followed by a minor third), while between two third-related scalar triads, the upper third of the lower triad will necessarily be the lower third of the upper triad. If the lower triad is a consonant triad, its lower third will differ from its upper third, and therefore from the lower third of the upper triad. Similarly, if the upper triad is a consonant triad, its upper third will differ from its lower third, and therefore from the upper third of the lower triad.

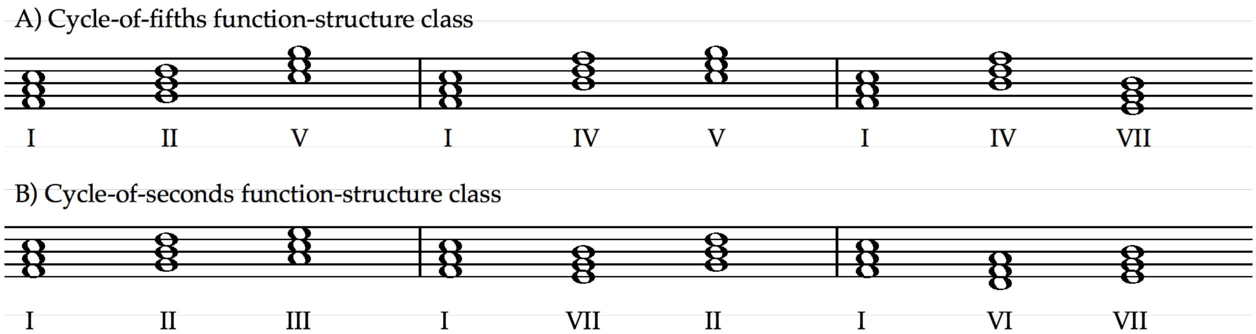
representative of at least one function. Therefore, every scale degree, should it occur in the melody, can carry functional weight. As a somewhat coarse illustration of this fact, the web apps associated with this chapter (see section 4.4 below) use these function structures in a simple harmonization heuristic, where every note of a melody is assigned a harmony automatically according to its functional membership.[100] (It would be interesting to examine to what extent composers' actual behavior in harmonizing melodies can be captured by a somewhat more sophisticated version of such a heuristic.)

In practice, our choice among these structures is constrained by the interaction of the norms of a given tonal style and the structure of the chosen mode. For instance, in common-practice style, {I, IV, VII} would not be a plausible choice of function structure for the major mode because of the diminished triad upon VII. On the other hand, note that, while common-practice music unquestionably features I and V as functions, the two other members of the cycle-of-fifths function-structure class—{I, II, V} and {I, IV, V}—capture between them some ambiguity as to the identity of the third function. (We will return to this observation below.)

[100] The exact functioning of the heuristic is as follows:
- $\hat{1}$, $\hat{3}$, and $\hat{5}$ are always harmonized by the tonic triad.
- All other scale degrees are harmonized according to the function that contains them (e.g., $\hat{6}$ is harmonized with IV, if IV is present).
- If a scale degree is present in more than one non-tonic function, it is randomly assigned to one or the other function. This only occurs with the {I, II, VII} function structure, where $\hat{4}$ and $\hat{2}$ may be randomly assigned to either II and VII.

The heuristic makes no attempt to avoid parallel fifths or octaves, or to take any other considerations of voice-leading into account.

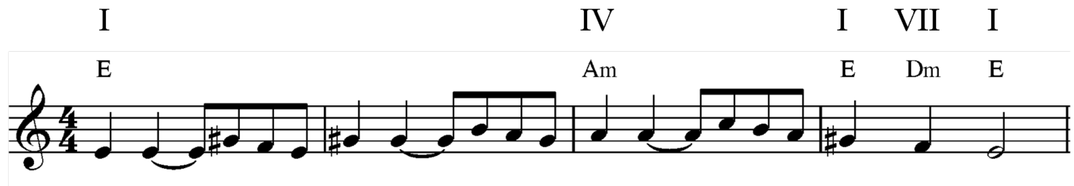
Figure 65. Complete function structures in staff notation. (Note the absence of clef—the pitch-class level is arbitrary.)



Moreover, in other musical styles, {I, IV, VII} *is* a plausible function structure in certain modes. For instance, in Klezmer music, the "Freygish" mode (i.e., h:5, which would more typically be called Phrygian ♯3 in this thesis) is often accompanied with these functions. The best-known example of these functions in this mode would surely be the typical harmonization of *Hava Nagila* given in Figure 66A. Similarly, a great deal of Mixolydian pop music relies on these three functions. As an example of the type, consider the rhythm-section vamp from Kool & the Gang's *Celebration,* in Figure 66B. This sort of use of these three functions serves two musical purposes: 1) favouring the primary triads of the Mixolydian mode and 2) favouring the subdominant-side harmonic progressions characteristic of a great deal of pop music.

Figure 66. {I, IV, VII} function structures in two well-known masterworks.[101]

A) *Hava Nagila*, opening with typical harmonization.



B) Kool & the Gang, *Celebration*. Rhythm section vamp.

It is less apparent that any members of the cycle-of-seconds function-structure class can be musically useful. I am not aware that of any music that convincingly presents any type of cycle-of-seconds function structure. One reason for this lack of usefulness may be that the two third-related chords maximally intersect (two 3-chords can share at most 2 pitch-classes without being identical), which tends to make them sound relatively undifferentiated. This is much like the ambiguity between II and IV already noted in common practice harmony.

Observing this ambiguity between II and IV—the fact that these chords seem to be able to operate as functional synonyms for one another[102]—and extrapolating this potential synonymy to other third-related triads, we might posit a *functional substitution*

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[101] The Kool & the Gang vamp, out of context, is liable to sound like D-flat major. A-flat Mixolydian is firmly established by the surrounding music, however, as the reader can hear for herself at https://www.youtube.com/watch?v=3GwjfUFyY6M. Another famous example of the use of these three chords in a pop tune context is the coda from the Beatles' *Hey Jude*.

[102] It is interesting to note that the ambiguity between II and IV has been shown to be psychologically real: as Patel (2007, 261, citing Hutchins 2003) has written, "listeners can easily distinguish a IV chord from a II^6 chord in isolation. Yet the same listeners find it very difficult to distinguish a I-IV-V chord sequence from a I-II^6-V sequence." This is presumably because, in a tonal context, both chords play a very similar role and are therefore assimilated to the same pre-dominant (or subdominant) function.

operation that permits third-related triads to function as substitutes for one another. This operation permits us to take the segment of the diatonic cycle of fifths from IV to II and curl it into a single circle of three chords (see Figure 67A).[103] Although this particular formulation of this procedure may be new, the idea behind it has a long and hallowed pedigree: it is simply Rameau's *double emploi*. We can generalize *double emploi* from the specific case of IV and II as follows: a procedure that, by functional substitution of third-related chords, allows a small number of functions to behave "as if" they formed a complete diatonic interval cycle.

This functional substitution also permits other function structures as well, such as that in Figure 67B, where V and VII occur as functional synonyms—and here we might take a moment to recall that VII is very often interpreted as a representative of dominant function. The operation of both the IV/II and V/VII functional synonyms at once would allow us to curl the cycle of fifths from VII to II into a functional structure, as in Figure 67C.[104] Treating III and VI as substitutes for I allows us to take the entire diatonic fifths-cycle of seven roots, cut it between III and VI, and curl it into a more tightly wound cycle of three groups of functional synonyms, as in Figure 68. (The resulting structure is much like that permitted by Riemann's parallel transformation.)

[103] The functional substitution of third-related chords proposed here resembles an equivalence relation, but it is not in fact such a relation, because it is not transitive (otherwise V could substitute for I, since III can substitute for I and V can substitute for III). It is not necessarily symmetric either, since, while we may well wish to allow IV to substitute for II and vice versa, it is less clear that (say) if we wish to allow III to substitute for I, we should also wish to allow I to substitute for III.

[104] Proceeding in a similar way, it is also possible to produce I ≅ VI, II, and V and I ≅ III, IV, and VII function structures. The common-practice antecedents of these are less apparent, which is not to say that they may not be musically useful.

Figure 67. Function substitution in cycle-of-fifths function structures 1.

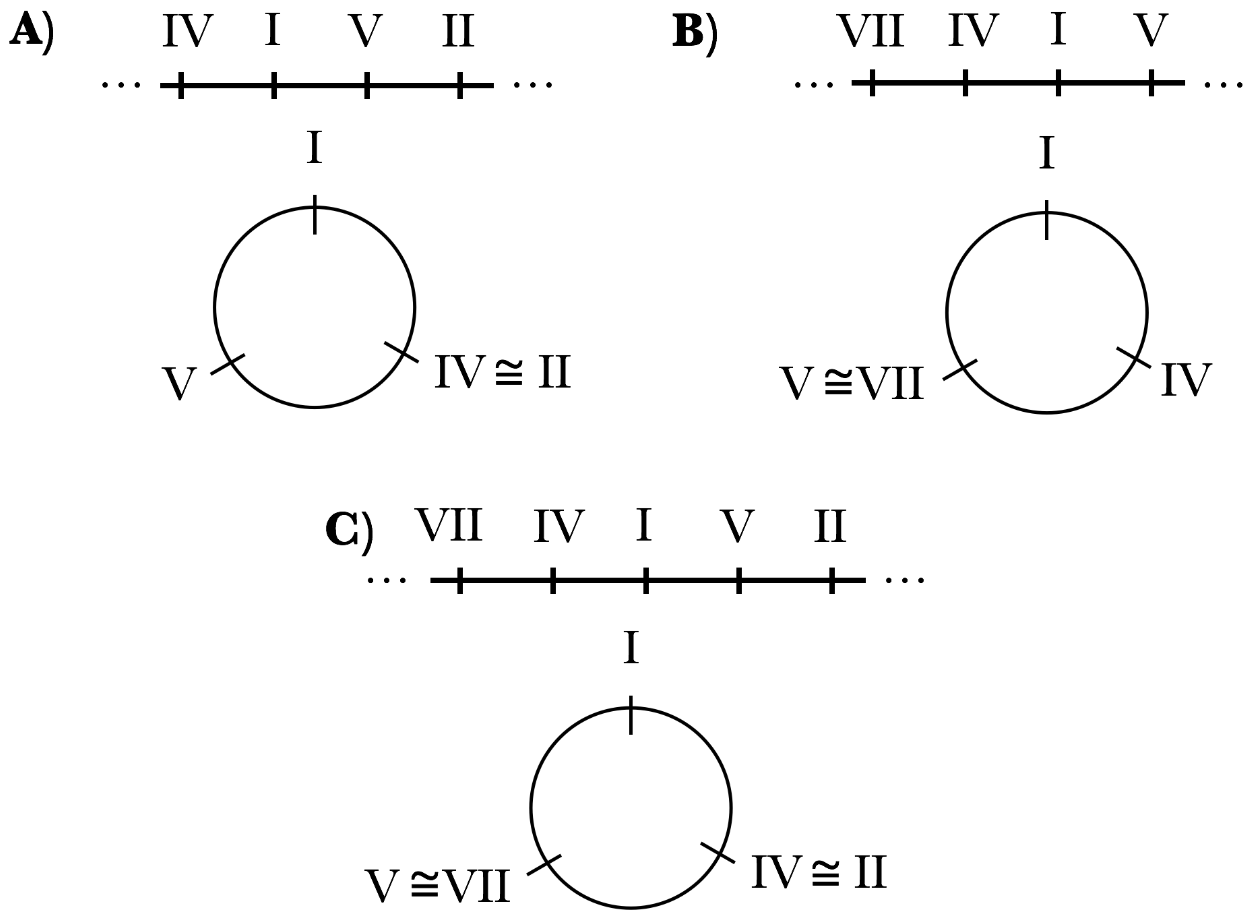
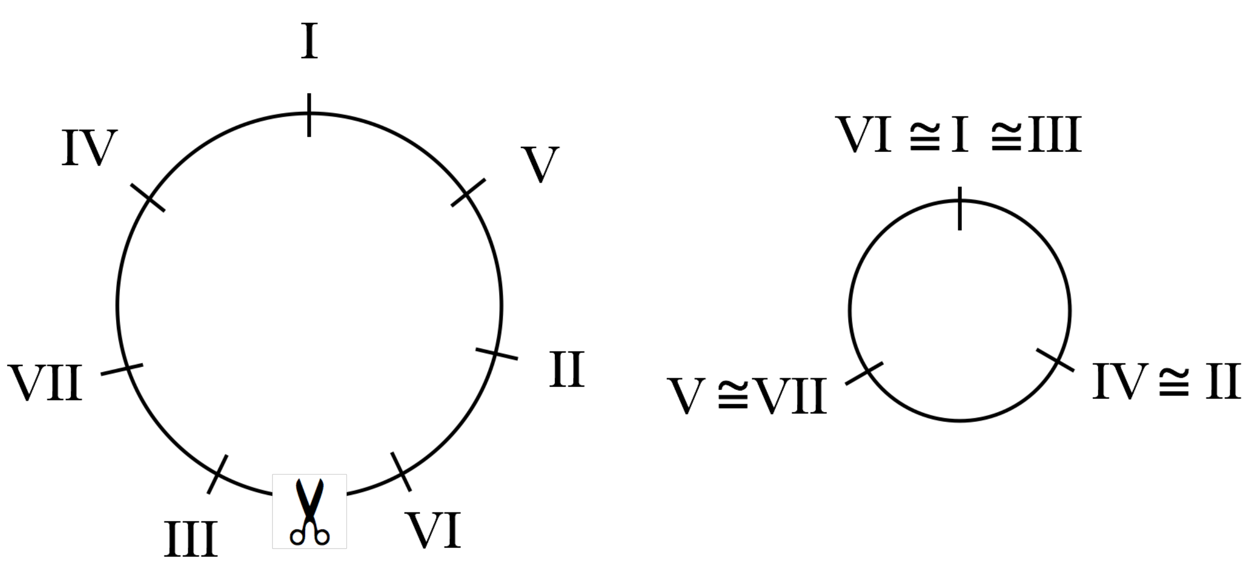


Figure 68. Function substitution in cycle-of-fifths function structures 2.



4.2 Synonym-cycles

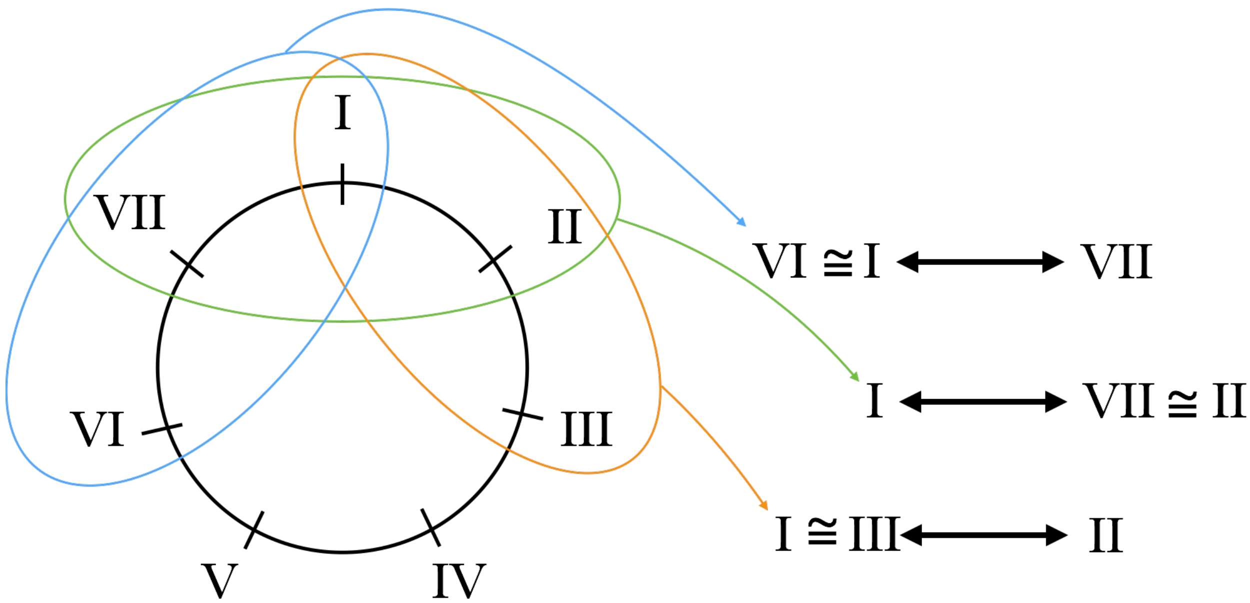
Our discussion of function structures began with the requirement that function structures be complete. As noted above, this has the virtue of permitting every scale-degree to carry functional weight. Nevertheless, we might wonder whether it is really necessary that every scale degree have a harmonization that occupies the rank of a preeminent function in the tonal dynamics of the mode. If we wish to avoid beginning from the requirement of complete function structures, functional substitution of third-related chords allows an alternative strategy for defining function structures, a strategy that starts from different assumptions but gives similar results. In this strategy, we take it as given that third-related chords are functional synonyms, due to their maximal endowment with common tones. We therefore define a function structure as a segment of a scalar interval cycle that is delimited by such functional synonyms. Using a scalar cycle of fifths, for instance, we might begin on IV and continue through I and then V before arriving at II, which, as a third-related triad, is a synonym for IV. This gives the {I, IV ≅ II, V} function structure already given above in Figure 67A. Or we might begin on VII, producing a {I, IV, VII ≅ V} function structure seen in Figure 67B. Because these structures form cycles delimited by functional synonyms, I will call them *synonym cycles*.

In constructing function structures from synonym cycles, the use of a scalar interval cycle may appear somewhat arbitrary. We can justify this use, however, by appealing to the fact that a style of music is likely to suggest certain intervallic norms for chord succession (e.g., succession by fifth). Within such a style, the structural harmonies of the music may function best if related by such an interval. In common-practice music, for instance, root motion tends to proceed by fifth. It is Rameau's observation of this fact that informs his assertion of the *double emploi*.

In the initial discussion of function structures, cycle-of-thirds function structures were excluded because they cannot be minimally complete. If we accept synonym cycles as a plausible means of constructing function structures, they suggest a new reason that scalar cycles of thirds may be less than ideal as function structures. Namely, because third-related triads are functional synonyms, a synonym-cycle function structure built from a cycle of thirds would consist of only two, synonymous chords, either I ≅ VI or I ≅ III. In other words, the seemingly minimal requirement for a meaningful functional syntax of chord progressions—that there be both a tonic chord and a non-tonic chord—cannot be fulfilled by a synonym cycle-of-thirds.

Cycles of seconds are more promising than cycles of thirds for constructing synonym-cycle function structures. Moreover, the notion of functional synonyms provides a musically useful manner of conceiving of the minimally complete cycle-of-seconds function-structure class introduced above. By recognizing functional synonyms, we effectively parse these function structures into binary systems (i.e., {I ≅ III, II} rather than {I, II, III}, {I, II ≅ VII} rather than {I, II, VII}, and {I ≅ VI, VII} rather than {I, VI, VII}). These binary systems consist either of 1) a tonic chord and its substitute, together with a non-tonic chord (as in the case of {I ≅ III, II} or {VI ≅ I, VII}), or 2) a tonic chord, together with two non-tonic chords that can substitute for each other (in the case of {I, VII ≅ II}). These possibilities are illustrated in Figure 69.

Figure 69. Synonym cycle-of-seconds function structures.



4.3 Function structures as realized in actual scales

We will now consider how the abstract function structures discussed above may be realized in actual scales. (In this section, we revert to the minimally complete function-structure classes introduced in Section 4.1.) We will limit our investigations to those cases where the functions are consonant triads,[105] and begin with the cycle-of-fifths (i.e., {0, 1, 4}) function-structure class. Because the cycle of fifths is scalar, it might initially appear that function structures with chord roots related by diminished and augmented fifths are possible in addition to those related by perfect fifth. But any two consonant triads belonging to the same scale and related by fifth must be related by perfect fifth,[106] and scales realizing a cycle-of-fifths function structure of consonant triads must therefore have functions related by perfect fifths. As we have already seen in chapter one, the set of scales that can be formed from the combination of three consonant triads a perfect fifth

[105] To be precise, where the functions are regular tertian consonant triads (i.e., root position consonant triads formed out of consecutive thirds within the scale).

[106] This is simply because the fifth of the first chord must be the root of the second, and the fifth of the first chord is by definition a perfect fifth.

apart is the quasi-diatonic scales (Meredith 2011, 167). The scales and associated modes that can be produced in this way are listed in Figure 70. (Schenker (1954, 87) was perhaps the first to follow a similar procedure, by creating modes through systematically combining minor and major third, sixth, and seventh degrees. Since these degrees are the thirds of the I, IV, and V chords, the effect is to construct the modes with {I, IV, V} function structure for every combination of major and minor triads.) Moreover, every stable mode of the quasi-diatonic scales is found at least once in Figure 70, with the exception of h:6, Lydian ♯2, and H:3, Phrygian b4. This follows from the fact that all the consonant triads of the quasi-diatonic modes are related to each other by fifth, with the exception of the triads at the tonic of the inversionally related modes h:6 and H:3.

Figure 70. Scales and modes associated with cycle-of-fifths function structures of consonant triads.

		Function structure		
Triad qualities	**Scale type**	**I V II**	**IV I V**	**VII IV I**
M M M	Diatonic	M:4, Lydian	M:1, major	M:5, Mixolydian
M M m	Diatonic	M:1, major	M:5, Mixolydian	M:2, Dorian
M m M	Melodic minor	m:4, Lydian b7	m:1, melodic minor	m:5, Mixolydian b6
M m m	Diatonic	M:5, mixolydian	M:2, Dorian	M:6, natural minor
m M M	Harmonic major	H:4, Lydian b3	H:1, harmonic major	H:5, Mixolydian b2
m m M	Harmonic minor	h:4, Dorian #4	h:1, harmonic minor	h:5, Phrygian #3
m M m	Melodic minor	m:1, melodic minor	m:5, Mixolydian b6	m:2, Phrygian #6
m m m	Diatonic	M:2, Dorian	M:6, natural minor	M:3, Phrygian

In the case of consonant triads combining in cycle-of-seconds (i.e., {0, 1, 2}) function structures, the two third-related triads must be of differing quality.[107] If the lower triad is major, the upper triad will be minor, and if the lower triad is minor, the upper triad will be major. Either way, the union of the two chords can be seen as forming a seventh-chord

[107] The reason that they must be of different quality is explained in footnote 99.

superstructure, and the pitch-classes of the remaining chord must each lie within the thirds of this superstructure. The various possibilities for doing so are listed in Figure 71. (Rather than devise some sort of novel notation for conveying chord quality and root interval without reference to any specific pitch-class level, I have simply listed the various forms as realized with the root of the lowest triad C.) Because the diatonic, harmonic major and minor, and Hungarian minor are the only scales containing three or more adjacent scalar consonant triads, every mode in the table belongs to one of these four scales. In fact, the table contains every stable mode in each of these scales, with the exception of h:1, harmonic minor, and H:1, harmonic major. This follows from the fact that all the consonant triads of these four scales are related to each other by second, with the exception of the triads at the tonic of the modes h:1 and H:1.

Figure 71. Scales and modes associated with cycle-of-seconds function structures of consonant triads.

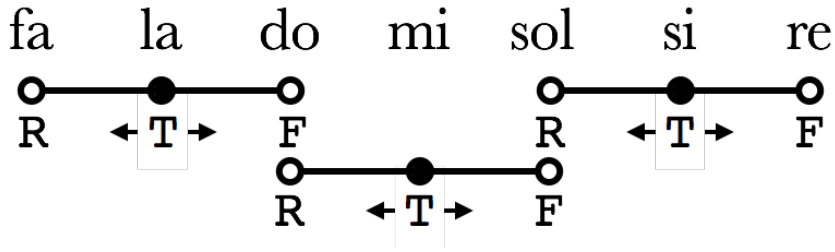
Third-related triads	Triads		Function Structure		
		Scale Type	I II III	VII I II	VI VII I
C and E-	C Db E-	Hungarian minor	Hungarian minor:5	Hungarian minor:6	Hungarian minor:7
	C D- E-	Diatonic	M:1, major	M:2, Dorian	M:3, Phrygian
	C D E-	Diatonic	M:4, Lydian	M:5, Mixolydian	M:6, natural minor
	C D#- E-	Hungarian minor	Hungarian minor:6	Hungarian minor:7	Hungarian minor:1
C- and Eb	C- Db- Eb	Harmonic major	H:3, Phrygian b4	H:4, Lydian b3	H:5, Mixolydian b2
	C- Db Eb	Diatonic	M:3, Phrygian	M:4, Lydian	M:5, Mixolydian
	C- D- Eb	Diatonic	M:2, Dorian	M:3, Phrygian	M:4, Lydian
	C- D Eb	Harmonic minor	h:4, Dorian #4	h:5, Phrygian #3	h:6, Lydian #2

Daniel Harrison's *Harmonic Function in Chromatic Music* (1994) expounds a theory of how the various scale degrees contained in a function participate in the expression of that function—or, put differently, the functional role played by each of the constituent chord factors of a function. (As part of the exposition of this theory he develops a novel nomenclature for each chord factor—"base", "agent", and "associate" rather than root, third and fifth. These terms are not without a certain logic and, indeed, panache. Nevertheless, I find myself internally translating them into the familiar numerical chord factors overlearned through decades of musical experience. I suspect I am not the only reader to do so, and so I will use the more familiar chord factors in the present thesis.) Harrison's "renewed dualist" theory considers only the standard complement of tonic, subdominant, and dominant functions, but the functional roles that he attributes to the different chord factors follow from the structure of the cycle-of-fifths function-structure class more broadly. Having introduced the scales and modes of this function-structure class,[108] it is now useful to consider these functional roles and how their expression might shape the usefulness and musical character of each mode.

Figure 72 will be helpful in considering the chord factor structure of the cycle-of-fifths function-structure class. Because any one of the three triads can be the tonic function, I have chosen to label the nodes not with scale degree numbers (e.g., $\hat{1}$ and $\hat{3}$) but instead with solfège syllables. Any of *fa*, *do*, or *sol* can be the tonic pitch-class, depending on the choice of function structure. In the following discussion, I will refer to the flatmost function as the "*fa* function," and the other functions as the "*do* function" and "*sol* function," respectively. (Readers familiar with moveable *do* systems of solmization are asked to temporarily suspend the association between *do* and tonic.) Whether the *fa* function is I, IV, or VII, will depend on the particular function structure (i.e., whether {I, II, V}, {I, IV, V}, or {I, IV, VII}), and the *do* and *sol* functions vary in a similar way.

[108] In fact, we have only introduced the scales and modes where all three functions are consonant triads.

Figure 72. Arrangement of chord factors in a cycle-of-fifths function structure.



Harrison (1994, 49) has observed that thirds are the only chord factor always "attached to one function and one function only" and therefore "unequivocal communicators of their function." From Figure 72, it can be readily observed that this fact follows immediately from the structure of the cycle-of-fifths function-structure class. In contrast, roots and fifths are most often shared between two functions, but the root of the *fa* function and the fifth of the *sol* function also belong uniquely to a single function. Harrison argues that "a kind of historical oppression of Subdominant function has militated against" the root of the *fa* function being an effective communicator of subdominant function (1994, 48-49), especially because of the ubiquity of the dominant seventh chord. (Consider the first pitch of the repetition of the basic idea at the opening of *Eine Kleine Nachtmusik*—does this pitch have even a hint of "subdominant-ness"?) The situation is similar with the fifth of the *sol* function, so often conjoined to the subdominant in a ii6/5 formation.[109] In other styles of music, however, it is possible that these chord factors may behave as effective communicators of their function. For instance, in a {I, IV, VII} function structure (i.e., where the *sol* of Figure 72 is tonic), it is easy to imagine that the root of the *fa* function (i.e., VII) can be an effective communicator of this function. (For example, consider the pitch B-flat, as root of a B-flat major triad, within a C Mixolydian piece predominately using the chords C, F, and B-flat major.)

The roots and fifths of Figure 72 are fixed by the requirement that each triad be a consonant triad. This fixity is represented by illustrating these chord factors with open

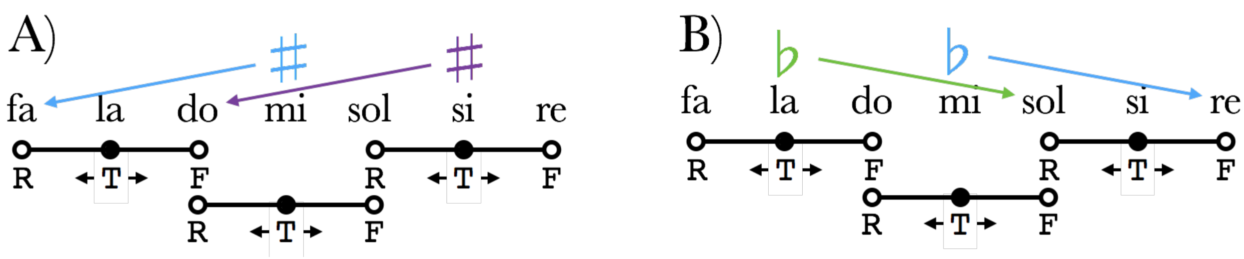
[109] See Harrison 1994, 54–56 for further discussion of these questions.

nodes. The thirds of each triad are thus the only variable pitch-classes, and they are therefore represented with a filled-in node together with arrows indicating that it may be higher (i.e., a major third) or lower (a minor third). (The eight different combinations of two qualities and three thirds give the eight scales of Figure 70.) Harrison (1994, 50) notes that thirds are "differentiated from [roots] in that they communicate not only functional attitudes but also the modal character" of each function and, by extension, the resulting scale and mode.[110] (In fact, their communication of modal character differentiates them not only from roots but also from fifths, but for Harrison, fifths are only minimally capable of communicating function.) The mutability of thirds is the reason for this unique contribution of thirds to mode within a cycle-of-fifths function structure.

The fixity of root and fifth and mutability of third in a cycle-of-fifths function structure also has implications concerning the tendency-tone structure of the resultant modes. For instance, if the third of the two sharp-side functions (the *do* and *sol* functions) is major, then it will be a minor second below the root of the adjacent flatwards function (consider Figure 73A). In other words, when the *sol* function has a major-third *si*, it will be a minor second below the root of the *do* function—this is the familiar leading tone of the major mode—and likewise, when the *do* function has a major-third *mi*, it will be a minor second below fa. On the other hand, if the third of the two flat-side functions (the *fa* and *do* functions) is minor, then it will be a minor second above the fifth of the adjacent sharpwards function (consider Figure 73B). In this case, when the *fa* function has a minor-third *la*, it will be a minor second above the fifth *sol* of the *do* function—this is the "dual leading tone" $\hat{6}$ of the minor mode—and likewise, when *mi* is a minor third above do, it will be a minor second above *re*.

[110] I find Harrison's term "functional attitudes" somewhat obscure here. I understand him to mean that they simply communicate the function to which they belong (e.g., tonic, subdominant, etc.).

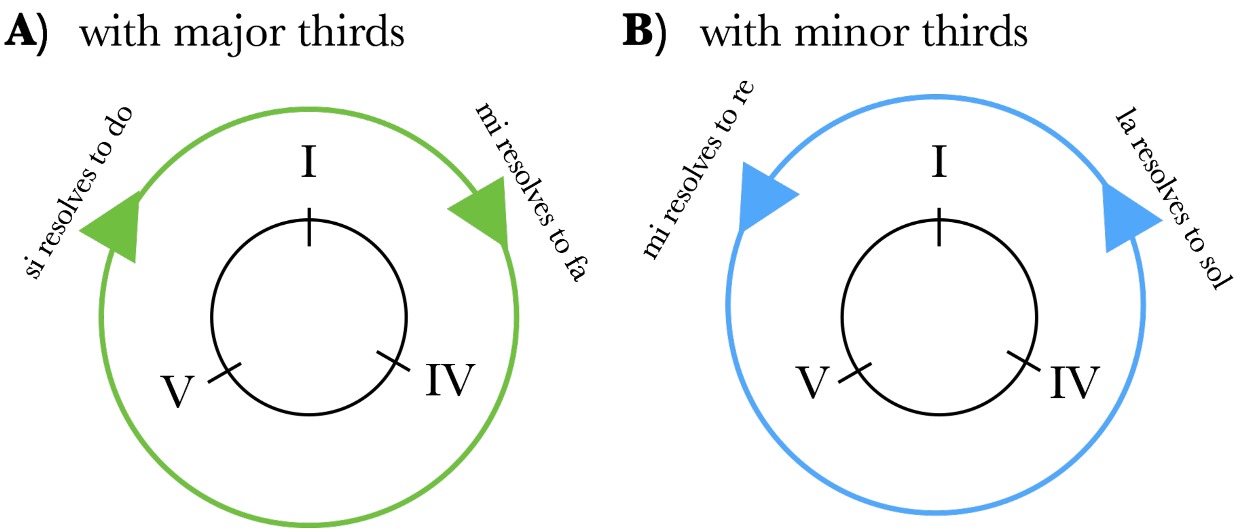
Figure 73. Resolution tendencies of third qualities in a cycle-of-fifths function structure.[111]



Function structures, as realized in actual music, often function cyclically, and this cycle often has a polarity, tending to proceed in one direction rather than the other. (Such cycles can be established by the operation of functional substitutions, as done in Section 4.1 above.) For instance, the {I, IV, V} function structure of common-practice tonality typically proceeds from I to IV, from IV to V, and from V to I, and not the other way around. If thirds are understood as tending to resolve to roots or fifths (as in Figure 73), then the polarity of such cycles is perhaps implicit in the quality of third. For example, if the thirds of all three functions are major, as in the major mode, then V will tend to move to I, with *si* resolving to do, and I will tend to move to IV, with *mi* resolving to *fa* (see Figure 74A). Therefore, the cyclical polarity of the function structure of common-practice tonality can be seen as implicit in the major mode, given certain assumptions. On the other hand, in the minor mode, where all three functions are minor, the "natural" polarity is reversed, under the same assumptions (see Figure 74B). From this perspective, the fact that common-practice music preserves the characteristic polarity of the major mode even in minor-mode music suggests a motivation for the greater chromaticism often found in the latter mode. In an alternate musical universe where the minor mode polarity was the norm, perhaps it would be the major mode that was more associated with chromatic alteration.

[111] The sharps above *mi* and *si* in the first half of this figure are meant to indicate that these pitches are relatively raised (i.e., are major, rather than minor, thirds), and not to literally inflect the pitches *mi* (= E-natural) and *si* (= B-natural).

Figure 74. Polarity of {I, IV, V} function structures, as inferred from the minor second resolution of chordal thirds.



4.4 Web apps 4.1, 4.2, and 4.3

Three web apps of similar functionality are relevant to the discussion in this chapter:

Web app 4.1, "Function Structures," allows the user to combine any scale and complete function structure, and hear the application of this function structure in a note-against-note harmonization using the heuristic described in footnote 100 above.[112] The user can either apply 1) a totally random scale and function structure, 2) a random scale and function structure using only major and minor triads, but permitting enharmonics such as {C, F♭, G}, or 3) a random scale and function structure using only correctly-spelled major and minor triads. The user can also construct a scale (only 7-scales are permitted, however) and choose a function structure herself.

Web apps 4.2, "Cycle-of-Fifths Function Structures," and 4.3, "Cycle-of-Seconds Function Structures," work very similarly.[113] The difference is that they reproduce in an

[112] http://sailor.music.mcgill.ca/funcstructs
[113] http://sailor.music.mcgill.ca/cycleoffifthsmodes
http://sailor.music.mcgill.ca/cycleofsecondsmodes

interactive form the tables from Figure 70 and Figure 71, respectively. The user can select any of the modes in the table, and then hear the “musical” results when this is applied to the harmonization of a melody.

Conclusion

I often visualize the range of possible tonalities as a landscape created by the many varied parameters that combine to define a musical style. On this landscape, different styles correspond to different locales. For instance, in the territory explored over the course of Western music history, we find locales (i.e., musical styles) that do not have a stable background collection but maintain a triadic basis and conservative dissonance treatment, such as Gesualdo. But we also find other locales that maintain a stable background collection and a triadic basis but do not have conservative dissonance treatment, such as the "melodic-harmonic divorce" (Temperley 2007) of much recent popular music. If, upon this landscape, we set off from Chopin in one direction, we may end up at Wagner, but if we set off in another, we may arrive at Fauré.

In examining some ways in which we might extend the scalar resources of Western music, the aims of this thesis have been twofold. First, to explore the abstract structure of these traditional scalar resources in the hope that we may, second, discover some areas of the "scalar landscape" that lie beyond the boundaries of traditional, common-practice tonality. Within the scope of a Master's thesis, I have only been able to point the way to a few of these territories, or sketch out their bare outlines, rather than give a complete account of them. I have not, moreover, had the space to apply the ideas to the actual analysis of existing musical works in late- or extended-tonal styles. This sort of analysis is, of course, one of the main activities of the normal practice of music theory, and it is an important check and balance upon our theorizing. One of the ways in which I intend to develop this research in the near future is through such analytical applications. (A first such foray in this direction is Sailor 2017).

Nevertheless, I believe that such analytical applications are limited by the fact that there simply does not exist much music—or, at the very least, much well-known music—exploring many of the new territories I have sketched out. Because I come to music theory as a composer, I have not shied away from describing these unexplored territories.

Nevertheless, we might well wish to ask: why have these territories remained so little explored? Why do we—at least as Western art musicians—make so little music along these alternative paths? I would like to propose two possible explanations (although I by no means wish to suggest that these are the only possibilities):

1) There may be a strong inertial constraint upon the evolution of tonal systems. This could mean that, even if the initial choice of scale is relatively arbitrary, once a scale—such as the diatonic—becomes established at the center of the musical practice of a culture, it becomes very difficult to dislodge from this position. An analogy can be made to a marble poised upon the edge of a bowl. Such a marble can easily be dislodged in either direction by a small disturbance. If it falls one way, it will roll down into the center of the bowl, while if it falls in the other, it will end up outside of the bowl. The location in which the marble ultimately finds itself may be a product of accident, but it is nevertheless permanent: the marble shall never, of its own accord, pick itself up and move from the inside of the bowl to the outside, or vice versa. In the case of music, such inertia might occur either for aesthetic reasons—i.e., once we have learned to enjoy a certain scale or a chord, we hear deviations from it as less pleasing, regardless of their intrinsic characteristics. Or the inertia might occur for practical reasons—i.e., once musicians have learned to hear and perform a certain scale or tuning, hearing and performing another requires a considerable, perhaps immense, effort. Moreover, the combination of aesthetic and practical reasons may exceed the sum of their parts: if learning to perform novel tonalities requires an immense effort on the part of musicians *and* we perceive the result as less pleasing due to our habituation to the materials of Western tonality, these novel tonalities may be doubly discouraged.
2) The expansion of tonality may be constrained by certain more-or-less intrinsic factors. For instance, all non-diatonic 7-scales contain more augmented and

diminished intervals than the diatonic. It has been my experience as a musician that such augmented and diminished intervals strongly "colour" any music that contains them. This coloration may constrain the use of these scales, meaning that the very small number of possible tonal objects that contain only the more neutrally coloured major and minor intervals will have a certain intrinsic advantage. In other words, there may be a small range of tonal objects capable of a relatively wide range of coloration and expression, and a wider range of tonal objects capable of a relatively narrower range. I am aware that, to the extent that this sort of argument suggests that certain musical objects are more "natural," it is problematic to say the least. I would never presume to place hard boundaries around the borders of possible musical styles, nor do I mean to suggest that it is at all easy to determine which sorts of musical stimulus are more "natural" (or that our intuitions about such questions cannot lead us astray). Nevertheless, I think it would be equally foolish to dismiss out of hand the idea that there are at least some natural constraints that tend to bias the sort of musical objects that humans find themselves making use of. For instance, the ubiquity of the just fifth in the world's music seems hardly likely to be a coincidence, and the ubiquity of octave equivalence still less so.

Insofar as such explanations tend to suggest that the alternative means of tonal organization explored in this thesis are unlikely to be of wider use, they can be somewhat discouraging for those of us interested in writing music that exhibits such novel organization. Nevertheless, as a creative musician, the process of exploring musical possibilities from the perspective of this thesis has been very inspiring. As a composer, the possibilities of alternative functional harmony stand before me like a clouded distant mountain range whose peaks I am only too eager to discover whether I have the mettle to ascend.

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