Numerical Methods Assignment 2

Malcolm Watt

March 6, 2018

Chaper 3 Exercises

Question 3.3

The linear least squares system is given as follows:

$$Ax = \begin{bmatrix} 1 & e \\ 2 & e^2 \\ 3 & e^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = b$$

Question 3.18

Consider the vector

$$a = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

.

(a)

We need to annihilate the third component of a, using an elementary elimination matrix. The easiest is to subtract twict the first column from the last column. Therefore the elementary row elimination matrix is given as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(b)

Now we need to do the same thing but with a Householder transformation. We want to apply two Householder transformations, the first H_1 eliminating all but the first element, and the second H_2 eliminating all but the second element. Then it stand that the Householder matrix H which eliminates only the third element is simply the matrix multiplication of $H = H_1 H_2$.

Let's find H_1

$$v = a - \alpha(e_1) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha = \pm||a||_2 = \pm\sqrt{29}$. Since a_1 is positive, we should use the negative α to avoid cancellation.

Following this we have

$$v = a - \sqrt{29}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2-\sqrt{29}\\3\\4\end{bmatrix}$$

and Householder matrix

$$H_1 = I - 2\frac{vv^T}{v^Tv} = \begin{bmatrix}0.6885828 & 0.37139068 & -0.62283441\\0.37139068 & 0.55708601 & 0.74278135\\-0.62283441 & 0.74278135 & -0.24566881\end{bmatrix}$$

by a similar process we find

$$H_2 = \begin{bmatrix}1.45796221 & 0.64642473 & -1.19938586\\0.64642473 & 1.22908525 & 1.43036648\\-1.19938586 & 1.43036648 & -0.68704746\end{bmatrix}$$

and the desired

$$H = H_1 + H_2 = \begin{bmatrix}1.37716559 & 0.74278135 & -1.24566881\\0.74278135 & 1.11417203 & 1.48556271\\-1.24566881 & 1.48556271 & -0.49133762\end{bmatrix}$$

(c)

We pick the Given's Rotation matrix such that G is of the form

$$\begin{bmatrix}1 & 0 & 0\\0 & c & s\\0 & -s & c\end{bmatrix}$$

Since $|a_3| > |a_2|$ we use the cotangent formulation to find c and s. We have $\tau = c/s = \frac{a_3}{a_2} = 0.75$ and therefore

$$s = \frac{1}{\sqrt{1+\tau^2}} = \frac{4}{5}$$

and

$$c = \frac{3}{5}.$$

so the givens rotation matrix is

$$G = \begin{bmatrix}1 & 0 & 0\\0 & \frac{3}{5} & \frac{4}{5}\\0 & \frac{-4}{5} & \frac{3}{5}\end{bmatrix}$$

Question 3.24

Alright, well the obvious first step is to figure out what is the QR factorization by Householder...

We expect an $m * n$ matrix to have dominant term $n^2 m - \frac{n^3}{3}$.

Let's go down the diagonal: The selection of v_i for the i^{th} diagonal term requires the calculation of the 2-norm. Starting at the i^{th} position this requires $m - i$ multiplications, one square root (which has the same complexity as a division) and 1 addition (that is the norm vector plus the column we are annihilating).

The algorithm requires

$$\sum_{k=1}^{n}(2(m-k) + \sum_{j=k}^{n} 2(m-k)) = \sum_{k=1}^{n}(2(m-k) + 2(m-k)(n-k))$$

$$= \sum_{k=1}^{n} 2(mn+m)n - (m+n+1)n(n+1) + n(n+1)(2n+1)/3$$

from which the dominant term is $mn^2 - n^3/3$ multiply-add operations.

Question 3.25

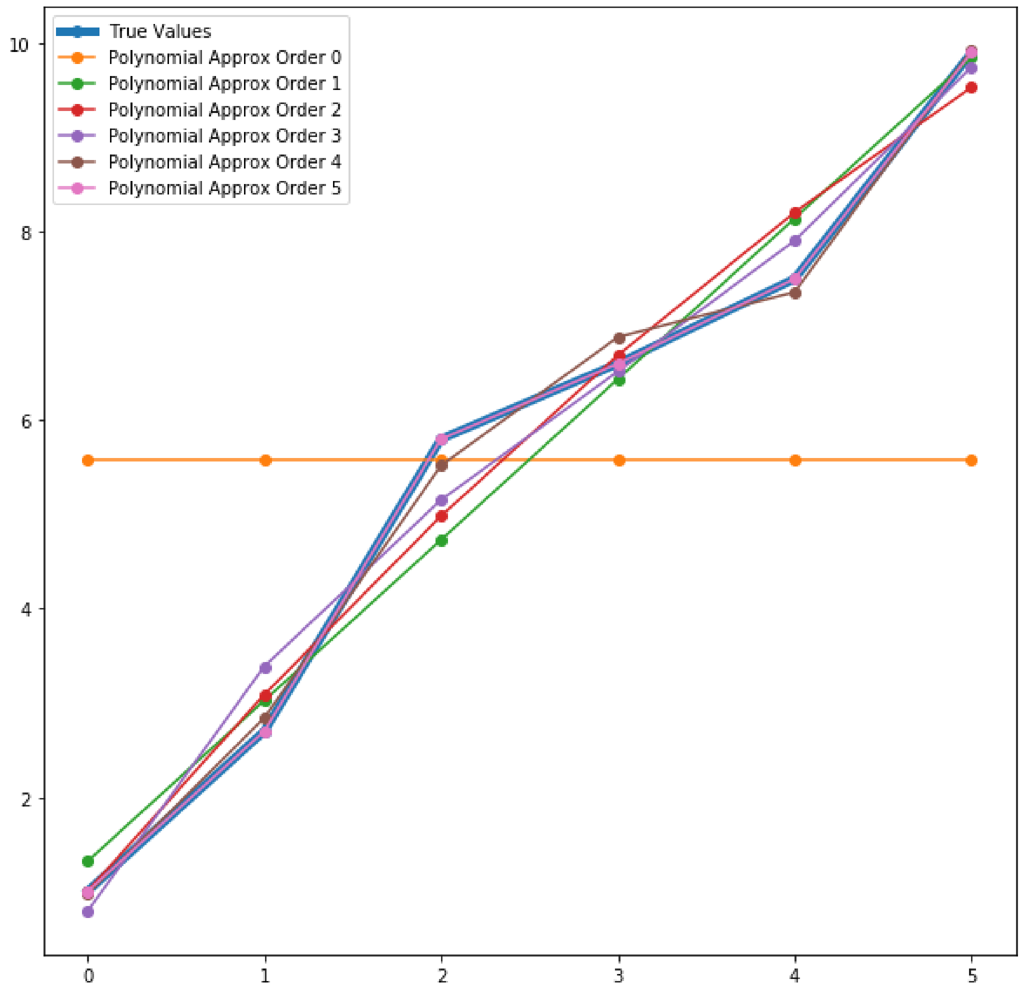
Chapter 3 Programming Questions

Question 3.1

The code for this question can be found in assignment2.ipynb.

The results are shown in the following figure for different orders of polynomials. Which polynomial fits this the best is the question. Now, if there is absolutely no error, and we know these are the correct points, then the order 5 polynomial is the best fit. Obviously the 0 order polynomial is basically garbage. If we know the data is relatively noisy, then we'll want to pick one of the least aggressive approximations (i.e. the linear fit). Really it depends on the data what the actual correct answer is, the key thing to keep in mind is that the higher the order of the polynomial, the more it will fit to the particular examples given, and perhaps not generalize well to a noisier set of results that are not included in this "training set".

Polynomial approximations from question 3.1



Question 3.3

The results and discussion can be found in the ipython notebook.

Question 3.4

The results and discussion can be found in the ipython notebook.

Question 3.5

The results and discussion can be found in the ipython notebook.

Chapter 4 Exercises

Question 4.3

Find the eigenvalues and corresponding eigenvectors for the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A - \lambda I = 0$$
$$\begin{pmatrix} 1-\lambda & 2 & -4 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix} X = 0$$

$$det(\begin{pmatrix} 1-\lambda & 2 & -4 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix}) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Which has roots at $\lambda = 3, \lambda = 2, \lambda = 1$. The corresponding eigenvectors are for $\lambda = 1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, for $\lambda = 2 \rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, for $\lambda = 3 \rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Question 4.9

PART A

Let $Ax = \lambda x$.

$$\begin{aligned} \lambda \bar{x}^T x &= \bar{x}^T (\lambda x) \\ &= \bar{x}^T A x \\ &= (A^T \bar{x})^T x \\ &= (A\bar{x})^T x \\ &= (\bar{A}\bar{x})^T x \\ &= (\bar{\lambda}\bar{x})^T x \\ &= \bar{\lambda}\bar{x}^T x \end{aligned}$$

Since $x \neq 0$, $\bar{x}^T x \neq 0$ and $\lambda = \bar{\lambda}$ which means λ is real.

PART B Again, we have $Ax = \lambda x$.

Multiplying by x^H, that is $\bar{x}^T$, from the left, we get

$$
\begin{aligned}
x^H(Ax) &= x^H(\lambda x) \\
&= \lambda x^H x \\
&= \lambda ||x||.
\end{aligned}
$$

Now if we take the Hermitian of both sides:

$$x^H A^H x = \bar{\lambda}||x||.$$

Since A is a Hermitian matrix, we have $A^H = A$, so then the lhs becomes

$$
\begin{aligned}
x^H A^H x &= x^H Ax \\
&= x^H \lambda x \\
&= \lambda ||x||.
\end{aligned}
$$

and we then obtain

$$\lambda ||x|| = \bar{\lambda}||x||.$$

Since x is an eigenvector, it is by definition not the zero vector and so the length of the vector is not zero. If we divide both sides of the above equation by the length (we don't have to worry about division by zero), we get

$$\lambda = \bar{\lambda}$$

therefore λ is a real number.

Since λ is any eigenvalue, we can conclude that for all eigenvalues of a hermitian matrix A, we have real valued eigenvalues.

Question 4.10

This can be proven by counter-argument. If the eigenvalue is to be positive, it can not be negative or 0.

Let's consider the case where the eigenvalue is 0. If $\lambda = 0$, then there must exist an eigenvector x so that $Ax = 0$, however, if we have $Ax = 0$, then $x^T Ax = 0$, so A is not positive definite.

Now let's consider the case where the eigenvalue is negative. If we have $\lambda < 0$, then there is some eigenvector x so that $Ax = \lambda x$, which then means we have $x^T Ax = \lambda |x|^2$. Since we have a negative eigenvalue, and the squared eigenvector is positive, we necessarily have a negative right hand side, and so the matrix A does not satisfy the condition for being positive definite.

Therefore, if a matrix A is positive definite, it necessarily has positive eigenvalues.

Question 4.15

In order for A to be invertible in the first place, we need the eigenvalues of A to be non-zero. So with

$$Ax = \lambda x$$

we can multiply by A^{-1} from the left to get

$$A^{-1}Ax = \lambda A^{-1}x$$
$$x = \lambda A^{-1}x$$

and rearranging it to look more like the standard eigenvalue equality we then have

$$A^{-1}x = \frac{1}{\lambda}x$$

so all eigenvalues of a non-singular matrix A are the reciprocal of corresponding eigenvalues for the inverse matrix A^{-1}.

Note that this equality has not changed the x in the eigenvalue equation, therefore for each eigenvalue λ and eigenvector x of A, we have a corresponding eigenvalue $\frac{1}{\lambda}$ and unchanged eigenvector x for A^{-1}.

Question 4.19

Question 4.25

Chapter 4 Programming Problems

Question 4.2

The results and discussion can be found in the ipython notebook.

Question 4.5

The results and discussion can be found in the ipython notebook.

Question 4.7

The results and discussion can be found in the ipython notebook.

Question 4.9

The results and discussion can be found in the ipython notebook.

Question 4.15

The results and discussion can be found in the ipython notebook.